

SOBOLEV REGULARITY FOR OPTIMAL TRANSPORT MAPS OF
NONCONVEX PLANAR DOMAINS*

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Abstract. We prove a sharp global $W^{2,p}$ estimate for potentials of optimal transport maps that take a certain class of nonconvex planar domains to convex ones.

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1. Introduction. Optimal transport maps play an important role in physics, geometry, economics, and meteorology. The regularity of optimal transport maps is a delicate matter that for the most part has focused on the case that the source and target domains are convex. However, this condition is not satisfied in many applications. In this paper, we initiate the study of the Sobolev regularity of optimal transport maps in the plane, where the source domain is nonconvex.

The setting is as follows. Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^2 of unit area. We assume that Ω_2 is convex. Then the optimal transport map from Ω_1 to Ω_2 is the gradient map of a convex function u on \mathbb{R}^2 which satisfies (see [7])

$$(1) \quad \det D^2u = \chi_{\Omega_1}$$

in the Alexandrov sense, u is smooth and locally uniformly convex in Ω_1 , and

$$(2) \quad \nabla u(\Omega_1) = \Omega_2.$$

We assume further that Ω_1 is a convex domain Ω_0 with a finite number of disjoint, $C^{1,1}$, uniformly convex holes a positive distance δ from $\partial\Omega_0$ and from each other removed. Our main result is the following.

THEOREM 1.1. *We have $u \in C^{1,1/2}(\overline{\Omega_1})$ with norm depending only on the diameters of Ω_1 and Ω_2 , δ , and the lower and upper bounds for the boundary curvatures of the holes in Ω_1 . We also have $u \in W^{2,p}(\Omega_1)$ for any $p < 2$, with norm depending only on the same quantities and p .*

Theorem 1.1 is sharp. To see this, consider the radially symmetric example where Ω_1 is an annulus with inner radius r , Ω_2 is a disk, and the potential is

$$(3) \quad u(x) = \int_0^{|x|} (s^2 - r^2)_+^{1/2} ds.$$

Below, we will refer to (3) as the model example.

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One motivation for Theorem 1.1 comes from the semigeostrophic equations (SGEs) from meteorology. The SGEs lead one to consider optimal transport maps that take a bounded density on the torus to the uniform one (see, e.g., [15]). When the source density is bounded between positive constants, $W^{2,1}$ estimates for the potential are available [13], [14], [24], which lead to long-time existence results for the SGEs [2], [3], [15]. However, in physically interesting cases, the SGEs involve optimal transport maps where the source density is allowed to vanish. In this case, $W^{2,1}$ estimates for the potential do not always hold (see [22]). An important special situation is when the source density is the characteristic function of a domain (which need not be convex), as in the situation of Theorem 1.1. For the SGEs, this corresponds to a fully nonlinear analogue of the vortex patch problem for the two-dimensional Euler equations. Global $W^{2,1}$ estimates for optimal transport maps of nonconvex domains may be useful for extending long-time existence results for the SGEs to this situation.

More generally, Theorem 1.1 can be viewed as a step toward obtaining global regularity results for optimal transport maps of general nonconvex source domains. The global regularity of optimal transport maps in the case of convex source and target domains is well studied. Caffarelli proved that, in this case, the potentials are $C^{1,\alpha}$ up to the boundary and $C^{2,\alpha}$ up to the boundary provided that the domains are C^2 and uniformly convex [6]. Here α is small. The conditions on the domains required for global $C^{2,\alpha}$ regularity of the potential were recently relaxed to $C^{1,1}$ and merely convex [9] and even slightly nonconvex but close to convex in the $C^{1,1}$ sense [10]. In two dimensions, Savin and Yu showed that convexity of the domains is enough to get global $W^{2,p}$ regularity for any $p < \infty$ [23]. As for the case of nonconvex source domain, in [4], the authors obtain global $C^{1,\alpha}$ estimates for potentials of optimal transport maps in any dimension when the densities are bounded between positive constants, the target domain is convex, and the source domain is a convex set with finitely many convex holes removed, using ideas from [5]. (Again, here α is small.) Our methods (described below) are quite different from those in [4], and the smoothness of the densities and the regularity properties of the holes play a delicate role in our analysis. We remark that our methods in fact apply near any “uniformly concave” part of the boundary of a general smooth planar source domain.

Our strategy is as follows. First, we may focus our attention on a neighborhood of the holes in Ω_1 (the “concave part” of the boundary of Ω_1), thanks to the work of Savin and Yu, which shows the $W^{2,p}$ regularity of u (for any p) near the “convex part” $\partial\Omega_0$ of the boundary of Ω_1 [23]. We carefully analyze the geometry of the sections of u (defined in section 2), which are centered at concave boundary points. We show that there are three possible cases, all of which are “good.” The first case is that the complement of Ω_1 fills only a tiny fraction of the section. In this case, morally speaking, u solves $\det D^2u = 1$ in the whole section, and we can control section geometry at smaller scales using the regularity theory for the Monge–Ampère equation. The second case is that the complement of Ω_1 fills a positive universal fraction of the section, and the long axis of the section is transversal to the boundary. In this case, we show that renormalization by an affine transformation flattens the boundary, and we are again in a good situation where section geometry can be controlled at smaller scales using Pogorelov-type estimates for the Monge–Ampère equation. The last case is that the complement of Ω_1 fills a positive universal fraction of the section, and the long axis of the section is roughly tangent to the boundary. In this case, we are in a situation that resembles what happens at every inner boundary point for the model example, which has the desired regularity properties. Our analysis near the holes is valid for any solution of the Monge–Ampère equation and does not use the convexity of Ω_2 .

In the course of the proof, we also prove a new interior second derivative estimate for solutions to $\det D^2w = \chi_{\{x_2 > 0\}}$, which is special to two dimensions. The classical Pogorelov estimate bounds the tangential second derivative w_{11} . Although this suffices for our application, using the partial Legendre transform, we are also able to bound the ratio $|w_{12}|/w_{11}$ from above (see Proposition 2.3). As a result, the sections of w centered on $\{x_2 = 0\}$ are well approximated by ellipsoids whose axes are aligned with the coordinate axes. This result simplifies our proof and may be useful for future applications.

The paper is organized as follows. In section 2, we discuss some preliminary results about the geometry of centered sections as well as some Pogorelov-type estimates (including the one mentioned in the previous paragraph). In section 3, we prove several key lemmas, corresponding to the three scenarios mentioned above. In section 4, we prove Theorem 1.1. In section 5, we discuss some future directions. Finally, in Appendix A, we prove some of the preliminary results.

2. Preliminaries. For the remainder of the paper, we fix a constant $\delta > 0$. We let \mathcal{F}_δ denote the space of convex functions on \mathbb{R}^2 that satisfy (1) and (2), where Ω_1 and Ω_2 have unit area and are contained in $B_{\delta^{-1}}$, Ω_2 is convex, and the source domain Ω_1 consists of a convex domain Ω_0 with convex holes removed, where the holes are separated a distance at least δ from one another and from the boundary of Ω_0 and the boundaries of the holes have lower and upper bounds δ, δ^{-1} on their curvature. (Here Ω_i are not fixed; they are any domains satisfying the above conditions.) We note that \mathcal{F}_δ is a compact family; namely, any sequence in \mathcal{F}_δ contains (after possibly adding constants) a subsequence that converges locally uniformly on \mathbb{R}^2 to a function in \mathcal{F}_δ . The local uniform convergence follows from the fact that the gradients lie in $B_{\delta^{-1}}$ and the Arzelà–Ascoli theorem. The fact that the limit lies in \mathcal{F}_δ uses the weak convergence of Monge–Ampère measures under local uniform convergence [20]. We call constants depending only on δ universal, and we say that positive quantities a and b satisfy $a \sim b$ if their ratio is trapped between positive universal constants. We call constants “absolute” if they are fixed positive numbers independent of δ . (Generally, these are dimensional constants which become absolute in view of the fact that we are working in the plane.) Finally, we say that $a \approx b$ if their ratio is trapped between absolute constants.

Let $u \in \mathcal{F}_\delta$. For any $x \in \overline{\Omega}_1$ and $h > 0$, there exists an affine function $L_{x,h}$ such that

$$L_{x,h}(x) = u(x) + h$$

and such that the set $\{u < L_{x,h}\}$ is bounded and has center of mass x (see [6]). We call $\{u < L_{x,h}\}$ the centered section of height h at x , and we denote it by $S_h^u(x)$. One can show that u is not linear when restricted to any line segment centered at a point in $\overline{\Omega}_1$ (see the appendix). Combined with a compactness argument, this shows that there exists a universal modulus of continuity ω such that for any $x \in \overline{\Omega}_1$ and any $h < 1$,

$$(4) \quad \text{diam}(S_h^u(x)) \leq \omega(h).$$

In particular, for $h < c_0$ universal, we have that $S_h^u(x)$ intersects at most one connected component of Ω_1^c for any $x \in \overline{\Omega}_1$. Below, we will always assume that $h \in (0, c_0)$, and we will only consider sections centered in $\overline{\Omega}_1$ that are either contained in Ω_1 or intersect a hole in Ω_1 .

Such centered sections satisfy the area estimate

$$(5) \quad |S_h^u(x)| \approx h.$$

This estimate follows from the absolute positive density of Ω_1 in such sections; see Lemma A.3 in the appendix for a proof of (5). By a version of John's lemma, there exist rectangles $R_h^u(x)$ centered at 0 of area $4h$ and $d_0 \approx 1$ such that

$$(6) \quad x + d_0 R_h^u(x) \subset S_h^u(x) \subset x + (d_0)^{-1} R_h^u(x).$$

We denote the short and long side lengths of $R_h^u(x)$ by $2\lambda_h^u(x)$ and $2\Lambda_h^u(x)$, respectively (note that $\lambda_h^u(x)\Lambda_h^u(x) = h$), and we define the eccentricity of $R_h^u(x)$ by the quantity

$$\eta_h^u(x) = \frac{\Lambda_h^u(x)}{\lambda_h^u(x)}.$$

Finally, we have the following engulfing property (see the appendix), which allows us to compare sections in Ω_1 tangent to a hole with a section centered on the boundary of the hole.

PROPOSITION 2.1. *If $u \in \mathcal{F}_\delta$, $y \in \overline{S_h^u(x)} \cap (\partial\Omega_1 \setminus \partial\Omega_0)$, and $S_h^u(x) \subset \Omega_1$, then*

$$S_h^u(x) \subset y + R_{c^{-1}h}^u(y)$$

for some universal $c > 0$.

We now state some Pogorelov-type estimates. The first result is Pogorelov's interior C^2 estimate (see, e.g., [20]).

PROPOSITION 2.2. *If $\det D^2w = 1$ in $S_1^w(0)$ and $B_\lambda \subset S_1^w(0)$ for some $\lambda > 0$, then $|D^2w| < \kappa^{-1}$ in $\frac{1}{2}S_1^w(0)$ and*

$$B_{\kappa t^{1/2}} \subset S_h^w(0) \subset B_{\kappa^{-1}t^{1/2}}$$

for all $t < 1$ and some $\kappa(\lambda) > 0$.

Combining Proposition 2.2 with the affine invariance of the Monge–Ampère equation and the area estimate (5), we have

$$(7) \quad |D^2u(x)| \approx \eta_h(x)$$

whenever $S_h^u(x) \subset \Omega_1$. To see this, assume after a translation and a rotation that $x = 0$ and that the long side of $R_h^u(0)$ is horizontal. Then

$$u(x_1, x_2) := hw(x_1/\Lambda_h^u(0), x_2/\lambda_h^u(0)),$$

where w satisfies the conditions of Proposition 2.2 with λ absolute. Thus, $|D^2u(0)| \leq h\lambda_h^u(0)^{-2}|D^2w(0)| \leq \kappa^{-1}h\lambda_h^u(0)^{-2}$ for some $\kappa > 0$ absolute. Using that $h = \lambda_h^u(0)\Lambda_h^u(0)$ gives the upper bound in (7). The lower bound in (7) follows similarly using that $w_{22}(0) \approx 1$, which comes from the equation solved by w and the absolute upper bound on its Hessian at 0.

The next estimate is a variant of Pogorelov's interior C^2 estimate with flat boundary, which to our knowledge is new.

PROPOSITION 2.3. *If $\det D^2w = \chi_{\{x_2 > 0\}}$ in $S_1^w(0)$ and $B_\lambda \subset S_1^w(0) \subset B_{\lambda^{-1}}$ for some $\lambda > 0$, then*

$$\sup_{\frac{1}{2}S_1^w(0) \cap \{x_2 > 0\}} \left(w_{11} + \frac{|w_{12}|}{w_{11}} \right) \leq \kappa^{-1}$$

for some $\kappa(\lambda) > 0$.

The upper bound on w_{11} is the classical Pogorelov estimate (see [6]) and does not use that we are working in the plane. The upper bound on $|w_{12}|/w_{11}$ uses that we are in the plane.

Proof. We may assume after subtracting a linear function that $w|_{\partial S_t^w(0)} = 0$. Let w^* denote the partial Legendre transform of w (the Preliminaries section of [11] discusses the definition and relevant properties, and we recall these in section A.5 of the appendix for the reader's convenience). The function w^* is convex in the first variable and concave in the second and formally solves

$$\chi_{\{x_2 > 0\}} w_{11}^* + w_{22}^* = 0$$

in $B_{\kappa_1}(0)$ for some $\kappa_1(\lambda) > 0$. More precisely, w^* is harmonic in $\{x_2 > 0\}$ and linear on vertical segments in $\{x_2 < 0\}$, and, moreover, w_2^* has the same limit from above and below on $\{x_2 = 0\}$ along vertical lines. It is not hard to verify the first two properties by approximating $\chi_{\{x_2 > 0\}}$ with smooth positive functions of x_2 . The third property can be verified using that $w \in C^1$ [1], [18]. Since w^* is linear on vertical segments in $\{x_2 \leq 0\}$, we have

$$w_2^*(x_1, 0) = a^{-1}(w^*(x_1, 0) - w^*(x_1, -a))$$

for any $a > 0$. Choosing $a = \kappa_2(\lambda) > 0$ and using that w^* is convex (hence locally Lipschitz with the Lipschitz constant bounded by a constant depending only on λ) in the horizontal directions, we conclude that w_2^* is Lipschitz on $\{x_2 = 0\}$. In particular, w_2^* is harmonic in $B_{\kappa_1} \cap \{x_2 > 0\}$ and Lipschitz on $\{x_2 = 0\}$ (with the Lipschitz constant bounded above by a constant depending only on λ). It follows from harmonic function theory that $|w_{12}^*| < \kappa_3^{-1}$ in $B_{\kappa_1/2} \cap \{x_2 > 0\}$ for some $\kappa_3(\lambda) > 0$. To see this, one can apply Lemma A.6 of the appendix, appropriately rescaled. Using the relation

$$w_{12} = -w_{12}^* w_{11}$$

(see (18)), we obtain the desired estimate on $|w_{12}|/w_{11}$. \square

As a result of Proposition 2.3, under the same assumptions, we can say that $S_t^w(0)$ is approximated by (contains and is contained in dilations by constants depending only on λ of) a rectangle with axes that are aligned with the coordinate axes for all $t < 1$. Indeed, if not, then $S_t^w(0)$ is approximated by (contains and is contained in dilations by absolute constants times) an ellipsoid of the form $A_t B_1$, where

$$A_t(x_1, x_2) = \left(A\sqrt{t}(x_1 + Kx_2), A^{-1}\sqrt{t}x_2 \right),$$

$A \geq \kappa > 0$ (this follows from the upper bound on w_{11}), and $|K| \gg 1$. (Here and for the rest of this paragraph, κ denotes a small positive constant depending on λ .) The function

$$v(x) = \frac{1}{t}w(A_t x)$$

satisfies the conditions of Proposition 2.3 with λ absolute (call it $2r$). In addition, we have

$$w_{12} = v_{12} - Kv_{11}.$$

In $B_r(re_2)$, we can find points where $v_{11} \geq k$ and $|v_{12}| < k^{-1}$ for some absolute constant $k > 0$ to arrive at a contradiction of Proposition 2.3 when $|K|$ is sufficiently large depending on λ .

3. Key lemmas. In this section, we prove some lemmas that control the geometry of sections of $u \in \mathcal{F}_\delta$ centered at concave boundary points in various scenarios. Below, we will use several times the standard fact that if w_k are convex functions with $B_\lambda \subset S_1^{w_k}(0) \subset B_{\lambda-1}$ for some $\lambda > 0$, $w_k|_{\partial S_1^{w_k}(0)} = 0$, and $\det D^2 w_k$ are uniformly bounded above, then a subsequence of w_k converges uniformly to a convex function w satisfying the same properties, and the Monge–Ampère measures converge weakly to that of the limit.

The first lemma deals with the case that Ω_1^c bites only a small fraction of the section.

LEMMA 3.1. *There is an absolute constant $c_1 > 0$ such that the following holds. Given $M > 1$, there exists $\epsilon > 0$ depending on δ and M such that if $u \in \mathcal{F}_\delta$, $x \in \partial\Omega_1 \setminus \partial\Omega_0$, and*

$$\frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} \leq \epsilon,$$

then

$$\eta_{h/M}^u(x) \leq c_1^{-1} \eta_h^u(x)$$

and

$$R_{h/M}^u(x) \subset c_1^{-1} M^{-1/2} R_h^u(x).$$

Proof. Assume by way of contradiction that the lemma is false. Then there exist a sequence $u_k \in \mathcal{F}_\delta$, points x_k on the boundaries of the holes in the source domains, and $h_k > 0$ such that the area fraction ϵ_k of the complements of the source domains in $x_k + R_{h_k}^{u_k}(x_k)$ tends to zero, but the conclusions do not hold for a choice of absolute constant c_1 to be explained below and all k . After performing a rigid motion, we may assume that $x_k = 0$ and that $R_{h_k}^{u_k}(0)$ have short side vertical. Then up to adding affine functions and taking a subsequence, the rescalings $w_k := h_k^{-1} u_k(\Lambda_{h_k}^{u_k}(0)x_1, \Lambda_{h_k}^{u_k}(0)x_2)$ converge locally uniformly to a function w that satisfies the conditions of Proposition 2.2, with $\lambda = a_0 > 0$ absolute. Applying Proposition 2.2, we conclude that $B_{2a_1 M^{-1/2}} \subset S_{1/M}^w(0) \subset B_{(2a_1)^{-1} M^{-1/2}}$ for some $a_1 > 0$ absolute. In particular, $B_{a_1 M^{-1/2}} \subset S_{1/M}^{w_k}(0) \subset B_{(a_1)^{-1} M^{-1/2}}$ for k large. It follows after scaling back that

$$a_2 M^{-1/2} R_{h_k}^{u_k}(0) \subset R_{h_k/M}^{u_k}(0) \subset (a_2)^{-1} M^{-1/2} R_{h_k}^{u_k}(0)$$

for some $a_2 > 0$ absolute and all k large; hence,

$$\lambda_{h_k/M}^{u_k}(0) \geq a_2 M^{-1/2} \lambda_{h_k}^{u_k}(0) \text{ and } \Lambda_{h_k/M}^{u_k}(0) \leq 2(a_2)^{-1} M^{-1/2} \Lambda_{h_k}^{u_k}(0).$$

Define $c_1 := a_2^2/4$, and note that c_1 is an absolute positive constant. For this choice, we get the desired contradiction when k is large. \square

We define

$$(8) \quad M_1 := c_1^{-6},$$

where c_1 is the absolute constant in Lemma 3.1, and we let ϵ_1 be the volume fraction from the lemma corresponding to the choice $M = M_1$. (In particular, ϵ_1 is universal.)

The next lemma deals with the case that Ω_1^c bites a positive fraction of the section, and the long axis is transversal to the boundary. We let $l_h^u(x)$ denote the length of the intersection of the tangent line to $\partial\Omega_1$ at $x \in \partial\Omega_1$ with $x + R_h^u(x)$. This lemma uses the regularity and uniform convexity of the boundary.

LEMMA 3.2. *There exists a universal constant $c_2 > 0$ such that the following holds. Given $M > 1$, there exists $\epsilon > 0$ depending on δ and M such that if $u \in \mathcal{F}_\delta$, $x \in \partial\Omega_1 \setminus \partial\Omega_0$, and, in addition,*

$$\eta_h^u(x) > c_2^{-1}M, \quad \frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} > \epsilon_1, \quad \text{and} \quad \frac{l_h^u(x)}{\Lambda_h^u(x)} \leq \epsilon,$$

then

$$\eta_{h/M}^u(x) \leq c_2^{-1}\eta_h^u(x).$$

Furthermore, if

$$\eta_{h/M}^u(x) > \eta_h^u(x),$$

then

$$R_{h/M}^u(x) \subset c_2^{-1}M^{-1/2}R_h^u(x).$$

Proof. Assume by way of contradiction that the lemma is false for a choice of universal constant c_2 explained below. Then there exist a sequence $u_k \in \mathcal{F}_\delta$, points x_k on the boundaries of the holes in the source domains Ω_{1k} , and $h_k > 0$ such that the first two inequalities hold for $R_{h_k}^{u_k}(x_k)$ and

$$(9) \quad l_{h_k}^{u_k}(x_k)/\Lambda_{h_k}^{u_k}(x_k) < 1/k,$$

but the conclusion is false for all k . After performing a rigid motion, we may assume that $x_k = 0$, that $R_{h_k}^{u_k}(0)$ have short side vertical, and that te_2 and se_1 are contained in the source domain for $t \in (0, \lambda_{h_k}^{u_k}(0))$ and $s \in (-\Lambda_{h_k}^{u_k}(0), 0)$. From here on out, we write $R_k = R_{h_k}^{u_k}(0)$, $\lambda_k = \lambda_{h_k}^{u_k}(0)$, and $\Lambda_k = \Lambda_{h_k}^{u_k}(0)$. We claim that up to taking a subsequence and adding affine functions, the rescalings $h_k^{-1}u_k(\Lambda_k x_1, \lambda_k x_2)$ converge locally uniformly to a function w such that $S_1^w(0)$ contains and is contained in balls of absolute radius, and Proposition 2.3 applies to w up to swapping x_1 and x_2 (see Figure 1).

Before proceeding, we note that (9) implies that $\lambda_k \Lambda_k^{-1} \rightarrow 0$, which in turn implies that $h_k \rightarrow 0$ (by the area estimate for centered sections) and thus that $\Lambda_k \rightarrow 0$ (by the universal modulus of continuity for the diameters of centered sections).

Let $s_k \geq 0$ be the slope of the tangent line to the boundary of the source domain at 0 (we allow $s_k = \infty$). Inequality (9) implies that

$$(10) \quad s_k \Lambda_k \lambda_k^{-1} > k.$$

We first treat the case that $s_k > 1$ for some subsequence. Then for k large, the fact that Λ_k tend to zero and elementary geometry imply that

$$(11) \quad \{x_2 < |x_1|/2, x_1 > 0\} \cap S_{h_k}^{u_k}(0) \subset \Omega_1^c.$$

After the change of variable $x_1 := \Lambda_k \tilde{x}_1$, $x_2 := \lambda_k \tilde{x}_2$, the domains $\{x_2 < |x_1|/2, x_1 > 0\}$ become $\{\tilde{x}_2 < \Lambda_k \lambda_k^{-1}|\tilde{x}_1|/2, \tilde{x}_1 > 0\}$. Since the latter domains converge to the right half-space, we conclude that w solves $\det D^2w = \chi_{\{x_2 < 0\}}$ in $S_1^w(0)$, as desired.

The alternative is that $s_k \leq 1$ for all k large. Uniform concavity of the boundary implies that $\{x_2 < s_k x_1 - a_0 x_1^2\}$ contain the holes with 0 in their boundary for some $a_0 > 0$ universal. Since

$$R_k \cap \{x_2 < s_k x_1 - a_0 x_1^2\} \subset R_k \cap \{|x_1| < a_1^{-1}(s_k^2 + \lambda_k)^{1/2}\}$$

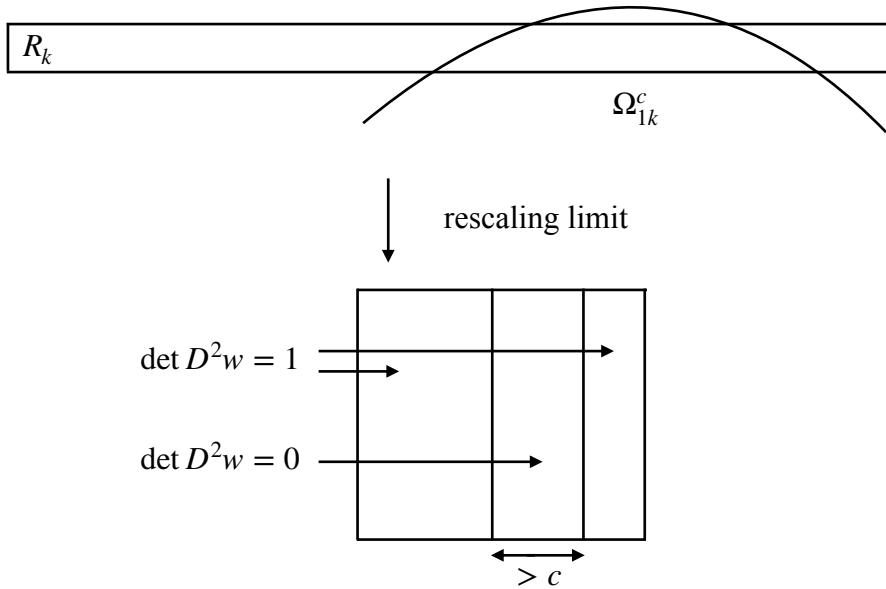


FIG. 1. *Rescaling limit in the case of nontrivial exterior area and transversal boundary.*

for some $a_1 > 0$ universal, the lower bound on the volume of the complement implies that

$$\Lambda_k^2 \leq a_2^{-1}(s_k^2 + \lambda_k)$$

for some $a_2 > 0$ universal. We claim that $s_k^2 \geq \lambda_k$ for k large universal. Indeed, if not, then the previous inequality implies that $\Lambda_k^2 \leq 2a_2^{-1}\lambda_k$, which, combined with (10), gives $k < \sqrt{2}a_2^{-1/2}s_k\lambda_k^{-1/2}$, and since we assumed that $s_k\lambda_k^{-1/2} < 1$, we get a contradiction for k large universal. We conclude that

$$(12) \quad \Lambda_k \leq \sqrt{2}a_2^{-1/2}s_k := a_3^{-1}s_k$$

when k is large universal. By the $C^{1,1}$ regularity of the holes, the complements of the domains contain $\{x_2 < s_k x_1 - a_4^{-1}x_1^2\} \cap S_{h_k}^{u_k}(0)$ for some $a_4 > 0$ universal. In the new coordinates defined above, the parabolic domains $\{x_2 < s_k x_1 - a_4^{-1}x_1^2\}$ become

$$\{\tilde{x}_2 < s_k \Lambda_k \lambda_k^{-1} \tilde{x}_1 - a_4^{-1} \Lambda_k^2 \lambda_k^{-1} \tilde{x}_1^2\}.$$

Using the bound (12) on Λ_k , we see that these domains contain

$$\{\tilde{x}_2 < s_k \Lambda_k \lambda_k^{-1} (\tilde{x}_1 - a_5^{-1} \tilde{x}_1^2)\}$$

for some $a_5 > 0$ universal. Using (10), we see that these domains converge to the slab $\{0 < \tilde{x}_1 < a_5\}$ as k tends to infinity. We conclude that w satisfies $\det D^2w = 1$ in $\{x_1 < 0\} \cap S_1^w(0)$ and that $\det D^2w = 0$ in $\{0 < x_1 < a_5\} \cap S_1^w(0)$, as desired.

In either case, a small modification of Proposition 2.3 implies that $S_{1/M}^w(0)$ contains and is contained in dilations by a universal constant of a rectangle with axes aligned with the coordinate axes. Moreover, the upper bound on the vertical second derivative of w implies that the horizontal length l and vertical length L of this rectangle satisfy $l \leq a_6^{-1}L$ for $a_6 > 0$ universal. For k large, we conclude that

$$\eta_{h_k/M}^{u_k}(0) \sim \eta_{h_k}^{u_k}(0)l/L$$

provided that $\eta_{h_k}^{u_k}(0)l/L > 1$. (Here and below, we use that if a rectangle R_1 centered at 0 is approximated by (contains and is contained in dilations by universal constants times) another rectangle R_2 centered at 0, then their side lengths λ_i, Λ_i ($i = 1, 2$) satisfy $\lambda_1 \sim \lambda_2, \Lambda_1 \sim \Lambda_2$.) Since $L/l \leq a_7^{-1}M$ for some $a_7 > 0$ universal by the local universal Lipschitz regularity of w , the first inequality we assumed about $\eta_{h_k}^{u_k}(0)$ guarantees this is satisfied (for a sufficiently small universal choice of c_2). Thus, the eccentricity $\eta_{h_k/M}^{u_k}(0)$ increased by at most a universal factor compared to $\eta_{h_k}^{u_k}(0)$. Moreover, if $l < a_8L$ for $a_8 > 0$ universal sufficiently small, then we have $\eta_{h_k/M}^{u_k}(0) < \eta_{h_k}^{u_k}(0)$. The alternative is that $S_{1/M}^w(0)$ is approximated by (contains and is contained in dilations by universal constants times) $B_{M^{-1/2}}$; thus, for k large, the rectangles $R_{h_k/M}^{u_k}(0)$ are contained in universal dilations of $M^{-1/2}R_k$. This gives the desired contradiction (namely, the conclusion is true for k large) provided that c_2 was chosen sufficiently small (but still universal). \square

We now define

$$(13) \quad M_2 := c_2^{-6},$$

where c_2 is the universal constant from Lemma 3.2, and we let $\epsilon_2 > 0$ be the length ratio from that lemma corresponding to the choice $M = M_2$. Note that ϵ_2 is universal.

Finally, the remaining lemma is purely geometric and also uses the regularity and convexity properties of the holes in Ω_1 . Below, d denotes the distance function from Ω_1^c .

LEMMA 3.3. *There exists $c_3 > 0$ universal such that the following holds. Assume that $u \in \mathcal{F}_\delta$, that $x \in \partial\Omega_1 \setminus \partial\Omega_0$, and that*

$$\frac{|(x + R_h^u(x)) \cap \Omega_1^c|}{|R_h^u(x)|} > \epsilon_1, \quad \frac{l_h^u(x)}{\Lambda_h^u(x)} > \epsilon_2.$$

Then

$$[\Lambda_h^u(x)]^2 + \sup_{x + R_h^u(x)} d \leq c_3^{-1} \lambda_h^u(x).$$

Proof. Perform a rigid motion as in the proof of Lemma 3.2 so that $x = 0$, the short side of $R_h^u(0)$ is vertical, and the hole lies beneath its tangent line at 0, which has slope $s \geq 0$. Below, we will denote $R_h^u(0)$ by R_h , and we will similarly drop the superscript u and the point (0) in the notation from the other relevant quantities.

Elementary geometry implies that $l_h \leq 4 \max\{\lambda_h, \lambda_h/s\}$, so the second inequality in the hypothesis gives $\epsilon_2 < 4 \max\{\lambda_h/\Lambda_h, \lambda_h/(s\Lambda_h)\}$. In the case that $\Lambda_h < 4\epsilon_2^{-1}\lambda_h$, the lemma is obvious since then $\lambda_h \sim \Lambda_h$. We can thus assume otherwise. This gives

$$(14) \quad s \leq 4\epsilon_2^{-1}\lambda_h/\Lambda_h \leq 1.$$

We may further assume that h is small enough that in $\{|x_1| \leq \Lambda_h\}$, the top part of the boundary of the hole is a graph with slope bounded by 2. Indeed, this is guaranteed for $h < c$ universal by the universal diameter bound on sections, and for $h \geq c$, we have $\lambda_h \sim \Lambda_h$ (in particular, that the lemma holds) by the area estimate for sections and the universal Lipschitz bound on u . Here and below, c denotes a positive universal constant that may change from line to line.

The uniform concavity of $\partial\Omega_1$ implies that

$$R_h \cap \Omega_1^c \subset R_h \cap \{x_2 < sx_1 - cx_1^2\} \subset R_h \cap \{|x_1| < c^{-1}(s^2 + \lambda_h)^{1/2}\}.$$

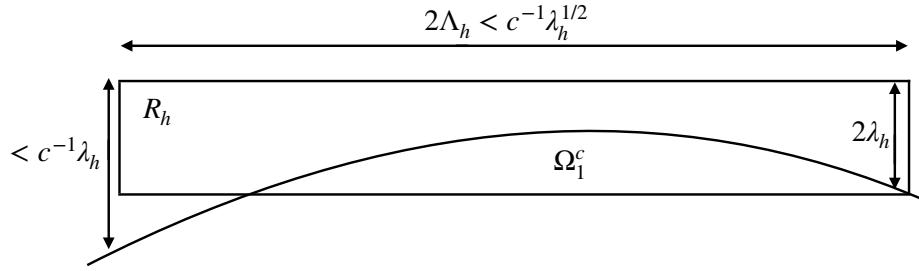


FIG. 2. “Model example geometry” in the case of nontrivial exterior area and roughly tangential boundary.

Using the first inequality in the hypothesis, we conclude that

$$\Lambda_h^2 < c^{-1}(s^2 + \lambda_h).$$

Using (14) in the previous inequality gives

$$\Lambda_h^2 \leq c^{-1}(\lambda_h^2 \Lambda_h^{-2} + \lambda_h),$$

and it follows that

$$(15) \quad \Lambda_h^2 \leq c^{-1} \lambda_h.$$

Furthermore, the $C^{1,1}$ regularity of $\partial\Omega_1$ implies that the hole has a boundary portion that lies above $\{x_2 = sx_1 - c^{-1}x_1^2\}$ in $\{|x_1| < \Lambda_h\}$. The distance of points in R_h from Ω_1^c is thus bounded above by

$$c^{-1}(\lambda_h + |s|\Lambda_h + c^{-1}\Lambda_h^2).$$

Using (14) and (15), we arrive at the desired estimate. \square

See Figure 2 for a summary of the results from Lemma 3.3.

4. Proof of Theorem 1.1.

Proof of Theorem 1.1. For each $x \in \Omega_1$, let $S_{\bar{h}(x)}^u(x)$ be the “maximal section contained in Ω_1 ” centered at x so that $S_{\bar{h}(x)}^u(x)$ is contained in Ω_1 and tangent to $\partial\Omega_1$. The existence of such a section follows from the continuity of the sections in h ; see [8]. By (4) and the universal Lipschitz bound on u , there exists $\bar{c} > 0$ universal such that for all x in the \bar{c} -neighborhood $\mathcal{N}_{\bar{c}}$ of the union of the holes in Ω_1 , the section $S_{\bar{h}(x)}^u(x)$ is tangent to a hole and not $\partial\Omega_0$.

The arguments in the proof of Theorem 1.1 of [23] show that

$$u \in W^{2,p}(\Omega_1 \setminus \mathcal{N}_{\bar{c}}) \cap C^{1,\alpha}(\overline{\Omega_1 \setminus \mathcal{N}_{\bar{c}}})$$

for any $p > 1$ and $\alpha \in (0, 1)$, with corresponding estimates in these spaces depending on δ , p , and α . More precisely, the estimates used to prove Theorem 1.1 of [23] are local in nature and thus apply in our setting to sections centered in $\overline{\Omega_1 \setminus \mathcal{N}_{\bar{c}}}$ of height smaller than some $h_1 > 0$ universal chosen so that these sections do not intersect $\mathcal{N}_{\bar{c}/2}$. In particular, in our setting, Theorem 1.1 of [23] has the form $|D^2u| \leq C(\epsilon, \delta)d^{-\epsilon}$ in $\Omega_1 \setminus \mathcal{N}_{\bar{c}}$ for any $\epsilon > 0$, from which the claim follows. Here and below, d denotes the distance from Ω_1^c . (Alternatively, one can note that all arguments in [23] work equally

well if the source density is smooth and positive in the closure of the source domain and 1 in a neighborhood of the boundary. In our setting, we can reduce to this case by considering the function that is u near $\partial\Omega_0$, glued to $\rho_\tau * (u + \tau|x|^2)$ for $\tau > 0$ small universal, a positive universal distance from $\partial\Omega_0$. Here ρ_τ is a standard mollifier.)

We may thus focus our attention on $\mathcal{N}_{\bar{c}}$. To that end, let $x \in \mathcal{N}_{\bar{c}}$, and let d be the distance from x to the boundary. We will prove that

$$(16) \quad |D^2u(x)| \leq c^{-1}d^{-1/2}.$$

Here and for the remainder of the proof, c denotes a positive universal constant that may change from line to line. The $W^{2,p}$ estimate from Theorem 1.1 follows immediately from (16), and the $C^{1,1/2}$ estimate comes from integrating (16) along line segments.

Assume after a translation that $S_{h(x)}^u(x)$ is tangent to a hole at the origin. Applying Proposition 2.1, we engulf $S_{h(x)}^u(x)$ by $R_h^u(0)$, with $h \sim \bar{h}(x)$. We will prove that

$$\eta_h^u(0) \leq c^{-1}d^{-1/2}.$$

Using that $|S_{h(x)}^u(x)| \sim |R_h^u(0)|$, it is easy to see that $\eta_{h(x)}^u(x) \leq c^{-1}\eta_h^u(0)$. Combining this with the above inequality and (7) gives (16).

In what follows, we will use that if $h_1 \sim h_2$, then $R_{h_1}^u(0)$ is approximated by (contains and is contained in dilations by universal constants times) $R_{h_2}^u(0)$ (see section A.4 of the appendix), whence $\lambda_{h_1}^u(0) \sim \lambda_{h_2}^u(0)$, $\Lambda_{h_1}^u(0) \sim \Lambda_{h_2}^u(0)$, and $\eta_{h_1}^u(0) \sim \eta_{h_2}^u(0)$. We will also denote $\eta_h^u(0)$ by η_h , and we will similarly drop the notation (0) and the superscript u from the other relevant quantities.

If either $\eta_h \leq c_2^{-1}M_2$ or the conditions of Lemma 3.3 are satisfied, then we are done. Indeed, in the first case, use that $c_2^{-1}M_2$ is universal; hence, the desired bound on η_h follows provided that c universal is sufficiently small. In the second case, Lemma 3.3 gives $\Lambda_h^2 \leq c^{-1}\lambda_h$; hence, $\eta_h \leq c^{-1}\lambda_h^{-1/2}$. Moreover, the distance from the boundary in R_h is at most $c^{-1}\lambda_h$. In particular, $\lambda_h^{-1/2} \leq c^{-1}d^{-1/2}$, and we are also done in this case.

So assume that neither is satisfied, and let \hat{t} be the supremum of heights t such that $\eta_s > c_2^{-1}M_2$ and the conditions of Lemma 3.3 are not satisfied at height s for all $s \in [h, t]$.

We first assume that $\hat{t} \leq 1$. Then there exist $\hat{h} \in [h, \hat{t}]$ and $\kappa \in (1, 2)$ such that $\kappa\hat{h} \geq \hat{t}$ and either $\eta_{\kappa\hat{h}} \leq c_2^{-1}M_2$ or the conditions of Lemma 3.3 are satisfied at height $\kappa\hat{h}$ but neither holds at height \hat{h} . At height \hat{h} , we can apply either Lemma 3.1 with the choice $M = M_1$ or Lemma 3.2 with the choice $M = M_2$, giving a conclusion at height $M_1^{-1}\hat{h}$ or $M_2^{-1}\hat{h}$. Repeat applying Lemma 3.1 or 3.2 (say, the former k times and the latter l times) until the first time $M_1^{-k}M_2^{-l}\hat{h} < h$. Assume that eccentricity increased in the application of Lemma 3.2 $l' \leq l$ times. We then have

$$\eta_h \leq c^{-1}c_1^{-k}c_2^{-l'}\eta_{\hat{h}} \leq c^{-1}r^{-1/2}\eta_{\kappa\hat{h}},$$

where

$$r := c_1^{-k}M_1^{-k/2}c_2^{-l'}M_2^{-l'/2} = c_1^{2k}c_2^{2l'} < 1.$$

By Lemmas 3.1 and 3.2, we also have that

$$R_h \subset c^{-1}rR_{\kappa\hat{h}}.$$

We now consider the case that at height $\kappa\hat{h}$, the conditions of Lemma 3.3 are satisfied. The lemma implies that

$$\eta_{\kappa\hat{h}} \leq c^{-1}\lambda_{\kappa\hat{h}}^{-1/2};$$

thus,

$$(17) \quad \eta_h \leq c^{-1}r^{-1/2}\lambda_{\kappa\hat{h}}^{-1/2}.$$

Assume after a rigid motion that the picture is oriented as in the proof of Lemma 3.3 so that $R_{\kappa\hat{h}}$ has vertical short axis. Arguing as in the proof of Lemma 3.3, we have either that $\lambda_{\kappa\hat{h}} \sim \Lambda_{\kappa\hat{h}}$, in which case the desired estimate $\eta_h \leq c^{-1}r^{-1/2}\lambda_{\kappa\hat{h}}^{-1/2} \leq c^{-1}d^{-1/2}$ follows from the inclusion $R_h \subset c^{-1}rR_{\kappa\hat{h}}$, or that the boundary of Ω_1 contains a portion that lies above $\{x_2 = sx_1 - c^{-1}x_1^2\}$ in $\{|x_1| < \Lambda_{\kappa\hat{h}}\}$, where $0 \leq s \leq c^{-1}\lambda_{\kappa\hat{h}}/\Lambda_{\kappa\hat{h}}$ and $\Lambda_{\kappa\hat{h}}^2 \leq c^{-1}\lambda_{\kappa\hat{h}}$. Recall that R_h is contained in the $c^{-1}r$ times dilation of $R_{\kappa\hat{h}}$. Thus, in R_h , the distance from the boundary is at most

$$c^{-1}(r\lambda_{\kappa\hat{h}} + sr\Lambda_{\kappa\hat{h}} + r^2\Lambda_{\kappa\hat{h}}^2).$$

Using the previous inequalities for s and $\Lambda_{\kappa\hat{h}}$, we see that the second and third terms are bounded by $c^{-1}r\lambda_{\kappa\hat{h}}$ and $c^{-1}r^2\lambda_{\kappa\hat{h}}$, respectively, giving a bound on the distance between the boundary in R_h of the size

$$d \leq c^{-1}r\lambda_{\kappa\hat{h}}.$$

Rearranging gives

$$\lambda_{\kappa\hat{h}}^{-1/2} \leq c^{-1}r^{1/2}d^{-1/2}.$$

Using this in (17) gives

$$\eta_h \leq c^{-1}d^{-1/2},$$

and we are done with this case.

In the case that $\eta_{\kappa\hat{h}} \leq c_2^{-1}M_2$, we have

$$\eta_h \leq c^{-1}r^{-1/2}.$$

Furthermore, we have that R_h is contained in a universal dilation of B_r since $R_{\hat{h}}$ is contained in $B_{c^{-1}}$; thus, in R_h , the distance from the boundary is at most $c^{-1}r$, and hence

$$\eta_h \leq c^{-1}d^{-1/2}$$

in this case as well.

Finally, we deal with the case that $\hat{t} > 1$. Since η_1 is still bounded by a universal constant, we can take $\hat{h} = 1$ and repeat exactly the same arguments as above. More precisely, by repeated application of Lemma 3.1 or 3.2 starting from height $\hat{h} = 1$, we get

$$\eta_h \leq c^{-1}r^{-1/2}$$

and

$$R_h \subset B_{c^{-1}r},$$

where r is defined in the same way as above. Combining these two conclusions, we get $\eta_h \leq c^{-1}d^{-1/2}$, and this completes the proof. \square

5. Future directions. To conclude the paper, we list a few questions to be investigated in future work:

1. Establish Sobolev regularity for optimal transport maps of a natural class of nonconvex domains in higher dimensions. As noted above, the Pogorelov-type estimate Proposition 2.3 is convenient but not required for the result in this paper, so there is hope for such an extension.
2. In two dimensions, enlarge the class of source domains being considered, e.g., to arbitrary smooth domains. Our arguments handle concave parts of the boundary since we only use the equation and not the convexity of the target domain. However, convex parts of the boundary that lie inside the convex hull of the source domain may be tricky to handle since at such points, we do not have duality, which played an important role in the works of Caffarelli [6] and Savin and Yu [23].
3. Investigate applications of our results to the existence theory for the SGEs in cases where the source density is allowed to vanish. In previous works dealing with the case where the source density is bounded from below by a positive constant, $W^{2,1}$ estimates played a central role [2], [3], [15].
4. Investigate applications of the ideas in this paper to the partial regularity theory of optimal transport maps when the domains are not convex. In this case, optimal transport maps can have singularities, and interesting results have been obtained about the size of the singular set [12], [17], [16], [19]. However, the fine geometric measure-theoretic structure of the singular set is not well understood, even in two dimensions with smooth domains and quadratic cost. In that case, a reasonable conjecture seems to be that the one-dimensional Hausdorff measure of the singular set is bounded.

Appendix A. In this section, we provide some of the details that we omitted for simplicity of presentation above.

A.1. No segments. We start with a simple lemma that will be used in some of the subsequent proofs.

LEMMA A.1. *There is no convex function w on $B_1 \subset \mathbb{R}^2$ that satisfies*

$$\det D^2w \geq \chi_{\{x_2 > 0\}}, \quad w|_{\{x_2=0\}} \text{ linear.}$$

Proof. After subtracting an affine function, we may assume that $w \geq 0$ and $w|_{\{x_2=0\}} = 0$. After subtracting a multiple of x_2 , we may assume further that $w(0, t) = o(t)$ for $t > 0$. It follows that for any $k > 0$, we can choose $h > 0$ small such that

$$R := [-1/2, 1/2] \times [0, 2kh] \subset \{w < h\}.$$

The quadratic polynomial $Q = 8hx_1^2 + 2k^{-2}h^{-1}(x_2 - kh)^2$ thus lies above w on the boundary of R and for k large satisfies $\det D^2Q < 1$. The comparison principle implies that $Q > w \geq 0$ in R , contradicting that Q vanishes at the center of R . \square

We now prove the claim that functions in F_δ are not linear along line segments centered in the closure of the source domain, from which the universal bound on diameters of sections followed.

LEMMA A.2. *Let $u \in \mathcal{F}_\delta$. Then u is not linear along any line segment centered at a point in $\overline{\Omega_1}$.*

Proof. By Lemma A.1 (appropriately rescaled), u is not linear along any segment in $\overline{\Omega_1}$. The only remaining possibility is that u is linear along a segment that is tangent

to $\partial\Omega_0$ at a single point that lies inside the segment. The function u cannot be linear along the whole line l containing this segment. Indeed, if it was, then the monotonicity of the subgradient of u implies that $\partial u(\mathbb{R}^2)$ (the subgradient image of u on the whole plane) lies in some line orthogonal to l , whence $\det D^2u \equiv 0$, a contradiction. We conclude that the agreement set between u and a linear function has extremal points outside of Ω_0 . This contradicts that $\det D^2u = 0$ outside of Ω_0 . \square

A.2. Area estimate. Next we prove the area estimate (5). Note that if $S_h^u(x)$ is centered at a point $x \in \overline{\Omega_1}$ and intersects at most one hole, then the hole lies on one side of some line through x . Since x is the center of mass of $S_h^u(x)$, it follows that $|S_h^u(x) \cap \Omega_1| \geq d|S_h^u(x)|$ for some $d > 0$ absolute. The area estimate (5) is thus a consequence of the following general result.

LEMMA A.3. *Assume that $S_t^w(x)$ is a bounded domain in \mathbb{R}^n , that $\det D^2w = \chi_U$ for some $U \subset S_t^w(x)$, and that $|U| \geq \lambda|S_t^w(x)|$ for some $\lambda > 0$. Then there exists $\kappa(n, \lambda) > 0$ such that*

$$\kappa t^{n/2} \leq |S_t^w(x)| \leq \kappa^{-1} t^{n/2}.$$

Proof. After translating and subtracting a linear function, we may assume that $x = 0$ and that $w = 0$ on $\partial S_t^w(0)$. By John's lemma, there exists an ellipsoid E_t centered at zero such that $c(n)E_t \subset S_t^w(0) \subset E_t$. In this proof, $c(n)$ will denote a constant depending on n that may change from line to line. Write $E_t = A_t(B_1)$, where A_t is a linear map, and let $\tilde{w}(x) = |\det A_t|^{-2/n}w(A_tx)$, $\tilde{U} = A_t^{-1}U$, and $H = |\det A_t|^{-2/n}t$. Then $c(n)B_1 \subset S_H^{\tilde{w}}(0) \subset B_1$, $\det D^2\tilde{w} = \chi_{\tilde{U}}$, and $|\tilde{U}| \geq c(n)\lambda$. In particular, provided that $\alpha(n, \lambda)$ is sufficiently close to one, we have $|S_H^{\tilde{w}}(0) \setminus \alpha S_H^{\tilde{w}}(0)| \leq |\tilde{U}|/2$, giving $|\tilde{U}| \leq 2|\tilde{U} \cap \alpha S_H^{\tilde{w}}(0)|$. Proposition 1.1 of [21] implies that $H \sim 1$ up to multiplication by constants depending on n, λ . Since $|\det A_t|$ is equivalent to $|S_t^w(0)|$ up to multiplication by dimensional constants, we conclude that $t^{n/2}$ is equivalent to $|S_t^w(0)|$ up to multiplication by constants depending on n and λ , as desired. \square

A.3. Engulfing properties. The engulfing property Proposition 2.1 follows from the following pair of lemmas. Before proceeding, we recall a standard renormalization procedure. For $u \in \mathcal{F}_\delta$, $x \in \overline{\Omega_1}$, and $S_h^u(x)$ contained in Ω_1 or intersecting one hole, let

$$u_h(y) = \frac{1}{h}u(A_h y) + L_h,$$

where A_h is an affine transformation of determinant h that takes a square centered at 0 to $x + R_h(x)$ and L_h is a linear function chosen so that $u_h = 0$ on $\partial S_1^{u_h}(0)$. Then u_h solves $\det D^2u_h = \chi_{\Omega_h}$ in the normalized domain $S_1^{u_h}(0)$ (equivalent to B_1 up to dilations by absolute constants), where Ω_h^c is convex and $\overline{\Omega_h}$ contains 0. We call this procedure “renormalizing in the section $S_h^u(x)$.”

LEMMA A.4. *If $u \in \mathcal{F}_\delta$, $y \in \overline{S_h^u(x)} \cap (\partial\Omega_1 \setminus \partial\Omega_0)$, and $S_h^u(x) \subset \Omega_1$, then*

$$S_h^u(x) \subset S_{c^{-1}h}^u(y).$$

Here c is universal.

Proof. Assume by way of contradiction that the lemma is false. Then there exists a sequence $u_k \in \mathcal{F}_\delta$ such that $S_{h_k}^{u_k}(x_k)$ are contained in the source domains and y_k are in their closures and the boundary of a hole (say, $y_k = 0$ after a translation) but $S_{kh_k}^{u_k}(0)$ do not contain $S_{h_k}^{u_k}(x_k)$. Note that we may assume that $h_k \leq c^{-1}k^{-1}$ by the

uniform Lipschitz bound on u_k , so $h_k \rightarrow 0$, and the sections localize close to the holes. After renormalizing in the sections $S_{kh_k}^{u_k}(0)$, we get a subsequence of rescalings w_k of u_k that converge to a function w which satisfies that $S_1^w(0)$ is normalized, $\det D^2w = 1$ in a domain Ω with Ω^c convex, and w is linear along a segment from 0 to $\partial S_1^w(0)$ contained in $\bar{\Omega}$. This last fact comes from the observation that the sections $S_{h_k}^{u_k}(x_k)$, after renormalization, become sections of height $1/k$ in which $\det D^2w_k = 1$, which are not contained in the renormalized sections $S_1^{w_k}(0)$ but have 0 in their boundaries. These sections contain line segments with endpoints at 0 and $\partial S_1^{w_k}(0)$ that tend to a line segment from 0 to $\partial S_1^w(0)$ in $\bar{\Omega}$ on which w is linear since the heights of the sections tend to zero. This contradicts Lemma A.1, appropriately rescaled. \square

LEMMA A.5. *Let $u \in \mathcal{F}_\delta$. Then for all $\alpha \in (0, 1)$, there exists $\beta(\delta, \alpha) > 0$ such that*

$$S_{\beta h}^u(y) \subset \alpha S_h^u(y)$$

for all $y \in \partial\Omega_1 \setminus \partial\Omega_0$, where $\alpha S_h^u(y)$ is the α -times dilation of $S_h^u(y)$ around y .

Proof. The argument is similar to the one above. If the lemma is false, there is a sequence $u_k \in \mathcal{F}_\delta$ such that (up to translations) 0 is in the boundary of a hole and $S_{h_k/k}^{u_k}(0)$ are not contained in $\alpha S_{h_k}^{u_k}(0)$. After renormalizing in the sections $S_{h_k}^{u_k}(0)$, we get a subsequence of rescalings w_k that converge to a function w which satisfies that $S_1^w(0)$ is normalized, $\det D^2w = 1$ in a domain Ω with Ω^c convex and $0 \in \partial\Omega$, and w is linear along a line segment passing through the origin with an endpoint on $\partial(\alpha S_1^w(0))$. (The reasoning is similar to above; $S_{1/k}^{w_k}(0)$ are not contained in $\alpha S_1^{w_k}(0)$, so they contain segments passing through 0 with one endpoint on $\partial(\alpha S_1^{w_k}(0))$ and the other, by the fact that the sections are centered at 0, a positive distance depending on α from 0. These segments converge to the one claimed above.) Again, we contradict Lemma A.1. \square

A.4. Rectangle containments. Now we sketch the proof that if $u \in \mathcal{F}_\delta$, $0 \in \partial\Omega_1 \setminus \partial\Omega_0$, and $h_1 \sim h_2$, then $R_{h_1}^u(0)$ is approximated by (contains and is contained in dilations by universal constants of) $R_{h_2}^u(0)$. This fact was used in the proof of Theorem 1.1 in section 4. It suffices to show that $S_{h_1}^u(0)$ is approximated by $S_{h_2}^u(0)$. After renormalizing in $S_{h_1}^u(0)$ as in section A.3, we get a convex function w such that $S_1^w(0) = \{w < 0\}$ is normalized, and we need to show that $S_c^w(0)$ is approximated by B_1 , where $c \sim 1$. Since

$$|S_c^w(0)| \sim 1,$$

we just need to show that $S_c^w(0)$ contains a ball centered at 0 that has small universal radius. If not, then by the local universal Lipschitz bound on w , the slope of the linear function defining $S_c^w(0)$ is extremely large, say, after a rotation, Ke_1 with $K \gg 1$. But in this case, the line segment in $S_c^w(0)$ through the origin parallel to e_1 would intersect $\partial S_c^w(0)$ at a distance much shorter from the origin on the left than on the right, contradicting that 0 is the center of mass of $S_c^w(0)$ and completing the proof.

A.5. Partial Legendre transform. Let u be a convex function on a domain $\Omega \subset \mathbb{R}^2$. We define its partial Legendre transform by taking the usual Legendre transform along horizontal lines:

$$u^*(p, x_2) := \sup_{x_1} (px_1 - u(x_1, x_2)).$$

The partial Legendre transformed is defined on $\{((\partial_1 u(x_1, x_2), x_2) : (x_1, x_2) \in \Omega)\}$, where ∂_1 denotes the subgradient of u understood as a one-variable function on the horizontal line with x_2 constant. The function u^* is convex in the horizontal directions and concave in the vertical directions.

When u is smooth and uniformly convex, we have

$$u^*(u_1(x_1, x_2), x_2) = x_1 u_1(x_1, x_2) - u(x_1, x_2).$$

By taking various derivatives of this identity, one obtains

$$(18) \quad u_{11}^* = \frac{1}{u_{11}}, \quad u_{12}^* = -\frac{u_{12}}{u_{11}}, \quad u_{22}^* = -\frac{\det D^2 u}{u_{11}}.$$

In particular, if $\det D^2 u = 1$, then u^* is harmonic.

Finally, we recall that if u and v are convex on Ω , then $|u^* - v^*| \leq \|u - v\|_{L^\infty(\Omega)}$ on the common domain of definition of u^* and v^* . In particular, C^0 convergence of convex functions in Ω implies C^0 convergence of their partial Legendre transforms.

A.6. Boundary estimate for harmonic functions. We denote by B_r^+ the half-ball $B_r \cap \{x_n > 0\}$ in \mathbb{R}^n .

LEMMA A.6. *Let $u \in C(\overline{B_1^+})$ be a harmonic function such that $u|_{\{x_n=0\}}$ is Lipschitz, with Lipschitz constant L . Then*

$$\|u_i\|_{L^\infty(B_{1/2}^+)} \leq C(n)(L + \|u\|_{L^\infty(B_1^+)})$$

for all $i < n$.

Proof. We may assume that $L + \|u\|_{L^\infty(B_1^+)} \leq 1$ after dividing by this quantity. We may also assume that $u \in C^\infty(\overline{B_1^+})$; the general result follows by approximation. Let ψ be a smooth cutoff function that is 1 in $B_{3/4}$ and 0 outside of $B_{7/8}$, and let g be the function on $\{x_n = 0\}$ defined by $u\psi$ on $\{x_n = 0\} \cap B_1$ and extended to be zero outside. Note that $|g_i|, i < n$, is bounded by an absolute constant. Let K be the upper half-space Poisson kernel, and let $h = K * g$ be the harmonic function in the upper half-space with boundary data g . Since $h_i = g_i$ on $\{x_n = 0\}$ and h_i tends to zero at infinity, the maximum principle implies that on the upper half-space, $|h_i| \leq \|g_i\|_{L^\infty(\mathbb{R}^{n-1})}$, which in turn is bounded by an absolute constant. To conclude, we note that in $B_{3/4}^+$, we have $u = h + h'$, where h' is a harmonic function vanishing on $\{x_n = 0\} \cap B_{3/4}$ and $|h'| \leq |u| + |h| \leq 2$. The Schwarz reflection principle and the interior gradient estimate for harmonic functions give $\|\nabla h'\|_{L^\infty(B_{1/2}^+)} \leq C(n)$, so the result follows. \square

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