Improving student understanding of the number of distinct many-particle states for a system of identical particles with a fixed number of available single-particle states

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We examine students' challenges in determining the number of distinct many-particle stationary states for a system of non-interacting identical particles, focusing on how these insights guided the design, validation, and evaluation of a Quantum Interactive Learning Tutorial (QuILT) to aid students' understanding. Specifically, we focus on systems with a fixed number of available single-particle states and particles, where the total energy is not fixed. The QuILT is designed to provide scaffolding support to help students learn these complex concepts more effectively. This study was conducted in advanced quantum mechanics courses, where written questions were administered to students in class following traditional instruction on the relevant concepts. Additionally, individual interviews were conducted with students to gain deeper insights. Our findings reveal that both upper-level undergraduate and graduate students face similar challenges in understanding these concepts. Additionally, difficulty with basic concepts in combinatorics that are necessary to answer the questions correctly were also found. The QuILT offers scaffolding support to help undergraduate and graduate students systematically reason through these concepts.

I. INTRODUCTION AND FRAMEWORK

Quantum mechanics (QM) is challenging to learn, even for advanced students, in part because its formalism is counterintuitive and contrasts sharply with classical mechanics [1–4]. Previous studies indicate that many upper-level undergraduate and graduate students face difficulties when learning QM [3, 4] and research-based methods and learning tools can enhance students' understanding of quantum concepts [5–9]. Researchers have focused on developing conceptual surveys to assess students' understanding of various quantum concepts [10–16] while others have investigated other aspects of learning QM [17, 18]. Furthermore, to build expertise in quantum mechanics, students need a good understanding of various representations commonly used in the field [19–28]. Furthermore, the quantum information revolution underscores the importance of workforce development in this field. Preparing students with a deep understanding of quantum concepts is essential to meet the demands of this rapidly evolving area so that they can help make e.g., fault tolerant quantum computers [29–37].

Previous investigations have aimed at helping students visualize quantum concepts, recognizing that visual representations can play a crucial role in building a more intuitive and accessible understanding of quantum mechanics [38–43] as well as learn QM via games [44]. Other studies have explored instructor views [45, 46]. These investigations shed light on how educator perspectives influence instructional methods and evaluation in quantum mechanics education. Some researchers have concentrated on identifying student difficulties that persist after traditional lecture-based instruction, particularly challenges related to quantum measurement [47–52], probability distributions for measuring physical observables, expectation values and their time dependence as well as student understanding of relative phase in the quantum states [23, 53–58], addition of angular momentum [59, 60], as well as quantum experiments [41, 61–69]. These studies highlight areas where students struggle to learn foundational quantum mechanics concepts after conventional teaching methods. Other research studies have examined cognitive issues particularly in relation to the unique challenges posed by the novel paradigm of quantum mechanics [70–78]. These studies aim to understand students' cognitive processes and their ability to invoke and apply QM concepts in light of the novel paradigm, emphasizing the need for research-based instructional strategies in this field. Some researchers have investigated student difficulties with additional quantum concepts, e.g., bound and scattering states [79–81]. Other researchers have explored student difficulties pertaining to several advanced quantum concepts [82–89]. Many researchers in previous studies have also created research-based instructional materials and teaching strategies to improve student comprehension of quantum concepts. These resources are designed to facilitate deeper learning in quantum mechanics.

Despite some studies on improving student understanding of identical particles [86, 87], except for a conference proceedings [90], there are no other papers that specifically investigate student difficulties with issues related to the number of distinct many-particle states in systems of identical particles with a fixed number of available single-particle states, nor has there been research on how to improve student learning of these concepts. This paper expands upon the difficulties in Ref. [90] and then discusses a QuILT that was developed and validated to improve student understanding of systems with these constraints.

For the topic we focus on, students must be proficient in combinatorics as well as identical particles. However, it has been found in a number of different contexts in introductory and advanced physics that students struggle to apply mathematics correctly in the context of physics [91–101] even if they can solve similar mathematics problems without the physics context. If the students' level of expertise is not commensurate with the complexity of the problem at hand and they have not been provided appropriate scaffolding support, they may not be able to solve the problem correctly [102].

As described in some of the prior papers on QuILTs [55], this research is guided by the framework of Zone of Proximal Development (ZPD) attributed to Vygotsky [103]. The ZPD is defined as the difference between what a learner can achieve on their own without support and what they can achieve under the guidance of an expert or in collaboration with peers. This framework emphasizes that for meaningful learning to occur, the activities students should engage in to learn must be within their ZPD, which itself is dynamic and grows as learners progress in their level of expertise. Thus, effective instruction must be commensurate with students' prior knowledge and build on their knowledge at a given time. Engaging with carefully designed instructional tasks such as the guided inquiry-based teaching learning sequences in the QuILT in collaboration with peers or with guidance from instructors, can stretch student ZPD and help them develop expertise in quantum mechanics. For instruction to incorporate students' initial knowledge appropriately, one must investigate student difficulties with relevant concepts and use this research as a guide in developing and validating the learning tools such as the QuILT discussed here. To ensure that the learning activities students engage with while working on the QuILT were in their ZPD, our investigation identified common student difficulties and used them as resources in the design of the QuILT.

Supplemental materials for this paper provide a short background of pertinent concepts, summarize difficulties discussed in Ref.[90] before we expand upon them in the main text below. Supplemental materials also include the standard aspects of the development, validation and implementation of the QuILT as well as the structure of the QuILT that have been described earlier, e.g., see [86, 87]. Below, the main text focuses on the learning objectives of the QuILT and how it focuses on improving student learning. Then, we describe the in-class assessment of the QuILT and summarize the findings.

Although the background on relevant issues is discussed in Ref.[90] and included in supplementary materials of this paper, it is important to recognize that in order to determine the number of distinct many-particle states in a given situation, one must make appropriate connections between the relevant physics and math conceptual and procedural knowledge. For example, for correctly determining the number of distinct many particle states in a given situation, one must recognize relevant properties of fermions, bosons and distinguishable particles, and also be able to reason about and use the combinatorics correctly consistent with the given situation. For example, in a given situation, students should be able to first recognize the consequences of the particles being distinguishable vs. indistinguishable. Then, in the case of indistinguishable particles, they must recognize the consequences of the many-particle wavefunction being completely symmetric for the bosonic case and being completely anti-symmetric for the fermionic case. For example, for the fermionic case, if one only recalls the Pauli exclusion principle as memorized knowledge (but did not realize that the many-particle wavefunction must be completely anti-symmetric consistent with the exclusion principle) and not the fact that the identical fermions are indistinguishable, there is lack of all relevant physics conceptual knowledge and one will not be able to do the correct combinatorics even if they knew how to do the combinatorics correctly in a given situation. In particular, indistinguishability entails how many unique many-particle states there are in a given situation while the difference between the bosonic (many-particle states being completely symmetric) and fermionic (many-particle states being completely antisymmetric) cases entails how you are allowed to place different particles in the available single-particle states (e.g., in the fermionic case, there is only zero or one particle in each single-particle state often referred to as the Pauli exclusion principle whereas there is no restriction on the number of particles that can be placed in any given single-particle state for the bosonic case). Thus, for the fermionic case, if there are four states and four particles, taking into account only Pauli exclusion principle as a memorized knowledge (without considering indistinguishability which is central) would incorrectly imply that there are $4 \times 3 \times 2 \times 1 = 24$ distinct many-particles states. However, there is only one distinct four-particle state for indistinguishable fermions in this situation with each fermion in a different single-particle state because indinstinguishably ensures that all permutations of the particles are the same (if the particles were distinguishable, there would be 24 permutations but for fermionic system, we must divide by the number of permutations due to indistinguishability, so there is only one possibility). We also note that even if one has the conceptual knowledge, they may not have the relevant procedural knowledge to be able to correctly apply the conceptual knowledge in various situations.

One should also realize that the following are all equivalent ways to express a basis state for a system of three non-interacting identical particles in three single-particle states because the single-particle wavefunctions in the product can be written in any order to construct the three-particle basis state using standard notation (see Supplementary materials for details): $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, $\psi_{n_2}(x_2)\psi_{n_1}(x_1)\psi_{n_3}(x_3)$, $\psi_{n_2}(x_2)\psi_{n_3}(x_3)\psi_{n_1}(x_1)$, $\psi_{n_3}(x_3)\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, and $\psi_{n_3}(x_3)\psi_{n_2}(x_2)\psi_{n_1}(x_1)$. In other words, these are not six different basis states and if one were to count it as such, they will count the number of distinct many particle states incorrectly even if they had all other relevant conceptual and procedural knowledge.

Determining the number of distinct many-particle states for a system in which the number of single-particle states is fixed is an important concept for students to help prepare them, e.g., for quantum mechanics leading to quantum statistical mechanics.

II. METHODOLOGY FOR INVESTIGATING STUDENT DIFFICULTIES

The participants in this study were undergraduate and graduate students enrolled in the second semester of a two-semester upper-level undergraduate or first-year core graduate quantum mechanics course. A majority of the students in the undergraduate course are juniors and seniors (typically greater than 75% are seniors). We note that the investigation presented here is part of a larger study focusing on improving student understanding of identical particles pertaining to concepts typically covered in advanced quantum mechanics courses so the methodology is the same [86, 87]. The data from these earlier administration of the questions in previous years were useful for gradually refining the questions into the final validated versions of the pre/posttest. Additional insight was gained concerning difficulties with identical particles and the extent to which a particular version of the QuILT helped improve student understanding via responses of 14 paid student volunteers (after they had traditional instruction in relevant concepts) during a total of 81 hours of individual think-aloud interviews [104]. Here we focus specifically on difficulties pertaining to the number of distinct many-particle states for a system of identical fermions, bosons or distinguishable particles for a fixed number of single-particle states and how the research was used as a guide to develop and validate the corresponding QuILT (along with the pre/posttest). This subset of the interview was typically one hour long with each student.

To probe difficulties for a given system with the given constraints, two questions were posed to the students, one in the pretest and one in the posttest. As noted, other QuILTs on identical particles focused on other aspects (e.g., counting number of distinct states with fixed energy of the system, impact of incorporating degeneracy in the single-particle states on counting in various situations and incorporation of spin degree of freedom in addition to the spatial degree of freedom) have other associated pre/posttest questions and here we will focus only on the two questions relevant for the specific topic under discussion. Question Q1 [90] (see Supplementary materials) was posed during the individual interviews as well as on the pretest for the QuILT after traditional instruction in relevant topics. Q2 was posed on the posttest following traditional instruction on identical particles as well as after students engaged with the QuILT. In the last in-class administration of the pre/posttest discussed here, Q1 and Q2 were posed to 30 graduate students and 25 undergraduate students.

Q2. For a system of two non-interacting identical particles, there are five distinct single-particle states $\psi_{n_1}(x)$, $\psi_{n_2}(x)$, $\psi_{n_3}(x)$, $\psi_{n_4}(x)$, and $\psi_{n_5}(x)$ available to each particle. How many different two-particle states can you construct if the particles are

(a) Fermions? (Ignore spin).

(b) Bosons? (Ignore spin).

(c) Distinguishable particles? (Ignore spin).

In Q2(a), there are $\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$ distinct two-particle states. In Q2(b), for a system of identical bosons, a single-particle state can have more than one boson. There are $\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15$ distinct two-particle states for a system of two identical bosons in Q2(b). In Q2(c), for the contrasting case of identical particles that can be treated as distinguishable, there are $5^2 = 25$ distinct two-particle states for a system of two identical particles that can be treated as distinguishable.

The pretest question Q1 [90] in Supplementary materials is completely analogous to posttest question Q2 but involves 3 particles and 4 states.

III. STUDENT DIFFICULTIES

The goal of investigating student difficulties was to use them as a guide in the development and validation of the QuILT to help students develop a functional understanding of relevant concepts and procedures. Both written responses and individual one-on-one interviews were useful for diagnosing student difficulties. However, interviews with 14 students throughout the development of different versions of the QuILT in which students worked through the pretest, QuILT and posttest while thinking-aloud were particularly helpful for additional clarification. Student difficulties were often due to either not having all relevant knowledge of the concepts in physics and math or having relevant knowledge but not knowing how to apply it correctly in the given situation. We divide student difficulties related to the number of distinct many-particle states for a system of non-interacting identical particles in the given situations into four categories. These categories include (1) difficulties with relevant conceptual knowledge, (2) difficulty with procedural knowledge, (3) difficulty with mathematical sense-making in the context of physics, and (4) reliance on memorized formulas. We note that this categorization is only one of the many ways to do it and there are many other ways to categorize student difficulties. Also, some of the difficulties discussed here can be placed in more than one of these four categories, but we have often placed them into only one category (unless the researchers concluded that there is a point to be made in more than one category with the same example) to illustrate these broader classifications. Some of the difficulties discussed in Ref. [90] are in supplementary materials while others are discussed below. Also, in a given broad category, any text in boldface separates different difficulties.

A. Difficulty with relevant conceptual knowledge

We find that students often had conceptual difficulties with indistinguishability and symmetrization requirement pertaining to identical particles. Conceptual difficulties described in Ref. [90] are summarized in the supplementary materials.

B. Difficulties with procedural knowledge

Some students appeared to have the correct conceptual understanding but struggled to connect their conceptual knowledge with the procedural knowledge for determining the number of distinct many-particle states correctly and in systematic reasoning about it. Interview gave a closer look at how some students had difficulty determining the number of distinct many-particle states even for a system with a small number of particles and available single-particle states while others only had difficulty generalizing to a system with a large number of particles and available single-particle states. Below are some of the difficulties students had in interviews and written responses in formulating a systematic approach for determining the number of distinct many-particle states in the given situation while others are in Suplementary materials [90].

Attempting to explicitly list all of the possible many-particle states but omitting at least one possible combination: Both in interviews and written responses, students often attempted to list all states, e.g., by considering many-particle states constructed from placing particles in various single-particle states or writing many-particle wavefunctions from single-particle wavefunctions or both. Nearly all the interviewed students after traditional instruction began by attempting to list all of the possible many-particle states for a system of indistinguishable bosons in Q1(b). Most of the students continued to list as many of the distinct many-particle states as they could. However, some of them omitted at least one of the possible many-particle states partly because they were not systematic. For example, in Q1(b), some students began by listing several states in one type of arrangement (e.g., all the bosons in the same single-particle state) and then moved on to listing states in another type of arrangement (e.g., all the bosons in different single-particle states) without listing all the many-particle states in each arrangement before moving on to the next arrangement. They would often continue to list states in various arrangements and then stop when they could not identify any new many-particle states that they had not already listed. Some students missed at least one of the three-particle states in Q1(b) in which two of the bosons are in one single-particle state and one boson is in a different single-particle state.

Difficulty counting the different arrangements correctly for a system of distinguishable particles: Sometimes students were using correct combinatorics for the case involving a system of distinguishable particles but they missed one important aspect of the combinatorics. For example, one difficulty with Q1 for the system with distinguishable particles was that some students who correctly determined that there were a total of $4 \times 3 \times 2 = 24$ distinct many-particle states for the case in which each distinguishable particle is in different single-particle state and 4 many-particle states in which the distinguishable particles were all in the same single-particle states, but they had difficulty with the many-particle states in which two distinguishable particles were in one single-particle state and another distinguishable particle was in another single-particle state. For example, some students stated that two distinguishable particles can be placed in the same single-particle state in 4 ways and then the third distinguishable particle can be placed in any of the other three single-particle states so the total possibilities in this case are $4 \times 3 = 12$. Thus, they incorrectly came up with the total number of many particle states as $24 + 4 + 12$. However, they overlooked that when two distinguishable particles are in one single-particle state and the third in another, the two particles out of three can be chosen in 3 ways so the total possibilities in that case are not 12 but $12 \times 3 = 36$ and then the total number of many-particle states is $24 + 4 + 36 = 4^3$.

Some students who claimed that the single-particle wavefunctions of different particles in the basis states in the product space do not "commute" had difficulty generating a many-particle wavefunction with the appropriate number of terms and in determining the normalization constant. For example, students with this type of difficulty often claimed that the many-particle wavefunction for a system of three identical bosons in which all the bosons are in the same single-particle state is $\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)+\psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1)+\psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)]$ or $\frac{1}{\sqrt{6}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3) +\psi_{n_1}(x_1)\psi_{n_1}(x_3)\psi_{n_1}(x_2)+\psi_{n_1}(x_2)\psi_{n_1}(x_1)\psi_{n_1}(x_3)+\psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1) +\psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)+\psi_{n_1}(x_3)\psi_{n_1}(x_2)\psi_{n_1}(x_1)]$. They had difficulty realizing that all terms in both expressions are equivalent and can be simplified to a single term $\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$. Additionally, in the interview situation, when students with this type of response were asked for the normalization constant, they often had difficulty in correctly determining the normalization constant. For example, the expression $\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3) + \psi_{n_1}(x_2)\psi_{n_1}(x_3)\psi_{n_1}(x_1) + \psi_{n_1}(x_3)\psi_{n_1}(x_1)\psi_{n_1}(x_2)]$ reduces to $\sqrt{3}\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$, which is not the properly normalized many-particle wavefunction for a system of three identical bosons in the single-particle state ψ_{n_1}.

Difficulty with the bin and divider method for determining the number of distinct many-particle states for a system of identical bosons: Students often had difficulty determining the number of distinct many-particle states for a system of indistinguishable bosons using the "bin and divider" method (see Supplementary materials). For example, some students who used the bin and divider method had difficulty realizing that one should be using the number of dividers (number of available single-particle states minus 1) as opposed to the number of bins (number of available single-particle states) to determine the number of distinct states for a system of identical bosons. One interviewed student incorrectly claimed that "we can either count the number of ways to arrange the bosons or the states" among the total number of indistinguishable objects. This student and others with this type of difficulty incorrectly claimed that the number of distinct many-particle states was $\binom{N+M}{N}=\binom{N+M}{M}=\frac{(N+M)!}{N!M!}$.

C. Difficulty with mathematical sense-making in the context of determining the number of distinct many-particle states

Some students had difficulty integrating physics and mathematics concepts correctly in order to determine the number of distinct many-particle states for a system of identical particles. Below, we discuss some difficulties students had in determining the number of distinct many-particle states due to difficulty in applying an underlying mathematical concept correctly in different quantum mechanical context.

Incorrectly adding the number of available single-particle states for each identical particle: Some students stated that each indistinguishable particle can be placed in any of the available single-particle states and that the total number of distinct many-particle states is the sum of the number of available single-particle states for each boson.

For a system of identical fermions in Q1(a), one interviewed student incorrectly claimed that there are $4+3+2=9$ distinct many-particle states for a system of three fermions and four available single-particle states. At least some students with this type of response correctly applied their memorized knowledge of Pauli exclusion principle and determined the number of distinct many-particle states such that no two fermions are in the same single-particle state, but incorrectly added the number of ways to arrange the fermions in each single-particle state rather than multiplying. This is an interesting way of incorrectly applying the Pauli exclusion principle or justifying the procedure for determining the number of distinct many-particle states.

For a system of identical bosons in Q1(b), one interviewed student stated that "there are four available (single-particle) states for the first boson to go in and there are four available (single-particle) states for the second, since bosons can occupy the same (single-particle) state. The same for the third. So there are four (available single-particle states) for the first (boson), four (available single-particle states) for the second (boson), and four (available single-particle states) for the third (boson)." The student then jotted down $4+4+4=12$ and claimed there were 12 distinct three-particle states for Q1 for a system of identical bosons.

Difficulty counting the different arrangements correctly for a system of indistinguishable bosons: In Q1, for a system of three identical bosons and four available single-particle states, many students attempted to determine the number of ways: (1) all three particles could be arranged in the same single particle state, (2) two bosons could be in the same state and the other boson is in a different state, (3) all three bosons could be in different single-particle states to determine the total number of distinct many particle states. For example, one common incorrect response in Q1 was $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}=4+6+4=14$. One interviewed student with this type of response stated that "when all the bosons are in the same state, there are four states and we need to choose which one has the bosons. There are $\binom{4}{1}$ ways to arrange all the bosons in one state. If two of the bosons are in the same state and one is in another, then we need to choose which two states have the bosons. That makes $\binom{4}{2}$ ways to arrange the bosons. And then, if all three bosons are in different states, then we need to choose which three states have the bosons. There are $\binom{4}{3}$ ways to do that." The student then jotted down that the total number of distinct many-particle states in Q1(b) was $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}=4+6+4=14$.

D. Reliance on memorized formulas

Some students calculated the number of distinct many-particle states in Q1 using a memorized formula rather than formulating a systematic reasoning for a given system. This was more clear in the interviews in which students simply stated that they remembered something from what they had been taught instead of reasoning about it when asked to do so. In some cases, students recalled one or more expressions which were correct for a particular type of system but applied these expressions to the wrong system of identical particles.

Mixing up the fermionic and bosonic cases: Some students answered Q1(a) in a manner which would have been correct for a system of identical bosons and Q1(b) in a manner which would have been correct for a system of identical fermions. These students often wrote the formula for the number of many-particle states from memory.

Incorrectly multiplying (as opposed to dividing) by the number of indistinguishable combinations: Some students attempted to determine the number of distinct many-particle states for a system of indistinguishable particles by determining the number of arrangements of identical particles in the single-particle states and then adjusting this number based upon the number of indistinguishable permutations. However, in Q1(a) and Q1(b), some students incorrectly multiplied the number of distinct many-particle states by the number of permutations of the indistinguishable particles (as opposed to dividing it). One interviewed student who was unsure about whether to multiply or divide by the number of permutations of the indistinguishable particles decided to multiply stating that is what he remembers from the course. Even when prodded explicitly, he did not explicitly reason about whether indistinguishability should give rise to more or less many-particle states compared to the case when particles are distinguishable.

Similarly, for a system of identical fermions in Q1(a), another interviewed student incorrectly determined that there are $4! \cdot 3!$ distinct many-particle states for the system of three indistinguishable fermions in four single-particle states. When asked, this student stated how he obtained $4!$ by noting that "we can put the first fermion in any of the four states. The second fermion can go in any of the three states that the first fermion didn't go in. And the third fermion can be in either of the two remaining states." The student then went on to try to account for the indistinguishability of the three fermions. "Then we need to multiply by the number of arrangements that are the same for these three identical fermions. There are 3! ways to arrange these three fermions so we need to multiply by this factor." The student then jotted down his answer as $4! \cdot 3!$. This student used rote memory to multiply rather than divide and claimed that there are more distinct states when taking into account the indistinguishability of the fermions. Even when questioned about it, he did not correct his response and realize that there are fewer distinct many-particle states for the system of indistinguishable fermions than there are for a system of three distinguishable particles all in different single-particle states, e.g., there are $\binom{4}{3} = \frac{4!}{3!1!} = 4$ distinct ways to arrange the three indistinguishable fermions among the four single-particle states.

Similar differences were observed in Q1(b) as well both in written responses and interviews. For example, an interviewed student in Q1(b) stated that there are $4^3 \times 3!$ distinct many-particle states for a system of three indistinguishable bosons in four single-particle states. The student incorrectly multiplied 4^3 for the distinguishable particle case by 3! and even when asked to explain why one should multiply did not want to reason about it and claimed that this is what he remembers.

Incorrectly determining the number of distinct many-particle states for a different case in which the number of particles in the system was not fixed: Some students answered Q1 as though the number of particles in the system was not fixed when the number of identical particles for a given system is specified in the problem. During Interviews, some students with this type of response attempted to recall an example they had seen in class in which the number of particles in the system was not fixed and instead they were asked to determine the number of distinct many-particle states for the system with different conditions specified by the problem. Below, we give two such examples.

For a system of identical fermions in Q1(a), one interviewed student correctly stated that "each single-particle state can have either zero or one fermion, so there are two possibilities for the first single-particle state, two for the second, and two for the third and fourth. There are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ distinct many-particle states." This student and others with this type of response answered a completely different question from the one posed (and calculated all the possible many-particle states for fermions in these four single-particle states ranging from zero to four fermions) and failed to recognize that the system in Q1 had three indistinguishable fermions. When asked by the interviewer to clarify his answer, the student noted that this is what he remembers from what he learned.

For a system of identical particles that can be treated as distinguishable in Q1(c), one interviewed student incorrectly claimed that "there are three particles that can be put in the first state, three particles that can be put in the second state, three in the third, and three in the fourth." The student then wrote $3^4 = 81$ as the total number of distinct three-particle states. When asked to clarify his response, the student noted that this is what he remembers about the case of distinguishable particles. This student failed to realize that his method for counting the total number of distinct states was not consistent for a system with only three particles in Q1. He did not consider that if there are three identical particles that can be treated as distinguishable in the first single-particle state, then there are none remaining to be placed in the other single-particle states. He also did not realize that for a system in which there are three particles in each of the four single-particle states, the system would have 12 particles not 3. This student and others with this type of reasoning failed to do a consistency check for the fact that they were determining the number of distinct many-particle states for a system restricted to only the specified number of particles and that specified number of particles was not 12 but 3.

Using the formula N^M instead of M^N without considering whether it makes sense for the given situation: Some students did not explicitly reason about the number of distinct many-particle states for a system of identical particles that can be treated as distinguishable for a system with N particles and M available single-particle states and wrote N^M for their final answer instead of N^M. For example, for a system of identical particles that can be treated as distinguishable in Q1, some of the interviewed students who answered that there are $3^4 = 81$ distinct many-particle states, noted that this is what they remember when asked to reason about it. In particular, while some of the students with this type of response were recalling a different case in which the total number of particles was not fixed as discussed in the preceding difficulty, others did not even try to reason about why their answer should be $3^4 = 81$ instead of 4^3 even when explicitly asked during interview. When asked how they arrived at the answer, they could have checked the reasonability of the formulas M^N or N^M for a system with a small number of particles and available single-particle states. For example, they could have considered a system of one particle and two available single-particle states in which the particle can be in either of the two single-particle states and there are two distinct many-particle states and thus, $M^N = 2^1 = 2$ gives the correct answer but using the formula $N^M = 1^2 = 1$, one incorrectly obtains only one distinct many-particle state for the system. However, even prodding did not get students to give any reasoning except state that this is what they remember.

We note that even though the last few difficulties in this section have focused on using the formula $\binom{M}{N}$, they are somewhat different, e.g., the student may be reasoning correctly in at least one case but using this formula incorrectly.

IV. METHODOLOGY FOR DEVELOPMENT OF THE QUILT

A. Development and Validation of the QuILT

Based upon our learning objectives (see the next section) and research on student difficulties with fundamental concepts for systems of identical particles discussed earlier, we developed and validated a QuILT (along with the corresponding pre-/posttests) that strives to help students learn relevant concepts. The development and structure of the QuILT was inspired by Vygotsky's zone of proximal development (ZPD) [103]. The QuILT strives to help students learn relevant concepts by providing appropriate scaffolding. The types of problems that many students were unable to solve successfully at the onset of the QuILT after traditional lecture-based instruction are scaffolded using guided inquiry-based teaching-learning sequences that build on each other. The amount of support provided to students via the QuILT is gradually decreased to help them develop self-reliance. The QuILT strives to scaffold student learning using a guided inquiry-based approach. It incorporates hypothetical student conversations and sets of inquiry-based sequences designed to help them focus on inconsistencies in their initial reasoning and provide scaffolding to help them resolve the inconsistencies. Please see Supplementary materials for additional issues [86, 87].

B. Learning Objective

After working through the QuILT, students should be able to do the following:

- Determine whether a system of identical fermions is possible based upon the given information about a hypothetical situation and justify their reasoning, e.g., using the Pauli exclusion principle.
- Use a systematic approach when counting the number of many-particle states for a system of identical fermions, bosons, or a system of distinguishable particles and support this approach in words or using diagrams displaying the particles and energy levels.
- Compare the number of different many-particle states for systems of identical fermions or bosons to a system in which the particles can be treated as distinguishable.
- Write all of the possible many-particle wavefunctions for a given system and compare to the calculated number of many-particle states for the same system.
- Calculate the number of different three particle states for a system of N non-interacting identical particles and M distinct single particle states available to each particle, if the particles were fermions, bosons or distinguishable (for $M > N$).

C. IMPROVING STUDENT UNDERSTANDING VIA THE QUILT

In the guided inquiry-based teaching-learning sequences in the QuILT, students actively engage with examples focusing on concepts in a given situation consistent with the learning objectives. These guided teaching-learning sequences use common student difficulties found in our investigation as resources, e.g., how to determine the number of distinct many-particle states in a given situation. In particular, the QuILT scaffolds student learning and helps them develop a systematic approach for determining the number of many-particle states for a system of identical particles and connect the number of distinct many-particle states to the possible number of many-particle stationary state wavefunctions. In the QuILT, students consider the systems of identical particles in the following order: (1) indistinguishable fermions, (2) indistinguishable bosons, and (3) identical particles that can be treated as distinguishable. To help students learn to deduce complicated cases starting from simple ones, for each system, students begin by determining the number of distinct many-particle states for a system of two identical particles. They then consider a system of three identical particles and determine the number of distinct many-particle states. Finally, students are presented with systems in which the number of particles becomes very large and they are provided guidance and support in learning to determine the number of distinct many-particle states. Since not recognizing indistinguishability and its consequences was a common difficulty, for the systems of indistinguishable fermions and indistinguishable bosons, students also work with diagrammatic representations for the system that strive to help students recognize why care must be taken to ensure that one is determining these particles as indistinguishable particles. These diagrammatic representations are intended to help them develop a systematic reasoning for determining the number of distinct many-particle states for a system with a large number of particles and available single-particle states. Below are several examples from the QuILT that strives to provide scaffolding support intended to help students with these concepts and address some of the common difficulties discussed earlier.

Helping students determine the number of distinct many-particle states for a system of fermions: As we noted in the preceding section, students often had difficulties with determining the number of distinct many-particle states for a system of fermions. Therefore, the QuILT strives to take into account these difficulties via the guided inquiry-based learning sequences.

In particular, it focuses on helping students learn to identify that a system of identical fermions is made up of indistinguishable particles and one must be careful to only count distinct many-particle stationary states (since students often had difficulties with this). The QuILT also strives to help students learn that in addition to the indistinguishability requirement, the many-particle states for a system of indistinguishable fermions are consistent with the Pauli exclusion principle since this was also a difficulty. One consequence of the Pauli exclusion principle is that a system cannot have more fermions than the number of available single-particle states.

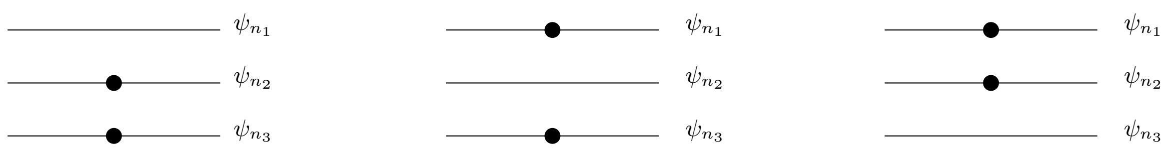
The following is an example of a hypothetical student conversation from the QuILT that focuses on providing an opportunity for reflection of some common difficulties in which students must consider each statement and explain why they agree or disagree with each. This conversation is part of a guided inquiry-based learning sequence that strives to help students determine the number of distinct two-particle states for a system of two indistinguishable fermions and three distinct single-particle states while not overcounting states by treating the particles as distinguishable.

Student 1: *For a system of two fermions and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}, there are three available single-particle states for the first fermion. That leaves two single-particle states for the second fermion since the second fermion cannot occupy the same single-particle state as the first fermion. The number of two-particle states is* $3 \times 2 = 6$.

Student 2: *I disagree with Student 1. Since the fermions are indistinguishable, we cannot distinguish which fermion is in which single-particle state. For example, we can only tell that one fermion is in single-particle state ψ_{n_2} and another fermion in single-particle state ψ_{n_3}. But, there is no way to tell which fermion is in which single-particle state. This indistinguishability is reflected in the antisymmetrized wavefunction.*

Student 3: *I agree with Student 2. Figure 1 shows the diagrammatic representation for the 3 distinct two-particle states:*

FIG. 1. Diagramatic representation for two fermions in three single-particle states.



Student 1 is not correct while Students 2 and 3 are correct in the preceding conversation. This conversation is designed to help students reflect upon the fact that the fermions are indistinguishable. After considering this hypothetical conversation, as part of the guided inquiry-based sequence, students are asked to write all the possible stationary state wavefunctions for a system of two fermions and three available single-particles states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} for the case when the two fermions are in the same single-particle state and when the two fermions are in different single-particle states. The students are then asked to reflect upon the number of distinct many-particle states and the number of possible many-particle stationary state wavefunctions. Further scaffolding is provided that strives to help students realize that the number of distinct many-particle states is the same as the number of possible many-particle stationary state wavefunctions for a given system.

The following statement is an excerpt from a hypothetical conversation between students that strives to help them reflect upon how to determine the number of distinct many-particle states and connect this reasoning to a mathematical expression for counting the states (this was a common difficulty). Students are asked to explain why they agree or disagree with each student such as the following:

Student 2: *There are three distinct single-particle states available to the fermions and we must choose any two for the fermions to occupy. The number of distinct two-particle states for a system of two indistinguishable fermions and three distinct single-particle states is* $\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$.

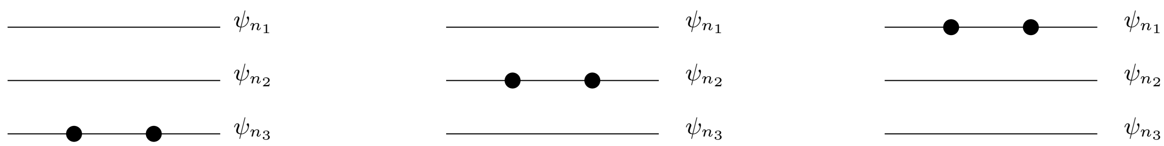
Student 2 is correct. After students consider these types of examples of determining the number of distinct two-particle states for a system of two fermions, they then work through guided inquiry-based sequences for a system of three identical fermions. Then, they consider systems for a large number of fermions and a large number of available single-particle states. Students are provided further scaffolding support that strives to help them generalize the results from the systems of two and three fermions and become proficient in determining the number of distinct many-particle states for a system with a large number of fermions (something that was challenging for many students as discussed in the preceding section).

Helping students determine the number of distinct many-particle states for a system of bosons: To address the difficulty in distinguishing between bosons and distinguishable particles, the QuILT strives to help students learn that a system of identical bosons must be treated as a system of indistinguishable particles and develop a systematic approach for determining the number of distinct many-particle states in a given situation.

The following hypothetical conversation is part of a guided inquiry-based learning sequence that aims to help students with the fact that a system of identical bosons cannot be treated as a system of distinguishable particles and provides a diagrammatic representation to help them reflect upon the distinct many-particle states. In this conversation, students consider a system of two indistinguishable bosons and three distinct single-particle states and are asked to explain why they agree or disagree with each:

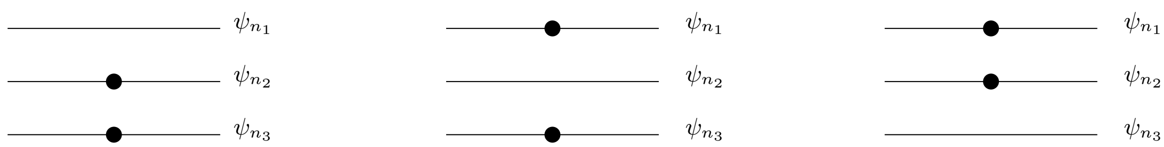
Student 1: *For a system of two bosons and three distinct single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}, there are three available states for the first boson and three available states for the second boson. The number of two-particle states is* $3 \times 3 = 9$.
Student 2: *I disagree with Student 1. You are overcounting since you are not taking into account the fact that bosons are indistinguishable. If the bosons are in the same single-particle state, there are three possibilities as shown in Figure 2.*
But, if the bosons are in different single-particle states, there are three possibilities since bosons are indistinguishable and

FIG. 2. Diagramatic representation for two bosons in the same single-particle state.



swapping the two bosons in the two single-particle states in each of the following situations does not produce a new two-particle state as depicted in Figure 3.
There are 6 distinct two-particle states for a system of two bosons and three distinct single-particle states.

FIG. 3. Diagramatic representation for two bosons in which the bosons are in different single-particle states.



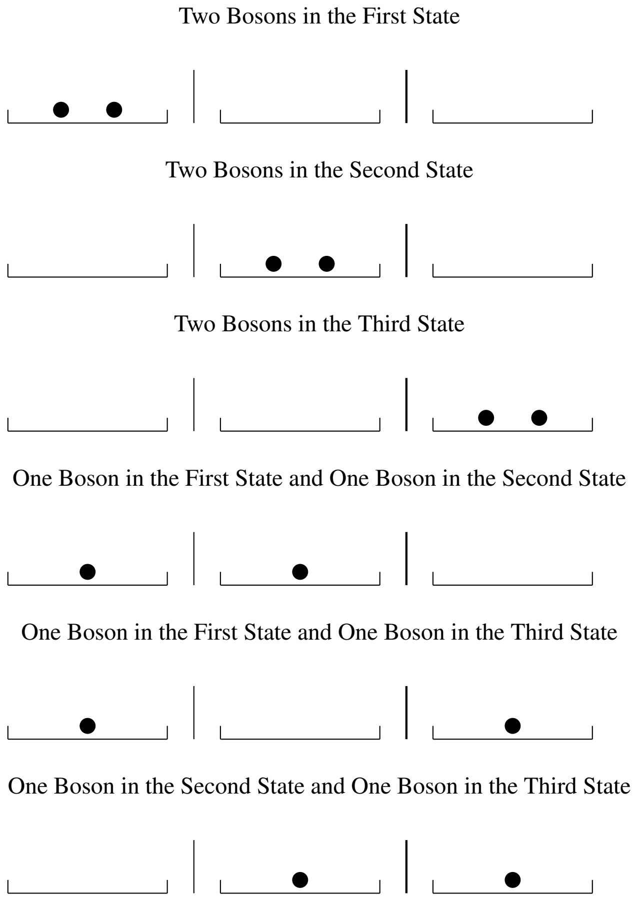
Student 1 is incorrect and Student 2 is correct in the preceding conversation. If one treats the identical bosons as distinguishable, as Student 1 has, then one is overcounting the case in which the two identical bosons are in different single-particle states. Student 2's statement regarding the particles being indistinguishable under the exchange of the particles strives to draw students' attention to the fact that these two bosons cannot be distinguished. After considering this hypothetical conversation, as part of the guided inquiry-based sequence, students are asked to write all of the possible stationary state wavefunctions for a system of two bosons and three available single-particles states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3}. The students are then asked to reflect upon the number of distinct many-particle states and the number of possible many-particle stationary state wavefunctions. Further scaffolding is provided that strives to help students realize that one must obtain the same number of distinct many-particle states from the combinatorics as the number of possible many-particle stationary state wavefunctions for a given system.

The next hypothetical conversation in the guided inquiry-based learning sequence strives to help students learn a method for determining the number of distinct ways two indistinguishable bosons can be arranged in the three distinct single-particle states by introducing the bin and divider method (a method that was challenging for students as discussed in the preceding section).

Student 1: *For a system of two bosons, there can be more than one boson in a given single-particle state. We can treat the single-particle states as bins to be filled with bosons and dividers to separate the different single-particle states or bins. For example, if the system had two bosons in the first single-particle state then the first bin would have two bosons. For a system with three single-particle states available, we would need two dividers between the three single-particle states. In the case of three single-particle states and two bosons, we must find the number of possible arrangements of the two bosons and two dividers.*
Student 2: *I agree with Student 1. Furthermore, since the two dividers cannot be distinguished from one another and the bosons cannot be distinguished from one another, we can permute the indistinguishable dividers with the indistinguishable bosons to find all the possible ways to permute two bosons in the three single-particle states as shown in Figure 4:*
Student 3: *The number of distinct many-particle states comes from the number of ways the two bosons and two dividers can be permuted. We have a total of four objects (two bosons and two dividers) and we can find the number of ways to permute the two bosons or equivalently the number of ways to permute the two dividers among the four objects. The number of distinct two-particle states is* $\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$.

All three students in the preceding conversation are correct. Student 1 is describing the bin and divider method and Student 2 is providing a diagrammatical representation of different arrangements of the two bosons in the bins representing the single-particle

FIG. 4. Diagramatic representation of the bin and divider method for a system of identical bosons.



states. Student 3 provides a mathematical expression for the total number of distinct two-particle states.

After students consider examples that strive to help them learn how to determine the distinct two-particle states for a system of two bosons, they then work through several guided inquiry-based sequences for a system of three identical bosons. Then, they consider systems for a large number of bosons and a large number of available single-particle states. Students are provided scaffolding support that strives to help them generalize the results from the systems of two and three bosons to be able to determine the number of distinct many-particle states for a system with a large number of bosons. The following is a hypothetical student conversation aimed at helping students develop a systematic approach for determining the number of distinct ways N indistinguishable bosons can be arranged in the M distinct single-particle states.

Student 1: *Using the bin and divider method, there are $N + M - 1$ total objects that should be permuted, out of which N bosons are indistinguishable from each other and the $M - 1$ dividers are indistinguishable from each other. We must calculate the number of distinct arrangements.*
Student 2: *When we choose the number of ways to place the $M - 1$ indistinguishable dividers between the N bosons, we get $\binom{N+M-1}{M-1} = \frac{(N+M-1)!}{(M-1)![(N+M-1)-(M-1))]!} = \frac{(N+M-1)!}{(M-1)!N!}$. If instead we choose the number of ways to place the N bosons between $M - 1$ dividers, we get $\binom{N+M-1}{N} = \frac{(N+M-1)!}{N![(N+M-1)-N)]!} = \frac{(N+M-1)!}{N!(M-1)!}$. Either way it is the same!*

Both students in the previous conversation are correct and are drawing attention to the fact that one must focus on the number of bosons and the number of dividers (as opposed to the number of available single-particle states).

The QuILT also asks students to reflect upon and compare the number of distinct many-particle states for a system of indistinguishable fermions, indistinguishable bosons, and identical particles that could be treated as distinguishable since this was a common difficulty. In particular, they are asked to rank the number of distinct many-particle states for each system with the same number of particles and the same number of single-particle states. The goal is to have students understand that for the

TABLE I. Average pretest and posttest scores for Q1 and Q2 for the given system on the pretest and posttest for undergraduates (number of students $N = 25$) and graduate students ($N = 30$).

Question	Type of Particle	Graduate		Undergraduate	
		Pre (%)	Post (%)	Pre (%)	Post (%)
Q1	Fermions	48	-	56	-
	Bosons	28	-	27	-
	Distinguishable	28	-	39	-
Q2	Fermions	-	100	-	100
	Bosons	-	92	-	96
	Distinguishable	-	93	-	86

TABLE II. The percentages of undergraduate students who answered questions Q1(a) and Q1(b) correctly for the given system on the midterm examination four weeks after completing the Quantum Interactive Learning Tutorial (number of students $N = 12$).

Question	Type of Particle	Answered Correctly (%)
Q1(a)	Fermions	75
Q1(b)	Bosons	75

same number of particles and available single-particle states, a system of distinguishable particles has the largest number of distinct many-particle states and that the indistinguishability of the identical fermions and bosons results in fewer distinct states (unless the system of identical bosons has only one available single-particle state, in which case this system will have the same number of distinct many-particle states as a system of distinguishable particles). A system of identical fermions must satisfy the Pauli exclusion principle which reduces the number of possible many-particle states compared to identical bosons. The QuILT also strives to help students learn that the number of distinct many-particles states for a given number of particles and available single-particle states increase by particle type in the order: indistinguishable fermions, indistinguishable bosons, and identical particles that can be treated as distinguishable and be able to reason why that is the case.

V. IN-CLASS EVALUATION OF THE QUILT

Details of in class implementation are similar to Ref. [86, 87] and described in supplementary materials. Table I shows the performance of undergraduate and graduate students on the pretest and posttest. The results are encouraging and suggest that the QuILT is effective in helping students be able to count the number of distinct many-particle states for systems of identical fermions or bosons, as well as the contrasting case in which the identical particles could be treated as distinguishable. Q2 was given on the posttest and was intended to be a similar question to Q1 on the pretest. There are a different number of identical particles and available single-particle states in the two questions. Overall, the students did very well on the posttest with more than 80% of the graduate students and 75% of the undergraduates answering all three parts of Q2 correctly for the given system of identical particles.

As a measure of retention, the students in the undergraduate course in one of the years were given questions Q1(a) and Q1(b) on their midterm examination four weeks after completing the posttest. Table II summarizes the percentages of students who determined the number of distinct many-particle states in Q1(a) and Q1(b) correctly on the midterm examination. These findings are encouraging.

VI. SUMMARY

We investigated students' difficulties with a system of identical particles in a context in which there is a fixed number of available single-particle states. We used the research as a guide to develop a research-validated QuILT commensurate with the learning objectives. The QuILT provides scaffolding support to help students learn to reason and determine the number of distinct many-particle states for a system of identical particles in which the total number of particles and available single-particles states is fixed. We find that students who are still developing expertise have difficulty in integrating the physics with combinatorics. This involves how to count objects with different properties (e.g., in cases in which particles are distinguishable vs. indistinguishable and the overall wavefunction is symmetric or antisymmetric) and accounting for appropriate restrictions on the ways in which these objects can be arranged. The math-physics connection makes these issues even more challenging, e.g., students must learn the consequences of the indistinguishability of bosons and fermions as well as the consequences of symmetrization or anti-symmetrization requirement of the wave function in these cases on how the counting process would differ for these and how these two cases would be different from the corresponding case of distinguishable particles. In particular, the fact that students must understand the implication of all of the physics constraints in the fermion and boson situations on the appropriate way to do the combinatorics makes the task significantly challenging. Since it can be challenging for students to do appropriate physics and math connections to solve these complex problems, it is important that they are provided appropriate scaffolding support to learn. For example, scaffolding support is provided via reflections on hypothetical student conversations that focus on the common difficulties and provide students opportunities to compare and contrast cases. In the QuILT, problems focusing on each learning objective, start with small number of particles and available single-particle states so that students can check the correctness of their predictions using trial and error method and then figure out heuristics to generalize different cases with scaffolding support provided. The checkpoints are provided to help students go back and reconcile any differences between their predictions and hints provided. The findings indicate that the QuILT effectively enhances students' understanding of these concepts.

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