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RESEARCH ARTICLE

Rigorous derivation of a binary-ternary Boltzmann equation for a non ideal gas of hard spheres

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Abstract
This paper focuses on dynamics of systems of particles that allow interactions beyond binary, and their behavior as the number of particles goes to infinity. More precisely, the paper provides the first rigorous derivation of a binary-ternary Boltzmann equation describing the kinetic properties of a gas consisting of hard spheres, where particles undergo either binary or ternary instantaneous interactions, while preserving momentum and energy. An important challenge we overcome in deriving this equation is related to providing a mathematical framework that allows us to detect both binary and ternary interactions. Furthermore, this paper introduces new algebraic and geometric techniques in order to eventually decouple binary and ternary interactions and understand the way they could succeed one another in time. We expect that this paper can serve as a guideline for deriving a generalized Boltzmann equation that incorporates higher-order interactions among particles.

Contents

1. Introduction

The Boltzmann equation, introduced by L. Boltzmann [11] and J.C. Maxwell [27], describes the time evolution of the probability density of a rarefied, monoatomic gas in thermal non-equilibrium in $\mathbb{R}^d$, for $d \geq 2$. The Boltzmann equation accurately describes very dilute gases since only **binary** interactions

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between particles are taken into account. However, in certain situations, higher-order interactions are much more likely to happen; therefore, they produce a significant effect in the time evolution of the gas. A relevant example is a colloid, which is a homogeneous non-crystalline substance consisting of either large molecules or ultramicroscopic particles of one substance dispersed through a second substance. As pointed out in [29], multi-particle interactions, which are modeled by a sum of higher-order interaction terms, significantly contribute to the grand potential of the colloidal gas. A surprising result of [29], but of invaluable computational importance in numerical simulations, is that interactions among three particles are actually characterized by the sum of the distances between particles, as opposed to depending on different geometric configurations among interacting particles. The results of [29] have been further verified experimentally (e.g., [16]) and numerically (e.g., [23]).

1.1. Previous work and the goal of this paper

Motivated by the fact that the Boltzmann equation is valid only for very dilute gases and by the observations of [29] in [5], we suggested a kinetic model which goes beyond binary interactions incorporating sums of higher-order interaction terms. In particular, we introduced a generalized equation, which could serve as a toy model for incorporating higher-order interactions among particles and is of the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \displaystyle\sum_{k=2}^{m} Q_k(\underbrace{f, f, \cdots, f}_{k\text{-times}}), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0, x, v) = f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where, for $k = 1, ..., m$, the expression $Q_k(f, ..., f)$ is the k-th order collisional operator and $m \in \mathbb{N}$ is the highest order collisions allowed. Notice that for $m = 2$, equation (1.1) reduces to the classical Boltzmann equation. We note that equations similar to (1.1) were studied for Maxwell molecules in the works of Bobylev, Gamba and Cercignani [8, 7] using Fourier transform methods.

The task of rigorously deriving an equation of the form (1.1) from a classical many particle system, even for the case $m = 2$ (i.e., the Boltzmann equation), is a challenging problem that has been first settled for short times and hard sphere interactions in the pioneering work of Lanford [26], and for short range potentials by King [25]. This program was revisited by Gallagher, Saint-Raymond, Texier in [18], where important quantitative information on the convergence was provided. See also [12, 28, 30, 31, 19] and the references mentioned in these papers. More recent works related to derivation of the Boltzmann equation itself have been carried out using the notion of fluctuations in, for example, [9, 10, 20]. Regarding longer times, the equation was derived for hard spheres for long times originally only for initial data near vacuum in [24]. However, recently, a different derivation has been carried out by Deng, Hani and Ma [15] as long as the Boltzmann equation itself is well-posed.

A relevant step towards rigorously deriving (1.1) for $m = 3$ has been recently obtained in [5], where we considered a certain type of three-particle interactions that lead us to derive a purely ternary kinetic equation, which we called a ternary Boltzmann equation. However, the derivation of (1.1) for $m = 3$ has not been addressed yet, and that is exactly what we do in this paper. Furthermore, we expect that this paper can serve as a guideline for rigorously deriving generalized Boltzmann equation.

We start by describing challenges that we faced when introducing a framework that allows detection of binary and ternary interactions, while also accommodating a decoupling of such interactions so that it is clear which one is responsible for a creation of a binary or ternary collision terms in the nonlinear equation (1.1).

1.2. Challenges of detecting both binary and ternary interactions

The first challenge we face in deriving (1.1) for $m = 3$ is to provide a mathematical framework allowing us to detect both binary and ternary interactions among particles. We achieve that by assuming the following:

- Binary interactions are modeled as elastic collisions of hard spheres of diameter ϵ (i.e., two particles interact when the distance of their centers defined as

$$d_2(x_i, x_j) := |x_i - x_j|$$

becomes equal to the diameter ϵ). We call this an (i, j) interaction. As known, the relevant scaling to observe binary interactions is the Boltzmann-Grad scaling [21, 22]

$$N\epsilon^{d-1} \simeq 1, \tag{1.2}$$

as the number of particles $N \longrightarrow \infty$ and their diameter $\epsilon \longrightarrow 0^+$.
- Ternary interactions that we consider in this paper are going to be of an interaction zone type as in [5], by which we mean a particle i interacts with the pair of uncorrelated particles (j, k) when the non-symmetric ternary distance

$$d_3(x_i; x_j, x_k) := \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2}$$

becomes $\sqrt{2}\epsilon$. We call this an $(i; j, k)$ interaction. The particle i is called the central particle of the interaction, and the particles j, k are called adjacent particles.In terms of scaling, one could interpret an $(i; j, k)$ interaction of interaction zone ϵ as a special hard sphere interaction of radius $\sqrt{2}\epsilon$ in $\mathbb{R}^{2d}$ since the collisional condition $d_3(x_i; x_j, x_k) = \sqrt{2}\epsilon$ can be equivalently written as

$$|\boldsymbol{x_{i,i}} - \boldsymbol{x_{j,k}}|_{2d} = \sqrt{2}\epsilon,$$

where $\boldsymbol{x_{i,i}} = \begin{pmatrix} x_i \\ x_i \end{pmatrix}$ and $\boldsymbol{x_{j,k}} = \begin{pmatrix} x_j \\ x_k \end{pmatrix}$. Then a $2d$-particle with position $\boldsymbol{x_{i,i}}$ would span a volume of order ϵ^{2d-1} in a unit of time. Assuming there are N-particles in the system, in order to observe $O(1)$ interaction per unit of time, there are $N^2 - 1$ options for the $2d$-particle positioned at $\boldsymbol{x_{j,k}}$. We obtain that $N^2\epsilon^{2d-1} = O(1)$, or equivalently,

$$N\epsilon^{d-1/2} \simeq 1, \tag{1.3}$$

as the number of particles $N \longrightarrow \infty$ and the interaction zone $\epsilon \longrightarrow 0^+$, which is the scaling used in [5] to control ternary interactions.

Simultaneous consideration of both binary and ternary interactions brings the first crucial obstacle which is of conceptual nature; the apparent incompatibility of the Boltzmann-Grad scaling (1.2) dictated by binary interactions and the scaling (1.3) of ternary interactions, if both of them are of order ϵ. This incompatibility creates major difficulties even at the formal level. We overcome this scaling obstacle by assuming that, at the N-particle level, hard spheres of diameter ϵ_2 can participate in binary interactions as well as in ternary interactions via an interaction zone ϵ_3. Imposing scalings (1.2) with $\epsilon := \epsilon_2$ and (1.3) with $\epsilon := \epsilon_3$, we obtain the common scaling

$$N\epsilon_2^{d-1} \simeq N\epsilon_3^{d-1/2} \simeq 1, \tag{1.4}$$

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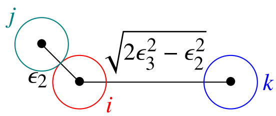

Figure 1. *Both binary and ternary interactions at the same time.*

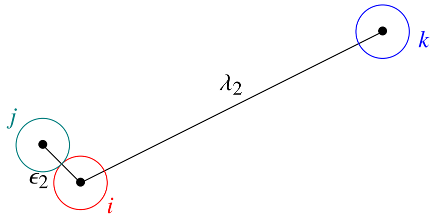

Figure 2. *Binary interaction:* $\epsilon_2^2 + \lambda_2^2 > 2\epsilon_3^2, \quad \lambda_2 > \epsilon_2.$

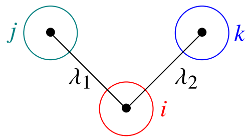

Figure 3. *Ternary interaction:* $\lambda_1^2 + \lambda_2^2 = 2\epsilon_3^2, \quad \lambda_1, \lambda_2 > \epsilon_2.$

as $N \longrightarrow \infty$ and $\epsilon_2, \epsilon_3 \longrightarrow 0^+$. Notice that the scaling (1.4) implies that for sufficiently large N, we have

$$\epsilon_2 << \epsilon_3, \tag{1.5}$$

which will have a prominent role in this paper.

The next challenge we address is the need to decouple binary and ternary interactions for a system of finitely many particles. More precisely, our framework a-priori allows that particles i and j interact as hard spheres:

$$d_2(x_i, x_j) = \epsilon_2,$$

while at the same time there is another particle k such that the particle i interacts with the particles j and k:

$$d_3(x_i; x_j, x_k) = \sqrt{2}\epsilon_3.$$

Such a configuration is illustrated in Figure 1. Pathological configurations, including the one we just described, are going to be shown to be negligible. This is far from trivial, and for more details on the microscopic dynamics, see Subsection 1.3 and Section 3. In particular, we shall show that as long as $0 < \epsilon_2 < \epsilon_3 < 1$, only the following two interaction scenarios are possible with nontrivial probability under time evolution:

1. Two particles interact as hard spheres, while all other particles are not involved in any binary or ternary interactions at the same time. This type of configurations generates the binary collisional operator. It is illustrated in Figure 2.
2. Three particles interact via an interaction zone, while none of them is involved in a binary interaction with either of the other two particles of the interaction zone at the same time. The rest of the particles are not involved in any binary or ternary interactions. This type of configurations is responsible for generating the ternary collisional operator. It is illustrated in Figure 3.

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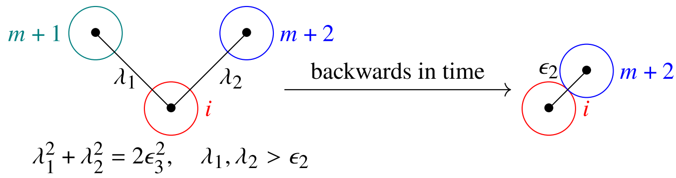

Figure 4.

Finally, since we will eventually let the number of particles $N \to \infty$, the main challenge we need to address is the stability of a good configuration[1] under the adjunction of one or two collisional particles. Assume, for a moment, that we have a good configuration of m-particles and we add σ particles to the system, where $\sigma \in \{1, 2\}$, such that a binary or ternary interaction is formed among one of the existing particles and the σ new particles. In general, under backwards time evolution, the system could run into another binary or ternary interaction; see, for example, Figure 4, which illustrates the mathematically most difficult case where the newly formed $(m+2)$-configuration runs into a binary interaction. To the best of our knowledge, this is the first time there was the need to address the possibility of a newly formed interacting configuration running into an interaction of a different type (binary to ternary or ternary to binary) backwards in time. However, in Section 8 and Section 9, we develop novel algebraic and geometric tools which help us eliminate pathological scenarios, including the one described in Figure 4, by showing that outside of a small measure set, negligible in the limit, the newly formed configuration does not run into any additional interactions backwards in time. For more details on the technical difficulties faced, see Subsection 1.6.

In the next subsection, we investigate more precisely what happens when a binary or a ternary interactions occurs and describe the time evolution of such a system.

1.3. Dynamics of finitely many particles

Let us describe the evolution in $\mathbb{R}^d$, $d \geq 2$, of a system of N hard spheres of diameter ϵ_2 and interaction zone ϵ_3, where $0 < \epsilon_2 < \epsilon_3 < 1$. The assumption $\epsilon_2 < \epsilon_3$ is necessary for ternary interactions to be of non trivial probability; see Remark 3.1 for more details.

1.3.1. Interactions considered

We first define the interactions considered in this paper.

Definition 1.1. Let $N \in \mathbb{N}$, with $N \geq 3$, and $0 < \epsilon_2 < \epsilon_3 < 1$. We define binary and ternary interactions, also referred to as collisions, as follows:

- Consider two particles $i, j \in \{1, \ldots, N\}$ with positions $x_i, x_j \in \mathbb{R}^d$. We say that the particles i, j are in an (i, j) binary interaction if the following geometric condition holds:

$$d_2(x_i, x_j) := |x_i - x_j| = \epsilon_2. \tag{1.6}$$

- Consider three particles $i, j, k \in \{1, \ldots, N\}$, with positions $x_i, x_j, x_k \in \mathbb{R}^d$. We say that the particles i, j, k are in an $(i; j, k)$ interaction[2] if the following geometric condition holds:

$$d_3(x_i; x_j, x_k) := \sqrt{|x_i - x_j|^2 + |x_i - x_k|^2} = \sqrt{2}\epsilon_3. \tag{1.7}$$

[1]By which we mean a configuration which does not run into any kind of interactions under backwards time evolution.
[2]We use the notation $(i; j, k)$ because the interaction condition is not symmetric. The particle i is the central particle of the interaction (i.e., the one interacting with the particles j and k, respectively).

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When an (i, j) interaction occurs, the velocities v_i, v_j of the i-th and j-th particles instantaneously transform according to the binary collisional law:

$$\begin{aligned} v_i' &= v_i + \langle \omega_1, v_j - v_i \rangle \omega_1, \\ v_j' &= v_j - \langle \omega_1, v_j - v_i \rangle \omega_1, \end{aligned} \tag{1.8}$$

where

$$\omega_1 := \frac{x_j - x_i}{\epsilon_2}. \tag{1.9}$$

Thanks to (1.6), we have $\omega_1 \in \mathbb{S}_1^{d-1}$. The vector ω_1 is called binary impact direction and it represents the scaled relative position of the colliding particles. Moreover, one can see that the binary momentum-energy system

$$\begin{aligned} v' + v_1' &= v + v_1, \\ |v'|^2 + |v_1'|^2 &= |v|^2 + |v_1|^2, \end{aligned} \tag{1.10}$$

is satisfied.

When an $(i; j, k)$ interaction happens, the velocities v_i, v_j, v_k of the i-th, j-th and k-th particles instantaneously transform according to the ternary collisional law derived in [5]

$$\begin{aligned} v_i^* &= v_i + \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle}(\omega_1 + \omega_2), \\ v_j^* &= v_j - \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle}\omega_1, \\ v_k^* &= v_k - \frac{\langle \omega_1, v_j - v_i \rangle + \langle \omega_2, v_k - v_i \rangle}{1 + \langle \omega_1, \omega_2 \rangle}\omega_2, \end{aligned} \tag{1.11}$$

where

$$(\omega_1, \omega_2) := \left(\frac{x_j - x_i}{\sqrt{2}\epsilon_3}, \frac{x_k - x_i}{\sqrt{2}\epsilon_3} \right). \tag{1.12}$$

Thanks to (1.7), we have $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$. The vectors (ω_1, ω_2) are called ternary impact directions, and they represent the scaled relative positions of the interacting particles. Moreover, it has been shown that the ternary momentum-energy system

$$\begin{aligned} v^* + v_1^* + v_2^* &= v + v_1 + v_2, \\ |v^*|^2 + |v_1^*|^2 + |v_2^*|^2 &= |v|^2 + |v_1|^2 + |v_2|^2, \end{aligned} \tag{1.13}$$

is satisfied. In particular, expression (1.11) provides the unique solution to (1.13) equipped with the extra condition

$$v_2^* = v_2 + c\omega_1, \quad v_3^* = v_3 + c\omega_2, \quad c \in \mathbb{R}.$$

We note that we had a choice in selecting the additional condition to uniquely solve (1.13). However, the one we chose in this work expresses the uncorrelation of the adjacent particles since their velocities are transformed uniformly with respect to the impact directions.

Remark 1.2. We note that both binary and ternary interactions are involutionary (i.e., reversible and measure-preserving). For more details, see Proposition 2.2 and Proposition 2.5 for binary and ternary interactions, respectively.

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1.3.2. Phase space and description of the flow

Let $N \in \mathbb{N}$, with $N \geq 3$, and $0 < \epsilon_2 < \epsilon_3 < 1$. The natural phase space[3] to capture both binary and ternary interactions is

$$\begin{aligned}\mathcal{D}_{N,\epsilon_2,\epsilon_3} = \big\{Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} : d_2(x_i, x_j) \geq \epsilon_2,\ \forall (i,j) \in \mathcal{I}_N^2,\\ \text{and } d_3(x_i; x_j, x_k) \geq \sqrt{2}\epsilon_3,\ \forall (i,j,k) \in \mathcal{I}_N^3\big\},\end{aligned} \tag{1.14}$$

where $X_N = (x_1, x_2, \dots, x_N)$, $V_N = (v_1, v_2, \dots, v_N)$, represent the positions and velocities of the N-particles, and the index sets $\mathcal{I}_N^2, \mathcal{I}_N^3$ are given by

$$\mathcal{I}_N^2 = \{(i,j) \in \{1,\dots,N\}^2 : i < j\}, \quad \mathcal{I}_N^3 = \{(i,j) \in \{1,\dots,N\}^3 : i < j < k\}.$$

Let us describe the evolution in time of such a system. Consider an initial configuration $Z_N \in \mathcal{D}_{N,\epsilon_2,\epsilon_3}$. The motion is described as follows:

1. Particles are assumed to perform rectilinear motion as long as there is no interaction

$$\dot{x}_i = v_i, \quad \dot{v}_i = 0, \quad \forall i \in \{1,\dots,N\}.$$

2. Assume now that an initial configuration $Z_N = (X_N, V_N)$ has evolved until time $t > 0$, reaching $Z_N(t) = (X_N(t), V_N(t))$, and that there is an interaction at time t. We have the following cases:
   - The interaction is binary: Assuming there is an (i,j) interaction, the velocities of the interacting particles instantaneously transform velocities according to the binary collisional law $(v_i(t), v_j(t)) \to (v_i'(t), v_j'(t))$ given in (1.8).
   - The interaction is ternary: Assuming there is an $(i;j,k)$ interaction, the velocities of the interacting particles instantaneously transform velocities according to the ternary collisional law

   $$(v_i(t), v_j(t), v_k(t)) \to (v_i^*(t), v_j^*(t), v_k^*(t)),$$

   given in (1.11).

Let us note that (I)–(II) are not sufficient to generate a global in time flow for the particle system since the velocity transformations are not smooth. In general, pathologies might arise as time evolves, meaning more than one type of interactions happening at the same time, grazing interaction, or infinitely many interactions in finite time. Although well-defined dynamics were shown to exist in [1] for hard spheres and in [5] for the purely ternary case, those results do not imply well-posedness of the flow for the mixed case, where both binary and ternary interactions are taken into account. The reason for that is that a binary interaction can be succeeded by a ternary interaction and vice versa, a situation which was not addressed in [1] or [5]. However, we are showing that a non-grazing interaction cannot be succeeded by the same interaction. In other words, when two particles (i,j) interact, the next interaction could be anything, binary or ternary, except a binary recollision of the particles (i,j). Similarly, when three particles run into an $(i;j,k)$ interaction, the next interaction can be anything except a ternary $(i;j,k)$[4] interaction. This observation allows us to define the flow locally a.e. and then run some combinatorial covering arguments to geometrically exclude a zero Lebesgue measure set such that the flow is globally in time defined on the complement.

Let us informally state this result. For a detailed statement, see Theorem 3.23.

Existence of a global flow: *Let $N \in \mathbb{N}$ and $0 < \epsilon_2 < \epsilon_3 < 1$. There is a global in time measure-preserving flow $(\Psi_m^t)_{t\in\mathbb{R}} : \mathcal{D}_{N,\epsilon_2,\epsilon_3} \to \mathcal{D}_{N,\epsilon_2,\epsilon_3}$ described a.e. by (I)-(II) which preserves kinetic energy and and is time reversible*. This flow is called the N-particle (ϵ_2, ϵ_3)-interaction flow.

[3]Upon symmetrization, one could define the phase space without ordering the particles and obtain a symmetrized version of ternary operator (see [2] for more details). For simplicity, we opt to work upon ordering the particles.

[4]Any other permutation of the particle i, j, k cannot form an interaction since $i < j < k$. In case one does not order the particles, a subsequent $(j; i, k)$ interaction, for instance, could possibly happen.

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The global measure-preserving interaction flow yields the Liouville equation[5] for the evolution f_N of an initial N-particle probability density $f_{N,0}$.

$$\begin{aligned} &\partial_t f_N + \sum_{i=1}^{N} v_i \nabla_{x_i} f_N = 0, \quad (t, Z_N) \in (0,\infty) \times \mathring{D}_{N,\epsilon_2,\epsilon_3}, \\ &f_N(t, Z_N') = f(t, Z_N), \quad t \in [0,\infty), \quad Z_N \text{ is a simple binary interaction}^6, \\ &f_N(t, Z_N^*) = f(t, Z_N), \quad t \in [0,\infty), \quad Z_N \text{ is a simple ternary interaction}^7, \\ &f_N(0, Z_N) = f_{N,0}(Z_N), \quad Z_N \in \mathring{D}_{N,\epsilon_2,\epsilon_3}. \end{aligned} \tag{1.15}$$

The Liouville equation provides a complete deterministic description of the system of N-particles. Although Liouville's equation is a linear transport equation, efficiently solving it is almost impossible in the case where the particle number N is very large. This is why an accurate kinetic description is welcome, and to obtain it, one wants to understand the limiting behavior of it as $N \to \infty$ and $\epsilon_2, \epsilon_3 \to 0^+$, with the hope that qualitative properties will be revealed for a large but finite N.

1.4. The binary-ternary Botzmann equation

To obtain such a kinetic description, we let the number of particles $N \to \infty$ and the diameter and interaction zone of the particles $\epsilon_2, \epsilon_3 \to 0^+$ in the **common scaling** (1.4):

$$N\epsilon_2^{d-1} \simeq N\epsilon_3^{d-\frac{1}{2}} \simeq 1,$$

which will lead the binary-ternary Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_2(f,f) + Q_3(f,f,f), & (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(t=0) = f_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{1.16}$$

The operator $Q_2(f,f)$ (see, for example, [13]) is the classical hard sphere binary collisional operator given by

$$Q_2(f,f) = \int_{\mathbb{S}_1^{d-1} \times \mathbb{R}^d} b_2^+ \big(f' f_1' - f f_1\big)\, d\omega_1\, dv_1, \tag{1.17}$$

where

$$\begin{aligned} &b_2 = \langle \omega_1, v_1 - v \rangle, \quad b_2^+ = \max\{b_2, 0\}, \\ &f' = f(t,x,v'), \quad f = f(x,t,v), \quad f_1' = f_1(t,x,v_1'), \quad f_1 = f(t,x,v_1). \end{aligned}$$

The operator $Q_3(f,f,f)$, introduced for the first time in [5], is the ternary hard interaction zone operator given by

$$Q_3(f,f,f) = \int_{\mathbb{S}_1^{2d-1} \times \mathbb{R}^{2d}} b_3^+ \big(f^* f_1^* f_2^* - f f_1 f_2\big)\, d\omega_1\, d\omega_2\, dv_1\, dv_2, \tag{1.18}$$

where

$$\begin{aligned} b_3(\omega_1, \omega_2, v_1 - v, v_2 - v) &:= \langle \omega_1, v_1 - v \rangle + \langle \omega_2, v_2 - v \rangle, \quad b_3^+ = \max\{b_3, 0\}, \\ f^* = f(t,x,v^*), \quad f = f(x,t,v), &\quad f_i^* = f_i^*(t,x,v_i^*), \quad f_i = f(t,x,v_i),\ i \in \{1,2\}. \end{aligned} \tag{1.19}$$

[5]In case $N = 2$, the ternary boundary condition is not present in (1.15), while if $N = 1$, equation (1.15) is just the transport equation.

[6]By simple binary interaction, we mean the only interaction happening is an (i,j) interaction. In this case, we write $Z_N' = (X_N, V_N')$, where $V_N' = (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_N)$.

[7]By simple ternary interaction, we mean the only interaction happening is an $(i;j,k)$ interaction. In this case, we write $Z_N^* = (X_N, V_N^*)$, where $V_N^* = (v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_{j-1}, v_j^*, v_{j+1}, \ldots, v_{k-1}, v_k^*, v_{k+1}, \ldots, v_N)$.

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We should mention that in [3], global well-posedness near vacuum has been shown for (1.16) for potentials ranging from moderately soft to hard in spaces of functions bounded by Maxwellian. In fact, in [3], it is seen that the ternary collisional operator allows consideration of softer potentials that the binary operator. In other words, the ternary correction to the Boltzmann equation does not behave worse than the classical Boltzmann equation.

It is important to point out that, upon symmetrization of the ternary collisional operator (see [2], [4]), the corresponding binary-ternary Boltzmann equation enjoys similar statistical and entropy production properties, as well as conservation laws, as the classical Boltzmann equation. Inspired by this fact, in collaboration with Gamba, Tasković [4], we studied the generation and propagation of polynomial and exponential moments, as well as the global well-posedness, of the space homogeneous binary-ternary Boltzmann equation. Interestingly, the results of [4] show that the co-existence of binary and ternary collisions yields better generation properties and time decay than when only binary or ternary collisions are considered. This suggests that such a model could potentially serve as a correction of the classical Boltzmann equation.

Recently, in collaboration with Warner [6], based on ideas introduced in the current paper, we were able to derive an equation of the type (1.1) for arbitrary order collisions based on a *symmetric* distance/collisional law among the particles. In other words, unlike the asymmetry present in the definition of the ternary distance (1.7), in [6] particles are fully interchangeable.

1.5. Strategy of the derivation and statement of the main result

In order to pass from the N-particle system dynamics to the kinetic equation (1.16), we implement the program of constructing linear finite and infinite hierarchies of equations, pioneered by Lanford [26] and refined by Gallagher, Saint-Raymond, Texier [18], and connecting them to the new binary-ternary Boltzmann equation. In [5], we extended this program to include ternary interactions, which led to the rigorous derivation of a purely ternary kinetic equation for particles with hard interaction zone in the scaling (1.3). However, rigorous derivation of (1.16) does not follow from [26, 18] or the ternary work [5]. As mentioned in Subsection 1.2, the first difficulty is the apparent incompatibility of scalings (1.2)-(1.3), which we overcome by introducing the common scaling (1.4). The most challenging task is to make the argument rigorous, though, is the analysis of all the possible recollisions[8] of the backwards (ϵ_2, ϵ_3)-flow. In contrast to the binary or the ternary case where each binary or ternary interaction is succeeded by a binary or ternary interaction, respectively, here we can have any possible interaction sequence of binary or ternary interactions. We keep track of this combinatorics using the set

$$S_k = \{\sigma = (\sigma_1, \dots, \sigma_k) : \sigma_i \in \{1, 2\}, \quad \forall i = 1, \dots, k\}. \tag{1.20}$$

In addition to more involved combinatorics, careful analysis of all the possible interaction sequences requires development of novel geometric and algebraic tools, which we discuss in details in Subsection 1.6. For now, we continue to discuss the process of derivation.

More specifically, we first derive a finite, linear, coupled hierarchy of equations for the marginal densities

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(Z_N) \mathbb{1}_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}(Z_N)\, dx_{s+1} \dots dx_N\, dv_{s+1} \dots dv_N, \quad s \in \{1, \dots, N-1\}$$

of the solution f_N to the Liouville equation, which we call the BBGKY.[9] This hierarchy is given by

$$\partial_t f_N^{(s)} + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}^N f_N^{(s+1)} + \mathcal{C}_{s,s+2}^N f_N^{(s+2)}, \quad s \in \{1, \dots, N-1\}. \tag{1.21}$$

[8]By recollisions we mean the possible divergence of the backwards (ϵ_2, ϵ_3)-interaction flow from the backwards free flow.
[9]Bogoliubov, Born, Green, Kirkwood, Yvon

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For the precise form of the operators $\mathcal{C}^N_{s,s+1}$, $\mathcal{C}^N_{s,s+2}$, see (4.15)–(4.16). Duhamel's Formula yields that the BBGKY hierarchy can be written in mild form as follows:

$$f_N^{(s)}(t,Z_s) = T_s^t f_{N,0}(Z_s) + \int_0^t T_s^{t-\tau}(\mathcal{C}^N_{s,s+1} f_N^{(s+1)} + \mathcal{C}^N_{s,s+2} f_N^{(s+2)})(\tau, Z_s)\, d\tau, \quad s \in \mathbb{N}, \tag{1.22}$$

where for any continuous function $g_s : \mathcal{D}_{s,\epsilon_2,\epsilon_3} \to \mathbb{R}$, we write $T_s^t g_s(Z_s) := g_s(\Psi_s^{-t} Z_s)$, and Ψ_s^t is the (ϵ_2, ϵ_3)-interaction zone flow of s-particles.

We then formally let $N \to \infty$ and $\epsilon_2, \epsilon_3 \to 0^+$ in the scaling (1.4) to obtain an infinite, linear, coupled hierarchy of equations, which we call the Boltzmann hierarchy. This hierarchy is given by

$$\partial_t f^{(s)} + \sum_{i=1}^s v_i \cdot \nabla_{x_i} f^{(s)} = \mathcal{C}^\infty_{s,s+1} f^{(s+1)} + \mathcal{C}^\infty_{s,s+2} f^{(s+2)}, \quad s \in \mathbb{N}. \tag{1.23}$$

For the precise form of the operators $\mathcal{C}^\infty_{s,s+1}$, $\mathcal{C}^\infty_{s,s+2}$, see (4.28), (4.32), respectively. Duhamel's Formula yields that the Boltzmann hierarchy can be written in mild form as follows:

$$f^{(s)}(t,Z_s) = S_s^t f_0(Z_s) + \int_0^t S_s^{t-\tau}(\mathcal{C}^\infty_{s,s+1} f^{(s+1)} + \mathcal{C}^\infty_{s,s+2} f^{(s+2)})(\tau, Z_s)\, d\tau, \quad s \in \mathbb{N}, \tag{1.24}$$

where for any continuous function $g_s : \mathbb{R}^{2ds} \to \mathbb{R}$, we write $S_s^t g_s(Z_s) := g_s(\Phi_s^{-t} Z_s)$, and Φ_s^t is the s-particle free flow of s-particles defined by $S_s^t Z_s = S_s^t(X_s, V_s) = (X_s - tV_s, V_s)$.

It can be observed that for factorized initial data and assuming that the solution remains factorized in time,[10] the Boltzmann hierarchy reduces to the binary-ternary Boltzmann equation (1.16). This observation connects the Boltzmann hierarchy with the binary-ternary Boltzmann equation (1.16).

To make this argument rigorous, we first show that the BBGKY and Boltzmann hierarchy are well-posed in the scaling (1.4), at least for short times, and then that the convergence of the BBGKY hierarchy initial data to the Boltzmann hierarchy initial data propagates in the time interval of existence of the solutions. Showing convergence is a very challenging task, and is the heart of our contribution. We describe details in Subsection 1.6.

Now, we informally state our main result. For a rigorous statement, see Theorem 6.8 and Corollary 6.10.

***Statement of the main result*:** *Let F_0 be initial data for the Boltzmann hierarchy (1.23), and $F_{N,0}$ be some BBGKY hierarchy (1.23) initial data which 'approximate'*[11] *F_0 as $N \to \infty$, $\epsilon \to 0^+$ under the scaling (1.4). Let $\boldsymbol{F_N}$ be the mild solution to the BBGKY hierarchy (1.21) with initial data $F_{N,0}$, and $\boldsymbol{F}$ the mild solution to the Boltzmann hierarchy (1.23), with initial data F_0, up to short time $T > 0$. Then $\boldsymbol{F_N}$ converges in observables*[12] *to $\boldsymbol{F}$ in $[0,T]$ as $N \to \infty$, $\epsilon \to 0^+$, under the scaling (1.4). In the case of Hölder continuous $C^{0,\gamma}$, $\gamma \in (0,1]$ tensorized Boltzmann hierarchy initial data and approximation by conditioned BBGKY hierarchy initial data, we obtain convergence to the solution of the binary-ternary Boltzmann equation (1.16) with a rate $O(\epsilon^r)$ for any $0 < r < \min\{1/2, \gamma\}$.*

1.6. Difficulties faced in the proof of the main result

The main idea to obtain convergence (Theorem 6.8) is to inductively use mild forms (1.22), (1.24) of the BBGKY hierarchy and Boltzmann hierarchy, respectively, to formally obtain series expansions with

[10]This is typically called propagation of chaos assumption.
[11]See Section 6 for details.
[12]For a precise definition of convergence in observables, see Subsection 6.2.

respect to the initial data:

$$f_N^{(s)}(t,Z_s) = T_s^t f_{N,0}^{(s)}(Z_s)+ \sum_{k=1}^{\infty}\sum_{\sigma\in S_k}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} T_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^N T_{s+\widetilde{\sigma}_1}^{t_1-t_2}\ldots\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^N T_{s+\widetilde{\sigma}_k}^{t_k} f_{N,0}^{(s+\widetilde{\sigma}_k)}(Z_s)\,dt_k\ldots dt_1, \tag{1.25}$$

$$f^{(s)}(t,Z_s) = S_s^t f_0^{(s)}(Z_s)+ \sum_{k=1}^{\infty}\sum_{\sigma\in S_k}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} S_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^\infty S_{s+\widetilde{\sigma}_1}^{t_1-t_2}\ldots\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^\infty S_{s+\widetilde{\sigma}_k}^{t_k} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\,dt_k\ldots dt_1, \tag{1.26}$$

where S_k is defined in (1.20), and given $\sigma\in S_k$, $\ell=1,\ldots,k$, we write $\widetilde{\sigma}_\ell := \sum_{i=1}^{\ell}\sigma_i$. We note that the summation over S_k in (1.25)-(1.26) allows us to keep track of the possible interaction sequences occurring by 'adding' one or two particles in each time step. For more details, see Section 7.

Comparing expressions (1.25)-(1.26), we expect to obtain the required convergence under the scaling (1.4) as long as $f_{N,0}^{(s)}$ 'approximates' $f_0^{(s)}$ under the same scaling. However it is not possible to directly compare (1.25)-(1.26) because of the possible divergence of the backwards interaction flow from the free flow, which we call recollisions. Although recollisions were also faced in [18] and [5], the mixed case, where both binary and ternary interactions are considered, requires different conceptual treatment in many instances and is not implied by the results of these works. The reason for that is that a binary interaction can be succeeded by a ternary interaction and vice versa, a situation which was not addressed in [18, 5]. The key to overcome these difficulties is that the diameter of the particles is much smaller than the interaction zone, as implied by the common scaling (1.4). This fact allows us to develop certain delicate algebraic and geometric arguments to extract a small measure set of pathological initial data which lead to recollisions. On the complement of this set, expansions (1.25)-(1.26) are comparable and the required convergence is obtained.

The main idea for eliminating recollisions is an inductive application in each time step of Proposition 9.2 and Proposition 9.4, which treat the binary adjunction, or Proposition 9.6 and Proposition 9.7, which treat the ternary adjunction. More precisely, we face the following different cases:

1. **Binary adjunction**: One particle is added forming a binary interaction with one of the existing particles. The pathological situations that might arise under backwards time evolution are the following:
   - The newly formed binary collisional configuration runs to a binary interaction under time evolution. This pathological situation is eliminated using arguments inspired by [18]. This is actually the only case which is similar to the cases covered in [18].
   - The newly formed binary collisional configuration runs to a ternary interaction under time evolution. This pathological situation did not appear in any of the previous works since merely binary or ternary interactions were studied. However, due to the fact that $\epsilon_2 << \epsilon_3$, which comes from the scaling (1.4), this pathological situation can be treated using techniques inspired by [5] and adapting them to the binary case.

   Proposition 9.2 and Proposition 9.4 are the relevant results controlling recollisions after a binary adjunction.
2. **Ternary adjunction**: Two particles are added forming a ternary interaction with one of the existing particles. The pathological situations that might arise under backwards time evolution are the following:
   - The newly formed ternary collisional configuration runs to a ternary interaction under time evolution. This case was studied in depth in [5]. We eliminate this pathological situation using Proposition 9.5. For its proof, we refer to [5].
   - The newly formed ternary collisional configuration runs to a binary interaction under time evolution. This is the most challenging case to treat and is the heart of the technical contribution because the scaling (1.4) does not directly help as in the case of the binary adjunction where

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one of the collisional particles enters an interaction zone. To treat this case, we need to use new algebraic tools (see Proposition 9.6) to exclude sets of initial data which lead to these pathological trajectories and develop elaborate geometric estimates to control its measure. The geometric estimates needed are thoroughly presented in Section 8. In particular, Subsection 8.2 is devoted to developing novel tools which rely on an appropriate representation of $(2d-1)$-spheres (see (8.1)). More specifically, in 8.2.1, we perform some initial truncations to the impact directions, while in 8.2.2, we establish certain spherical cap and conic region estimates needed to control the precollisional case, while 8.2.3 focuses on developing the necessary annuli estimates enabling us to control the postcollisional case using precollisional arguments. After establishing the necessary geometric tools, we employ them in Proposition 9.7 to show that the corresponding set constructed in Proposition 9.6 is negligible.

1.7. Notation

For convenience, we introduce some basic notation which will be frequently used throughout the manuscript:

- $d \in \mathbb{N}$ will be a fixed dimension with $d \geq 2$.
- Given $x, y \in \mathbb{R}$, we write $x \lesssim y$ if there is a constant $C_d > 0$ such that $x \leq C_d y$. Similarly, we write $x \simeq y$ if there is a constant $C_d > 0$ such that $x = C_d y$.
- Given $n \in \mathbb{N}$, $\rho > 0$ and $w \in \mathbb{R}^n$, we write $B^n_\rho(w)$ for the n-closed ball of radius $\rho > 0$, centered at $w \in \mathbb{R}^n$. In particular, we write $B^n_\rho := B^n_\rho(0)$ for the ρ-ball centered at the origin.
- Given $n \in \mathbb{N}$ and $\rho > 0$, we write $\mathbb{S}^{n-1}_\rho$ for the $(n-1)$-sphere of radius $\rho > 0$.
- When we write $x << y$, we mean that there is a small enough constant $0 < c < 1$ such that $x < cy$.

2. Collisional transformations

In this section, we define the collisional transformations of two and three interacting particles, respectively. In the two-particle case, particles will interact as regular hard spheres, while in the three-particle case, particles will interact as triplets of particles with an interaction zone.

2.1. Binary interaction

Here, we define the binary collisional tranformation of two interacting hard spheres, induced by an impact direction $\omega_1 \in \mathbb{S}^{d-1}_1$. This will be the law under which the velocities (v_1, v_2) of two interacting hard spheres, with impact direction $\omega_1 \in \mathbb{S}^{d-1}_1$, instantaneously transform. The impact direction will represent the scaled relative position of the colliding hard spheres.

Definition 2.1. Consider a binary impact direction $\omega_1 \in \mathbb{S}^{d-1}_1$. We define the binary collisional transformation induced by $\omega_1 \in \mathbb{S}^{d-1}_1$ as the map $T_{\omega_1} : (v_1, v_2) \in \mathbb{R}^{2d} \to (v_1', v_2') \in \mathbb{R}^{2d}$, where

$$\begin{aligned} v_1' &= v_1 + \langle \omega_1, v_2 - v_1 \rangle \omega_1, \\ v_2' &= v_2 - \langle \omega_1, v_2 - v_1 \rangle \omega_1. \end{aligned} \tag{2.1}$$

Let us introduce some notation we will be constantly using. We define the binary cross-section

$$b_2(\omega_1, v_1) := \langle \omega_1, v_1 \rangle, \quad (\omega_1, v_1) \in \mathbb{S}^{d-1}_1 \times \mathbb{R}^d. \tag{2.2}$$

One can verify that the binary momentum-energy conservation system

$$\begin{aligned} v_1' + v_2' &= v_1 + v_2, \\ |v_1'|^2 + |v_2|^2 &= |v_1|^2 + |v_2|^2 \end{aligned} \tag{2.3}$$

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is satisfied. Given a binary impact direction $\omega_1 \in \mathbb{S}_1^{d-1}$, the binary collisional transformation T_{ω_1} satisfies the following properties (see, for example, [13]).

Proposition 2.2. *Consider a binary impact direction $\omega_1 \in \mathbb{S}_1^{d-1}$. The induced binary collisional transformation T_{ω_1} has the following properties:*

1. *Conservation of momentum*

$$v_1' + v_2' = v_1 + v_2. \tag{2.4}$$

2. *Conservation of energy*

$$|v_1'|^2 + |v_2'|^2 = |v_1|^2 + |v_2|^2. \tag{2.5}$$

3. *Conservation of relative velocities magnitude*

$$|v_1' - v_2'| = |v_1 - v_2|. \tag{2.6}$$

4. *Micro-reversibility of the binary cross-section*

$$b_2(\omega_1, v_2' - v_1') = -b_2(\omega_1, v_2 - v_1). \tag{2.7}$$

5. *T_{ω_1} is a linear involutio (i.e., T_{ω_1} is linear and $T_{\omega_1}^{-1} = T_{\omega_1}$). In particular, $|\det T_{\omega_1}| = 1$, so T_{ω_1} is measure-preserving.*

2.2. Ternary interaction

Now we define the ternary collisional tranformation, induced by a given pair of impact directions, and investigate its properties. The interaction considered will be an instantaneous interaction of three particles with an interaction zone (for more details, see [5]). This will be the law under which the velocities (v_1, v_2, v_3) of three interacting particles, with impact directions $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$, instanteously transform. The impact directions will represent the scaled relative positions of the three particles in the interaction zone setting.

Definition 2.3. Consider a pair of impact directions $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$. We define the ternary collisional transformation induced by $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$ as the map $T_{\omega_1,\omega_2} : (v_1, v_2, v_3) \in \mathbb{R}^{3d} \longrightarrow (v_1^*, v_2^*, v_3^*) \in \mathbb{R}^{3d}$, where

$$\begin{cases} v_1^* = v_1 + \dfrac{\langle \omega_1, v_2 - v_1\rangle + \langle \omega_2, v_3 - v_1\rangle}{1 + \langle \omega_1, \omega_2\rangle}(\omega_1 + \omega_2), \\ v_2^* = v_2 - \dfrac{\langle \omega_1, v_2 - v_1\rangle + \langle \omega_2, v_3 - v_1\rangle}{1 + \langle \omega_1, \omega_2\rangle}\omega_1, \\ v_3^* = v_3 - \dfrac{\langle \omega_1, v_2 - v_1\rangle + \langle \omega_2, v_3 - v_1\rangle}{1 + \langle \omega_1, \omega_2\rangle}\omega_2. \end{cases} \tag{2.8}$$

We also define the ternary cross-section as

$$b_3(\omega_1, \omega_2, v_1, v_2) := \langle \omega_1, v_1\rangle + \langle \omega_2, v_2\rangle, \quad (\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}, \quad (v_1, v_2) \in \mathbb{R}^{2d}. \tag{2.9}$$

Remark 2.4. Cauchy-Schwartz inequality and the fact that $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$ yield

$$\frac{2}{3} \leq \frac{1}{1 + \langle \omega_1, \omega_2\rangle} \leq 2. \tag{2.10}$$

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One can verify that the ternary momentum-energy conservation system

$$\begin{aligned} v_1^* + v_2^* + v_3^* &= v_1 + v_2 + v_3, \\ |v_1^*|^2 + |v_2^*|^2 + |v_3^*|^2 &= |v_1|^2 + |v_2|^2 + |v_3|^2, \end{aligned} \tag{2.11}$$

is satisfied. The main properties of the ternary collisional tranformation are summarized in the following Proposition. For the proof, see Proposition 2.3. from [5].

Proposition 2.5. *Consider a pair of impact directions* $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$. *The induced collisional transformation* T_{ω_1,ω_2} *has the following properties:*

1. *Conservation of momentum*

$$v_1^* + v_2^* + v_3^* = v_1 + v_2 + v_3. \tag{2.12}$$

2. *Conservation of energy*

$$|v_1^*|^2 + |v_2^*|^2 + |v_3^*|^2 = |v_1|^2 + |v_2|^2 + |v_3|^2. \tag{2.13}$$

3. *Conservation of relative velocities magnitude*

$$|v_1^* - v_2^*|^2 + |v_1^* - v_3^*|^2 + |v_2^* - v_3^*|^2 = |v_1 - v_2|^2 + |v_1 - v_3|^2 + |v_2 - v_3|^2. \tag{2.14}$$

4. *Micro-reversibility of the ternary cross-section*

$$b_3(\omega_1, \omega_2, v_2^* - v_1^*, v_3^* - v_1^*) = -b_3(\omega_1, \omega_2, v_2 - v_1, v_3 - v_1). \tag{2.15}$$

5. T_{ω_1,ω_2} *is a linear involution i.e.* T_{ω_1,ω_2} *is linear and* $T_{\omega_1,\omega_2}^{-1} = T_{\omega_1,\omega_2}$. *In particular,* $|\det T_{\omega_1,\omega_2}| = 1$, *so* T_{ω_1,ω_2} *is measure-preserving.*

3. Dynamics of *m*-particles

In this section, we rigorously define the dynamics of m hard spheres of diameter σ_2 and interaction zone σ_3, where $0 < \sigma_2 < \sigma_3 < 1$. Heuristically speaking, particles perform rectilinear motion as long as there is no interaction (binary or ternary) and they interact through the binary or ternary collision law when a binary or ternary interaction occurs, respectively. However, it is far from obvious that a global dynamics can be defined since the system might run into pathological configurations (e.g., more than one type of interaction at a time, infinitely many interactions in finite time or interactions which graze under time evolution). Although this problem was present is [1, 5] as well, here we need to decouple binary and ternary interaction sequences since both types of interactions are allowed in each time step. The goal of this section is to extract a set of measure zero such that on the complement a global in time, measure-preserving flow can be defined.

Throughout this section, we consider $m \in \mathbb{N}$ and $0 < \sigma_2 < \sigma_3 < 1$.

3.1. Phase space definitions

For convenience, we define the following index sets:

$$\text{For } m \geq 2\text{: } \mathcal{I}_m^2 = \left\{(i,j) \in \{1,\dots,m\}^2 : i < j\right\}. \tag{3.1}$$

$$\text{For } m \geq 3\text{: } \mathcal{I}_m^3 = \left\{(i,j,k) \in \{1,\dots,m\}^3 : i < j < k\right\}. \tag{3.2}$$

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Given positions $(x_1, x_2) \in \mathbb{R}^{2d}$, we define the binary distance:

$$d_2(x_1, x_2) := |x_1 - x_2|, \tag{3.3}$$

and given positions $(x_1, x_2, x_3) \in \mathbb{R}^{3d}$, we define the ternary distance:

$$d_3(x_1; x_2, x_3) = \sqrt{|x_1 - x_2|^2 + |x_1 - x_3|^2}. \tag{3.4}$$

For $m \geq 3$, we define the phase space of m-particles of diameter $\sigma_2 > 0$ and interaction zone $\sigma_3 > 0$, with $\sigma_2 < \sigma_3 < 1$ as

$$\begin{aligned} \mathcal{D}_{m,\sigma_2,\sigma_3} = \big\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d_2(x_i, x_j) \geq \sigma_2,\ \forall (i,j) \in \mathcal{I}_m^2, \\ \text{and } d_3(x_i; x_j, x_k) \geq \sqrt{2}\sigma_3,\ \forall (i,j,k) \in \mathcal{I}_m^3 \big\}, \end{aligned} \tag{3.5}$$

where $X_m = (x_1, ..., x_m) \in \mathbb{R}^{dm}$ represents the positions of the m-particles, while $V_m = (v_1, ..., v_m) \in \mathbb{R}^{dm}$ represents the velocities of the m-particles. For convenience, we also define

$$\mathcal{D}_{2,\sigma_2,\sigma_3} = \big\{ Z_2 = (X_2, V_2) \in \mathbb{R}^{2d} : |x_1 - x_2| \geq \sigma_2 \big\}, \quad \mathcal{D}_{1,\sigma_2,\sigma_3} = \mathbb{R}^{2d}. \tag{3.6}$$

For $m \geq 3$, the phase space $\mathcal{D}_{m,\sigma_2,\sigma_3}$ decomposes as $\mathcal{D}_{m,\sigma_2,\sigma_3} = \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3} \cup \partial \mathcal{D}_{m,\sigma_2,\sigma_3}$, where the interior is given by

$$\begin{aligned} \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3} = \big\{ Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : d_2(x_i, x_j) > \sigma_2,\ \forall (i,j) \in \mathcal{I}_m^2, \\ \text{and } d_3(x_i; x_j, x_k) > \sqrt{2}\sigma_3,\ \forall (i,j,k) \in \mathcal{I}_m^3 \big\}, \end{aligned} \tag{3.7}$$

and the boundary is given by

$$\partial \mathcal{D}_{m,\sigma_2,\sigma_3} = \partial_2 \mathcal{D}_{m,\sigma_2,\sigma_3} \cup \partial_3 \mathcal{D}_{m,\sigma_2,\sigma_3}, \tag{3.8}$$

where $\partial_2 \mathcal{D}_{m,\sigma_2,\sigma_3}$ is the binary boundary

$$\partial_2 \mathcal{D}_{m,\sigma_2,\sigma_3} = \big\{ Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma_2,\sigma_3} : \exists (i,j) \in \mathcal{I}_m^2 \text{ with } d_2(x_i, x_j) = \sigma_2 \big\}, \tag{3.9}$$

and $\partial_3 \mathcal{D}_{m,\sigma_2,\sigma_3}$ is the ternary boundary

$$\partial_3 \mathcal{D}_{m,\sigma_2,\sigma_3} = \Big\{ Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma_2,\sigma_3} : \exists (i,j,k) \in \mathcal{I}_m^3 \text{ with } d_3(x_i; x_j, x_k) = \sqrt{2}\sigma_3 \Big\}. \tag{3.10}$$

Elements of $\mathcal{D}_{m,\sigma_2,\sigma_3}$ are called configurations, elements of $\mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}$ are called noncollisional configurations, and elements of $\partial_2 \mathcal{D}_{m,\sigma_2,\sigma_3}$ are called collisional configurations, or just collisions. Elements of $\partial \mathcal{D}_{m,\sigma_2,\sigma_3}$ are called binary collisions, while elements of $\partial_3 \mathcal{D}_{m,\sigma_2,\sigma_3}$ are called ternary collisions. When we refer to a collision, it will be either binary or ternary.

Clearly, the binary boundary can be written as $\partial_2 \mathcal{D}_{m,\sigma_2,\sigma_3} = \bigcup_{(i,j) \in \mathcal{I}_m^2} \Sigma_{ij}^2$, where Σ_{ij}^2 are the binary collisional surfaces given by

$$\Sigma_{ij}^2 := \big\{ Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3} : d_2(x_i, x_j) = \sigma_2 \big\}. \tag{3.11}$$

In the same spirit, the ternary boundary can be written as $\partial_3 \mathcal{D}_{m,\sigma_2,\sigma_3} = \bigcup_{(i,j,k) \in \mathcal{I}_m^3} \Sigma_{ijk}^3$, where Σ_{ijk}^3 are the ternary collisional surfaces given by

$$\Sigma_{ijk}^3 := \Big\{ Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3} : d_3(x_i; x_j, x_k) = \sqrt{2}\sigma_3 \Big\}. \tag{3.12}$$

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We now further decompose collisions to simple binary collisions, simple ternary collisions and multiple collisions. In particular, we define simple binary collisions as

$$\begin{aligned}\partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} :=\big\{&Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma_2,\sigma_3} : \exists (i,j) \in \mathcal{I}_m^2 \text{ with } Z_m \in \Sigma_{ij}^2,\\ &Z_m \notin \Sigma^2_{i'j'},\ \forall (i',j') \in \mathcal{I}_m^2 \setminus \{(i,j)\},\ Z_m \notin \Sigma^3_{i'j'k'},\ \forall (i',j',k') \in \mathcal{I}_m^3\big\}.\end{aligned} \tag{3.13}$$

We also define simple ternary collisions as

$$\begin{aligned}\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} :=\big\{&Z_m = (X_m, V_m) \in \mathcal{D}_{m,\sigma_2,\sigma_3} : \exists (i,j,k) \in \mathcal{I}_m^3 \text{ with } Z_m \in \Sigma_{ijk}^3,\\ &Z_m \notin \Sigma^3_{i'j'k'},\ \forall (i',j',k') \in \mathcal{I}_m^3 \setminus \{(i,j,k)\}, Z_m \notin \Sigma^2_{i'j'},\ \forall (i',j') \in \mathcal{I}_m^2\big\}.\end{aligned} \tag{3.14}$$

Remark 3.1. The assumption $\sigma_2 < \sigma_3$ made at the beginning of the section is necessary for $\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}$ to be nonempty. Indeed, let $\sigma_2 \geq \sigma_3$ and assume that $\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \neq \emptyset$. Consider $Z_m \in \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}$. Then, by (3.14), there is $(i,j,k) \in \mathcal{I}_m^3$ such that

$$|x_i - x_j|^2 + |x_i - x_j|^2 = 2\epsilon_3^2, \tag{3.15}$$

and

$$|x_i - x_j| > \epsilon_2, \quad |x_i - x_k| > \epsilon_2. \tag{3.16}$$

By (3.15), at least one of $|x_i - x_j|$ or $|x_i - x_k|$ has to be smaller than or equal to ϵ_3. Assume, without loss of generality, that $|x_i - x_j| \leq \epsilon_3$. Since $\epsilon_2 \geq \epsilon_3$, we obtain $|x_i - x_j| \leq \epsilon_2$, which contradicts (3.16). Therefore, if $\sigma_2 \geq \sigma_3$, we have $\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} = \emptyset$.

A simple collision will be a binary or ternary simple collision; that is,

$$\partial_{sc}\mathcal{D}_{m,\sigma_2,\sigma_3} := \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \cup \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}. \tag{3.17}$$

Multiple collisions are configurations which are not simple; that is,

$$\partial_{mu}\mathcal{D}_{m,\sigma_2,\sigma_3} := \partial\mathcal{D}_{m,\sigma_2,\sigma_3} \setminus \partial_{sc}\mathcal{D}_{m,\sigma_2,\sigma_3}. \tag{3.18}$$

Remark 3.2. For $m = 2$, there is only binary boundary.

For the binary case, we give the following definitions:

Definition 3.3. Let $m \geq 2$ and $Z_m \in \partial_{2,sc}D_{m,\sigma_2,\sigma_3}$. Then there is a unique $(i,j) \in \mathcal{I}_m^2$ such that $Z_m \in \Sigma_{ij}^2$ and $Z_m \notin \Sigma^3_{i'j'k'}$, for all $(i',j',k') \in \mathcal{I}_m^3$. In this case, we will say Z_m is an (i,j) collision, and we will write

$$\Sigma_{ij}^{2,sc} = \big\{Z_m \in \mathcal{D}_{m,\sigma_1,\sigma_2} : Z_m \text{ is } (i,j) \text{ collision}\big\}. \tag{3.19}$$

Clearly, $\Sigma_{ij}^{2,sc} \cap \Sigma_{i'j'}^{2,sc} = \emptyset$, for all $(i,j) \neq (i',j') \in \mathcal{I}_m^2$, and $\partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}$ decomposes to

$$\partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} = \bigcup_{(i,j)\in\mathcal{I}_m^2} \Sigma_{ij}^{2,sc}. \tag{3.20}$$

Remark 3.4. Let $m \geq 2$, $(i,j) \in \mathcal{I}_m^2$ and $Z_m \in \Sigma_{ij}^{2,sc}$. Then

$$\omega_1 := \frac{x_j - x_i}{\sigma_2} \in \mathbb{S}_1^{d-1}. \tag{3.21}$$

Therefore, each (i,j) collision naturally induces a binary impact direction $\omega_1 \in \mathbb{S}_1^{d-1}$ and consequently a binary collisional transformation T_{ω_1}.

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Definition 3.5. Let $m \geq 2$, $(i, j) \in \mathcal{I}_m^2$ and $Z_m = (X_m, V_m) \in \Sigma_{ij}^{2,sc}$. We write $Z_m' = (X_m, V_m')$, where

$$V_m' = (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_m),$$

and $(v_i', v_j') = T_{\omega_1}(v_i, v_j), \quad \omega_1 \in \mathbb{S}_1^{d-1}$ is given by (3.21).

In the same spirit, for the ternary case, we give the following definitions:

Definition 3.6. Let $m \geq 3$ and $Z_m \in \partial_{3,sc} D_{m,\sigma_2,\sigma_3}$. Then there is a unique $(i; j, k) \in \mathcal{I}_m^3$ such that $Z_m \in \Sigma_{ijk}^3$ and $Z_m \notin \Sigma_{i'j'}^2$, for all $(i', j') \in \mathcal{I}_m^2$. In this case, we will say Z_m is an $(i; j, k)$ collision, and we will write

$$\Sigma_{ijk}^{3,sc} = \left\{Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3} : Z_m \text{ is } (i; j, k) \text{ collision}\right\}. \tag{3.22}$$

Clearly, $\Sigma_{ijk}^{3,sc} \cap \Sigma_{i'j'k'}^{3,sc} = \emptyset$, for all $(i, j, k) \neq (i', j', k') \in \mathcal{I}_m^3$ and $\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}$ decomposes to

$$\partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} = \bigcup_{(i,j,k)\in\mathcal{I}_m^3} \Sigma_{ijk}^{3,sc}. \tag{3.23}$$

Remark 3.7. Let $m \geq 3$, $(i, j, k) \in \mathcal{I}_m^3$ and $Z_m \in \Sigma_{ijk}^{3,sc}$. Then

$$(\omega_1, \omega_2) := \frac{1}{\sqrt{2}\sigma_3}\left(x_j - x_i, x_k - x_i\right) \in \mathbb{S}_1^{2d-1}. \tag{3.24}$$

Therefore, each $(i; j, k)$ collision naturally induces ternary impact directions $(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$ and consequently a collisional transformation T_{ω_1,ω_2}.

Definition 3.8. Let $m \geq 3$, $(i, j, k) \in \mathcal{I}_m^3$ and $Z_m = (X_m, V_m) \in \Sigma_{ijk}^{3,s}$. We write $Z_m^* = (X_m, V_m^*)$, where

$$V_m^* = (v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_{j-1}, v_j^*, v_{j+1}, \ldots, v_{k-1}, v_k^*, v_{k+1}, \ldots, v_m),$$

and $(v_i^*, v_j^*, v_k^*) = T_{\omega_1,\omega_2}(v_i, v_j, v_k), \quad (\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}$ are given by (3.24).

3.2. *Classification of simple collisions*

We will now classify simple collisions in order to eliminate collisions which graze in time. For this purpose, we come across the following definitions for the binary and the ternary case, respectively.

For the binary case:

Definition 3.9. Let $m \geq 2$, $(i, j) \in \mathcal{I}_m^2$ and $Z_m \in \Sigma_{ij}^{2,s}$. The configuration Z_m is called

- binary precollisional when $b_2(\omega_1, v_j - v_i) < 0$,
- binary postcollisional when $b_2(\omega_1, v_j - v_i) > 0$,
- binary grazing when $b_2(\omega_1, v_j - v_i) = 0$,

where $\omega_1 \in \mathbb{S}_1^{d-1}$ is given by (3.21) and b_2 is given by (2.2).

Remark 3.10. Let $m \geq 2$, $(i, j) \in \mathcal{I}_m^2$ and $Z_m \in \Sigma_{ij}^{2,s}$. Using (2.7), we obtain the following:

1. Z_m is binary precollisional iff Z_m' is binary postcollisional.
2. Z_m is binary postcollisional iff Z_m' is binary precollisional.
3. $Z_m = Z_m'$ iff Z_m is binary grazing.

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For the ternary case:

Definition 3.11. Let $m \geq 3$, $(i,j,k) \in \mathcal{I}_m^3$ and $Z_m \in \Sigma_{ijk}^{3,s}$. The configuration Z_m is called

- ternary precollisional when $b_3(\omega_1,\omega_2,v_j-v_i,v_k-v_i) < 0$,
- ternary postcollisional when $b_3(\omega_1,\omega_2,v_j-v_i,v_k-v_i) > 0$,
- ternary grazing when $b_3(\omega_1,\omega_2,v_j-v_i,v_k-v_i) = 0$,

where $(\omega_1,\omega_2) \in \mathbb{S}_1^{2d-1}$ is given by (3.24) and b is given by (2.9).

Remark 3.12. Let $m \geq 3$, $(i,j,k) \in \mathcal{I}_m^3$ and $Z_m \in \Sigma_{ijk}^{3,s}$. Using (2.15), we obtain the following:

1. Z_m is ternary precollisional iff Z_m^* is ternary postcollisional.
2. Z_m is ternary postcollisional iff Z_m^* is ternary precollisional.
3. $Z_m = Z_m^*$ iff Z_m is ternary grazing.

We will just say precollisional, postcollisional or grazing configuration when it is implied whether a simple collision is binary or ternary.

For $m \geq 2$, we refine the phase space defining

$$\mathcal{D}_{m,\sigma_2,\sigma_3}^* := \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3} \cup \partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3}, \tag{3.25}$$

where $\partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3}$ denotes the part of $\partial\mathcal{D}_{m,\sigma_2,\sigma_3}$ consisting of simple, non-grazing collisions – that is, defined as

$$\partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3} := \left\{Z_m \in \partial_{sc}\mathcal{D}_{m,\sigma_2,\sigma_3} : Z_m \text{ is non-grazing}\right\}. \tag{3.26}$$

It is immediate that $\mathcal{D}_{m,\sigma_2,\sigma_3}^*$ is a full measure subset of $\mathcal{D}_{m,\sigma_2,\sigma_3}$ and $\partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3}$ is a full surface measure subset of $\partial\mathcal{D}_{m,\sigma_2,\sigma_3}$, since its complement constitutes of lower dimension submanifolds of $\partial\mathcal{D}_{m,\sigma_2,\sigma_3}$ which have zero surface measure.

3.3. *Construction of the local flow*

The next Lemma shows that the flow can be locally defined for any initial configuration $Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3}^*$ up to the time of the first collision.

Lemma 3.13. *Let $m \geq 3$ and $Z_m \in \mathcal{D}_{m,\sigma_2,\sigma_3}^*$. Then there is a time $\tau_{Z_m}^1 \in (0,\infty]$ such that defining $Z_m(\cdot) : [0,\tau_{Z_m}^1] \to \mathbb{R}^{2dm}$ by*

$$Z_m(t) = \begin{cases} (X_m + tV_m, V_m) & \text{if } Z_m \text{ is noncollisional or postcollisional,} \\ (X_m + tV_m', V_m'), & \text{if } Z_m \text{ is binary precollisional,} \\ (X_m + tV_m^*, V_m^*), & \text{if } Z_m \text{ is ternary precollisional,} \end{cases}$$

the following hold:

1. $Z_m(t) \in \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}, \quad \forall t \in (0,\tau_{Z_m}^1)$.
2. *if* $\tau_{Z_m}^1 < \infty$, *then* $Z_m(\tau_{Z_m}^1) \in \partial\mathcal{D}_{m,\sigma_2,\sigma_3}$.
3. *If* $Z_m \in \Sigma_{ij}^{2,sc}$ *for some* $(i,j) \in \mathcal{I}_m^2$, *then* $Z_m(\tau_{Z_m}^1) \notin \Sigma_{ij}^2$.
4. *If* $Z_m \in \Sigma_{ijk}^{3,sc}$ *for some* $(i,j,k) \in \mathcal{I}_m^3$, *then* $Z_m(\tau_{Z_m}^1) \notin \Sigma_{ijk}^3$.

An analogous statement holds in the case $m = 2$, where we just neglect the ternary terms.

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Proof. Let us make the convention $\inf\emptyset=+\infty$. We define

$$\tau^1_{Z_m}=\begin{cases}\inf\big\{t>0: X_m+tV_m\in\partial\mathcal{D}_{m,\sigma_2,\sigma_3}\big\}, & \text{if } Z_m \text{ is noncollisional or postcollisional,}\\ \inf\big\{t>0: X_m+tV_m'\in\partial\mathcal{D}_{m,\sigma_2,\sigma_3}\big\}, & \text{if } Z_m \text{ is binary precollisional,}\\ \inf\big\{t>0: X_m+tV_m^*\in\partial\mathcal{D}_{m,\sigma_2,\sigma_3}\big\}, & \text{if } Z_m \text{ is ternary precollisional.}\end{cases}$$

Since $\mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}$ is open, we get $\tau^1_{Z_m}>0,\quad\forall Z_m\in\mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}$, and claims *(i)–(ii)* follow immediately for $Z_m\in\mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}$.

Assume $Z_m\in\partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3}$ which yields that Z_m is non-grazing. Therefore, we may distinguish the following cases:

- Z_m is an (i,j) binary postcollisional configuration: For any $t>0$, we have

$$\begin{aligned}|x_i-x_j+(v_i-v_j)t|^2&=|x_i-x_j|^2+t^2|v_i-v_j|^2+2t\langle x_i-x_j,v_i-v_j\rangle\\&\geq\sigma_2^2+2tb_2(x_j-x_i,v_j-v_i)\\&>\sigma_2^2,\end{aligned}$$

  since $b_2(\omega_1,v_j-v_i)>0$. This inequality and the fact that Z_m is a simple binary collision imply that $\tau^1_{Z_m}>0$ and claims (i), (ii), (iii) as well.
- Z_m is (i,j) binary precollisional configuration: We use the same argument for Z_m' which is (i,j) binary postcollisional.
- Z_m is an $(i;j,k)$ ternary postcollisional configuration: For any $t>0$, we have

$$\begin{aligned}&|x_i-x_j+(v_i-v_j)t|^2+|x_i-x_k+(v_i-v_k)t|^2\\&=|x_i-x_j|^2+|x_i-x_k|^2+t^2\Big(|v_i-v_j|^2+|v_i-v_k|^2\Big)+2t\big(\langle x_i-x_j,v_i-v_j\rangle+\langle x_i-x_k,v_i-v_k\rangle\big)\\&\geq 2\sigma_3^2+2tb_3(x_j-x_i,x_k-x_i,v_j-v_i,v_k-v_i)\\&>2\sigma_3^2,\end{aligned}$$

  since $b_3(\omega_1,\omega_2,v_j-v_i,v_k-v_i)>0$. This inequality and the fact that Z_m is a simple ternary collision imply that $\tau^1_{Z_m}>0$ and claims (i), (ii), (iv) as well.
- Z_m is an $(i;j,k)$ ternary precollisional configuration: We use the same argument for Z_m^* which is $(i;j,k)$ ternary postcollisional.

□

Let us make an elementary but crucial remark.

Remark 3.14. Clearly, for configurations with $\tau^1_{Z_m}=\infty$, the flow is globally defined as the free flow. In the case where $\tau^1_{Z_m}<\infty$ and $Z_m(\tau^1_{Z_m})$ is a non-grazing (i,j) collision or non-grazing $(i;j,k)$ collision, we may apply Lemma 3.13 once more and get a corresponding time $\tau^2_{Z_m}$ with the property that $Z_m(\tau^2_{Z_m})\notin\Sigma^2_{ij}$ or $Z_m(\tau^2_{Z_m})\notin\Sigma^3_{ijk}$, respectively, if $\tau^2_{Z_m}<\infty$. Therefore, in this case, the flow can be defined up to time $\tau^2_{Z_m}$.

Remark 3.15. Note that Lemma 3.13 implies that given a non-grazing (i,j) collision, the next collision (if it happens) will not be (i,j). Similarly, given a non-grazing $(i;j,k)$ collision, the next collision (if it happens) will not be $(i;j,k)$. However, Lemma 3.13 does not imply that the same particles are not involved in a collision of a different type. For instance, one could have the sequence of collisions (i,j) and $(i;j,k)$, or $(i;j,k)$ and (i,j), etc. All these cases will be taken into account when establishing a global flow in Subsection 3.4.

Remark 3.16. Similar results hold for the case $m=2$ where there are no ternary interactions.

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3.4. Extension to a global flow

Now, we extract a zero measure set from $\mathcal{D}^*_{m,\sigma_2,\sigma_3}$ such that the flow is globally defined on the complement. For this purpose, we will first truncate positions and velocities using two parameters $1 << R < \rho$ and then perform time truncation with a small parameter δ in the scaling:

$$0 < \delta R << \sigma_2 < \sigma_3 < 1 << R < \rho. \tag{3.27}$$

Throughout this subsection, we consider parameters satisfying the scaling (3.27).

Recall that given $r > 0$, we denote the dm-ball of radius $r > 0$, centered at the origin as B_r^{dm}. We first assume initial positions are in B_ρ^{dm} and initial velocities in B_R^{dm}.

For $m \geq 2$, we decompose $D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm})$ in the following subsets:

$$\begin{aligned}
I_{free} &= \left\{Z_m = (X_m, V_m) \in D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} > \delta\right\},\\
I^1_{sc,ng} &= \left\{Z_m = (X_m, V_m) \in D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} \leq \delta,\ Z_m(\tau^1_{Z_m}) \in \partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3},\ \tau^2_{Z_m} > \delta\right\},\\
I^1_{sc,g} &= \Big\{Z_m = (X_m, V_m) \in D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} \leq \delta,\ Z_m(\tau^1_{Z_m}) \in \partial_{sc}\mathcal{D}_{m,\sigma_2,\sigma_3},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } Z_m(\tau^1_{Z_m}) \text{ is grazing}\Big\},\\
I^1_{mu} &= \left\{Z_m = (X_m, V_m) \in D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} \leq \delta,\ Z_m(\tau^1_{Z_m}) \in \partial_{mu}\mathcal{D}_{m,\sigma_2,\sigma_3}\right\},\\
I^2_{sc,ng} &= \left\{Z_m = (X_m, V_m) \in D^*_{m,\sigma_2,\sigma_3} \cap (B_\rho^{dm} \times B_R^{dm}) : \tau^1_{Z_m} \leq \delta,\ Z_m(\tau^1_{Z_m}) \in \partial_{sc,ng}\mathcal{D}_{m,\sigma_2,\sigma_3},\ \tau^2_{Z_m} \leq \delta\right\}.
\end{aligned}$$

We remark that there is a well-defined flow up to time δ for $Z_m \in I_{free} \cup I^1_{sc,ng}$, since in such cases, one has at most one simple non-grazing collision in $[0, \delta]$. We aim to estimate the measure of the pathological set $I^1_{sc,g} \cup I^1_{mu} \cup I^2_{sc,ng}$, with respect to the truncation parameters.

Before proceeding to the next result, let us note that conservation of energy (2.5), (2.13) imply the following elementary but useful remark:

Remark 3.17. The following hold:

- For $m \geq 2$: $Z_m \in \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \cap (\mathbb{R}^{dm} \times B_R^{dm}) \Leftrightarrow Z'_m \in \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \cap (\mathbb{R}^{dm} \times B_R^{dm})$.
- For $m \geq 3$: $Z_m \in \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \cap (\mathbb{R}^{dm} \times B_R^{dm}) \Leftrightarrow Z^*_m \in \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \cap (\mathbb{R}^{dm} \times B_R^{dm})$.

Lemma 3.18. *For $m \geq 3$, the following inclusion holds:*

$$I^1_{mu} \cup I^2_{sc,ng} \subseteq U_{22} \cup U_{23} \cup U_{32} \cup U_{33}, \tag{3.28}$$

where

$$U_{22} := \bigcup_{(i,j)\neq(i',j')\in\mathcal{I}^2_m} (U^2_{ij} \cap U^2_{i'j'}), \tag{3.29}$$

$$U_{23} := \bigcup_{(i,j)\in\mathcal{I}^2_m,(i',j',k')\in\mathcal{I}^3_m} (U^2_{ij} \cap U^3_{i'j'k'}), \tag{3.30}$$

$$U_{32} := \bigcup_{(i,j,k)\in\mathcal{I}^3_m,(i',j')\in\mathcal{I}^2_m} (U^3_{ijk} \cap U^2_{i'j'}), \tag{3.31}$$

$$U_{33} := \bigcup_{(i,j,k)\neq(i',j',k')\in\mathcal{I}^3_m} (U^3_{ijk} \cap U^3_{i'j'k'}), \tag{3.32}$$

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and given $(i,j)\in\mathcal{I}_m^2$, $(i,j,k)\in\mathcal{I}_m^3$, *we denote*

$$U_{ij}^2 := \{Z_m=(X_m,V_m)\in B_\rho^{dm}\times B_R^{dm} : \sigma_2\le d_2(x_i,x_j)\le \sigma_2+2\delta R\}. \tag{3.33}$$

$$U_{ijk}^3 := \Big\{Z_m=(X_m,V_m)\in B_\rho^{dm}\times B_R^{dm} : 2\sigma_3^2\le d_3^2(x_i;x_j,x_k)\le (\sqrt{2}\sigma_3+4\delta R)^2\Big\}. \tag{3.34}$$

For $m=2$, *we have* $I_1^{mu}=I_{sc,ng}^2=\emptyset$.

Proof. For $m=2$, we have that $\partial_{mu}\mathcal{D}_{2,\sigma_2,\sigma_3}=\emptyset$, and hence, $I_{mu}^1=\emptyset$. Also, since $m=2$, we trivially obtain $\mathcal{I}_2=\{(1,2)\}$, and hence, Remark 3.14 implies that $\tau_{Z_m}^2=\infty$ (i.e., $I_{sc,ng}^2=\emptyset$).

Assume now that $m\ge 3$. We prove the inclusion only for $I_{sc,ng}^2$; the inclusion for I_{mu}^1 is similar but simpler. We first assume that either $Z_m\in\mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}$ or Z_m is postcollisional. Therefore, up to time $\tau_{Z_m}^1$, we have free flow (i.e., $Z_m(t)=(X_m+tV_m,V_m)$, for all $t\in[0,\tau_{Z_m}^1]$). Remark 3.14 guarantees that

$$\begin{cases} Z_m(\tau_{Z_m}^1)\in\Sigma_{ij}^2 \Rightarrow Z_m(\tau_{Z_m}^2)\notin\Sigma_{ij}^2,\\ Z_m(\tau_{Z_m}^1)\in\Sigma_{ijk}^3 \Rightarrow Z_m(\tau_{Z_m}^2)\notin\Sigma_{ijk}^3.\end{cases} \tag{3.35}$$

We claim the following:

1. $Z_m(\tau_{Z_m}^1)\in\Sigma_{ij}^2,\ Z_m(\tau_{Z_m}^2)\in\Sigma_{i'j'}^2 \Rightarrow Z_m\in U_{ij}^2\cap U_{i'j'}^2,\quad \forall (i,j),(i',j')\in\mathcal{I}_m^2.$
2. $Z_m(\tau_{Z_m}^1)\in\Sigma_{ij}^2,\ Z_m(\tau_{Z_m}^2)\in\Sigma_{i'j'k'}^3 \Rightarrow Z_m\in U_{ij}^2\cap U_{i'j'k'}^3,\quad \forall (i,j)\in\mathcal{I}_m^3,\quad \forall (i',j,k')\in\mathcal{I}_m^3.$
3. $Z_m(\tau_{Z_m}^1)\in\Sigma_{ijk}^3,Z_m(\tau_{Z_m}^2)\in\Sigma_{i'j'}^2 \Rightarrow Z_m\in U_{ijk}^3\cap U_{i'j'}^2,\quad \forall (i,j,k)\in\mathcal{I}_m^3,\quad \forall (i',j')\in\mathcal{I}_m^2.$
4. $Z_m(\tau_{Z_m}^1)\in\Sigma_{ijk}^3,Z_m(\tau_{Z_m}^2)\in\Sigma_{i'j'k'}^3 \Rightarrow Z_m\in U_{ij}^2\cap U_{i'j'k'}^3,\quad \forall (i,j,k),(i',j',k')\in\mathcal{I}_m^3.$

By (3.35), proving claims (I)–(IV) imply inclusion (3.28) for $I_{sc,ng}^2$.

Without loss of generality, we prove claim (III). We have $Z_m(\tau_{Z_m}^1)\in\Sigma_{ijk}^3\cap\Sigma_{i'j'}^2$; therefore,

$$d_3^2\Big(x_i\Big(\tau_{Z_m}^1\Big);x_j\Big(\tau_{Z_m}^1\Big),x_k\Big(\tau_{Z_m}^1\Big)\Big)=2\sigma_3^2,\quad d_2\Big(x_{i'}\Big(\tau_{Z_m}^1\Big),x_{j'}\Big(\tau_{Z_m}^1\Big)\Big)=\sigma_2. \tag{3.36}$$

Since there is free motion up to $\tau_{Z_m}^1$, triangle inequality implies

$$|x_i-x_j|\le |x_i(\tau_{Z_m}^1)-x_j(\tau_{Z_m}^1)|+\delta|v_i-v_j|\le |x_i(\tau_{Z_m}^1)-x_j(\tau_{Z_m}^1)|+2\delta R. \tag{3.37}$$

Since there is an $(i;j,k)$ ternary collision at $\tau_{Z_m}^1$, we have

$$|x_i(\tau_{Z_m}^1)-x_j(\tau_{Z_m}^1)|^2+|x_i(\tau_{Z_m}^1)-x_k(\tau_{Z_m}^1)|^2=2\sigma_3^2\Rightarrow |x_i(\tau_{Z_m}^1)-x_j(\tau_{Z_m}^1)|\le\sqrt{2}\sigma_3. \tag{3.38}$$

Combining (3.37)–(3.38), we obtain

$$|x_i-x_j|^2\le |x_i(\tau_{Z_m}^1)-x_j(\tau_{Z_m}^1)|^2+4\sqrt{2}\sigma_3\delta R+4\delta^2R^2. \tag{3.39}$$

Using the same argument for the pair (i,k), adding and recalling the fact that there is $(i;j,k)$ collision at $\tau_{Z_m}^1$, we obtain

$$2\sigma_3^2\le d_3^2(x_i;x_j,x_k)\le 2\sigma_3^2+8\sqrt{2}\sigma_3R\delta+8\delta R^2\le 2\sigma_3^2+8\sqrt{2}\sigma_3R\delta+16\delta R^2=(\sqrt{2}\sigma_3+4\delta R)^2,$$

where the lower inequality holds trivially since $Z_m\in\mathcal{D}_{m,\sigma_2,\sigma_3}$. Hence, $Z_m\in U_{ijk}^3$.

We wish to prove as well $Z_m\in U_{i'j'}^2$; that is,

$$\sigma_2\le d_2(x_{i'},x_{j'})\le\sigma_2+2\delta R.$$

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The first inequality trivially holds because of the phase space. To prove the second inequality, we distinguish the following cases:

1. $i', j' \notin \{i, j, k\}$: Since particles (i', j') perform free motion up to $\tau^2_{Z_m}$, triangle inequality and the facts that $Z_m(\tau^2_{Z_m}) \in \Sigma^2_{i',j'}$, $\tau^2_{Z_m} \leq \delta$ imply

$$|x_{i'} - x_{j'}| \leq |x_{i'}(\tau^2_{Z_m}) - x_{i'}(\tau^2_{Z_m})| + 2\tau^2_{Z_m} R \leq \sigma_2 + 2\delta R,$$

and thus, $Z_m \in U_{i'j'}$.

2. There is at least one recollision (i.e., at least one of i', j' belongs to $\{i, j, k\}$): The argument is similar to (i), the only difference being that velocities of the recolliding particles transform at $\tau^1_{Z_m}$. Since the argument is similar for all cases, let us provide a detailed proof only for one recollisional case – for instance, $(i', j') = (i, k)$. We have

$$x_i(\tau^2_{Z_m}) = x_i(\tau^1_{Z_m}) + (\tau^2_{Z_m} - \tau^1_{Z_m})v_i^* = x_i + \tau^1_{Z_m} v_i + (\tau^2_{Z_m} - \tau^1_{Z_m})v_i^*,$$
$$x_k(\tau^2_{Z_m}) = x_k(\tau^1_{Z_m}) + (\tau^2_{Z_m} - \tau^1_{Z_m})v_k^* = x_k + \tau^1_{Z_m} v_k + (\tau^2_{Z_m} - \tau^1_{Z_m})v_k^*,$$

so

$$x_i - x_k = x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m}) - \tau^1_{Z_m}(v_i - v_k) - (\tau^2_{Z_m} - \tau^1_{Z_m})(v_i^* - v_k^*).$$

Therefore, triangle inequality, conservation of energy and the facts that $Z_m(\tau^2_{Z_m}) \in \Sigma^2_{i,k}$, $\tau^2_{Z_m} \leq \delta$ imply

$$\begin{aligned}|x_i - x_k| &\leq |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + \tau^1_{Z_m}|v_i - v_k| + (\tau^2_{Z_m} - \tau^1_{Z_m})|v_i^* - v_k^*| \\ &\leq |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + 2\tau^1_{Z_m} R + 2(\tau^2_{Z_m} - \tau^1_{Z_m})R \\ &= |x_i(\tau^2_{Z_m}) - x_k(\tau^2_{Z_m})| + 2\tau^2_{Z_m} R \\ &\leq \sigma_2 + 2\delta R,\end{aligned}$$

and hence, $Z_m \in U^2_{i,k}$. All the other recollisional cases are proved similarly.

Therefore, $Z_m \in U^3_{ijk} \cap U^2_{i'j'}$, and claim (III) follows. The rest of the claims are proved in the same spirit. We conclude that

$$I^2_{sc,ng} \subseteq U_{22} \cup U_{23} \cup U_{32} \cup U_{33}. \tag{3.40}$$

Assume now that Z_m is precollisional. Therefore, we obtain

$$Z_m(t) = \begin{cases}(X_m + tV'_m, V'_m), & \forall t \in [0, \tau^1_{Z_m}],\ \text{if } Z_m \in \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3} \\ (X_m + tV^*_m, V^*_m), & \forall t \in [0, \tau^1_{Z_m}],\ \text{if } Z_m \in \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3},\end{cases}$$

where the collisional transformation is taken with respect to the initial collisional particles. The proof follows the same lines, using Remark 3.17 for the initial collisional particles whenever needed. □

Now we wish to estimate the measure of $I^1_{sc,g} \cup I^1_{mu} \cup I^2_{sc,ng}$ in order to show that outside of a small measure set, we have a well defined flow. Let us first introduce some notation.

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For $m \geq 2$, $(i,j) \in \mathcal{I}_m^2$, a permutation $\pi : \{i,j\} \to \{i,j\}$ and $x_{\pi_j} \in \mathbb{R}^d$, we define the set

$$S_{\pi_i}(x_{\pi_j}) = \{x_{\pi_i} \in \mathbb{R}^d : (x_i, x_j) \in U_{ij}^2\}. \tag{3.41}$$

For $m \geq 3$, $(i,j,k) \in \mathcal{I}_m^3$, a permutation $\pi : \{i,j,k\} \to \{i,j,k\}$ and $(x_{\pi_j}, x_{\pi_k}) \in \mathbb{R}^{2d}$, we define the set

$$S_{\pi_i}(x_{\pi_j}, x_{\pi_k}) = \{x_{\pi_i} \in \mathbb{R}^d : (x_i, x_j, x_k) \in U_{ijk}^3\}. \tag{3.42}$$

Lemma 3.19. *The following hold*

1. *Let $m \geq 2$, $(i,j,k) \in \mathcal{I}_m^2$, a permutation $\pi : \{i,j\} \to \{i,j\}$ and $x_{\pi_j} \in \mathbb{R}^d$. Then*

$$|S_{\pi_i}(x_{\pi_j})|_d \leq C_{d,R}\delta. \tag{3.43}$$

2. *Let $m \geq 3$, $(i,j,k) \in \mathcal{I}_m^3$, a permutation $\pi : \{i,j,k\} \to \{i,j,k\}$ and $(x_{\pi_j}, x_{\pi_k}) \in \mathbb{R}^{2d}$. Then*

$$|S_{\pi_i}(x_{\pi_j}, x_{\pi_k})|_d \leq C_{d,R}\delta. \tag{3.44}$$

Proof. For proof of estimate (3.44), we refer to Lemma 3.10. in [5].

Let us prove (3.43). Consider $(i,j) \in \mathcal{I}_m^2$, and assume without loss of generality that $\pi(i,j) = (i,j)$. Let $x_j \in \mathbb{R}^d$. Recalling (3.41), we obtain

$$S_i(x_j) = \left\{x_i \in \mathbb{R}^d : \sigma_2 \leq |x_i - x_j| \leq \sigma_2 + 2\delta R\right\},$$

and thus, $S_i(x_j)$ is a spherical shell in $\mathbb{R}^d$ of inner radius σ_2 and outer radius $\sigma_2 + 2\delta R$. Therefore, by scaling (3.27), we obtain

$$|S_i(x_j)|_d \simeq (\sigma_2 + 2\delta R)^d - \sigma_2^d = 2\delta R \sum_{\ell=0}^{d-1} (\sigma_2 + 2\delta R)^{d-1-\ell} \sigma_2^\ell \leq C_{d,R}\delta.$$

□

Remark 3.20. Estimates of Lemma 3.19 are not sufficient to generate a global flow because δ represents the length of an elementary time step; therefore iterating, we cannot eliminate pathological sets. We will derive a better estimate of order δ^2 to achieve this elimination.

Lemma 3.21. *Let $m \geq 2$, $1 < R < \rho$ and $0 < \delta R < \sigma_2 < \sigma_3 < 1$. Then the following estimate holds:*

$$|I_{sc,g}^1 \cup I_{mu}^1 \cup I_{sc,ng}^2|_{2dm} \leq C_{m,d,R}\rho^{d(m-2)}\delta^2. \tag{3.45}$$

Proof. We first note that $I_{sc,g}$ is of zero measure since it is covered by lower codimension submanifolds of the phase spase; therefore, it suffices to estimate the measure of $I_{mu}^1 \cup I_{sc,ng}^2$. For $m = 2$, the result comes trivially from Lemma 3.18. For $m \geq 3$, we have

$$I_{mu}^1 \cup I_{sc,ng}^2 = U_{22} \cup U_{23} \cup U_{32} \cup U_{33},$$

where $U_{22}, U_{23}, U_{32}, U_{33}$ are given by (3.29)–(3.32). Therefore, it suffices to estimate the measure of $U_{22}, U_{23}, U_{32}, U_{33}$. We will strongly rely on Lemma 3.19.

∘ **Estimate of U_{22}:** By (3.29), we have

$$U_{22} = \bigcup_{(i,j)\neq(i',j')\in\mathcal{I}_m^2} (U_{ij}^2 \cap U_{i'j'}^2).$$

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Consider $(i,j) \neq (i',j') \in \mathcal{I}_m^2$. We distinguish the following possible cases:

1. $i', j' \notin \{i,j\}$: By (3.33), followed by Fubini's Theorem and part *(i)* of Lemma 3.19, we have

$$
\begin{aligned}
|U_{ij}^2 \cap U_{i'j'}^2|_{2dm} &\lesssim R^{dm}\rho^{d(m-4)} \int_{B_\rho^{4d}} \mathbb{1}_{S_i^2(x_j)\cap S_{i'}^2(x_{j'})}\, dx_i\, dx_{i'}\, dx_j\, dx_{j'} \\
&\le R^{dm}\rho^{d(m-4)}\left(\int_{B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_i^2(x_j)}\, dx_i\, dx_j\right)\left(\int_{B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_{i'}^2(x_{j'})}\, dx_{i'}\, dx_{j'}\right) \\
&\le C_{d,R}\rho^{d(m-2)}\delta^2.
\end{aligned}
$$

2. Exactly one of i', j' belongs to $\{i,j\}$: Without loss of generality, we consider the case $(i',j') = (j,j')$, for some $j' > j$, and all other cases follow similarly. Fubini's Theorem and part *(i)* of Lemma 3.19 imply

$$
\begin{aligned}
|U_{ij}^2 \cap U_{jj'}^2|_{2dm} &\lesssim R^{dm}\rho^{d(m-3)} \int_{B_\rho^{3d}} \mathbb{1}_{S_i^2(x_j)\cap S_j^2(x_{j'})}\, dx_j\, dx_{j'}\, dx_i \\
&\le R^{dm}\rho^{d(m-3)} \int_{B_\rho^d}\left(\int_{\mathbb{R}^d} \mathbb{1}_{S_i^2(x_j)}\, dx_i\right)\left(\int_{\mathbb{R}^d} \mathbb{1}_{S_{j'}^2(x_j)}\, dx_{j'}\right) dx_j \\
&\le C_{d,R}\rho^{d(m-2)}\delta^2.
\end{aligned}
$$

Combining cases (I)–(II), we obtain

$$
|U_{22}|_{2dm} \le C_{m,d,R}\rho^{d(m-2)}\delta^2. \tag{3.46}
$$

∘ **Estimate of U_{23}:** By (3.30), we have

$$
U_{23} = \bigcup_{(i,j)\in\mathcal{I}_m^2,\,(i',j',k')\in\mathcal{I}_m^3} (U_{ij}^2 \cap U_{i'j'k'}^3).
$$

Consider $(i,j)\in\mathcal{I}_m^2$, $(i',j',k')\in\mathcal{I}_m^3$. We distinguish the following possible cases:

1. $i', j', k' \notin \{i,j\}$: By Fubini's Theorem and parts *(i)–(ii)* of Lemma 3.19, we obtain

$$
\begin{aligned}
|U_{ij}^2 \cap U_{i'j'k'}^3|_{2dm} &\lesssim R^{dm}\rho^{d(m-5)} \int_{B_\rho^{5d}} \mathbb{1}_{S_j^2(x_i)\cap S_{k'}^3(x_{i'},x_{j'})}\, dx_i\, dx_j\, dx_{i'}\, dx_{j'}\, dx_{k'} \\
&\le R^{dm}\rho^{d(m-5)}\left(\int_{B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_j^2(x_i)}\, dx_i\, dx_j\right)\left(\int_{B_\rho^d\times B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_{k'}^3(x_{i'},x_{j'})}\, dx_{i'}\, dx_{j'}\, dx_{k'}\right) \\
&\le C_{d,R}\rho^{d(m-2)}\delta^2.
\end{aligned}
$$

2. Exactly one of i', j', k' belongs in $\{i,j\}$: Without loss of generality, we consider the case $(i',j',k') := (i',i,k')$, for some $i' < i < k'$, and all other cases follow similarly. Using Fubini's Theorem and parts *(i)–(ii)* of Lemma 3.19, we obtain

$$
\begin{aligned}
|U_{ij}^2 \cap U_{i'ik'}|_{2dm} &\lesssim R^{dm}\rho^{d(m-4)} \int_{B_\rho^{4d}} \mathbb{1}_{S_j^2(x_i)\cap S_{i'}^3(x_i,x_{k'})}\, dx_i\, dx_j\, dx_{i'}\, dx_{k'} \\
&\le R^{dm}\rho^{d(m-4)} \int_{B_\rho^d}\left(\int_{\mathbb{R}^d} \mathbb{1}_{S_j^2(x_i)}\, dx_j\right)\left(\int_{B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_{i'}^3(x_i,x_{k'})}\, dx_{i'}\, dx_{k'}\right) dx_i \\
&\le C_{d,R}\rho^{d(m-2)}\delta^2.
\end{aligned}
$$

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3. Exactly two of i', j', k' belongs in $\{i, j\}$: Without loss of generality, we consider the case $(i', j', k') = (i', i, j)$, for some $i' < i$, and all other cases follow similarly. Using Fubini's Theorem and parts *(i)*–*(ii)* of Lemma 3.19, we obtain

$$\begin{aligned}
|U_{ij}^2 \cap U_{i'ij}^3|_{2dm} &\lesssim R^{dm}\rho^{d(m-3)} \int_{B_\rho^{3d}} \mathbb{1}_{S_i^2(x_j)\cap S_{i'}^3(x_i,x_j)}\, dx_i\, dx_j\, dx_{i'} \\
&\leq R^{dm}\rho^{d(m-3)} \int_{B_\rho^d\times B_\rho^d} \left(\int_{\mathbb{R}^d} \mathbb{1}_{S_i^2(x_j)} \mathbb{1}_{S_{i'}^3(x_i,x_j)}\, dx_{i'}\right) dx_i\, dx_j \\
&= R^{dm}\rho^{d(m-3)} \int_{B_\rho^d\times B_\rho^d} \mathbb{1}_{S_i^2(x_j)} \Big(\int_{\mathbb{R}^d} \mathbb{1}_{S_{i'}^3(x_i,x_j)}\, dx_{i'}\Big)\, dx_i\, dx_j \\
&\leq C_{d,R}\rho^{d(m-3)}\delta \int_{B_\rho^d}\int_{\mathbb{R}^d} \mathbb{1}_{S_i(x_j)}\, dx_i\, dx_j \\
&\leq C_{d,R}\rho^{d(m-2)}\delta^2.
\end{aligned}$$

Combining cases (I)–(III), we obtain

$$|U_{23}|_{2dm} \leq C_{m,d,R}\rho^{d(m-2)}\delta^2. \tag{3.47}$$

∘ **Estimate of U_{32}:** We use a similar argument to the estimate for U_{23} to obtain

$$|U_{32}|_{2dm} \leq C_{m,d,R}\rho^{d(m-2)}\delta^2. \tag{3.48}$$

∘ **Estimate of U_{33}:** We refer to Lemma 3.11 from [5] for a detailed proof. We obtain

$$|U_{33}|_{2dm} \leq C_{m,d,R}\rho^{d(m-2)}\delta^2. \tag{3.49}$$

Combining (3.46)–(3.49), we obtain (3.45), and the proof is complete. □

We inductively use Lemma 3.21 to define a global flow which preserves energy for almost all configuration. For this purpose, given $Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}$, we define its kinetic energy as

$$E_m(Z_m) = \frac{1}{2}\sum_{i=1}^m |v_i|^2. \tag{3.50}$$

For convenience, let us define the m-particle free flow:

Definition 3.22. Let $m \in \mathbb{N}$. We define the m-particle free flow as the family of measure-preserving maps $(\Phi_m^t)_{t\in\mathbb{R}} : \mathbb{R}^{2dm} \to \mathbb{R}^{2dm}$, given by

$$\Phi_m^t Z_m = \Phi_m^t(X_m, V_m) = (X_m + tV_m, V_m). \tag{3.51}$$

We are now in the position to state the Existence Theorem of the m-particle (σ_2, σ_3)-flow.

Theorem 3.23. *Let $m \in \mathbb{N}$ and $0 < \sigma_2 < \sigma_3 < 1$. There exists a full measure subset $\Gamma_{m,\sigma_2,\sigma_3} \subseteq \mathcal{D}^*_{m,\sigma_2,\sigma_3}$ which is a countable intersection of dense open sets, and a measure-preserving family of diffeomorphisms $(\Psi_m^t)_{t\in\mathbb{R}} : \Gamma_{m,\sigma_2,\sigma_3} \to \Gamma_{m,\sigma_2,\sigma_3}$ such that*

$$\Psi_m^{t+s} Z_m = (\Psi_m^t \circ \Psi_m^s)(Z_m) = (\Psi_m^s \circ \Psi_m^t)(Z_m), \quad \text{a.e. in } \Gamma_{m,\sigma_2,\sigma_3}, \quad \forall t, s \in \mathbb{R}, \tag{3.52}$$

$$E_m\big(\Psi_m^t Z_m\big) = E_m(Z_m), \quad \text{a.e. in } \Gamma_{m,\sigma_2,\sigma_3}, \quad \forall t \in \mathbb{R}, \text{ where } E_m \text{ is given by (3.50)}. \tag{3.53}$$

Moreover, the flow is defined a.e. on $\Gamma_{m,\sigma} \cap \partial_{sc,ng}\mathcal{D}_{m,\sigma}$ with respect to the induced measure $d\sigma$ and preserves energy; that is,

$$\Psi_m^t Z_m' = \Psi_m^t Z_m, \quad \sigma - a.e. \text{ on } \Gamma_{m,\sigma_2,\sigma_3} \cap \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}, \quad \forall t \in \mathbb{R}, \tag{3.54}$$

$$\Psi_m^t Z_m^* = \Psi_m^t Z_m, \quad \sigma - a.e. \text{ on } \Gamma_{m,\sigma_2,\sigma_3} \cap \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}, \quad \forall t \in \mathbb{R}, \tag{3.55}$$

This family of maps is called the m-particle (σ_2, σ_3)-flow. For $m = 1$, the flow is just the free flow.

Proof. The proof follows the same steps as the proof of Theorem 4.9.1 in [2], using the corresponding estimates. For an outline of the proof, see Theorem 3.14 in [5] as well. □

3.5. *The Liouville equation*

Here, we introduce the flow operators used throughout the paper and formally derive the Liouville equation for $m \geq 2$.

Definition 3.24. For $t \in \mathbb{R}$, we define the σ-interaction zone flow of m-particles operator $T_m^t : L^\infty(\mathcal{D}_{m,\sigma}) \to L^\infty(\mathcal{D}_{m,\sigma})$ as

$$T_m^t g_m(Z_m) = g_m(\Psi_m^{-t} Z_m). \tag{3.56}$$

Definition 3.25. For $t \in \mathbb{R}$ and $m \in \mathbb{N}$, we define the free flow of m-particles operator $S_m^t : L^\infty(\mathbb{R}^{2dm}) \to L^\infty(\mathbb{R}^{2dm})$ as

$$S_m^t g_m(Z_m) = g_m(\Phi_m^{-t} Z_m) = g_m(X_m - tV_m, V_m). \tag{3.57}$$

Given a symmetric with respect to the particles initial probability density $f_{m,0}$ in $\mathcal{D}_{m,\sigma_2,\sigma_3}$, we define its evolution as $f_m(t, Z_m) := T_m^t f_{m,0}$. Clearly, f_m is symmetric and by Theorem 3.23 it formally satisfies the m-particle Liouville equation

$$\begin{cases} \partial_t f_m + \sum_{i=1}^m v_i \cdot \nabla_{x_i} f_m = 0, & (t, Z_m) \in (0, \infty) \times \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3} \\ f_m(t, Z_m') = f_m(t, Z_m), & (t, Z_m) \in [0, \infty) \times \partial_{2,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}, \\ f_m(t, Z_m^*) = f_m(t, Z_m), & (t, Z_m) \in [0, \infty) \times \partial_{3,sc}\mathcal{D}_{m,\sigma_2,\sigma_3}, \\ f_m(0, Z_m) = f_{m,0}(Z_m), & Z_m \in \mathring{\mathcal{D}}_{m,\sigma_2,\sigma_3}. \end{cases} \tag{3.58}$$

Let us note that in the case $m = 2$, the equation has only binary boundary conditions.

4. BBGKY hierarchy, Boltzmann hierarchy and the binary-ternary Boltzmann equation

4.1. *The BBGKY hierarchy*

Consider N-particles of diameter $0 < \epsilon_2 < 1$ and interaction zone $0 < \epsilon_3 < 1$, where $N \geq 3$ and $\epsilon_2 < \epsilon_3$. For $s \in \mathbb{N}$, we define the s-marginal of a symmetric probability density f_N, supported in $\mathcal{D}_{N,\epsilon_2,\epsilon_3}$, as

$$f_N^{(s)}(Z_s) = \begin{cases} \int_{\mathbb{R}^{2d(N-s)}} f_N(Z_N)\, dx_{s+1} \ldots dx_N\, dv_{s+1} \ldots dv_N, & 1 \leq s < N, \\ f_N, & s = N, \\ 0, & s > N, \end{cases} \tag{4.1}$$

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where for $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, we write $Z_N = (X_s, x_{s+1}, \dots, x_N, V_s, v_{s+1}, \dots, v_N)$. One can see, for all $1 \le s \le N$, the marginals $f_N^{(s)}$ are symmetric probability densities, supported in $\mathcal{D}_{s,\epsilon_2,\epsilon_3}$ and

$$f_N^{(s)}(Z_s) = \int_{\mathbb{R}^{2d}} f_N^{(s+1)}(X_N, V_N)\, dx_{s+1}\, dv_{s+1}, \quad \forall 1 \le s \le N-1.$$

Assume now that f_N is formally the solution to the N-particle Liouville equation (3.58) with initial data $f_{N,0}$. We seek to formally find a hierarchy of equations satisfied by the marginals of f_N. For $s \ge N$, by definition, we have

$$f_N^{(N)} = f_N, \text{ and } f_N^{(s)} = 0, \text{ for } s > N. \tag{4.2}$$

We observe that $\partial\mathcal{D}_{N,\epsilon_2,\epsilon_3}$ is equivalent up to surface measure zero to $\Sigma^X \times \mathbb{R}^{dN}$ where

$$\Sigma^X := \bigcup_{(i,j)\in\mathcal{I}_N^2} \Sigma_{ij}^{2,sc,X} \cup \bigcup_{(i,j,k)\in\mathcal{I}_N^3} \Sigma_{ijk}^{3,sc,X}, \tag{4.3}$$

$$\begin{aligned}\Sigma_{ij}^{2,sc,X} := \Big\{X_N \in \mathbb{R}^{dN} :& d_2(x_i, x_j) = \epsilon_2,\ d_2(x_{i'}, x_{j'}) > \epsilon_2, \quad \forall (i', j') \in \mathcal{I}_N^2 \setminus \{(i,j)\} \\ &\text{and } d_3(x_{i'};, x_{j'}, x_{k'}) > \sqrt{2}\epsilon_3, \quad \forall (i', j', k') \in \mathcal{I}_N^3\Big\},\end{aligned}$$

$$\begin{aligned}\Sigma_{ijk}^{3,sc,X} := \Big\{X_N \in \mathbb{R}^{dN} :& d_3(x_i; x_j, x_k) = \sqrt{2}\epsilon_3,\ d_2(x_{i'}, x_{j'}) > \epsilon_2, \quad \forall (i', j') \in \mathcal{I}_N^2 \\ &\text{and } d_3(x_{i'};, x_{j'}, x_{k'}) > \sqrt{2}\epsilon_3, \quad \forall (i', j', k') \in \mathcal{I}_N^3\} \setminus \{(i,j,k)\}\Big\}.\end{aligned}$$

Notice that (4.3) is a pairwise disjoint union.

Remark 4.1. The assumption $\epsilon_2 < \epsilon_3$ made at at the beginning of the section is necessary for the ternary contribution to be visible. Indeed, if $\epsilon_2 \ge \epsilon_3$, Remark 3.1 and (3.23) would imply that $\Sigma_{ijk}^{3,sc,X} = \emptyset$ for all $(i, j, k) \in \mathcal{I}_m^3$, and therefore, there would not be a ternary collisional term.

The hierarchy for $s < N$ will come after integrating by parts the Liouville equation (3.58). Consider $1 \le s \le N-1$. The boundary and initial conditions can be easily recovered integrating Liouville's equation boundary and initial conditions, respectively; that is,

$$\begin{cases} f_N^{(s)}(t, Z_s') = f_N^{(s)}(t, Z_s), & (t, Z_s) \in [0,\infty) \times \partial_{2,sc}\mathcal{D}_{s,\epsilon_2,\epsilon_3}, \quad s \ge 2, \\ f_N^{(s)}(t, Z_s^*) = f_N^{(s)}(t, Z_s), & (t, Z_s) \in [0,\infty) \times \partial_{3,sc}\mathcal{D}_{s,\epsilon_2,\epsilon_3}, \quad s \ge 3, \\ f_N^{(s)}(0, Z_s) = f_{N,0}^{(s)}(Z_s), & Z_s \in \mathring{\mathcal{D}}_{s,\epsilon_2,\epsilon_3}. \end{cases} \tag{4.4}$$

Notice that for $s = 2$, there is no ternary boundary condition, while for $s = 1$, there is no boundary condition at all.

Consider now a smooth test function ϕ_s compactly supported in $(0,\infty) \times \mathcal{D}_{s,\epsilon_2,\epsilon_3}$ such that the following hold:

- For any $(i, j) \in \mathcal{I}_N^2$ with $j \le s$, we have

$$\phi_s(t, p_s Z_N') = \phi_s(t, p_s Z_N) = \phi_s(t, Z_s), \quad \forall (t, Z_N) \in (0,\infty) \times \Sigma_{i,j}^{sc,2}, \tag{4.5}$$

- For any $(i, j, k) \in \mathcal{I}_N^3$ with $j \le s$, we have

$$\phi_s(t, p_s Z_N^*) = \phi_s(t, p_s Z_N) = \phi_s(t, Z_s), \quad \forall (t, Z_N) \in (0,\infty) \times \Sigma_{i,j,k}^{sc,3}, \tag{4.6}$$

where $p_s : \mathbb{R}^{2dN} \to \mathbb{R}^{2ds}$ denotes the natural projection in space and velocities, given by $p_s(Z_N) = Z_s$.

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Multiplying the Liouville equation by ϕ_s and integrating, we obtain its weak form

$$\int_{(0,\infty)\times\mathcal{D}_{N,\epsilon_2,\epsilon_3}}\left(\partial_t f_N(t,Z_N)+\sum_{i=1}^N v_i\nabla_{x_i}f_N(t,Z_N)\right)\phi_s(t,Z_s)\,dX_N\,dV_N\,dt=0. \tag{4.7}$$

For the time derivative in (4.7), we use Fubini's Theorem, integration by parts in time, the fact that f_N is supported in $(0,\infty)\times\mathcal{D}_{N,\epsilon_2,\epsilon_3}$ and the fact that ϕ_s is compactly supported in $(0,\infty)\times\mathcal{D}_{s,\epsilon_2,\epsilon_3}$ to obtain

$$\int_{(0,\infty)\times\mathcal{D}_{N,\epsilon_2,\epsilon_3}}\partial_t f_N(t,Z_N)\phi_s(t,Z_s)\,dX_N\,dV_N\,dt=\int_{(0,\infty)\times\mathcal{D}_{s,\epsilon_2,\epsilon_3}}\partial_t f_N^{(s)}(t,Z_s)\phi_s(t,Z_s)\,dX_s\,dV_s\,dt. \tag{4.8}$$

For the material derivative term in (4.7), the Divergence Theorem implies that

$$\begin{aligned}\int_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}\sum_{i=1}^N v_i\nabla_{x_i}f_N(t,Z_N)\phi_s(t,Z_s)\,dX_N\,dV_N&=\int_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}\mathbf{div}_{X_N}[f_N(t,Z_N)V_N]\phi_s(t,Z_s)\,dX_N\,dV_N\\ =-\int_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}V_N\cdot\nabla_{X_N}\phi_s(t,Z_s)f_N(t,Z_N)\,dX_N\,dV_N+&\\ \int_{\Sigma^X\times\mathbb{R}^{dN}}\hat{n}(X_N)\cdot V_N f_N(t,Z_N)\phi_s(t,Z_s)\,dV_N\,d\sigma,&\end{aligned} \tag{4.9}$$

where Σ^X is given by (4.3), $\hat{n}(X_N)$ is the outwards normal vector on Σ^X at $X_N\in\Sigma^X$ and $d\sigma$ is the surface measure on Σ^X. Using the fact that f_N is supported in $\mathcal{D}_{N,\epsilon_2,\epsilon_3}$, Divergence Theorem and the fact that ϕ_s is compactly supported in $(0,\infty)\times\mathcal{D}_{s,\epsilon_2,\epsilon_3}$, we obtain

$$\int_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}V_N\cdot\nabla_{X_N}\phi_s(t,Z_s)f_N(t,Z_N)\,dX_N\,dV_N=-\int_{\mathcal{D}_{s,\epsilon_2,\epsilon_3}}\sum_{i=1}^s v_i\nabla_{x_i}f_N^{(s)}(t,Z_s)\phi_s(t,Z_s)\,dX_s\,dV_s. \tag{4.10}$$

Combining (4.7)–(4.10), and recalling the space boundary decomposition (4.3), we obtain

$$\begin{aligned}&\int_{(0,\infty)\times\mathcal{D}_{s,\epsilon_2,\epsilon_3}}\left(\partial_t f_N^{(s)}(t,Z_s)+\sum_{i=1}^s v_i\nabla_{x_i}f_N^{(s)}(t,Z_s)\right)\phi_s(t,Z_s)\,dX_s\,dV_s\,dt\\ &=-\int_{(0,\infty)\times\Sigma^X\times\mathbb{R}^{dN}}\hat{n}(X_N)\cdot V_N f_N(t,Z_N)\phi_s(t,Z_s)\,dV_N\,d\sigma\,dt,\\ &=:\int_0^\infty\sum_{(i,j)\in\mathcal{I}_N^2}C_{ij}^2(t)+\sum_{(i,j,k)\in\mathcal{I}_N^3}C_{ijk}^3(t)\,dt,\end{aligned} \tag{4.11}$$

where for $(i,j)\in\mathcal{I}_N^2$, $t>0$, we denote

$$C_{ij}^2(t)=-\int_{\Sigma_{i,j}^{2,sc,X}\times\mathbb{R}^{dN}}\hat{n}_{ij}^2(X_N)\cdot V_N f_N(t,Z_N)\phi_s(t,Z_s)\,dV_N\,d\sigma_{ij}^2, \tag{4.12}$$

for $(i,j,k)\in\mathcal{I}_N^3$, $t>0$, we denote

$$C_{ijk}^3(t)=-\int_{\Sigma_{i,j,k}^{3,sc,X}\times\mathbb{R}^{dN}}\hat{n}_{ijk}^3(X_N)\cdot V_N f_N(t,Z_N)\phi_s(t,Z_s)\,dV_N\,d\sigma_{ijk}^3, \tag{4.13}$$

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and $\hat{n}_{ij}^2(X_N)$ is the outwards normal vector on $\Sigma_{ij}^{2,sc,X}$ at $X_N \in \Sigma_{ij}^{2,sc,X}$, $d\sigma_{ij}^2$ is the surface measure on $\Sigma_{ij}^{2,sc,X}$, while $\hat{n}_{ijk}^3(X_N)$ is the outwards normal vector on $\Sigma_{ijk}^{3,sc,X}$ at $X_N \in \Sigma_{ijk}^{3,sc,X}$ and $d\sigma_{ijk}^3$ is the surface measure on $\Sigma_{ijk}^{3,sc,X}$.

Following similar calculations to [18] which treats the binary case, and [5] which treats the ternary case, we formally obtain the BBGKY hierarchy:

$$\begin{cases} \partial_t f_N^{(s)} + \sum_{i=1}^s v_i \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}^N f_N^{(s+1)} + \mathcal{C}_{s,s+2}^N f_N^{(s+2)}, & (t,Z_s) \in (0,\infty) \times \mathring{\mathcal{D}}_{s,\epsilon_2,\epsilon_3}, \\ f_N^{(s)}(t,Z_s') = f_N^{(s)}(t,Z_s), \quad (t,Z_s) \in [0,\infty) \times \partial_{2,sc}\mathcal{D}_{s,\epsilon_2,\epsilon_3}, & \text{whenever } s \geq 2, \\ f_N^{(s)}(t,Z_s^*) = f_N^{(s)}(t,Z_s), \quad (t,Z_s) \in [0,\infty) \times \partial_{3,sc}\mathcal{D}_{s,\epsilon_2,\epsilon_3}, & \text{whenever } s \geq 3, \\ f_N^{(s)}(0,Z_s) = f_{N,0}^{(s)}(Z_s), \quad Z_s \in \mathring{\mathcal{D}}_{s,\epsilon_2,\epsilon_3}, \end{cases} \tag{4.14}$$

where

$$\mathcal{C}_{s,s+1}^N = \mathcal{C}_{s,s+1}^{N,+} - \mathcal{C}_{s,s+1}^{N,-}, \tag{4.15}$$

$$\mathcal{C}_{s,s+2}^N = \mathcal{C}_{s,s+2}^{N,+} - \mathcal{C}_{s,s+2}^{N,-}, \tag{4.16}$$

and we use the following notation:

∘ **Binary notation:** For $1 \leq s \leq N-1$, we denote

$$\mathcal{C}_{s,s+1}^{N,+} f_N^{(s+1)}(t,Z_s) = A_{N,\epsilon_2,s}^2 \sum_{i=1}^s \int_{\mathbb{S}_1^{d-1}\times\mathbb{R}^d} b_2^+(\omega_1, v_{s+1}-v_i) f_N^{(s+1)}\left(t, Z_{s+1,\epsilon_2,i}'\right) d\omega_1\, dv_{s+1}, \tag{4.17}$$

$$\mathcal{C}_{s,s+1}^{N,-} f_N^{(s+2)}(t,Z_s) = A_{N,\epsilon_2,s}^2 \sum_{i=1}^s \int_{\mathbb{S}_1^{d-1}\times\mathbb{R}^d} b_2^+(\omega_1, v_{s+1}-v_i) f_N^{(s+1)}\left(t, Z_{s+1,\epsilon_2,i}\right) d\omega_1\, dv_{s+1}, \tag{4.18}$$

where

$$\begin{aligned} &b_2(\omega_1, v_{s+1}-v_i) = \langle \omega_1, v_{s+1}-v_i\rangle, \\ &b_2^+ = \max\{b_2, 0\}, \\ &A_{N,\epsilon_2,s}^2 = (N-s)\epsilon_2^{d-1}, \\ &Z_{s+1,\epsilon_2,i} = (x_1,\dots,x_i,\dots,x_s,x_i-\epsilon_2\omega_1, v_1,\dots v_{i-1}, v_i, v_{i+1},\dots,v_s,v_{s+1}), \\ &Z_{s+1,\epsilon_2,i}' = (x_1,\dots,x_i,\dots,x_s,x_i+\epsilon_2\omega_1, v_1,\dots v_{i-1}, v_i', v_{i+1},\dots,v_s,v_{s+1}'). \end{aligned} \tag{4.19}$$

For $s \geq N$, we trivially define $\mathcal{C}_{s,s+1}^N \equiv 0$.

∘ **Ternary notation:** For $1 \leq s \leq N-2$, we denote

$$\begin{aligned} \mathcal{C}_{s,s+2}^{N,+} f_N^{(s+2)}(t,Z_s) = A_{N,\epsilon_3,s}^3 \sum_{i=1}^s \int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}} &\frac{b_3^+(\omega_1,\omega_2,v_{s+1}-v_i,v_{s+2}-v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} \\ &\times f_N^{(s+2)}\left(t, Z_{s+2,\epsilon_3,i}^*\right) d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} \mathcal{C}_{s,s+2}^{N,-} f_N^{(s+2)}(t,Z_s) = A_{N,\epsilon_3,s}^3 \sum_{i=1}^s \int_{\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d}} &\frac{b_3^+(\omega_1,\omega_2,v_{s+1}-v_i,v_{s+2}-v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} \\ &\times f_N^{(s+2)}\left(t, Z_{s+2,\epsilon_3,i}\right) d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2}, \end{aligned} \tag{4.21}$$

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where

$$\begin{aligned}
&A^3_{N,\epsilon_3,s} = 2^{d-2}(N-s)(N-s-1)\epsilon_3^{2d-1},\\
&b_3(\omega_1,\omega_2,v_{s+1}-v_i,v_{s+2}-v_i) = \langle\omega_1,v_{s+1}-v_i\rangle + \langle\omega_2,v_{s+2}-v_i\rangle,\\
&b_3^+ = \max\{b_3,0\},\\
&Z_{s+2,\epsilon_3,i} = (x_1,\ldots,x_i,\ldots,x_s,x_i-\sqrt{2}\epsilon_3\omega_1,x_i-\sqrt{2}\epsilon_3\omega_2,v_1,\ldots v_{i-1},v_i,v_{i+1},\ldots,v_s,v_{s+1},v_{s+2}),\\
&Z^*_{s+2,\epsilon_3,i} = (x_1,\ldots,x_i,\ldots,x_s,x_i+\sqrt{2}\epsilon_3\omega_1,x_i+\sqrt{2}\epsilon_3\omega_2,v_1,\ldots v_{i-1},v_i^*,v_{i+1},\ldots,v_s,v^*_{s+1},v^*_{s+2}).
\end{aligned} \tag{4.22}$$

For $s \geq N-1$, we trivially define $\mathcal{C}^N_{s,s+2} \equiv 0$.

Duhamel's formula implies that the BBGKY hierarchy can be written in mild form as follows:

$$f_N^{(s)}(t,Z_s) = T_s^t f_{N,0}^{(s)}(Z_s) + \int_0^t T_s^{t-\tau}\Big(\mathcal{C}^N_{s,s+1}f_N^{(s+1)} + \mathcal{C}^N_{s,s+2}f_N^{(s+2)}\Big)(\tau,Z_s)\,d\tau, \quad s\in\mathbb{N}, \tag{4.23}$$

where T_s^t is the s-particle (ϵ_2,ϵ_3)-flow operator given in (3.56).

4.2. *The Boltzmann hierarchy*

We will now derive the Boltzmann hierarchy as the formal limit of the BBGKY hierarchy as $N\to\infty$ and $\epsilon_2,\epsilon_3\to 0^+$ under the scaling

$$N\epsilon_2^{d-1} \simeq N\epsilon_3^{d-1/2} \simeq 1. \tag{4.24}$$

This scaling implies that ϵ_2,ϵ_3 satisfy

$$\epsilon_2^{d-1} \simeq \epsilon_3^{d-1/2}. \tag{4.25}$$

Remark 4.2. Using the scaling (4.24), we obtain

$$\epsilon_2 \simeq N^{-\frac{1}{d-1}} \overset{N\to\infty}{\longrightarrow} 0, \quad \epsilon_3 \simeq N^{-\frac{2}{2d-1}} \overset{N\to\infty}{\longrightarrow} 0, \tag{4.26}$$

and thus,

$$\frac{\epsilon_2}{\epsilon_3} \simeq N^{-\frac{1}{(d-1)(2d-1)}} \overset{N\to\infty}{\longrightarrow} 0. \tag{4.27}$$

Therefore, for N large enough, we have $\epsilon_2 << \epsilon_3$.

Remark 4.3. The scaling (4.24) guarantees that for a fixed $s\in\mathbb{N}$, we have

$$\begin{aligned}
&A^2_{N,\epsilon_2,s} = (N-s)\epsilon_2^{d-1} \longrightarrow 1, \quad \text{as } N\to\infty,\\
&A^3_{N,\epsilon_3,s} = 2^{d-2}(N-s)(N-s-1)\epsilon_3^{2d-1} \longrightarrow 1, \quad \text{as } N\to\infty.
\end{aligned}$$

Formally taking the limit under the scaling imposed, we may define the following collisional operators:

∘ **Binary Boltzmann operator:**

$$\mathcal{C}^\infty_{s,s+1} = \mathcal{C}^{\infty,+}_{s,s+1} - \mathcal{C}^{\infty,-}_{s,s+1}, \tag{4.28}$$

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where

$$\mathcal{C}^{\infty,+}_{s,s+1} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int_{(\mathbb{S}_1^{d-1}\times\mathbb{R}^d)} b_2^+(\omega_1, v_{s+2} - v_i) f^{(s+1)}\left(t, Z'_{s+1,i}\right) \times d\omega_1\, dv_{s+1}, \tag{4.29}$$

$$\mathcal{C}^{\infty,-}_{s,s+1} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int_{(\mathbb{S}_1^{d-1}\times\mathbb{R}^d)} b_2^+(\omega_1, v_{s+2} - v_i) \times f^{(s+1)}\left(t, Z_{s+1,i}\right) \times d\omega_1\, dv_{s+1}, \tag{4.30}$$

$$\begin{aligned} &b_2(\omega_1, v_{s+1} - v_i) = \langle \omega_1, v_{s+1} - v_i\rangle, \\ &b_2 = \max\{0, b_2\}, \\ &Z_{s+1,i} = (x_1, ..., x_i, ..., x_s, x_i, v_1, ...v_{i-1}, v_i, v_{i+1}, ..., v_s, v_{s+1}), \\ &Z'_{s+1,i} = (x_1, ..., x_i, ..., x_s, x_i, v_1, ...v_{i-1}, v'_i, v_{i+1}, ..., v_s, v'_{s+1}). \end{aligned} \tag{4.31}$$

∘ **Ternary Boltzmann operator:**

$$\mathcal{C}^{\infty}_{s,s+2} = \mathcal{C}^{\infty,+}_{s,s+2} - \mathcal{C}^{\infty,-}_{s,s+2}, \tag{4.32}$$

where

$$\begin{aligned} \mathcal{C}^{\infty,+}_{s,s+2} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int_{(\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d})} \frac{b_3^+(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i)}{\sqrt{1 + \langle \omega_1, \omega_2\rangle}} f^{(s+2)}\left(t, Z^*_{s+2,i}\right) \\ \times d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} \mathcal{C}^{\infty,-}_{s,s+2} f^{(s+2)}(t, Z_s) = \sum_{i=1}^{s} \int_{(\mathbb{S}_1^{2d-1}\times\mathbb{R}^{2d})} \frac{b_3^+(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i)}{\sqrt{1 + \langle \omega_1, \omega_2\rangle}} \times f^{(s+2)}\left(t, Z_{s+2,i}\right) \\ \times d\omega_1\, d\omega_2\, dv_{s+1}\, dv_{s+2}, \end{aligned} \tag{4.34}$$

$$\begin{aligned} &b_3(\omega_1, \omega_2, v_{s+1} - v_i, v_{s+2} - v_i) = \langle \omega_1, v_{s+1} - v_i\rangle + \langle \omega_2, v_{s+2} - v_i\rangle, \\ &b_3^+ = \max\{b_3, 0\}, \\ &Z_{s+2,i} = (x_1, ..., x_i, ..., x_s, x_i, x_i, v_1, ...v_{i-1}, v_i, v_{i+1}, ..., v_s, v_{s+1}, v_{s+2}), \\ &Z^*_{s+2,i} = (x_1, ..., x_i, ..., x_s, x_i, x_i, v_1, ...v_{i-1}, v^*_i, v_{i+1}, ..., v_s, v^*_{s+1}, v^*_{s+2}). \end{aligned} \tag{4.35}$$

Now we are ready to introduce the Boltzmann hierarchy. More precisely, given an initial probability density f_0, the Boltzmann hierarchy for $s \in \mathbb{N}$ is given by

$$\begin{cases} \partial_t f^{(s)} + \sum_{i=1}^{s} v_i \nabla_{x_i} f^{(s)} = \mathcal{C}^{\infty}_{s,s+1} f^{(s+1)} + \mathcal{C}^{\infty}_{s,s+2} f^{(s+2)}, & (t, Z_s) \in (0,\infty) \times \mathbb{R}^{2ds}, \\ f^{(s)}(0, Z_s) = f_0^{(s)}(Z_s), & \forall Z_s \in \mathbb{R}^{2ds}. \end{cases} \tag{4.36}$$

Duhamel's formula implies that the Boltzmann hierarchy can be written in mild form as follows:

$$f^{(s)}(t, Z_s) = S_s^t f_0^{(s)}(Z_s) + \int_0^t S_s^{t-\tau}\left(\mathcal{C}^{\infty}_{s,s+1} f^{(s+1)} + \mathcal{C}^{\infty}_{s,s+2} f^{(s+2)}\right)(\tau, Z_s)\, d\tau, \quad s \in \mathbb{N}, \tag{4.37}$$

where S_s^t denotes the $s-$particle free flow operator given in (3.57).

5. Local well-posedness

In this section, we show that the BBGKY hierarchy, the Boltzmann hierarchy and the binary-ternary Boltzmann equation are well-posed for short times in Maxwellian weighted L^∞-spaces. To obtain these

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results, we combine the continuity estimates on the binary and ternary collisional operators, obtained in [18] and [5], respectively.

5.1. LWP for the BBGKY hierarchy

Consider $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24), with $N \geq 3$.

For $\beta > 0$, we define the Banach space

$$X_{N,\beta,s} := \big\{g_{N,s} \in L^\infty(\mathcal{D}_{m,\epsilon_2,\epsilon_3}) \text{ and } |g_{N,s}|_{N,\beta,s} < \infty\big\},$$

with norm $|g_{N,s}|_{N,\beta,s} = \sup_{Z_s \in \mathbb{R}^{2ds}} |g_{N,s}(Z_s)| e^{\beta E_s(Z_s)}$, where $E_s(Z_s)$ is the kinetic energy of the s-particles given by (3.50). For $s > N$, we trivially define $X_{N,\beta,s} := \{0\}$.

Remark 5.1. Given $t \in \mathbb{R}$ and $s \in \mathbb{N}$, conservation of energy under the flow (3.53) implies that the s-particle of (ϵ_2, ϵ_3)-flow operator $T_s^t : X_{N,\beta,s} \to X_{N,\beta,s}$, given in (3.56) is an isometry; that is,

$$|T_s^t g_{N,s}|_{N,\beta,s} = |g_{N,s}|_{N,\beta,s}, \quad \forall g_{N,s} \in X_{N,\beta,s}.$$

Proof. Let $g_{N,s} \in X_{N,\beta,s}$ and $Z_s \in \mathbb{R}^{2ds}$. If $Z_s \notin \mathcal{D}_{s,\epsilon_2,\epsilon_3}$, the result is trivial since $g_{N,s}$ is supported in $\mathcal{D}_{s,\epsilon_2,\epsilon_3}$. Assume $Z_s \in \mathcal{D}_{s,\epsilon_2,\epsilon_3}$. Then Theorem 3.23 yields

$$e^{\beta E_s(Z_s)} |T_s^t g_{N,s}| = e^{\beta E_s(Z_s)} |(g_{N,s} \circ \Psi_s^{-t})(Z_s)| = e^{\beta E_s(\Psi_s^{-t} Z_s)} |g_{N,s}(\Psi_s^{-t} Z_s)| \leq |g_{N,s}|_{N,s,\beta},$$

and hence, $|T_s^t g_{N,s}|_{N,s,\beta} \leq |g_{N,s}|_{N,s,\beta}$. The other side of the inequality comes similarly using the fact that $Z_s = \Psi_s^{-t}(\Psi_s^t Z_s)$. □

Consider as well $\mu \in \mathbb{R}$. We define the Banach space

$$X_{N,\beta,\mu} := \big\{G_N = (g_{N,s})_{s\in\mathbb{N}} : \|G_N\|_{N,\beta,\mu} < \infty\big\},$$

with norm $\|G_N\|_{N,\beta,\mu} = \sup_{s\in\mathbb{N}} e^{\mu s} |g_{N,s}|_{N,\beta,s} = \max_{s\in\{1,\dots,N\}} e^{\mu s} |g_{N,s}|_{N,\beta,s}$.

Remark 5.2. Given $t \in \mathbb{R}$, Remark 5.1 implies that the map $\mathcal{T}^t : X_{N,\beta,\mu} \to X_{N,\beta,\mu}$ given by

$$\mathcal{T}^t G_N := \big(T_s^t g_{N,s}\big)_{s\in\mathbb{N}} \tag{5.1}$$

is an isometry; that is, $\|\mathcal{T}^t G_N\|_{N,\beta,\mu} = \|G_N\|_{N,\beta,\mu}$, for any $G_N \in X_{N,\beta,\mu}$.

Finally, given $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$, we define the Banach space

$$\boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}} := C^0\big([0,T], X_{N,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\big),$$

with norm $|||\boldsymbol{G_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} = \sup_{t\in[0,T]} \|\boldsymbol{G_N}(t)\|_{N,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}$. Similarly as in Proposition 6.2. from [2], one can obtain the following bounds:

Proposition 5.3. *Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \boldsymbol{\beta}(0)$, $\boldsymbol{\beta}(T) > 0$ $\mu_0 = \boldsymbol{\mu}(0)$. Then for any $G_N = \big(g_{N,s}\big)_{s\in\mathbb{N}} \in X_{N,\beta_0,\mu_0}$, the following estimates hold:*

1. $|||G_N|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \|G_N\|_{N,\beta_0,\mu_0}$.
2. $\left|\left|\left|\displaystyle\int_0^t \mathcal{T}^\tau G_N \, d\tau\right|\right|\right|_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \leq T \|G_N\|_{N,\beta_0,\mu_0}$.

From Proposition 5.3.1. in [18] and Lemma 5.1. in [5], we have the following continuity estimates for the binary and ternary collisional operators, respectively:

Lemma 5.4. *Let $m \in \mathbb{N}$, $\beta > 0$. For any $Z_m \in \mathcal{D}_{m,\epsilon_2,\epsilon_3}$ and $k \in \{1,2\}$, the following estimate holds:*

$$\left|\mathcal{C}^N_{m,m+k} g_{N,m+k}(Z_m)\right| \lesssim \beta^{-kd/2}\left(m\beta^{-1/2} + \sum_{i=1}^m |v_i|\right) e^{-\beta E_m(Z_m)} |g_{N,m+k}|_{N,\beta,m+k}.$$

Let us now define mild solutions to the BBGKY hierarchy:

Definition 5.5. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $G_{N,0} = \left(g_{N,s,0}\right) \in X_{N,\beta_0,\mu_0}$. A map $\boldsymbol{G_N} = \left(g_{N,s}\right)_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution of the BBGKY hierarchy in $[0,T]$, with initial data $G_{N,0}$, if it satisfies

$$\boldsymbol{G_N}(t) = \mathcal{T}^t G_{N,0} + \int_0^t \mathcal{T}^{t-\tau} \mathcal{C}_N \boldsymbol{G_N}(\tau)\, d\tau,$$

where, given $\beta > 0$, $\mu \in \mathbb{R}$ and $G_N = (g_{N,s})_{s\in\mathbb{N}} \in X_{N,\beta,\mu}$, we write

$$\mathcal{C}_N G_N := (\mathcal{C}^2_N + \mathcal{C}^3_N) G_N, \quad \mathcal{C}^2_N G_N := \left(\mathcal{C}^N_{s,s+1} g_{N,s+1}\right)_{s\in\mathbb{N}}, \quad \mathcal{C}^3_N G_N := \left(\mathcal{C}^N_{s,s+2} g_{N,s+2}\right)_{s\in\mathbb{N}},$$

and $\mathcal{T}^t$ is given by (5.1).

Using Lemma 5.4, we obtain the following a-priori bounds:

Lemma 5.6. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\boldsymbol{\beta}_\lambda, \boldsymbol{\mu}_\lambda : [0,T] \to \mathbb{R}$ given by*

$$\boldsymbol{\beta}_\lambda(t) = \beta_0 - \lambda t, \quad \boldsymbol{\mu}_\lambda(t) = \mu_0 - \lambda t. \tag{5.2}$$

Then for any $\mathcal{F}(t) \subseteq [0,t]$ measurable, $\boldsymbol{G_N} = \left(g_{N,s}\right)_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}$ and $k \in \{1,2\}$, the following bounds hold:

$$\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau} \mathcal{C}^{k+1}_N \boldsymbol{G_N}(\tau)\, d\tau\right|\right|\right|_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} \leq C_{k+1} |||\boldsymbol{G_N}|||_{N,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}, \tag{5.3}$$

$$C_{k+1} = C_{k+1}(d, \beta_0, \mu_0, T, \lambda) = C_d \lambda^{-1} e^{-k\boldsymbol{\mu}_\lambda(T)} \boldsymbol{\beta}_\lambda^{-kd/2}(T)\left(1 + \boldsymbol{\beta}_\lambda^{-1/2}(T)\right). \tag{5.4}$$

Proof. For the proof of (5.3) for $k = 1$, see Lemma 5.3.1 from [18], and for the proof for $k = 2$, see Lemma 6.4 from [2]. □

Choosing $\lambda = \beta_0/2T$, Lemma 5.6 implies well-posedness of the BBGKY hierarchy up to short time. The proof follows similar steps to the proof of Theorem 6 from [18] and Theorem 6.4.1 from [2].

Theorem 5.7. *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial datum $F_{N,0} = (f^{(s)}_{N,0})_{s\in\mathbb{N}} \in X_{N,\beta_0,\mu_0}$, there is unique mild solution $\boldsymbol{F_N} = (f^{(s)}_N)_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the BBGKY hierarchy in $[0,T]$ for the functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ given by*

$$\boldsymbol{\beta}(t) = \beta_0 - \frac{\beta_0}{2T} t, \quad \boldsymbol{\mu}(t) = \mu_0 - \frac{\beta_0}{2T} t. \tag{5.5}$$

The solution $\boldsymbol{F_N}$ satisfies the bound

$$|||\boldsymbol{F_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|F_{N,0}\|_{N,\beta_0,\mu_0}. \tag{5.6}$$

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Moreover, for any $\mathcal{F}(t) \subseteq [0,t]$ measurable and $k \in \{1,2\}$, the following bound holds:

$$\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{T}^{t-\tau} C_N^{k+1} \boldsymbol{G}_N(\tau)\,d\tau\right|\right|\right|_{N,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{16}|||G_N|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}}, \quad \forall G_N \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}. \tag{5.7}$$

The time T is explicitly given by

$$T \simeq \beta_0\left(e^{-\mu_0-\frac{\beta_0}{2}}(\frac{\beta_0}{2})^{-d/2} + e^{-2\mu_0-\beta_0}(\frac{\beta_0}{2})^{-d}\right)^{-1}\left(1+(\frac{\beta_0}{2})^{-1/2}\right)^{-1}. \tag{5.8}$$

5.2. *LWP for the Boltzmann hierarchy*

Similary to Subsection 5.1, here we establish a-priori bounds and local well-posedness for the Boltzmann hierarchy. Without loss of generality, we will omit the proofs since they are identical to the BBGKY hierarchy case. Given $s \in \mathbb{N}$ and $\beta > 0$, we define the Banach space

$$X_{\infty,\beta,s} := \left\{g_s \in L^\infty(\mathbb{R}^{2ds}) : |g_s|_{\infty,\beta,s} < \infty\right\},$$

with norm $|g_s|_{\infty,\beta,s} = \sup_{Z_s\in\mathbb{R}^{2ds}} |g_s(Z_s)|e^{\beta E_s(Z_s)}$, where $E_s(Z_s)$ is the kinetic energy of the s-particles given by (3.50).

Remark 5.8. Given $t \in \mathbb{R}$ and $s \in \mathbb{N}$, conservation of energy under the free flow implies that the s-particle free flow operator $S_s^t : X_{\infty,\beta,s} \to X_{\infty,\beta,s}$, given in (3.57), is an isometry; that is,

$$|S_s^t g_s|_{\infty,\beta,s} = |g_s|_{\infty,\beta,s}, \quad \forall g_s \in X_{\infty,\beta,s}.$$

Consider as well $\mu \in \mathbb{R}$. We define the Banach space

$$X_{\infty,\beta,\mu} := \left\{G = (g_s)_{s\in\mathbb{N}} : \|G\|_{\infty,\beta,\mu} < \infty\right\},$$

with norm $\|G\|_{\infty,\beta,\mu} = \sup_{s\in\mathbb{N}} e^{\mu s}|g_s|_{\infty,\beta,s}$.

Remark 5.9. Given $t \in \mathbb{R}$, Remark 5.8 implies that the map $\mathcal{S}^t : X_{\infty,\beta,\mu} \to X_{\infty,\beta,\mu}$ given by

$$\mathcal{S}^t G := \left(S_s^t g_s\right)_{s\in\mathbb{N}}, \tag{5.9}$$

is an isometry; that is, $\|\mathcal{S}^t G\|_{\infty,\beta,\mu} = \|G\|_{\infty,\beta,\mu}$, for any $G \in X_{\infty,\beta,\mu}$.

Finally, given $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta},\boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$, we define the Banach space

$$\boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} := C^0\left([0,T], X_{\infty,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\right),$$

with norm $|||\boldsymbol{G}|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} = \sup_{t\in[0,T]} \|\boldsymbol{G}(t)\|_{\infty,\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}$.

Proposition 5.10. *Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta},\boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \boldsymbol{\beta}(0)$, $\boldsymbol{\beta}(T) > 0$ $\mu_0 = \boldsymbol{\mu}(0)$. Then for any $G = (g_s)_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$, the following estimates hold:*

1. $|||G|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \|G\|_{\infty,\beta_0,\mu_0}$.
2. $\left|\left|\left|\int_0^t \mathcal{S}^\tau G\,d\tau\right|\right|\right|_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \leq T\|G\|_{\infty,\beta_0,\mu_0}$.

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Similarly to Lemma 5.4, we obtain the following:

Lemma 5.11. *Let $m \in \mathbb{N}$ and $\beta > 0$. For any $Z_m \in \mathbb{R}^{2dm}$ and $k \in \{1,2\}$, the following continuity estimate holds:*

$$\left|\mathcal{C}^{\infty}_{m,m+k}g_{m+k}(Z_m)\right| \lesssim \beta^{-kd/2}\left(m\beta^{-1/2} + \sum_{i=1}^{m}|v_i|\right)e^{-\beta E_m(Z_m)}|g_{m+k}|_{\infty,\beta,m+k}. \tag{5.10}$$

Let us now define mild solutions to the Boltzmann hierarchy:

Definition 5.12. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and the decreasing functions $\boldsymbol{\beta},\boldsymbol{\mu} : [0,T] \to \mathbb{R}$ with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $G_0 = \left(g_{s,0}\right) \in X_{\infty,\beta_0,\mu_0}$. A map $\boldsymbol{G} = (g_s)_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution of the Boltzmann hierarchy in $[0,T]$, with initial data G_0, if it satisfies

$$\boldsymbol{G}(t) = \mathcal{S}^t G_0 + \int_0^t \mathcal{S}^{t-\tau}\mathcal{C}_\infty \boldsymbol{G}(\tau)\,d\tau,$$

where, given $\beta > 0$, $\mu \in \mathbb{R}$ and $\widetilde{G} = (\widetilde{g}_s)_{s\in\mathbb{N}} \in X_{\infty,\beta,\mu}$, we write

$$\mathcal{C}_\infty G := (\mathcal{C}^2_\infty + \mathcal{C}^3_\infty)G, \quad \mathcal{C}^2_\infty G := \left(\mathcal{C}^{\infty}_{s,s+1}g_{s+1}\right)_{s\in\mathbb{N}}, \quad \mathcal{C}^3_\infty G := \left(\mathcal{C}^{\infty}_{s,s+2}g_{s+2}\right)_{s\in\mathbb{N}},$$

and $\mathcal{S}^t$ is given by (5.9).

Using Lemma 5.11, we obtain the following a-priori bounds:

Lemma 5.13. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0,\beta_0/T)$. Consider the functions $\boldsymbol{\beta}_\lambda, \boldsymbol{\mu}_\lambda : [0,T] \to \mathbb{R}$ given by (5.2). Then for any $\mathcal{F}(t) \subseteq [0,t]$ measurable, $\boldsymbol{G} = (g_s)_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}$ and $k \in \{1,2\}$, the following bound holds:*

$$\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{S}^{t-\tau}\mathcal{C}^{k+1}_\infty \boldsymbol{G}(\tau)\,d\tau\right|\right|\right|_{\infty,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} \le C_{k+1}|||\boldsymbol{G}|||_{\infty,\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}, \tag{5.11}$$

where the constant $C_{k+1} = C_{k+1}(d,\beta_0,\mu_0,T,\lambda)$ is given by (5.4).

Choosing $\lambda = \beta_0/2T$, Lemma 5.13 directly implies well-posedness of the Boltzmann hierarchy up to short time.

Theorem 5.14. *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d,\beta_0,\mu_0) > 0$ such that for any initial datum $F_0 = (f_0^{(s)})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$, there is unique mild solution $\boldsymbol{F} = (f^{(s)})_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the Boltzmann hierarchy in $[0,T]$ for the functions $\boldsymbol{\beta},\boldsymbol{\mu} : [0,T] \to \mathbb{R}$ given by (5.5). The solution $\boldsymbol{F}$ satisfies the bound*

$$|||\boldsymbol{F}|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \le 2\|F_0\|_{\infty,\beta_0,\mu_0}. \tag{5.12}$$

Moreover, for any $\mathcal{F}(t) \subseteq [0,t]$ measurable and $k \in \{1,2\}$, the following bound holds:

$$\left|\left|\left|\int_{\mathcal{F}(t)} \mathcal{S}^{t-\tau}\mathcal{C}^{k+1}_\infty \boldsymbol{G}(\tau)\,d\tau\right|\right|\right|_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}} \le \frac{1}{16}|||G|||_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}, \quad \forall G \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}, \tag{5.13}$$

and the time T is explicitly given by (5.8).

5.3. LWP for the binary-ternary Boltzmann equation and propagation of chaos

Now, we show local well-posedness for the binary-ternary Boltzmann equation and that, for chaotic initial data, their tensorized product produces the unique mild solution of the Boltzmann hierarchy.

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Therefore, uniqueness implies that the mild solution to the Boltzmann hierarchy remains factorized under time evolution, and hence, chaos is propagated in time.

For $\beta > 0$, let us define the Banach space

$$X_{\beta,\mu} := \{g \in L^\infty(\mathbb{R}^{2d}) : |g|_{\beta,\mu} < \infty\},$$

with norm $|g|_{\beta,\mu} = \sup_{(x,v)\in\mathbb{R}^{2d}} |g(x,v)|e^{\mu+\frac{\beta}{2}|v|^2}$. Notice that for any $t \in [0,T]$, the map $S_1^t : X_{\beta,\mu} \to X_{\beta,\mu}$ is an isometry.

Consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions of time with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$ and $\boldsymbol{\mu}(0) = \mu_0$. We define the Banach space

$$\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}} := C^0\big([0,T], X_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}\big),$$

with norm $\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} = \sup_{t\in[0,T]} |\boldsymbol{g}(t)|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)}$. One can see that the following estimate holds:

Remark 5.15. Let $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions with $\beta_0 = \boldsymbol{\beta}(0)$, $\boldsymbol{\beta}(T) > 0$ $\mu_0 = \boldsymbol{\mu}(0)$. Then for any $g \in X_{\beta_0,\mu_0}$, the following estimate holds:

$$\|g\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \le |g|_{\beta_0,\mu_0}.$$

To prove LWP for the binary-ternary Boltzmann equation (1.16), we will need certain continuity estimates on the binary and ternary collisional operators. The binary estimate we provide below is the bilinear analogue of Proposition 5.3.2 in [18]. For the ternary operator, continuity estimates have been derived in [2], Lemma 6.10. Combining these results, we derive continuity estimates for the binary-ternary collisional operator $Q_2 + Q_3$:

Lemma 5.16. *Let $\beta > 0$, $\mu \in \mathbb{R}$. Then for any $g, h \in X_{\beta,\mu}$ and $(x,v) \in \mathbb{R}^{2d}$, the following nonlinear continuity estimate holds:*

$$\begin{aligned}
&\big|[Q_2(g,g) + Q_3(g,g,g)](x,v) - [Q_2(h,h) + Q_3(h,h,h)](x,v)\big| \\
&\quad \lesssim \Big(e^{-2\mu}\beta^{-d/2} + e^{-3\mu}\beta^{-d}\Big)\Big(\beta^{-1/2} + |v|\Big)e^{-\frac{\beta}{2}|v|^2}\big(|g|_{\beta,\mu} + |h|_{\beta,\mu}\big)\big(1 + |g|_{\beta,\mu} + |h|_{\beta,\mu}\big)|g - h|_{\beta,\mu}.
\end{aligned}$$

We define mild solutions to the binary-ternary Boltzmann equation (1.16) as follows:

Definition 5.17. Consider $T > 0$, $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ decreasing functions of time, with $\boldsymbol{\beta}(0) = \beta_0$, $\boldsymbol{\beta}(T) > 0$, $\boldsymbol{\mu}(0) = \mu_0$. Consider also initial data $g_0 \in X_{\beta_0,\mu_0}$. A map $\boldsymbol{g} \in \boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ is a mild solution to the binary-ternary Boltzmann equation (1.16) in $[0,T]$, with initial data $g_0 \in X_{\beta_0,\mu_0}$, if it satisfies

$$\boldsymbol{g}(t) = S_1^t g_0 + \int_0^t S_1^{t-\tau}[Q_2(\boldsymbol{g},\boldsymbol{g}) + Q_3(\boldsymbol{g},\boldsymbol{g},\boldsymbol{g})](\tau)\,d\tau, \tag{5.14}$$

where S_1^t denotes the free flow of one particle given in (3.57).

A similar proof to Lemma 5.6 gives the following:

Lemma 5.18. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ and $\lambda \in (0, \beta_0/T)$. Consider the functions $\boldsymbol{\beta}_\lambda, \boldsymbol{\mu}_\lambda : [0,T] \to \mathbb{R}$ given by (5.2). Then for any $\boldsymbol{g}, \boldsymbol{h} \in \boldsymbol{X}_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}$, the following bounds hold:*

$$\begin{aligned}
&\left\|\int_0^t S_1^{t-\tau}[Q_2(\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h}) + Q_3(\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h})](\tau)\,d\tau\right\|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} \\
&\quad \le C\big(|\boldsymbol{g}|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} + |\boldsymbol{h}|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}\big)\big(1 + |\boldsymbol{g}|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} + |\boldsymbol{h}|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda}\big)|\boldsymbol{g}-\boldsymbol{h}|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda},
\end{aligned}$$

where $C = C(d, \beta_0, \mu_0, T, \lambda) = C_2 + C_3$ and C_2, C_3 are given by (5.4) for $k = 1, 2$, respectively.

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Choosing $\lambda = \beta_0/2T$, this estimate implies local well-posedness of the binary-ternary Boltzmann equation up to short times. Let us write $B_{\boldsymbol{X}_{\beta,\mu}}$ for the unit ball of $\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$.

Theorem 5.19 (LWP for the binary-ternary Boltzmann equation). *Let $\beta_0 > 0$ and $\mu_0 \in \mathbb{R}$. Then there is $T = T(d, \beta_0, \mu_0) > 0$ such that for any initial data $f_0 \in X_{\beta_0,\mu_0}$, with $|f_0|_{\beta_0,\mu_0} \leq 1/2$, there is a unique mild solution $\boldsymbol{f} \in B_{\boldsymbol{X}_{\beta,\mu}}$ to the binary-ternary Boltzmann equation in $[0,T]$ with initial data f_0, where $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ are the functions given by (5.5). The solution $\boldsymbol{f}$ satisfies the bound*

$$\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq 4|f_0|_{\beta_0,\mu_0}. \tag{5.15}$$

Moreover, for any $\boldsymbol{g}, \boldsymbol{h} \in \boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$, the following estimates hold:

$$\begin{aligned}\left\|\int_0^t S_1^{t-\tau}[Q_2(\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h}) + Q_3(\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h},\boldsymbol{g}-\boldsymbol{h})](\tau)\,d\tau\right\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \\ \leq \frac{1}{8}\left(\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} + \|\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}\right)\left(1 + |\boldsymbol{g}|_{\boldsymbol{\beta},\boldsymbol{\mu}} + |\boldsymbol{h}|_{\boldsymbol{\beta},\boldsymbol{\mu}}\right)\|\boldsymbol{g}-\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}.\end{aligned} \tag{5.16}$$

The time T is explicitly given by (5.8).

Proof. Choosing T as in (5.8), we obtain $C(d, \beta_0, \mu_0, T, \beta_0/2T) = 1/8$. Thus, Lemma 5.18 implies estimate (5.16). Therefore, for any $g \in B_{\boldsymbol{X}_{\beta,\mu}}$, using (5.16) for $\boldsymbol{h} = 0$, we obtain

$$\left\|\int_0^t S_1^{t-\tau}[Q_2(\boldsymbol{g},\boldsymbol{g}) + Q_3(\boldsymbol{g},\boldsymbol{g},\boldsymbol{g})](\tau)\,d\tau\right\|_{\boldsymbol{\beta}_\lambda,\boldsymbol{\mu}_\lambda} \leq \frac{1}{8}(1 + \|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}})\|\boldsymbol{g}\|^2_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{4}\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}. \tag{5.17}$$

Let us define the nonlinear operator $\mathcal{L} : \boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}} \to \boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}$ by

$$\mathcal{L}\boldsymbol{g}(t) = S_1^t f_0 + \int_0^t S_1^{t-\tau} Q(\boldsymbol{g},\boldsymbol{g},\boldsymbol{g})(\tau)\,d\tau.$$

By triangle inequality, the fact that the free flow is isometric, Remark 5.15, bound (5.17) and the assumption $|f_0|_{\beta_0,\mu_0} \leq 1/2$, for any $\boldsymbol{g} \in B_{\boldsymbol{X}_{\beta,\mu}}$ and $t \in [0,T]$, we have

$$|\mathcal{L}\boldsymbol{g}|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} \leq |S_1^t f_0|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} + \frac{1}{4}\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} = |f_0|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} + \frac{1}{4}\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq |f_0|_{\beta_0,\mu_0} + \frac{1}{4}\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Thus, $\mathcal{L} : B_{\boldsymbol{X}_{\beta,\mu}} \to B_{\boldsymbol{X}_{\beta,\mu}}$. Moreover, for any $\boldsymbol{g}, \boldsymbol{h} \in B_{\boldsymbol{X}_{\beta,\mu}}$, using (5.16), we obtain

$$\|\mathcal{L}\boldsymbol{g} - \mathcal{L}\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{8}\left(\|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} + \|\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}\right)\left(1 + \|\boldsymbol{g}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} + \|\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}\right)\|\boldsymbol{g}-\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{3}{4}\|\boldsymbol{g}-\boldsymbol{h}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}. \tag{5.18}$$

Therefore, the operator $\mathcal{L} : B_{\boldsymbol{X}_{\beta,\mu}} \to B_{\boldsymbol{X}_{\beta,\mu}}$ is a contraction, so it has a unique fixed point $\boldsymbol{f} \in B_{\boldsymbol{X}_{\beta,\mu}}$ which is clearly the unique mild solution of the binary-ternary Boltzmann equation in $[0,T]$ with initial data f_0.

To prove (5.15), we use the fact that $\boldsymbol{f} = \mathcal{L}\boldsymbol{f}$. Then for any $t \in [0,T]$, triangle inequality, definition of $\mathcal{L}$, estimate (5.18) (for $\boldsymbol{g} = \boldsymbol{f}$ and $\boldsymbol{g} = 0$), free flow being isometric, and Remark 5.15 yield

$$\begin{aligned}|\boldsymbol{f}|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} = |\mathcal{L}\boldsymbol{f}|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} &\leq |\mathcal{L}\boldsymbol{0}|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} + |\mathcal{L}\boldsymbol{f} - \mathcal{L}\boldsymbol{0}|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} \leq |S_1^t f_0|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} + \frac{3}{4}\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \\ &= |f_0|_{\boldsymbol{\beta}(t),\boldsymbol{\mu}(t)} + \frac{3}{4}\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq |f_0|_{\beta_0,\mu_0} + \frac{3}{4}\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}},\end{aligned}$$

and thus, $\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq |f_0|_{\beta_0,\mu_0} + \frac{3}{4}\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}}$, and (5.15) follows. ◻

We can now prove that chaos is propagated by the Boltzmann hierarchy.

Theorem 5.20 (Propagation of chaos). *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ be the time given in (5.8), and $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ the functions defined by (5.5). Consider $f_0 \in X_{\beta_0,\mu_0}$ with $|f_0|_{\beta_0,\mu_0} \leq 1/2$. Assume $\boldsymbol{f} \in B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ is the corresponding mild solution of the binary-ternary Boltzmann equation in $[0,T]$, with initial data f_0 given by Theorem 5.19. Then the following hold:*

1. $F_0 = (f_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$.
2. $\boldsymbol{F} = (\boldsymbol{f}^{\otimes s})_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$.
3. $\boldsymbol{F}$ *is the unique mild solution of the Boltzmann hierarchy in* $[0,T]$*, with initial data* F_0.

Proof. *(i)* is trivially verified by the bound on the initial data (5.15) and the definition of the norms. By the same bound again, we may apply Theorem 5.19 to obtain the unique mild solution $\boldsymbol{f} \in B_{\boldsymbol{X}_{\boldsymbol{\beta},\boldsymbol{\mu}}}$ of the corresponding binary-ternary Boltzmann equation. Since $\|\boldsymbol{f}\|_{\boldsymbol{\beta},\boldsymbol{\mu}} \leq 1$, the definition of the norms directly implies *(ii)*. It is also straightforward to verify that $\boldsymbol{F}$ is a mild solution of the Boltzmann hierarchy in $[0,T]$, with initial data F_0. Uniqueness of the mild solution to the Boltzmann hierarchy, obtained by Theorem 5.14, implies that $\boldsymbol{F}$ is the unique mild solution. □

6. Convergence Statement

In this section, we define an appropriate notion of convergence – namely, convergence in observables – and we state the main result of this paper. While our convergence result is valid for a general type of Boltzmann initial data and approximation by BBGKY hierarchy initial data (see Definition 6.1), we also provide a rate of convergence in the case of chaotic Boltzmann initial data and initial approximation by conditioned BBGKY hierarchy initial data (introduced in Definition 6.4).

Throughout this section, we consider $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24). We will also use the phase space $\mathcal{D}_{m,\epsilon_2,\epsilon_3}$ of m-particles of radius ϵ_2 and of interaction zone ϵ given by (3.5) and the functional spaces of Section 5.

6.1. Approximation of Boltzmann initial data

This subsection focuses on introducing relevant types of initial data. First, we define the general notion of BBGKY hierarchy sequences approximating Boltzmann hierarchy initial data. Then we show that chaotic initial data produced by tensorized probability densities are approximated by conditioned BBGKY hierarchy sequences in the scaling (4.24).

Definition 6.1. Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and $G_0 = (g_{s,0})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$. A sequence $G_{N,0} = (g_{N,s,0})_{s\in\mathbb{N}} \in X_{N,\beta_0,\mu_0}$ is called a BBGKY hierarchy sequence approximating G_0 if the following conditions hold:

1. $\sup_{N\in\mathbb{N}} \|G_{N,0}\|_{N,\beta_0,\mu_0} < \infty$.
2. For any $s \in \mathbb{N}$, there holds $\lim_{N\to\infty} \|g_{N,s,0} - g_{s,0}\|_{L^\infty(\mathcal{D}_{s,\epsilon_2,\epsilon_3})} = 0$.

Remark 6.2. Every $G_0 = (g_{s,0})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$ has a BBGKY hierarchy approximating sequence. Indeed, it is straightforward to verify that the sequence $G_{N,0} = (g_{N,s,0})_{s\in\mathbb{N}}$ given by $g_{N,s,0} = \mathbb{1}_{\mathcal{D}_{s,\epsilon_2,\epsilon_3}} g_{s,0}$ satisfies the properties stated above in the scaling (4.24).

Especially meaningful initial data, corresponding to initial independence between particles, are given below:

Remark 6.3. Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density, that is, $g_0 > 0$ a.e. and $\int_{\mathbb{R}^{2d}} g_0(x,v)\,dx\,dv = 1$ and assume that $\|g_0\|_{\beta_0,\mu_0+1} \leq 1$. Then one can easily see that the chaotic configuration $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1} \subseteq X_{\infty,\beta_0,\mu_0}$. This type of initial data, corresponding to tensorized initial measures, will lead to the binary-ternary Boltzmann equation (1.16). In fact, we will see that one can approximate tensorized initial data in the scaling (4.24) by conditioned BBGKY hierarchy initial data which are defined below.

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Definition 6.4. Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density and denote $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1}$. We define the conditioned BBGKY hierarchy sequence $G_{N,0} = (g_{N,0}^{(s)})_{s\in\mathbb{N}}$ of G_0 as

$$g_{N,0}^{(s)}(X_s,V_s) = \begin{cases} \mathcal{Z}_N^{-1} \displaystyle\int_{\mathbb{R}^{2d(N-s)}} \mathbb{1}_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}} g_0^{\otimes N}(X_s, x_{s+1},\ldots,x_N, V_s, v_{s+1},\ldots,v_N) \\ \qquad dx_{s+1}\,dv_{s+1}\ldots dx_N\,dv_N, \quad 1 \le s < N \\ \mathcal{Z}_N^{-1}\mathbb{1}_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}} g_0^{\otimes N}(Z_N), \quad s = N, \\ 0, \quad s > N. \end{cases} \tag{6.1}$$

where the normalization is preserved by the introduction of the partition function

$$\mathcal{Z}_m = \int_{\mathbb{R}^{2dm}} \mathbb{1}_{\mathcal{D}_{m,\epsilon_2,\epsilon_3}} g_0^{\otimes m}(X_m,V_m)\,dX_m\,dV_m, \quad m \in \mathbb{N}.$$

Notice that since g_0 is a.e. positive and integrates to 1, we have $0 < Z_m < 1$ for all $m \in \mathbb{N}$.

In fact, for tensorized initial data, the conditioned BBGKY hierarchy sequence is an approximating sequence (according to Definition 6.1). This will be important to obtain a rate of convergence to the solution of the binary-ternary Boltzmann equation (1.16) (see Corollary 6.10 for more details). For the binary Boltzmann equation, such a result was proved in, for example, [18], obtaining an $O(\epsilon_2)$ rate of convergence, where ϵ_2 is the radius of the hard spheres. In [5], a similar result with rate of convergence $O(\epsilon_3^{1/2})$ was proved when merely ternary interactions of interaction zone ϵ_3 were taken into account. We note that the slower convergence rate of the ternary model is due to the scaling $N\epsilon^{d-\frac{1}{2}} \simeq 1$ which is different that the Boltzmann-Grad scaling $N\epsilon_2^{d-1} \simeq 1$ of the hard spheres. In this paper, where binary and ternary interactions coexist in the scaling (4.24), we are able to deduce the slower rate of convergence $O(\epsilon_3^{1/2})$. The absence of ϵ_2 in the estimates is due to the fact $\epsilon_2 << \epsilon_3$.

Proposition 6.5. *Let $g_0 \in X_{\beta_0,\mu_0+1}$ be a positive probability density with $|g_0|_{\beta_0,\mu_0+1} \le 1$ and $G_0 = (g_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1} \subseteq X_{\infty,\beta_0,\mu_0}$. Let $G_{N,0} = (g_{N,0}^{(s)})_{s\in\mathbb{N}}$ be the conditioned BBGKY hierarchy sequence of the tensorized initial data G_0 given in Definition 6.4. Then $G_{N,0}$ is a BBGKY hierarchy sequence approximating G_0 (in the sense of Definition 6.1) in the scaling (4.24). In particular, for all (N,ϵ) in the scaling (4.24) with N large enough (or equivalently ϵ small enough), there holds the estimate*

$$\|g_{N,0}^{(s)} - g_0^{\otimes s}\|_{L^\infty(\mathcal{D}_{s,\epsilon_2,\epsilon_3})} \le C_{d,s,\beta_0,\mu_0}\epsilon_3^{1/2}\|G_0\|_{\infty,\beta_0,\mu_0}. \tag{6.2}$$

Proof. The proof comes by following a similar argument as in Section 6 of [5] to estimate first the partition functions and then the rate of convergence. The only difference is that one has to incorporate binary interactions in the phase space, which is achieved by decomposing the phase space as

$$\begin{aligned} \mathbb{1}_{\mathcal{D}_{N,\epsilon_2,\epsilon_3}}(Z_N) =& \mathbb{1}_{\mathcal{D}_{s,\epsilon_2,\epsilon_3}}(Z_s) \prod_{1\le i\le s<j\le N} \mathbb{1}_{|x_i-x_j|>\epsilon_2}(x_i,x_j) \prod_{1\le i<j\le s<k\le N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon_3^2}(x_i,x_j,x_k) \\ & \prod_{1\le i\le s<j<k\le N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon_3^2}(x_i,x_j,x_k) \prod_{s+1\le i<j<k\le N} \mathbb{1}_{|x_i-x_j|^2+|x_i-x_k|^2>2\epsilon_3^2}(x_i,x_j,x_k), \end{aligned}$$

and using scaling (4.24). □

6.2. Convergence in observables

Now, we define the convergence in observables. Given $s \in \mathbb{N}$, we use the space of test continuous and compactly supported functions in velocities $C_c(\mathbb{R}^{ds})$.

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Definition 6.6. Consider $T > 0$, $s \in \mathbb{N}$ and $g_s \in C^0([0,T], L^\infty(\mathbb{R}^{2ds}))$. Given a test function $\phi_s \in C_c(\mathbb{R}^{ds})$, we define the s-observable functional as $I_{\phi_s} g_s(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) g_s(t, X_s, V_s)\, dV_s$.

Before giving the definition of convergence in observables, we start with some definitions on the configurations we are using. Given $m \in \mathbb{N}$ and $\sigma > 0$, we define the set of well-separated spatial configurations

$$\Delta_m^X(\sigma) = \{\widetilde{X}_m \in \mathbb{R}^{dm} : |\widetilde{x}_i - \widetilde{x}_j| > \sigma, \quad \forall 1 \le i < j \le m\}, \quad m \ge 2, \quad \Delta_1^X(\sigma) = \mathbb{R}^{2d}, \tag{6.3}$$

and the set of well separated configurations

$$\Delta_m(\sigma) = \Delta_m^X(\sigma) \times \mathbb{R}^{dm}. \tag{6.4}$$

Definition 6.7. Let $T > 0$. For each $N \in \mathbb{N}$, consider $\boldsymbol{G_N} = (g_{N,s})_{s\in\mathbb{N}} \in \prod_{s=1}^\infty C^0([0,T], L^\infty(\mathbb{R}^{2ds}))$ and $\boldsymbol{G} = (g_s)_{s\in\mathbb{N}} \in \prod_{s=1}^\infty C^0([0,T], L^\infty(\mathbb{R}^{2ds}))$. We say that the sequence $(\boldsymbol{G_N})_{N\in\mathbb{N}}$ converges in observables to $\boldsymbol{G}$, and write

$$\boldsymbol{G_N} \overset{\sim}{\longrightarrow} \boldsymbol{G},$$

if for any $\sigma > 0$, $s \in \mathbb{N}$, and $\phi_s \in C_c(\mathbb{R}^{ds})$, we have

$$\lim_{N\to\infty} \| I_{\phi_s} g_{N,s}(t) - I_{\phi_s} g_s(t) \|_{L^\infty(\Delta_s^X(\sigma))} = 0, \quad \text{uniformly in } [0,T].$$

6.3. *Statement of the main result*

We are now in the position to state our main result.

Theorem 6.8 (Convergence). *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$ and T be given by (5.8). Consider the Boltzmann hierarchy initial data $F_0 = (f_0^{(s)})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0}$, and let $(F_{N,0})_{N\in\mathbb{N}}$ be a BBGKY hierarchy sequence approximating F_0. Assume that*

∘ *For each N, $\boldsymbol{F_N} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$ is the mild solution of the BBGKY hierarchy (4.14) with initial data $F_{N,0}$ in $[0,T]$.*

∘ *$\boldsymbol{F} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ is the mild solution of the Boltzmann hierarchy (4.36) with initial data F_0 in $[0,T]$.*

∘ *F_0 satisfies the following uniform continuity condition: There exists $C > 0$ such that, for any $\zeta > 0$, there is $q = q(\zeta) > 0$ such that for all $s \in \mathbb{N}$, and for all $Z_s, Z_s' \in \mathbb{R}^{2ds}$ with $|Z_s - Z_s'| < q$, we have*

$$|f_0^{(s)}(Z_s) - f_0^{(s)}(Z_s')| < C^{s-1}\zeta. \tag{6.5}$$

Then $\boldsymbol{F_N} \overset{\sim}{\longrightarrow} \boldsymbol{F}$.

Remark 6.9. To prove Theorem 6.8, it suffices to prove

$$\| I_s^N(t) - I_s^\infty(t) \|_{L^\infty(\Delta_s^X(\sigma))} \overset{N\to\infty}{\longrightarrow} 0, \text{ uniformly in } [0,T],$$

for any $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ and $\sigma > 0$, where

$$I_s^N(t)(X_s) := I_{\phi_s} f_N^{(s)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_N^{(s)}(t, X_s, V_s)\, dV_s, \tag{6.6}$$

$$I_s^\infty(t)(X_s) := I_{\phi_s} f^{(s)}(t)(X_s) = \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s)}(t, X_s, V_s)\, dV_s. \tag{6.7}$$

The following Corollary of Theorem 6.8 justifies the derivation of the binary-ternary Boltzmann equation from finitely many particle systems.

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Corollary 6.10. *Let $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, and T be given by (5.8). Let $f_0 \in X_{\beta_0,\mu_0+1}$ be a Hölder continuous $C^{0,\gamma}$, $\gamma \in (0,1]$ probability density with $|f_0|_{\beta_0,\mu_0+1} \leq 1/2$. Let us write $F_0 = (f_0^{\otimes s})_{s\in\mathbb{N}} \in X_{\infty,\beta_0,\mu_0+1}$ and let $F_{N,0} = (f_{N,0}^{(s)})_{s\in\mathbb{N}}$ be the conditioned BBGKY hierarchy sequence given in Definition 6.4 approximating the tensorized data F_0. Then for any $\sigma > 0$, $s \in \mathbb{N}$ and $\phi_s \in C_c(\mathbb{R}^{ds})$, we have the rate of convergence*

$$\|I_{\phi_s} f_N^{(s)}(t) - I_{\phi_s} f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\sigma))} = O(\epsilon^r), \quad \textit{uniformly in } [0,T], \tag{6.8}$$

for any $0 < r < \min\{1/2, \gamma\}$, where $\boldsymbol{F_N} = (f_N^{(s)})_{s\in\mathbb{N}} \in \boldsymbol{X_{N,\beta,\mu}}$ is the mild solution of the BBGKY hierarchy (4.14) in $[0,T]$ with initial data $F_{N,0}$ and f is the mild solution to the ternary Boltzmann equation (1.16) in $[0,T]$, with initial data f_0.

7. Reduction to term by term convergence

In this section, we reduce the proof of Theorem 6.8 to term by term convergence after truncating the observables. After introducing the necessary combinatorial notation to take care of all the possible collision sequences occurring, the idea of the truncation is essentially the same as in [18, 2], and it relies on the local estimates developed in Section 5. For this reason, we illustrate the similarities by providing the proof of the first estimate and omit the proofs of the rest of the estimates.

Throughout this section, we consider $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, the functions $\boldsymbol{\beta}, \boldsymbol{\mu} : [0,T] \to \mathbb{R}$ defined by (5.5), $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) and initial data $F_{N,0} \in X_{N,\beta_0,\mu_0}$, $F_0 \in X_{\infty,\beta_0,\mu_0}$. Let $\boldsymbol{F_N} = (f_N^{(s)})_{s\in\mathbb{N}} \in \boldsymbol{X}_{N,\boldsymbol{\beta},\boldsymbol{\mu}}$, $\boldsymbol{F} = (f^{(s)})_{s\in\mathbb{N}} \in \boldsymbol{X}_{\infty,\boldsymbol{\beta},\boldsymbol{\mu}}$ be the mild solutions of the corresponding BBGKY and Boltzmann hierarchies, respectively, in $[0,T]$, given by Theorems 5.7 and Theorem 5.14. Let us note that by (5.5), we obtain

$$\boldsymbol{\beta}(T) = \frac{\beta_0}{2}, \quad \boldsymbol{\mu}(T) = \mu_0 - \frac{\beta_0}{2}, \tag{7.1}$$

and thus, $\boldsymbol{\beta}(T), \boldsymbol{\mu}(T)$ do not depend on T.

For convenience, we introduce the following notation. Given $k \in \mathbb{N}$ and $t \geq 0$, we denote

$$\mathcal{T}_k(t) := \left\{(t_1, ..., t_k) \in \mathbb{R}^k : 0 \leq t_k < ... \leq t_1 \leq t\right\}. \tag{7.2}$$

Since the collisions happening can be either binary or ternary, we will introduce some additional notation to keep track of the collision sequences. In particular, given $k \geq 1$, we denote

$$S_k := \{\sigma = (\sigma_1, ..., \sigma_k) : \sigma_i \in \{1,2\}, \quad \forall i = 1, ..., k\}. \tag{7.3}$$

Notice that the cardinality of S_k is given by

$$|S_k| = 2^k, \quad \forall k \geq 1. \tag{7.4}$$

Given $k \in \mathbb{N}$ and $\sigma \in S_k$, for any $1 \leq \ell \leq k$, we write

$$\widetilde{\sigma}_\ell = \sum_{i=1}^{\ell} \sigma_i. \tag{7.5}$$

We also write $\widetilde{\sigma}_0 := 0$. Notice that

$$k \leq \widetilde{\sigma}_k \leq 2k, \quad \forall k \in \mathbb{N}. \tag{7.6}$$

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7.1. Series expansion

Now, we make a series expansion for the mild solution $\boldsymbol{F_N} = (f_N^{(s)})_{s\in\mathbb{N}}$ of the BBGKY hierarchy with respect to the initial data $F_{N,0}$. By Definition 5.5, for any $\in \mathbb{N}$, we have Duhamel's formula:

$$f_N^{(s)}(t) = T_s^t f_{N,0}^{(s)} + \int_0^t T_s^{t-t_1}\left[\mathcal{C}_{s,s+1}^N f_N^{(s+1)} + \mathcal{C}_{s,s+2}^N f_N^{(s+2)}\right](t_1)\,dt_1.$$

Let $n \in \mathbb{N}$. Iterating n-times Duhamel's formula, we obtain

$$f_N^{(s)}(t) = \sum_{k=0}^n f_N^{(s,k)}(t) + R_N^{(s,n+1)}(t), \tag{7.7}$$

where we use the notation

$$f_N^{(s,k)}(t) := \sum_{\sigma\in S_k} f_N^{(s,k,\sigma)}(t), \text{ for } 1 \le k \le n, \quad f_N^{(s,0)}(t) := T_s^t f_{N,0}^{(s)}. \tag{7.8}$$

$$f_N^{(s,k,\sigma)}(t) = \int_{\mathcal{T}_k(t)} T_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^N T_{s+\widetilde{\sigma}_1}^{t_1-t_2}\mathcal{C}_{s+\widetilde{\sigma}_1,s+\widetilde{\sigma}_2}^N T_{s+\widetilde{\sigma}_2}^{t_2-t_3}\ldots T_{s+\widetilde{\sigma}_{k-1}}^{t_{k-1}-t_k}\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^N T_{s+\widetilde{\sigma}_k}^{t_k} f_{N,0}^{(s+\widetilde{\sigma}_k)}\,dt_k\ldots dt_1, \tag{7.9}$$

$$R_N^{(s,n+1)}(t) := \sum_{\sigma\in S_{n+1}} R_N^{(s,n+1,\sigma)}(t), \tag{7.10}$$

$$\begin{aligned} R_N^{(s,n+1,\sigma)}(t) := \int_{\mathcal{T}_{n+1}(t)} & T_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^N T_{s+\widetilde{\sigma}_1}^{t_1-t_2}\mathcal{C}_{s+\widetilde{\sigma}_1,s+\widetilde{\sigma}_2}^N T_{s+\widetilde{\sigma}_2}^{t_2-t_3}\ldots \\ T_{s+\widetilde{\sigma}_{n-1}}^{t_{n-1}-t_n}\mathcal{C}_{s+\widetilde{\sigma}_{n-1},s+\widetilde{\sigma}_n}^N T_{s+\widetilde{\sigma}_n}^{t_n-t_{n+1}}\mathcal{C}_{s+\widetilde{\sigma}_n,s+\widetilde{\sigma}_{n+1}}^N & f_N^{(s+\widetilde{\sigma}_{n+1})}(t_{n+1})\,dt_{n+1}\,dt_n\ldots dt_1. \end{aligned} \tag{7.11}$$

One can make a similar series expansion for the Boltzmann hierarchy. By Definition 5.5, for any $\in \mathbb{N}$, we have Duhamel's formula:

$$f^{(s)}(t) = S_s^t f_0^{(s)} + \int_0^t S_s^{t-t_1}\left[\mathcal{C}_{s,s+1}^\infty f^{(s+1)} + \mathcal{C}_{s,s+2}^\infty f^{(s+2)}\right](t_1)\,dt_1.$$

Iterating n-times Duhamel's formula, we obtain

$$f^{(s)}(t) = \sum_{k=0}^n f^{(s,k)}(t) + R^{(s,n+1)}(t), \tag{7.12}$$

where we use the notation

$$f^{(s,k)}(t) := \sum_{\sigma\in S_k} f^{(s,k,\sigma)}(t), \text{ for } 1 \le k \le n, \quad f^{(s,0)}(t) := S_s^t f_0^{(s)}. \tag{7.13}$$

$$f^{(s,k,\sigma)}(t) := \int_{\mathcal{T}_k(t)} S_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^\infty S_{s+\widetilde{\sigma}_1}^{t_1-t_2}\mathcal{C}_{s+\widetilde{\sigma}_1,s+\widetilde{\sigma}_2}^\infty S_{s+\widetilde{\sigma}_2}^{t_2-t_3}\ldots S_{s+\widetilde{\sigma}_{k-1}}^{t_{k-1}-t_k}\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^\infty S_{s+\widetilde{\sigma}_k}^{t_k} f_0^{(s+\widetilde{\sigma}_k)}\,dt_k\ldots dt_1, \tag{7.14}$$

$$R^{(s,n+1)}(t) := \sum_{\sigma\in S_{n+1}} R^{(s,n+1,\sigma)}(t), \tag{7.15}$$

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$$R^{(s,n+1,\sigma)}(t) := \int_{\mathcal{T}_{n+1}(t)} S_s^{t-t_1} \mathcal{C}^\infty_{s,s+\widetilde{\sigma}_1} S_{s+\widetilde{\sigma}_1}^{t_1-t_2} \mathcal{C}^\infty_{s+\widetilde{\sigma}_1,s+\widetilde{\sigma}_2} S_{s+\widetilde{\sigma}_2}^{t_2-t_3} \cdots$$
$$S_{s+\widetilde{\sigma}_{n-1}}^{t_{n-1}-t_n} \mathcal{C}^\infty_{s+\widetilde{\sigma}_{n-1},s+\widetilde{\sigma}_n} S_{s+\widetilde{\sigma}_n}^{t_n-t_{n+1}} \mathcal{C}^\infty_{s+\widetilde{\sigma}_n,s+\widetilde{\sigma}_{n+1}} f^{(s+\widetilde{\sigma}_{n+1})}(t_{n+1})\, dt_{n+1}\, dt_n \ldots dt_1. \tag{7.16}$$

Given $\phi_s \in C_c(\mathbb{R}^{ds})$ and $k \in \mathbb{N}$, let us denote

$$I_{s,k}^N(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_N^{(s,k)}(t, X_s, V_s)\, dV_s, \tag{7.17}$$

$$I_{s,k}^\infty(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f^{(s,k)}(t, X_s, V_s)\, dV_s. \tag{7.18}$$

We obtain the following estimates:

Lemma 7.1. *For any $s, n \in \mathbb{N}$ and $t \in [0, T]$, the following estimates hold:*

$$\|I_s^N(t) - \sum_{k=0}^n I_{s,k}^N(t)\|_{L^\infty_{X_s}} \le C_{s,\beta_0,\mu_0} \|\phi_s\|_{L^\infty_{V_s}} 4^{-n} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\|I_s^\infty(t) - \sum_{k=0}^n I_{s,k}^\infty(t)\|_{L^\infty_{X_s}} \le C_{s,\beta_0,\mu_0} \|\phi_s\|_{L^\infty_{V_s}} 4^{-n} \|F_0\|_{\infty,\beta_0,\mu_0},$$

where the observables I_s^N, I_s^∞ are defined in (6.6)–(6.7).

Proof. Fix $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $t \in [0, T]$ and $\sigma \in S_{n+1}$. We repeatedly use estimate (5.7) of Theorem 5.7, for $k = 1$ if $\sigma_i = 1$ or for $k = 2$ if $\sigma_i = 2$, to obtain

$$e^{\beta(t)E_s(Z_s)+s\mu(t)} |R_N^{(s,n+1,\sigma)}(t, X_s, V_s)| \le 8^{-(n+1)} |||\boldsymbol{F_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}},$$

so adding for all $\sigma \in S_{n+1}$, using (7.4), (5.6) and the definition of the norms, we take

$$|\phi_s(V_s) R_N^{(s,n+1)}(t, X_s, V_s)| \lesssim 4^{-(n+1)} e^{-s\mu(t)} \|\phi_s\|_{L^\infty_{V_s}} |||\boldsymbol{F_N}|||_{N,\boldsymbol{\beta},\boldsymbol{\mu}} e^{-\beta(t)E_s(Z_s)}$$
$$\le 4^{-n} e^{-s\mu(T)} \|\phi_s\|_{L^\infty_{V_s}} \|F_{N,0}\|_{N,\beta_0,\mu_0} e^{-\beta(T)E_s(Z_s)}.$$

Thus, integrating with respect to velocities and recalling (7.7), (7.17), (7.1), we obtain

$$|I_s^N(t)(X_s) - \sum_{k=0}^n I_{s,k}^N(t)(X_s)| \le C_{s,\mu_0} \|\phi_s\|_{L^\infty_{V_s}} 4^{-n} \|F_{N,0}\|_{N,\beta_0,\mu_0} \int_{\mathbb{R}^{ds}} e^{-\beta(T)E_s(Z_s)}\, dV_s$$
$$\le C_{s,\beta_0,\mu_0} \|\phi_s\|_{L^\infty_{V_s}} 4^{-n} \|F_{N,0}\|_{N,\beta_0,\mu_0}.$$

For the Boltzmann hierarchy, we follow a similar argument using estimates (5.13) and (5.12) instead. □

7.2. High energy truncation

We will now truncate energies, so that we can focus on bounded energy domains. Let us fix $s, n \in \mathbb{N}$ and $R > 1$. As usual, we denote B_R^{2d} to be the $2d$-ball of radius R centered at the origin.

We first define the truncated BBGKY hierarchy and Boltzmann hierarchy collisional operators. For $\ell \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{C}^{N,R}_{\ell,\ell+1} g_{l+1} &:= \mathcal{C}^N_{\ell,\ell+1}(g_{l+1} \mathbb{1}_{[E_{\ell+1} \le R^2]}), & \mathcal{C}^{N,R}_{\ell,\ell+2} g_{l+2} &:= \mathcal{C}^N_{\ell,\ell+2}(g_{l+2} \mathbb{1}_{[E_{\ell+2} \le R^2]}), \\ \mathcal{C}^{\infty,R}_{\ell,\ell+1} g_{l+1} &:= \mathcal{C}^\infty_{\ell,\ell+1}(g_{l+1} \mathbb{1}_{[E_{\ell+1} \le R^2]}), & \mathcal{C}^{\infty,R}_{\ell,\ell+2} g_{l+2} &:= \mathcal{C}^\infty_{\ell,\ell+2}(g_{l+2} \mathbb{1}_{[E_{\ell+2} \le R^2]}). \end{aligned} \tag{7.19}$$

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For the BBGKY hierarchy, we define

$$f_{N,R}^{(s,k)}(t,Z_s) := \sum_{\sigma\in S_k} f_{N,R}^{(s,k,\sigma)}(t,Z_s), \text{ for } 1\leq k\leq n, \quad f_{N,R}^{(s,0)}(t,Z_s) := T_s^t(f_{N,0}\mathbb{1}_{[E_s\leq R^2]})(Z_s),$$

where given $k\geq 1$ and $\sigma\in S_k$, we denote

$$f_{N,R}^{(s,k,\sigma)}(t,Z_s) := \int_{\mathcal{T}_k(t)} T_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^{N,R} T_{s+\widetilde{\sigma}_1}^{t_1-t_2}\ldots\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^{N,R} T_{s+\widetilde{\sigma}_k}^{t_k} f_{N,0}^{(s+\widetilde{\sigma}_k)}(Z_s)\,dt_k\ldots dt_1.$$

For the Boltzmann hierarchy, we define

$$f_{R}^{(s,k)}(t,Z_s) := \sum_{\sigma\in S_k} f_{R}^{(s,k,\sigma)}(t,Z_s), \text{ for } 1\leq k\leq n, \quad f_{R}^{(s,0)}(t,Z_s) := S_s^t(f_0\mathbb{1}_{[E_s\leq R^2]})(Z_s),$$

where given $k\geq 1$ and $\sigma\in S_k$, we denote

$$f_{R}^{(s,k,\sigma)}(t,Z_s) := \int_{\mathcal{T}_k(t)} S_s^{t-t_1}\mathcal{C}_{s,s+\widetilde{\sigma}_1}^{\infty,R} S_{s+\widetilde{\sigma}_1}^{t_1-t_2}\ldots\mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^{\infty,R} S_{s+\widetilde{\sigma}_k}^{t_k} f_{0}^{(s+\widetilde{\sigma}_k)}(Z_s)\,dt_k\ldots dt_1.$$

Given $\phi_s\in C_c(\mathbb{R}^{ds})$ and $k\in\mathbb{N}$, let us denote

$$I_{s,k,R}^N(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_{N,R}^{(s,k)}(t,X_s,V_s)\,dV_s = \int_{B_R^{ds}} \phi_s(V_s) f_{N,R}^{(s,k)}(t,X_s,V_s)\,dV_s, \tag{7.20}$$

$$I_{s,k,R}^\infty(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_{R}^{(s,k)}(t,X_s,V_s)\,dV_s = \int_{B_R^{ds}} \phi_s(V_s) f_{R}^{(s,k)}(t,X_s,V_s)\,dV_s. \tag{7.21}$$

Recalling the observables $I_{s,k}^N$, $I_{s,k}^\infty$, defined in (7.17)–(7.18), we obtain the following estimates:

Lemma 7.2. *For any $s,n\in\mathbb{N}$, $R>1$ and $t\in[0,T]$, the following estimates hold:*

$$\sum_{k=0}^n \|I_{s,k,R}^N(t) - I_{s,k}^N(t)\|_{L^\infty_{X_s}} \leq C_{s,\beta_0,\mu_0,T}\|\phi_s\|_{L^\infty_{V_s}} e^{-\frac{\beta_0}{3}R^2}\|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\sum_{k=0}^n \|I_{s,k,R}^\infty(t) - I_{s,k}^\infty(t)\|_{L^\infty_{X_s}} \leq C_{s,\beta_0,\mu_0,T}\|\phi_s\|_{L^\infty_{V_s}} e^{-\frac{\beta_0}{3}R^2}\|F_{0}\|_{\infty,\beta_0,\mu_0}.$$

Proof. For the proof, we use the same ideas as in Lemma 8.4. from [2], and we also use (7.4) to sum over all possible collision sequences. □

7.3. Separation of collision times

We will now separate the time intervals we are integrating at, so that collisions occuring are separated in time. For this purpose, consider a small time parameter $\delta>0$.

For convenience, given $t\geq 0$ and $k\in\mathbb{N}$, we define

$$\mathcal{T}_{k,\delta}(t) := \{(t_1,\ldots,t_k)\in\mathcal{T}_k(t): \quad 0\leq t_{i+1}\leq t_i-\delta, \quad \forall i\in[0,k]\}, \tag{7.22}$$

where we denote $t_{k+1}=0$, $t_0=t$.

For the BBGKY hierarchy, we define

$$f_{N,R,\delta}^{(s,k)}(t,Z_s) := \sum_{\sigma\in S_k} f_{N,R,\delta}^{(s,k,\sigma)}(t,Z_s), \text{ for } 1\leq k\leq n, \quad f_{N,R,\delta}^{(s,0)}(t,Z_s) := T_s^t(f_{N,0}\mathbb{1}_{[E_s\leq R^2]})(Z_s),$$

where, given $k \geq 1$ and $\sigma \in S_k$, we denote

$$f_{N,R,\delta}^{(s,k,\sigma)}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} T_s^{t-t_1} \mathcal{C}_{s,s+\widetilde{\sigma}_1}^{N,R} T_{s+\widetilde{\sigma}_1}^{t_1-t_2} \dots \mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^{N,R} T_{s+\widetilde{\sigma}_k}^{t_k} f_{N,0}^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k, \dots dt_1.$$

In the same spirit, for the Boltzmann hierarchy, we define

$$f_{N,R,\delta}^{(s,k)}(t, Z_s) := \sum_{\sigma \in S_k} f_{N,R,\delta}^{(s,k,\sigma)}(t, Z_s), \text{ for } 1 \leq k \leq n, \quad f_{R,\delta}^{(s,0)}(t, Z_s) := S_s^t(f_0 \mathbb{1}_{[E_s \leq R^2]})(Z_s),$$

where, given $k \geq 1$ and $\sigma \in S_k$, we denote

$$f_{R,\delta}^{(s,k,\sigma)}(t, Z_s) := \int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1} \mathcal{C}_{s,s+\widetilde{\sigma}_1}^{\infty,R} S_{s+\widetilde{\sigma}_1}^{t_1-t_2} \dots \mathcal{C}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k}^{\infty,R} S_{s+\widetilde{\sigma}_k}^{t_m} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k, \dots dt_1.$$

Given $\phi_s \in C_c(\mathbb{R}^{ds})$ and $k \in \mathbb{N}$, we define

$$I_{s,k,R,\delta}^N(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_{N,R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s = \int_{B_R^{ds}} \phi_s(V_s) f_{N,R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s, \quad (7.23)$$

$$I_{s,k,R,\delta}^\infty(t)(X_s) := \int_{\mathbb{R}^{ds}} \phi_s(V_s) f_{R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s = \int_{B_R^{ds}} \phi_s(V_s) f_{R,\delta}^{(s,k)}(t, X_s, V_s)\, dV_s. \quad (7.24)$$

Remark 7.3. For $0 \leq t \leq \delta$, we trivially obtain $\mathcal{T}_{k,\delta}(t) = \emptyset$. In this case, the functionals $I_{s,k,R,\delta}^N(t), I_{s,k,R,\delta}^\infty(t)$ are identically zero.

Recalling the observables $I_{s,k,R}^N, I_{s,k,R}^\infty$ defined in (7.20)–(7.21), we obtain the following estimates:

Lemma 7.4. *For any $s, n \in \mathbb{N}$, $R > 0$, $\delta > 0$ and $t \in [0, T]$, the following estimates hold:*

$$\sum_{k=0}^n \|I_{s,k,R,\delta}^N(t) - I_{s,k,R}^N(t)\|_{L_{X_s}^\infty} \leq \delta \|\phi_s\|_{L_{V_s}^\infty} C_{d,s,\beta_0,\mu_0,T}^n \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\sum_{k=0}^n \|I_{s,k,R,\delta}^\infty(t) - I_{s,k,R}^\infty(t)\|_{L_{X_s}^\infty} \leq \delta \|\phi_s\|_{L_{V_s}^\infty} C_{d,s,\beta_0,\mu_0,T}^n \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proof. For the proof, we follow similar ideas as in Lemma 8.7. from [2], and we also use bound (7.6) to control the combinatorics occurring. □

Combining Lemma 7.1, Lemma 7.2 and Lemma 7.4, we obtain the following:

Proposition 7.5. *For any $s, n \in \mathbb{N}$, $R > 1$, $\delta > 0$ and $t \in [0, T]$, the following estimates hold:*

$$\|I_s^N(t) - \sum_{k=1}^n I_{s,k,R,\delta}^N(t)\|_{L_{X_s}^\infty} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\|I_s^\infty(t) - \sum_{k=1}^n I_{s,k,R,\delta}^\infty(t)\|_{L_{X_s}^\infty} \leq C_{s,\beta_0,\mu_0,T} \|\phi_s\|_{L_{V_s}^\infty} \left(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C_{d,s,\beta_0,\mu_0,T}^n\right) \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proposition 7.5 implies that, given $0 \leq k \leq n$, $R > 1$, $\delta > 0$, the convergence proof reduces to controlling the differences $I_{s,k,R,\delta}^N(t) - I_{s,k,R,\delta}^\infty(t)$, where the observables $I_{s,k,R,\delta}^N, I_{s,k,R,\delta}^\infty$ are given by (7.23)–(7.24). However, this is not immediate since the backwards (ϵ_2, ϵ_3)-flow and the backwards free flow do not coincide in general. The goal is to eliminate some small measure set of initial data, negligible in the limit, such that the backwards (ϵ_2, ϵ_3)-flow and the backwards free flow are comparable.

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8. Geometric estimates

In this section, we present some geometric results which will be essential for estimating the measure of the pathological sets leading to recollisions of the backwards (ϵ_2, ϵ_3) flow (see Section 9). First, we review some of the results we used in [5] which are useful here as well. We then present certain novel results – namely, Lemma 8.3, Lemma 8.6, Lemma 8.7 and, most importantly, Lemma 8.8 – which crucially rely on the following symmetric representation of the $(2d-1)$ sphere of radius $r > 0$:

$$\mathbb{S}_r^{2d-1} = \left\{(\omega_1,\omega_2) \in B_r^d \times B_r^d : \omega_2 \in \mathbb{S}^{d-1}_{\sqrt{r^2-|\omega_1|^2}}\right\} = \left\{(\omega_1,\omega_2) \in B_r^d \times B_r^d : \omega_1 \in \mathbb{S}^{d-1}_{\sqrt{r^2-|\omega_2|^2}}\right\} \quad (8.1)$$

Representation (8.1) is very useful when one wants to estimate the intersection of $\mathbb{S}_r^{2d-1}$ with sets of the form $S \times \mathbb{R}^d$ or $\mathbb{R}^d \times S$, where $S \subseteq \mathbb{R}^d$ is of small measure.

8.1. Cylinder-Sphere estimates

Here, we present certain estimates based on the intersection of a sphere with a given solid cylinder. These estimates were used in [5] as well. Similar estimates can be found in [14, 18].

Lemma 8.1. *Let $\rho, r > 0$ and $K_\rho^d \subseteq \mathbb{R}^d$ be a solid cylinder. Then the following estimate holds for the $(d-1)$-spherical measure:*

$$\int_{\mathbb{S}_r^{d-1}} \mathbb{1}_{K_\rho^d}\, d\omega \lesssim r^{d-1} \min\left\{1, \left(\frac{\rho}{r}\right)^{\frac{d-1}{2}}\right\}.$$

Proof. After re-scaling, we may clearly assume that $r = 1$. Then, we refer to the work of R. Denlinger [14], p.30, for the rest of the proof. □

Applying Lemma 8.1, we obtain the following geometric estimate, which will be crucially used in Section 9.

Corollary 8.2. *Given $0 < \rho \leq 1 \leq R$, the following estimate holds:*

$$|B_R^d \cap K_\rho^d|_d \lesssim R^d \rho^{\frac{d-1}{2}}.$$

Proof. The co-area formula and Lemma 8.1 imply

$$\begin{aligned} |B_R^d \cap K_\rho^d|_d &= \int_0^R \int_{\mathbb{S}_r^{d-1}} \mathbb{1}_{K_\rho^d}\, d\omega\, dr \\ &\lesssim \int_0^R r^{d-1} \min\left\{1, (\frac{\rho}{r})^{\frac{d-1}{2}}\right\} dr \\ &\leq \int_0^\rho r^{d-1}\, dr + \rho^{\frac{d-1}{2}} \int_0^R r^{\frac{d-1}{2}}\, dr \\ &\simeq \rho^d + \rho^{\frac{d-1}{2}} R^{\frac{d+1}{2}}, \quad \text{since } d \geq 2 \\ &\lesssim R^d \rho^{\frac{d-1}{2}}, \quad \text{since } 0 < \rho \leq 1 \leq R. \end{aligned} \quad (8.2)$$

□

8.2. Estimates relying on the $(2d-1)$-sphere representation

Here, we present certain geometric estimates relying on the representation (8.1). In particular, up to our knowledge, Lemma 8.3, Lemma 8.6, Lemma 8.7 and, most importantly, Lemma 8.8 are novel results. Lemma 8.4 is a special case of a result proved in [5].

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8.2.1. Truncation of impact directions

We first estimate the intersection of $\mathbb{S}_1^{2d-1}$ with sets of the form $B_\rho^d \times \mathbb{R}^d$ or $\mathbb{R}^d \times B_\rho^d$.

Lemma 8.3. *Consider* $\rho > 0$. *We define the sets*

$$M_1(\rho) = B_\rho^d \times \mathbb{R}^d = \left\{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1| \leq \rho\right\}, \tag{8.3}$$

$$M_2(\rho) = \mathbb{R}^d \times B_\rho^d = \left\{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_2| \leq \rho\right\}. \tag{8.4}$$

Then, the following holds:

$$\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{M_1(\rho)} \, d\omega_1 \, d\omega_2 = \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{M_2(\rho)} \, d\omega_1 \, d\omega_2 \lesssim \min\{1, \rho^d\}.$$

Proof. By symmetry, it suffices to estimate the first term. Using (8.3) and representation (8.1), we obtain

$$\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{M_1(\rho)} \, d\omega_1 \, d\omega_2 = \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{B_\rho^d \times \mathbb{R}^d} \, d\omega_1 \, d\omega_2 \lesssim \int_{B_\rho^d \cap B_1^d} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} d\omega_2 \, d\omega_1 \lesssim \min\{1, \rho^d\}.$$

□

The following result is a special case of Lemma 8.4. from [5]. For the proof, see Lemma 9.5. in [2].

Lemma 8.4. *Consider* $\rho > 0$. *Let us define the strip*

$$W_\rho^{2d} = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1 - \omega_2| \leq \rho\}. \tag{8.5}$$

Then, the following estimate holds:

$$\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{W_\rho^{2d}} \, d\omega_1 \, d\omega_2 \lesssim \min\left\{1, \rho^{\frac{d-1}{2}}\right\}.$$

Proof. For the proof, see Lemma 9.5. in [2]. The main idea is to first use representation (8.1) and then apply Lemma 8.1. □

8.2.2. Conic estimates

Now we establish estimates related to conic regions. We first present a well-known spherical cap estimate.

Lemma 8.5. *Consider* $0 \leq \alpha \leq 1$ *and* $\nu \in \mathbb{R}^d \setminus \{0\}$. *Let us define*

$$S(\alpha, \nu) = \left\{\omega \in \mathbb{R}^d : |\langle \omega, \nu \rangle| \geq \alpha |\omega||\nu|\right\}. \tag{8.6}$$

Then, for $\rho > 0$, *the following estimate holds:*

$$\int_{\mathbb{S}_r^{d-1}} \mathbb{1}_{S(\alpha,\nu)} \, d\omega = r^{d-1} |\mathbb{S}_1^{d-2}| \int_0^{2 \arccos \alpha} \sin^{d-2}(\theta) \, d\theta \lesssim r^{d-1} \arccos \alpha.$$

Proof. After re-scaling, it suffices to prove the result for $r = 1$. Notice that $\mathbb{S}_1^{d-1} \cap S(\alpha, \nu)$ is a spherical cap of angle $2 \arccos \alpha$ and direction $\nu \neq 0$ on the unit sphere. Therefore, integrating in spherical coordinates, we obtain

$$\int_{\mathbb{S}_1^{d-1}} \mathbb{1}_{S(\alpha,\nu)} \, d\omega = |\mathbb{S}_1^{d-2}| \int_0^{2 \arccos \alpha} \sin^{d-2} \theta \, d\theta \lesssim \arccos \alpha.$$

□

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We apply Lemma 8.5 to obtain the following result:

Lemma 8.6. *Consider* $0 \leq \alpha \leq 1$ *and* $\nu \in \mathbb{R}^d \setminus \{0\}$. *Let us define*

$$N(\alpha,\nu) = \left\{(\omega_1,\omega_2) \in \mathbb{R}^{2d} : \langle \omega_1 - \omega_2, \nu\rangle \geq \alpha|\omega_1-\omega_2||\nu|\right\}. \tag{8.7}$$

Then, we have the estimate

$$\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{N(\alpha,\nu)}\, d\omega_1\, d\omega_2 \lesssim \arccos\alpha.$$

Proof. Recalling (8.6)–(8.7), we have

$$N(\alpha,\nu) = \{(\omega_1,\omega_2) \in \mathbb{R}^{2d} : \omega_1 - \omega_2 \in S(\alpha,\nu)\}. \tag{8.8}$$

Let us define the linear map $T : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}$ by

$$(u_1,u_2) = T(\omega_1,\omega_2) := (\omega_1+\omega_2, \omega_1-\omega_2).$$

Clearly,

$$|u_1|^2+|u_2|^2 = |\omega_1+\omega_2|^2 + |\omega_1-\omega_2|^2 = 2|\omega_1|^2+2|\omega_2|^2 = 2, \quad \forall(\omega_1,\omega_2)\in\mathbb{S}_1^{2d-1},$$

and hence, $T : \mathbb{S}_1^{2d-1} \longrightarrow \mathbb{S}_{\sqrt{2}}^{2d-1}$. Therefore, using (8.8) and changing variables under T, we have

$$\begin{aligned}
\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{N(\alpha,\nu)}(\omega_1,\omega_2)\, d\omega_1\, d\omega_2 &= \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{S(\alpha,\nu)}(\omega_1-\omega_2)\, d\omega_1\, d\omega_2 \\
&\simeq \int_{\mathbb{S}_2^{2d-1}} \mathbb{1}_{S(\alpha,\nu)}(u_2)\, du_1\, du_2 \\
&= \int_{B_{\sqrt{2}}^d} \int_{\mathbb{S}^{d-1}_{\sqrt{2-|u_1|^2}}} \mathbb{1}_{S(\alpha,\nu)}(u_2)\, du_2\, du_1 \qquad (8.9) \\
&\lesssim \arccos\alpha, \qquad (8.10)
\end{aligned}$$

where to obtain (8.9), we use the representation of the sphere (8.1), and to obtain (8.10), we use Lemma 8.5. □

8.2.3. Annuli estimates

We present some estimates based on the intersection of the unit sphere with appropriate annuli.

Lemma 8.7. *Let* $0 < \beta < 1/2$, *and consider the sets*

$$I_1 = \left\{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \left|1-2|\omega_1|^2\right| \leq 2\beta\right\}, \tag{8.11}$$

$$I_2 = \left\{(\omega_1,\omega_2)\in\mathbb{R}^{2d} : \left|1-2|\omega_2|^2\right| \leq 2\beta\right\}. \tag{8.12}$$

There hold the estimates

$$\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{I_1}\, d\omega_1\, d\omega_2 = \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{I_2}\, d\omega_1\, d\omega_2 \lesssim \beta.$$

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Proof. By symmetry, it suffices to prove the estimate for I_1. Since $0 < \beta < 1/2$, we may write

$$I_1 = \left\{(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1} : \sqrt{\frac{1}{2} - \beta} \le |\omega_1| \le \sqrt{\frac{1}{2} + \beta}\right\}.$$

Using the representation (8.1) of the $(2d-1)$-unit sphere, we obtain

$$\begin{aligned}
\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{I_1}\, d\omega_1\, d\omega_2 &\le \int_{\sqrt{\frac{1}{2}-\beta} \le |\omega_1| \le \sqrt{\frac{1}{2}+\beta}} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} d\omega_2\, d\omega_1 \\
&\lesssim \left(\frac{1}{2} + \beta\right)^{d/2} - \left(\frac{1}{2} - \beta\right)^{d/2} \\
&\overset{d\ge 2}{=} \left(\sqrt{\frac{1}{2} + \beta} - \sqrt{\frac{1}{2} - \beta}\right) \sum_{j=0}^{d-1} \left(\frac{1}{2} + \beta\right)^{j/2} \left(\frac{1}{2} - \beta\right)^{\frac{d-1-j}{2}} \\
&= \frac{2\beta}{\sqrt{\frac{1}{2} + \beta} + \sqrt{\frac{1}{2} - \beta}} \sum_{j=0}^{d-1} \left(\frac{1}{2} + \beta\right)^{j/2} \left(\frac{1}{2} - \beta\right)^{\frac{d-1-j}{2}} \\
&\le 2\sqrt{2}\beta \sum_{j=0}^{d-1} \left(\frac{1}{2} + \beta\right)^{j/2} \left(\frac{1}{2} - \beta\right)^{\frac{d-1-j}{2}} \\
&\lesssim \beta,
\end{aligned}$$

since $0 < \beta < 1/2$. The proof is complete. □

Lemma 8.8. *Consider* $0 < \beta < 1/4$. *Let us define the hemispheres*

$$\mathcal{S}_{1,2} = \{(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1} : |\omega_1| < |\omega_2|\}, \tag{8.13}$$

$$\mathcal{S}_{2,1} = \{(\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1} : |\omega_2| < |\omega_1|\}, \tag{8.14}$$

and the annuli

$$I_{1,2} = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : \big||\omega_1|^2 + 2\langle\omega_1, \omega_2\rangle\big| \le \beta\}, \tag{8.15}$$

$$I_{2,1} = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : \big||\omega_2|^2 + 2\langle\omega_1, \omega_2\rangle\big| \le \beta\}. \tag{8.16}$$

Then, there holds

$$\int_{\mathcal{S}_{1,2}} \mathbb{1}_{I_{1,2}}\, d\omega_1\, d\omega_2 = \int_{\mathcal{S}_{2,1}} \mathbb{1}_{I_{2,1}}\, d\omega_1\, d\omega_2 \lesssim \beta.$$

Proof. By symmetry, it suffices to prove

$$\int_{\mathcal{S}_{2,1}} \mathbb{1}_{I_{2,1}}\, d\omega_1\, d\omega_2 \lesssim \beta. \tag{8.17}$$

Recalling notation from (8.3)–(8.4), let us define

$$U_\beta = M_1^c(2\sqrt{\beta}) \cap M_2^c(2\sqrt{\beta}) = \{(\omega_1, \omega_2) \in \mathbb{R}^{2d} : |\omega_1| > 2\sqrt{\beta} \text{ and } |\omega_2| > 2\sqrt{\beta}\}.$$

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Clearly, $U_\beta^c = M_1(2\sqrt{\beta}) \cup M_2(2\sqrt{\beta})$. Writing $A := I_{2,1} \cap U_\beta$, we have

$$\int_{\mathcal{S}_{2,1}} \mathbb{1}_{I_{2,1}}\, d\omega_1\, d\omega_2 \le \int_{\mathcal{S}_{2,1}} \mathbb{1}_{U_\beta^c}\, d\omega_1\, d\omega_2 + \int_{\mathcal{S}_{2,1}} \mathbb{1}_A\, d\omega_1\, d\omega_2 \lesssim \beta^{d/2} + \int_{\mathcal{S}_{2,1}} \mathbb{1}_A\, d\omega_1\, d\omega_2, \tag{8.18}$$

where to obtain (8.18), we used Lemma 8.3. Notice that we may write

$$A = \{(\omega_1,\omega_2) \in \mathbb{R}^{2d} : |\omega_1| > 2\sqrt{\beta},\ |\omega_2| > 2\sqrt{\beta} \text{ and } \sqrt{|\omega_1|^2-\beta} \le |\omega_1+\omega_2| \le \sqrt{|\omega_1|^2+\beta}\}. \tag{8.19}$$

By (8.18), the representation of the sphere (8.1) and (8.19), we have

$$\int_{\mathcal{S}_{2,1}} \mathbb{1}_{I_{2,1}}\, \omega_1\, d\omega_2 \lesssim \beta^{d/2} + \int_{2\sqrt{\beta}<|\omega_1|\le 1} \int_{\mathcal{S}_{2,1,\omega_1}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2\, d\omega_1, \tag{8.20}$$

where given $2\sqrt{\beta} < |\omega_1| \le 1$, we denote

$$\mathcal{S}_{2,1,\omega_1} = \{\omega_2 \in \mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}} : |\omega_2| < |\omega_1|\}, \tag{8.21}$$

$$\begin{aligned} A_{\omega_1} &= \{\omega_2 \in \mathbb{R}^d : (\omega_1,\omega_2) \in A\} \\ &= \{\omega_2 \in \mathbb{R}^d : |\omega_2| > 2\sqrt{\beta} \text{ and } \sqrt{|\omega_1|^2-\beta} \le |\omega_1+\omega_2| \le \sqrt{|\omega_1|^2+\beta}\}. \end{aligned} \tag{8.22}$$

Since $\beta < 1/4$, it suffices to control the term:

$$I' = \int_{2\sqrt{\beta}<|\omega_1|\le 1} \int_{\mathcal{S}_{2,1,\omega_1}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2\, d\omega_1. \tag{8.23}$$

Now we shall prove that, in fact,

$$I' = \int_{2\sqrt{\beta}<\sqrt{1-|\omega_1|^2}<|\omega_1|\le 1} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2\, d\omega_1. \tag{8.24}$$

Indeed, assume ω_1 does not satisfy

$$2\sqrt{\beta} < \sqrt{1-|\omega_1|^2} < |\omega_1|. \tag{8.25}$$

Since we are integrating in the region $2\sqrt{\beta} < |\omega_1| \le 1$, exactly one of the following holds:

$$|\omega_1| \le \sqrt{1-|\omega_1|^2}, \tag{8.26}$$

$$\sqrt{1-|\omega_1|^2} \le 2\sqrt{\beta}. \tag{8.27}$$

Recalling (8.21), condition (8.26) implies that $\mathcal{S}_{2,1,\omega_1} = \emptyset$, while recalling (8.22), condition (8.27) implies $\mathcal{S}_{2,1,\omega_1} \cap A_{\omega_1} = \emptyset$. Therefore,

$$I' = \int_{2\sqrt{\beta}<\sqrt{1-|\omega_1|^2}<|\omega_1|\le 1} \int_{\mathcal{S}_{2,1,\omega_1}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2\, d\omega_1,$$

and (8.24) follows from (8.21).

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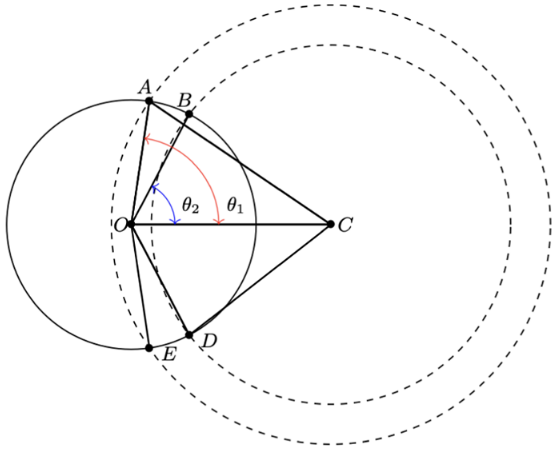

$(OA) = (OB) = \sqrt{1-|\omega_1|^2}, \quad \overrightarrow{OC} = -\omega_1,$
$(AC) = \sqrt{|\omega_1|^2+\beta}, \quad (CD) = \sqrt{|\omega_1|^2-\beta}.$

Figure 5.

Fix any ω_1 satisfying (8.25). We first estimate the inner integral:

$$\int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2. \tag{8.28}$$

Notice that (8.25) also yields

$$|\omega_1| - \sqrt{|\omega_1|^2-\beta} = \frac{\beta}{|\omega_1| + \sqrt{|\omega_1|^2-\beta}} < \frac{\beta}{|\omega_1|} \leq \frac{1}{2}\sqrt{\beta} \leq \frac{1}{4}\sqrt{1-|\omega_1|^2}. \tag{8.29}$$

Condition (8.25) guarantees that the vector[13] $-\omega_1$ lays outside of the sphere $\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}$, while condition (8.29) guarantees that the sphere is not contained in the annulus A_{ω_1}. Therefore, the projection of $\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}} \cap A_{\omega_1}$ on any plane containing the origin and the vector $-\omega_1$ can be visualized as follows:

We conclude that

$$\mathbb{S}^{d-1}_{\sqrt{1-|\omega|^2}} \cap A_{\omega_1} = \mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}} \cap \left(S(\cos\theta_1, -\omega_1) \setminus S(\cos\theta_2, -\omega_1)\right), \tag{8.30}$$

where recalling the notation introduced in (8.6),

$$\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}} \cap S(\cos\theta_1, -\omega_1), \qquad \mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}} \cap S(\cos\theta_2, -\omega_1),$$

are the spherical shells on $\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}$, of direction $-\omega_1$ and angles $2\theta_1$, $2\theta_2$ respectively, where

$$\theta_1 = \widehat{AOC}, \quad \theta_2 = \widehat{BOC}.$$

[13]Understood as a point in $\mathbb{R}^d$.

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Therefore, by (8.30), we have

$$\int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2 = \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{S(\cos\theta_1,-\omega_1)\setminus S(\cos\theta_2,-\omega_1)}(\omega_2)\, d\omega_2$$
$$= (1-|\omega_1|^2)^{\frac{d-1}{2}} |\mathbb{S}_1^{d-2}| \int_{2\theta_2}^{2\theta_1} \sin^{d-2}\theta\, d\theta \tag{8.31}$$
$$\lesssim \theta_1 - \theta_2, \tag{8.32}$$

where to obtain (8.31), we use Lemma 8.5, and to obtain (8.32), we use the fact that $d \geq 2$.

Let us calculate $\alpha_1 = \cos\theta_1$, $\alpha_2 = \cos\theta_2$. By the cosine law on the triangle AOC, we obtain

$$\alpha_1 = \cos\theta_1 = \frac{(OA)^2 + (OC)^2 - (AC)^2}{2(OA)(OC)} = \frac{1 - |\omega_1|^2 - \beta}{2|\omega_1|\sqrt{1-|\omega_1|^2}}, \tag{8.33}$$

and by the cosine law on the triangle BOC, we obtain

$$\alpha_2 = \cos\theta_2 = \frac{(OB)^2 + (OC)^2 - (CB)^2}{2(OB)(OC)} = \frac{1 - |\omega_1|^2 + \beta}{2|\omega_1|\sqrt{1-|\omega_1|^2}}. \tag{8.34}$$

Then, expression (8.33) implies

$$|\alpha_1| \leq \frac{\sqrt{1-|\omega_1|^2}}{2|\omega_1|} + \frac{\beta}{2|\omega_1|\sqrt{1-|\omega_1|^2}} < \frac{5}{8}, \tag{8.35}$$

since by (8.25) we have $|\omega_1| > \sqrt{1-|\omega_1|^2} > 2\sqrt{\beta}$. In the same spirit, expression (8.34) yields

$$|\alpha_2| < \frac{5}{8}. \tag{8.36}$$

The inverse cosine is smooth in $(-1, 1)$, so it is Lipschitz in $[-\frac{5}{8}, \frac{5}{8}]$; thus, by (8.35)–(8.36) and (8.25), we have

$$|\arccos\alpha_1 - \arccos\alpha_2| \lesssim |\alpha_1 - \alpha_2| = \frac{\beta}{|\omega_1|\sqrt{1-|\omega_1|^2}}.$$

Therefore, (8.32) implies

$$\int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2 \lesssim \theta_1 - \theta_2 = \arccos\alpha_1 - \arccos\alpha_2 \lesssim \frac{\beta}{|\omega_1|\sqrt{1-|\omega_1|^2}}. \tag{8.37}$$

Using (8.37), and recalling (8.24), we have

$$I' = \int_{2\sqrt{\beta}<\sqrt{1-|\omega_1|^2}|\omega_1|<1} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_1|^2}}} \mathbb{1}_{A_{\omega_1}}(\omega_2)\, d\omega_2\, d\omega_1$$
$$\lesssim \beta \int_{B_1^d} \frac{1}{|\omega_1|\sqrt{1-|\omega_1|^2}}\, d\omega_1$$
$$\simeq \beta \int_0^1 \frac{r^{d-2}}{\sqrt{1-r^2}}\, dr \tag{8.38}$$

$$\leq \beta \int_0^1 \frac{1}{\sqrt{1-r^2}}\,dr \tag{8.39}$$

$$= \frac{\pi}{2}\beta, \tag{8.40}$$

where to obtain (8.38), we use integration in polar coordinates, and to obtain (8.39), we use the fact that $d \geq 2$. Using (8.20) and (8.40), we obtain

$$\int_{\mathcal{S}_{2,1}} \mathbb{1}_{I_{2,1}}\,d\omega_1\,d\omega_2 \lesssim \beta^{d/2} + \beta \lesssim \beta,$$

since $\beta < 1/4$. The proof is complete. □

9. Good configurations and stability

9.1. Adjunction of new particles

In this section, we investigate stability of good configurations under adjunctions of collisional particles. Subsection 9.2 investigates binary adjunctions, while Subsection 9.3 investigates ternary adjunctions. To perform the measure estimates needed, we will strongly rely on the results of Section 8.

We start with some definitions on the configurations we are using. Consider $m \in \mathbb{N}$ and $\theta > 0$, and recall from (6.3)-(6.4) the set of well-separated configurations

$$\Delta_m(\theta) = \{\widetilde{Z}_m = (\widetilde{X}_m, \widetilde{V}_m) \in \mathbb{R}^{2dm} : |\widetilde{x}_i - \widetilde{x}_j| > \theta, \quad \forall 1 \leq i < j \leq m\}, \quad m \geq 2, \quad \Delta_1(\theta) = \mathbb{R}^{2d}.$$

Roughly speaking, a good configuration is a configuration which remains well-separated under backwards time evolution. More precisely, given $\theta > 0$, $t_0 > 0$, we define the set of good configurations as

$$G_m(\theta, t_0) = \left\{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : Z_m(t) \in \Delta_m(\theta), \quad \forall t \geq t_0\right\}, \tag{9.1}$$

where $Z_m(t)$ denotes the backwards in time free flow of $Z_m = (X_m, V_m)$, given by

$$Z_m(t) = ((X_m(t), V_m(t)) := (X_m - tV_m, V_m), \quad t \geq 0. \tag{9.2}$$

Notice that Z_m is the initial point of the trajectory (i.e., $Z_m(0) = Z_m$). In other words for $m \geq 2$, we have

$$G_m(\theta, t_0) = \left\{Z_m = (X_m, V_m) \in \mathbb{R}^{2dm} : |x_i(t) - x_j(t)| > \theta, \quad \forall t \geq t_0, \quad \forall i < j \in \{1, \dots, m\}\right\}. \tag{9.3}$$

From now on, we consider parameters $R >> 1$ and $0 < \delta, \eta, \epsilon_0, \alpha << 1$ satisfying

$$\alpha << \epsilon_0 << \eta\delta, \quad R\alpha << \eta\epsilon_0. \tag{9.4}$$

For convenience, we choose the parameters in (9.4) in the very end of the paper; see (11.23), (11.24). Throughout this section, we will write K_η^d for a cylinder of radius η in $\mathbb{R}^d$.

The following Lemma is useful for the adjunction of particles to a given configuration. For the proof, see Lemma 12.2.1 from [18] or Lemma 10.2. from [2].

Lemma 9.1. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_3 << \alpha$. Let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}^d$, with $|\bar{y}_1 - \bar{y}_2| > \epsilon_0$ and $v_1 \in B_R^d$. Then there is a d-cylinder $K_\eta^d \subseteq \mathbb{R}^d$ such that for any $y_1 \in B_\alpha^d(\bar{y}_1)$, $y_2 \in B_\alpha^d(\bar{y}_2)$ and $v_2 \in B_R^d \setminus K_\eta^d$, we have*

1. $(y_1, y_2, v_1, v_2) \in G_2(\sqrt{2}\epsilon_3, 0)$,
2. $(y_1, y_2, v_1, v_2) \in G_2(\epsilon_0, \delta)$.

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9.2. Stability under binary adjunction

The main results of this subsection are stated in Proposition 9.2, which will be the inductive step of adding a colliding particle, and Proposition 9.4, which presents the measure estimate of the bad set that appears in this process. The proofs of the Propositions presented below are in part inspired by arguments in [18] and [5] with a caveat that the new scenario needs to be addressed, in the case when the binary collisional configuration formed runs to a ternary interaction under time evolution.

9.2.1. Binary adjunction

For convenience, given $v \in \mathbb{R}^d$, let us denote

$$\left(\mathbb{S}_1^{d-1} \times B_R^d\right)^+(v) = \left\{(\omega_1, v_1) \in \mathbb{S}_1^{d-1} \times B_R^d : b_2(\omega_1, v_1 - v) > 0\right\}, \tag{9.5}$$

where $b_2(\omega)1, v_1 - v) = \langle \omega_1, v_1 - v\rangle$. Recall from (9.2) that given $m \in \mathbb{N}$ and $Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}$, we denote the backwards in time free flow as $Z_m(t) = (X_m - tV_m, V_m)$, $t \geq 0$. Recall also the notation from (3.7)

$$\begin{aligned}\mathring{\mathcal{D}}_{m+1,\epsilon_2,\epsilon_3} = \Big\{Z_{m+1} = (X_{m+1}, V_{m+1}) \in \mathbb{R}^{2d(m+1)} :\ & d_2(x_i, x_j) > \epsilon_2, \quad \forall (i,j) \in \mathcal{I}_{m+1}^2, \\ & \text{and } d_3(x_i; x_j, x_k) > \sqrt{2}\epsilon_3, \quad \forall (i,j,k) \in \mathcal{I}_{m+1}^3\Big\},\end{aligned}$$

where $\mathcal{I}_{m+1}^2, \mathcal{I}_{m+1}^3$ are given by (3.1)–(3.2), respectively.

Proposition 9.2. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_2 << \epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \ldots, m\}$, $\bar{V}_m \in B_R^{dm}$ and $X_m \in B_{\alpha/2}^{dm}(\bar{X}_m)$. Then there is a subset $\mathcal{B}_\ell^2(\bar{Z}_m) \subseteq (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell)$ such that*

1. *For any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell) \setminus \mathcal{B}_\ell^2(\bar{Z}_m)$, one has*

$$Z_{m+1}(t) \in \mathring{\mathcal{D}}_{m+1,\epsilon_2,\epsilon_3}, \quad \forall t \geq 0, \tag{9.6}$$

$$Z_{m+1} \in G_{m+1}(\epsilon_0/2, \delta), \tag{9.7}$$

$$\bar{Z}_{m+1} \in G_{m+1}(\epsilon_0, \delta). \tag{9.8}$$

where

$$\begin{aligned} Z_{m+1} &= (x_1, \ldots, x_\ell, \ldots, x_m, x_{m+1}, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}), \\ x_{m+1} &= x_\ell - \epsilon_2\omega_1, \\ \bar{Z}_{m+1} &= (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_m, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}), \end{aligned} \tag{9.9}$$

2. *For any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_\ell) \setminus \mathcal{B}_\ell^2(\bar{Z}_m)$, one has*

$$Z'_{m+1}(t) \in \mathring{\mathcal{D}}_{m+1,\epsilon_2,\epsilon_3}, \quad \forall t \geq 0, \tag{9.10}$$

$$Z'_{m+1} \in G_{m+1}(\epsilon_0/2, \delta), \tag{9.11}$$

$$\bar{Z}'_{m+1} \in G_{m+1}(\epsilon_0, \delta), \tag{9.12}$$

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where

$$\begin{aligned}
&Z'_{m+1} = (x_1, ..., x_\ell, ..., x_m, x_{m+1}, \bar{v}_1, ..., \bar{v}'_\ell, ..., \bar{v}_m, v'_{m+1}),\\
&x_{m+1} = x_\ell + \epsilon_2\omega_1,\\
&\bar{Z}'_{m+1} = (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_m, \bar{v}_1, ..., \bar{v}'_\ell, ..., \bar{v}_m, v'_{m+1}),\\
&(\bar{v}'_\ell, v'_{m+1}) = T_{\omega_1}(\bar{v}_\ell, v_{m+1}).
\end{aligned} \tag{9.13}$$

Proof. By symmetry, we may assume that $\ell = m$. For convenience, let us define the set

$$\mathcal{F}_{m+1} = \{(i,j) \in \{1, ..., m+1\} \times \{1, ..., m+1\} : i < \min\{j, m\}\}.$$

Proof of *(i)*: Here, we use notation from (9.9). We start by formulating the following claim, which will imply (9.6).

Lemma 9.3. *Under the assumptions of Proposition 9.2, there is a subset $\mathcal{B}_m^{2,0,-}(\bar{Z}_m) \subseteq \mathbb{S}_1^{d-1} \times B_R^d$ such that for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \setminus \mathcal{B}_m^{2,0,-}(\bar{Z}_m)$, there holds*

$$d_2\big(x_i(t), x_j(t)\big) > \sqrt{2}\epsilon_3, \quad \forall t \geq 0, \quad \forall (i,j) \in \mathcal{F}_{m+1}, \tag{9.14}$$

$$d_2(x_m(t), x_{m+1}(t)) > \epsilon_2, \quad \forall t \geq 0. \tag{9.15}$$

Notice that (9.14)–(9.15) trivially imply (9.6), since $\epsilon_2 << \epsilon_3$.

Proof of Lemma 9.3
Step 1: The proof of (9.14): We distinguish the following cases:
∘ $j \leq m$: Since $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $j \leq m$, we have $|\bar{x}_i(t) - \bar{x}_j(t)| > \epsilon_0$, for all $t \geq 0$. Therefore, triangle inequality implies that

$$|x_i(t) - x_j(t)| = |x_i - x_j - t(\bar{v}_i - \bar{v}_j)| \geq |\bar{x}_i - \bar{x}_j - t(\bar{v}_i - \bar{v}_j)| - \alpha \geq \epsilon_0 - \alpha > \frac{\epsilon_0}{2} > \sqrt{2}\epsilon_3, \tag{9.16}$$

since $\epsilon_3 << \alpha << \epsilon_0$.
∘ $j = m+1$: Since $(i, m+1) \in \mathcal{F}_{m+1}$, we have $i \leq m-1$. Since $\bar{Z}_m \in G_m(\epsilon_0, 0)$ and $X_m \in B_{\alpha/2}^{dm}(\bar{X}_m)$, we conclude

$$|\bar{x}_i - \bar{x}_m| > \epsilon_0, \quad |x_i - \bar{x}_i| \leq \frac{\alpha}{2} < \alpha, \quad |x_{m+1} - \bar{x}_m| \leq |x_m - \bar{x}_m| + \epsilon_2|\omega_1| \leq \frac{\alpha}{2} + \epsilon_2 < \alpha,$$

since $\epsilon_2 << \alpha$. Applying part *(i)* of Lemma 9.1 for $\bar{y}_1 = \bar{x}_i$, $\bar{y}_2 = \bar{x}_m$, $y_1 = x_i$, $y_2 = x_{m+1}$, we may find a cylinder $K_\eta^{d,i}$ such that for any $v_{m+1} \in B_R^d \setminus K_\eta^{d,i}$, we have $|x_i(t) - x_{m+1}(t)| > \sqrt{2}\epsilon_3$, for all $t \geq 0$. Hence, the inequality in (9.14) holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \setminus V_{m+1}^i$, where

$$V_{m+1}^i = \mathbb{S}_1^{d-1} \times K_\eta^{d,i}. \tag{9.17}$$

We conclude that (9.14) holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus \bigcup_{i=1}^{m-1} V_{m+1}^i$.
Step 2: The proof of (9.15): We recall notation from (9.9). Considering $t \geq 0$ and $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m)$. Using the fact that $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m)$, we obtain

$$|x_m(t) - x_{m+1}(t)|^2 = |\epsilon_2\omega_1 - t(\bar{v}_m - v_{m+1})|^2 \geq \epsilon_2^2|\omega_1|^2 + 2\epsilon_2 t b_2(\omega_1, v_{m+1} - \bar{v}_m) > \epsilon_2^2. \tag{9.18}$$

Therefore, (9.15) holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m)$.

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Defining

$$\mathcal{B}_m^{2,0,-}(\bar{Z}_m) = \bigcup_{i=1}^{m-1} V_{m+1}^i, \tag{9.19}$$

the claim of Lemma 9.3 follows.

Now we go back to the proof of part *(i)* of Proposition 9.2. We will find a set $\mathcal{B}_m^{2,\delta,-}(\bar{Z}_m) \subseteq \mathbb{S}_1^{d-1} \times B_R^d$ such that (9.7) holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_m^{2,\delta,-}(\bar{Z}_m)$.

Let us fix $i, j \in \{1, ..., m+1\}$ with $i < j$. We distinguish the following cases:

∘ $j \leq m$: We use the same argument as in (9.16), to obtain $|x_i(t) - x_j(t)| > \frac{\epsilon_0}{2}$, for all $t \geq 0$.

∘ $(i, j) \in \mathcal{F}_{m+1}$, $j = m+1$: Since $(i, m+1) \in \mathcal{F}_{m+1}$, we have $i \leq m-1$. Applying a similar argument to the corresponding case in the proof of (9.14), using part *(ii)* of Lemma 9.1 instead, we obtain that the inequality $|x_i(t) - x_{m+1}(t)| > \epsilon_0$, for all $t \geq \delta$, holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus V_{m+1}^i$, where V_{m+1}^i is given by (9.17). Notice that the lower bound is in fact ϵ_0.

∘ $i = m$, $j = m+1$: Triangle inequality and the fact that $\epsilon_2 << \epsilon_0 << \eta\delta$ imply that for any $t \geq \delta$ and $(\omega_1, v_{m+1}) \in \mathbb{S}_1^{d-1} \times B_R^d$ with $|v_{m+1} - \bar{v}_m| > \eta$, we have

$$|x_m(t) - x_{m+1}(t)| = |\epsilon_2\omega_1 - t(\bar{v}_m - v_{m+1})| \geq |\bar{v}_m - v_{m+1}|\delta - \epsilon_2 > \eta\delta - \epsilon_2 > \epsilon_0.$$

Therefore, the inequality $|x_m(t) - x_{m+1}(t)| > \epsilon_0$, for all $t \geq \delta$, holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus V_{m,m+1}$, where

$$V_{m,m+1} = \mathbb{S}_1^{d-1} \times B_\eta^d(\bar{v}_m). \tag{9.20}$$

Notice that the lower bound is ϵ_0 again.

Defining

$$\mathcal{B}_m^{2,\delta,-}(\bar{Z}_m) = \mathcal{B}_m^{2,0,-}(\bar{Z}_m) \cup V_{m,m+1}, \tag{9.21}$$

we conclude that (9.7) holds for any $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_m^{2,\delta,-}(\bar{Z}_m)$.

Let us note that the only case which prevents us from having $Z_{m+1} \in G_{m+1}(\epsilon_0, \delta)$ is the case $1 \leq i < j \leq m$, where we obtain a lower bound of $\epsilon_0/2$. In all other cases, we can obtain lower bound ϵ_0.

More precisely, for $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_m^{2,\delta,-}(\bar{Z}_m)$, the inequality $|\bar{x}_i(t) - \bar{x}_j(t)| > \epsilon_0$, for all $t \geq \delta$, holds for all $1 \leq i < j \leq m+1$ except the case $1 \leq i < j \leq m$. However, in this case, for any $1 \leq i < j \leq m$, we have $|\bar{x}_i(t) - \bar{x}_j(t)| > \epsilon_0$, for all $t > 0$, since $\bar{Z}_m \in G_m(\epsilon_0, 0)$. Therefore, (9.8) holds for $(\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d) \setminus \mathcal{B}_m^{2,\delta,-}(\bar{Z}_m)$.

We conclude that the set

$$\mathcal{B}_m^{2,-}(\bar{Z}_m) = (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap \left(\mathcal{B}_m^{2,0,-}(\bar{Z}_m) \cup \mathcal{B}_m^{2,\delta,-}(\bar{Z}_m)\right) \tag{9.22}$$

is the set we need for the precollisional case.

Proof of *(ii)*: Here, we use the notation from (9.13). The proof follows the steps of the precollisional case, but we replace the velocities $(\bar{v}_m, v_{m+1})$ by the transformed velocities $(\bar{v}_m', v_{m+1}')$ and then pull-back. It is worth mentioning that the m-th particle needs special treatment since its velocity is transformed to $\bar{v}_m'$. Following similar arguments to the precollisional case, we conclude that the appropriate set for the postcollisional case is given by

$$\mathcal{B}_m^{2,+}(\bar{Z}_m) := (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap \left[V_{m,m+1} \cup \bigcup_{i=1}^{m-1}\left(V_m^{i'} \cup V_{m+1}^{i'}\right)\right], \tag{9.23}$$

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where

$$V_m^{i'} = \{(\omega_1, v_{m+1}) \in \mathbb{S}_1^{d-1} \times B_R^d : \bar{v}_m' \in K_\eta^{d,i}\}, \tag{9.24}$$

$$V_{m+1}^{i'} = \{(\omega_1, v_{m+1}) \in \mathbb{S}_1^{d-1} \times B_R^d : v_{m+1}' \in K_\eta^{d,i}\}, \tag{9.25}$$

$$V_{m,m+1} = \mathbb{S}_1^{d-1} \times B_\eta^d(\bar{v}_m). \tag{9.26}$$

The set

$$\mathcal{B}_m^2(\bar{Z}_m) = \mathcal{B}_m^{2,-}(\bar{Z}_m) \cup \mathcal{B}_m^{2,+}(\bar{Z}_m) \tag{9.27}$$

is the one we need to conclude the proof. □

9.2.2. Measure estimate for binary adjunction

We now estimate the measure of the pathological set $\mathcal{B}_\ell^2(\bar{Z}_m)$ appearing in Proposition 9.2. To control postcollisional configurations, we will strongly rely on the binary transition map introduced in the Appendix (see Proposition 12.2).

Proposition 9.4. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_2 << \epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m \in G_m(\epsilon_0, 0)$, $\ell \in \{1, ..., m\}$ and $\mathcal{B}_\ell^2(\bar{Z}_m)$ the set given in the statement of Proposition 9.2. Then the following measure estimate holds:*

$$\left|\mathcal{B}_\ell^2(\bar{Z}_m)\right| \lesssim mR^d \eta^{\frac{d-1}{2d+2}},$$

where $|\cdot|$ denotes the product measure on $\mathbb{S}_1^{d-1} \times B_R^d$.

Proof. Without loss of generality, we may assume that $\ell = m$. By (9.27), it suffices to estimate the measure of $\mathcal{B}_m^{2,-}(\bar{Z}_m)$ and $\mathcal{B}_m^{2,+}(\bar{Z}_m)$.

Estimate of $\mathcal{B}_m^{2,-}(\bar{Z}_m)$: Recalling (9.5), (9.22), (9.21), (9.19), we have

$$\mathcal{B}_m^{2,-}(\bar{Z}_m) = (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap \left[V_{m,m+1} \cup \bigcup_{i=1}^{m-1} V_{m+1}^i\right], \tag{9.28}$$

where $V_{m,m+1}$ is given by (9.20) and V_{m+1}^i are given by (9.17). By sub-additivity, it suffices to estimate the measure of each term in (9.28).

∘ Estimate of the term corresponding to $V_{m,m+1}$: By (9.20), we have $V_{m,m+1} = \mathbb{S}_1^{d-1} \times B_\eta^d(\bar{v}_m)$, and therefore,

$$|(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap V_{m,m+1}| \leq |\mathbb{S}_1^{d-1} \times (B_R^d \cap B_\eta^d(\bar{v}_m))| \leq |\mathbb{S}_1^{d-1}|_{\mathbb{S}_1^{d-1}} |B_\eta^d(\bar{v}_m)|_d \lesssim \eta^d. \tag{9.29}$$

∘ Estimate of the term corresponding to V_{m+1}^i: By (9.17), we have $V_{m+1}^i = \mathbb{S}_1^{d-1} \times K_\eta^{d,i}$; therefore, by Corollary 8.2, we obtain

$$|(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \times V_{m+1}^i| \leq |\mathbb{S}_1^{d-1} \times (B_R^d \cap K_\eta^{d,i})| \simeq |\mathbb{S}_1^{d-1}|_{\mathbb{S}_1^{d-1}} |B_R^d \cap K_\eta^{d,i}|_d \lesssim R^d \eta^{\frac{d-1}{2}}. \tag{9.30}$$

Using (9.28)–(9.30), subadditivity, and the fact that $\eta << 1$, $m \geq 1$, we obtain

$$|\mathcal{B}_m^{2,-}(\bar{Z}_m)| \lesssim mR^d \eta^{\frac{d-1}{2}}. \tag{9.31}$$

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Estimate of $\mathcal{B}_m^{2,+}(\bar{Z}_m)$: Recalling (9.23), we have

$$\mathcal{B}_m^{2,+}(\bar{Z}_m) = (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap \left[V_{m,m+1} \cup \bigcup_{i=1}^{m-1}\left(V_m^{i'} \cup V_{m+1}^{i'}\right)\right], \tag{9.32}$$

where $V_{m,m+1}$ is given by (9.20) and $V_m^{i'}$, $V_{m+1}^{i'}$ are given by (9.24)–(9.25). By subadditivity, it suffices to estimate the measure of each term in (9.32). The term corresponding to $V_{m,m+1}$ has already been estimated in (9.29). We have

$$|(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m) \cap V_{m,m+1}| \lesssim \eta^d. \tag{9.33}$$

To estimate the measure of the remaining terms, we will strongly rely on the properties of the binary transition map defined in Proposition 12.2. We first introduce some notation. Given $0 < r \leq 2R$, let us define the r-sphere, centered at $\bar{v}_m$:

$$S_r^{d-1}(\bar{v}_m) = \left\{v_{m+1} \in \mathbb{R}^d : |\bar{v}_m - v_{m+1}| = r\right\}.$$

Also, given $v_{m+1} \in \mathbb{R}^d$, we define the set

$$\mathcal{S}_{\bar{v}_m, v_{m+1}}^+ = \left\{\omega_1 \in \mathbb{S}_1^{d-1} : b_2(\omega_1, v_{m+1} - \bar{v}_m) > 0\right\} = \left\{\omega_1 \in \mathbb{S}_1^{d-1} : (\omega_1, v_{m+1}) \in (\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m)\right\}. \tag{9.34}$$

Since $\bar{v}_m \in B_R^d$, triangle inequality implies $B_R^d \subseteq B_{2R}^d(\bar{v}_m)$. Under this notation, Fubini's Theorem, the co-area formula, and relations (9.32)–(9.33) yield

$$\begin{aligned}
|\mathcal{B}_m^{2+}(\bar{Z}_m)| &= \int_{(\mathbb{S}_1^{d-1} \times B_R^d)^+(\bar{v}_m)} \mathbb{1}_{\mathcal{B}_m^{2+}(\bar{Z}_m)}\, d\omega_1\, dv_{m+1} \\
&= \int_{B_R^d} \int_{\mathcal{S}_{\bar{v}_m, v_{m+1}}^+} \mathbb{1}_{\mathcal{B}_m^{2+}(\bar{Z}_m)}\, d\omega_1\, dv_{m+1} \\
&\lesssim \eta^d + \int_0^{2R} \int_{S_r^{d-1}(\bar{v}_m)} \int_{\mathcal{S}_{\bar{v}_m, v_{m+1}}^+} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i'} \cup V_{m+1}^{i'})}(\omega_1)\, d\omega_1\, dv_{m+1}\, dr.
\end{aligned} \tag{9.35}$$

Let us estimate the integral

$$\int_{\mathcal{S}_{\bar{v}_m, v_{m+1}}^+} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i,'} \cup V_{m+1}^{i,'})}(\omega_1)\, d\omega_1,$$

for fixed $0 < r \leq 2R$ and $v_{m+1} \in S_r^{d-1}(\bar{v}_m)$. We introduce a parameter $0 < \beta << 1$, which will be chosen later in terms of η, and decompose $\mathcal{S}_{\bar{v}_m, v_{m+1}}^+$ as follows:

$$\mathcal{S}_{\bar{v}_m, v_{m+1}}^+ = \mathcal{S}_{\bar{v}_m, v_{m+1}}^{1,+} \cup \mathcal{S}_{\bar{v}_m, v_{m+1}}^{2,+}, \tag{9.36}$$

where

$$\mathcal{S}_{\bar{v}_m, v_{m+1}}^{1,+} = \left\{\omega_1 \in \mathcal{S}_{\bar{v}_m, v_{m+1}}^+ : b_2(\omega_1, v_{m+1} - \bar{v}_m) > \beta|v_{m+1} - \bar{v}_m|\right\}, \tag{9.37}$$

and

$$\mathcal{S}_{\bar{v}_m, v_{m+1}}^{2,+} = \left\{\omega_1 \in \mathcal{S}_{\bar{v}_m, v_{m+1}}^+ : b_2(\omega_1, v_{m+1} - \bar{v}_m) \leq \beta|v_{m+1} - \bar{v}_m|\right\}. \tag{9.38}$$

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Notice that $\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1}}$ is the union of two unit $(d-1)$-spherical caps of angle $\pi/2 - \arccos\beta$. Thus, integrating in spherical coordinates, we may estimate its measure as follows:

$$\int_{\mathbb{S}_1^{d-1}} \mathbb{1}_{\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1}}}(\omega_1)\,d\omega_1 \lesssim \int_{\arccos\beta}^{\pi/2} \sin^{d-2}(\theta)\,d\theta \leq \frac{\pi}{2} - \arccos\beta = \arcsin\beta.$$

Thus,

$$\int_{\mathcal{S}^{2,+}_{\bar{v}_m,v_{m+1}}} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i'}\cup V_{m+1}^{i'})}(\omega_1)\,d\omega_1 \lesssim \arcsin\beta. \tag{9.39}$$

We now wish to estimate

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1}}} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i'}\cup V_{m+1}^{i'})}(\omega_1)\,d\omega_1. \tag{9.40}$$

We will use the binary transition map $\mathcal{J}_{\bar{v}_m,m_{m+1}} : \mathcal{S}^{+}_{\bar{v}_m,v_{m+1}} \longrightarrow \mathbb{S}_1^{d-1}$, which is given by

$$v_1 := \mathcal{J}_{\bar{v}_m,v_{m+1}}(\omega_1) = r^{-1}(\bar{v}_m' - v_{m+1}'), \tag{9.41}$$

to change variables in the above integral. For details on the transition map, see Proposition 12.2 in the Appendix. By Proposition 12.2, for $\omega_1 \in \mathcal{S}^{+}_{\bar{v}_m,v_{m+1}}$, the Jacobian matrix of the transition map is

$$\operatorname{Jac}(\mathcal{J}_{\bar{v}_m,v_{m+1}})(\omega_1) \simeq r^{-d} b_2^d(\omega_1, v_{m+1} - \bar{v}_m) > 0.$$

Therefore, for $\omega_1 \in \mathcal{S}^{1+}_{\bar{v}_m,v_{m+1}}$, we have

$$\operatorname{Jac}^{-1}(\mathcal{J}_{\bar{v}_m,v_{m+1}})(\omega_1) \simeq r^d b_2^{-d}(\omega_1, v_{m+1} - \bar{v}_m) \leq r^d \beta^{-d}|v_{m+1} - \bar{v}_m|^{-d} \lesssim \beta^{-d}, \tag{9.42}$$

since $|v_{m+1} - \bar{v}_m| = r$.

For convenience, we express $\bar{v}_m'$, v_{m+1}' in terms of the precollisional velocities $\bar{v}_m$, v_{m+1} and v_1 given by (9.41). Since $|v_{m+1} - \bar{v}_m| = r$, expressions (2.1) yield

$$\bar{v}_m' = \frac{\bar{v}_m + v_{m+1}}{2} + \frac{r}{2}v_1, \tag{9.43}$$

$$v_{m+1}' = \frac{\bar{v}_m + v_{m+1}}{2} - \frac{r}{2}v_1. \tag{9.44}$$

We are now in the position to estimate the integral in (9.40). We first estimate for the term corresponding to $V_m^{i'}$: Recalling (9.24), we have $V_m^{i'} = \left\{(\omega_1, v_{m+1}) \in \mathbb{S}_1^{d-1} \times B_R^d : \bar{v}_m' \in K_\eta^{d,i}\right\}$. By (9.43),

$$\bar{v}_m' \in K_\eta^{d,i} \Leftrightarrow v_1 = \mathcal{J}_{\bar{v}_m,v_{m+1}}(\omega_1) \in \widetilde{K}^{d,i}_{2\eta/r}, \tag{9.45}$$

where $\widetilde{K}^{d,i}_{2\eta/r}$ is a cylinder of radius $2\eta/r$. Therefore, we obtain

$$\begin{aligned}\int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1}}} \mathbb{1}_{V_m^{i'}}(\omega_1)\,d\omega_1 &= \int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1}}} \mathbb{1}_{\bar{v}_m' \in K^{d,i}_{2\eta}}(\omega_1)\,d\omega_1 \\ &= \int_{\mathcal{S}^{1,+}_{\bar{v}_m,v_{m+1}}} (\mathbb{1}_{\widetilde{K}^{d,i}_{2\eta/r}} \circ \mathcal{J}_{\bar{v}_m,v_{m+1}})(\omega_1)\,d\omega_1 \end{aligned} \tag{9.46}$$

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$$\lesssim \beta^{-d} \int_{\mathbb{S}_1^{d-1}} \mathbb{1}_{\widetilde{K}^{d,i}_{2\eta/r}}(v)\, dv \tag{9.47}$$

$$\lesssim \beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}, \tag{9.48}$$

where to obtain (9.46), we use (9.45), to obtain (9.47), we use part *(iv)* of Proposition 12.2 and estimate (9.42), and to obtain (9.48), we use Lemma 8.1.

Hence, for fixed $v_{m+1} \in S_r^{d-1}(\bar{v}_m)$, we have

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}}} \mathbb{1}_{V_m^{i,\prime}}(\omega_1)\, d\omega_1 \lesssim \beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \tag{9.49}$$

Recalling also $V^{i'}_{m+1}$ from (9.25), we obtain in an analogous way the estimate

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}}} \mathbb{1}_{V^{i'}_{m+1}}(\omega_1)\, d\omega_1 \lesssim \beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \tag{9.50}$$

Combining (9.49)–(9.50) and adding for $i = 1, \ldots, m-1$, we obtain

$$\int_{\mathcal{S}^{1,+}_{\bar{v}_m, v_{m+1}}} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i,\prime} \cup V_{m+1}^{i,\prime})}(\omega_1)\, d\omega_1 \lesssim m\beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \tag{9.51}$$

Therefore, recalling (9.36) and using estimates (9.39), (9.51), we obtain the estimate

$$\int_{\mathcal{S}^{+}_{\bar{v}_m, v_{m+1}}} \mathbb{1}_{\bigcup_{i=1}^{m-1}(V_m^{i'} \cup V_{m+1}^{i'})}(\omega_1)\, d\omega_1 \lesssim \arcsin\beta + m\beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}. \tag{9.52}$$

Hence, (9.35) yields

$$\begin{aligned} |\mathcal{B}_m^{2+}(\bar{Z}_m)| &\lesssim \eta^d + \int_0^{2R} \int_{S_r^{d-1}(\bar{v}_m)} \arcsin\beta + m\beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\} dv_{m+1}\, dr \\ &\lesssim \eta^d + \int_0^{2R} r^{d-1}\left(\arcsin\beta + m\beta^{-d} \min\left\{1, \left(\frac{\eta}{r}\right)^{\frac{d-1}{2}}\right\}\right) dr \\ &\lesssim \eta^d + mR^d\left(\arcsin\beta + \beta^{-d}\eta^{\frac{d-1}{2}}\right) \\ &\lesssim mR^d\left(\beta + \beta^{-d}\eta^{\frac{d-1}{2}}\right), \end{aligned} \tag{9.53}$$

after using an estimate similar to (8.2) and the fact that $\eta << 1$, $m \geq 1$, $\beta << 1$. Choosing $\beta = \eta^{\frac{d-1}{2d+2}}$, we obtain

$$|\mathcal{B}_m^{2+}(\bar{Z}_m)| \lesssim mR^d \eta^{\frac{d-1}{2d+2}}. \tag{9.54}$$

Combining (9.27), (9.31), (9.54), and the fact $\eta << 1$, we obtain the required estimate. □

9.3. *Stability under ternary adjunction*

Now, we prove Proposition 9.6 and Proposition 9.7 which will be the inductive step for controlling ternary adjunction of particles. To derive Proposition 9.6 and Proposition 9.7, in addition to results from [5], we develop new algebraic and geometric techniques, thanks to which we can treat the newly formed ternary collisional configuration runs to a binary collision under time evolution.

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9.3.1. Ternary adjunction

For convenience, given $v \in \mathbb{R}^d$, let us denote

$$\left(\mathbb{S}_1^{2d-1} \times B_R^{2d}\right)^+(v) = \left\{(\omega_1, \omega_2, v_1, v_2) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : b_3(\omega_1, \omega_2, v_1 - v, v_2 - v) > 0\right\}, \tag{9.55}$$

where b_3 is the ternary cross-section given in (2.9).

Recall from (9.2) that given $m \in \mathbb{N}$ and $Z_m = (X_m, V_m) \in \mathbb{R}^{2dm}$, we denote the backwards in time free flow as $Z_m(t) = (X_m - tV_m, V_m), \quad t \geq 0$.

Proposition 9.5. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0)$, $\ell \in \{1, ..., m\}$, and $X_m \in B_{\alpha/2}^{dm}(\bar{X}_m)$. Let us denote*

$$\mathcal{F}_{m+2}^{\ell} = \{(i, j) \in \{1, ..., m+2\} \times \{1, ..., m+2\} : i \neq \ell,\ i \leq \min\{j, m\}\}.$$

Then there is a subset $\widetilde{\mathcal{B}}_\ell^3(\bar{Z}_m) \subseteq (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m)$ such that

1. *For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \setminus \widetilde{\mathcal{B}}_\ell^3(\bar{Z}_m)$, one has*

$$\begin{aligned} d_2(x_i(t), x_j(t)) &> \sqrt{2}\epsilon_3, \quad \forall (i, j) \in \mathcal{F}_{m+2}^{\ell}, \quad \forall t \geq 0, \\ d_3(x_\ell(t); x_{m+1}(t), x_{m+2}(t)) &> \sqrt{2}\epsilon_3, \quad \forall t \geq 0, \\ Z_{m+2} &\in G_{m+2}(\epsilon_0/2, \delta), \\ \bar{Z}_{m+2} &\in G_{m+2}(\epsilon_0, \delta), \end{aligned} \tag{9.56}$$

where

$$\begin{aligned} Z_{m+2} &= (x_1, ..., x_\ell, ..., x_m, x_{m+1}, x_{m+2}, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v_{m+1}, v_{m+2}), \\ x_{m+i} &= x_\ell + \sqrt{2}\epsilon_3\omega_i, \quad \forall i \in \{1, 2\}, \\ \bar{Z}_{m+2} &= (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_m, \bar{x}_m, \bar{v}_1, ..., \bar{v}_\ell, ..., \bar{v}_m, v_{m+1}, v_{m+2}). \end{aligned}$$

2. *For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_\ell) \setminus \widetilde{\mathcal{B}}_\ell^3(\bar{Z}_m)$, one has*

$$\begin{aligned} d_2(x_i(t), x_j(t)) &> \sqrt{2}\epsilon_3, \quad \forall (i, j) \in \mathcal{F}_{m+2}^{\ell}, \quad \forall t \geq 0, \\ d_3(x_\ell(t); x_{m+1}(t), x_{m+2}(t)) &> \sqrt{2}\epsilon_3, \quad \forall t \geq 0, \\ Z_{m+2}^* &\in G_{m+2}(\epsilon_0/2, \delta), \\ \bar{Z}_{m+2}^* &\in G_{m+2}(\epsilon_0, \delta), \end{aligned} \tag{9.57}$$

where

$$\begin{aligned} Z_{m+2}^* &= (x_1, ..., x_\ell, ..., x_m, x_{m+1}, x_{m+2}, \bar{v}_1, ..., \bar{v}_\ell^*, ..., \bar{v}_m, v_{m+1}^*, v_{m+2}^*), \\ x_{m+i} &= x_\ell + \sqrt{2}\epsilon_3\omega_i, \quad \forall i \in \{1, 2\}, \\ \bar{Z}_{m+2}^* &= (\bar{x}_1, ..., \bar{x}_\ell, ..., \bar{x}_m, \bar{x}_m, \bar{x}_m, \bar{v}_1, ..., \bar{v}_\ell^*, ..., \bar{v}_m, v_{m+1}^*, v_{m+2}^*), \\ (\bar{v}_\ell^*, v_{m+1}^*, v_{m+2}^*) &= T_{\omega_1, \omega_2}(\bar{v}_\ell, v_{m+1}, v_{m+2}). \end{aligned}$$

There also holds the measure estimate

$$|\widetilde{\mathcal{B}}_\ell^3(\bar{Z}_m)| \lesssim m R^{2d} \eta^{\frac{d-1}{4d+2}}, \tag{9.58}$$

where $|\cdot|$ denotes the product measure on $\mathbb{S}_1^{2d-1} \times B_R^{2d}$.

Proof. This Proposition follows from the statement and the proof of Proposition 9.2 and the statement of Proposition 9.4 from [5]. ◻

We rely on Proposition 9.5 to derive Proposition 9.6 and Proposition 9.7. Recall the notation from (3.7)

$$\mathring{\mathcal{D}}_{m+2,\epsilon_2,\epsilon_3} = \Big\{Z_{m+2} = (X_{m+2}, V_{m+2}) \in \mathbb{R}^{2d(m+2)} : \ d_2(x_i, x_j) > \epsilon_2, \quad \forall (i,j) \in \mathcal{I}^2_{m+2}, \\ \text{and } d_3(x_i; x_j, x_k) > \sqrt{2}\epsilon_3, \quad \forall (i,j,k) \in \mathcal{I}^3_{m+2}\Big\},$$

where $\mathcal{I}^2_{m+2}, \mathcal{I}^3_{m+2}$ are given by (3.1)–(3.2), respectively.

Proposition 9.6. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_2 << \eta^2\epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m = (\bar{X}_m, \bar{V}_m) \in G_m(\epsilon_0, 0)$, $\ell \in \{1, \ldots, m\}$ and $X_m \in B^{dm}_{\alpha/2}(\bar{X}_m)$. Then there is a subset $\mathcal{B}^3_\ell(\bar{Z}_m) \subseteq (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+(\bar{v}_\ell)$ such that*

1. *For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+(\bar{v}_\ell) \setminus \mathcal{B}^3_\ell(\bar{Z}_m)$, one has*

$$Z_{m+2}(t) \in \mathring{\mathcal{D}}_{m+2,\epsilon_2,\epsilon_3}, \quad \forall t \geq 0, \tag{9.59}$$

$$Z_{m+2} \in G_{m+2}(\epsilon_0/2, \delta) \tag{9.60}$$

$$\bar{Z}_{m+2} \in G_{m+2}(\epsilon_0, \delta), \tag{9.61}$$

where

$$\begin{aligned} &Z_{m+2} = (x_1, \ldots, x_\ell, \ldots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}, v_{m+2}), \\ &x_{m+i} = x_\ell - \sqrt{2}\epsilon_3\omega_i, \quad \forall i \in \{1,2\}, \\ &\bar{Z}_{m+2} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_m, \bar{x}_m, \bar{v}_1, \ldots, \bar{v}_\ell, \ldots, \bar{v}_m, v_{m+1}, v_{m+2}). \end{aligned} \tag{9.62}$$

2. *For any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+(\bar{v}_\ell) \setminus \mathcal{B}^3_\ell(\bar{Z}_m)$, one has*

$$Z^*_{m+2}(t) \in \mathring{\mathcal{D}}_{m+2,\epsilon_2,\epsilon_3}, \quad \forall t \geq 0, \tag{9.63}$$

$$Z^*_{m+2} \in G_{m+2}(\epsilon_0/2, \delta), \tag{9.64}$$

$$\bar{Z}^*_{m+2} \in G_{m+2}(\epsilon_0, \delta), \tag{9.65}$$

where

$$\begin{aligned} &Z^*_{m+2} = (x_1, \ldots, x_\ell, \ldots, x_m, x_{m+1}, x_{m+2}, \bar{v}_1, \ldots, \bar{v}^*_\ell, \ldots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}), \\ &x_{m+i} = x_\ell + \sqrt{2}\epsilon_3\omega_i, \quad \forall i \in \{1,2\}, \\ &\bar{Z}^*_{m+2} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_m, \bar{x}_m, \bar{x}_m, \bar{v}_1, \ldots, \bar{v}^*_\ell, \ldots, \bar{v}_m, v^*_{m+1}, v^*_{m+2}), \\ &(\bar{v}^*_\ell, v^*_{m+1}, v^*_{m+2}) = T_{\omega_1,\omega_2}(\bar{v}_\ell, v_{m+1}, v_{m+2}). \end{aligned} \tag{9.66}$$

Proof. By symmetry, we may assume that $\ell = m$. Recall the set $\widetilde{\mathcal{B}}^3_m(\bar{Z}_m)$ from Proposition 9.5 satisfying (9.56)–(9.57).

We will construct a set $\mathcal{A}_m(\bar{Z}_m) \subseteq (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+(\bar{v}_m)$, such that for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}^{2d-1}_1 \times B^{2d}_R)^+(\bar{v}_m) \setminus \mathcal{A}_m(\bar{Z}_m)$.

- Using notation from (9.62) for the precollisional case, we have

$$|x_i(t) - x_j(t)| > \epsilon_2, \quad \forall t \geq 0, \quad \forall i,j \in \{m, m+1, m+2\} \text{ with } i < j. \tag{9.67}$$

- Using notation from (9.66) for the postcollisional case, we have

$$|x_i(t) - x_j(t)| > \epsilon_2, \quad \forall t \geq 0, \quad \forall i,j \in \{m, m+1, m+2\} \text{ with } i < j. \tag{9.68}$$

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Then thanks to Proposition 9.5 and (9.67)–(9.68), the set

$$\mathcal{B}_m^3(\bar{Z}_m) := \widetilde{\mathcal{B}}_m^3(\bar{Z}_m) \cup \mathcal{A}_m(\bar{Z}_m)$$

will satisfy (9.59)–(9.61), (9.63)–(9.65). Let us introduce the following notation:

$$\gamma := \frac{\epsilon_2}{\epsilon_3} << \eta^2, \quad \text{since } \epsilon_2 << \eta^2 \epsilon_3, \text{ by assumption,} \tag{9.69}$$

and

$$\gamma' = \left(1 - \frac{\gamma}{2}\right)^{1/2} < 1. \tag{9.70}$$

Construction of the set satisfying (9.67): Here, we use notation from (9.62). We distinguish the following cases:

∘ Case $(i, j) = (m, m+1)$: Consider $t \geq 0$. We have

$$\begin{aligned} |x_i(t) - x_j(t)|^2 &= |x_m(t) - x_{m+1}(t)|^2 \\ &= |\sqrt{2}\epsilon_3\omega_1 + (v_{m+1} - \bar{v}_m)t|^2 \\ &= 2\epsilon_3^2|\omega_1|^2 + 2\sqrt{2}\epsilon_3\langle\omega_1, v_{m+1} - \bar{v}_m\rangle t + |v_{m+1} - \bar{v}_m|^2 t^2. \end{aligned} \tag{9.71}$$

We define the sets

$$\Omega_1 = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : |\omega_1| \leq \sqrt{\gamma}\}, \tag{9.72}$$

$$A_{m,m+1} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : |\langle\omega_1, v_{m+1} - \bar{v}_m\rangle| \geq \gamma'|\omega_1||v_{m+1} - \bar{v}_m|\}. \tag{9.73}$$

Consider the second degree polynomial in t:

$$P(t) = (2-\gamma)\epsilon_3^2|\omega_1|^2 + 2\sqrt{2}\epsilon_3\langle\omega_1, v_{m+1} - \bar{v}_m\rangle t + |v_{m+1} - \bar{v}_m|^2 t^2. \tag{9.74}$$

Let $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_1 \cup A_{m,m+1})$. The polynomial P has discriminant

$$\begin{aligned} \Delta &= 8\epsilon_3^2|\langle\omega_1, v_{m+1} - \bar{v}_m\rangle|^2 - 4(2-\gamma)\epsilon_3^2|\omega_1|^2|v_{m+1} - \bar{v}_m|^2 \\ &= 8\epsilon_3^2|\langle\omega_1, v_{m+1} - \bar{v}_m\rangle|^2 - 8\gamma'^2\epsilon_3^2|\omega_1|^2|v_{m+1} - \bar{v}_m|^2 \\ &= 8\epsilon_3^2\left(|\langle\omega_1, v_{m+1} - \bar{v}_m\rangle|^2 - \gamma'^2|\omega_1|^2|v_{m+1} - \bar{v}_m|^2\right) \\ &< 0 \end{aligned}$$

since $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \notin A_{m,m+1}$. Since $\gamma << 1$, we obtain $P(t) > 0$, for all $t \geq 0$, or in other words,

$$2\epsilon_3^2|\omega_1|^2 + 2\sqrt{2}\epsilon_3\langle\omega_1, v_{m+1} - \bar{v}_m\rangle t + |v_{m+1} - \bar{v}_m|^2 t^2 > \gamma\epsilon_3^2|\omega_1|^2. \tag{9.75}$$

Since $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \notin \Omega_1$, expressions (9.71), (9.75) yield

$$|x_m(t) - x_{m+1}(t)|^2 > \gamma\epsilon_3^2|\omega_1|^2 > \gamma^2\epsilon_3^2 = \epsilon_2^2. \tag{9.76}$$

Therefore, for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_1 \cup A_{m,m+1})$, we have

$$|x_m(t) - x_{m+1}(t)| > \epsilon_2, \quad \forall t \geq 0.$$

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∘ Case $(i, j) = (m, m+2)$: We follow a similar argument using the sets

$$\Omega_2 = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : |\omega_2| \leq \sqrt{\gamma}\}, \tag{9.77}$$

$$A_{m,m+2} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : |\langle \omega_2, v_{m+2} - \bar{v}_m \rangle| \geq \gamma' |\omega_2| |v_{m+2} - \bar{v}_m|\} \tag{9.78}$$

to conclude that for all $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup A_{m,m+2})$, we have

$$|x_{m+2}(t) - x_m(t)| > \epsilon_2, \quad \forall t \geq 0.$$

∘ Case $(i, j) = (m+1, m+2)$: We follow a similar argument using the sets

$$\Omega_{1,2} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : |\omega_1 - \omega_2| \leq \sqrt{\gamma}\}, \tag{9.79}$$

$$\begin{aligned} B_{m+1,m+2} = \{&(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \\ &|\langle \omega_1 - \omega_2, v_{m+1} - v_{m+2} \rangle| \geq \gamma' |\omega_1 - \omega_2| |v_{m+1} - v_{m+2}|\} \end{aligned} \tag{9.80}$$

to conclude that for all $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_{1,2} \cup B_{m+1,m+2})$, we have

$$|x_{m+1}(t) - x_{m+2}(t)| > \epsilon_2, \quad \forall t \geq 0.$$

Defining

$$\mathcal{A}_m^-(\bar{Z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1} \cup A_{m,m+2} \cup B_{m+1,m+2}, \tag{9.81}$$

we obtain that (9.67) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{A}_m^-(\bar{Z}_m)$.

Construction of the set satisfying (9.68): Here, we use notation from (9.66). We distinguish the following cases:

∘ Case $(i, j) = (m, m+1)$: We follow a similar argument to the precollisional case, using the set Ω_1, defined in (9.72), and the set

$$A^*_{m,m+1} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \left|\langle \omega_1, v^*_{m+1} - \bar{v}^*_m \rangle\right| \geq \gamma' |\omega_1| |v^*_{m+1} - \bar{v}^*_m|\} \tag{9.82}$$

to conclude that for all $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup A^*_{m,m+1})$, we have

$$|x_{m+1}(t) - x_m(t)| > \epsilon_2, \quad \forall t \geq 0.$$

∘ Case $(i, j) = (m, m+2)$: We follow a similar argument to the precollisional case, using the set Ω_2, defined in (9.77), and the set

$$A^*_{m,m+2} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \left|\langle \omega_2, v^*_{m+2} - \bar{v}^*_m \rangle\right| \geq \gamma' |\omega_2| |v^*_{m+2} - \bar{v}^*_m|\} \tag{9.83}$$

to conclude that for all $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup A^*_{m,m+2})$, we have

$$|x_{m+2}(t) - x_m(t)| > \epsilon_2, \quad \forall t \geq 0.$$

∘ Case $(i, j) = (m+1, m+2)$: We follow a similar argument to the precollisional case, using the set $\Omega_{1,2}$, defined in (9.79), and the set

$$B^*_{m+1,m+2} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \tag{9.84}$$

$$\left|\langle \omega_1 - \omega_2, v^*_{m+1} - v^*_{m+2} \rangle\right| \geq \gamma' |\omega_1 - \omega_2| |v^*_{m+1} - v^*_{m+2}|\} \tag{9.85}$$

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to conclude that for all $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus (\Omega_2 \cup B^*_{m+1,m+2})$, we have

$$|x_{m+1}(t) - x_{m+2}(t)| > \epsilon_2, \quad \forall t \geq 0.$$

Defining

$$\mathcal{A}_m^+(\bar{Z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A^*_{m,m+1} \cup A^*_{m,m+2} \cup B^*_{m+1,m+2}, \tag{9.86}$$

we obtain that (9.68) holds for $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d}) \setminus \mathcal{A}_m^+(\bar{Z}_m)$.

Defining

$$\mathcal{A}_m(\bar{Z}_m) = (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \cap \left(\mathcal{A}_m^-(\bar{Z}_m) \cup \mathcal{A}_m^+(\bar{Z}_m)\right), \tag{9.87}$$

(9.67)–(9.68) hold for any $(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in (\mathbb{S}_1^{2d-1} \times B_R^{2d})^+(\bar{v}_m) \setminus \mathcal{A}_m(\bar{Z}_m)$.

The set

$$\mathcal{B}_m^3(\bar{Z}_m) = \widetilde{\mathcal{B}}_m^3(\bar{Z}_m) \cup \mathcal{A}_m(\bar{Z}_m) \tag{9.88}$$

satisfies (9.59)–(9.61), (9.63)–(9.65); thus, it is the set we need to conclude the proof. □

9.3.2. Measure estimate for ternary adjunction

We now provide the corresponding measure estimate for the set $\mathcal{B}_\ell^3(\bar{Z}_m)$ appearing in Proposition 9.6. To estimate the measure of this set, we will strongly rely on the results of Section 8.

Proposition 9.7. *Consider parameters $\alpha, \epsilon_0, R, \eta, \delta$ as in (9.4) and $\epsilon_2 << \eta^2 \epsilon_3 << \alpha$. Let $m \in \mathbb{N}$, $\bar{Z}_m \in G_m(\epsilon_0, 0)$, $\ell \in \{1, ..., m\}$ and $\mathcal{B}_\ell^3(\bar{Z}_m)$ be the set appearing in the statement of Proposition 9.6. Then the following measure estimate holds:*

$$\left|\mathcal{B}_\ell^3(\bar{Z}_m)\right| \lesssim mR^{2d} \eta^{\frac{d-1}{4d+2}},$$

where $|\cdot|$ denotes the product measure on $\mathbb{S}_1^{2d-1} \times B_R^{2d}$.

Proof. By symmetry, we may assume $\ell = m$. Recall that

$$\mathcal{B}_m^3(\bar{Z}_m) = \widetilde{\mathcal{B}}_m^3(\bar{Z}_m) \cup \mathcal{A}_m(\bar{Z}_m), \tag{9.89}$$

where $\widetilde{\mathcal{B}}_m^3(\bar{Z}_m)$ is given by Proposition 9.5 and $\mathcal{A}_m(\bar{Z}_m)$ is given by (9.87). Estimate (9.58) yields

$$|\widetilde{\mathcal{B}}_m^3(\bar{Z}_m)| \lesssim mR^{2d} \eta^{\frac{d-1}{4d+2}}, \tag{9.90}$$

so it suffices to estimate the measure of $\mathcal{A}_m(\bar{Z}_m)$. By (9.87), it suffices to estimate the measure of $\mathcal{A}_m^-(\bar{Z}_m)$ and $\mathcal{A}_m^+(\bar{Z}_m)$ which are given by (9.81), (9.86), respectively.

Let us recall the notation from (9.69)–(9.70):

$$\gamma = \frac{\epsilon_2}{\epsilon_3} << \eta^2, \quad \gamma' = \sqrt{1 - \frac{\gamma}{2}}.$$

Estimate of $\mathcal{A}_m^-(\bar{Z}_m)$: Recall from (9.81) that

$$\mathcal{A}_m^-(\bar{Z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A_{m,m+1} \cup A_{m,m+2} \cup B_{m+1,m+2}, \tag{9.91}$$

where $\Omega_1, A_{m,m+1}$ are given by (9.72)–(9.73), $\Omega_2, A_{m,m+2}$ by (9.77)–(9.78) and $\Omega_{1,2}, B_{m+1,m+2}$ are given by (9.79)–(9.80).

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∘ Estimate for Ω_1, Ω_2: Without loss of generality, it suffices to estimate the measure of Ω_1. Recalling notation from (8.3), Fubini's Theorem and Lemma 8.3 yield

$$|\Omega_1| = \int_{B_R^{2d}} \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{M_1(\sqrt{\gamma})} \, d\omega_1 \, d\omega_2 \, dv_{m+1} \, dv_{m+2} \lesssim R^{2d} \gamma^{d/2}, \tag{9.92}$$

A symmetric argument yields

$$|\Omega_2| \lesssim R^{2d} \gamma^{d/2}. \tag{9.93}$$

∘ Estimate for $\Omega_{1,2}$: Recalling notation from (8.5), (9.79) yields

$$\Omega_{1,2} = (\mathbb{S}_1^{2d-1} \cap W_{\sqrt{\gamma}}^{2d}) \times B_R^{2d}.$$

Therefore, Fubini's Theorem and Lemma 8.4 imply

$$|\Omega_{1,2}| = \int_{B_R^{2d}} \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{W_{\sqrt{\gamma}}^{2d}} \, d\omega_1 \, d\omega_2 \, dv_{m+1} \, dv_{m+2} \lesssim R^{2d} \gamma^{\frac{d-1}{4}}. \tag{9.94}$$

∘ Estimate for $A_{m,m+1}$: Recalling notation from (8.6), the set $A_{m,m+1}$, which was defined in (9.73), can be written as

$$A_{m,m+1} = \left\{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : \omega_1 \in S(\gamma', v_{m+1} - \bar{v}_m)\right\}.$$

Therefore, the representation of the $(2d-1)$- unit sphere (8.1) and Lemma 8.5 yield

$$\begin{aligned} |A_{m,m+1}| &\le \int_{B_R^{2d}} \int_{B_1^d} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_2|^2}}} \mathbb{1}_{S(\gamma', v_{m+1} - \bar{v}_m)} \, d\omega_1 \, d\omega_2 \, dv_{m+1} \, dv_{m+2} \\ &\lesssim R^{2d} \arccos \gamma' \\ &= R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}. \end{aligned} \tag{9.95}$$

∘ Estimate for $A_{m,m+2}$: We follow a similar argument as in the previous case to obtain

$$|A_{m,m+2}| \lesssim R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}. \tag{9.96}$$

∘ Estimate for $B_{m+1,m+2}$: Recalling notation from (8.7), (9.80) yields

$$B_{m+1,m+2} = \{(\omega_1, \omega_2, v_{m+1}, v_{m+2}) \in \mathbb{S}_1^{2d-1} \times B_R^{2d} : (\omega_1, \omega_2) \in N(\gamma', v_{m+1} - v_{m+2})\}.$$

Therefore, using Lemma 8.6, we obtain

$$\begin{aligned} |B_{m+1,m+2}| &= \int_{B_R^{2d}} \int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{N(\gamma', v_{m+1} - v_{m+2})}(\omega_1, \omega_2) \, d\omega_1 \, d\omega_2 \, dv_{m+1} \, dv_{m+2} \\ &\lesssim R^{2d} \arccos \gamma' \\ &= R^{2d} \arccos \sqrt{1 - \frac{\gamma}{2}}. \end{aligned} \tag{9.97}$$

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Using (9.91) and estimates (9.92)–(9.97), we obtain

$$|\mathcal{A}_m^-(\bar{Z}_m)| \lesssim R^{2d}\left(\gamma^{d/2} + \gamma^{\frac{d-1}{4}} + \arccos\sqrt{1-\frac{\gamma}{2}}\right). \tag{9.98}$$

Estimate of $\mathcal{A}_m^+(\bar{Z}_m)$: Recall from (9.86) that

$$\mathcal{A}_m^+(\bar{Z}_m) = \Omega_1 \cup \Omega_2 \cup \Omega_{1,2} \cup A^*_{m,m+1} \cup A^*_{m,m+2} \cup B^*_{m+1,m+2}, \tag{9.99}$$

where Ω_1, Ω_2, $\Omega_{1,2}$, $A^*_{m,m+1}$, $A^*_{m,m+2}$, $B^*_{m+1,m+2}$ are given by (9.72), (9.77), (9.79), (9.82)–(9.85), respectively. We already have estimates for Ω_1, Ω_2, $\Omega_{1,2}$ from (9.92)–(9.94); hence, it suffices to derive estimates for $A^*_{m,m+1}$, $A^*_{m,m+2}$, $B^*_{m+1,m+2}$.

For the rest of the proof, we consider a parameter $0 < \beta << 1$ which will be chosen later in terms of η, see (9.149).

∘ Estimate for $A^*_{m,m+1}$: Recall from (9.82) the set

$$A^*_{m,m+1} = \left\{(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d} : |\langle\omega_1, v^*_{m+1} - \bar{v}^*_m\rangle| \geq \gamma'|\omega_1||v^*_{m+1} - \bar{v}^*_m|\right\}. \tag{9.100}$$

But for any $(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d}$, the ternary collisional law (2.8) implies

$$v^*_{m+1} - \bar{v}^*_m = v_{m+1} - \bar{v}_m - 2c_{\omega_1,\omega_2,\bar{v}_m,v_{m+1},v_{m+2}}\omega_1 - c_{\omega_1,\omega_2,\bar{v}_m,v_{m+1},v_{m+2}}\omega_2,$$

where

$$c_{\omega_1,\omega_2,\bar{v}_m,v_{m+1},v_{m+2}} = \frac{\langle\omega_1, v_{m+1}-\bar{v}_m\rangle + \langle\omega_2, v_{m+2}-\bar{v}_m\rangle}{1+\langle\omega_1,\omega_2\rangle}. \tag{9.101}$$

For convenience, we denote

$$c := c_{\omega_1,\omega_2,\bar{v}_m,v_{m+1},v_{m+2}}.$$

Therefore, by (9.100), we may write

$$\begin{aligned} A^*_{m,m+1} = \{&(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d} : \\ &|\langle\omega_1, v_{m+1}-\bar{v}_m - 2c\omega_1 - c\omega_2\rangle| \geq \gamma'|\omega_1||v_{m+1}-\bar{v}_m - 2c\omega_1 - c\omega_2|\}. \end{aligned}$$

By Fubini's Theorem, we have

$$|A^*_{m,m+1}| \leq \int_{\mathbb{S}_1^{2d-1}\times B_R^d}\int_{B_R^d} \mathbb{1}_{V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}}}(v_{m+1})\,dv_{m+1}\,d\omega_1\,d\omega_2\,dv_{m+2}, \tag{9.102}$$

where given $(\omega_1,\omega_2,v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^d$, we write

$$V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}} = \left\{v_{m+1} \in B_R^d : (\omega_1,\omega_2,v_{m+1},v_{m+2}) \in A^*_{m,m+1}\right\}. \tag{9.103}$$

Recall from (8.11) the set

$$I_1 = \left\{(\omega_1,\omega_2) \in \mathbb{R}^{2d} \big| 1 - 2|\omega_1|^2\big| \leq 2\beta\right\}. \tag{9.104}$$

Using (9.102), we obtain

$$|A^*_{m,m+1}| = \widetilde{I}_1 + \widetilde{I}_1', \tag{9.105}$$

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where

$$\widetilde{I}_1 = \int_{(\mathbb{S}_1^{2d-1}\cap I_1)\times B_R^d}\int_{B_R^d} \mathbb{1}_{V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}}}(v_{m+1})\, dv_{m+1}\, d\omega_1\, d\omega_2\, dv_{m+2}, \tag{9.106}$$

$$\widetilde{I}_1' = \int_{(\mathbb{S}_1^{2d-1}\setminus I_1)\times B_R^d}\int_{B_R^d} \mathbb{1}_{V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}}}(v_{m+1})\, dv_{m+1}\, d\omega_1\, d\omega_2\, dv_{m+2}. \tag{9.107}$$

We treat each of the terms in (9.105) separately.

Estimate for $\widetilde{I}_1$: By (9.106), Fubini's Theorem and Lemma 8.7, we obtain

$$\widetilde{I}_1 \lesssim R^{2d}\int_{\mathbb{S}_1^{2d-1}} \mathbb{1}_{I_1}\, d\omega_1\, d\omega_2 \lesssim R^{2d}\beta. \tag{9.108}$$

Estimate for $\widetilde{I}_1'$: Let us fix $(\omega_1,\omega_2,v_{m+2})\in(\mathbb{S}_1^{2d-1}\setminus I_1)\times B_R^d$. We define the smooth map $F^1_{\omega_1,\omega_2,v_{m+2}}: B_R^d\to\mathbb{R}^d$, by

$$F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1}) := v^*_{m+1} - \bar{v}^*_m = v_{m+1} - \bar{v}_m - 2c\omega_1 - c\omega_2, \tag{9.109}$$

where c is given by (9.101).

We are showing that we may change variables under $F^1_{\omega_1,\omega_2,v_{m+2}}$, as long as $(\omega_1,\omega_2,v_{m+1})\in(\mathbb{S}_1^{2d-1}\setminus I_1)\times B_R^d$ (i.e., we are showing that $F^1_{\omega_1,\omega_2,v_{m+2}}$ has nonzero Jacobian and is injective). In, particular we will see that the Jacobian is bounded from below by β.

We first show the Jacobian has a lower bound β. Differentiating with respect to v_{m+1}, we obtain

$$\frac{\partial F^1_{\omega_1,\omega_2,v_{m+2}}}{\partial v_{m+1}} = I_d + (-2\omega_1-\omega_2)\nabla^T_{v_{m+1}}c.$$

Recalling (9.101), we have

$$\nabla^T_{v_{m+1}}c = \frac{1}{1+\langle\omega_1,\omega_2\rangle}\omega_1^T.$$

Using Lemma 12.1 from the Appendix, we get

$$\begin{aligned}\operatorname{Jac} F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1}) &= \det\left(I_d + \frac{1}{1+\langle\omega_1,\omega_2\rangle}(-2\omega_1-\omega_2)\omega_1^T\right)\\ &= 1 + \frac{-2|\omega_1|^2 - \langle\omega_1,\omega_2\rangle}{1+\langle\omega_1,\omega_2\rangle}\\ &= \frac{1-2|\omega_1|^2}{1+\langle\omega_1,\omega_2\rangle}.\end{aligned}$$

Since $(\omega_1,\omega_2)\notin I_1$, we have $\left|1-2|\omega_1|^2\right| > 2\beta$, and hence,

$$\left|\operatorname{Jac} F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})\right| = \frac{\left|1-2|\omega_1|^2\right|}{1+\langle\omega_1,\omega_2\rangle} > \frac{2\beta}{1+\langle\omega_1,\omega_2\rangle} \geq \frac{4\beta}{3} > \beta, \tag{9.110}$$

since $\frac{1}{2}\leq 1+\langle\omega_1,\omega_2\rangle\leq\frac{3}{2}$, by (2.10). Thus,

$$\left|\operatorname{Jac} F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})\right|^{-1} < \beta^{-1}, \quad \forall v_{m+1}\in B_R^d. \tag{9.111}$$

We now show that $F^1_{\omega_1,\omega_2,v_{m+2}}$ is injective. For this purpose, consider $v_{m+1},\xi_{m+1}\in B_R^d$ such that

$$\begin{aligned} &F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1}) = F^1_{\omega_1,\omega_2,v_{m+2}}(\xi_{m+1}) \\ &\Leftrightarrow v_{m+1}-\xi_{m+1} = \frac{\langle v_{m+1}-\xi_{m+1},\omega_1\rangle}{1+\langle\omega_1,\omega_2\rangle}(2\omega_1+\omega_2), \end{aligned} \tag{9.112}$$

thanks to (9.101). Therefore, there is $\lambda\in\mathbb{R}$ such that

$$v_{m+1}-\xi_{m+1} = \lambda(2\omega_1+\omega_2), \tag{9.113}$$

so replacing $v_{m+1}-\xi_{m+1}$ in (9.112) with the right-hand side of (9.113), we obtain

$$\lambda(1-2|\omega_1|^2)=0,$$

which yields $\lambda=0$, since we have assumed $(\omega_1,\omega_2)\notin I_1$. Therefore, $v_{m+1}=\xi_{m+1}$, thus $F^1_{\omega_1,\omega_2,v_{m+2}}$ is injective.

Since $(\omega_1,\omega_2,v_{m+2})\in\mathbb{S}_1^{2d-1}\times B_R^d$ and $\bar{v}_m\in B_R^d$, Cauchy-Schwartz inequality yields that, for any $v_{m+1}\in B_R^d$, we have

$$|F_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})| \le |v_{m+1}|+|\bar{v}_m| + \frac{|\omega_1|(|v_{m+1}|+|\bar{v}_m|)+|\omega_2|(|\bar{v}_m|+|v_{m+2}|)}{1+\langle\omega_1,\omega_2\rangle}(2|\omega_1|+|\omega_2|) \le 26R,$$

by the fact that $(\omega_1,\omega_2,v_{m+2})\in\mathbb{S}_1^{2d-1}\times B_R^d$ and (2.10). Therefore,

$$F^1_{\omega_1,\omega_2,v_{m+2}}[B_R^d]\subseteq B_{26R}^d. \tag{9.114}$$

Additionally, recalling (9.103), (9.100) and (9.109), we have

$$V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}} = \{v_{m+1}\in B_R^d : \langle\omega_1, F_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})\rangle \ge \beta|\omega_1||F_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})|\},$$

and thus,

$$v_{m+1}\in V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}} \Leftrightarrow F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})\in U_{\omega_1}, \tag{9.115}$$

where

$$U_{\omega_1}=\left\{\nu\in\mathbb{R}^d : \langle\omega_1,\nu\rangle \ge \gamma'|\omega_1||\nu|\right\}. \tag{9.116}$$

Hence,

$$\mathbb{1}_{V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}}}(v_{m+1}) = \mathbb{1}_{U_{\omega_1}}(F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})), \quad \forall v_{m+1}\in B_R^d. \tag{9.117}$$

Therefore, performing the substitution $\nu := F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1})$, and using (9.111), we obtain

$$\int_{B_R^d}\mathbb{1}_{V^{m,m+1}_{\omega_1,\omega_2,v_{m+2}}}(v_{m+1})\,dv_{m+1} = \int_{B_R^d}\mathbb{1}_{U_{\omega_1}}(F^1_{\omega_1,\omega_2,v_{m+2}}(v_{m+1}))\,dv_{m+1} \le \beta^{-1}\int_{B_{26R}^d}\mathbb{1}_{U_{\omega_1}}(\nu)\,d\nu.$$

Recalling notation from (8.6) and (9.116), we have

$$\mathbb{1}_{U_{\omega_1}}(\nu) = \mathbb{1}_{S(\gamma',\nu)}(\omega_1), \quad \forall\omega_1\in B_1^d, \quad \forall\nu\in B_{26R}^d. \tag{9.118}$$

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Therefore, using (9.107), (9.118), Fubini's Theorem and (9.118), we obtain

$$\begin{aligned} I_1' &\leq \beta^{-1} \int_{(\mathbb{S}_1^{2d-1}\setminus I_1)\times B_R^d} \int_{B_{26R}^d} \mathbb{1}_{U_{\omega_1}}(v)\, dv\, d\omega_1\, d\omega_2\, dv_{m+2} \\ &\leq \beta^{-1} \int_{B_{26R}^d\times B_R^d} \int_{B_1^d} \int_{\mathbb{S}^{d-1}_{\sqrt{1-|\omega_2|^2}}} \mathbb{1}_{S(\gamma',v)(\omega_1)}\, d\omega_1\, d\omega_2\, dv\, dv_{m+2} \\ &\lesssim R^{2d}\beta^{-1} \arccos \gamma' \qquad (9.119) \\ &= R^{2d}\beta^{-1} \arccos \sqrt{1-\frac{\gamma}{2}}, \qquad (9.120) \end{aligned}$$

where to obtain (9.119), we use Lemma 8.5. Combining (9.105), (9.108), (9.120), we obtain

$$|A^*_{m,m+1}| \leq R^{2d}\left(\beta + \beta^{-1} \arccos \sqrt{1-\frac{\gamma}{2}}\right). \qquad (9.121)$$

∘ Estimate for $A^*_{m,m+2}$: The argument is entirely symmetric, using the set

$$V^{m,m+2}_{\omega_1,\omega_2,v_{m+1}} = \left\{v_{m+2} \in B_R^d : (\omega_1,\omega_2,v_{m+1},v_{m+2}) \in A^*_{m,m+2}\right\},$$

for fixed $(\omega_1,\omega_2,v_{m+1}) \in \mathbb{S}_1^{2d-1}\times B_R^d$ and the map

$$F^2_{\omega_1,\omega_2,v_{m+1}}(v_{m+2}) = v_{m+2} - \bar{v}_m - c\omega_1 - 2c\omega_2.$$

We obtain the estimate

$$|A^*_{m,m+2}| \lesssim R^{2d}\left(\beta + \beta^{-1} \arccos \sqrt{1-\frac{\gamma}{2}}\right). \qquad (9.122)$$

∘ Estimate for $B^*_{m+1,m+2}$: The estimate for $B^*_{m+1,m+2}$ is in the same spirit as the previous estimates; however, we will need to distinguish cases depending on the size of the impact directions. The reason for that is that we rely on Lemma 8.8 from Section 8 which provides estimates on hemispheres of the $(2d-1)$-unit sphere.

Recall from (9.85) the set

$$\begin{aligned} B^*_{m+1,m+2} = \{&(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d} : \\ &|\langle\omega_1-\omega_2, v^*_{m+1}-v^*_{m+2}\rangle| \geq \gamma'|\omega_1-\omega_2||v^*_{m+1}-v^*_{m+2}|\}. \end{aligned} \qquad (9.123)$$

The ternary collisional law (2.8) yields $v^*_{m+1}-v^*_{m+2} = v_{m+1}-v_{m+2}-c(\omega_1-\omega_2)$, where c is given by (9.101). Thus, we may write

$$\begin{aligned} B^*_{m+1,m+2} =\{&(\omega_1,\omega_2,v_{m+1},v_{m+2}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d} : \\ &|\langle\omega_2-\omega_1, v_{m+2}-v_{m+1}-c(\omega_2-\omega_1)\rangle| \geq \gamma'|\omega_2-\omega_1||v_{m+2}-v_{m+1}-c(\omega_2-\omega_1)|\}. \end{aligned}$$

Recall from (8.13)–(8.14), the sets

$$\mathcal{S}_{1,2} = \left\{(\omega_1,\omega_2) \in \mathbb{S}_1^{2d-1} : |\omega_1| < |\omega_2|\right\}, \quad \mathcal{S}_{2,1} = \left\{(\omega_1,\omega_2) \in \mathbb{S}_1^{2d-1} : |\omega_2| < |\omega_1|\right\}.$$

We also recall from (8.15)–(8.16) the sets

$$I_{1,2} = \{(\omega_1,\omega_2) \in \mathbb{R}^{2d}\big||\omega_1|^2 + 2\langle\omega_1,\omega_2\rangle\big| \leq \beta\}, \quad I_{2,1} = \{(\omega_1,\omega_2) \in \mathbb{R}^{2d}\big||\omega_2|^2 + 2\langle\omega_1,\omega_2\rangle\big| \leq \beta\}.$$

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We clearly have

$$\begin{aligned}|B^*_{m+1,m+2}| &= \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\\ &= \int_{\mathcal{S}_{1,2}\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2} + \int_{\mathcal{S}_{2,1}\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}\\ &= \widetilde{I}_{1,2} + \widetilde{I}'_{1,2} + \widetilde{I}_{2,1} + \widetilde{I}'_{2,1}, \end{aligned} \tag{9.124}$$

where

$$\widetilde{I}_{1,2} = \int_{(\mathcal{S}_{1,2}\cap I_{1,2})\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}, \tag{9.125}$$

$$\widetilde{I}'_{1,2} = \int_{(\mathcal{S}_{1,2}\setminus I_{1,2})\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}, \tag{9.126}$$

$$\widetilde{I}_{2,1} = \int_{(\mathcal{S}_{2,1}\cap I_{2,1})\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}, \tag{9.127}$$

$$I'_{2,1} = \int_{(\mathcal{S}_{2,1}\setminus I_{2,1})\times B_R^{2d}} \mathbb{1}_{B^*_{m+1,m+2}}\,d\omega_1\,d\omega_2\,dv_{m+1}\,dv_{m+2}. \tag{9.128}$$

We treat each of the terms in (9.124) separately.

Estimate for $\widetilde{I}_{1,2}$: By (9.125), Fubini's Theorem and Lemma 8.8, we obtain

$$\widetilde{I}_{1,2} \lesssim R^{2d}\int_{\mathcal{S}_{1,2}} \mathbb{1}_{I_{1,2}}\,d\omega_1\,d\omega_2 \lesssim R^{2d}\beta. \tag{9.129}$$

Estimate for $\widetilde{I}_{2,1}$: Similarly, we obtain

$$\widetilde{I}_{2,1} \lesssim R^{2d}\beta. \tag{9.130}$$

Estimate for $I'_{1,2}$: From (9.126), we obtain

$$I'_{1,2} \le \int_{\mathcal{S}_{1,2}\setminus I_{1,2}}\int_{B_R^d}\int_{B_R^d} \mathbb{1}_{V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+1}}}(v_{m+2})\,dv_{m+2}\,dv_{m+1}\,d\omega_1\,d\omega_2, \tag{9.131}$$

where given $(\omega_1,\omega_2,v_{m+1})\in(\mathcal{S}_{1,2}\setminus I_{1,2})\times B_R^d$, we denote

$$V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+1}} = \left\{v_{m+2}\in B_R^d : (\omega_1,\omega_2,v_{m+1},v_{m+2})\in B^*_{m+1,m+2}\right\}. \tag{9.132}$$

Let us fix $(\omega_1,\omega_2,v_{m+1})\in(\mathcal{S}_{1,2}\setminus I_{1,2})\times B_R^d$. We define the map $F^{1,2}_{\omega_1,\omega_2,v_{m+1}} : B_R^d \to \mathbb{R}^d$ by

$$F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2}) = v_{m+2} - v_{m+1} - c(\omega_2-\omega_1),$$

where c is given by (9.101).

In a similar way as in the estimate of of $|A^*_{m,m+1}|$, for any $(\omega_1,\omega_2)\notin I_{1,2}$, we have

$$\left|\mathrm{Jac}\,F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2})\right| = \frac{\left||\omega_1|^2 + 2\langle\omega_1,\omega_2\rangle\right|}{1+\langle\omega_1,\omega_2\rangle} > \frac{\beta}{1+\langle\omega_1,\omega_2\rangle} \ge \frac{2\beta}{3}. \tag{9.133}$$

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Thus,

$$\left|\operatorname{Jac} F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2})\right|^{-1} \le \frac{3\beta^{-1}}{2}, \quad \forall v_{m+2} \in B^d_R. \tag{9.134}$$

Similarly to the estimate for $|A^*_{m,m+1}|$, we show also that $F^{1,2}_{\omega_1,\omega_2,v_{m+1}}$ is injective.

Since $(\omega_1,\omega_2,v_{m+1}) \in \mathbb{S}^{2d-1}_1 \times B^d_R$ and $\bar{v}_m \in B^d_R$, Cauchy-Schwartz inequality yields that, for any $v_{m+2} \in B^d_R$, we have

$$|F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2})| \le |v_{m+2}| + |v_{m+1}| + \frac{|\omega_1|(|v_{m+1}| + |\bar{v}_m|) + |\omega_2|(|v_{m+2}| + |\bar{v}_m|)}{1 + \langle \omega_1,\omega_2\rangle}(|\omega_2| + |\omega_1|) \le 18R,$$

since $\frac{1}{2} \le 1 + \langle \omega_1,\omega_2\rangle \le \frac{3}{2}$. Therefore,

$$F^{1,2}_{\omega_1,\omega_2,v_{m+1}}[B^d_R] \subseteq B^d_{18R}. \tag{9.135}$$

Additionally,

$$v_{m+2} \in V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+1}} \Leftrightarrow F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2}) \in U_{\omega_1,\omega_2},$$

where

$$U_{\omega_1,\omega_2} = \left\{v \in \mathbb{R}^d : \langle \omega_2 - \omega_1, v\rangle \ge \gamma'|\omega_2 - \omega_1||v|\right\}. \tag{9.136}$$

Hence,

$$\mathbb{1}_{V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+1}}}(v_{m+2}) = \mathbb{1}_{U_{\omega_1,\omega_2}}(F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2})), \quad \forall v_{m+2} \in B^d_R. \tag{9.137}$$

Therefore, performing the substitution $\nu := F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2})$, and using (9.134), we obtain

$$\int_{B^d_R} \mathbb{1}_{V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+1}}}(v_{m+2})\,dv_{m+2} = \int_{B^d_R} \mathbb{1}_{U_{\omega_1,\omega_2}}(F^{1,2}_{\omega_1,\omega_2,v_{m+1}}(v_{m+2}))\,dv_{m+2} \le \beta^{-1}\int_{B^d_{18R}} \mathbb{1}_{U_{\omega_1,\omega_2}}(\nu)\,d\nu. \tag{9.138}$$

Recalling the set $N(\gamma',\nu) = \left\{(\omega_1,\omega_2) \in \mathbb{R}^{2d} : \langle \omega_1 - \omega_2, \nu\rangle \ge \gamma'|\omega_1 - \omega_2||\nu|\right\}$, from (8.7) and (9.136), we have

$$\mathbb{1}_{U_{\omega_1,\omega_2}}(\nu) = \mathbb{1}_{N(\gamma',\nu)}(\omega_1,\omega_2), \quad \forall(\omega_1,\omega_2) \in \mathbb{S}^{2d-1}_1, \quad \forall \nu \in B^d_{18R}. \tag{9.139}$$

Therefore, using (9.131), (9.138), Fubini's Theorem and (9.139), we obtain

$$\begin{aligned}
I'_{1,2} &\le \beta^{-1}\int_{(\mathcal{S}_{1,2}\setminus I_{1,2})\times B^d_R}\int_{B^d_{18R}} \mathbb{1}_{U_{\omega_1,\omega_2}}(\nu)\,d\nu\,d\omega_1\,d\omega_2\,dv_{m+1} \\
&\le \beta^{-1}\int_{B^d_R\times B^d_{18R}}\int_{\mathbb{S}^{2d-1}_1} \mathbb{1}_{N(\gamma',\nu)}(\omega_1,\omega_2)\,d\omega_1\,d\omega_2\,d\nu\,dv_{m+1} \\
&\lesssim R^{2d}\beta^{-1}\arccos\gamma' \\
&= R^{2d}\beta^{-1}\arccos\sqrt{1-\frac{\gamma}{2}},
\end{aligned} \tag{9.140}$$

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where to obtain (9.140), we use Lemma 8.6. Therefore,

$$I'_{12} \le R^{2d}\beta^{-1} \arccos\sqrt{1-\frac{\gamma}{2}}. \tag{9.141}$$

Estimate for $I'_{2,1}$: The argument is entirely symmetric, using the set

$$V^{m+1,m+2}_{\omega_1,\omega_2,v_{m+2}} = \left\{v_{m+1} \in B^d_R : (\omega_1,\omega_2,v_{m+1},v_{m+2}) \in B^*_{m+1,m+2}\right\}.$$

for given $(\omega_1,\omega_2,v_{m+2}) \in (\mathcal{S}_{2,1}\setminus I_{2,1})\times B^d_R$ and the map $F^{2,1}_{\omega_1,\omega_2,v_{m+2}}(v_{m+1}) = v_{m+1}-v_{m+2}-c(\omega_1-\omega_2)$. We obtain

$$I'_{21} \le R^{2d}\beta^{-1} \arccos\sqrt{1-\frac{\gamma}{2}}. \tag{9.142}$$

Recalling (9.124) and using (9.129)–(9.130), (9.141)–(9.142), we obtain

$$|B^*_{m+1,m+2}| \lesssim R^{2d}\left(\beta+\beta^{-1}\arccos\sqrt{1-\frac{\gamma}{2}}\right). \tag{9.143}$$

Recalling (9.99) and using (9.92)–(9.94), (9.121), (9.122), (9.143), we obtain

$$|\mathcal{A}^+_m(\bar{Z}_m)| \lesssim R^{2d}\left(\gamma^{d/2}+\gamma^{\frac{d-1}{4}}+\beta+\beta^{-1}\arccos\sqrt{1-\frac{\gamma}{2}}\right). \tag{9.144}$$

Recalling (9.87), using (9.98), (9.144) and using the fact that $\gamma << 1$, we obtain

$$|\mathcal{A}_m(\bar{Z}_m)| \lesssim R^{2d}\left(\gamma^{\frac{d-1}{4}}+\beta+\beta^{-1}\arccos\sqrt{1-\frac{\gamma}{2}}\right). \tag{9.145}$$

Choice of β: Let us now choose β in terms of η. Recalling that $\epsilon_2 << \eta^2\epsilon_3$ and (9.69), we have

$$\gamma^{\frac{d-1}{4}} << \eta^{\frac{d-1}{2}}. \tag{9.146}$$

Moreover, since $\eta << 1$, we may assume

$$\frac{\eta}{\sqrt{2}} \le \sin\eta \le \eta. \tag{9.147}$$

Since $\gamma << \eta^2$, (9.147) implies

$$\gamma << 2\sin^2\eta \Rightarrow \arccos\sqrt{1-\frac{\gamma}{2}} < \eta. \tag{9.148}$$

Choosing

$$\beta = \eta^{1/2} << 1, \tag{9.149}$$

estimates (9.145)–(9.146), (9.148) imply

$$|\mathcal{A}_m(\bar{Z}_m)| \lesssim R^{2d}\left(\eta^{\frac{d-1}{2}}+\eta^{1/2}\right) \lesssim R^{2d}\eta^{\frac{d-1}{4d+2}}, \tag{9.150}$$

since $\eta << 1$ and $d \ge 2$. The claim comes from (9.89)–(9.90) and (9.150). □

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10. Elimination of recollisions

In this section, we reduce the convergence proof to comparing truncated elementary observables. We first restrict to good configurations and provide the corresponding measure estimate. This is happening in Proposition 10.2. We then inductively apply Proposition 9.2 and Proposition 9.4 or Proposition 9.6 and Proposition 9.7 (depending on whether the adjunction is binary or ternary) to reduce the convergence proof to truncated elementary observables. The convergence proof, completed in Section 11, will then follow naturally, since the backwards (ϵ_2, ϵ_3)-flow and the backwards free flow will be comparable out of a small measure set. Throughout this section, $s \in \mathbb{N}$ will be fixed, $(N, \epsilon_2, \epsilon_3)$ are given in the scaling (4.24) with N large enough such that $\epsilon_2 << \epsilon_3$, and the parameters $n, R, \epsilon_0, \alpha, \eta, \delta$ satisfy (9.4).

10.1. Restriction to good configurations

Inductively using Lemma 9.1, we are able to reduce the convergence proof to good configurations, up to a small measure set. The measure of the complement will be negligible in the limit.

For convenience, given $m \in \mathbb{N}$, let us define the set

$$G_m(\epsilon_3, \epsilon_0, \delta) := G_m(\epsilon_3, 0) \cap G_m(\epsilon_0, \delta). \tag{10.1}$$

For $s \in \mathbb{N}$, we also recall from (6.3) the set $\Delta_s^X(\epsilon_0)$ of well-separated spatial configurations.

Lemma 10.1. *Let $s \in \mathbb{N}$. Let $s \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.4) and $\epsilon_2 << \epsilon_3 << \alpha$. Then for any $X_s \in \Delta_s^X(\epsilon_0)$, there is a subset of velocities $\mathcal{M}_s(X_s) \subseteq B_R^{ds}$ of measure*

$$|\mathcal{M}_s(X_s)|_{ds} \le C_{d,s} R^{ds} \eta^{\frac{d-1}{2}}, \tag{10.2}$$

such that

$$Z_s \in G_s(\epsilon_3, \epsilon_0, \delta), \quad \forall V_s \in B_R^{ds} \setminus \mathcal{M}_s(X_s). \tag{10.3}$$

Proof. We use Proposition 10.1. from [5] for $\epsilon = \epsilon_3$. □

For $s \in \mathbb{N}$ and $X_s \in \Delta_s^X(\epsilon_0)$, let us denote $\mathcal{M}_s^c(X_s) = B_R^{ds} \setminus \mathcal{M}_s(X_s)$. Consider $1 \le k \le n$ and let us recall the observables $I_{s,k,R,\delta}^N$, $I_{s,k,R,\delta}^\infty$ defined in (7.23)–(7.24). We restrict the domain of integration to velocities giving good configurations.

In particular, we define

$$\widetilde{I}_{s,k,R,\delta}^N(t)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) f_{N,R,\delta}^{(s,k)}(X_s, V_s)\, dV_s, \tag{10.4}$$

$$\widetilde{I}_{s,k,R,\delta}^\infty(t)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) f_{R,\delta}^{(s,k)}(X_s, V_s)\, dV_s. \tag{10.5}$$

Let us apply Proposition 10.1 to restrict to initially good configurations. To keep track of all the possible adjuctions, we recall recall the notation from (7.3)–(7.5): given $k \in \mathbb{N}$, we write

$$S_k = \{\sigma = (\sigma_1, ..., \sigma_k) : \sigma_i \in \{1, 2\}\},$$

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and given $\sigma \in S_k$, we write

$$\widetilde{\sigma}_\ell = \sum_{i=1}^{\ell} \sigma_i, \quad 1 \le \ell \le k, \quad \widetilde{\sigma}_0 = 0.$$

Proposition 10.2. *Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.4), $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $\epsilon_2 << \epsilon_3 << \alpha$, and $t \in [0, T]$. Then, the following estimates hold:*

$$\sum_{k=1}^{n} \|I_{s,k,R,\delta}^N(t) - \widetilde{I}_{s,k,R,\delta}^N(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \le C_{d,s,\mu_0,T} R^{ds} \eta^{\frac{d-1}{2}} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\sum_{k=1}^{n} \|I_{s,k,R,\delta}^\infty(t) - \widetilde{I}_{s,k,R,\delta}^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \le C_{d,s,\mu_0,T} R^{ds} \eta^{\frac{d-1}{2}} \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proof. We present the proof for the BBGKY hierarchy case only. The proof for the Boltzmann hierarchy case is similar. Let us fix $X_s \in \Delta_s^X(\epsilon_0)$.

We first assume that $k \in \{1, ..., n\}$. Triangle inequality, an inductive application of estimate (5.7), estimate (5.6) and part (ii) of Proposition 5.3 yield

$$\begin{aligned} |I_{s,k,R,\delta}^N(t)(X_s) - \widetilde{I}_{s,k,R,\delta}^N(t)(X_s)| &\le \sum_{\sigma \in S_k} \int_{\mathcal{M}_s(X_s)} |\phi_s(V_s) f_{N,R,\delta}^{(s,k,\sigma)}(t, X_s, V_s)| \, dV_s \\ &\le 2T \|\phi_s\|_{L^\infty_{V_s}} e^{-s\mu(T)} \left(\frac{1}{8}\right)^{k-1} \|F_{N,0}\|_{N,\beta_0,\mu_0} \int_{\mathcal{M}_s(X_s)} e^{-\beta(T) E_s(Z_s)} \, dV_s \qquad (10.6) \\ &\le 2T \|\phi_s\|_{L^\infty_{V_s}} e^{-s\mu(T)} \left(\frac{1}{8}\right)^{k-1} |\mathcal{M}_s(X_s)|_{ds} \|F_{N,0}\|_{N,\beta_0,\mu_0}, \qquad (10.7) \end{aligned}$$

where to obtain (10.6), we use (7.4).

For $k = 0$, part (i) of Proposition 5.3 and Remark 5.1 similarly yield

$$|I_{s,0,R,\delta}^N(t)(X_s) - \widetilde{I}_{s,0,R,\delta}^N(t)(X_s)| \le \|\phi_s\|_{L^\infty_{V_s}} e^{-s\mu(T)} |\mathcal{M}_s(X_s)|_{ds} \|F_{N,0}\|_{N,\beta_0,\mu_0}. \qquad (10.8)$$

The claim comes after using (10.7)–(10.8), adding over $k = 0, ..., n$, and using the measure estimate of Proposition 10.1. □

Remark 10.3. Given $s \in \mathbb{N}$ and $X_s \in \Delta_s^X(\epsilon_0)$, the definition of $\mathcal{M}_s(X_s)$ implies that

$$\widetilde{I}_{s,0,R,\delta}^N(t)(X_s) = \widetilde{I}_{s,0,R,\delta}^\infty(t)(X_s).$$

Therefore, by Proposition 10.2, convergence reduces to controlling the differences $\widetilde{I}_{s,k,R,\delta}^N(t) - \widetilde{I}_{s,k,R,\delta}^\infty(t)$, for $k = 1, ..., n$, in the scaled limit.

10.2. Reduction to elementary observables

Here, given $s \in \mathbb{N}$ and $1 \le k \le n$, we express the observables $\widetilde{I}_{s,k,R,\delta}^N(t)$, $\widetilde{I}_{s,k,R,\delta}^\infty(t)$, defined in (10.4)–(10.5), as a superposition of elementary observables.

For this purpose, given $\ell \in \mathbb{N}$, and recalling (7.19), (4.15), we decompose the BBGKY hierarchy binary truncated collisional operator as

$$\mathcal{C}_{\ell,\ell+1}^{N,R} = \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+1}^{N,R,+,i} - \sum_{i=1}^{\ell} \mathcal{C}_{\ell,\ell+1}^{N,R,-,i},$$

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where

$$\mathcal{C}^{N,R,+,i}_{\ell,\ell+1} g_{\ell+1}(Z_\ell) = A^2_{N,\epsilon_2,\ell} \int_{\mathbb{S}_1^{d-1}\times B_R^d} b_2^+(\omega_1, v_{\ell+1} - v_i) g_{\ell+1}(Z^{i'}_{\ell+1,\epsilon_2})\, d\omega_1\, dv_{\ell+1},$$

$$\mathcal{C}^{N,R,-,i}_{\ell,\ell+1} g_{\ell+1}(Z_\ell) = A^2_{N,\epsilon_2,\ell} \int_{\mathbb{S}_1^{d-1}\times B_R^d} b_2^+(\omega_1, v_{\ell+1} - v_i) g_{\ell+1}(Z^{i}_{\ell+1,\epsilon_2})\, d\omega_1\, dv_{\ell+1},$$

and the ternary truncated collisional operator as

$$\mathcal{C}^{N,R}_{\ell,\ell+2} = \sum_{i=1}^{\ell} \mathcal{C}^{N,R,+,i}_{\ell,\ell+2} - \sum_{i=1}^{\ell} \mathcal{C}^{N,R,-,i}_{\ell,\ell+2},$$

where

$$\mathcal{C}^{N,R,+,i}_{\ell,\ell+2} g_{\ell+2}(Z_\ell) = A^3_{N,\epsilon_3,\ell} \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} \frac{b_3^+(\omega_1,\omega_2, v_{\ell+1} - v_i, v_{\ell+2} - v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} g_{\ell+2}(Z^{i*}_{\ell+2,\epsilon_3})\, d\omega_1\, d\omega_2\, dv_{\ell+1}\, dv_{\ell+2},$$

$$\mathcal{C}^{N,R,-,i}_{\ell,\ell+2} g_{\ell+2}(Z_\ell) = A^3_{N,\epsilon_3,\ell} \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} \frac{b_3^+(\omega_1,\omega_2, v_{\ell+1} - v_i, v_{\ell+2} - v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} g_{\ell+2}(Z^{i}_{\ell+2,\epsilon_3})\, d\omega_1\, d\omega_2\, dv_{\ell+1}\, dv_{\ell+2}.$$

In order to expand the observable $\widetilde{I}^N_{s,k,R,\delta}(t)$ to elementary observables, we need to take into account all the possible particle adjuctions occurring by adding one or two particles to the system in each step. More precisely, given $\sigma \in S_k$, and $i \in \{1, \dots, k\}$, we are adding $\sigma_i \in \{1, 2\}$ particle(s) to the existing $s + \widetilde{\sigma}_{i-1}$ particles in either precollisional or postcollisional way. In order to keep track of this process, given $1 \le k \le n$, $\sigma \in S_k$, we introduce the notation

$$\mathcal{M}_{s,k,\sigma} = \big\{M = (m_1, \dots, m_k) \in \mathbb{N}^k : m_i \in \{1, \dots, s + \widetilde{\sigma}_{i-1}\}, \quad \forall i \in \{1, \dots, k\}\big\}, \tag{10.9}$$

$$\mathcal{J}_{s,k,\sigma} = \big\{J = (j_1, \dots, j_k) \in \mathbb{N}^k : j_i \in \{-1, 1\}, \quad \forall i \in \{1, \dots, k\}\big\}. \tag{10.10}$$

$$\mathcal{U}_{s,k,\sigma} = \mathcal{J}_{s,k,\sigma} \times \mathcal{M}_{s,k,\sigma}. \tag{10.11}$$

Under this notation, the BBGKY hierarchy observable functional $\widetilde{I}^N_{s,k,R,\delta}(t)$ can be expressed, for $1 \le k \le n$, as a superposition of elementary observables

$$\widetilde{I}^N_{s,k,R,\delta}(t)(X_s) = \sum_{\sigma\in S_k} \sum_{(J,M)\in\mathcal{U}_{s,k,\sigma}} \left(\prod_{i=1}^k j_i\right) \widetilde{I}^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s), \tag{10.12}$$

where the elementary observables are defined by

$$\begin{aligned} \widetilde{I}^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} T_s^{t-t_1} \mathcal{C}^{N,R,j_1,m_1}_{s,s+\widetilde{\sigma}_1} T^{t_1-t_2}_{s+\widetilde{\sigma}_1} \dots \\ \dots T^{t_{k-1}-t_k}_{s+\widetilde{\sigma}_{k-1}} \mathcal{C}^{N,R,j_k,m_k}_{s+\widetilde{\sigma}_{k-1}, s+\widetilde{\sigma}_k} T^{t_m}_{s+\widetilde{\sigma}_k} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k \dots dt_1 dV_s. \end{aligned} \tag{10.13}$$

Similarly, given $\ell \in \mathbb{N}$, and recalling (4.31), (4.35), we decompose the Boltzmann hierarchy binary and ternary collisional operators as

$$\mathcal{C}^{\infty,R}_{\ell,\ell+1} = \sum_{i=1}^{\ell} \mathcal{C}^{\infty,R,+,i}_{\ell,\ell+1} - \sum_{i=1}^{\ell} \mathcal{C}^{\infty,R,-,i}_{\ell,\ell+1},$$

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where

$$\mathcal{C}^{\infty,R,+,i}_{\ell,\ell+1} g_{\ell+1}(Z_\ell) = \int_{\mathbb{S}_1^{d-1}\times B_R^d} b_2^+(\omega_1, v_{\ell+1}-v_i) g_{\ell+1}(Z^{i'}_{\ell+1})\, d\omega_1\, dv_{\ell+1},$$

$$\mathcal{C}^{\infty,R,-,i}_{\ell,\ell+1} g_{\ell+1}(Z_\ell) = \int_{\mathbb{S}_1^{d-1}\times B_R^d} b_2^+(\omega_1, v_{\ell+1}-v_i) g_{\ell+1}(Z^{i}_{\ell+1})\, d\omega_1\, dv_{\ell+1},$$

$$\mathcal{C}^{\infty,R}_{\ell,\ell+2} = \sum_{i=1}^{\ell} \mathcal{C}^{\infty,R,+,i}_{\ell,\ell+2} - \sum_{i=1}^{\ell} \mathcal{C}^{\infty,R,-,i}_{\ell,\ell+2},$$

where

$$\mathcal{C}^{\infty,R,+,i}_{\ell,\ell+2} g_{\ell+2}(Z_\ell) = \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} \frac{b_3^+(\omega_1,\omega_2, v_{\ell+1}-v_i, v_{\ell+2}-v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} g_{\ell+2}(Z^{i*}_{\ell+2})\, d\omega_1\, d\omega_2\, dv_{\ell+1}\, dv_{\ell+2},$$

$$\mathcal{C}^{\infty,R,-,i}_{\ell,\ell+2} g_{\ell+2}(Z_\ell) = \int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} \frac{b_3^+(\omega_1,\omega_2, v_{\ell+1}-v_i, v_{\ell+2}-v_i)}{\sqrt{1+\langle\omega_1,\omega_2\rangle}} g_{\ell+2}(Z^{i}_{\ell+2})\, d\omega_1\, d\omega_2\, dv_{\ell+1}\, dv_{\ell+2}.$$

Under this notation, the Boltzmann hierarchy observable functional $\widetilde{I}^{\infty}_{s,k,R,\delta}(t)$ can be expressed, for $1 \le k \le n$, as a superposition of elementary observables

$$\widetilde{I}^{\infty}_{s,k,R,\delta}(t)(X_s) = \sum_{\sigma\in S_k} \sum_{(J,M)\in\mathcal{U}_{s,k,\sigma}} \left(\prod_{i=1}^{k} j_i\right) \widetilde{I}^{\infty}_{s,k,R,\delta,\sigma}(t,J,M)(X_s), \tag{10.14}$$

where the elementary observables are defined by

$$\begin{aligned}\widetilde{I}^{\infty}_{s,k,R,\delta,\sigma}(t,J,M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1} \mathcal{C}^{\infty,R,j_1,m_1}_{s,s+\widetilde{\sigma}_1} S^{t_1-t_2}_{s+\widetilde{\sigma}_1} \dots \\ \dots S^{t_{k-1}-t_k}_{s+\widetilde{\sigma}_{k-1}} \mathcal{C}^{\infty,R,j_k,m_k}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k} S^{t_m}_{s+\widetilde{\sigma}_k} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k \dots dt_1 dV_s.\end{aligned} \tag{10.15}$$

10.3. Boltzmann hierarchy pseudo-trajectories

We introduce the following notation which we will be constantly using from now on. Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $1 \le k \le n$, $\sigma \in S_k$ and $t \in [0,T]$. Let us recall from (7.2) the set

$$\mathcal{T}_k(t) = \left\{(t_1,\dots,t_k)\in\mathbb{R}^k : 0 = t_{k+1} < t_k < \dots < t_1 < t_0 = t\right\}, \quad t_0 = t,\ t_{k+1} = 0.$$

Consider $(t_1,\dots,t_k)\in\mathcal{T}_k(t)$, $J=(j_1,\dots,j_k)$, $M=(m_1,\dots,m_k)$, $(J,M)\in\mathcal{U}_{s,k,\sigma}$. For each $i = 1,\dots,k$, we distinguish two possible situations:

$$\text{If } \sigma_i = 1, \text{ we consider } (\omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i}) \in \mathbb{S}_1^{d-1}\times B_R^d. \tag{10.16}$$

$$\text{If } \sigma_i = 2, \text{ we consider } (\omega_{s+\widetilde{\sigma}_i-1}, \omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i-1}, v_{s+\widetilde{\sigma}_i}) \in \mathbb{S}_1^{2d-1}\times B_R^{2d}. \tag{10.17}$$

For convenience, for each $i = 1,\dots,k$, we will write $(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}) \in \mathbb{S}_1^{d\sigma_i-1}\times B_R^{d\sigma_i}$ where $(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i})$ is of the form (10.16) if $\sigma_i = 1$ and of the form (10.17) if $\sigma_i = 2$.

We inductively define the Boltzmann hierarchy pseudo-trajectory of Z_s. Roughly speaking, the Boltzmann hierarchy pseudo-trajectory forms the configurations on which particles are adjusted during backwards in time evolution.

Intuitively, assume we are given a configuration $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ at time $t_0 = t$. Z_s evolves under backwards free flow until the time t_1 when the configuration $(\boldsymbol{\omega}_{\sigma,1}, \boldsymbol{v}_{\sigma,1})$ is added, neglecting positions,

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to the m_1-particle, the adjunction being precollisional if $j_1 = -1$ and postcollisional if $j_1 = 1$. We then form an $(s+\widetilde{\sigma}_1)$-configuration and continue this process inductively until time $t_{k+1} = 0$. More precisely, we inductively construct the Boltzmann hierarchy pseudo-trajectory of $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ as follows:

Time $t_0 = t$: We initially define $Z_s^\infty(t_0^-) = \left(x_1^\infty(t_0^-), ..., x_s^\infty(t_0^-), v_1^\infty(t_0^-), ..., v_s^\infty(t_0^-)\right) := Z_s$.

Time t_i, $i \in \{1, ..., k\}$: Consider $i \in \{1, ..., k\}$ and assume we know

$$Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-) = \left(x_1^\infty(t_{i-1}^-), ..., x^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-), v_1^\infty(t_{i-1}^-), ..., v^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-)\right).$$

We define $Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+) = \left(x_1^\infty(t_i^+), ..., x^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+), v_1^\infty(t_i^+), ..., v^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right)$ as:

$$Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+) := \left(X^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-) - (t_{i-1} - t_i)V^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-), V^\infty_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-)\right).$$

We also define $Z^\infty_{s+\widetilde{\sigma}_i}(t_i^-) = \left(x_1^\infty(t_i^-), ..., x^\infty_{s+\widetilde{\sigma}_i}(t_i^-), v_1^\infty(t_i^-), ..., v^\infty_{s+\widetilde{\sigma}_i}(t_i^-)\right)$ as:

$$\left(x_j^\infty(t_i^-), v_j^\infty(t_i^-)\right) := (x_j^\infty(t_i^+), v_j^\infty(t_i^+)), \quad \forall j \in \{1, ..., s+\widetilde{\sigma}_{i-1}\} \setminus \{m_i\}.$$

For the rest of the particles, we distiguish the following cases, depending on σ_i:

- $\sigma_i = 1$: If $j_i = -1$:

$$\begin{aligned}
\left(x^\infty_{m_i}(t_i^-), v^\infty_{m_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^\infty_{m_i}(t_i^+)\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i}\right),
\end{aligned}$$

  while if $j_i = 1$:

$$\begin{aligned}
\left(x^\infty_{m_i}(t_i^-), v^\infty_{m_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^{\infty\prime}_{m_i}(t_i^+)\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v'_{s+\widetilde{\sigma}_i}\right),
\end{aligned}$$

  where $(v^{\infty\prime}_{m_i}(t_i^-), v'_{s+\widetilde{\sigma}_i}) = T_{\omega_{s+\widetilde{\sigma}_i}}\left(v^\infty_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i}\right)$.
- $\sigma_i = 2$: If $j_i = -1$:

$$\begin{aligned}
\left(x^\infty_{m_i}(t_i^-), v^\infty_{m_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^\infty_{m_i}(t_i^+)\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i-1}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i-1}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i-1}\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i}\right),
\end{aligned}$$

  while if $j_i = 1$:

$$\begin{aligned}
\left(x^\infty_{m_i}(t_i^-), v^\infty_{m_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^{\infty *}_{m_i}(t_i^+)\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i-1}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i-1}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^*_{s+\widetilde{\sigma}_i-1}\right),\\
\left(x^\infty_{s+\widetilde{\sigma}_i}(t_i^-), v^\infty_{s+\widetilde{\sigma}_i}(t_i^-)\right) &:= \left(x^\infty_{m_i}(t_i^+), v^*_{s+\widetilde{\sigma}_i}\right),
\end{aligned}$$

  where $(v^{\infty *}_{m_i}(t_i^-), v^*_{s+\widetilde{\sigma}_i-1}, v^*_{s+\widetilde{\sigma}_i}) = T_{\omega_{s+\widetilde{\sigma}_i-1}, \omega_{s+\widetilde{\sigma}_i}}\left(v^\infty_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i-1}, v_{s+\widetilde{\sigma}_i}\right)$.

Time $t_{k+1} = 0$: We finally obtain

$$Z^\infty_{s+\widetilde{\sigma}_k}(0^+) = Z^\infty_{s+\widetilde{\sigma}_k}(t_{k+1}^+) = \left(X^\infty_{s+\widetilde{\sigma}_k}(t_k^-) - t_k V^\infty_{s+\widetilde{\sigma}_k}(t_k^-), V^\infty_{s+\widetilde{\sigma}_k}(t_k^-)\right).$$

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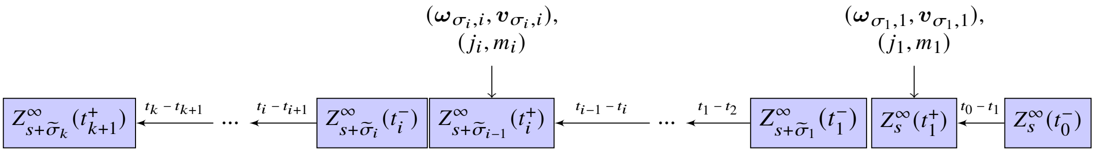
The process is illustrated in the following diagram (to be read from right to left):



We give the following definition:

Definition 10.4. Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $(t_1, ..., t_k) \in \mathcal{T}_k(t)$, $J = (j_1, ..., j_k)$, $M = (m_1, ..., m_k)$, $(J, M) \in \mathcal{U}_{s,k}$, and for each $i = 1, ..., k$, $\sigma \in S_k$, we consider $(\omega_{\sigma_i,i}, v_{\sigma_i,i}) \in \mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i}$. The sequence $\{Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t^+_i)\}_{i=0,...,k+1}$ constructed above is called the Boltzmann hierarchy pseudo-trajectory of Z_s.

10.4. Reduction to truncated elementary observables

We will now use the Boltzmann hierarchy pseudo-trajectory to define the BBGKY hierarchy and Boltzmann hierarchy truncated observables. The convergence proof will then be reduced to the convergence of the corresponding truncated elementary observables.

Given $\ell \in \mathbb{N}$, recall the notation from (10.1):

$$G_\ell(\epsilon_3, \epsilon_0, \delta) = G_\ell(\epsilon_3, 0) \cap G_\ell(\epsilon_0, \delta).$$

Given $t \in [0, T]$, we also recall from (7.22) the set $\mathcal{T}_{k,\delta}(t)$ of separated collision times:

$$\mathcal{T}_{k,\delta}(t) := \{(t_1, ..., t_k) \in \mathcal{T}_k(t) : \quad 0 \le t_{i+1} \le t_i - \delta, \quad \forall i \in [0, k]\}, \quad t_{k+1} = 0, \quad t_0 = t.$$

Consider $t \in [0, T]$, $X_s \in \Delta^X_s(\epsilon_0)$, $1 \le k \le n$, $\sigma \in S_k$ and $(J, M) \in \mathcal{U}_{s,k,\sigma}$ and $(t_1, ..., t_k) \in \mathcal{T}_{k,\delta}$. By Proposition 10.1, for any $V_s \in \mathcal{M}^c_s(X_s)$, we have $Z_s = (X_s, V_s) \in G_s(\epsilon_3, \epsilon_0, \delta)$, which in turn implies $Z^\infty_s(t^+_1) \in G_s(\epsilon_0, 0)$ since $t_0 - t_1 > \delta$. Now we observe that either (9.8), (9.12) from Proposition 9.2 (if the adjunction is binary), or (9.61), (9.65) from Proposition 9.6 (if the adjunction is ternary) yield that there is a set $\mathcal{B}_{m_1}(Z^\infty_s(t^+_1)) \subseteq \mathbb{S}_1^{d\sigma_1 - 1} \times B_R^{d\sigma_1}$ such that

$$Z^\infty_{s+\widetilde{\sigma}_1}(t^+_2) \in G_{s+\widetilde{\sigma}_1}(\epsilon_0, 0), \quad \forall (\omega_{\sigma_1,1}, v_{\sigma_1,1}) \in \mathcal{B}^c_{m_1}(Z^\infty_s(t^+_1)),$$

$$\mathcal{B}^c_{m_1}(Z^\infty_s(t^+_1)) := (\mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i})^+(v^\infty_{m_1}(t^+_1)) \setminus \mathcal{B}_{m_1}(Z^\infty_s(t^+_1)).$$

Clearly, this process can be iterated. In particular, given $i \in \{2, ..., k\}$, we have

$$Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t^+_i) \in G_{s+\widetilde{\sigma}_{i-1}}(\epsilon_0, 0),$$

so there exists a set $\mathcal{B}_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t^+_i)\right) \subseteq \mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i}$ such that

$$Z^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) \in G_{s+\widetilde{\sigma}_i}(\epsilon_0, 0), \quad \forall (\omega_{\sigma_i,i}, v_{\sigma_i,i}) \in \mathcal{B}^c_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t^+_i)\right), \tag{10.18}$$

where

$$\mathcal{B}^c_{m_i}(Z^\infty_s(t^+_i)) := (\mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i})^+(v^\infty_{m_i}(t^+_i)) \setminus \mathcal{B}_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_i}(t^+_i)\right).$$

We finally obtain $Z^\infty_{s+\widetilde{\sigma}_k}(0^+) \in G_{s+\widetilde{\sigma}_k}(\epsilon_0, 0)$.

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Let us now define the truncated elementary observables. Heuristically we will truncate the domains of adjusted particles in the definition of the observables $\widetilde{I}^N_{s,k,R,\delta}$, $\widetilde{I}^\infty_{s,k,R,\delta}$, defined in (10.4)–(10.5).

More precisely, consider $1 \leq k \leq n$, $\sigma \in S_k$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$ and $t \in [0, T]$. For $X_s \in \Delta_s^X(\epsilon_0)$, Proposition 10.1 implies there is a set of velocities $\mathcal{M}_s(X_s) \subseteq B_R^{2d}$ such that $Z_s = (X_s, V_s) \in G_s(\epsilon_3, \epsilon_0, \delta)$, $\forall V_s \in \mathcal{M}_s^c(X_s)$. Following the reasoning above, we define the BBGKY hierarchy truncated observables as

$$J^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} T_s^{t-t_1} \widetilde{C}^{N,R,j_1,m_1}_{s,s+\widetilde{\sigma}_1} T^{t_1-t_2}_{s+\widetilde{\sigma}_1} \dots \\ \dots \widetilde{C}^{N,R,j_k,m_k}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k} T^{t_m}_{s+\widetilde{\sigma}_k} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k, \dots dt_1 dV_s, \tag{10.19}$$

where for each $i = 1, \dots, k$, we denote

$$\widetilde{C}^{N,R,j_i,m_i}_{s+\widetilde{\sigma}_{i-1},s+\widetilde{\sigma}_i} g_{N,s+\widetilde{\sigma}_i} = C^{N,R,j_i,m_i}_{s+\widetilde{\sigma}_{i-1},s+\widetilde{\sigma}_i}\left[g_{N,s+\widetilde{\sigma}_i} \mathbb{1}_{(\omega_{\sigma_i,i}, v_{\sigma_i,i}) \in \mathcal{B}^c_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right)}\right].$$

In the same spirit, for $X_s \in \Delta_s^X(\epsilon_0)$, we define the Boltzmann hierarchy truncated elementary observables as

$$J^\infty_{s,k,R,\delta,\sigma}(t, J, M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} S_s^{t-t_1} \widetilde{C}^{\infty,R,j_1,m_1}_{s,s+\widetilde{\sigma}_1} S^{t_1-t_2}_{s\widetilde{\sigma}_1} \dots \\ \dots \widetilde{C}^{\infty,R,j_k,m_k}_{s+\widetilde{\sigma}_{k-1},s+\widetilde{\sigma}_k} S^{t_m}_{s+\widetilde{\sigma}_k} f_0^{(s+\widetilde{\sigma}_k)}(Z_s)\, dt_k, \dots dt_1 dV_s, \tag{10.20}$$

where for each $i = 1, \dots, k$, we denote

$$\widetilde{C}^{\infty,R,j_i,m_i}_{s+\widetilde{\sigma}_{i-1},s+\widetilde{\sigma}_i} g_{s+\widetilde{\sigma}_i} = C^{\infty,R,j_i,m_i}_{s+\widetilde{\sigma}_{i-1},s+\widetilde{\sigma}_i}\left[g_{s+\widetilde{\sigma}_i} \mathbb{1}_{(\omega_{\sigma_i,i}, v_{\sigma_i,i}) \in \mathcal{B}^c_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right)}\right].$$

Recalling the observables $\widetilde{I}^N_{s,k,R,\delta,\sigma}$, $\widetilde{I}^\infty_{s,k,R,\delta,\sigma}$ from (10.13), (10.15) and using Proposition 9.4 or Proposition 9.7, we obtain the following:

Proposition 10.5. *Let $s, n \in \mathbb{N}$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.4), $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $\epsilon_2 << \epsilon_3 << \alpha$ and $t \in [0, T]$. Then the following estimates hold:*

$$\sum_{k=1}^n \sum_{\sigma \in S_k} \sum_{(J,M) \in \mathcal{U}_{s,k,\sigma}} \|\widetilde{I}^N_{s,k,R,\delta,\sigma}(t, J, M) - J^N_{s,k,R,\delta,\sigma}(t, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq \\ \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0},$$

$$\sum_{k=1}^n \sum_{\sigma \in S_k} \sum_{(J,M) \in \mathcal{U}_{s,k,\sigma}} \|\widetilde{I}^\infty_{s,k,R,\delta,\sigma}(t, J, M) - J^\infty_{s,k,R,\delta,\sigma}(t, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq \\ \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \eta^{\frac{d-1}{4d+2}} \|F_0\|_{\infty,\beta_0,\mu_0}.$$

Proof. As usual, it suffices to prove the estimate for the BBGKY hierarchy case, and the Boltzmann hierarchy case follows similarly. Fix $k \in \{1, \dots, n\}$, $\sigma \in S_k$ and $(J, M) \in \mathcal{U}_{s,k,\sigma}$. We first estimate the difference:

$$\widetilde{I}^N_{s,k,R,\delta}(t, J, M)(X_s) - J^N_{s,k,R,\delta}(t, J, M)(X_s). \tag{10.21}$$

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Cauchy-Schwartz inequality and triangle inequality imply

$$|\langle \omega_1, v_1 - v\rangle| \leq 2R, \quad \forall \omega_1 \in \mathbb{S}_1^{d-1}, \quad \forall v, v_1 \in B_R^d, \tag{10.22}$$

$$\left|b_3(\omega_1, \omega_2, v_1 - v, v_2 - v)\right| \leq 4R, \quad \forall (\omega_1, \omega_2) \in \mathbb{S}_1^{2d-1}, \quad \forall v, v_1, v_2 \in B_R^d, \tag{10.23}$$

so

$$\int_{\mathbb{S}_1^{d-1}\times B_R^d} |\langle \omega_1, v_1 - v\rangle|\, d\omega_1\, dv_1 \leq C_d R^{d+1} \leq C_d R^{3d}, \quad \forall v \in B_R^d, \tag{10.24}$$

$$\int_{\mathbb{S}_1^{2d-1}\times B_R^{2d}} |b_3(\omega_1, \omega_2, v_1 - v, v_2 - v_2)|\, d\omega_1\, d\omega_2\, dv_1\, dv_2 \leq C_d R^{2d+1} \leq C_d R^{3d}, \quad \forall v \in B_R^d, \tag{10.25}$$

since $R >> 1$. But in order to estimate the difference (10.21), we integrate at least once over $\mathcal{B}_{m_i}\left(Z^\infty_{s+2i-2}(t_i^+)\right)$ for some $i \in \{1, \ldots, k\}$. For convenience, given $v \in \mathbb{R}^d$, let us write

$$b_{\sigma_i}(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}, v) := \begin{cases} b_2(\omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i} - v), & \text{if } \sigma_i = 1, \\ b_3(\omega_{s+\widetilde{\sigma}_i-1}, \omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i-1} - v, v_{s+\widetilde{\sigma}_i} - v), & \text{if } \sigma_i = 2. \end{cases} \tag{10.26}$$

Under this notation, (10.22)–(10.23) together with Proposition 9.4 or Proposition 9.7, depending on whether the adunction is binary or ternary, yield the estimate

$$\begin{aligned} \int_{\mathcal{B}_{m_i}\left(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right)} |b_{\sigma_i}(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}, v)|\, d\boldsymbol{\omega}_{\sigma_i,i} \boldsymbol{v}_{\sigma_i,i} &\leq C_d(s + \widetilde{\sigma}_{i-1}) R^{d\sigma_i+1} \eta^{\frac{d-1}{2d\sigma_i+2}} \\ &\leq C_d(s+2k) R^{3d} \eta^{\frac{d-1}{4d+2}}, \quad \forall v \in B_R^d, \end{aligned} \tag{10.27}$$

since $R >> 1$ and $\eta << 1$.

Moreover, we have the elementary inequalities

$$\|f_{N,0}^{(s+\widetilde{\sigma}_k)}\|_{L^\infty} \leq e^{-(s+\widetilde{\sigma}_k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0} \leq e^{-(s+k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0}, \tag{10.28}$$

$$\int_{\mathcal{T}_{k,\delta}(t)} dt_1 \ldots dt_k \leq \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} dt_1 \ldots dt_k = \frac{t^k}{k!} \leq \frac{T^k}{k!}. \tag{10.29}$$

Therefore, (10.24)–(10.29) imply

$$\begin{aligned} &\left|\widetilde{I}^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s) - J^N_{s,k,R,\delta,\sigma}(t, J, M)(X_s)\right| \\ &\leq \|\phi_s\|_{L^\infty_{V_s}} e^{-(s+k)\mu_0} \|F_{N,0}\|_{N,\beta_0,\mu_0} C_d R^{ds} C_d^{k-1} R^{3d(k-1)} (s+2k) C_d R^{3d} \eta^{\frac{d-1}{4d+2}} \frac{T^k}{k!} \\ &\leq C^k_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} \frac{(s+2k)}{k!} R^{d(s+3k)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0}. \end{aligned}$$

Adding for all $(J, M) \in \mathcal{U}_{s,k,\sigma}$, we have $2^k s(s+\widetilde{\sigma}_1)\ldots(s+\widetilde{\sigma}_{k-1}) \leq 2^k(s+2k)^k$ contributions, and thus,

$$\begin{aligned} &\sum_{(J,M)\in\mathcal{U}_{s,k,\sigma}} \|\widetilde{I}^N_{s,k,R,\delta,\sigma}(t, J, M) - J^N_{s,k,R,\delta,\sigma}(t, J, M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\ &\leq C^k_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)} \frac{(s+2k)^{k+1}}{k!} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0} \\ &\leq C^k_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)} \eta^{\frac{d-1}{4d+2}} \|F_{N,0}\|_{N,\beta_0,\mu_0}, \end{aligned}$$

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since

$$\frac{(s+2k)^{k+1}}{k!} \leq \frac{2^{k+1}(s+k)(s+k)^k}{k!} \leq 2^{k+1}(s+k)e^{s+k} \leq C_s^k.$$

Summing over $\sigma \in S_k$, $k = 1, ..., n$, we get the required estimate. □

In the next section, in order to conclude the convergence proof, we will estimate the differences of the corresponding BBGKY hierarchy and Boltzmann hierarchy truncated elementary observables in the scaled limit.

11. Convergence proof

Recall from Subsection 10.4 that given $s \in \mathbb{N}$, $t \in [0, T]$, and parameters satisfying (9.4), we have reduced the convergence proof to controlling the differences:

$$J^N_{s,k,R,\delta}(t, J, M) - J^\infty_{s,k,R,\delta}(t, J, M)$$

for given $1 \leq k \leq n$ and $(J, M) \in \mathcal{U}_{s,k}$, where $J^N_{s,k,R,\delta}(t, J, M)$, $J^\infty_{s,k,R,\delta}(t, J, M)$ are given by (10.19), (10.20). This will be the aim of this section.

Throughout this section, $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ will be fixed, $(N, \epsilon_2, \epsilon_3)$ are in the scaling (4.24), $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$, $T > 0$ are given by the statements of Theorem 5.7 and Theorem 5.14, and the parameters $n, \delta, R, \eta, \epsilon_0, \alpha$ satisfy (9.4).

11.1. BBGKY hierarchy pseudo-trajectories and proximity to the Boltzmann hierarchy pseudo-trajectories

In the same spirit as in Subsection 10.3, we may define the BBGKY hierarchy pseudo-trajectory. Consider $s \in \mathbb{N}$, $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24), $k \in \mathbb{N}$ and $t \in [0, T]$. Let us recall from (7.2) the set

$$\mathcal{T}_k(t) = \left\{(t_1, ..., t_k) \in \mathbb{R}^k : 0 = t_{k+1} < t_k < ... < t_1 < t_0 = t\right\},$$

where we use the convention $t_0 = t$ and $t_{k+1} = 0$. Consider $(t_1, ..., t_k) \in \mathcal{T}_k(t)$, $\sigma \in S_k$, $J = (j_1, ..., j_k)$, $M = (m_1, ..., m_k)$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$, and for each $i = 1, ..., k$, we consider $(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}) \in \mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i}$.

The process followed is similar to the construction of the Boltzmann hierarchy pseudo-trajectory. The only difference is that we take into account the diameter ϵ_2 or the interaction zone ϵ_3 of the adjusted particles in each step.

More precisely, we inductively construct the BBGKY hierarchy pseudo-trajectory of $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$ as follows:

Time $t_0 = t$: We initially define $Z_s^N(t_0^-) = \left(x_1^N(t_0^-), ..., x_s^N(t_0^-), v_1^N(t_0^-), ..., v_s^N(t_0^-)\right) := Z_s$.

Time t_i, $i \in \{1, ..., k\}$: Consider $i \in \{1, ..., k\}$ and assume we know

$$Z^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-) = \left(x_1^N(t_{i-1}^-), ..., x^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-), v_1^N(t_{i-1}^-), ..., v^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-)\right).$$

We define $Z^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+) = \left(x_1^N(t_i^+), ..., x^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+), v_1^N(t_i^+), ..., v^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right)$ as

$$Z^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+) := \left(X^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-) - (t_{i-1} - t_i)V^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-), V^N_{s+\widetilde{\sigma}_{i-1}}(t_{i-1}^-)\right).$$

We also define $Z^N_{s+\widetilde{\sigma}_i}(t_i^-) = \left(x_1^N(t_i^-), ..., x^N_{s+\widetilde{\sigma}_i}(t_i^-), v_1^N(t_i^-), ..., v^N_{s+\widetilde{\sigma}_i}(t_i^-)\right)$ as

$$\left(x_j^N(t_i^-), v_j^N(t_i^-)\right) := (x_j^N(t_i^+), v_j^N(t_i^+)) \quad \forall j \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},$$

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For the rest of the particles, we distiguish the following cases, depending on σ_i:

- $\sigma_i = 1$: If $j_i = -1$:

$$\left(x^N_{m_i}(t_i^-), v^N_{m_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+), v^N_{m_i}(t_i^+)\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i}(t_i^-), v^N_{s+\widetilde{\sigma}_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) - \epsilon_2\omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i}\right),$$

while if $j_i = 1$:

$$\left(x^N_{m_i}(t_i^-), v^N_{m_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+), v^{N\prime}_{m_i}(t_i^+)\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i}(t_i^-), v^N_{s+\widetilde{\sigma}_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) + \epsilon_2\omega_{s+\widetilde{\sigma}_i}, v'_{s+\widetilde{\sigma}_i}\right),$$

where $(v^{N\prime}_{m_i}(t_i^-), v'_{s+\widetilde{\sigma}_i}) = T_{\omega_{s+\widetilde{\sigma}_i}}\left(v^N_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i}\right)$.

- $\sigma_i = 2$: If $j_i = -1$:

$$\left(x^N_{m_i}(t_i^-), v^N_{m_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+), v^N_{m_i}(t_i^+)\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i-1}(t_i^-), v^N_{s+\widetilde{\sigma}_i-1}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) - \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i-1}, v_{s+\widetilde{\sigma}_i-1}\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i}(t_i^-), v^N_{s+\widetilde{\sigma}_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) - \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i}, v_{s+\widetilde{\sigma}_i}\right),$$

while if $j_i = 1$:

$$\left(x^N_{m_i}(t_i^-), v^N_{m_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+), v^{N*}_{m_i}(t_i^+)\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i-1}(t_i^-), v^N_{s+\widetilde{\sigma}_i-1}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) + \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i-1}, v^*_{s+\widetilde{\sigma}_i-1}\right),$$
$$\left(x^N_{s+\widetilde{\sigma}_i}(t_i^-), v^N_{s+\widetilde{\sigma}_i}(t_i^-)\right) := \left(x^N_{m_i}(t_i^+) + \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i}, v^*_{s+\widetilde{\sigma}_i}\right),$$

where $(v^{N*}_{m_i}(t_i^-), v^*_{s+\widetilde{\sigma}_i-1}, v^*_{s+\widetilde{\sigma}_i}) = T_{\omega_{s+\widetilde{\sigma}_i-1},\omega_{s+\widetilde{\sigma}_i}}\left(v^N_{m_i}(t_i^+), v_{s+\widetilde{\sigma}_i-1}, v_{s+\widetilde{\sigma}_i}\right)$.

Time $t_{k+1} = 0$: We finally obtain

$$Z^N_{s+\widetilde{\sigma}_k}(0^+) = Z^N_{s+\widetilde{\sigma}_k}(t_{k+1}^+) = \left(X^N_{s+\widetilde{\sigma}_k}(t_k^-) - t_k V^N_{s+\widetilde{\sigma}_k}(t_k^-), V^N_{s+\widetilde{\sigma}_k}(t_k^-)\right).$$

We give the following definition:

Definition 11.1. Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$, $J = (j_1, \dots, j_k)$, $M = (m_1, \dots, m_k)$, $(J, M) \in \mathcal{U}_{s,k}$, and for each $i = 1, \dots, k$, $\sigma \in S_k$, we consider $(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}) \in \mathbb{S}_1^{d\sigma_i - 1} \times B_R^{d\sigma_i}$. The sequence $\{Z^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\}_{i=0,\dots,k+1}$ constructed above is called the BBGKY hierarchy pseudo-trajectory of Z_s.

We now state the following elementary proximity result of the corresponding BBGKY hierarchy and Boltzmann hierarchy pseudo-trajectories.

Lemma 11.2. *Let $s \in \mathbb{N}$, $Z_s = (X_s, V_s) \in \mathbb{R}^{2ds}$, $1 \leq k \leq n$, $\sigma \in S_k$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $t \in [0, T]$ and $(t_1, \dots, t_k) \in \mathcal{T}_k(t)$. For each $i = 1, \dots, k$, consider $(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}) \in \mathbb{S}_1^{d\sigma_i - 1} \times \mathbb{R}^{d\sigma_i}$. Then for all $i = 1, \dots, k$ and $\ell = 1, \dots, s + \widetilde{\sigma}_{i-1}$, we have*

$$|x^N_\ell(t_i^+) - x^\infty_\ell(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1), \quad v^N_\ell(t_i^+) = v^\infty_\ell(t_i^+). \tag{11.1}$$

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Moreover, if $s < n$, then for each $i \in \{1, ..., k\}$, there holds

$$\left|X^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+) - X^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)\right| \leq n^{3/2}\epsilon_3. \tag{11.2}$$

Proof. We first prove (11.1) by induction on $i \in \{1, ..., k\}$. For $i = 1$, the result is trivial since the pseudo-trajectories initially coincide by construction. Assume the conclusion holds for $i \in \{1, ..., k-1\}$; that is, for all $\ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\}$, there holds

$$|x^N_\ell(t_i^+) - x^\infty_\ell(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1) \quad \text{and} \quad v^N_\ell(t_i^+) = v^\infty_\ell(t_i^+). \tag{11.3}$$

We prove the conclusion holds for $(i+1) \in \{2, ..., k\}$. We need to take different cases for $j_i \in \{-1, 1\}$ and $\sigma_i \in \{1, 2\}$.

- $\sigma_i = 1, j_i = -1$: For the Boltzmann pseudo-trajectory, we get

$$\begin{aligned}
&x^\infty_\ell(t^+_{i+1}) = x^\infty_\ell(t_i^+) - (t_i - t_{i+1})v^\infty_\ell(t_i^+), \quad v^\infty_\ell(t^+_{i+1}) = v^\infty_\ell(t_i^+), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^\infty_{m_i}(t^+_{i+1}) = x^\infty_{m_i}(t_i^+) - (t_i - t_{i+1})v^\infty_\ell(t_i^+), \quad v^\infty_{m_i}(t^+_{i+1}) = v^\infty_{m_i}(t_i^+),\\
&x^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^\infty_{m_i}(t_i^+) - (t_i - t_{i+1})v_{s+\widetilde{\sigma}_i}, \quad v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v_{s+\widetilde{\sigma}_i},
\end{aligned}$$

while for the BBGKY hierarchy pseudo-trajectory, we get

$$\begin{aligned}
&x^N_\ell(t^+_{i+1}) = x^N_\ell(t_i^+) - (t_i - t_{i+1})v^N_\ell(t_i^+), \quad v^N_\ell(t^+_{i+1}) = v^N_\ell(t_i^-), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^N_{m_i}(t^+_{i+1}) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v^N_{m_i}(t_i^+), \quad v^N_{m_i}(t^+_{i+1}) = v^N_{m_i}(t_i^-),\\
&x^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v_{s+\widetilde{\sigma}_i} - \epsilon_2\omega_{s+\widetilde{\sigma}_i}, \quad v^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v_{s+\widetilde{\sigma}_i}.
\end{aligned}$$

So, for any $\ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\}$, the induction assumption (11.3) implies

$$\begin{aligned}
&v^N_\ell(t^+_{i+1}) = v^N_\ell(t_i^+) = v^\infty_\ell(t_i^+) = v^\infty_\ell(t^+_{i+1}),\\
&|x^N_\ell(t^+_{i+1}) - x^\infty_\ell(t^+_{i+1})| = |x^N_\ell(t_i^+) - x^\infty_\ell(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1).
\end{aligned}$$

Moreover, since $\epsilon_2 << \epsilon_3$, for $\ell = s + \widetilde{\sigma}_i$, we get

$$\begin{aligned}
&v^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v_{s+\widetilde{\sigma}_i} = v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}),\\
&|x^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) - x^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1})| \leq |x^N_{m_i}(t_i^+) - x^\infty_{m_i}(t_i^+)| + \epsilon_2|\omega_{s+\widetilde{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \epsilon_2 < \sqrt{2}\epsilon_3 i.
\end{aligned}$$

- $\sigma_i = 1, j_i = 1$: For the Boltzmann hierarchy pseudo-trajectory, we get

$$\begin{aligned}
&x^\infty_\ell(t^+_{i+1}) = x^\infty_\ell(t_i^+) - (t_i - t_{i+1})v^\infty_\ell(t_i^+), \quad v^\infty_\ell(t^+_{i+1}) = v^\infty_\ell(t_i^+), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^\infty_{m_i}(t^+_{i+1}) = x^\infty_{m_i}(t_i^+) - (t_i - t_{i+1})v^{\infty'}_{m_i}(t_i^+), \quad v^\infty_{m_i}(t^+_{i+1}) = v^{\infty'}_{m_i}(t_i^+),\\
&x^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^\infty_{m_i}(t_i^+) - (t_i - t_{i+1})v'_{s+\widetilde{\sigma}_i}, \quad v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v'_{s+\widetilde{\sigma}_i}.
\end{aligned}$$

and for the BBGKY hierarchy pseudo-trajectory, we obtain

$$\begin{aligned}
&x^N_\ell(t^+_{i+1}) = x^N_\ell(t_i^+) - (t_i - t_{i+1})v^N_\ell(t_i^+), \quad v^N_\ell(t^+_{i+1}) = v^N_\ell(t_i^+), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^N_{m_i}(t^+_{i+1}) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v^{N'}_{m_i}(t_i^+), \quad v^N_{m_i}(t^+_{i+1}) = v^{N'}_{m_i}(t_i^+),\\
&x^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^N_{m_i}(t_i^+) - (t_i - t_{i+1})v'_{s+\widetilde{\sigma}_i} + \epsilon_2\omega_{s+\widetilde{\sigma}_i}, \quad v^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v'_{s+\widetilde{\sigma}_i}.
\end{aligned}$$

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For $\ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\}$, the induction assumption (11.3) yields

$$\begin{aligned} v_\ell^N(t_{i+1}^+) &= v_\ell^N(t_i^+) = v_\ell^\infty(t_i^+) = v_\ell^\infty(t_{i+1}^+), \\ |x_\ell^N(t_{i+1}^+) - x_\ell^\infty(t_{i+1}^+)| &= |x_\ell^N(t_i^+) - x_\ell^\infty(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1), \end{aligned}$$

and for $\ell = m_i$, it yields

$$\begin{aligned} v_{m_i}^N(t_{i+1}^+) &= v_{m_i}^{N'}(t_i^+) = v_{m_i}^{\infty'}(t_i^+) = v_\ell^\infty(t_{i+1}^+), \\ |x_{m_i}^N(t_{i+1}^+) - x_{m_i}^\infty(t_{i+1}^+)| &= |x_{m_i}^N(t_i^+) - x_{m_i}^\infty(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1). \end{aligned}$$

Moreover, since $\epsilon_2 << \epsilon_3$, for $\ell = s + \widetilde{\sigma}_i$, we obtain

$$\begin{aligned} &v_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) = v'_{s+\widetilde{\sigma}_i} = v_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+), \\ &|x_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) - x_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+)| \leq |x_{m_i}^N(t_i^+) - x_{m_i}^\infty(t_i^+)| + \epsilon_2|\omega_{s+\widetilde{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \epsilon_2 < \sqrt{2}\epsilon_3 i. \end{aligned}$$

- $\sigma_i = 2$, $j_i = -1$: For the Boltzmann hierarchy pseudo-trajectory, we get

$$\begin{aligned} &x_\ell^\infty(t_{i+1}^+) = x_\ell^\infty(t_i^+) - (t_i - t_{i+1})v_\ell^\infty(t_i^+), \quad v_\ell^\infty(t_{i+1}^+) = v_\ell^\infty(t_i^+), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\}, \\ &x_{m_i}^\infty(t_{i+1}^+) = x_{m_i}^\infty(t_i^+) - (t_i - t_{i+1})v_{m_i}^\infty(t_i^+), \quad v_{m_i}^\infty(t_{i+1}^+) = v_{m_i}^\infty(t_i^+), \\ &x_{s+\widetilde{\sigma}_i-1}^\infty(t_{i+1}^+) = x_{m_i}^\infty(t_i^+) - (t_i - t_{i+1})v_{\widetilde{\sigma}_i-1}, \quad v_\ell^\infty(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i-1}, \\ &x_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+) = x_{m_i}^\infty(t_i^+) - (t_i - t_{i+1})v_{s+\widetilde{\sigma}_i}, \quad v_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i}, \end{aligned}$$

while for the BBGKY hierarchy pseudo-trajectory, we get

$$\begin{aligned} &x_\ell^N(t_{i+1}^+) = x_\ell^N(t_i^+) - (t_i - t_{i+1})v_\ell^N(t_i^+), \quad v_\ell^N(t_{i+1}^+) = v_\ell^N(t_i^-), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\}, \\ &x_{m_i}^N(t_{i+1}^+) = x_{m_i}^N(t_i^+) - (t_i - t_{i+1})v_{m_i}^N(t_i^+), \quad v_{m_i}^N(t_{i+1}^+) = v_{m_i}^N(t_i^-), \\ &x_{s+\widetilde{\sigma}_i-1}^N(t_{i+1}^+) = x_{m_i}^N(t_i^+) - (t_i - t_{i+1})v_{s+\widetilde{\sigma}_i-1} - \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i-1}, \quad v_{s+\widetilde{\sigma}_i-1}^N(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i-1}, \\ &x_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) = x_{m_i}^N(t_i^+) - (t_i - t_{i+1})v_{s+\widetilde{\sigma}_i} - \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i}, \quad v_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i}. \end{aligned}$$

So, for any $\ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\}$, the induction assumption (11.3) implies

$$\begin{aligned} v_\ell^N(t_{i+1}^+) &= v_\ell^N(t_i^+) = v_\ell^\infty(t_i^+) = v_\ell^\infty(t_{i+1}^+), \\ |x_\ell^N(t_{i+1}^+) - x_\ell^\infty(t_{i+1}^+)| &= |x_\ell^N(t_i^+) - x_\ell^\infty(t_i^+)| \leq \sqrt{2}\epsilon_3(i-1), \end{aligned}$$

Moreover, for $\ell = s + \widetilde{\sigma}_i - 1$, we get

$$\begin{aligned} &v_{s+\widetilde{\sigma}_i-1}^N(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i-1} = v_{s+\widetilde{\sigma}_i-1}^\infty(t_{i+1}^+), \\ &|x_{s+\widetilde{\sigma}_i-1}^N(t_{i+1}^+) - x_{s+\widetilde{\sigma}_i-1}^\infty(t_{i+1}^+)| \leq |x_{m_i}^N(t_i^+) - x_{m_i}^\infty(t_i^+)| + \sqrt{2}\epsilon_3|\omega_{s+\widetilde{\sigma}_i-1}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3 i. \end{aligned}$$

and for $\ell = s + \widetilde{\sigma}_i$, we get

$$\begin{aligned} &v_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) = v_{s+\widetilde{\sigma}_i} = v_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+), \\ &|x_{s+\widetilde{\sigma}_i}^N(t_{i+1}^+) - x_{s+\widetilde{\sigma}_i}^\infty(t_{i+1}^+)| \leq |x_{m_i}^N(t_i^+) - x_{m_i}^\infty(t_i^+)| + \sqrt{2}\epsilon_3|\omega_{s+\widetilde{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3 i. \end{aligned}$$

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- $\sigma_i = 2, j_i = 1$: For the Boltzmann hierarchy pseudo-trajectory, we get

$$\begin{aligned}
&x^\infty_\ell(t^+_{i+1}) = x^\infty_\ell(t^+_i) - (t_i - t_{i+1})v^\infty_\ell(t^+_i), \quad v^\infty_\ell(t^+_{i+1}) = v^\infty_\ell(t^+_i), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^\infty_{m_i}(t^+_{i+1}) = x^\infty_{m_i}(t^+_i) - (t_i - t_{i+1})v^{\infty *}_{m_i}(t^+_i), \quad v^\infty_{m_i}(t^+_{i+1}) = v^{\infty *}_{m_i}(t^+_i),\\
&x^\infty_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) = x^\infty_{m_i}(t^+_i) - (t_i - t_{i+1})v^*_{s+\widetilde{\sigma}_i-1},\\
&v^\infty_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i-1},\\
&x^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^\infty_{m_i}(t^+_i) - (t_i - t_{i+1})v^*_{s+\widetilde{\sigma}_i}, \quad v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i},
\end{aligned}$$

and for the BBGKY hierarchy pseudo-trajectory, we obtain

$$\begin{aligned}
&x^N_\ell(t^+_{i+1}) = x^N_\ell(t^+_i) - (t_i - t_{i+1})v^N_\ell(t^+_i), \quad v^N_\ell(t^+_{i+1}) = v^N_\ell(t^+_i), \quad \forall \ell \in \{1, ..., s + \widetilde{\sigma}_{i-1}\} \setminus \{m_i\},\\
&x^N_{m_i}(t^+_{i+1}) = x^N_{m_i}(t^+_i) - (t_i - t_{i+1})v^{N*}_{m_i}(t^+_i), \quad v^N_{m_i}(t^+_{i+1}) = v^{N*}_{m_i}(t^+_i),\\
&x^N_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) = x^N_{m_i}(t^+_i) - (t_i - t_{i+1})v^*_{s+\widetilde{\sigma}_i-1} + \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i-1},\\
&v^N_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i-1},\\
&x^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = x^N_{m_i}(t^+_i) - (t_i - t_{i+1})v^*_{s+\widetilde{\sigma}_i} + \sqrt{2}\epsilon_3\omega_{s+\widetilde{\sigma}_i},\\
&v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i}.
\end{aligned}$$

For $\ell \in \{1, ..., \widetilde{\sigma}_{i-1}\} \setminus \{m_i\}$, the induction assumption (11.3) yields

$$\begin{aligned}
&v^N_\ell(t^+_{i+1}) = v^N_\ell(t^+_i) = v^\infty_\ell(t^+_i) = v^\infty_\ell(t^+_{i+1}),\\
&|x^N_\ell(t^+_{i+1}) - x^\infty_\ell(t^+_{i+1})| = |x^N_\ell(t^+_i) - x^\infty_\ell(t^+_i)| \leq \sqrt{2}\epsilon_3(i-1).
\end{aligned}$$

Thus, for $\ell = m_i$,

$$\begin{aligned}
&v^N_{m_i}(t^+_{i+1}) = v^{N*}_{m_i}(t^+_i) = v^{\infty *}_{m_i}(t^+_i) = v^\infty_\ell(t^+_{i+1}),\\
&|x^N_{m_i}(t^+_{i+1}) - x^\infty_{m_i}(t^+_{i+1})| = |x^N_{m_i}(t^+_i) - x^\infty_{m_i}(t^+_i)| \leq \sqrt{2}\epsilon_3(i-1),
\end{aligned}$$

for $\ell = s + \widetilde{\sigma}_i - 1$,

$$\begin{aligned}
&v^N_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i-1} = v^\infty_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}),\\
&|x^N_{s+\widetilde{\sigma}_i-1}(t^+_{i+1}) - x^\infty_{s+\widetilde{\sigma}_i-1}(t^+_{i+1})| \leq |x^N_{m_i}(t^+_i) - x^\infty_{m_i}(t^+_i)| + \sqrt{2}\epsilon_3|\omega_{s+\widetilde{\sigma}_i-1}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3 i,
\end{aligned}$$

and for $\ell = s + \widetilde{\sigma}_i$,

$$\begin{aligned}
&v^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) = v^*_{s+\widetilde{\sigma}_i} = v^\infty_{s+\widetilde{\sigma}_i}(t^+_{i+1}),\\
&|x^N_{s+\widetilde{\sigma}_i}(t^+_{i+1}) - x^\infty_{m_i}(t^+_{i+1})| \leq |x^N_{m_i}(t^+_i) - x^\infty_{s+\widetilde{\sigma}_i}(t^+_i)| + \sqrt{2}\epsilon_3|\omega_{s+\widetilde{\sigma}_i}| \leq \sqrt{2}\epsilon_3(i-1) + \sqrt{2}\epsilon_3 = \sqrt{2}\epsilon_3 i.
\end{aligned}$$

Combining all cases, (11.1) is proved by induction.

To prove (11.2), it suffices to add for $\ell = 1, ..., s + \widetilde{\sigma}_{i-1}$, and use the facts $1 \leq i \leq k-1, \widetilde{\sigma}_{i-1} < \widetilde{\sigma}_i \leq \widetilde{\sigma}_{k-1} < 2k \leq 2n$, from (7.6), and the assumption $s < n$. □

11.2. Reformulation in terms of pseudo-trajectories

We will now re-write the BBGKY hierarchy and Boltzmann hierarchy truncated elementary observables in terms of pseudo-trajectories.

Let $s \in \mathbb{N}$ and assume $s < n$. For the Boltzmann hierarchy case, there is always free flow between the collision times. Therefore, recalling (10.20) and (10.26), for $X_s \in \Delta^X_s(\epsilon_0)$, $1 \leq k \leq n$, $\sigma \in S_k$,

$(J, M) \in \mathcal{U}_{s,k,\sigma}$, $t \in [0, T]$ and $(t_1, ..., t_k) \in \mathcal{T}_{k,\delta}(t)$, the Boltzmann hierarchy truncated elementary observable can be equivalently written as

$$\begin{aligned} J^{\infty}_{s,k,R,\delta,\sigma}(t, J, M)(X_s) &= \int_{\mathcal{M}^c_s(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}^c_{m_1}(Z^\infty_s(t_1^+))} \cdots \int_{\mathcal{B}^c_{m_k}\left(Z^\infty_{s+\widetilde{\sigma}_{k-1}}(t_k^+)\right)} \\ &\times \prod_{i=1}^k b^+_{\sigma_i}\left(\omega_{\sigma_i,i}, v_{\sigma_i,i}, v^\infty_{m_i}(t_i^+)\right) f_0^{(s+\widetilde{\sigma}_k)}\left(Z^\infty_{s+\widetilde{\sigma}_k}(0^+)\right) \prod_{i=1}^k \left(d\omega_{\sigma_i,i}\, dv_{\sigma_i,i}\right) dt_k \dots dt_1\, dV_s. \end{aligned} \tag{11.4}$$

Now we shall see that due to Lemma 11.2, it is possible to make a similar expansion for the BBGKY hierarchy truncated elementary observables as well.

More precisely, fix $X_s \in \Delta^X_s(\epsilon_0)$, $1 \le k \le n$, $\sigma \in S_k$, $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $t \in [0, T]$ and $(t_1, ..., t_k) \in \mathcal{T}_{k,\delta}(t)$. Consider $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) such that $\epsilon_2 << \eta^2\epsilon_3$ and $n^{3/2}\epsilon_3 << \alpha$. By Lemma 10.1, given $V_s \in \mathcal{M}^c_s(X_s)$, we have $Z_s \in G_s(\epsilon_3, \epsilon_0, \delta)$. By the definition of the set $G_s(\epsilon_3, \epsilon_0, \delta)$, see (10.1), and the fact that $\epsilon_2 << \epsilon_3$, we have

$$Z_s \in G_s(\epsilon_3, \epsilon_0, \delta) \Rightarrow Z_s(\tau) \in \mathring{\mathcal{D}}_{s,\epsilon_2,\epsilon_3}, \quad \forall \tau \ge 0,$$

and thus,

$$\Psi_s^{\tau - t_0} Z^N_s(t_0^-) = \Phi_s^{\tau - t_0} Z^N_s(t_0^-), \quad \forall \tau \in [t_1, t_0], \tag{11.5}$$

where Ψ_s, given in (3.56), denotes the s-particle (ϵ_2, ϵ_3)-interaction zone flow and Φ_s, given in (3.57), denotes the s-particle free flow respectively. We also have

$$Z_s = (X_s, V_s) \in G_s(\epsilon_3, \epsilon_0, \delta) \Rightarrow Z^\infty_s(t_1^+) \in G_s(\epsilon_0, 0).$$

For all $i \in \{1, ..., k\}$, inductive application of Proposition 9.2 or Proposition 9.6, depending on whether the adjunction is binary or ternary, implies that

$$Z^\infty_{s+\widetilde{\sigma}_i}(t_{i+1}^+) \in G_{s+\widetilde{\sigma}_i}(\epsilon_0, 0), \quad \forall (\omega_{\sigma_i,i}, v_{\sigma_i,i}) \in \mathcal{B}^c_{m_i}(Z^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+)). \tag{11.6}$$

Since we have assumed $n^{3/2}\epsilon_3 << \alpha$ and $s < n$, (11.2) from Lemma 11.2 implies

$$\left| X^N_{s+\widetilde{\sigma}_{i-1}}(t_i^+) - X^\infty_{s+\widetilde{\sigma}_{i-1}}(t_i^+) \right| \le \frac{\alpha}{2}, \quad \forall i = 1, ..., k. \tag{11.7}$$

Then, (9.6), (9.10) from Proposition 9.2, or (9.59), (9.63) from Proposition 9.6, depending on whether the adjunction is binary or ternary, yield that for any $i = 1, ..., k$, we have

$$\Psi^{\tau - t_i}_{s+\widetilde{\sigma}_i} Z^N_{s+\widetilde{\sigma}_i}(t_i^-) = \Phi^{\tau - t_i}_{s+\widetilde{\sigma}_i} Z^N_{s+\widetilde{\sigma}_i}(t_i^-), \quad \forall \tau \in [t_{i+1}, t_i],$$

where $\Psi_{s+\widetilde{\sigma}_i}$ and $\Phi_{s+\widetilde{\sigma}_i}$ denote the $(s + \widetilde{\sigma}_i)$-particle (ϵ_2, ϵ_3)-flow and the $(s + \widetilde{\sigma}_i)$-particle free flow, given in (3.56) and (3.57), respectively. In other words, the backwards (ϵ_2, ϵ_3)-flow coincides with the free flow in $[t_{i+1}, t_i]$. Finally, Lemma 11.2 also implies that

$$v^N_{m_i}(t_i^+) = v^\infty_{m_i}(t_i^+), \quad \forall i = 1, ..., k.$$

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Therefore, for $X_s \in \Delta_s^X(\epsilon_0)$, and $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $n\epsilon_3^{3/2} << \alpha$ and $\epsilon_2 << \eta^2\epsilon_3$, the BBGKY hierarchy truncated elementary observable can be equivalently written as

$$J^N_{s,k,R,\delta,\sigma}(t,J,M)(X_s) = \boldsymbol{A}^{\boldsymbol{s,k}}_{\boldsymbol{N,\epsilon_2,\epsilon_3}} \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}^c_{m_1}(Z_s^\infty(t_1^+))} \cdots \int_{\mathcal{B}^c_{m_k}\left(Z^\infty_{s+\widetilde{\sigma}_{k-1}}(t_k^+)\right)} \times \prod_{i=1}^k b^+_{\sigma_i}(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}, v^\infty_{m_i}(t_i^+)) f_{N,0}^{(s+\widetilde{\sigma}_k)}\left(Z^N_{s+\widetilde{\sigma}_k}(0^+)\right) \times \prod_{i=1}^k \left(d\boldsymbol{\omega}_{\sigma_i,i}\, d\boldsymbol{v}_{\sigma_i,i}\right) dt_k \ldots dt_1\, dV_s, \tag{11.8}$$

where, recalling (4.19), (4.22), we denote

$$\boldsymbol{A}^{\boldsymbol{s,k,\sigma}}_{\boldsymbol{N,\epsilon_2,\epsilon_3}} = \prod_{i\in\{1,\ldots,k\}:\sigma_i=1} A^2_{N,\epsilon_2,s+\widetilde{\sigma}_{i-1}} \prod_{i\in\{1,\ldots,k\}:\sigma_i=2} A^3_{N,\epsilon_3,s+\widetilde{\sigma}_{i-1}}. \tag{11.9}$$

Remark 11.3. Notice that for fixed $s \in \mathbb{N}$ and $k \geq 1$ and $\sigma \in S_k$, the scaling (4.24) implies

$$1 - \boldsymbol{A}^{\boldsymbol{s,k,\sigma}}_{\boldsymbol{N,\epsilon_2,\epsilon_3}} \lesssim \frac{k(s+2k)}{N} \simeq k(s+2k)\epsilon_2^{d-1} \simeq k(s+2k)\epsilon_3^{d-1/2}. \tag{11.10}$$

In particular, $\boldsymbol{A}^{\boldsymbol{s,k,\sigma}}_{\boldsymbol{N,\epsilon_2,\epsilon_3}} \nearrow 1$ as $N \to \infty$ and $\epsilon_2, \epsilon_3 \to 0^+$ in the scaling (4.24).

Let us approximate the BBGKY hierarchy truncated elementary observables by Boltzmann hierarchy truncated elementary observables defining some auxiliary functionals. Let $s \in \mathbb{N}$ and $X_s \in \Delta_s^X(\epsilon_0)$. For $1 \leq k \leq n$, $\sigma \in S_k$ and $(J,M) \in \mathcal{U}_{s,k,\sigma}$, we define

$$\widehat{J}^N_{s,k,R,\delta,\sigma}(t,J,M)(X_s) = \int_{\mathcal{M}_s^c(X_s)} \phi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \int_{\mathcal{B}^c_{m_1}(Z_s^\infty(t_1^+))} \cdots \int_{\mathcal{B}^c_{m_k}\left(Z^\infty_{s+\widetilde{\sigma}_{k-1}}(t_k^+)\right)} \times \prod_{i=1}^k b^+_{\sigma_i}(\boldsymbol{\omega}_{\sigma_i,i}, \boldsymbol{v}_{\sigma_i,i}, v^\infty_{m_i}(t_i^+)) f_0^{(s+\widetilde{\sigma}_k)}\left(Z^N_{s+\widetilde{\sigma}_k}(0^+)\right) \prod_{i=1}^k \left(d\boldsymbol{\omega}_{\sigma_i,i}\, d\boldsymbol{v}_{\sigma_i,i}\right) dt_k \ldots dt_1\, dV_s. \tag{11.11}$$

We conclude that the auxiliary functionals approximate the BBGKY hierarchy truncated elementary observables $J^N_{s,k,R,\delta}$, defined in (11.8)

Proposition 11.4. *Let $s, n \in \mathbb{N}$, with $s < n$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.4), and $t \in [0,T]$. Then for any $\zeta > 0$, there is $N_1 = N_1(\zeta, n, \alpha, \eta, \epsilon_0) \in \mathbb{N}$, such that for all $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $N > N_1$, there holds*

$$\sum_{k=1}^n \sum_{\sigma\in S_k} \sum_{(J,M)\in\mathcal{U}_{s,k}} \|J^N_{s,k,R,\delta,\sigma}(t,J,M) - \widehat{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \zeta^2. \tag{11.12}$$

In the case of tensorized initial data and approximation by conditioned BBGKY initial data (see Proposition 6.5), the estimate can be improved to

$$\sum_{k=1}^n \sum_{\sigma\in S_k} \sum_{(J,M)\in\mathcal{U}_{s,k,\sigma}} \|J^N_{s,k,R,\delta,\sigma}(t,J,M) - \widehat{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C^n_{d,s,\beta_0,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \epsilon_3^{1/2}, \tag{11.13}$$

for all $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with N large enough.

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Proof. Fix $1 \leq k \leq n$, $\sigma \in S_k$ and $(J, M) \in \mathcal{U}_{s,k,\sigma}$. Consider $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24). Remark 4.2 guarantees that we can consider N large enough such that $\epsilon_2 << \eta^2\epsilon_3$ and $n^{3/2}\epsilon_3 << \alpha$. Triangle inequality and the inclusion $\Delta_s^X(\epsilon_0) \subseteq \Delta_s^X(\epsilon_0/2)$ yield

$$\begin{aligned}
&\|J^N_{s,k,R,\delta.\sigma}(t,J,M) - \widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \\
&\quad\leq \|J^N_{s,k,R,\delta,\sigma}(t,J,M) - \boldsymbol{A}^{s,k,\sigma}_{N,\epsilon_2,\epsilon_3}\widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0/2))} \\
&\qquad + (1 - \boldsymbol{A}^{s,k,\sigma}_{N,\epsilon_2,\epsilon_3})\|\widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))}.
\end{aligned} \tag{11.14}$$

We estimate each of the terms in (11.14) separately. For the first term, let us fix $(t_1, \ldots, t_k) \in \mathcal{T}_{k,\delta}(t)$. Applying (10.18) for $i = k - 1$, we obtain

$$Z^\infty_{s+\widetilde{\sigma}_{k-1}}(t_k^+) \in G_{s+\widetilde{\sigma}_{k-1}}(\epsilon_0, 0).$$

Since $s < n$ and $n^{3/2}\epsilon_3 << \alpha$, (11.2), applied for $i = k$, implies

$$|X^N_{s+\widetilde{\sigma}_{k-1}}(t_k^+) - X^\infty_{s+\widetilde{\sigma}_{k-1}}(t_k^+)| \leq \frac{\alpha}{2}.$$

Therefore, (9.7), (9.11) from Proposition 9.2, or (9.60), (9.64) from Proposition 9.6, depending on whether the adjunction is binary or ternary, imply

$$Z^N_{s+\widetilde{\sigma}_k}(0^+) \in G_{s+\widetilde{\sigma}_k}(\epsilon_0/2, 0) \subseteq \Delta_{s+\widetilde{\sigma}_k}(\epsilon_0/2). \tag{11.15}$$

Thus, (10.24)–(10.25), (10.29), (11.8)–(11.11) and crucially (11.15) imply

$$\begin{aligned}
&\|J^N_{s,k,R,\delta,\sigma}(t,J,M) - \boldsymbol{A}^{s,k,\sigma}_{N,\epsilon_2,\epsilon_3}\widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0/2))} \\
&\qquad\leq \frac{C^k_{d,s,T}}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|f_{N,0}^{(s+\widetilde{\sigma}_k)} - f_0^{(s+\widetilde{\sigma}_k)}\|_{L^\infty(\Delta_{s+\widetilde{\sigma}_k}(\epsilon_0/2))} \\
&\qquad\leq \frac{C^k_{d,s,T}}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|f_{N,0}^{(s+\widetilde{\sigma}_k)} - f_0^{(s+\widetilde{\sigma}_k)}\|_{L^\infty(\mathcal{D}_{s+\widetilde{\sigma}_k,\epsilon_2,\epsilon_3})},
\end{aligned} \tag{11.16}$$

as long as $\epsilon_3 < \epsilon_0/2\sqrt{2}$ (i.e., N large enough) so that $\Delta_{s+\widetilde{\sigma}_k}(\epsilon_0/2) \subseteq \mathcal{D}_{s+\widetilde{\sigma}_k,\epsilon_2,\cdot}$.

For the second term, by (10.28), we have $\|f_0^{(s+\widetilde{\sigma}_k)}\|_{L^\infty} \leq e^{-(s+k)\mu_0}\|F_0\|_{\infty,\beta_0,\mu_0}$. Therefore, using (10.24)– (10.25) and (10.29), we obtain

$$\|\widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq \frac{C^k_{d,s,\mu_0,T}}{k!}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3k)}\|F_0\|_{\infty,\beta_0,\mu_0}. \tag{11.17}$$

Adding over all $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $\sigma \in S_k$, $k = 1, \ldots, n$, using (11.16)–(11.17) and the scaling estimate (11.10), we obtain

$$\begin{aligned}
&\sum_{k=1}^n \sum_{\sigma\in S_k} \sum_{(J,M)\in\mathcal{U}_{s,k,\sigma}} \|J^N_{s,k,R,\delta,\sigma}(t,J,M) - \widetilde{J}^N_{s,k,R,\delta,\sigma}(t,J,M)\|_{L^\infty(\Delta_s(\epsilon_0))} \leq C^n_{d,s,\mu_0,T}\|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \\
&\times\left(\sup_{k\in\{1,\ldots,n\}}\sup_{\sigma\in S_k}\|f_{N,0}^{(s+\widetilde{\sigma}_k)} - f_0^{(s+\widetilde{\sigma}_k)}\|_{L^\infty(\mathcal{D}_{s+\widetilde{\sigma}_k,\epsilon_2,\epsilon_3})} + \frac{\|F_0\|_{\infty,\beta_0,\mu_0}}{N}\right).
\end{aligned}$$

Since n is fixed, (11.12) follows from convergence in the level of initial data.

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In the case of tensorized initial data and approximation by conditioned BBGKY initial data, the estimate can be improved to (11.13) using (6.2) and the fact that $N\epsilon_3^{d-1/2} \simeq 1$. □

By the proximity Lemma 11.2 and the uniform continuity assumption on the initial data, we also obtain the following estimate:

Proposition 11.5. *Let $s, n \in \mathbb{N}$ with $s < n$, $\alpha, \epsilon_0, R, \eta, \delta$ be parameters as in (9.4) and $t \in [0, T]$. Then for any $\zeta > 0$, there is $N_2 = N_2(\zeta, n) \in \mathbb{N}$, such that for all $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24) with $N > N_2$, there holds*

$$\sum_{k=1}^{n} \sum_{\sigma \in S_k} \sum_{(J,M) \in \mathcal{U}_{s,k,\sigma}} \|\widehat{J}^N_{s,k,R,\delta,\sigma}(t, J, M) - J^{\infty}_{s,k,R,\delta,\sigma}(t, J, M)\|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \zeta^2. \tag{11.18}$$

In the case of Hölder continuous $C^{0,\gamma}$, $\gamma \in (0, 1]$ tensorized initial data (see Remark 6.3), the estimate can be improved to

$$\sum_{k=1}^{n} \sum_{\sigma \in S_k} \sum_{(J,M) \in \mathcal{U}_{s,k,\sigma}} \|\widehat{J}^N_{s,k,R,\delta,\sigma}(t, J, M) - J^{\infty}_{s,k,R,\delta,\sigma}(t, J, M)\|_{L^\infty(\Delta^X_s(\epsilon_0))} \leq C^n_{d,s,\mu_0,T} \|\phi_s\|_{L^\infty_{V_s}} R^{d(s+3n)} \epsilon^{\gamma}, \tag{11.19}$$

for all $(N, \epsilon_2, \epsilon_3)$ in the scaling (4.24).

Proof. Let $\zeta > 0$. Fix $1 \leq k \leq n$, $\sigma \in S_k$ and $(J, M) \in \mathcal{U}_{s,k,\sigma}$. Since $s < n$, Lemma 11.2 yields

$$|Z^N_{s+\widetilde{\sigma}_k}(0^+) - Z^{\infty}_{s+\widetilde{\sigma}_k}(0^+)| \leq \sqrt{6} n^{3/2} \epsilon_3, \quad \forall Z_s \in \mathbb{R}^{2ds}. \tag{11.20}$$

Thus, the continuity assumption (6.5) on F_0, (11.20), the scaling (4.24), and (4.26) from Remark 4.2 imply that there exists $N_2 = N_2(\zeta, n) \in \mathbb{N}$, such that for all $N > N_2$, we have

$$|f_0^{(s+\widetilde{\sigma}_k)}(Z^N_{s+\widetilde{\sigma}_k}(0^+)) - f_0^{(s+\widetilde{\sigma}_k)}(Z^{\infty}_{s+\widetilde{\sigma}_k}(0^+))| \leq C^{s+\widetilde{\sigma}_k - 1} \zeta^2 \leq C^{s+2k-1} \zeta^2, \quad \forall Z_s \in \mathbb{R}^{2ds}. \tag{11.21}$$

In the same spirit as in the proof of Proposition 11.4, using (11.21), (10.24)–(10.25), (10.29), and summing over $(J, M) \in \mathcal{U}_{s,k,\sigma}$, $\sigma \in S_k$, $k = 1, \ldots, n$, we obtain the result.

In the case of tensorized $C^{0,\gamma}$ data, one can easily see by induction that for any $Z_{s+\widetilde{\sigma}_k}, Z'_{s+\widetilde{\sigma}_k} \in \mathbb{R}^{2d(s+\widetilde{\sigma}_k)}$, we have

$$\begin{aligned} |f_0^{\otimes(s+\widetilde{\sigma}_k)}(Z_{s+\widetilde{\sigma}_k}) - f_0^{\otimes(s+\widetilde{\sigma}_k)}(Z'_{s+\widetilde{\sigma}_k})| &\leq \|f_0\|^{s+\widetilde{\sigma}_k-1}_{L^\infty} [f_0]_{C^{0,\gamma}} \sqrt{2d(s+\widetilde{\sigma}_k)} |Z_{s+\widetilde{\sigma}_k} - Z'_{s+\widetilde{\sigma}_k}|^{\gamma} \\ &\leq C^{s+\widetilde{\sigma}_k-1} |Z_{s+\widetilde{\sigma}_k} - Z'_{s+\widetilde{\sigma}_k}|^{\gamma}. \end{aligned}$$

Thus, by (11.20), we have

$$|f_0^{(s+\widetilde{\sigma}_k)}(Z^N_{s+\widetilde{\sigma}_k}(0^+)) - f_0^{(s+\widetilde{\sigma}_k)}(Z^{\infty}_{s+\widetilde{\sigma}_k}(0^+))| \leq C^{s+\widetilde{\sigma}_k-1} \epsilon^{\gamma},$$

and the estimate (11.19) follows in a similar manner as estimate (11.18). □

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11.3. Proof of Theorem 6.8

We are now in the position to prove Theorem 6.8. Fix $\theta > 0$, $s \in \mathbb{N}$, $\phi_s \in C_c(\mathbb{R}^{ds})$ and $t \in [0, T]$. Consider $n \in \mathbb{N}$ with $s < n$, and parameters $\alpha, \epsilon_0, R, \eta, \delta$ satisfying (9.4). Let $\zeta > 0$ small enough. Triangle inequality, Propositions 7.5, 10.2, 10.5, Remark 10.3, estimates (11.12), (11.18) and part *(i)* of Definition 6.1, yield that there is $N^*(\zeta) \in \mathbb{N}$ such that for all $N > N^*$, we have

$$\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C\Big(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C^n\Big) + C^n R^{4dn}\eta^{\frac{d-1}{4d+2}} + C^n R^{4dn}\zeta^2, \tag{11.22}$$

where $C > 1$ is an appropriate constant.

We now choose parameters satisfying (9.4), depending only on ζ, such that the right-hand side of (11.22) becomes less than ζ.

Choice of parameters: For ζ sufficiently small, we choose $n \in \mathbb{N}$ and the parameters $\delta, \eta, R, \epsilon_0, \alpha$ in the following order:

$$\begin{aligned}
&\max\big\{s, \log_2(C\zeta^{-1})\big\} << n, \quad \delta << \zeta C^{-(n+1)},\\
&\max\Big\{1, \sqrt{3}\beta_0^{-1/2}\ln^{1/2}(C\zeta^{-1})\Big\} << R << \zeta^{-1/4dn}C^{-1/4d},\\
&\eta << \zeta^{\frac{8d+4}{d-1}}, \quad \epsilon_0 << \min\{\theta, \eta\delta\}, \quad \alpha << \epsilon_0\min\{1, R^{-1}\eta\}.
\end{aligned} \tag{11.23}$$

Relations (11.23) imply the parameters chosen satisfy (9.4) and depend only on ζ. Then, (11.22)–(11.23) imply that we may find $N_0(\zeta) \in \mathbb{N}$, such that for all (N, ϵ) in the scaling (4.24) with $N > N_0$, there holds

$$\|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\theta))} \overset{\epsilon_0<\theta}{\leq} \|I_s^N(t) - I_s^\infty(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} < \zeta,$$

and Theorem 6.8 is proved.

Proof of Corollary 6.10

By Theorem 5.20, we have that $\boldsymbol{F} = (f^{\otimes s})_{s\in\mathbb{N}}$, where f is the mild solution of the ternary Boltzmann equation. Therefore, in the same spirit as before (using estimates (11.13), (11.19) instead of (11.12), (11.18)), for N large enough, we have

$$\|I_{\phi_s}f_N^{(s)}(t) - I_{\phi_s}f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} \leq C\Big(2^{-n} + e^{-\frac{\beta_0}{3}R^2} + \delta C^n\Big) + C^n R^{4dn}\eta^{\frac{d-1}{4d+2}} + C^n R^{4dn}\epsilon^{\gamma_*}, \tag{11.24}$$

where $\gamma_* = \min\{1/2, \gamma\} \in (0, \frac{1}{2}]$ and γ is the Hölder regularity of f_0. Consider $0 < r < \gamma_*$.

Choice of parameters: For N large enough (or equivalently for ϵ small enough), we choose $n \in \mathbb{N}$ and the parameters $\delta, \eta, R, \epsilon_0, \alpha$ in the following order:

$$\begin{aligned}
&\max\big\{s, \log_2(C\epsilon^{\gamma_*})\big\} << n, \quad \delta << \epsilon^{\gamma_*}C^{-(n+1)},\\
&\max\Big\{1, \sqrt{3}\beta_0^{-1/2}\ln^{1/2}(C\epsilon^{-\gamma_*})\Big\} << R << \epsilon^{\frac{r-\gamma_*}{4dn}}C^{-1/4d},\\
&\eta << \epsilon^{\frac{4d+2)}{d-1}\gamma_*}, \quad \epsilon_0 << \min\{\theta, \eta\delta\}, \quad \alpha << \epsilon_0\min\{1, R^{-1}\eta\}.
\end{aligned} \tag{11.25}$$

Then by (11.24), for N large enough, we take

$$\|I_{\phi_s}f_N^{(s)}(t) - I_{\phi_s}f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\theta))} \overset{\epsilon_0<\theta}{\leq} \|I_{\phi_s}f_N^{(s)}(t) - I_{\phi_s}f^{\otimes s}(t)\|_{L^\infty(\Delta_s^X(\epsilon_0))} < \epsilon^r,$$

and Corollary 6.10 is proved.

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12. Appendix

In this appendix, we present some auxiliary results which are used throughout the paper.

12.1. Calculation of Jacobians

We first present an elementary Linear Algebra result, which will be useful throughout the manuscript for the calculation of Jacobians. For a proof, see Lemma A.1 from [2].

Lemma 12.1. *Let $n \in \mathbb{N}$, $\lambda \neq 0$ and $w, u \in \mathbb{R}^n$. Then*

$$\det(\lambda I_n + wu^T) = \lambda^n (1 + \lambda^{-1} \langle w, u \rangle),$$

where I_n is the $n \times n$ identity matrix.

12.2. The binary transition map

Here, we introduce the binary transition map, which will enable us to control binary postcollisional configurations. Recall from (2.2) the binary cross-section

$$b_2(\omega_1, v_1) = \langle \omega, v_1 \rangle, \quad (\omega_1, v_1) \in \mathbb{S}_1^{d-1} \times \mathbb{R}^d.$$

Given $v_1, v_2 \in \mathbb{R}^d$, we define the domain[14] $\Omega := \big\{\omega_1 \in \mathbb{R}^d : |\omega_1| \leq 2, \text{ and } b_2(\omega_1, v_2 - v_1) > 0\big\}$, and the set $\mathcal{S}_{v_1,v_2}^+ = \{\omega_1 \in \mathbb{S}_1^{d-1} : b_2(\omega_1, v_2 - v_1) > 0\} \subseteq \Omega$. We also define the smooth map $\Psi : \mathbb{R}^d \to \mathbb{R}$ by $\Psi(\omega_1) := |\omega_1|^2$. Notice that the unit $(d-1)$-sphere is given by level sets of Ψ i.e. $\mathbb{S}_1^{d-1} = [\Psi = 1]$.

Proposition 12.2. *Consider $v_1, v_2 \in \mathbb{R}^d$ and $r > 0$ such that $|v_1 - v_2| = r$. We define the binary transition map $\mathcal{J}_{v_1,v_2} : \Omega \to \mathbb{R}^d$ as follows:*[15]

$$\mathcal{J}_{v_1,v_2}(\omega_1) := r^{-1}(v_1' - v_2'), \quad \omega \in \Omega. \tag{12.1}$$

The map $\mathcal{J}_{v_1,v_2}$ has the following properties:

1. *$\mathcal{J}_{v_1,v_2}$ is smooth in Ω with bounded derivative uniformly in r; that is,*

$$\|D\mathcal{J}_{v_1,v_2}(\omega_1)\|_\infty \leq C_d, \quad \forall \omega_1 \in \Omega, \tag{12.2}$$

   where $\|\cdot\|_\infty$ denotes the maximum element matrix norm of $D\mathcal{J}_{v_1,v_2,v_3}(\omega_1)$.
2. *The Jacobian of $\mathcal{J}_{v_1,v_2}$ is given by*

$$\operatorname{Jac}(\mathcal{J}_{v_1,v_2})(\omega_1) \simeq r^{-d} b_2^d(\omega_1, v_2 - v_1) > 0, \quad \forall \omega_1 \in \Omega. \tag{12.3}$$

3. *The map $\mathcal{J}_{v_1,v_2} : \mathcal{S}_{v_1,v_2}^+ \to \mathbb{S}_1^{d-1} \setminus \{r^{-1}(v_1 - v_2)\}$ is bijective. Moreover, there holds*

$$\mathcal{S}_{v_1,v_2}^+ = [\Psi \circ \mathcal{J}_{v_1,v_2} = 1]. \tag{12.4}$$

4. *For any measurable $g : \mathbb{R}^d \to [0 + \infty]$, there holds the change of variables estimate*

$$\int_{\mathcal{S}_{v_1,v_2}^+} (g \circ \mathcal{J}_{v_1,v_2}(\omega_1) |\operatorname{Jac} \mathcal{J}_{v_1,v_2}(\omega_1)| \, d\omega_1 \lesssim \int_{\mathbb{S}_1^{d-1}} g(v_1) \, dv_1. \tag{12.5}$$

Proof. The proof is the binary analogue of the proof of Proposition 8.5. in [5]. □

[14] We trivially extend the binary cross-section for any $\omega \in \mathbb{R}^d$.
[15] We trivially extend the binary collisional operator for any $\omega \in \Omega$.

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Competing interest. The authors have no competing interest to declare.

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