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Counterexamples to maximal regularity for operators in divergence form

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Abstract. In this paper, we present counterexamples to maximal L^p-regularity for a parabolic PDE. The example is a second-order operator in divergence form with space and time-dependent coefficients. It is well-known from Lions' theory that such operators admit maximal L^2-regularity on H^{-1} under a coercivity condition on the coefficients, and without any regularity conditions in time and space. We show that in general one cannot expect maximal L^p-regularity on $H^{-1}(\mathbb{R}^d)$ or L^2-regularity on $L^2(\mathbb{R}^d)$.

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1. Introduction. Let V and H be complex Hilbert spaces such that $V \hookrightarrow H$ densely and continuously. Identifying H^* with its dual, we can view H as a subspace of V^*, which is the dual of V. We start with the abstract problem

$$\begin{cases} u' - \mathcal{A}u = f, & \text{on } (0,1), \\ u(0) = 0. \end{cases} \tag{CP}$$

Here $\mathcal{A} : (0,1) \to \mathcal{L}(V, V^*)$ is strongly measurable and $f \in L^2(0,1;V^*)$. Moreover, we suppose that there are $\Lambda, \lambda > 0$ such that for all $t \in (0,1)$ and $v \in V$, one has

$$\|\mathcal{A}(t)v\|_{V^*} \leq \Lambda \|v\|_V, \qquad \mathrm{Re}\langle \mathcal{A}(t)v, v\rangle \geq \lambda \|v\|_V^2.$$

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By Lions' theory [19], it is known that (CP) has a unique weak solution $u \in H^1(0,1;V^*) \cap L^2(0,1;V)$. More generally (see [8, XVIII.3.5]), if $f \in L^2(0,1;V^*)+L^1(0,1;H)$, then (CP) has a unique weak solution $u \in L^2(0,1;V)$ such that $u' \in L^2(0,1;V^*) + L^1(0,1;H)$.

In this paper, we study two problems concerning the regularity of u.

Problem 1. *Let $p \in (1,\infty)\backslash\{2\}$. Under what condition on $\mathcal{A}$ does the following hold: for all $f \in L^p(0,1;V^*)$, there is a unique weak solution u which is in $L^p(0,1;V)$?*

Using the equation, this also gives $u \in H^{1,p}(0,1;V^*)$. Problem 1 can equivalently be formulated as the question whether there is a constant $C > 0$ such that for all step functions f valued in H and with u the unique solution to (CP) given by Lions' result for $p = 2$, one has the estimate

$$\|u\|_{L^p(0,1;V)} \leq C\|f\|_{L^p(0,1;V^*)}.$$

It is well-known that regularity results fail for the endpoint cases $p = 1$ and $p = \infty$ unless $V = V^*$ and thus $\mathcal{A}$ is a family of bounded operators on V (see [6,14] and also [16, Thm. 17.4.4 & Cor. 17.4.5]). Hence, we can safely concentrate on the case $p \in (1,\infty)$ in this paper.

The second problem is a variation of Lions' problem [19, p. 68], who originally asked this question for symmetric $\mathcal{A}$.

Problem 2. *Under what condition on $\mathcal{A}$ does the following hold: for all $f \in L^2(0,1;H)$, the unique weak solution u satisfies $u' \in L^2(0,1;H)$?*

Again using the equation, one also has $\mathcal{A}u \in L^2(0,1;H)$. However, it is unclear what this tells about the regularity of u since the domain of the operators $\mathcal{A}(t)$ in H are not easy to describe in general.

Both problems have in common (at least if $p > 2$) that they imply regularity of u such as $u \in C^\alpha([0,1];[H,V]_\lambda)$ for some $\alpha > 0$ and $\lambda \in (0,1)$. Such properties are for instance useful in the study of non-linear problems. They follow from standard interpolation estimates and Sobolev embeddings. In Lions' general L^2-setting, one can only obtain $C([0,1];H)$ or Hölder regularity of small order, compare with Theorem 2.2.

If $\mathcal{A}$ is autonomous, that is to say, the family $\mathcal{A}(t)$ does not depend on t, then in both problems the answer is affirmative. Indeed, in Problem 1, this can be concluded from Lions' result and [16, Thm. 17.2.31]. Concerning Problem 2, this follows from a result of de Simon [9].

Otherwise, some conditions are needed. In the case of Problem 1, it is sufficient that the mapping $t \mapsto \mathcal{A}(t) \in \mathcal{L}(V,V^*)$ is (piecewise relatively) continuous [2,21]. Without any continuity, it is only known that there is some $\varepsilon > 0$ depending on Λ, λ such that Problem 1 holds with $p \in (2-\varepsilon, 2+\varepsilon)$, see Theorem 2.2. For Problem 2, the situation is even more delicate. Based on an abstract counterexample for the Kato square root problem due to McIntosh, Dier constructed a first non-symmetric counterexample to Problem 2 in his PhD thesis, see also [3, Sec. 5]. Moreover, Fackler constructed a counterexample that is symmetric and $C^{\frac{1}{2}}$-Hölder continuous [12]. To the contrary, with slightly more regularity (for instance $C^{\frac{1}{2}+\varepsilon}$-Hölder regularity), many positive results were given [1,4,7,10,13,15,21]. We also recommend the survey [3] for an overview of Problem 2.

The counterexamples due to Dier and Fackler are abstract and not differential operators. As is highlighted by the solution to the Kato square root problem [5], the extra structure of a differential operator can be beneficial compared to the general situation. It is hence of interest to find counterexamples to Problems 1 and 2 that are differential operators. For Problem 2, this was explicitly pointed out in [13, Problem 6.1] and [3, Prop. 12.1]. To be more precise with our setting, we work with $H = L^2(\mathbb{R}^d)$ and $V = H^1(\mathbb{R}^d)$. If $B\colon [0,1]\times\mathbb{R}^d \to \mathbb{C}^{d\times d}$ is elliptic in the sense that there exist constants $\Lambda, \lambda > 0$ such that for all $t \in [0,1]$ and $x \in \mathbb{R}^d$, one has

$$|B(t,x)\xi| \le \Lambda|\xi|, \qquad \mathrm{Re} B(t,x)\xi\cdot\overline{\xi} \ge \lambda|\xi|^2 \qquad (\xi \in \mathbb{C}^d),$$

then consider the problem

$$\begin{cases} u' - \mathrm{div}(B\nabla u) = f, & \text{on } (0,1), \\ u(0) = 0, \end{cases} \tag{P}$$

where $f \in L^2(0,1;H^{-1}(\mathbb{R}^d))$. In the notation of (CP), we have put $\mathcal{A}(t)v = -\mathrm{div}(B(t)\nabla v)$. Lions' theory yields a unique solution u in the regularity class $H^1(0,1;L^2(\mathbb{R}^d)) \cap L^2(0,1;H^1(\mathbb{R}^d))$. Problems 1 and 2 ask if for all elliptic coefficients, one has the regularity $u \in L^p(0,1;H^1(\mathbb{R}^d))$ or $u' \in L^2(0,1;L^2(\mathbb{R}^d))$ when the forcing term f is taken from $L^p(0,1;H^{-1}(\mathbb{R}^d))$ or $L^2(0,1;L^2(\mathbb{R}^d))$. The answer is negative in both cases and the respective counterexamples will be the content of the present article. More precisely, based on a construction of the second-named author [20], we will obtain equations whose solutions fail to have higher $L^r(L^s)$-integrability (Theorem 2.4). Based on this result, we produce a counterexample to Problem 1 for every $p \neq 2$ in Theorem 2.5. In particular, this shows sharpness of Theorem 2.2 which we mentioned briefly above. Concerning Problem 2, we will show in Theorem 2.7 that it fails in the worst possible way for general elliptic problems in divergence form.

2. Maximal regularity in Hilbert spaces.

2.1. Known positive results. We present the results regarding Problem 1 that were already discussed in the introduction.

A function $u \in L^2(0,1;V) \cap H^1(0,1;V^*)$ is called a solution to (CP) if for all $t \in [0,1]$,

$$u(t) - \int_0^t \mathcal{A}u\,ds = \int_0^t f\,ds,$$

where the integrals are defined as Bochner integrals in V^*.

One can check that u is a solution to (CP) if and only if for all $\phi \in C_c^\infty((0,1);V)$, one has

$$-\int_0^1 \langle u(s), \phi'(s)\rangle ds - \int_0^1 \langle \mathcal{A}(s)u(s), \phi(s)\rangle ds = \int_0^1 \langle f(s), \phi(s)\rangle ds. \qquad (2.1)$$

Theorem 2.1 (Lions). *In the situation of* (CP), *there is a unique solution* $u \in L^2(0,1;V) \cap H^1(0,1;V^*)$. *Constants in the maximal regularity estimate depend only on* Λ *and* λ.

Based on a perturbation principle for isomorphisms in complex interpolation scales due to Sneiberg [22], Lions' result can be extended to p close to 2, see [11, Thm. 4.2].

Theorem 2.2. *In the situation of Lions' result* (Theorem 2.1), *there exists* $\varepsilon > 0$ *depending on* Λ, λ, *and the pair* (V,H) *such that* Problem 1 *has a positive solution for* $p \in (2-\varepsilon, 2+\varepsilon)$.

It will be clear from Theorem 2.5 that one cannot improve the latter to all $p \in (1,\infty)$, even if $f \in L^p(0,1;H)$.

2.2. A preliminary counterexample from the literature. Our proof relies on an example by the second-named author from [20]. We recall some of the details of that example here for the reader's convenience. In [20], complex-valued solutions $\zeta : (-\infty, 1) \times \mathbb{R}^d \to \mathbb{C}$ to linear, uniformly parabolic PDEs with complex coefficients $B : (-\infty, 1) \times \mathbb{R}^d \to \mathbb{C}^{d\times d}$ of the form

$$\partial_t \zeta = \operatorname{div}(B(t, x)\nabla\zeta) \qquad (2.2)$$

are constructed, with the following properties. First, (2.2) holds in the sense of distributions. Moreover, ζ and B are smooth away from $(-\infty, 1) \times \{0\}$ (hence the equation holds classically away from $\{x = 0\}$), and ζ is locally Lipschitz on $(-\infty, 1) \times \mathbb{R}^d$. Second, the solutions are chosen to obey certain scaling symmetries. Solutions of the following form are constructed:

$$\zeta(t, x) = (1-t)^{-\mu/2} e^{-i\log(1-t)/2} w(x/(1-t)^{1/2}), \qquad (2.3)$$
$$B(t, x) = a(x/(1-t)^{1/2}), \qquad (2.4)$$

for appropriate choices of a parameter $\mu \in \mathbb{R}$, a function $w : \mathbb{R}^d \to \mathbb{C}$, and uniformly elliptic coefficients $a : \mathbb{R}^d \to \mathbb{C}^{d\times d}$. The parabolic PDE (2.2) is equivalent to the following elliptic PDE for w on $\mathbb{R}^d$:

$$\operatorname{div}(a(x)\nabla w) = \frac{1}{2}(iw + \mu w + x \cdot \nabla w). \qquad (2.5)$$

In [20], it is shown that for any choice of the parameter μ satisfying

$$0 \le \mu < d/2,$$

one can find a uniformly elliptic matrix field a on $\mathbb{R}^d$ that is smooth away from 0, and a solution w to (2.5) on $\mathbb{R}^d$ that is smooth away from 0 and locally Lipschitz on $\mathbb{R}^d$, such that the following estimates are satisfied: for all multi-indices α and all $|x| \geq 1$, we have

$$|\partial^\alpha w(x)| \leq C_\alpha |x|^{-|\alpha|-\mu}, \tag{2.6}$$

$$|\partial^\alpha a(x)| \leq C_\alpha |x|^{-|\alpha|}. \tag{2.7}$$

The desired solutions ζ and coefficients B for the parabolic problem (2.2) are then obtained through (2.3) and (2.4). We stress that the coefficients a satisfy the boundedness and uniform ellipticity conditions

$$|a(x)\xi| \leq \Lambda|\xi|, \quad \mathrm{Re}(a(x)\xi \cdot \overline{\xi}) \geq \lambda|\xi|^2$$

for some λ, $\Lambda > 0$ depending on μ and all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, so B satisfies the conditions (1.1) with the same λ and Λ.

We will only use the properties discussed above in the sequel. For the details of how w and a are chosen, the interested reader can consult Section 3 in [20]. Roughly speaking, for judicious choices of the form of w and a, one can reduce the PDE (2.5) to a system of ODEs. These can be solved by fixing a choice of w and solving for the coefficients in a.

Remark 2.3. When split into real and imaginary parts, the Equation (2.2) can be understood as a system of two equations, which are coupled through the imaginary part of B. An important philosophical point in [20] is that when the imaginary part of B is taken to be symmetric, that part of B does not contribute to the ellipticity condition. This allows strong coupling of the equations without breaking the ellipticity condition.

2.3. Failure of $L^r(L^s)$-integrability for variational solutions. The following counterexample is based on the construction from the previous subsection and is the basis for our subsequent counterexamples to Problems 1 and 2.

Theorem 2.4 (Failure of higher integrability). *Let $d \geq 2$. For every $s, r \in (1, \infty)$ with $\frac{2}{r} + \frac{d}{s} < \frac{d}{2}$, there exists an elliptic matrix $B\colon [0,1] \times \mathbb{R}^d \to \mathbb{C}^{d\times d}$ and $f \in L^\infty(0,1;L^2(\mathbb{R}^d))$ such that the unique solution u to (P) satisfies $u \notin L^r(0,1;L^s(\mathbb{R}^d))$.*

Proof. Pick $\frac{2}{r} + \frac{d}{s} < \mu < \frac{d}{2}$ and fix the corresponding w, a, ζ, and B as described in Section 2.2.

Step 1: General setup. It is clear from (2.2) and the properties of w and B that the weak derivative $\partial_t \zeta$ exists on $(0,T)$ as an $L^2(B_R)$-valued function for any $T < 1$ and $R > 0$, where B_R is a shorthand notation for the Euclidean ball of radius R in $\mathbb{R}^d$. Now let

$$\begin{aligned} u(t,x) &= t\zeta(t,x)\eta(x), \\ f(t,x) &= \eta(x)\zeta(t,x) - t\sum_{k,\ell=1}^{d} [B_{k,\ell}(t,x) + B_{\ell,k}(t,x)]\partial_k \zeta(t,x)\partial_\ell \eta(x) \\ &\qquad - t\zeta(t,x)\mathrm{div}(B(t,x)\nabla\eta(x)), \end{aligned}$$

where the cut-off function $\eta \in C_c^\infty(\mathbb{R}^d)$ is such that $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, and $0 \leq \eta \leq 1$ on $\mathbb{R}^d$.

We claim that $u \in L^2(0,1;H^1(\mathbb{R}^d)) \cap H^1(0,1;H^{-1}(\mathbb{R}^d))$ is the unique solution to (P), and $f \in L^\infty(0,1;L^2(\mathbb{R}^d))$. Then, using (2.2), it is elementary to check that for $t \in (0,1)$ and $x \in \mathbb{R}^d \backslash \{0\}$, one pointwise has

$$\begin{cases} \partial_t u(t,x) - \operatorname{div}(B(t,x)\nabla u(t,x)) = f(t,x), \\ \qquad\qquad\qquad\qquad\qquad u(t,0) = 0. \end{cases}$$

Moreover, it also holds in distributional sense on $(0,1)\times\mathbb{R}^d$. Hence, by density and (2.1), it is a solution to (P).

Thus, it remains to check $u \in L^2(0,1;H^1(\mathbb{R}^d))$ and $f \in L^\infty(0,1;L^2(\mathbb{R}^d))$ in order to find that u is the unique solution to (P) provided by Theorem 2.1.
Step 2: $u \in L^2(0,1;H^1(\mathbb{R}^d))$. Fix $t \in (0,1)$ and let α be a multi-index with $|\alpha| \le 1$. By definition of ζ and using (2.6), one has

$$\begin{aligned} \|\partial^\alpha \zeta(t,\cdot)\|_{L^2(B_2)} &\le (1-t)^{-\frac{\mu+|\alpha|}{2}+\frac{d}{4}} \|\partial^\alpha w\|_{L^2(B_{2/(1-t)^{1/2}})} \\ &\le (1-t)^{-\frac{\mu+|\alpha|}{2}+\frac{d}{4}} \Big[C_{d,w} + C_0 \| |\cdot|^{-\mu-|\alpha|} \|_{L^2(B_{2/(1-t)^{1/2}}\backslash B_1)} \Big] \\ &\le (1-t)^{-\frac{\mu+|\alpha|}{2}+\frac{d}{4}} \Big[C_{d,w} + C_0 C_d (1-t)^{\frac{\mu+|\alpha|}{2}-\frac{d}{4}} \Big] \\ &\le C_{d,w}(1-t)^{-\frac{\mu+|\alpha|}{2}+\frac{d}{4}} + C'_{d,w} \end{aligned} \tag{2.8}$$

for some constant $C'_{d,w}$ only depending on d and w and with

$$C_{d,w} = |B_1|^{1/2} \sup_{x\in B_1} (|w(x)| + |\nabla w(x)|).$$

Note that $C_{d,w}$ is finite since w is Lipschitz. Recall that $\frac{d}{4} - \frac{\mu}{2}$ is positive. Thus, by definition of u and using (2.8), we find

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \le t\|\zeta(t,\cdot)\|_{L^2(B_2)} \le C_{d,w} + C'_{d,w}.$$

Thus, u is bounded as an $L^2(\mathbb{R}^d)$-valued function and therefore in particular $u \in L^2(0,1;L^2(\mathbb{R}^d))$. Similarly,

$$\begin{aligned} \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} &\le \|\nabla\zeta(t,\cdot)\|_{L^2(B_2)} + \|\nabla\eta\|_\infty \|\zeta(t,\cdot)\|_{L^2(B_2\backslash B_1)} \\ &\le C_{d,w}(1-t)^{-\frac{\mu+1}{2}+\frac{d}{4}} + C'_{d,w} + C'_{d,w}\|\eta\|_\infty. \end{aligned}$$

By the choice of μ, we have $-\frac{\mu+1}{2} + \frac{d}{4} > -\frac{1}{2}$, therefore and also $\nabla u \in L^2(0,1;L^2(\mathbb{R}^d))$.
Step 3: $f \in L^\infty(0,1;L^2(\mathbb{R}^d))$. To estimate the norm of f, we consider all parts separately. The fact that $\eta\zeta \in L^\infty(0,1;L^2(\mathbb{R}^d))$ follows from (2.8). Due to the support properties of $\nabla\eta$ and the boundedness of B, it suffices to prove a uniform estimate for $\|\partial_m\zeta(t,\cdot)\|_{L^2(B_2\backslash B_1)}$ to estimate $B_{k,\ell}\partial_m\zeta\partial_n\eta$. For this, calculate with (2.6) that

$$\begin{aligned} \|\partial_m\zeta(t,\cdot)\|_{L^2(B_2\backslash B_1)} &= (1-t)^{-\frac{\mu}{2}-\frac{1}{2}} \|(\partial_m w)(\cdot/(1-t)^{1/2})\|_{L^2(B_2\backslash B_1)} \\ &\le C_1 \| |\cdot|^{-1-\mu} \|_{L^2(B_2\backslash B_1)}. \end{aligned} \tag{2.9}$$

It remains to estimate $\zeta \operatorname{div}(B\nabla\eta)$. Again, due to the support properties of $\nabla\eta$, it suffices to consider $x \in B_2\backslash B_1$. First, by (2.6) and definition of ζ,

$$|\zeta(t,x)| \le (1-t)^{-\mu/2}|w(x/(1-t)^{1/2})| \le C_{1,0}.$$

Thus it remains to estimate $\|\partial^\alpha B_{k,\ell}(t,\cdot)\partial^\beta \eta\|_{L^2(B_2\setminus B_1)}$ for $|\alpha| \le 1$ and $1 \le |\beta| \le 2$. Since $B_{k,\ell}$ and $\partial^\beta \eta$ are uniformly bounded, it is enough to show that $\|\partial^\alpha B_{k,\ell}(t,\cdot)\|_{L^2(B_2\setminus B_1)}$ is uniformly bounded for $|\alpha| = 1$. Indeed, this follows from the analogous calculation to (2.9) using (2.7) instead of (2.6).
Step 4: $u \notin L^r(0,1;L^s(\mathbb{R}^d))$. Similar to (2.8) but using $B_1 \subseteq B_{1/(1-t)^{1/2}}$,

$$\|\zeta(t,\cdot)\|_{L^s(B_1)} = (1-t)^{-\frac{\mu}{2}+\frac{d}{2s}}\|w\|_{L^s(B_{1/(1-t)^{1/2}})} \ge (1-t)^{-\frac{\mu}{2}+\frac{d}{2s}}\|w\|_{L^s(B_1)}.$$

Therefore, with $t \in (\frac{1}{2},1)$,

$$\|u(t,\cdot)\|_{L^s(\mathbb{R}^d)} \ge t\|\zeta(t,\cdot)\|_{L^s(B_1)} \ge \frac{1}{2}(1-t)^{-\frac{\mu}{2}+\frac{d}{2s}}\|w\|_{L^s(B_1)}.$$

By the choice of μ, r, and s, we have $-\frac{\mu}{2} + \frac{d}{2\,s} < -\frac{1}{r}$, and therefore $u \notin L^r(0,1;L^s(\mathbb{R}^d))$. □

2.4. Negative result concerning Problem 1. The following result shows that for time-dependent operators in the variational setting, maximal L^2-regularity cannot be extrapolated to maximal L^p-regularity for $p \ne 2$ besides the small interval given in Theorem 2.2. It answers Problem 1 in a negative way in the setting of elliptic differential operators.

Theorem 2.5 (Failure of extrapolation of maximal L^p-regularity). *Let $d \ge 2$. For every $p \in (1,\infty)\setminus\{2\}$, there exists $B : [0,1]\times\mathbb{R}^d \to \mathbb{C}^{d\times d}$ elliptic and $f \in L^p(0,1;H^{-1}(\mathbb{R}^d))$ such that the unique solution u to (P) satisfies $u \notin L^p(0,1;H^1(\mathbb{R}^d))$.*

Proof. We divide the proof into two cases.

Case 1: $p > 2$. We appeal to Theorem 2.4. If $d \ge 3$, put $r = p$ and $s = 2^* := \frac{2d}{d-2}$ and if $d = 2$, put $r = p$ and $s > \frac{2p}{p-2}$. In both cases, the condition $\frac{2}{r} + \frac{d}{s} < \frac{d}{2}$ is satisfied, so that Theorem 2.4 yields $B : [0,1]\times\mathbb{R}^d \to \mathbb{C}^{d\times d}$ elliptic and $f \in L^\infty(0,1;L^2(\mathbb{R}^d)) \subseteq L^p(0,1;L^2(\mathbb{R}^d))$ such that the unique solution u to (2.4) satisfies $u \notin L^r(0,1;L^s(\mathbb{R}^d))$. This implies $u \notin L^p(0,1;H^1(\mathbb{R}^d))$ since otherwise the Sobolev embedding yields a contradiction to the previous assertion.

Case 2: $p < 2$. We use a duality argument that resembles [17, Thm. 6.2]. By the first part, there are $f \in L^{p'}(0,1;H^{-1}(\mathbb{R}^d))$, elliptic coefficients $B\colon [0,1]\times\mathbb{R}^d \to \mathbb{C}^{d\times d}$, and a solution $u \in L^2(0,1;H^1(\mathbb{R}^d)) \cap H^1(0,1;H^{-1}(\mathbb{R}^d))$ to the equation

$$\begin{cases} \partial_t u - \operatorname{div}(B\nabla u) = f, \\ \qquad\qquad u(0) = 0, \end{cases} \tag{2.10}$$

such that $u \notin L^{p'}(0,1;H^1(\mathbb{R}^d))$. Assume for the sake of contradiction that for every $A\colon [-1,0]\times\mathbb{R}^d \to \mathbb{C}^{d\times d}$ and $\widetilde{g} \in L^2(-1,0;H^{-1}(\mathbb{R}^d))$, there is a unique solution $\widetilde{v} \in L^2(-1,0;H^1(\mathbb{R}^d)) \cap H^1(-1,0;H^{-1}(\mathbb{R}^d))$ to the problem

$$\begin{cases} \partial_t \widetilde{v} - \operatorname{div}(A\nabla \widetilde{v}) = \widetilde{g}, \\ \qquad\qquad \widetilde{v}(-1) = 0, \end{cases} \tag{2.11}$$

satisfying the estimate

$$\|\partial_t \widetilde{v}\|_{L^p(-1,0;H^{-1}(\mathbb{R}^d))} + \|\widetilde{v}\|_{L^p(-1,0;H^1(\mathbb{R}^d))} \leq C\|\widetilde{g}\|_{L^p(-1,0;H^{-1}(\mathbb{R}^d))}, \quad (2.12)$$

where the constant $C > 0$ does not depend on $\widetilde{v}$ and $\widetilde{g}$. By translation, the question on $(-1, 0)$ is equivalent to that on $(0, 1)$. Both intervals are related by the transformation $v \mapsto -v$. We write for instance $\widetilde{u}(t) = u(-t)$ to translate u to a function on $(-1, 0)$ and vice versa. Specialize $A = (\widetilde{B})^*$ in (2.11). Since $u(0) = 0 = v(1)$, we can use integration by parts to obtain

$$\begin{aligned}\int_0^1 \langle g, u\rangle dt = \int_{-1}^0 \langle \widetilde{g}, \widetilde{u}\rangle dt &= \int_{-1}^0 \langle (\widetilde{v})', \widetilde{u}\rangle + ((\widetilde{B})^* \nabla\widetilde{v}|\nabla\widetilde{u})_2 dt\\ &= \int_0^1 -\langle v', u\rangle + (\nabla v|B\nabla u)_2 dt\\ &= \int_0^1 \langle v, u'\rangle + (\nabla v|B\nabla u)_2 dt.\end{aligned}$$

Plug in (2.10) to deduce

$$\int_0^1 \langle g, u\rangle dt = \int_0^1 \langle v, f\rangle dt.$$

Hence, using (2.12), we can estimate

$$\begin{aligned}\left|\int_0^1 \langle g, u\rangle dt\right| &\leq \|v\|_{L^p(0,1;H^1(\mathbb{R}^d))}\|f\|_{L^{p'}(0,1;H^{-1}(\mathbb{R}^d))}\\ &\leq C\|g\|_{L^p(0,1;H^{-1}(\mathbb{R}^d))}\|f\|_{L^{p'}(0,1;H^{-1}(\mathbb{R}^d))}.\end{aligned}$$

Since g was arbitrary, by duality, we obtain $u \in L^{p'}(0, 1; H^1(\mathbb{R}^d))$, which gives a contraction. □

Remark 2.6. For the case $p > 2$, we saw that although $f \in L^\infty(0, 1; L^2(\mathbb{R}^d))$, the unique solution u to (P) satisfies $u \notin L^p(0, 1; H^1(\mathbb{R}^d))$.

2.5. Negative result concerning Problem 2. We show that Problem 2 fails in the worst possible way in the case of elliptic operators in divergence form: for every $\nu \in (1/2, 1]$, there exist coefficients B and a forcing term $f \in L^\infty(0, 1; L^2(\mathbb{R}^d))$ such that the unique solution u to (P) satisfies $u \notin H^\nu(0, 1; L^2(\mathbb{R}^d))$. This solves Lions' problem in a negative way also for elliptic operators. It would still be interesting to find a counterexample for $\nu = 1$ where the coefficients are Hölder continuous of order α for some $\alpha \leq 1/2$, and this would then be an optimal counterexample due to the positive results for which we refer to the survey [3]. The present coefficient function cannot even be continuous, see Remark 2.8. There seems to be some room for improvement in the regularity of B as the regularity is much worse than H^1 in time.

Theorem 2.7 (Failure of maximal L^2-regularity on $L^2(\mathbb{R}^d)$). *Let $d \geq 2$. For every $\nu \in (1/2, 1]$, there exists $B : [0,1] \times \mathbb{R}^d \to \mathbb{C}^{d\times d}$ elliptic and $f \in L^\infty(0,1; L^2(\mathbb{R}^d))$ such that the unique solution to (P) satisfies $u \notin H^\nu(0,1; L^2)$.*

Proof. The strategy is as for the case $p > 2$ in Theorem 2.5. Let $\nu \in (1/2, 1]$ and let $0 < \theta < \frac{1}{2\nu}$. Then define parameters r and s through $\frac{1}{r} = \frac{1}{2} - \nu\theta$ and $\frac{1}{s} = \frac{1}{2} - \frac{1-\theta}{d}$. They satisfy the condition in Theorem 2.5, consequently there is $B : [0,1] \times \mathbb{R}^d \to \mathbb{C}^{d\times d}$ elliptic and $f \in L^\infty(0,1; L^2(\mathbb{R}^d))$ such that the unique solution u to (P) satisfies $u \notin L^r(0,1; L^s(\mathbb{R}^d))$. Suppose that $u \in H^\nu(0,1; L^2(\mathbb{R}^d))$. Since $u \in L^2(0,1; H^1(\mathbb{R}^d))$ by Theorem 2.1, complex interpolation (see [16, Theorem 14.7.12]) yields $u \in H^{\nu\theta}(0,1; H^{1-\theta}(\mathbb{R}^d))$. By choice of θ, the Sobolev embedding is applicable and gives $u \in L^r(0,1; L^s(\mathbb{R}^d))$, a contradiction. □

Remark 2.8. By construction, the given counterexample is at the same time a counterexample for Problem 1. We have discussed in the introduction that if $t \mapsto B(t) \in L^\infty(\mathbb{R}^d)$ is (piecewise relatively) continuous, then Problem 1 is automatically true for all $p \in (1, \infty)$. Therefore, the function B in Theorem 2.7 is not continuous as a map into $L^\infty(\mathbb{R}^d)$.

The examples show some limitations of what regularity estimates can hold for elliptic operators in divergence form. However, several issues remain for Problems 1 and 2. For instance, the operator $\mathcal{A}$ used in the above counterexamples is not symmetric/hermitian. We do not know what can be said for the case $d = 1$. The only reference we found on the one-dimensional setting is [18], where counterexamples are given in case of L^p-integrability in time and space for both the divergence and non-divergence setting for the range $p \in (1, 3/2) \cup (3, \infty)$. Finally, it would be interesting to see what can be said about Problem 2 if $\mathcal{A}$ is more regular in time (i.e., continuous or Hölder continuous).

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