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# On the Numerical Solution of the Near Field Refractor Problem

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**Abstract**
A numerical scheme is presented to solve the one source near field refractor problem to arbitrary precision and it is proved that for a given error, the scheme terminates in a finite number of iterations. The convergence of the algorithm depends upon proving appropriate Lipschitz estimates for the refractor measure. The algorithm is presented in general terms and has independent interest.

**Keywords** Geometric optics · Iterative methods · Descartes ovals

**Mathematics Subject Classification** 35J96 · 78A05 · 52A15 · 35R30

## 1 Introduction

Let $\Omega \subset S^{n-1}$ be a domain and suppose for each point $x \in \Omega$, a light ray with direction $x$ emanates from a punctual source at the origin $O$, with intensity density function $f(x)$, where $f > 0\, a.e.$ on $\Omega$ and $f \in L^1(\Omega)$ . Suppose $D \subset \mathbb{R}^n$, the target we want to illuminate, is a domain contained in an $n-1$ dimensional hyper-surface, with $\overline{D}$ compact and $O \notin \overline{D}$. Let $\mu$ be a Radon measure on $\overline{D}$ satisfying the mass balance condition

$$\mu(\overline{D}) = \int_\Omega f(x)\, dx.$$

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Given two homogeneous and isotropic media $I$ and $II$ with refractive indices $n_1, n_2$, respectively, so that the point source at $O$ is surrounded by medium $I$ and the target domain $D$ is surrounded by medium $II$, the *near field refractor problem* is to find an interface $\mathcal{S}$ between media $I$ and $II$ parametrized by $\mathcal{S} = \{\rho(x)x : x \in \overline{\Omega}\}$ so that each ray with direction $x \in \Omega$ is refracted into $D$ according to Snell law and so that the energy conservation condition

$$\int_{\mathcal{T}_{\mathcal{S}}(F)} f(x)\,dx = \mu(F)$$

holds for all $F \subset D$, where $\mathcal{T}_{\mathcal{S}}(F)$ represents the directions $x \in \Omega$ that are refracted into $F$, see Definition 2.2. Existence of solutions to this problem is obtained in [8].

The purpose of this paper is to present an iterative scheme to find approximate solutions for this problem with arbitrary precision when $\mu$ is a discrete measure and prove that the scheme gives the desired result in finite number of iterations. The physical problem is three dimensional, but we carry out the analysis in $n$ dimensions.

A similar iterative scheme was developed in [4] to solve the far field refractor problem, extended in [1] to deal with generated Jacobian equations, and in particular, used in [9] for mass transport problems with cost functions satisfying the MTW condition given in [10]. The algorithm arises in works by Caffarelli et al. [3] for far field reflectors, in Bertsekas [2] for the assignment problem, and in Oliker and Prussner [12] for the 2d Monge–Ampère equation. In [11] a general description of the method along with some numerical simulations is carried out for the near field refractor problem. However, no rigorous analysis of the numerical solution to the problem is described there. A major advance and simplification to solve these kind of problems numerically is introduced in [1,4] by showing that an appropriate mapping satisfies a Lipschitz condition. This essential step guarantees that the algorithm converges in a finite number of iterations, and we stress that this does not require smoothness of the density function $f$.

The difficulty in extending these ideas when dealing with the near field refractor problem is that it has a complicated geometrical structure given by Descartes ovals requiring non trivial analytical estimates for the derivatives of these ovals, and it does not have a mass transport structure. Moreover, we present an abstract form of the algorithm having independent interest, that slightly extends the one in [1], and might be useful to solve other problems with similar features.

The model proposed for near field refraction, as well as several other models in optics, fits under the general class of generated Jacobian equations introduced by Trudinger [13]; see also [6] for further applications and extensions. In the present case, the generating function is defined on the unit sphere and given by $G(x, P, b) = 1/h(x, P, b)$, where $h(x, P, b)$ is the polar radius of a Descartes oval given by (2.2).

The plan of the paper is as follows. Section 2 contains preliminary results, the set up, and definitions needed in the rest of the paper. In Sect. 3 we present estimates of the derivatives of Descartes ovals and lower gradient estimates under structural conditions on the discretization of the target $D$, Proposition 3.1. We will use these estimates in Section 4 to prove a one sided Lipschitz estimate of the refractor measure, Theorem 4.1. A Lipschitz estimate is a crucial ingredient to prove that the abstract algorithm introduced in Sect. 6, terminates in finitely many steps as shown in Sect. 6.3; in particular, when applied to the near field refractor problem. In Sect. 5, we introduce a class of admissible vectors that will be used to apply the abstract algorithm to the near field refractor. Finally, in Sect. 7 we show the application of the algorithm to solve the near field refractor problem.

## 2 Set up and Definitions

In this section, we recall Snell's law of refraction, discuss some properties of the building blocks from which we construct near field refractors, and state geometric conditions between the set of incident directions and target to guarantee existence of solutions. In addition, we give a precise statement of the problem solved in the paper.

### 2.1 Snell's Law of Refraction

If from a point source of light located at the origin and surrounded by media $I$, with refractive index $n_1$, a ray of light emanates with unit direction $x$ and strikes an interface $\mathcal{S}$ between medium I and medium II at a point $P$, then this light ray is refracted into the unit direction $m$ in medium $II$, with refractive index $n_2$, according to Snell's law given in vector form as

$$x - \kappa m = \lambda \nu \tag{2.1}$$

with $\kappa = n_2/n_1$ and $\lambda = x \cdot \nu - \kappa\sqrt{1 - \kappa^{-2}(1 - (x \cdot \nu)^2)}$, where $\nu$ is the unit normal at $P$ pointing towards medium $II$. From this the standard Snell's law follows: $n_1 \sin\theta_i = n_2 \sin\theta_r$, with $\theta_i$ the angle of incidence between $x$ and $\nu$, and $\theta_r$ the angle of refraction between $m$ and $\nu$. When $\kappa < 1$, waves propagate in medium I slower than in medium II and depending on the angle of incidence total internal reflection may occur, i.e., incident light may be totally reflected back into medium $I$ and not transmitted to medium II. If $x \cdot \nu \geq \sqrt{1 - \kappa^2}$ or equivalently, $x \cdot m \geq \kappa$, then there is no total internal reflection; see [7, Sect. 2.1].

We will assume throughout the paper that $\kappa < 1$. The analysis for $\kappa > 1$ is similar, requiring the properties of ovals proved in [8, Sect. 4.2]. When $\kappa > 1$, waves propagate in medium I faster than in medium II, and the physical differences between these two settings are explained in detail in [7, Sect. 2.1] and summarized in Lemma 2.1 there.

### 2.2 Descartes Ovals

The treatment of the near field refractor problem requires the use of Descartes ovals, which have a special refraction property. For $P \in \mathbb{R}^n$ and $\kappa|P| < b < |P|$, a refracting Descartes oval (see Fig. 1) is the surface

$$\mathcal{O}_b(P) = \left\{ h(x, P, b)x : x \in S^{n-1},\ x \cdot P \geq b \right\}$$

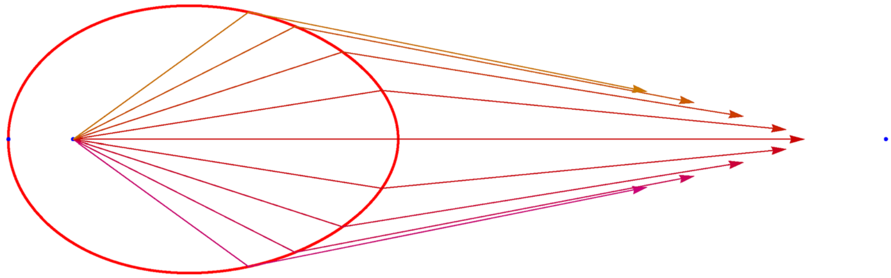

**Fig. 1** Refracting oval

where

$$h(x, P, b) = \frac{(b - \kappa^2 x \cdot P) - \sqrt{(b - \kappa^2 x \cdot P)^2 - (1 - \kappa^2)(b^2 - \kappa^2 |P|^2)}}{1 - \kappa^2}. \tag{2.2}$$

If the region inside the oval $\{h(x, P, b)x : x \in S^{n-1}\}$ is made of a material with refractive index $n_1$ and the outside made of material with refractive index $n_2$, then using Snell's law it can be shown that each light ray emanating from the origin $O$ and having direction $x \in S^{n-1}$ with $x \cdot P \geq b$ is refracted by the oval $\mathcal{O}_b(P)$ into the point $P$. See [8, Sect. 4] for detailed discussion.

### 2.3 Statement of the Problem

As in [8] we will impose the following two geometric configuration conditions on $\Omega$ and $D$ to formulate the main problem; we assume the surface measure of the boundary $\partial\Omega$ in $S^{n-1}$ is zero. The first condition is to avoid total internal reflection and the second is to guarantee that the target doesn't block itself:

**H.1** there exists $\tau$, with $0 < \tau < 1 - \kappa$, such that $x \cdot P \geq (\kappa + \tau)|P|$ for all $x \in \Omega$ and $P \in D$;

**H.2** let $0 < r_0 < \dfrac{\tau}{1+\kappa} \operatorname{dist}(O, D)$ and $Q_{r_0} = \{t\,x : x \in \Omega, 0 \leq t < r_0\}$. Then given $X \in Q_{r_0}$ each ray emanating from $X$ intersects $D$ in at most one point, that is, $\bar{D} \cap \{X + t\,m : m \in S^{n-1}, t \geq 0\}$ is at most one point.

Note that if $r_0$ satisfies **H.2**, then $r_0 < \dfrac{1-\kappa}{1+\kappa}|P|$ for $P \in D$. We shall prove that if

$$\kappa|P| < b \leq (1+\kappa)\,r_0 + \kappa|P| \tag{2.3}$$

then the oval $\mathcal{O}_b(P)$ refracts all directions $x \in \Omega$ into $P$. For this, we only need to verify that there is no internal reflection, that is, $x \cdot P \geq b$ for all $x \in \Omega$. Indeed,

$$\begin{aligned} b &\leq (1+\kappa)\,r_0 + \kappa|P| < (1+\kappa)\,\frac{\tau}{1+\kappa} \operatorname{dist}(0, D) + \kappa|P| \quad \text{from } \mathbf{H.2} \\ &\leq (\tau+\kappa)\,|P| \leq x \cdot P \quad \text{from } \mathbf{H.1}. \end{aligned} \tag{2.4}$$

Near field refractors are then defined as follows.

**Definition 2.1** A surface $\mathcal{S} = \{x\rho(x) : x \in \overline{\Omega}\} \subset Q_{r_o}$ is said to be a near field refractor if for any point $y\rho(y) \in \mathcal{S}$ there exists a point $P \in \overline{D}$ and $\kappa|P| < b < |P|$ such that the refracting oval $\mathcal{O}_b(P)$ supports $\mathcal{S}$ at $y\rho(y)$, i.e. $\rho(x) \leq h(x, P, b)$ for all $x \in \overline{\Omega}$ with equality at $x = y$.

We remark that if (**H.1**) and (**H.2**) hold, then each near field refractor is Lipschitz [8, Lemma 5.3].

The refractor map is as follows.

**Definition 2.2** Given a near field refractor $\mathcal{S}$, the near field refractor mapping of $\mathcal{S}$ is the multi-valued map defined for $x \in \overline{\Omega}$ by

$$\mathcal{R}_{\mathcal{S}}(x) = \{P \in \overline{D} : \text{ there exists a supporting oval } \mathcal{O}_b(P) \text{ to } \mathcal{S} \text{ at } \rho(x)x\}.$$

Given $P \in \overline{D}$ the near field tracing mapping of $\mathcal{S}$ is defined by

$$\mathcal{T}_{\mathcal{S}}(P) = \mathcal{R}_{\mathcal{S}}^{-1}(P) = \{x \in \overline{\Omega} : P \in \mathcal{R}_{\mathcal{S}}(x)\}.$$

$\mathcal{T}_{\mathcal{S}}(P)$ is also known as the visibility set of $P$.

Suppose that we are given a nonnegative $f \in L^1(\overline{\Omega})$, $f(x)$ represents the intensity of the light ray emanating from $O$ with direction $x$. We recall the definition of near field refractor measure, based on conservation of energy; see [8, Formula (5.5)].

**Definition 2.3** The near field refractor measure associated with the near field refractor $\mathcal{S}$ and the function $f \in L^1(\overline{\Omega})$ is the finite Borel measure given by

$$\mathcal{M}_{\mathcal{S},f}(F) = \int_{\mathcal{T}_{\mathcal{S}}(F)} f\,dx$$

for every Borel set $F \subset \overline{D}$.

Given a Radon measure $\mu$ defined on $D$ and the energy conservation condition $\int_\Omega f\,dx = \mu(D)$, *the near field refractor problem* is to find a near field refractor $\mathcal{S}$ such that

$$\mathcal{M}_{\mathcal{S},f}(F) = \mu(F)$$

for any Borel set $F \subset \overline{D}$. Under conditions **H.1** and **H.2** above, the existence of such a surface $\mathcal{S}$ is proved in [8]. In particular, when $\mu$ is discrete, $\mu = \sum_{i=1}^N g_i \delta_{P_i}$ with $P_1, \ldots, P_N$ distinct points in $\overline{D}$ and $g_1, \ldots, g_N$ positive numbers, it is proved in [8, Thm. 5.7] that if $f \in L^1(\Omega)$ is positive a.e. in $\Omega$ such that

$$\int_\Omega f(x)dx = \sum_{i=1}^N g_i \tag{2.5}$$

then given $\kappa|P_1| < b_1 < \kappa|P_1| + r_0\dfrac{(1-\kappa)^2}{1+\kappa}$, ($r_0$ in Hypothesis **H.2** above) there exists a unique vector $\mathbf{b} = (b_1, b_2, \ldots, b_N) \in \prod_{i=1}^{N}(\kappa|P_i|, |P_i|)$ such that the poly-oval

$$\mathcal{S}(\mathbf{b}) = \{\rho(x)x : x \in \overline{\Omega} \text{ and } \rho(x) = \min_{1\le i\le N} h(x, P_i, b_i)\} \tag{2.6}$$

is a near field refractor satisfying

$$\mathcal{M}_{\mathcal{S}(\mathbf{b}),f}(P_i) = g_i, \quad 1 \le i \le N,$$

and therefore is a solution to the near field refractor problem when $\mu$ is discrete.

*The main purpose of this paper is* to discuss an iterative scheme to approximate this unique vector $\mathbf{b}$ and consequently the refractor $\mathcal{S}(\mathbf{b})$ with arbitrary precision and to show that for a given error the scheme converges in a finite number of steps. That is, given $f$; $g_1, \ldots, g_N$; $P_1, \ldots, P_N$, $b_1$ satisfying $\kappa|P_1| < b_1 \le \kappa|P_1| + r_0\dfrac{(1-\kappa)^2}{1+\kappa}$ and $\epsilon > 0$, we demonstrate a scheme to seek a vector $\mathbf{b} = (b_1, b_2, \cdots, b_N) \in \prod_{i=1}^{N}(\kappa|P_i|, |P_i|)$, which depends on $\epsilon$, such that the poly-oval refractor $\mathcal{S}(\mathbf{b})$ satisfies

$$|\mathcal{M}_{\mathcal{S}(\mathbf{b}),f}(P_i) - g_i| \le \epsilon, \quad 1 \le i \le N. \tag{2.7}$$

### 2.4 Properties of the Refractor Mapping

In this subsection we prove some results that will be needed to apply the algorithm from Sect. 6 to solve the main problem. In the proof of these results, and in the subsequent sections, the following part of [8, Lemma 4.1] will be used frequently.

**Lemma 2.4** *Let* $0 < \kappa < 1$, $h(x, P, b)$ *given by* (2.2)*, and assume that* $\kappa|P| < b < |P|$. *Then*

$$\min_{x\in S^{n-1}} h(x, P, b) = \frac{b-\kappa|P|}{1+\kappa}, \quad \text{and} \quad \max_{x\in S^{n-1}} h(x, P, b) = \frac{b-\kappa|P|}{1-\kappa}. \tag{2.8}$$

We begin with the following monotonicity property.

**Lemma 2.5** *Let* $\boldsymbol{b} = (b_1, \ldots, b_N)$ *and* $\boldsymbol{b}^* = (b_1^*, \ldots, b_N^*)$ *be in* $\prod_{i=1}^{N}(\kappa|P_i|, (1+\kappa) r_0 + \kappa|P_i|)$. *Suppose that for some* $l$, $b_l^* \le b_l$ *and for all* $i \ne l$, $b_i^* = b_i$, *where* $1 \le l, i \le N$. *Then*

$$\mathcal{T}_{\mathcal{S}(\boldsymbol{b})}(P_l) \subseteq \mathcal{T}_{\mathcal{S}(\boldsymbol{b}^*)}(P_l) \tag{2.9}$$

*and*

$$\mathcal{T}_{\mathcal{S}(\boldsymbol{b}^*)}(P_i) \subseteq \mathcal{T}_{\mathcal{S}(\boldsymbol{b})}(P_i) \quad \text{for } i \ne l, \tag{2.10}$$

*where the inclusions are up to a set of measure zero. Consequently*

$$\mathcal{M}_{\mathcal{S}(\boldsymbol{b}),f}(P_l) \leq \mathcal{M}_{\mathcal{S}(\boldsymbol{b}^*),f}(P_l) \text{ and } \mathcal{M}_{\mathcal{S}(\boldsymbol{b}),f}(P_i) \geq \mathcal{M}_{\mathcal{S}(\boldsymbol{b}^*),f}(P_i) \quad \text{for } i \neq l.$$

***Proof*** From its definition, $\mathcal{S}(\mathbf{b})$ is differentiable a.e., so the set of singular points has surface measure zero. We use here that if $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_l)$ is not a singular point, then the oval $\mathcal{O}_{b_l}(P_l)$ supports $\mathcal{S}(\mathbf{b})$ at $x_0$; this holds for any near field refractor $\mathcal{S}(\mathbf{b})$ and any $1 \leq l \leq N$; see [8, Proof of Lemma 5.4], we are using here that the surface measure $|\partial\Omega| = 0$.

We first prove (2.10) when $x_0$ is not a singular point of $\mathcal{S}(\mathbf{b}^*)$. Since $b_l^* \leq b_l$, from (2.4), and differentiating (3.1) with respect to $b$, it follows that $\Delta(x \cdot P_l, b_l^*, |P_l|) \geq \Delta(x \cdot P_l, b_l, |P_l|)$ so $h(x, P_l, b_l) \geq h(x, P_l, b_l^*)$ and therefore $\rho^*(x) \leq \rho(x)$ for all $x \in \Omega$, where $\rho^*$ is the parametrization of $\mathcal{S}(\mathbf{b}^*)$ and $\rho$ is the parametrization of $\mathcal{S}(\mathbf{b})$. Suppose $i \neq l$ and let $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b}^*)}(P_i)$. Then, since $x_0$ is not a singular point of $\mathcal{S}(\mathbf{b}^*)$, the oval with polar radius $h(x, P_i, b_i)$ supports $\mathcal{S}(\mathbf{b}^*)$ at $x_0$. We have $\rho(x) \leq h(x, P_i, b_i)$. Therefore

$$h(x_0, P_i, b_i) = \rho^*(x_0) \leq \rho(x_0) \leq h(x_0, P_i, b_i),$$

that is, $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)$.

We now prove (2.9). That is, we prove that if $x_0$ is neither a singular point of $\mathcal{S}(\mathbf{b})$ nor a singular point of $\mathcal{S}(\mathbf{b}^*)$, $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_l)$, and $x_0 \notin \partial\Omega$, then $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b}^*)}(P_l)$. We may assume $b_l^* < b_l$. We have that the oval $\mathcal{O}_{b_l}(P_l)$ supports $\mathcal{S}(\mathbf{b})$ at $x_0$. We claim that the oval with polar radius $h\left(x, P_l, b_l^*\right)$ supports $\mathcal{S}(\mathbf{b}^*)$ at $x_0$. Suppose this is not true. Since by definition $\rho^*(x) \leq h\left(x, P_l, b_l^*\right)$, we would have $\rho^*(x_0) < h\left(x_0, P_l, b_l^*\right)$. So $\rho^*(x_0) = h\left(x_0, P_j, b_j\right)$ for some $j \neq l$, and therefore $h\left(x, P_j, b_j\right)$ supports $\mathcal{S}(\mathbf{b}^*)$ at $x_0$. Since $x_0$ is not a singular point of $\mathcal{S}(\mathbf{b}^*)$, then by the inclusion previously proved we get that $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_j)$. Since $j \neq l$, $x_0 \notin \partial\Omega$, and $x_0$ is not a singular point of $\mathcal{S}(\mathbf{b})$, the tangent planes to the ovals with polar radii $h\left(x, P_j, b_j\right)$ and $h\left(x, P_l, b_l\right)$ must coincide. Then arguing as in [8, Proof of Lemma 5.4] using Snell's law, we obtain a contradiction with the visibility Condition **H.2**. □

**Lemma 2.6** *Let* $\boldsymbol{b} = (b_1, \ldots, b_N) \in \prod_{i=1}^N (\kappa|P_i|, |P_i|)$. *Consider the family of refractors obtained from* $\mathcal{S}(\boldsymbol{b}) = \{\rho(x)x : x \in \Omega\}$, *by changing only* $b_i$ *and fixing* $b_j$ *for all* $j \neq i$. *Then* $\mathcal{M}_{\mathcal{S}(\boldsymbol{b}),f}(P_i) \to \int_\Omega f(x)\,dx$ *as* $b_i \to (\kappa|P_i|)^+$.

***Proof*** We have $h(x, P_i, b_i) \leq \dfrac{b_i - \kappa|P_i|}{1-\kappa}$ and also $h(x, P_j, b_j) \geq \dfrac{b_j - \kappa|P_j|}{1+\kappa}$ for all $x \in \Omega$, from Lemma 2.4. Let $\delta = \min_{j \neq i} \dfrac{b_j - \kappa|P_j|}{1+\kappa}$. For $b_i$ with $\dfrac{b_i - \kappa|P_i|}{1-\kappa} < \delta$, we then get $h(x, P_i, b_i) < h(x, P_j, b_j)$ for all $x \in \Omega$ and $j \neq i$. So $\rho(x) = h(x, P_i, b_i)$ and so $\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i) = \Omega$ and (a) follows. □

## 3 Estimates for Derivatives of the Polar Radii of Ovals

In this section, we will obtain bounds for the derivatives of the polar radius $h(x, P, b)$. In particular, we prove the lower gradient estimate 3.11 which will be used in the next section to prove a Lipschitz property of the refractor measure. From (2.2) we have

$$h(x, P, b) = \frac{(b - \kappa^2 x \cdot P) - \sqrt{\Delta(x \cdot P, b, |P|)}}{1 - \kappa^2},$$

where

$$\Delta(t, b, |P|) := \left(b - \kappa^2 t\right)^2 - \left(1 - \kappa^2\right)\left(b^2 - \kappa^2 |P|^2\right).$$

By calculation

$$\Delta(t, b, |P|) = \kappa^2 \left((b - t)^2 + (1 - \kappa^2)\left(|P|^2 - t^2\right)\right). \tag{3.1}$$

As a function of $t$, $\Delta$ is increasing in the interval $(b/\kappa^2, +\infty)$ and decreasing in the interval $(-\infty, b/\kappa^2)$. Let $t = x \cdot P$ with $|x| \leq 1+\epsilon$, so $-(1+\epsilon)|P| \leq t \leq (1+\epsilon)|P|$, and we have

$$\min_{|x| \leq 1+\epsilon} \Delta(x \cdot P, b, |P|) = \min_{-(1+\epsilon)|P| \leq t \leq (1+\epsilon)|P|} \Delta(t, b, |P|).$$

Let $\epsilon > 0$ be such that $1 + \epsilon < 1/\kappa$. Since $b > \kappa |P|$, it follows that $[-(1+\epsilon)|P|, (1+\epsilon)|P|] \subset \left(-\infty, b/\kappa^2\right)$. Therefore

$$\min_{-(1+\epsilon)|P| \leq t \leq (1+\epsilon)|P|} \Delta(t, b, |P|) = \Delta\left((1+\epsilon)|P|, b, |P|\right),$$

and

$$\max_{-(1+\epsilon)|P| \leq t \leq (1+\epsilon)|P|} \Delta(t, b, |P|) = \Delta\left(-(1+\epsilon)|P|, b, |P|\right).$$

We have

$$\begin{aligned}\Delta\left(-(1+\epsilon)|P|, b, |P|\right) &= \kappa^2 \left(((1+\epsilon)|P| + b)^2 + (1-\kappa^2)|P|^2\left(1 - (1+\epsilon)^2\right)\right)\\ &\leq C(\kappa)\, |P|^2\end{aligned}$$

for $\kappa|P| < b < |P|$, so

$$\Delta(x \cdot P, b, |P|) \leq C(\kappa)\, |P|^2, \quad \text{for } |x| \leq 1 + \epsilon. \tag{3.2}$$

Let us estimate $\Delta\left((1+\epsilon)|P|, b, |P|\right)$ from below. For this, we assume $b$ satisfies (2.3) and recall assumptions **H.1** and **H.2** .

We write

$$\Delta((1+\epsilon)|P|, b, |P|) = \kappa^2 \left(((1+\epsilon)|P| - b)^2 + (1-\kappa^2)|P|^2\left(1 - (1+\epsilon)^2\right)\right)$$

$$= \kappa^2 \left( (|P| - b)^2 + 2\epsilon\,|P|\,(|P| - b) \right.$$
$$\left. +\epsilon \left( \epsilon - (1-\kappa^2)\ (2+\epsilon) \right) |P|^2 \right)$$
$$\geq \kappa^2 \left( (|P| - b)^2 + \epsilon \left( \epsilon - (1-\kappa^2)\ (2+\epsilon) \right) |P|^2 \right).$$

We have

$$\begin{aligned} |P| - b &> (1-\kappa)|P| - (1+\kappa)\, r_0 \qquad \text{from (2.3)} \\ &\geq \frac{1+\kappa}{\tau}(1-\kappa) r_0 - (1+\kappa) r_0 \qquad \text{from } \mathbf{H.2} \text{ since } P \in D \\ &= (1+\kappa)\left(\frac{1-\kappa}{\tau} - 1\right) r_0 := \delta > 0 \end{aligned}$$

since $\tau < 1-\kappa$ from **H.1**. Since $D$ is a bounded set, $|P| \leq M$ for all $P \in D$, and since $\epsilon - (1-\kappa^2)\ (2+\epsilon) < 0$ for $\epsilon > 0$ small, we get

$$\epsilon \left( \epsilon - (1-\kappa^2)\ (2+\epsilon) \right) |P|^2 \geq \epsilon \left( \epsilon - (1-\kappa^2)\ (2+\epsilon) \right) M^2 \geq -\delta^2/2$$

choosing $\epsilon > 0$ small enough.

Therefore we obtain that there are structural constants $C_0 > 0$ and $\epsilon > 0$ such that

$$\Delta(x \cdot P, b, |P|) \geq C_0 \tag{3.3}$$

for all $|x| \leq 1+\epsilon$, $P \in D$ and $b$ satisfying (2.3), consequently, formula (2.2) can be extended and is then well defined for all these values. In particular, (2.2) can be differentiated with respect to $x$ for all $|x| \leq 1+\epsilon$ obtaining

$$\nabla_x h(x, P, b) = \frac{\kappa^2 h(x, P, b)}{\sqrt{\Delta\,(x \cdot P, b, |P|)}}\, P \qquad \text{for } |x| \leq 1+\epsilon. \tag{3.4}$$

**Upper bound for the norm of** $\nabla_x h(x, P, b)$. From (3.4), we only need to estimate the extension $h$ from above and $\sqrt{\Delta}$ from below. To estimate the extension $h$ we have from (2.2) that

$$h(x, P, b) \leq C_\kappa\,|P|, \qquad \text{for } |x| \leq 1+\epsilon \text{ and } \kappa|P| < b < |P|,$$

and to estimate $\sqrt{\Delta}$ from below we use (3.3) obtaining

$$|\nabla_x h\,(x, P, b)| \leq \frac{C_k}{\sqrt{C_0}}\,|P|^2 \qquad \text{for all } |x| \leq 1+\epsilon \tag{3.5}$$

when $b$ satisfies (2.3) and conditions **H.1** and **H.2** hold.

**Bounds for $h_b$:**

$$h_b = \frac{1 - \frac{1}{2}\Delta^{-1/2}\Delta_b}{1-\kappa^2} = \frac{1 - \Delta^{-1/2}\kappa^2(b - x \cdot P)}{1-\kappa^2} = \frac{\Delta^{1/2} - \kappa^2(b - x \cdot P)}{\left(1-\kappa^2\right)\sqrt{\Delta}}$$
$$= \frac{1}{1-\kappa^2} + \frac{\kappa^2\ (x \cdot P - b)}{\left(1-\kappa^2\right)\sqrt{\Delta}}.$$

From (3.1), $\Delta(x \cdot P, b, |P|) \geq \kappa^2\,(x \cdot P - b)^2$ for $|x| \leq 1$. If $b$ satisfies (2.3), then from (2.4) $x \cdot P \geq b$ (that avoids internal reflection), and so $\sqrt{\Delta} \geq \kappa\ (x \cdot P - b)$. Therefore

$$\frac{1}{1-\kappa^2} \leq h_b \leq \frac{1}{1-\kappa^2} + \frac{\kappa}{1-\kappa^2} = \frac{1}{1-\kappa}, \quad \text{for } |x| \leq 1. \tag{3.6}$$

**Bounds for $\mathcal{G}_{ij}$:**

For $i \neq j$, $\kappa|P_i| < a \leq (1+\kappa)\, r_0 + \kappa|P_i|$, $\kappa|P_j| < b \leq (1+\kappa)\, r_0 + \kappa|P_j|$, and $|x| \leq 1+\epsilon$ let's define

$$\mathcal{G}_{ij}\,(x, a, b) := h\,(x, P_i, a) - h\left(x, P_j, b\right). \tag{3.7}$$

From the analysis above this function is differentiable with respect to $x$ for $|x| \leq 1+\epsilon$. Suppose at some $|x| \leq 1+\epsilon$ and for some $a, b$ with $\kappa|P_i| + \delta_i \leq a \leq (1+\kappa)\, r_0 + \kappa|P_i|$; $\kappa|P_j| + \delta_j \leq b \leq (1+\kappa)\, r_0 + \kappa|P_j|$ we would have

$$\nabla_x \mathcal{G}_{ij}\,(x, a, b) = 0. \tag{3.8}$$

Then from (3.4)

$$\frac{\kappa^2 h(x, P_i, a)}{\sqrt{\Delta\,(x \cdot P_i, a, |P_i|)}}\, P_i = \frac{\kappa^2 h(x, P_j, b)}{\sqrt{\Delta\left(x \cdot P_j, b, |P_j|\right)}}\, P_j$$

and since the coefficients in front of $P_i$ and $P_j$ are not zero it follows that $P_i$ is a multiple of $P_j$, violating the visibility Condition **H.2** taking $X = 0 \in Q_{r_0}$.

Therefore by continuity

$$\min_{\substack{\kappa|P_i|+\delta_i \leq a \leq (1+\kappa)\, r_0 + \kappa|P_i| \\ \kappa|P_j|+\delta_j \leq b \leq (1+\kappa)\, r_0 + \kappa|P_j|}} \left|\nabla_x \mathcal{G}_{ij}\,(x, a, b)\right| = \lambda > 0 \tag{3.9}$$

for all $x \in \Omega$.

The estimate in the following proposition will be used in the proof of Theorem 4.1 via Proposition 4.4. Its proof requires the structural condition (3.10). We notice that from Condition **H.1**, we get $\Omega \subset \cap_{i=1}^{N} \left\{x \in S^{n-1} : x \cdot \frac{P_i}{|P_i|} \geq \kappa + \tau\right\}$ where $0 < \kappa + \tau < 1$ which implies that $\bar{\Omega}$ is contained in the intersection of the half-spaces

$\{z : z \cdot P_i > 0\}$. Therefore, one can write bijectively points $x \in \Omega$ with coordinates $u = (u_1, \ldots, u_{n-1}), x = x(u)$, so that $\{x_{u_1}, \ldots, x_{u_{n-1}}, x\}$ is a local orthogonal frame in $S^{n-1}$; and $\Omega = x(F)$ for some $F \subset \mathbb{R}^{n-1}$.

**Proposition 3.1** *Assume that conditions* **H.1** *and* **H.2** *hold. Fix $i, j$ with $1 \le i, j \le N$ and $i \neq j$, $F \subset \mathbb{R}^{n-1}$ with $x(F) = \Omega$.*

*We assume the following structural condition: if $\Pi_{ij}$ is the plane containing the origin $O$ and the points $P_i, P_j$, then*

$$\Omega \cap \Pi_{ij} = \emptyset. \tag{3.10}$$

*If $\delta_i, \delta_j$ are positive, then there exists a constant $\lambda > 0$ depending on $\delta_i, \delta_j$ and $F$ such that*

$$\min_{\substack{\kappa|P_i|+\delta_i \le a \le (1+\kappa) r_0+\kappa|P_i| \\ \kappa|P_j|+\delta_j \le b \le (1+\kappa) r_0+\kappa|P_j|}} \left|\nabla_u \left(\mathcal{G}_{ij} (x(u), a, b)\right)\right| \ge \lambda \tag{3.11}$$

*for all $u \in F$.*[1]

***Proof*** By contradiction. Suppose at some $u \in F$ and for some $a, b$ with $\kappa|P_i| + \delta_i \le a \le (1+\kappa) r_0 + \kappa|P_i|$; $\kappa|P_j| + \delta_j \le b \le (1+\kappa) r_0 + \kappa|P_j|$ we have

$$\nabla_u \left(\mathcal{G}_{ij} (x(u), a, b)\right) = 0. \tag{3.12}$$

That is, $\frac{\partial}{\partial u_k} \left(\mathcal{G}_{ij} (x(u), a, b)\right) = \nabla_x \mathcal{G}_{ij} (x(u), a, b) \cdot x_{u_k} = 0$ for $1 \le k \le n-1$. Since $\{x_{u_1}, \ldots, x_{u_{n-1}}, x\}$ is an orthogonal frame, we get that the vector $\nabla_x \mathcal{G}_{ij} (x(u), a, b)$ is parallel to $x$. From (3.4) we then get that

$$\frac{\kappa^2 h(x(u), P_i, a)}{\sqrt{\Delta\left(x(u) \cdot P_i, a, |P_i|\right)}} P_i - \frac{\kappa^2 h(x(u), P_j, b)}{\sqrt{\Delta\left(x(u) \cdot P_j, b, |P_j|\right)}} P_j \text{ is parallel to } x(u),$$

which is a contradiction with (3.10). □

***Remark 3.2*** Therefore, if the target points $P_1, \ldots, P_N$ satisfy

$$\Omega \cap \Pi_{ij} = \emptyset, \quad \forall i \neq j, \tag{3.13}$$

then (3.11) holds for all $i \neq j$. To understand this condition, let $\nu_{ij}$ be a normal to $\Pi_{ij}$, that is, $\nu_{ij}$ is parallel to the vector $\overrightarrow{OP_i} \times \overrightarrow{OP_j}$. Given $\Omega \subset S^2$ in the upper sphere,

[1] Notice that the minimum in (3.11) over the full range $\kappa|P_i| < a \le |P_i|$; $\kappa|P_j| < b \le |P_j|$ is zero. Because, if for example, $b \to \left(\kappa|P_j|\right)^+$, then $\Delta(x(u) \cdot P_j, b) \to \kappa^2 \left(|P_j| - \kappa\, x(u) \cdot P_j\right)^2 > 0$ since $\kappa < 1$. On the other hand, $h(x(u), P_j, b) \to 0$ as $b \to \left(\kappa|P_j|\right)^+$. Therefore from (3.4), $\nabla_u h\left(x(u), P_j, b\right) \to 0$ as $b \to \left(\kappa|P_j|\right)^+$; and similarly $\nabla_u h\left(x(u), P_i, a\right) \to 0$ as $a \to (\kappa|P_i|)^+$. The reader can compare (3.11) with [13, Condition G1]; and concerning the limitations for $a, b$ in (3.11) see [6, Definition 4.1].

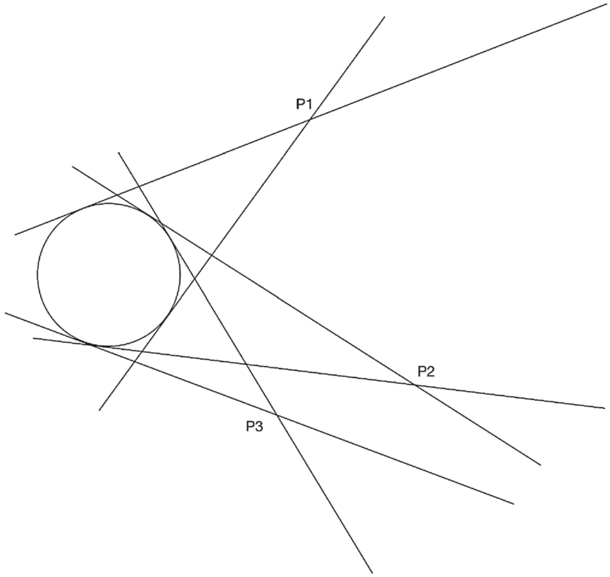


**Fig. 2** Illustration Remark 3.3

let $\Omega^{\perp} \subset S^2$ the orthogonal set of vectors

$$\Omega^{\perp} = \{y \in S^2 : \text{ there exists } x \in \Omega \text{ such that } y \cdot x = 0\}.$$

So (3.11) holds for all $i \neq j$ if the set of vectors $\nu_{ij}$ is contained in the complement of $\Omega^{\perp}$.

For example, if the points $P_1, \cdots, P_N$ lie on a plane through the origin that does not intersect $\Omega$, and so that any pair $(P_i, P_j)$ is not aligned with the origin, then (3.13) holds.

***Remark 3.3*** To illustrate (3.13), suppose the target $D$ is contained on the plane $z = a$. We can select points in $D$ in the following way so that (3.13) holds. Let $P_1 \in D$ so that the line $OP_1$ does not intersect $\Omega$ and consider $\mathcal{C}_1$ the collection of all planes containing the points $O$ and $P_1$ that intersect $\Omega$. Pick $P_2 \in D$ with $P_2 \notin \mathcal{C}_1$. Let $\mathcal{C}_2$ be the collection of all planes containing the points $O$ and $P_2$ that intersect $\Omega$. Pick $P_3 \in D$ such that $P_3 \notin \mathcal{C}_1 \cup \mathcal{C}_2$. Next let $\mathcal{C}_3$ be the collection of all planes containing the points $O$ and $P_3$ that intersect $\Omega$, and pick $P_4 \in D$ with $P_4 \notin \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Continuing in this way we choose points in $D$ so that (3.13) holds. This is illustrated in Fig. 2: the circle represents the radial projection of $\Omega$ over the plane $z = a$, the cone through $P_i$ represents the trace of $\mathcal{C}_i$ on the plane $z = a$.

## 4 Lipschitz Estimate of the Refractor Map

In this section, we will prove the one sided Lipschitz estimate 4.6, for the refractor measure. This result is a crucial ingredient to prove that the algorithm converges to the desired result in finitely many steps when applied to near field refractor problem.

Let $\mathbf{b} = (b_1, \dots, b_N) \in \prod_{i=1}^N (\kappa|P_i|, |P_i|)$ and let $\mathbf{e}_i$ be the unit direction in the $i$-th coordinate. For $\kappa|P_i| < b_i - t \le b_i$, define $\mathbf{b}^t = \mathbf{b} - t\,\mathbf{e}_i$. The domain $\Omega \subset S^{n-1}$ of incident directions is identified with $F \subset \mathbb{R}^{n-1}$ so that $x(F) = \Omega$ where $x = x(u)$ and $u$ the coordinates used in Proposition 3.1. Next we define the sets

$$V_{i,j}^{\mathbf{b}} := \left\{x \in \Omega : h(x, P_i, b_i) \le h(x, P_j, b_j)\right\}, \tag{4.1}$$

$$V_{i,j}^{\mathbf{b}^t} := \left\{x \in \Omega : h(x, P_i, b_i - t) \le h(x, P_j, b_j)\right\}, \tag{4.2}$$

$$V_i^{\mathbf{b}} := \bigcap_{j \ne i} V_{i,j}^{\mathbf{b}} = \{x \in \Omega : \rho_{\mathbf{b}}(x) = h(x, P_i, b_i)\}, \tag{4.3}$$

where $\rho_{\mathbf{b}}(x) = \min_{1 \le k \le N} h(x, P_k, b_k)$.

Since $t > 0$, from (3.6) $h$ is increasing in the last variable so $V_{i,j}^{\mathbf{b}} \subset V_{i,j}^{\mathbf{b}^t}$ for all $j \ne i$. Hence $V_i^{\mathbf{b}} \subset V_i^{\mathbf{b}^t}$. Since in the arguments in this section the vector $\mathbf{b}$ will be fixed, we adopt the short-hand

$$V_{i,j} := V_{i,j}^{\mathbf{b}}, \quad V_{i,j}^t := V_{i,j}^{\mathbf{b}^t}, \quad V_i := V_i^{\mathbf{b}}, \quad V_i^t := V_i^{\mathbf{b}^t}.$$

With this notation, for the refractor measure map given in Definition 2.3, we have

$$\mathcal{M}_{\mathcal{S}(\mathbf{b}), f}(P_i) = \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\, d\sigma(x) = \int_{V_i} f(x)\, d\sigma(x).$$

If for brevity we denote $\mathcal{M}_{\mathcal{S}(\mathbf{b}), f}(P_i)$ by $H_i(\mathbf{b})$ we have ,

$$H_i(\mathbf{b}) = \int_{V_i} f(x)\, d\sigma(x). \tag{4.4}$$

Our goal is to prove the following one-sided Lipschitz estimate for $H_i$.

**Theorem 4.1** *Assume that* **H.1** *and* **H.2** *in Sect.* 2.3 *hold, and the target points* $P_1, \dots, P_N$ *satisfy* (3.13). *Let* $\delta_1, \dots, \delta_N$ *be positive numbers and* $\mathbf{b} = (b_1, \dots, b_N)$ *satisfying*

$$\kappa\,|P_j| + \delta_j \le b_j \le (1+\kappa)\, r_0 + \kappa\,|P_j|, \quad j = 1, \dots, N. \tag{4.5}$$

*Then for each* $1 \le i \le N$ *we have*

$$0 \le H_i(\mathbf{b}^t) - H_i(\mathbf{b}) \le C_0\, \|f\|_{L^\infty(\Omega)}\, t \tag{4.6}$$

*for all $t$ with $\kappa\,|P_i| < b_i - t \le b_i \le (1+\kappa)\, r_0 + \kappa\,|P_i|$, where $\mathbf{b}^t = \mathbf{b} - t\,\mathbf{e}_i$. $C_0$ is a positive constant depending only on the bounds for the derivatives up to order two of the functions $h\,(x(u), P_i, b_i)$ over $u \in F$ $(\Omega = x(F))$ and over $b_i$ satisfying (2.3); in addition $C_0$ depends also on $\delta_i$, $N$, $\kappa$, and the constants in* **H.1** *and* **H.2** .

***Proof*** We have from Lemma 2.5 that

$$0 \le H_i(\mathbf{b}^t) - H_i(\mathbf{b}) = \int\limits_{V_i^t \setminus V_i} f(x)\, d\sigma(x).$$

Using (4.3) we obtain as in the proof of [1, Thm. 5.1] that

$$V_i^t \setminus V_i \subset \bigcup_{j \neq i} \left(V_{i,j}^t \setminus V_{i,j}\right). \tag{4.7}$$

It follows that

$$\begin{aligned} 0 \le H_i(\mathbf{b}^t) - H_i(\mathbf{b}) &= \int\limits_{V_i^t \setminus V_i} f(x)\, d\sigma(x) \le \|f\|_{L^\infty(\Omega)}\, \sigma\left(\bigcup_{j\neq i}\left(V_{i,j}^t \setminus V_{i,j}\right)\right) \\ &\le \|f\|_{L^\infty(\Omega)} \sum_{j \neq i} \sigma\left(V_{i,j}^t \setminus V_{i,j}\right), \end{aligned} \tag{4.8}$$

where $\sigma$ denotes the area measure in the sphere $S^{n-1}$. We proceed to estimate $\sigma\left(V_{i,j}^t \setminus V_{i,j}\right)$ for $j \neq i$. Notice that, by definition of $V_{i,j}$,

$$\begin{aligned} V_{i,j}^t \setminus V_{i,j} &= \left\{x \in \Omega : h(x, P_i, b_i) \ge h(x, P_j, b_j) \ge h(x, P_i, b_i - t)\right\} \\ &= \left\{x \in \Omega : 0 \le h(x, P_i, b_i) - h(x, P_j, b_j) \le h(x, P_i, b_i) - h(x, P_i, b_i - t)\right\}. \end{aligned}$$

If $E$ is a subset of a hemi-sphere in $S^{n-1}$, and $x(u) \in E$ with $u$ the coordinates used to prove Proposition 3.1, then there is $E' \subset \mathbb{R}^{n-1}$ such that $E = x(E')$ and from the formula of change of variables

$$\int_E f(x)\, d\sigma(x) = \int_{E'} f(x(u))\, \sqrt{\det Dx^T Dx}\, du,$$

where $Dx = \dfrac{\partial x_k}{\partial u_j}$, $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_{n-1})$. Recall $\Omega = x(F)$ with $F \subset \mathbb{R}^{n-1}$. Since $V_{i,j}^t \setminus V_{i,j} \subset \Omega \subset S^{n-1}$, let $F_{i,j}^t \subset \mathbb{R}^{n-1}$

$$V_{i,j}^t \setminus V_{i,j} = x\left(F_{i,j}^t\right),$$

and so the surface measure

$$\sigma\left(V_{i,j}^t \setminus V_{i,j}\right) \leq c\,|F_{i,j}^t|_{n-1},$$

where $|\cdot|_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure and $c$ a constant. If

$$\kappa|P_i| < b_i - t \leq (1+\kappa)\,r_0 + \kappa|P_i|, \tag{4.9}$$

then from the bound (3.6) for $h_b$—only depending on $\kappa$—and the mean value theorem we get

$$\begin{aligned} h(x, P_i, b_i) - h(x, P_i, b_i - t) &= h_b(x, P_i, \xi_i)\cdot t \\ &\leq C(\kappa)\,t, \end{aligned}$$

for all $x \in \Omega$. Therefore

$$F_{i,j}^t \subset \left\{u \in F : 0 < \mathcal{G}_{ij}(x(u), b_i, b_j) \leq C(\kappa)\,t\right\}, \tag{4.10}$$

for all $t$ satisfying (4.9) and $j \neq i$, with $\mathcal{G}_{ij}$ from (3.7).

The last set is a region contained between two level sets of the function $\mathcal{G}_{ij}$ and we now estimate the measure of this set. Let us first recall the co-area formula, [5, Sect. 3.4.2, Theorem 1].

**Proposition 4.2** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, and $\Sigma \subset \mathbb{R}^n$ measurable. Then*

$$\int_\Sigma |D\psi(x)|\,dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\Sigma \cap \psi^{-1}(s))\,ds, \tag{4.11}$$

*where $\mathcal{H}^{n-1}(\cdot)$ denotes $(n-1)$-dimensional Hausdorff measure.*

This has the following simple corollary.

**Corollary 4.3** *Let $\psi : \bar{\Omega} \to \mathbb{R}$ be Lipschitz, with $\inf_\Omega |D\psi| \geq \lambda > 0$, $-\infty \leq a \leq b \leq \infty$ and $\Omega \subset \mathbb{R}^n$ a bounded set. Then*

$$\mathcal{L}^n(\{x \in \Omega : a \leq \psi(x) \leq b\}) \leq \frac{1}{\lambda}\int_a^b \mathcal{H}^{n-1}\left(\Omega \cap \psi^{-1}(s)\right)\,ds, \tag{4.12}$$

*$\mathcal{L}^n$ being the $n$-dimensional Lebesgue measure.*

From (3.5) and the lower bound (3.11) we can apply the corollary to the function $\psi(u) = \mathcal{G}_{ij}(x(u), b_i, b_j)$, $j \neq i$, to conclude that

$$\mathcal{L}^{n-1}\left(\left\{u \in F : 0 < \mathcal{G}_{ij}(x(u), b_i, b_j) \leq C\,t\right\}\right)$$

$$\leq \frac{1}{\lambda} \int_0^{Ct} \mathcal{H}^{n-2}\left(F \cap \mathcal{G}_{ij}^{-1}(s)\right) ds; \quad C = C(\kappa). \tag{4.13}$$

We now show that the integrand on the right hand side of (4.13) is uniformly bounded for each $s$ in the range of $\mathcal{G}_{ij}$. For this, we need the following [1, Prop. 5.5].

**Proposition 4.4** *Suppose $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain and $\psi \in C^2(\bar{\Omega})$ satisfies $\min_{x\in\overline{\Omega}} |D\psi(x)| \geq \lambda > 0$ and $\|D\psi\|_{L^\infty(\Omega)}, \|D^2\psi\|_{L^\infty(\Omega)}$ are both finite. For any $s \in \text{Range}(\psi)$, let $\Gamma_s = \{x \in \Omega : \psi(x) = s\}$. Then there exists a constant $K = K\left(\lambda, \|D\psi\|_{L^\infty(\Omega)}, \|D^2\psi\|_{L^\infty(\Omega)}\right)$ such that*

$$\mathcal{H}^{n-1}(\Gamma_s) \leq \mathcal{H}^{n-1}(\partial\Omega) + K\,\mathcal{L}^n(\Omega). \tag{4.14}$$

We can now complete the proof of the desired Lipschitz estimate. Assuming (3.13), then (3.11) holds and so we can apply Proposition 4.4 when $n \rightsquigarrow n-1$, with $\Omega$ replaced by $F$, to the function $\psi(u) = \mathcal{G}_{ij}(x(u), b_i, b_j)$, for $b_i$ and $b_j$ satisfying $\kappa|P_i| + \delta_i \leq b_i \leq (1+\kappa)\,r_0 + \kappa|P_i|, \kappa|P_j| + \delta_j \leq b_j \leq (1+\kappa)\,r_0 + \kappa|P_j|$, provided $\|D\psi\|_{L^\infty(F)}, \|D^2\psi\|_{L^\infty(F)}$ are both finite. That $\|D\psi\|_{L^\infty(F)} < \infty$ follows from (4.5) and (3.5), and that $\|D^2\psi\|_{L^\infty(F)} < \infty$ follows computing $D^2\psi$ using (3.4), (3.3), and (3.5). Therefore (4.13) implies

$$\begin{aligned}&\mathcal{L}^{n-1}\left(\left\{u \in F : 0 < \mathcal{G}_{ij}(x(u), b_i, b_j) \leq C\,t\right\}\right)\\ &\quad\leq \frac{C\,t}{\lambda}\left(\mathcal{H}^{n-2}(\partial F) + K\mathcal{L}^{n-1}(F)\right), \quad j \neq i.\end{aligned}$$

Hence from (4.10)

$$\sigma\left(V_{i,j}^t \setminus V_{i,j}\right) \leq c\,|F_{i,j}^t|_{n-1} \leq \frac{C\,t}{\lambda}\left(\mathcal{H}^{n-2}(\partial F) + K\,\mathcal{L}^{n-1}(F)\right), \quad j \neq i$$

for each $t$ satisfying (4.9). Finally, adding these inequalities over $j \neq i$, from (4.8) we then obtain the desired Lipschitz estimate (4.6) with $C_0$ a constant depending on $F, N, \kappa, \delta_i$, and bounds for the derivatives up to order two of $h$. □

## 5 Admissible Vectors for the Iterative Method

Recall we have distinct points $P_1, \ldots, P_N$ in $D$, $g_i > 0$, $i = 1, \ldots, N$ satisfying the conservation condition (2.5) where $f > 0$ a.e. in $\Omega$. And also assume the configuration conditions **H.1** and **H.2** .

In the following proposition we introduce the set of admissible vectors that will be used in the iterative method. We remark that the Proposition gives that vectors in the admissible set $W_\delta$ have components bounded uniformly away from $\kappa|P|$.

**Proposition 5.1** *Suppose $b_1$ satisfies*[2]

$$\kappa|P_1| < b_1 \leq (1-\kappa)\, r_0 + \kappa|P_1|, \tag{5.1}$$

*and let* $0 < \delta < g_1/(N-1)$. *Consider the set*[3]

$$W_\delta = \{(b_2, \dots, b_N) : \kappa|P_i| < b_i < (1+\kappa)\, r_0 + \kappa|P_i| \\ \text{and} \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\, dx \leq g_i + \delta \text{ for } 2 \leq i \leq N \text{ with } \mathbf{b} = (b_1, b_2, \dots, b_N)\Big\}.$$

*Notice that from* **H.1** *and* **H.2** , $(1+\kappa)r_0 + \kappa|P_j| < |P_j|$.

*Then* $W_\delta \neq \emptyset$ *and for* $0 < \alpha := \dfrac{1-\kappa}{1+\kappa}\,(b_1 - \kappa|P_1|)\,(< (1+\kappa)\, r_0)$ *we have*

$$\text{if } (b_2, \dots, b_N) \in W_\delta \text{ then } b_i \geq \kappa|P_i| + \alpha \text{ for } 2 \leq i \leq N. \tag{5.2}$$

***Proof*** Let us first prove the second part of the proposition. Let $(b_2, \dots, b_N) \in W_\delta$ and consider $\mathbf{b} = (b_1, b_2, \dots, b_N)$. Since $\mathcal{T}_{\mathcal{S}(\mathbf{b})}\left(\cup_1^N P_i\right) = \Omega$ and $\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i) \cap \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_j)$ has surface measure zero for $i \neq j$, we have

$$\int_\Omega f(x)\, dx = \sum_1^N \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\, dx \leq \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_1)} f(x)\, dx + \sum_2^N (g_i + \delta)$$

which from (2.5) implies that

$$\int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_1)} f(x)\, dx \geq g_1 - (N-1)\,\delta > 0.$$

Since $f > 0$ a.e., we then get that the surface measure of the set $\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_1)$ is positive. From [8, Lemma 5.3], $\mathcal{S}(\mathbf{b})$ is Lipschitz and so the set of singular points has measure zero. Hence there exists a point $x_0 \in \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_1)$ non singular for $\rho$. That is, there exists $\kappa|P_1| < \bar{b} < |P_1|$ such that the oval with radius $h\left(x, P_1, \bar{b}\right)$ supports $\rho$ at $x_0$, that is, $\rho(x) \leq h\left(x, P_1, \bar{b}\right)$ for all $x \in \Omega$ with equality at $x = x_0$. On the other hand, by definition of $\rho$, $\rho(x) \leq h(x, P_1, b_1)$ and so $h\left(x_0, P_1, \bar{b}\right) = \rho(x_0) \leq h(x_0, P_1, b_1)$ implying $\bar{b} \leq b_1$. We claim that $\bar{b} = b_1$. If it were $\bar{b} < b_1$, then $\rho(x) \leq h\left(x, P_1, \bar{b}\right) < h\,(x, P_1, b_1)$ for all $x \in \Omega$. Hence $\rho(x) = \min_{2 \leq i \leq N} h(x, P_i, b_i)$, and so at $x_0$ there would exist $h(\cdot, P_i, b_i)$, for some $i \neq 1$, supporting $\rho$ at $x_0$. That is, the ovals $h\left(\cdot, P_1, \bar{b}\right)$ and $h(\cdot, P_i, b_i)$ with $i \neq 1$ would support $\rho$ at $x_0$ and therefore $x_0$ would be a singular point, a contradiction. The

[2] Notice this implies that $b = b_1$ satisfies the weaker inequality (2.3). A reason to assume (5.1) is to show the set $W_\delta$ is non empty.

[3] Notice that the bounds for $b_j$ imply from (2.3) that the oval $h(x, P_j, b_j)$ refracts all $x \in \Omega$ into $P_j$.

claim is then proved. Hence $h(x_0, P_1, b_1) = \rho(x_0) \leq h(x_0, P_i, b_i)$ for $i \neq 1$. From the estimates for the ovals in Lemma 2.4 we have

$$h(x_0, P_1, b_1) \geq \frac{b_1 - \kappa|P_1|}{1+\kappa} \quad \text{and} \quad h(x_0, P_i, b_i) \leq \frac{b_i - \kappa|P_i|}{1-\kappa}, \quad i \neq 1,$$

implying

$$b_i \geq \kappa|P_i| + \frac{1-\kappa}{1+\kappa}\,(b_1 - \kappa|P_1|) = \kappa|P_i| + \alpha, \quad i \neq 1,$$

with $\alpha := \frac{1-\kappa}{1+\kappa}\,(b_1 - \kappa|P_1|)$. From (5.1) and since $\frac{1-\kappa}{1+\kappa} < 1$, $\alpha \leq (1-\kappa)r_0 (< (1+\kappa)r_0)$, for $i \neq 1$, showing (5.2).

Finally, to show $W_\delta \neq \emptyset$, let $b_1$ satisfy (5.1) and construct $(b_2, \ldots, b_N) \in W_\delta$. For this, it is enough to show the existence of $\kappa|P_j| < b_j < (1+\kappa)r_0 + \kappa|P_j|$ such that $h(x, P_1, b_1) \leq h(x, P_j, b_j)$ for $2 \leq j \leq N$ and $x \in \Omega$. Because with this choice we would have that $\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_j)$ has surface measure zero for $j \geq 2$. We write $b_1 - \kappa|P_1| = \sigma(1-\kappa)r_0$ for some $0 < \sigma \leq 1$, and let $b_j = \kappa|P_j| + \sigma(1+\kappa)r_0$, for $j = 2, \ldots, N$. Then, once again from the estimates for the ovals Lemma 2.4,

$$h(x, P_1, b_1) \leq \frac{b_1 - \kappa|P_1|}{1-\kappa} = \sigma r_0 = \frac{\sigma(1+\kappa)r_0}{1+\kappa} = \frac{b_j - \kappa|P_j|}{1+\kappa} \leq h(x, P_j, b_j),$$

for $j \geq 2$ and we are done. □

## 6 Abstract Algorithm

We present here an algorithm, that in conjunction with the results previously obtained, will be applied in Sect. 7 to obtain a near field refractor satisfying (2.7). This type of algorithm has been used in [4] for the far field refractor; in [9] for optimal transport problems and is extended in [1] for generated Jacobian equations. Here the presentation is in an abstract setting so that it can be applied to solve other problems.

Let $G : \prod_{i=1}^N (\alpha_i, \beta_i) \to \mathbb{R}^N_{\geq 0}$ be a function, $G(\mathbf{b}) = (G_1(\mathbf{b}), \ldots, G_N(\mathbf{b}))$, $\mathbf{b} = (b_1, \ldots, b_N)$, satisfying the following properties:

(a) $G$ is continuous on $\prod_{i=1}^N (\alpha_i, \beta_i)$;
(b) for each $1 \leq i \leq N$, and $\alpha_i < s \leq t < \beta_i$

$$G_i(b_1, \ldots, b_{i-1}, t, b_{i+1}, \ldots, b_N) \leq G_i(b_1, \ldots, b_{i-1}, s, b_{i+1}, \ldots, b_N), \text{ and}$$
$$G_j(b_1, \ldots, b_{i-1}, t, b_{i+1}, \ldots, b_N) \geq G_j(b_1, \ldots, b_{i-1}, s, b_{i+1}, \ldots, b_N) \quad \forall j \neq i.$$

(c) for each $1 \leq i \leq N$ there is $C_i > 0$ such that

$$\lim_{t \to \alpha_i^+} G_i(b_1, \ldots, b_{i-1}, t, b_{i+1}, \ldots, b_N) = C_i$$

for all $\mathbf{b} = (b_1, \ldots, b_N)$.

Let $f_1, \ldots, f_N, \delta$ be positive numbers satisfying

$$f_i - \delta > 0 \text{ and } C_i > f_i \quad \text{for} \quad i = 2, \ldots, N;$$

and let us fix $b_1^0 \in (\alpha_1, \beta_1)$ and define the set

$$W = \left\{ \mathbf{b} = \left(b_1^0, b_2, \ldots, b_N\right) \in \prod_{i=1}^{N} (\alpha_i, \beta_i) : G_i(\mathbf{b}) \leq f_i + \delta \quad \text{for } i = 2, \ldots, N \right\}.$$

Our purpose is to present an iterative procedure to construct a vector $\mathbf{b} \in W$ so that

$$|G_i(\mathbf{b}) - f_i| < \delta \quad \text{for } 2 \leq i \leq N. \tag{6.1}$$

This will be done by successively decreasing the coordinates of the vectors involved. In addition, we will show also that if the function $G$ satisfies a Lipschitz condition, then the procedure terminates in a finite number of iterations.

### 6.1 Description of the Algorithm

Suppose $W \neq \emptyset$ and pick $\mathbf{b}^1 \in W$. We will construct $N - 1$ intermediate consecutive vectors $\mathbf{b}^2, \ldots, \mathbf{b}^N$ associated with $\mathbf{b}^1$ in the following way.

**Step 1** We first test if $\mathbf{b}^1 = \left(b_1^0, b_2, \ldots, b_N\right)$ satisfies the inequalities:

$$f_2 - \delta \leq G_2(\mathbf{b}^1) \leq f_2 + \delta. \tag{6.2}$$

Notice that the last inequality in (6.2) holds since $\mathbf{b}^1 \in W$. If $\mathbf{b}^1$ satisfies (6.2), then we set $\mathbf{b}^2 = \mathbf{b}^1$ and we proceed to Step 2 below. If $\mathbf{b}^1$ does not satisfy (6.2), then

$$G_2(\mathbf{b}^1) < f_2 - \delta. \tag{6.3}$$

We shall pick $b_2^* \in (\alpha_2, b_2)$, and leave all other components fixed, so that the new vector $\mathbf{b}^2 = \left(b_1^0, b_2^*, b_3, \ldots, b_N\right) \in W$, and satisfies

$$f_2 \leq G_2(\mathbf{b}^2) \leq f_2 + \delta. \tag{6.4}$$

Let us see this is possible. From (b) above, and since $\mathbf{b}^1 \in W$,

$$\begin{aligned} G_j(b_1^0, t, b_3, \ldots, b_N) &\leq G_j(b_1^0, b_2, b_3, \ldots, b_N) \\ &\leq f_j + \delta \quad \text{for } \alpha_2 < t < b_2 \text{ and } j \neq 2. \end{aligned}$$

From (c) above

$$\lim_{t \to \alpha_2^+} G_2(b_1^0, t, b_3, \ldots, b_N) = C_2.$$

From (a), $G_2(b_1^0, t, b_3, \ldots, b_N)$ is continuous for $t \in (\alpha_2, b_2)$. Since

$$C_2 > f_2 > f_2 - \delta,$$

then by the intermediate value theorem there is $b_2^* \in (\alpha_2, b_2)$ such that

$$G_2(b_1^0, b_2^*, b_3, \ldots, b_N) = f_2,$$

and therefore (6.4) holds and $\mathbf{b}^2 \in W$.

Therefore, if the vector $\mathbf{b}^1$ does not satisfy (6.2), we have then constructed a vector $\mathbf{b}^2 \in W$ that satisfies (6.4) which is stronger than (6.2).

**Step** $N-1$. We proceed to test the inequality

$$f_N - \delta \leq G_N(\mathbf{b}^{N-1}) \leq f_N + \delta, \tag{6.5}$$

where $\mathbf{b}^{N-1}$ is the vector from Step $N-2$. If this holds we set $\mathbf{b}^N = \mathbf{b}^{N-1}$. Otherwise, we have

$$G_N(\mathbf{b}^{N-1}) < f_N - \delta,$$

and proceeding as before, by decreasing the $N$th-component of $\mathbf{b}^{N-1}$, we obtain a vector $\mathbf{b}^N \in W$

$$f_N \leq G_N(\mathbf{b}^N) \leq f_N + \delta,$$

as long as

$$C_N > f_N > f_N - \delta.$$

In this way, if

$$C_j > f_j > f_j - \delta \quad j = 2, \ldots, N,$$

starting from a fixed vector $\mathbf{b}^1 \in W$, we have constructed intermediate vectors $\mathbf{b}^2, \ldots, \mathbf{b}^N$ all belonging to $W$ and satisfying the inequalities:

$$f_j - \delta \leq G_j(\mathbf{b}^j) \leq f_j + \delta \quad j = 2, \ldots, N.$$

Notice that if $\mathbf{b}^1 = \mathbf{b}^N$, then the vector $\mathbf{b}^1$ satisfies 6.1. If not, we repeat the above steps starting with the last vector $\mathbf{b}^N$.

It is important to notice that by construction, the $\ell$-th components of $\mathbf{b}^{j-1}$ and $\mathbf{b}^j$ are all equal for $\ell \neq j$. If for some $2 \leq j \leq N$, $\mathbf{b}^{j-1} \neq \mathbf{b}^j$, then the $j$-th component of $\mathbf{b}^j$ is strictly less than the $j$-th component of $\mathbf{b}^{j-1}$. And so if we needed to decrease the $j$-th component of $\mathbf{b}^{j-1}$ to construct $\mathbf{b}^j$ it's because

$$G_j(\mathbf{b}^{j-1}) < f_j - \delta,$$

and then by construction $\mathbf{b}^j$ satisfies

$$f_j \leq G_j(\mathbf{b}^j) \leq f_j + \delta.$$

Therefore combining the last two inequalities we obtain the following important inequality

$$\delta < G_j(\mathbf{b}^j) - G_j(\mathbf{b}^{j-1}), \quad \text{for intermediate consecutive vectors } \mathbf{b}^j \neq \mathbf{b}^{j-1}. \tag{6.6}$$

In summary, we started from a vector $\mathbf{b}^{1,1} \in W$ and constructed $N-1$ intermediate vectors $\mathbf{b}^{1,2}, \ldots, \mathbf{b}^{1,N}$ using the procedure described. So we obtain in the first stage the finite sequence of vectors

$$\mathbf{b}^{1,1}, \mathbf{b}^{1,2}, \ldots, \mathbf{b}^{1,N}.$$

For the second stage we repeat the construction now starting with the vector $\mathbf{b}^{1,N}$ and we get the finite sequence of vectors

$$\mathbf{b}^{2,1}, \mathbf{b}^{2,2}, \ldots, \mathbf{b}^{2,N}$$

with $\mathbf{b}^{2,1} = \mathbf{b}^{1,N}$. Continuing in this way we obtain a sequence of vectors, in principle infinite,

$$\begin{aligned} &\mathbf{b}^{1,1}, \ldots, \mathbf{b}^{1,N}; \mathbf{b}^{2,1}, \ldots, \mathbf{b}^{2,N}; \mathbf{b}^{3,1}, \ldots, \mathbf{b}^{3,N}; \ldots; \\ &\mathbf{b}^{n,1}, \ldots, \mathbf{b}^{n,N}; \mathbf{b}^{n+1,1}, \ldots, \mathbf{b}^{n+1,N}; \ldots \end{aligned} \tag{6.7}$$

with $\mathbf{b}^{2,1} = \mathbf{b}^{1,N}, \mathbf{b}^{3,1} = \mathbf{b}^{2,N}, \ldots, \mathbf{b}^{n+1,1} = \mathbf{b}^{n,N}, \ldots$. If for some $n$, the vectors in the $n$th-stage are all equal, i.e., $\mathbf{b}^{n,1} = \mathbf{b}^{n,2} = \cdots = \mathbf{b}^{n,N} := \mathbf{b}^n$, then from the construction

$$|G_j(\mathbf{b}^n) - f_j| \leq \delta, \quad \text{for } 2 \leq j \leq N.$$

Therefore, if we show that for some $n$ the intermediate vectors $\mathbf{b}^{n,1}, \mathbf{b}^{n,2}, \ldots, \mathbf{b}^{n,N}$ are all equal, we obtain the desired approximation (6.1).

So far we have used only conditions (a), (b), and (c). To get the error bound for $G_1$ suppose[4]

$$\sum_{i=1}^{N} G_i(\mathbf{b}^n) = \sum_{i=1}^{N} f_i. \tag{6.8}$$

Then

$$|f_1 - G_1(\mathbf{b}^n)| = \left|\sum_{j=2}^{N} G_j(\mathbf{b}^n) - f_j\right| \leq \sum_{j=2}^{N} |G_j(\mathbf{b}^n) - f_j| \leq N\,\delta.$$

Therefore the vector $\mathbf{b}^n$ satisfies

$$|G_j(\mathbf{b}^n) - f_j| \leq \delta, \quad \text{for } 2 \leq j \leq N, \text{ and } |G_1(\mathbf{b}^n) - f_1| < N\delta.$$

Summarizing, if $G_i$ satisfy (a), (b), (c), (6.8) and

$$C_j > f_j > f_j - \delta \quad j = 2, \dots, N,$$

choosing $\delta = \epsilon/N$, we then obtain

$$|G_j(\mathbf{b}^n) - f_j| \leq \epsilon, \quad \text{for } 1 \leq j \leq N.$$

### 6.2 Convergence of the Algorithm

We will show here that the procedure described will always give in an infinite number of steps, a vector $\mathbf{b} \in W$ satisfying (6.1) provided the following holds:[5]

$$\text{there exists } \alpha > 0 \text{ such that if } \left(b_1^0, b_2, \dots, b_N\right) \in W, \text{ then } b_j \geq \alpha_j + \alpha \text{ for all } 2 \leq j \leq N. \tag{6.9}$$

[4] For the application to the near field refractor $G_i(\mathbf{b}) = \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\,d\sigma(x)$ so (6.8) holds for each vector $\mathbf{b}$ because

$$\begin{aligned}\sum_{i=1}^{N} G_i(\mathbf{b}) &= \sum_{i=1}^{N} \int_{\mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\,d\sigma(x)\\ &= \int_{\cup_{i=1}^{N} \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i)} f(x)\,d\sigma(x) \quad \text{since } \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_i) \cap \mathcal{T}_{\mathcal{S}(\mathbf{b})}(P_j) \text{ has measure zero for } i \neq j \text{ and } f > 0 \text{ a.e.}\\ &= \int_{\Omega} f(x)\,d\sigma(x) = f_1 + \cdots + f_N \quad \text{from the energy conservation assumption.}\end{aligned}$$

[5] We remark that for the application to the near field refractor $\alpha = \dfrac{1-\kappa}{1+\kappa}\left(b_1^0 - \kappa|P_1|\right)$, see Proposition 5.1.

That is, in the limit as the number of iterations goes to infinity, the procedure will always give a vector $\mathbf{b} \in W$ satisfying (6.1) .

As pointed out in (6.7), by using the procedure described above we obtain a sequence of vectors

$$\mathbf{b}^{n,\ell} = \left(b_1^0, b_2^{n,\ell}, b_3^{n,\ell}, \dots, b_N^{n,\ell}\right)$$

$n \in \mathbb{N}$ and $1 \leq \ell \leq N$ which can be listed as

$$\mathbf{b}^{1,1}, \dots, \mathbf{b}^{1,N};\ \mathbf{b}^{2,1}, \dots, \mathbf{b}^{2,N};\ \mathbf{b}^{3,1}, \dots, \mathbf{b}^{3,N}; \dots;$$
$$\mathbf{b}^{n,1}, \cdots, \mathbf{b}^{n,N};\ \mathbf{b}^{n+1,1}, \dots, \mathbf{b}^{n+1,N}; \dots$$

Notice that in this listing, for a fixed $j$, $2 \leq j \leq N$ the sequence $\{b_j^{n,\ell}\}$ of the $j$th entries is non-increasing; that is,

$$b_j^{n,\ell} \geq b_j^{m,k}, \qquad \text{for } n \leq m \text{ or for } n = m \text{ and } \ell \leq k$$

and for $j = 1$, we have $b_j^{n,\ell} = b_1^0$ for all $n$ and for all $\ell$. Moreover, since the vectors belong to $W$, by assumption (6.9), each $j^{th}$ entry is bigger than or equal to $\alpha_j + \alpha$ for $2 \leq j \leq N$. Therefore for any $j$ the limit of the $j^{th}$ entries exists and the limit is strictly bigger than $\alpha_j$.

Let $b_j^\infty$ be the limit of the $j^{th}$ entries, $j \geq 2$.

Then the vector

$$\mathbf{b}^\infty = \left(b_1^0, b_2^\infty, b_3^\infty, \cdots, b_N^\infty\right) \in \{b_1^0\} \times \prod_{i=2}^{N} (\alpha_i, \beta_i)$$

satisfies

$$f_j - \delta \leq G_j(\mathbf{b}^\infty) \leq f_j + \delta,\ j = 2, \dots, N. \tag{6.10}$$

In fact, fix $2 \leq j \leq N$, the vector $\mathbf{b}^\infty$ is the limit of the vectors $\mathbf{b}^{i,j}$ as $i \to \infty$. But the vectors $\mathbf{b}^{i,j}$ verify

$$f_j - \delta \leq G_j(\mathbf{b}^{i,j}) \leq f_j + \delta, \qquad \text{for } i = 1, 2, \dots.$$

From assumption (a), $G_j$ is continuous for each $j$, taking the limit as $i \to \infty$ we obtain (6.10).

Assuming (6.8) for all vectors $\mathbf{b} \in W$, we conclude that (6.10) holds with $j = 1$ and with $\delta$ replaced by $N\delta$.

***Remark 6.1*** Notice that the argument above always gives a solution $\left(b_1^0, b_2, \dots, b_N\right)$ satisfying (6.10). To handle the case when $j = 1$ we need an extra condition. In fact, for (6.10) to hold for $j = 1$, the conservation of energy condition (6.8) is sufficient.

Also notice that if the conservation of energy condition (6.8) is assumed, then the second condition in (b) implies the first condition in (b). This is all applicable to the near field refractor in view of the Footnote 4 before (6.8).

### 6.3 If $G$ Satisfies a Lipschitz Estimate Then the Algorithm Terminates in a Finite Number of Steps

Suppose that given $\delta_1, \dots, \delta_N$ positive there is a constant $M > 0$ such that

$$G_i(\mathbf{b} - t\,\varepsilon_i) - G_i(\mathbf{b}) \leq M\,t, \tag{6.11}$$

for $\alpha_i < b_i - t \leq b_i \leq \beta_i$ and for each $\mathbf{b} \in \prod_{i=1}^N [\alpha_i + \delta_i, \beta_i]$ and for all $1 \leq i \leq N$. Notice that from assumption (b) above, $G_i(\mathbf{b} - t\,\varepsilon_i) - G_i(\mathbf{b}) \geq 0$.

We shall prove that the estimate (6.11) together with the assumption that $W$ satisfies (6.9), implies that there is $n$ such that the vectors in the $n$th group $\mathbf{b}^{n,1}, \mathbf{b}^{n,2}, \dots, \mathbf{b}^{n,N}$ are all equal, and we will also show an upper bound for the number of iterations. The argument is as in [1, Sect. 4] but since the notation there is different we include it here for completeness and convenience of the reader.

Suppose we originate the iteration at $\mathbf{b}^0 = \left(b_1^0, b_2^0, \dots, b_N^0\right) \in W$. Since by construction the coordinates of the vectors in the sequence (6.7) are decreased or kept constant, the $j$th coordinate of any vector in the sequence is less than or equal to $b_j^0$, $1 \leq j \leq N$. In addition, from (6.9), points in $W$ have first coordinate $b_1^0$ and their coordinates bounded below by $\alpha_j + \alpha$ for $j \geq 2$. Therefore, all terms in the sequence (6.7) are contained in the compact box $K = \{b_1^0\} \times \prod_{j=2}^N [\alpha_j + \alpha, b_j^0]$. We want to show that there is $n_0$ such that the intermediate vectors $\mathbf{b}^{n_0,1}, \mathbf{b}^{n_0,2}, \dots, \mathbf{b}^{n_0,N}$ are all equal. Otherwise, for each $n$ the intermediate vectors $\mathbf{b}^{n,1}, \mathbf{b}^{n,2}, \dots, \mathbf{b}^{n,N}$ are not all equal. This implies that for each $n$ there are two consecutive intermediate vectors $\left(b_1^0, b_2, b_3, \dots, b_N\right)$ and $\left(b_1^0, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_N\right)$, that are different. By construction of intermediate vectors, they can only differ in one coordinate, say that $b_j > \bar{b}_j$. Notice that $j$ depends on $n$, but there is $j$ and a subsequence $n_\ell$ such that there are two consecutive intermediate vectors $\left(b_1^0, b_2^{n_\ell}, b_3^{n_\ell}, \dots, b_N^{n_\ell}\right)$ and $\left(b_1^0, \bar{b}_2^{n_\ell}, \bar{b}_3^{n_\ell}, \dots, \bar{b}_N^{n_\ell}\right)$ in each group $\mathbf{b}^{n_\ell,1}, \dots, \mathbf{b}^{n_\ell,N}$ such that their $j$-th coordinates satisfy $b_j^{n_\ell} > \bar{b}_j^{n_\ell}$, and all other coordinates are equal. Also notice that since the coordinates are chosen in a decreasing manner we have $b_j^{n_\ell} > \bar{b}_j^{n_\ell} \geq b_j^{n_{\ell+1}} > \bar{b}_j^{n_{\ell+1}}$ for $\ell = 1, \dots$. From (6.6) we then get

$$\delta < G_j\left(b_1^0, \bar{b}_2^{n_\ell}, \bar{b}_3^{n_\ell}, \dots, \bar{b}_N^{n_\ell}\right) - G_j\left(b_1^0, b_2^{n_\ell}, b_3^{n_\ell}, \dots, b_N^{n_\ell}\right) = (*) \tag{6.12}$$

for each $\ell \geq 1$. We write

$$\left(b_1^0, \bar{b}_2^{n_\ell}, \bar{b}_3^{n_\ell}, \dots, \bar{b}_j^{n_\ell}, \dots \bar{b}_N^{n_\ell}\right) = \left(b_1^0, \bar{b}_2^{n_\ell}, \bar{b}_3^{n_\ell}, \dots, b_j^{n_\ell} + \bar{b}_j^{n_\ell} - b_j^{n_\ell}, \dots \bar{b}_N^{n_\ell}\right),$$

and let $t{:=}\bar{b}_j^{n_\ell} - b_j^{n_\ell} < 0$. Since the vectors belong to $W$, from (6.9) $\bar{b}_j^{n_\ell} \geq \alpha_j + \alpha$ for $2 \leq j \leq N$. Then from (6.11) we obtain

$$(*) \leq -\left(\bar{b}_j^{n_\ell} - b_j^{n_\ell}\right) M = M\,(b_j^{n_\ell} - \bar{b}_j^{n_\ell}), \quad \forall \ell. \tag{6.13}$$

On the other hand,

$$\sum_{\ell=1}^{\infty} \left(b_j^{n_\ell} - \bar{b}_j^{n_\ell}\right) \leq b_j^0 - (\alpha_j + \alpha), \tag{6.14}$$

which contradicts (6.12). Therefore, the intermediate vectors $\mathbf{b}^{n_0,1}, \mathbf{b}^{n_0,2}, \dots, \mathbf{b}^{n_0,N}$ are all equal for some $n_0$.

Let us now estimate the number of iterations used. Consider the groups of vectors (6.7) in the construction. We have proved the process stops at some $n_0$, i.e., all vectors in this group are equal. Let us estimate $n_0$. Fix a coordinate $2 \leq j \leq N$. Notice that in each group $k$, the $j$th coordinate of any vector in the group can decrease *at most only once* and only when passing from a vector $\mathbf{b}^{k,j-1}$ to the vector $\mathbf{b}^{k,j}$. Here $k$ denotes the group and $j$ the location in the group. The $j$th coordinate of all vectors are at most $b_j^0$, the $j$th coordinate of the initial vector, and since the vectors belong to $W$ and so (6.9) holds, the $j$th coordinates are at least $\alpha_j + \alpha$. So the change in the $j$th coordinate of a vector in the group one to the group $n_0$, is at most $b_j^0 - \left(\alpha_j + \alpha\right)$. On the other hand, on each group if the $j$th coordinate is decreased, from (6.13), it is decreased by at least $\dfrac{\delta}{M}$. Having $n_0$ groups, the total decrease of the $j$th coordinate in passing from group one to group $n_0$ is at least

$$n_0\,\frac{\delta}{M},$$

which is in turn smaller than the total possible decrease, that is, we have

$$n_0\,\frac{\delta}{M} \leq b_j^0 - \left(\alpha_j + \alpha\right).$$

Since this must hold for all the coordinates $2 \leq j \leq N$ we obtain the bound

$$n_0 \leq \frac{M}{\delta} \max_{2\leq j\leq N} \left(b_j^0 - \left(\alpha_j + \alpha\right)\right).$$

## 7 Application of the Algorithm to the Near Field Refractor

We set $G_i(\mathbf{b}) = H_i(\mathbf{b})$ with $H_i$ given in (4.4), $\alpha_i = \kappa\,|P_i|$, and $\beta_i = (1+\kappa)\,r_0 + \kappa\,|P_i|$, $1 \leq i \leq N$. The continuity property (a) for $H_i$ is contained in the proof of Step 2 in [8, Theorem 2.5]. Properties (b) and (c) follow from Lemmas 2.5 and 2.6 , with $C_i = \int_\Omega f\,dx$. The set $W = W_\delta$ in Proposition 5.1, and so assumption (6.9) is (5.2) for the near field refractor problem. Finally, the Lipschitz estimate (6.11) follows applying Theorem 4.1 with $\delta_1 = b_1^0 - \kappa\,|P_1|$ and $\delta_j = \alpha = \dfrac{1-\kappa}{1+\kappa}\,\left(b_1^0 - \kappa|P_1|\right)$ for $j = 2, \dots, N$. We then have everything in place to be able to apply the abstract algorithm to the near field refractor problem and we can obtain the poly-oval refractor $\mathcal{S}(\mathbf{b})$ that satisfies 2.7.

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