

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

Teaching Quantum Formalism and Postulates to First-Year Undergraduates

Jeremy Levy and Chandrakha Singh

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh,
PA 15260 USA.

(Dated: 18 November 2024)

Traditional approaches to undergraduate-level quantum mechanics require extensive mathematical preparation, preventing most students from enrolling in a quantum mechanics course until the third year of a physics major. Here we describe an approach to teaching quantum formalism and postulates that can be used with first-year undergraduate students and even high school students. The only pre-requisite is a familiarity with vector dot products. This approach enables students to learn Dirac notation and core postulates of quantum mechanics at a much earlier stage in their academic career, which can help students prepare for careers in quantum science and engineering and advance the Second Quantum Revolution.

I. INTRODUCTION

Quantum mechanics is arguably the most important course that an undergraduate physics major will take. It is central to understanding many aspects of physics, including astrophysics, particle physics, solid-state and materials science, photonics, and, more recently, quantum information science, engineering, and technology.

There is enormous enthusiasm for lowering the entry barriers for quantum science and engineering, in order to prepare a new generation of students to address the challenges of the Second Quantum Revolution^{1,2}. Many educational activities are aimed at younger and less-technically-advanced students, with the hope of preparing them for future careers in quantum industries.

In most university curricula, quantum mechanics is taught as a one-semester or two-semester course, preceded by modern physics and a full year of introductory physics. In preparation for learning quantum mechanics, students must first take courses in advanced calculus, complex analysis, differential equations, and linear algebra.

Why do we wait so long to teach quantum mechanics? Apart from curricular inertia, there is a broad consensus that students simply cannot learn quantum mechanics at an earlier stage of their academic career. In the early days of quantum theory (i.e., the dawn of the First Quantum Revolution), quantum mechanics was not taught to undergraduates, nor was it offered as a core graduate course³. One of the reasons why quantum education was delayed until the advanced graduate level was that the mathematical notation at the time was not well developed. Dirac provided a pedagogical service to the physics community by creating his namesake “Dirac notation”⁴, which helped to codify and simplify operations that are often obscured by representation-specific approaches or matrix-only representations of the Time-Dependent Schrödinger Equation (TDSE). However, despite this improved notation, a century after the introduction of quantum mechanics, it is still a highly challenging course for students.

Many introductory textbooks adopt a “spins-first” approach to quantum mechanics to help students appreci-

ate the mathematical structure of quantum mechanics^{5,6}, while setting the stage for learning about quantum information. Two-state quantum systems are not just spins, but also “qubits” which can be used to store quantum information. Dirac notation is generally not introduced in modern physics courses that cover quantum mechanics; introductory quantum textbooks have tended to use Dirac notation sparingly, although some more modern textbooks do employ Dirac notation more thoroughly^{7,8}.

Efforts to make quantum mechanics more accessible to students, so that they can be introduced to the subject at an earlier state, have been described by, for example, Zollman et al.⁹. Rudolf has developed an accessible guide to quantum concepts (*Q is for Quantum*, Ref. 10) which are necessary for quantum computing. Hands-on approaches often involve visualization of wavefunctions and de-emphasize the formal aspects of learning quantum mechanics. Other efforts to introduce quantum mechanics include the development of a variety of quantum-themed games¹¹⁻²⁴ designed to teach various aspects of quantum mechanics.

Here we describe an approach to teaching quantum mechanics to students who have no more than one semester of introductory physics. The material was developed as a one-credit course that gives students an introduction to how quantum mechanics works and provides a sound foundation for future studies. We create a bridge to learning Dirac formalism by extending the vector notation that students learn in an introductory calculus-based physics course. This extension enables a one-to-one map with Dirac notation, including the representation of operators. Elements of quantum theory are illustrated using a “Bloch cube”, which is a simplified version of the Bloch sphere. The Bloch cube enables introductory students to manipulate quantum states with their hands, thus building intuition about many advanced quantum concepts. This approach was developed and refined over a three-year period with three groups of 5-10 first-year undergraduate students by J.L. in collaboration with C.S. Students have consistently provided very positive feedback in post-instruction surveys regarding this approach.

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

II. BREAKING DOWN THE SCHRÖDINGER EQUATION

It is helpful to break down the TDSE (Eq. 1) into five components, which correspond roughly to five mathematical pre-requisites, summarized in Table I.

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle \quad (1)$$

TABLE I. Breaking down the TDSE.

TDSE Component	Mathematical Requirements
i	Complex numbers
\hbar	Algebra
$\partial/\partial t$	Calculus, differential equations
$ \psi\rangle$	Vectors
\hat{H}	Linear algebra

i : Complex arithmetic, multiplication, and complex conjugation are taught in calculus-based or honors introductory physics courses.

\hbar : The Planck constant is just a number (with units), emblematic of symbolic representation of numbers, and ordinary algebraic manipulations.

$\partial/\partial t$: Calculus is often a co-requisite in introductory physics courses. At a more advanced stage, students learn about coupled and partial differential equations.

$|\psi\rangle$: Vectors are used extensively in introductory physics to describe the position and dynamics of objects, and other fields (e.g., magnetic, electric) in three spatial dimensions. Non-calculus-based courses often skip the concept of unit vectors, whereas in calculus-based curricula they are well developed and used.

H : Operators, linear algebra and matrix manipulation are usually avoided in introductory physics courses.

To introduce quantum mechanics to introductory physics students, we need to choose what is essential and what can be left for more advanced instruction. Of the mathematics skill requirements just outlined, the skills least likely to be possessed by introductory students are linear algebra and differential equations.

The linear algebra required to use Dirac notation can be introduced by making an explicit analogy with unit vectors. This analogy can be used not only to teach about inner products, but also outer products and identity and rotation operators, as described in Sec. III-IV. The Bloch cube is introduced in Sec. V to provide a tangible representation of quantum, states and the Born rule.

The requirement for differential equations is eliminated because the TDSE can be “integrated out” to yield unitary operators $\hat{U} = \exp[-i\hat{H}t/\hbar]$ with the property that

$$\hat{U} |\psi(0)\rangle = |\psi(t)\rangle \quad (2)$$

assuming that \hat{H} is time-independent. Eq. 2 serves as a substitute for Eq. 1, which students do not need to learn how to solve until later in their academic career. While understanding the TDSE is essential to understanding the physical mechanisms by which states are transformed, students at earlier stages in their physics education can learn to work with the operators (in this work, rotation operators) that represent these transformations. Sec. VI shows how the rotation operator is introduced and illustrated with the Bloch cube. Then in Sec. VII-VIII, we outline the lecture sequence for this one-credit course and show how it compares to traditional instruction.

III. VECTORS AND DOT-VECTORS

Introductory students learn about unit vectors and scalar products, which are used to convert vector equations into a set of scalar equations. Here we review and then extend the notation to make it more compatible with Dirac “bra-ket” notation. We will restrict ourselves to two dimensions, which is sufficient for mapping onto two-state quantum systems. Unit vectors obey the standard set of orthogonality relations: $\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{y} = 0$, $\hat{y} \cdot \hat{x} = 0$, $\hat{y} \cdot \hat{y} = 1$. A general vector \vec{V} can be written as:

$$\vec{V} = V_x \hat{x} + V_y \hat{y} \quad (3)$$

where $V_x = \hat{x} \cdot \vec{V}$ and $V_y = \hat{y} \cdot \vec{V}$. Complete sets of orthonormal unit vectors define coordinate systems.

To help students understand bra-ket notation, we start by introducing a new type of vector, called the “dot-vector”, in which the “.” is attached to the unit vectors to define a new vector space. The dot-vector can be disarmingly introduced by making reference to the Dr. Seuss story about Sneetches²⁵. In that children’s story, there are two kinds of Sneetches: Star-Bellied Sneetches and Plain-Bellied Sneetches. Other than the fact that one sports a star on its belly, they look identical. By analogy, we can define, for every unit vector \hat{x} or \hat{y} , a corresponding dot-vector $\hat{x}\cdot$ and $\hat{y}\cdot$, as shown in Figure 1. A general dot-vector $\vec{W}\cdot$ in the two-dimensional dot-vector space can be written as:

$$\vec{W}\cdot = W_x \hat{x}\cdot + W_y \hat{y}\cdot \quad (4)$$

where W_x and W_y are scalar coefficients. The scalar product between $\vec{W}\cdot$ and \vec{V} is given by:

$$\vec{W}\cdot \cdot \vec{V} = (W_x \hat{x}\cdot + W_y \hat{y}\cdot)(V_x \hat{x} + V_y \hat{y}) = W_x V_x + W_y V_y \quad (5)$$

where we have taken advantage of the orthonormality relations defined above. The class of operations that students learn to manipulate vectors applies similarly to dot-vectors. In this sense, the distinction between vectors and dot-vectors is superficial, not unlike the stars that some Sneetches wear on their bellies.

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

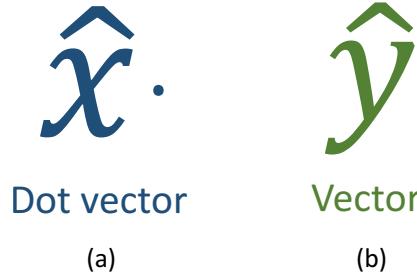


FIG. 1. Dot-Vec notation developed to help introduce Dirac notation. (a) Dot-vector $\hat{x}\cdot$. (b) vector \hat{y} . Products of dot-vectors and vectors can yield scalars or operators, depending on their relative ordering.

By combining vectors and dot-vectors in the opposite order, we can create operators which transform vectors into other vectors. An extremely useful operator is the identity operator, which can be written as follows:

$$\mathbf{I} = \hat{x}\hat{x}\cdot + \hat{y}\hat{y}\cdot \quad (6)$$

Acting the identity operator \mathbf{I} on the vector \vec{V} yields:

$$(\hat{x}\hat{x}\cdot + \hat{y}\hat{y}\cdot)\vec{V} = \hat{x}\hat{x}\cdot\vec{V} + \hat{y}\hat{y}\cdot\vec{V} = V_x\hat{x} + V_y\hat{y}. \quad (7)$$

where $V_x = \hat{x}\cdot\vec{V}$ and $V_y = \hat{y}\cdot\vec{V}$. The identity operator provides an explicit method for resolving a vector into components, which is something that introductory students are often asked to do when solving vector equations.

Another type of operator is one that produces a rotation in the xy plane. For example, the operator \mathbf{R} rotates vectors counterclockwise by 90° .

$$\mathbf{R} = \hat{y}\hat{x}\cdot - \hat{x}\hat{y}\cdot \quad (8)$$

The rotation properties can be seen by its effect on three test vectors, illustrated in Fig. 2.

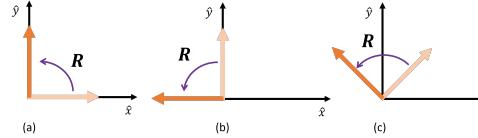


FIG. 2. Effect of \mathbf{R} operator on (a) \hat{x} , (b) \hat{y} , (c) $(\hat{x} + \hat{y})/\sqrt{2}$.

An important conceptual leap which is necessary for understanding quantum mechanics is the mapping between orthogonal vectors and distinct states of a system.

TABLE II. Dot-Vec / Bra-Ket notation analogy.

Dot-Vec	Bra-Ket
\hat{x}	$ +z\rangle$
\hat{y}	$ -z\rangle$
$\hat{x} \cdot \hat{x} = 1$	$\langle +z +z \rangle = 1$
$\hat{x} \cdot \hat{y} = 0$	$\langle +z -z \rangle = 0$
$\hat{y} \cdot \hat{x} = 0$	$\langle -z +z \rangle = 0$
$\hat{y} \cdot \hat{y} = 1$	$\langle -z -z \rangle = 1$
$\hat{x}\hat{x}\cdot$	$ +z\rangle\langle +z $
$\hat{x}\hat{y}\cdot$	$ +z\rangle\langle -z $
$\hat{y}\hat{x}\cdot$	$ -z\rangle\langle +z $
$\hat{y}\hat{y}\cdot$	$ -z\rangle\langle -z $

Unit vectors in quantum mechanics can represent mutually distinguishable states and not just directions in space. For example, we can imagine that \hat{x} represents the “heads” state of a coin, while \hat{y} represents the “tails” state.

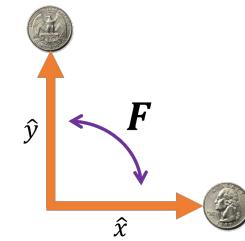


FIG. 3. Unit vectors can represent states of matter. Operator \mathbf{F} flips the state of a coin.

The action of flipping over a coin can be described by the operator $\mathbf{F} = \hat{y}\hat{x}\cdot + \hat{x}\hat{y}\cdot$.

IV. DIRAC NOTATION

After extending the existing vector notation to include dot-vectors (which we will refer to as “Dot-Vec” notation), the analogy with Dirac (bra-ket) notation can be introduced. Ket states are associated with unit vectors, and Bra states are associated with unit dot-vectors. The scalar products defined for \hat{x} and \hat{y} are mapped onto the scalar product relations for Dirac bras and kets. The analogy is summarized in Table II.

A. Operators

With this identification, we can use some of the tools developed for vectors to resolve a general state $|\psi\rangle$ into

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

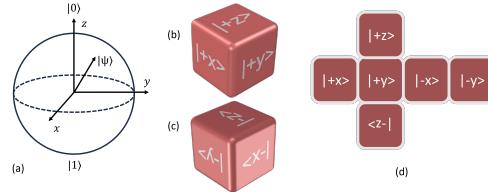


FIG. 4. Bloch sphere and Bloch cube. (a) Bloch sphere. (b) Top view of Bloch cube. (c) Bottom view of Bloch cube. (d) Fold-out pattern of Bloch cube.

components along $|+z\rangle$ and $| -z\rangle$ using the identity operator $\hat{1} = |+z\rangle\langle +z| + |-z\rangle\langle -z|$.

$$\hat{1}|\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle = \psi_+|+z\rangle + \psi_-|-z\rangle \quad (9)$$

where $\psi_+ = \langle +z|\psi\rangle$ and $\psi_- = \langle -z|\psi\rangle$.

When students are comfortable with scalar products and operators using Dirac notation, it is possible to discuss how it relates to quantum physics. In keeping with a “spins-first” approach, we restrict our attention to the two-dimensional space spanned by $|+z\rangle$ and $| -z\rangle$, which can also be labeled $|0\rangle$ and $|1\rangle$ in the context of quantum information and qubits, to describe a physical system that can be in one of two possible distinct states. We avoid describing any specific physical system, e.g., spin-1/2, since these concepts can increase the cognitive load of students. Additionally, in the first semester of college-level introductory physics, students have not been exposed to the idea of magnetic fields or spins. The abstract nature of the labels used to describe the two-state system is compensated by a “hands-on” approach involving a Bloch Cube, which is described in the next Section below.

V. BLOCH CUBE

Quantum states in two dimensions are often represented in terms of the Bloch sphere (Figure 4(a)). Here we simplify the Bloch sphere by fashioning it into a six-sided “Bloch cube” (Figure 4(b,c)). Opposite faces of the cube represent distinct states, and are labeled $|\pm n\rangle$, $n = x, y, z$. The allowed states are restricted to the six possible states represented on the Bloch cube. While only certain states are represented, the set is sufficiently rich to illustrate most of the important quantum concepts. Bloch cubes (Figure 5), like a deck of cards, can be used in multiple exercises or games to illustrate important quantum concepts.

It is important for students to understand that, even though we work with states labeled $|+x\rangle$, $|+y\rangle$, and $|+z\rangle$, none of these states are “orthogonal” to each other. Students should be told over and over that distinct states are those which are situated on opposite faces of the Bloch



FIG. 5. Bloch Cubes. The smaller ones are made from blank ceramic cubes and covered with laser-printed labels. The larger Bloch Cubes use a different “qubit” labeling in which $|+z\rangle = |0\rangle$ and $| -z\rangle = |1\rangle$, and other states are expressed in this basis.

Cube. There are many opportunities for instructors to emphasize this crucial point during instruction, but the Bloch Cube can help remind students of this.

The Bloch Cube can be used to represent the quantum state of two-state system. We will work with a convention that the state of a Bloch Cube is whatever side is facing upwards. This convention is useful for students who are working with the Bloch Cubes themselves. The convention can be adjusted if one is giving instruction to a class, in which the side of the Bloch Cube that is facing the students can serve as the Bloch Cube state.

A. Quantum Measurements and the Born Rule

The Born rule predicts the likelihood of a measurement outcome, and it can be presented as a postulate. If the system is in a state $|\psi\rangle$, the probability of measuring it to be $|\phi\rangle$ is given by

$$P(|\psi\rangle \rightarrow |\phi\rangle) = |\langle \phi|\psi\rangle|^2 \quad (10)$$

To illustrate this idea with the Bloch cube, we can define (according to standard conventions) the following superposition states (Eq. 11):

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle) \quad (11a)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle) \quad (11b)$$

$$|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle) \quad (11c)$$

$$|-y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - i|-z\rangle) \quad (11d)$$

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

If the question matches the state (e.g., Z question for $|+z\rangle$ state), then the state is unchanged. If the question type differs from the face-up state (e.g., X question for $|+z\rangle$ state), the Bloch Cube is rolled and the state changes to $|+x\rangle$ if the roll lands on a positive face; otherwise, the state changes to $|-x\rangle$. A flowchart is illustrated in Figure 6. An alternative to using a Bloch Cube to generate a probabilistic outcome would be to flip a coin.

VI. QUANTUM DYNAMICS

Solving the TDSE, even with a two-state quantum system, generally involves calculus, differential equations, and matrices, which are beyond the reach of most students taking introductory physics. However, the TDSE can be integrated to yield unitary operators that represent the formal solutions of the TDSE and correspond to physical rotations. We represent unitary operators as \hat{U} , where the “hat” is not to be confused with the unit vectors and dot-vectors from before. A

TABLE III. Born rule probabilities $|\langle \square | \square \rangle|^2$ for Bloch cube states.

		Initial State					
Question	Final State	$ +z\rangle$	$ -z \rangle$	$ +x\rangle$	$ -x \rangle$	$ +y\rangle$	$ -y \rangle$
Z	$ +z\rangle$	100%	0%	50%	50%	50%	50%
Z	$ -z \rangle$	0%	100%	50%	50%	50%	50%
X	$ +x\rangle$	50%	50%	100%	0%	50%	50%
X	$ -x \rangle$	50%	50%	0%	100%	50%	50%
Y	$ +y\rangle$	50%	50%	50%	50%	100%	0%
Y	$ -y \rangle$	50%	50%	50%	50%	0%	100%

If a system is in the $|+x\rangle$ state, students can calculate that the probability of measuring $+z$ is given by

$$P(|+x\rangle \rightarrow |+z\rangle) = |\langle +z | +x \rangle|^2 = \frac{1}{2} \quad (12)$$

The full set of probabilities for measurement is summarized in Table III. The rationale behind Eq. 11 is not provided to students by default, but students can be asked to work out the entries in Table III based on those definitions, and can be asked to think about alternative definitions that would be consistent with Table III.

To simulate the measurement process with a Bloch cube, students first choose an initial state, which by convention can be the face-up state of the Bloch cube after rolling. One of three possible questions can be asked (Eq. 13(a-c)).

$$X \text{ Question: } |+x\rangle \text{ state or } | -x \rangle \text{ state?} \quad (13a)$$

$$Y \text{ Question: } |+y\rangle \text{ state or } | -y \rangle \text{ state?} \quad (13b)$$

$$Z \text{ Question: } |+z\rangle \text{ state or } | -z \rangle \text{ state?} \quad (13c)$$

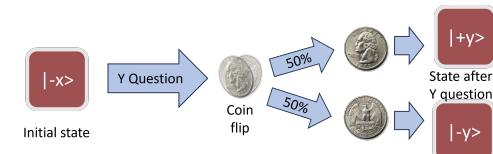


FIG. 6. Example illustrating measurement using the Bloch cube. The initial state is $| -x \rangle$, and the Y Question is asked. A coin is flipped and if it lands heads, the Bloch cube is oriented to the $|+y\rangle$ state; if it lands tails, the Bloch cube is oriented in the $| -y \rangle$ state.

unitary operator will in general take the form $\hat{U} = e^{i\phi_+} |+u\rangle\langle +z| + e^{i\phi_-} | -u \rangle\langle -z|$, so that $\hat{U} |+z\rangle = e^{i\phi_+} |+u\rangle$ and $\hat{U} | -z \rangle = e^{i\phi_-} | -u \rangle$, where $\{|+u\rangle, | -u \rangle\}$ form an orthonormal basis. However, if the initial and final states are restricted to the six faces of the Bloch cubes, this constrains the rotation angles to be integer multiples of 90° . Using this restriction, there is no need to teach students about the TDSE or to define unitary operators. The resulting system is simple enough to be readily understood and yet rich enough to illustrate how operators transform states.

We begin by defining rotation operators that rotate the Bloch cube about the X , Y , and Z axes by 90° in the clockwise direction (Figure 7). We can represent the \hat{Z} unitary operation corresponding to 90° rotation about the Z axis in two equivalent ways:

$$\hat{Z} = |+z\rangle\langle +z| + i | -z \rangle\langle -z| \quad (14a)$$

$$\hat{Z} = -i |+z\rangle\langle +z| + | -z \rangle\langle -z| \quad (14b)$$

The two forms of \hat{Z} differ only by a constant multiplicative factor, which we explain is an arbitrary mathematical choice with no physical effects, similar to the choice of zero potential energy in classical mechanics problems. Note that with these definitions, $\hat{Z}^4 = +1$; this is the rotation operator for a Bloch cube and not for a spin-1/2 state. Acting with \hat{Z} or its inverse \hat{Z}^{-1} on $|+z\rangle$ and $| -z \rangle$, does not change the state except for a factor of $\pm i$. The fact that $|+z\rangle$ and $| -z \rangle$ are eigenstates of \hat{Z} is intuitively seen because the Bloch cube is being rotated about the Z axis and hence does not transform into another state.

To understand the effect of \hat{Z} on the other four Bloch cube states, one can rotate the Bloch cube 90° clockwise about the Z axis. In general, \hat{Z} permutes the states as follows: $|+x\rangle \rightarrow |+y\rangle \rightarrow | -x \rangle \rightarrow | -y \rangle \rightarrow |+x\rangle$. Students can see this mathematically by rewriting the \hat{Z} operator in terms of its action on, for example, the $|+x\rangle$ and $| -x \rangle$ basis states. Students can also show mathematically that \hat{Z} can be expressed as follows:

$$\hat{Z} = |+y\rangle\langle +x| + | -y \rangle\langle -x| \quad (15a)$$

$$\hat{Z} = | -x \rangle\langle +y| + |+x\rangle\langle -y| \quad (15b)$$

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

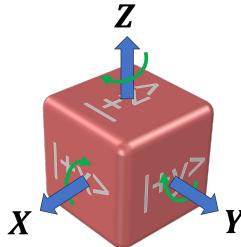


FIG. 7. Illustration of Bloch cube clockwise 90° rotations about X , Y , and Z axes.

Expressions like these are readily understood by examining the Bloch cube and remembering the definition for the \hat{Z} unitary rotation. Similar expressions can be found for rotations about the x and y directions.

In general, the Bloch cube serves as a “quantum abacus” for computing a series of unitary operations. Students can discover, for example, the non-commutative nature of these operations by writing down a sequence of operations in various orders and showing that the Bloch cube ends up in a different state depending on the order of operations.

VII. LECTURE SEQUENCE

This approach to teaching quantum mechanics was used in a one-credit course, but the level of detail can be expanded or compressed depending on the amount of time available. The development of this course took place with students who participated in a program titled “First Experiences in Quantum” (FEQ), held in the Spring semester at the University of Pittsburgh. Most students were first-year undergraduates who were also signed up to work on a project in the research group of a willing faculty member for 1-2 course credits. Because any grade was assigned by the research advisor, the only course assessment was a midterm test; there was no final exam and no out-of-class assignments. Below is a list of topics covered in the lectures.

1. Introduction and overview
2. Vectors and unit vectors in two dimensions
3. Dot products and introduction to Dot-Vec notation
4. More examples of Dot-Vec notation and coordinate transformations
5. Dirac notation and analogy with Dot-Vec notation
6. Bloch cube and representation of quantum states
7. Bloch cube and Born rule

8. Bloch cube and quantum dynamics
9. Proving that unitary operators are represented by Bloch cube rotation
10. Final assessment

Additional suggestions about how instructors may follow the approach described here are given in the Supplementary Material. Additional topics that cover more advanced topics can be taught using Bloch Cubes, including density matrix formalism, electron spin resonance, quantum key distribution, and elements of quantum field theory. These topics require the use of multiple Bloch Cubes. For example, the density matrix is represented by a set of Bloch cubes that represent the possible states and their respective probabilities. The completely mixed state would have half of the Bloch Cubes in a $|+z\rangle$ state and half in the $| -z\rangle$ state. Quantum key distribution (BBM92 type²⁶) would use Bloch Cubes to represent photon states measured Alice and Bob, as well as the outcome of joint measurements. Electron spin resonance involves rotations of states, which can be simulated using Bloch Cubes. Inhomogeneous dephasing can be represented “digitally” by allowing some states to rotate faster than others and advance by one quarter turn. A “pi pulse” applied to all the spins followed by more spinning will cause refocusing and spin-echo phenomena to become apparent. Quantum field theory requires an array of Bloch Cubes with particle existence represented by a $|z\rangle$ state and its absence represented by $| -z\rangle$ state.

VIII. COMPARISON WITH TRADITIONAL INSTRUCTION

The approach to teaching quantum mechanics described here carefully postpones some topics. Specifically, the TDSE is not explicitly solved, and the unitary operators that represent time evolution are not derived from the TDSE. Hamiltonians, which appear prominently during traditional instruction, do not appear anywhere. The main reason for this choice is that calculus, and the more complex mathematics surrounding exponentiation of operators and coupled ordinary differential equations, is generally too advanced for introductory students. However, the concept of eigenstates is introduced to students. In terms of the Bloch Cube, the eigenstates of the rotation operator are the two states that lie on the axis about which (unitary) rotation takes place. In the approach described here, unitary evolution takes intellectual precedence over the Hamiltonians that generate them. From a pedagogical perspective, it is fairly standard to introduce momentum conservation for a free particle and then later on “derive” it by showing that the momentum operator is the generator of translation in space. Similar pedagogical choices are made in introductory physics, where the forces involved in completely

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

ACKNOWLEDGMENTS

JL gratefully acknowledges funding support from NSF DMR-2225888. CS gratefully acknowledges funding support from NSF PHY-2309260. We thank Robert Devaty for helpful feedback in writing the manuscript.

inelastic collisions are “integrated out”, yielding a momentum conservation law that does not depend on the details of the collision. This approach to quantum mechanics is not a substitute for traditional approaches to teaching quantum mechanics—it is instead an introduction that can empower students to understand core quantum concepts years before they learn them in a standard physics sequence.

IX. CONCLUSION

Here we introduced an approach to teaching quantum mechanics to students who have some exposure to vectors and only limited experience with physics. Dirac bracket notation is introduced through a close analogy with unit vectors, requiring one extension that can be used to create dot-vec operators. The Bloch cube simplifies the number of quantum states, and eliminates the need for calculus. A number of exercises, more than are described here, can be paired with Bloch cubes, and calculations can be verified and made intuitive through manipulation of the Bloch cube. The knowledge and intuition derived from this approach can support students in more advanced introductory quantum courses, and can help democratize quantum science and engineering by broadening its appeal at an earlier stage of intellectual development.

Following are example comments from first-year college students on instructor evaluation:

- “I loved how we learned about different things using the cube. The class didn’t just give me information, it also taught me how to think about things differently and accept the weirdness of quantum ideas.”
- “Excellent introduction to quantum physics at an introductory level.”
- “The cubes are good.”
- “Material presented was not intimidating.”
- “I loved how [instructor] helped us learn using cubes and I loved the topic and I found it deeply interesting.”

X. AUTHOR DECLARATIONS

A. Conflict of Interest

All authors declare that they have no conflicts of interest.

¹A. Asfaw, A. Blais, K. R. Brown, J. Candelaria, C. Cantwell, L. D. Carr, J. Combes, D. M. Debroy, J. M. Donohue, S. E. Economou, E. Edwards, M. F. J. Fox, S. M. Girvin, A. Ho, H. M. Hurst, Z. Jacob, B. R. Johnson, E. Johnston-Halperin, R. Joynt, E. Kapit, J. Klein-Seetharaman, M. Laforest, H. J. Lewandowski, T. W. Lynn, C. R. H. McRae, C. Merzbacher, S. Michalakis, P. Narang, W. D. Oliver, J. Palsberg, D. P. Pappas, M. G. Raymer, D. J. Reilly, M. Saffman, T. A. Searles, J. H. Shapiro, and C. Singh, “Building a quantum engineering undergraduate program,” *IEEE transactions on education* **65**, 220–242 (2022).

²C. Singh, A. Asfaw, and J. Levy, “Preparing students to be leaders of the quantum information revolution,” *Physics Today* (2021).

³S. Coben, “The scientific establishment and the transmission of quantum mechanics to the united states, 1919–32,” *The American historical review* **76**, 442 (1971).

⁴P. A. M. Dirac, “A new notation for quantum mechanics,” *Mathematical proceedings of the Cambridge Philosophical Society* **35**, 416–418 (1939).

⁵J. S. Townsend, “A modern approach to quantum mechanics, second edition,” (2013).

⁶D. H. McIntyre, *Quantum mechanics: A paradigms approach* (Cambridge University Press, 2023).

⁷P. Kok, *A first introduction to quantum physics*, Undergraduate Lecture Notes in Physics (Springer International Publishing, 2018).

⁸A. I. Lvovsky, *Quantum Physics: An Introduction Based on Photons*, Undergraduate Lecture Notes in Physics (Springer, 2018).

⁹D. A. Zollman, N. S. Rebello, and K. Hogg, “Quantum mechanics for everyone: Hands-on activities integrated with technology,” *American journal of physics* **70**, 252–259 (2002).

¹⁰T. Rudolph, *Q is for quantum* (Terence Rudolph, 2017).

¹¹T. A. Corcovilos, “A simple game simulating quantum measurements of qubits,” *American journal of physics* **86**, 510–517 (2018).

¹²Z. C. Seskim, P. MigdaÅ, C. Weidner, A. Anupam, N. Case, N. Davis, C. Decaroli, Å. Ercan, C. Foti, P. Gora, K. Jankiewicz, B. R. Le Cour, J. Yago Malo, S. Maniscalco, A. Naeemi, L. Nita, N. Parvin, F. Scafirimuto, J. F. Sherson, E. Surer, J. Wootton, L. Yeh, O. Zabello, and M. Chiofalo, “Quantum games and interactive tools for quantum technologies outreach and education,” *Optical engineering* (Redondo Beach, Calif.) **61**, 081809 (2022).

¹³T. Liu, D. Gonzalez-Maldonado, D. B. Harlow, E. E. Edwards, and D. Franklin, “Qupcakery: A puzzle game that introduces quantum gates to young learners,” in *Proceedings of the 54th ACM Technical Symposium on Computer Science Education V. 1* (ACM, 2023).

¹⁴V. I. Yukalov, “Tossing quantum coins and dice,” *Laser physics* **31**, 055201 (2021).

¹⁵C. Cantwell, “Quantum chess: Developing a mathematical framework and design methodology for creating quantum games,” *arXiv [quant-ph]* (2019).

¹⁶R. D. Hoehn, N. Mack, and S. Kais, “Using quantum games to teach quantum mechanics, part 1,” *Journal of chemical education* **91**, 417–422 (2014).

¹⁷M. L. Chiofalo, C. Foti, M. Michelini, L. Santi, and A. Stefanelli, “Games for teaching/learning quantum mechanics: A pilot study with high-school students,” *Education sciences* **12**, 446 (2022).

¹⁸M. Gordon and G. Gordon, “Quantum computer games: quantum minesweeper,” *Physics education* **45**, 372–377 (2010).

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

¹⁹S. P. Gaunkar, N. Kawalek, D. Fischer, U. Bhatia, S. Verma, F. Rozpedek, and U. Zvi, "Game design inspired by scientific concepts of quantum physics," arXiv [physics.pop-ph] (2024).

²⁰A. La³pez-Incera, A. Hartmann, and W. D⁴fr, "Encrypt me! a game-based approach to bell inequalities and quantum cryptography," European journal of physics **41**, 065702 (2020).

²¹M. Sassoli de Bianchi, "Quantum dice," Annals of physics **336**, 56–75 (2013).

²²A. Goff, "Quantum tic-tac-toe: A teaching metaphor for superposition in quantum mechanics," American journal of physics **74**, 962–973 (2006).

²³A. La³pez-Incera and W. D⁴fr, "Entangle me! a game to demonstrate the principles of quantum mechanics," American journal of physics **87**, 95–101 (2019).

²⁴J. Marckwardt, A. Muller, D. Harlow, D. Franklin, and R. H. Landsberg, "Entanglement ball: Using dodgeball to introduce quantum entanglement," The physics teacher **59**, 613–616 (2021).

²⁵Seuss, *Sneetches, the: & other stories* (Random House USA Children's Books, 1961).

²⁶C. H. Bennett, G. Brassard, and N. D. Mermin, "Quantum cryptography without bell's theorem," Physical review letters **68**, 557–559 (1992).

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

Dot vector

(a)



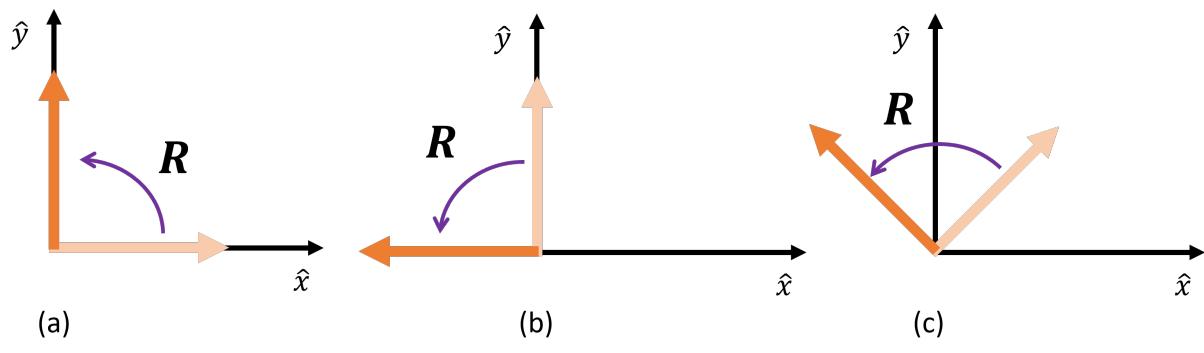
Vector

(b)

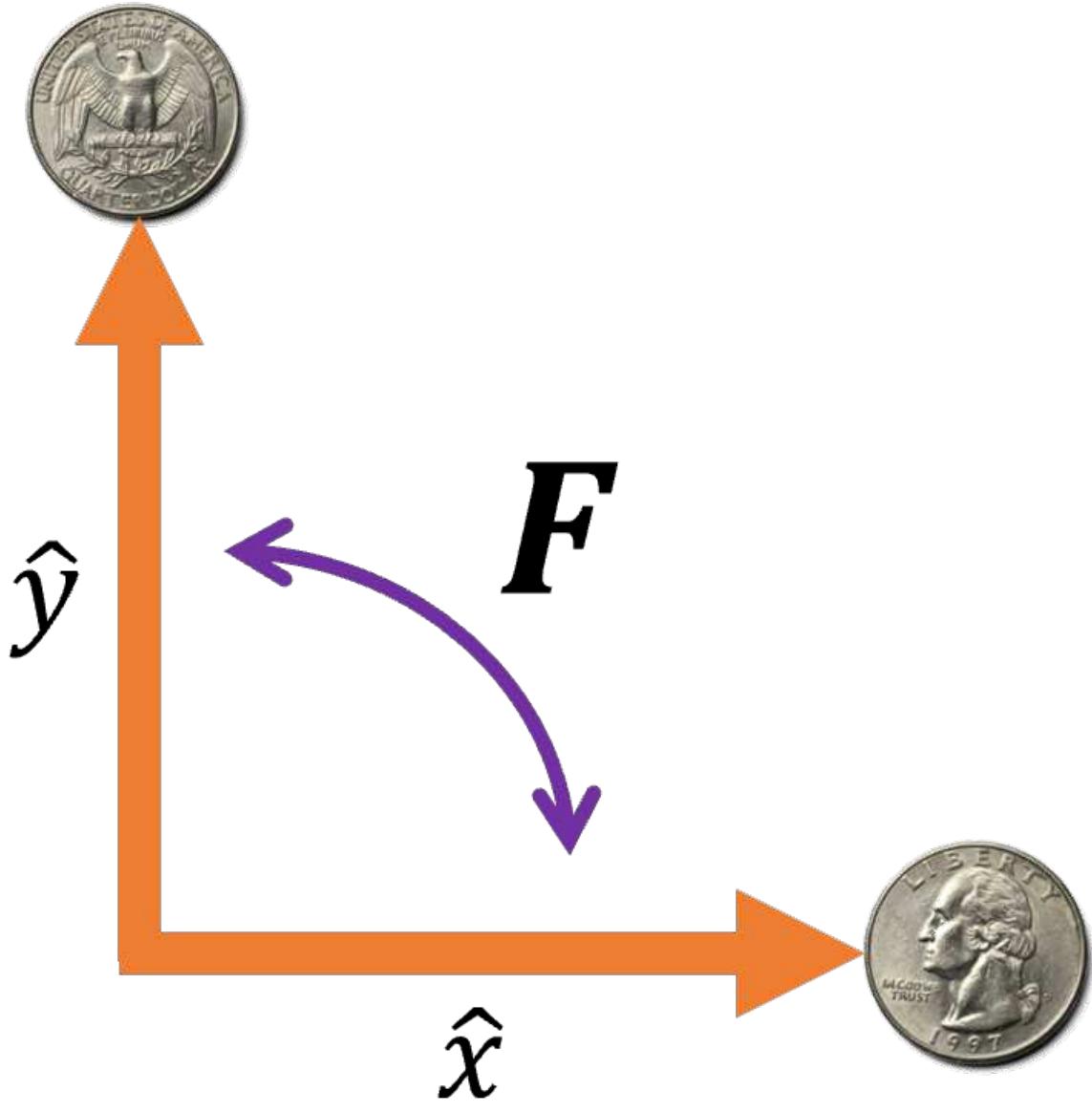


This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

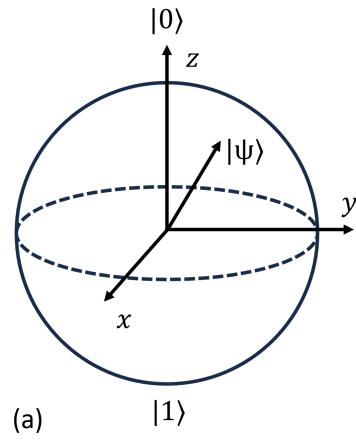


This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945



This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

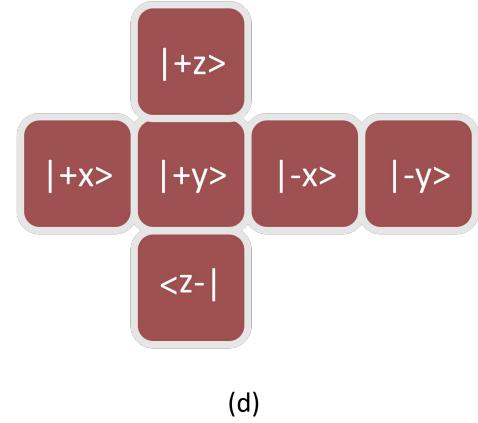
PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945



(b)



(c)



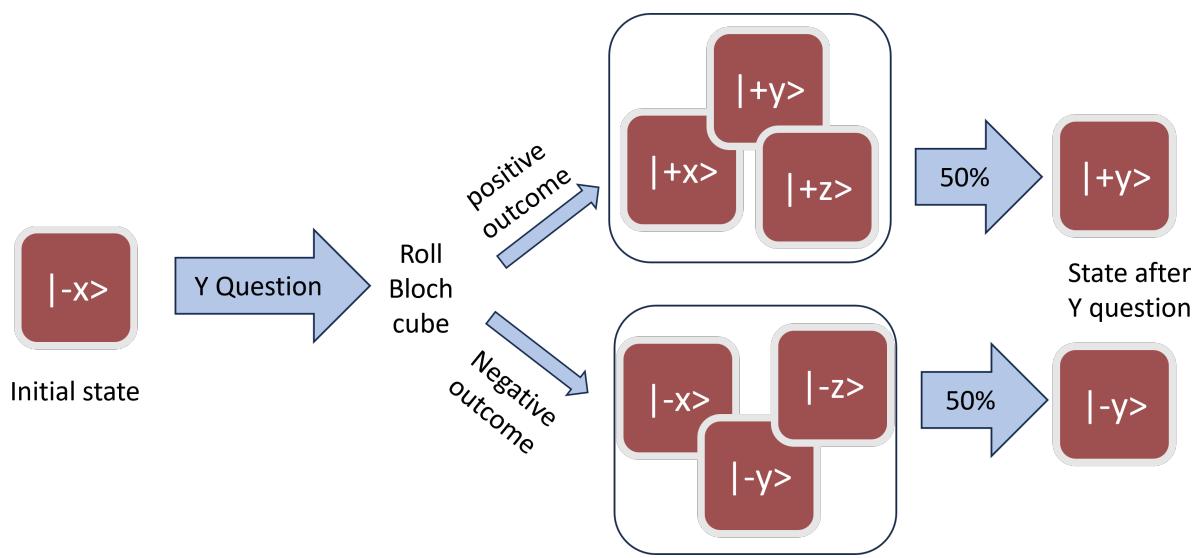
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945



This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945



This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1119/5.0209945

