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Transcendental Julia sets of minimal Hausdorff dimension

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Abstract. We show the existence of transcendental entire functions $f:\mathbb{C}\to\mathbb{C}$ with Hausdorff-dimension 1 Julia sets, such that every Fatou component of f has infinite inner connectivity. We also show that there exist singleton complementary components of any Fatou component of f, answering a question of Rippon and Stallard [Eremenko points and the structure of the escaping set. *Trans. Amer. Math. Soc.* **372**(5) (2019), 3083–3111]. Our proof relies on a quasiconformal-surgery approach developed by Burkart and Lazebnik [Interpolation of power mappings. *Rev. Mat. Iberoam.* **39**(3) (2023), 1181–1200].

Key words: complex dynamics, dimension, interpolation, Julia sets
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Contents

1. *Introduction*

The *Julia set* of an entire function $f : \mathbb{C} \to \mathbb{C}$, denoted by $\mathcal{J}(f)$, is the set of points at which the dynamical system $(f, \mathbb{C})$ behaves chaotically. The behavior of f near ∞ plays an important role, and one has the following dichotomy. Either ∞ is a removable singularity, in which case, f is a polynomial, or ∞ is an essential singularity, in which case, f is a transcendental entire function.

When f is a polynomial, $\mathcal{J}(f)$ is usually small in the sense of *Hausdorff dimension*. Hausdorff dimension is the most well-studied measure of size for Julia sets, and this is the measure we will study in this manuscript. For instance, one has that the quadratic polynomial $p_c(z) := z^2 + c$ satisfies $\dim(\mathcal{J}(p_c)) < 2$ for generic $c \in \mathbb{C}$ (see for instance [**Urb94**]), although there exist parameters c satisfying $\dim(\mathcal{J}(p_c)) = 2$ [**Shi98**] and even $\text{area}(\mathcal{J}(p_c)) > 0$ [**AL22**, **BC12**].

However, when f is a transcendental entire function, the generic situation is that $\dim(\mathcal{J}(f)) = 2$. For instance, in [**Mis81**], it was shown that $\mathcal{J}(e^z) = \mathbb{C}$, and in [**McM87**], it was shown that functions in certain standard exponential and sine families have Julia sets of dimension 2. Thus, in contrast with the polynomial setting, the difficulty in the transcendental setting is to find Julia sets of small dimension, a problem whose history we overview briefly now.

In [**Bak75**], it was proven that the Julia set of any transcendental f must contain a non-trivial continuum and, hence, we always have $\dim(\mathcal{J}(f)) \geq 1$. In the class of transcendental f with bounded singular set, denoted $\mathcal{B}$, it was shown in [**Sta91**, **Sta96**, **Sta00**] that

$$\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2],$$

(see also [**AB20**]). Finally, in [**Bis18**], it was proven that outside of the class $\mathcal{B}$, the lower bound of 1 in the inequality $\dim(\mathcal{J}(f)) \geq 1$ is actually attained (see also [**Bur21**, **Zha24**]). Our main result (see Theorem 1.1 below) also achieves this lower bound, but by different methods and with different resulting dynamical properties that we now discuss.

The Fatou set of the function in [**Bis18**] is in fact completely described: it consists of a collection of *multiply connected wandering domains*, abbreviated m.c.w.d. This class of Fatou components has been well studied [**Ber11**, **BRS13**, **BRS16**, **BZ11**, **Fer22**, **KS08**, **RS08**, **RS19**] and appears in several different contexts in transcendental dynamics. An m.c.w.d. U of [**Bis18**] consists of a topological annulus minus countably many discs that accumulate only on the outer boundary of U (see Figure 1(a)). This topological structure is aptly termed *infinite outer connectivity* (defined precisely in [**BRS13**]). The Fatou components of the function in our Theorem 1.1 are also all m.c.w.d.s; however, they have *infinite inner connectivity* (see Figure 1(b)), and we prove they have the following more intricate topological structure.

THEOREM 1.1. *There exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ satisfying:*

(1) $\dim(\mathcal{J}(f)) = 1$;

(2) *each Fatou component of f is a m.c.w.d. of infinite inner-connectivity; and*

(3) *each m.c.w.d. of f has uncountably many singleton complementary components.*

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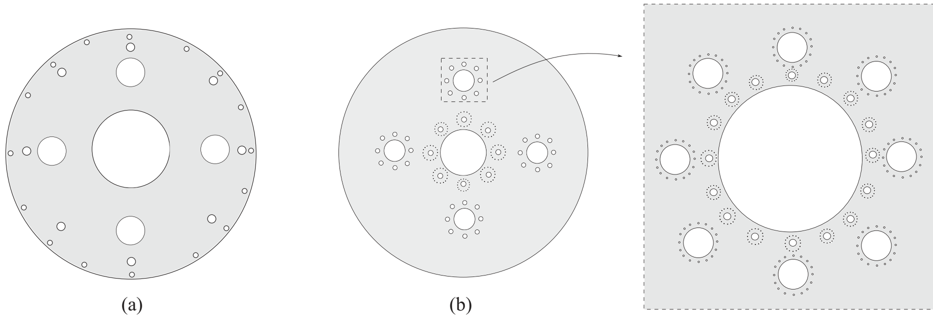

FIGURE 1. Panels (a) and (b) illustrate the concept of infinite outer-connectivity and infinite inner-connectivity, respectively. Panels (a) and (b) also accurately describe the topology of the m.c.w.d.s in [**Bis18**] and Theorem 1.1, respectively. As seen, the structure in panel (b) is more intricate.

Part (3) of Theorem 1.1 answers a question of [**RS19**] (see [**RS19**, Question 9.5]) on the structure of m.c.w.d.s. It is left open whether part (3) in fact must *always* occur for an m.c.w.d. of infinite inner-connectivity. Another intriguing question suggested by Theorem 1.1 is whether there exist transcendental f with $\dim(\mathcal{J}(f)) = 1$ and *doubly* connected m.c.w.d.: this is closely related to [**Bis18**, Question 7].

Much of the contribution of the present manuscript is in providing an alternative approach to the breakthrough result of [**Bis18**] (part (1) in Theorem 1.1), an approach that the authors find conceptual and readily adaptable to other settings. The function f of [**Bis18**] is defined by an infinite product that is roughly designed to behave as a monomial on large portions of $\mathbb{C}$. The technical work in describing the dynamics of f relies on formula-heavy estimates of the behavior of f by certain terms in the infinite product.

The approach in the present manuscript is similar in that it constructs f that is designed to behave as a monomial on large portions of $\mathbb{C}$; however, this is done by quasiconformal methods. Namely, a quasiregular $h : \mathbb{C} \to \mathbb{C}$ is constructed, so that by the measurable Riemann mapping theorem, there exists a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ such that $f := h \circ \phi^{-1}$ is the entire function of Theorem 1.1. One has freedom in prescribing the dynamics of h, and so the difficulty of describing the dynamics of f becomes a matter of estimating the 'correction' map ϕ, rather than on formula estimates as in [**Bis18**]. The details of this quasiconformal approach were detailed in [**BL23**], and has other applications besides that described in the present manuscript.

The advantages of the quasiconformal approach are usually technical in nature. For instance, a key aspect of the proof of Theorem 1.1 is in understanding the location of critical values of f. In [**Bis18**], this requires a delicate estimate involving the infinite product formula. In the quasiconformal approach, this is almost trivial since the critical values of h can be prescribed freely, and $f := h \circ \phi^{-1}$ and h share the same critical values. Another central difficulty in the proof of Theorem 1.1 is showing that the outer boundary of an m.c.w.d. of f is a C^1 curve (hence, one-dimensional). In both approaches, this involves studying pullbacks of circles. However, only in the quasiconformal approach is there an explicit parameterization (in terms of ϕ) for the pullback, and this provides a different

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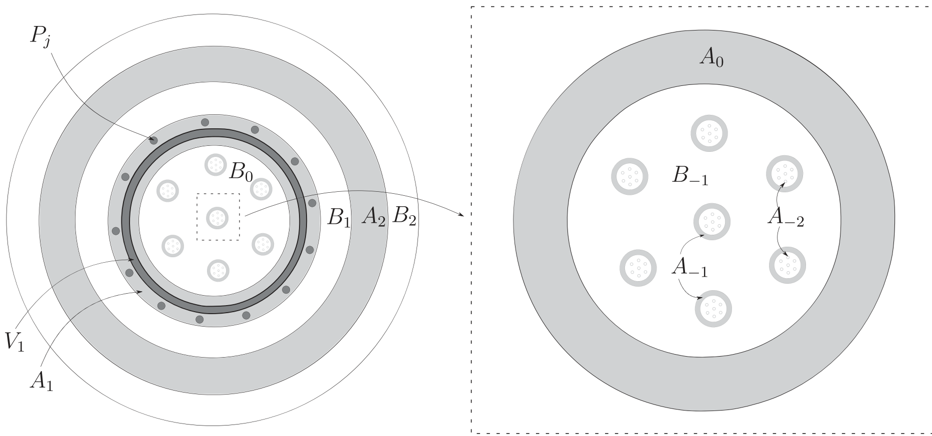

FIGURE 2. Definition of A_k, B_k for all $k \in \mathbb{Z}$. The annuli A_k are shaded light gray and the B_k are white. Also shown are $V_1 \subset A_1$ and the 'petals' $P_j \subset A_1$ (in dark gray).

approach to the question of the precise degree of regularity for these curves. We will discuss further technical advantages of quasiconformal methods throughout the paper.

We will outline the main arguments in the proof of Theorem 1.1 and the structure of the paper in §2, before filling in the details in §§3–11. Appendix A contains many classical theorems and definitions that we will make use of throughout the paper, along with a proof of an important lemma we need in §6.

2. *Outline of the proof*

We appeal to the main theorem of [**BL23**] (described in Appendix A) to produce the quasiregular function $h : \mathbb{C} \to \mathbb{C}$ as described in §1. The map h roughly behaves as $z \mapsto z^n$ for increasing n as $z \to \infty$. To be able to prove dynamical properties about $f := h \circ \phi^{-1}$, we need estimates on $|\phi(z) - z|$; these are proven in §3.

In §4, we define a sequence of annuli A_k, B_k for $k \geq 1$ (see Figure 2), and we prove the following mapping behavior. First, we show that

$$f(B_k) \subset B_{k+1}.$$

Thus, each B_k is contained in an m.c.w.d. of f. We define subannuli $V_k \subset A_k$ and prove that

$$A_{k+1} \subset f(V_k).$$

We also prove in §6 that there are balls $P_j \subset A_k$ such that $P_j \cap V_k = \emptyset$, which satisfy

$$A_{k+1} \subset f(P_j),$$

and $f|_{P_j}$ is conformal.

The definition of the annuli A_k, B_k are extended to negative indices k by pulling back under f (see Figure 2). Together, the annuli A_k and B_k cover $\mathbb{C}$ except for a Cantor set, which we denote by E. This Cantor set E is the Julia set of the polynomial-like mapping obtained by restricting the definition of f to a subdomain of $\mathbb{C}$, and we prove in §5 that

$\dim(E) \ll 1$. Similarly, denoting by E' the set of points that map to E, it is readily deduced that $\dim(E') = \dim(E)$.

As the B_k are contained in wandering components, we have that

$$\mathcal{J}(f) \setminus E' \subset \left\{ z \in \mathbb{C} : f^n(z) \in \bigcup_{k \in \mathbb{Z}} A_k \text{ for all } n \right\}. \tag{2.1}$$

We denote the set defined on the right-hand side of equation (2.1) by X, so that estimating $\dim(\mathcal{J}(f))$ reduces to estimating $\dim(X)$.

We partition X into two sets. For $z \in X$, we say that z *moves forwards* if $z \in A_k$ and $f(z) \in A_{k+1}$. If $z \in A_k$ and $f(z) \in A_j$ for $j \leq k$, we say that z *moves backwards*. We denote by Y the set of $z \in X$ that move backwards infinitely often, and $Z := X \setminus Y$ so that

$$X = Y \sqcup Z.$$

In §8, we construct a sequence of covers $\mathcal{C}_m$ of Y, such that $\mathcal{C}_m$ covers all those points that move backwards m times. This is done by simply pulling back the annuli A_k under iterates of f in regions where f is conformal. Standard distortion estimates apply when estimating the diameters of elements of $\mathcal{C}_m$, and we deduce that $\dim(Y) \ll 1$.

The set Z is further partitioned into those points that eventually always stay in $\bigcup_k V_k$, denoted by Z_1, and $Z_2 := Z \setminus Z_1$. The dimension of $\mathcal{J}(f)$ is supported on Z_1. We prove in §9 that Z_1 consists of Jordan curves and we prove in §10 that these curves are in fact C^1 (hence, have dimension 1). We prove in §11 that $\dim(Z_2) = 0$, and Z_2 is precisely the set of singleton complementary components in part (3) of Theorem 1.1.

3. *Quasiconformal mapping estimates*

In this section, we begin the proof of Theorem 1.1 by first applying Theorem A.17 (see Appendix A) to a specific sequence $(M_j)_{j=1}^{\infty}$, $(r_j)_{j=1}^{\infty}$ that we now define. This yields an entire function f, so that $f \circ \phi = h$ is the quasiregular function described in §2. As discussed, we have freedom in describing the mapping behavior and dynamics of the quasiregular map h, but transferring this behavior to the entire function $f := h \circ \phi^{-1}$ requires estimates on $|\phi(z) - z|$, and this is the main focus of this section.

Definition 3.1. We define an entire function f and a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ by applying Theorem A.17 to the parameters:

$$M_j := 2^j, r_1 := 16, c_1 := 1 \quad \text{and} \quad r_{j+1} := c_j \cdot (\tfrac{1}{2} r_j)^{M_j} \quad \text{for } j \geq 2. \tag{3.1}$$

We will need some rough estimates on how fast $(r_j)_{j=1}^{\infty}$ grows and $(c_j)_{j=1}^{\infty}$ decays. We first show that by assuming $(r_j)_{j=1}^{\infty}$ satisfies some mild growth conditions, we can show that $(r_j^{M_j})_{j=1}^{\infty}$ grows much faster than $(c_j)_{j=1}^{\infty}$ decays (see Table 1).

LEMMA 3.2. *Assume for all $k \geq 3$ that $\sqrt{r_k} \geq r_j$ for all $j < k$. Then, we have*

$$r_k^{M_k} \cdot c_k = r_k^{M_k} \cdot \prod_{j=1}^{k-1} r_j^{-M_j} \geq r_k^{M_{k-1}+1} \tag{3.2}$$

for all $k \geq 3$.

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TABLE 1. The values of M_k, c_k, and r_k for small values of k. The sequence M_k increases exponentially and r_k increases super-exponentially, while c_k decays super-exponentially.

k	0	1	2	3	4
M_k	1	2	4	8	16
c_k	undefined	1	2^{-8}	2^{-32}	2^{-128}
r_k	0	16	64	2^{12}	2^{56}

Remark 3.3. We will prove in Lemma 3.4 that the assumption of Lemma 3.2 does indeed hold.

Proof. This is just a calculation, making use of the fact that $M_j - M_{j+1} = -M_j$ and $2M_j = M_{j+1}$ for all $j \geq 0$, along with the definition of c_k given by equation (A.15). When $k = 1$, we verify equation (3.2) by checking that $r_1^{M_1} = c_1 \cdot r_1^{M_0+1}$. For the case of $k \geq 2$, we verify equation (3.2) by computing

$$\begin{aligned} r_k^{M_k} \cdot c_k &= r_k^{M_k} \cdot \prod_{j=2}^{k} r_{j-1}^{M_{j-1}-M_j} \\ &= r_k^{M_k} \cdot \prod_{j=2}^{k} r_{j-1}^{-M_{j-1}} \\ &\geq r_k^{M_k} \cdot \prod_{j=2}^{k} r_k^{-M_{j-2}} \qquad (\sqrt{r_k} \geq r_j) \\ &= r_k^{M_k} \cdot r_k^{-\sum_{j=0}^{k-2} M_j} \\ &= r_k^{M_k - M_{k-1}+1} = r_k^{M_{k-1}+1}. \end{aligned}$$

In the last line, we used the fact that $\sum_{j=0}^{k-2} M_j = M_{k-1} - 1$. □

LEMMA 3.4. *The sequence $(r_k)_{k=1}^{\infty}$ defined in Definition 3.1 satisfies:*

(1) $r_2 > r_1$;

(2) *for all* $k \geq 2$, $\sqrt{r_{k+1}} \geq r_k$.

In particular, $(r_k)_{k=1}^{\infty}$ is an increasing sequence, and if $k \geq 2$, we have $\sqrt{r_{k+1}} \geq r_j$ for all $j = 1, \ldots, k$.

Proof. The claim (1) is just a calculation:

$$r_2 = c_1 \left(\frac{r_1}{2}\right)^{M_1} = 8^2 = 64 > 16 = r_1. \tag{3.3}$$

We will prove the second claim by induction. First, we have

$$r_3 = c_2 \left(\frac{r_2}{2}\right)^{M_2} = 2^{-8}(2^5)^4 = 2^{12} = 64^2. \tag{3.4}$$

Therefore, $\sqrt{r_3} \geq r_2 > r_1$.

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Suppose that for some $k \geq 3$, we have $\sqrt{r_k} \geq r_j$ for all $j = 1, \ldots, k-1$. Then, by Lemma 3.2,

$$r_{k+1} = c_k r_k^{M_k} 2^{-M_k} \geq r_k^{M_{k-1}+1} 2^{-M_k} = r_k^{M_{k-2}+1} r_k^{M_{k-2}} 16^{-M_{k-2}} \geq r_k^{M_{k-2}+1}. \tag{3.5}$$

Since $k \geq 3$, we have $M_{k-2} \geq M_1 = 2$, so that

$$r_{k+1} \geq r_k \cdot r_k^2. \tag{3.6}$$

Therefore, by the inductive hypothesis, we must have $\sqrt{r_{k+1}} \geq r_k \geq r_j$ for all $j = 1, \ldots, k-1$. This proves the claim. □

We record the following important inequalities that follow from the proof of Lemma 3.4.

COROLLARY 3.5. *We have the following inequalities. For all $k \geq 3$,*

$$c_k r_k^{M_k} \geq r_k^{M_{k-1}+1} \quad \textit{and} \tag{3.7}$$

$$r_{k+1} \geq 2^{-M_k} r_k^{M_{k-1}+1}. \tag{3.8}$$

For all $k \geq 5$, we have

$$r_{k+1} \geq 2^{2^k} = 2^{M_k} \quad \textit{and} \tag{3.9}$$

$$r_{k+1} \geq 4 r_k^2. \tag{3.10}$$

Proof. Most of the work has already been done in the proof of Lemma 3.4. We first prove equation (3.7). When $k \geq 2$, By Lemma 3.4, we have $\sqrt{r_{k+1}} \geq r_j$ for all $j = 1, \ldots k$. Therefore, by Lemma 3.2, we obtain equation (3.7).

Equation (3.8) follows immediately. Indeed, the first two lines of equation (3.5) yields

$$r_{k+1} \geq r_k^{M_{k-1}+1} 2^{-M_k},$$

when $k \geq 3$.

When $k \geq 5$, we can refine the estimate $r_{k+1} \geq r_k^{M_{k-2}+1}$ from equation (3.5). We note that by Lemma 3.4, we have $r_k > 16$ for all $k \geq 5$. Therefore,

$$r_{k+1} \geq r_k^{M_{k-2}+1} > 16^{M_{k-2}} = (2^4)^{M_{k-2}} = 2^{M_k}.$$

Finally, since $r_k > 16$ for all $k \geq 1$, we certainly obtain equation (3.10) from equation (3.6). □

Corollary 3.5 concludes our discussion of some technical relations and inequalities we will need for the sequences $(r_j)_{j=1}^{\infty}$, $(c_j)_{j=1}^{\infty}$. As discussed in §1, one of the key advantages of the quasiconformal approach (over the infinite-product approach) is the relative simplicity of deducing the singular value structure of the constructed function. This is summarized in the following proposition. Although equations (3.11) and (3.12) are slightly technical, they simply say the critical points are radially equidistributed on each circle $|z| = r_j$, and the zeros are equidistributed on a slightly larger circle. Recall from Definition 3.1 that f is the entire function obtained by applying Theorem A.17 to the parameters in equation (3.1).

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PROPOSITION 3.6. *Let $(M_j)_{j=1}^{\infty}$, $(r_j)_{j=1}^{\infty}$ and $(c_j)_{j=1}^{\infty}$ be as in Definition 3.1. Then, the only critical points of f are* 0 *and the simple critical points given by*

$$\phi\left(r_j \cdot \exp\left(i\frac{(2k_j-1)\pi}{M_j}\right)\right), \tag{3.11}$$

where $j \in \mathbb{N}$ and $1 \le k_j \le M_j$. The only singular values of f are 0 *and the critical values $(\pm c_j r_j^{M_j})_{j=0}^{\infty}$. The zeros of f are given by*

$$0 \quad \textit{and} \quad \phi\left(r_j \cdot \exp\left(\frac{1}{4}\frac{\pi}{M_j} + i\cdot\frac{(2k_j-1)\pi}{M_j}\right)\right), \tag{3.12}$$

where $j \in \mathbb{N}$ and $1 \le k_j \le M_j$. All of the zeros of f are simple except for 0 *that is of multiplicity* 2.

Proof. This follows immediately from [**BL23**, Proposition 3.21]. □

We now move on to show how to modify f near the origin so that instead of being modeled by a function of the form z^n, it is modeled by a polynomial with a Cantor repeller Julia set. This will be advantageous because z^n has a Julia set of dimension 1, whereas the constructed Cantor repeller will have dimension $\ll 1$.

The main idea is that a monic, degree M_k polynomial $p(z)$ behaves like $z \mapsto z^{M_k}$ near ∞. We will show how to interpolate between $p(z)$ and $z \mapsto z^{M_k}$ in a way that is quasiconformal with dilatation bounded independent of k. Our strategy closely follows [**FJL19**, §3].

Definition 3.7. We define

$$b(x) = \begin{cases} \exp\left(1 + \dfrac{1}{x^2-1}\right) & \text{if } 0 \le x < 1, \\ 0 & \text{if } x \ge 1, \end{cases}$$

and, for $r \ge 1$, the smooth map

$$\widehat{\eta}_r(x) = \begin{cases} 1 & \text{if } x \le r-1, \\ b(x-r+1) & \text{if } r-1 \le x \le r, \\ 0 & \text{if } x \ge r. \end{cases}$$

We also set $\eta_r(z) = \widehat{\eta}_r(|z|)$.

PROPOSITION 3.8. *Let $g_k(z) := c_k z^{M_k} + r_k z \eta_{r_k}(z)$ and $\mu_k := (g_k)_{\overline{z}}/(g_k)_z$. Then, there exists $K' \in \mathbb{N}$ with $K' \ge 5$ such that*

$$\sup_{k \ge K'} \|\mu_k\|_{L^\infty(\mathbb{C})} < 1. \tag{3.13}$$

Proof. We abbreviate $\eta(z) := \eta_{r_k}(z)$. We use a similar strategy as in the proof of [**FJL19**, Lemma 3.1], and the initial steps of the proof are exactly the same. We have

$$(g_k)_z(z) = M_k c_k z^{M_k-1} + r_k\eta(z) + r_k z\eta_z(z) \quad \text{and} \quad (g_k)_{\overline{z}}(z) = r_k z \eta_{\overline{z}}(z).$$

Solving $b''(x) = 0$, one sees that $|b'(x)|$ has a maximum at $x_0 = (1/3)^{1/4}$ with $|b'(x_0)| < e$, so that $|b'(x)| \leq e$ for $x \in [0, 1]$. Thus, $|(\widehat{\eta})'(x)| \leq e$ for all $x > 0$. Using the chain rule again, we have

$$\left|\frac{\partial \eta}{\partial z}(z)\right| = |(\widehat{\eta})'(|z|)| \cdot \left|\frac{\partial |z|}{\partial z}\right| \leq \frac{e}{2} \quad \text{and} \quad \left|\frac{\partial \eta}{\partial \bar{z}}(z)\right| = |(\widehat{\eta})'(|z|)| \cdot \left|\frac{\partial |z|}{\partial \bar{z}}\right| \leq \frac{e}{2},$$

where we have used the fact that

$$\frac{\partial |z|}{\partial z} = \frac{\bar{z}}{2|z|} \quad \text{and} \quad \frac{\partial |z|}{\partial \bar{z}} = \frac{z}{2|z|}.$$

Hence,

$$\begin{aligned}\left|\frac{(g_k)_{\bar{z}}(z)}{(g_k)_z(z)}\right| &\leq \frac{r_k|z|e/2}{|M_k c_k |z|^{M_k-1} - r_k|\eta(z)| - r_k|z|e/2|} \\ &= \frac{e/2}{|M_k c_k |z|^{M_k-2}/r_k - |\eta(z)|/|z| - e/2|}.\end{aligned} \tag{3.14}$$

Let us consider the right-hand side of equation (3.14) for $|z| = r_k - 1$, recalling $M_k := 2^k$. We have that

$$\begin{aligned}\frac{c_k|z|^{M_k-2}}{r_k} &:= \frac{1}{r_k(r_k-1)^2}\left(\prod_{j=2}^{k}\frac{1}{r_{j-1}^{M_j-M_{j-1}}}\right)(r_k-1)^{2^k} \\ &= \frac{1}{r_k(r_k-1)^2}\cdot\frac{(r_k-1)^2\cdot(r_k-1)^{2^2}\cdot\cdots\cdot(r_k-1)^{2^{k-1}}}{r_1^2\cdot r_2^{2^2}\cdot\cdots\cdot r_{k-1}^{2^{k-1}}} \\ &= \frac{(r_k-1)^{2^2}}{r_k}\cdot\frac{(r_k-1)^{2^3}}{r_1^2\cdot r_2^{2^2}\cdot r_3^{2^3}}\cdot\frac{(r_k-1)^{2^4}\cdot\cdots\cdot(r_k-1)^{2^{k-1}}}{r_4^{2^4}\cdot\cdots\cdot r_{k-1}^{2^{k-1}}}.\end{aligned}$$

By Lemmas 3.4 and 3.5, we may deduce for all $k \geq 5$,

$$r_k - 1 > 2r_{k-1}, \tag{3.15}$$

$$\frac{(r_k-1)^{2^3}}{r_1^2\cdot r_2^{2^2}\cdot r_3^{2^3}} \geq 1, \quad \text{and} \tag{3.16}$$

$$(r_k-1)^2 > r_k. \tag{3.17}$$

Combining the above inequalities, it follows that when $|z| \geq r_k - 1$, we have

$$\frac{c_k|z|^{M_k-2}}{r_k} \geq r_{k-1}. \tag{3.18}$$

Thus, it follows from equation (3.14) that in fact $|(g_k)_{\bar{z}}/(g_k)_z| \to 0$ as $k \to \infty$. □

Definition 3.9. Let f, ϕ be as in Definition 3.1, and $h := f \circ \phi$. We define a family of entire functions $f_N := h_N \circ \phi_N^{-1}$ as follows. Let

$$h_N(z) := \begin{cases} g_N(z) & \text{if } |z| \leq r_N, \\ h(z) & \text{if } |z| \geq r_N, \end{cases}$$

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and $\phi_N : \mathbb{C} \to \mathbb{C}$ is the quasiconformal mapping such that:

(1) f_N is holomorphic;

(2) $\phi_N(0) = 0$; and

(3) $|\phi_N(z)/z - 1| \to 0$ as $z \to \infty$.

The fact that ϕ_N may be normalized so that condition (3) is satisfied follows from an argument similar to [**BL23**].

Remark 3.10. We will always assume that $N \geq 5$. Note that for $|z| = r_N$, we have $g_N(z) = h(z)$.

Remark 3.11. We will show that for all sufficiently large N, the function f_N satisfies the conclusions of Theorem 1.1. We will occasionally omit the subscript N and simply write f when convenient.

PROPOSITION 3.12. *For ϕ_N as in Definition 3.9,* $\sup_N K(\phi_N) < \infty$.

Proof. By Proposition 3.8, we have $\sup_N K(\phi_N|_{\{|z|\leq r_N\}}) < \infty$ and, by [**BL23**, Proposition 4.6], we have $\sup_N K(\phi_N|_{\{|z|\geq r_N\}}) < \infty$. □

In [**BL23**], the conclusion $|\phi(z)/z - 1| \xrightarrow{z\to\infty} 0$ of Theorem A.17 is deduced by an application of the Teichmüller–Wittich–Belinskii theorem (see [**LV73**, Theorem 6.1]). We will need a more quantitative statement for the purposes of proving Theorem 1.1, in particular, when we prove that the m.c.w.d.'s of Theorem 1.1 have smooth boundary. This quantitative statement is given in Theorem 3.14 below. The proof follows from the main arguments of [**Shi18**]. Indeed, Theorem 3.14 is quite analogous to the main result of [**Shi18**], but we will need to assume less than in [**Shi18**], and accordingly, we will obtain a weaker conclusion, which will nevertheless suffice to prove Theorem 1.1.

Definition 3.13. For $p \geq 1$ and $0 < r < 1$, we will denote

$$\omega_p(r) := \left(\tfrac{1}{2}\right)^{p^{-1}\sqrt{\log\log r^{-1}}}. \tag{3.19}$$

THEOREM 3.14. *Let $\psi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping, $\mu := \psi_{\bar{z}}/\psi_z$, and suppose that*

$$I(r) := \iint_{\{|z|<r\}} \frac{|\mu|}{1-|\mu|^2}\frac{dx\,dy}{|z|^2} \text{ is finite and has order } O(\omega_1(r)) \text{ as } r \searrow 0. \tag{3.20}$$

Then, ψ is conformal at 0 *and, for any $p > 2$, we have*

$$\psi(z) = \psi(0) + \psi'(0)z + O(\omega_p(|z|)) \quad \text{as } z \to 0. \tag{3.21}$$

Proof. By [**Shi18**, Lemma 10], we have that for any $p > 2$ and $0 < \rho < 1$, there exists $C' = C'(p, \rho)$ such that if $0 < |z_2| < \rho^2|z_1|$, then

$$\begin{aligned}\left|\iint_{\mathbb{C}} \frac{\mu(z)\phi_{z_1,z_2}(z)}{1-|\mu(z)|^2}\,dx\,dy\right| &\\ \leq \frac{1}{1-\rho^2}\left|\iint_{A(\rho^{-1}|z_2|,\rho|z_1|)} \frac{\mu(z)}{1-|\mu(z)|^2}\frac{dx\,dy}{z^2}\right| &+ C' I_{p,2}(\mu; |z_1|)^{1/p},\end{aligned} \tag{3.22}$$

and

$$\iint_{\mathbb{C}} \frac{|\mu(z)|^2|\phi_{z_1,z_2}(z)|}{1-|\mu(z)|^2}\,dx\,dy \\ \leq \frac{1}{1-\rho^2}\iint_{A(\rho^{-1}|z_2|,\rho|z_1|)} \frac{|\mu(z)|^2}{1-|\mu(z)|^2}\frac{dx\,dy}{|z|^2} + C' I_{p,2}(\mu;|z_1|)^{1/p}, \tag{3.23}$$

where

$$\phi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)} \quad \text{and} \quad I_{p,2} := \iint_{\mathbb{C}} \frac{|\mu(z)|^p}{(1-|\mu(z)|^2)^p}\frac{dx\,dy}{|z|^2(1+|z|/r)^2}. \tag{3.24}$$

Thus, by [Shi18, Theorem 8], it suffices to show that the $\liminf_{z_2\to 0}$ of the four terms on the right-hand sides of equations (3.22) and (3.23) are $O(\omega_p(|z_1|))$ as $z_1 \to 0$. In fact, we have better estimates on the two integral terms: they are $O(\omega_1(|z_1|))$ by assumption (3.20). For the remaining two terms, we use [Shi18, Lemma 11]: there exist constants C_2 and C_3 depending only on $K(\psi)$ such that for $0 < r < r'$,

$$I_{p,2}(\mu;r) \leq C_2 \iint_{|z|<r'} \frac{|\mu(z)|^2}{1-|\mu(z)|^2}\frac{dx\,dy}{|z|^2} + \frac{C_3}{2}\left(\frac{r}{r'}\right)^2. \tag{3.25}$$

Letting $r' = r^{1/2}$, we see the first term on the right-hand side of equation (3.25) has order $O(\omega_1(r')) = O(\omega_1(r))$, and the second term has order $O(r)$, so that $I_{p,2}$ has order $O(\omega_1(r))$. Thus, $I_{p,2}(\mu;|z_1|)^{1/p}$ has order $O(\omega_p(|z_1|))$, and so the result follows. □

Remark 3.15. One readily sees from the constants in the proof of Theorem 3.14 that the big-O constants in equation (3.21) depend only on $K(\psi)$ and the big-O constants in equation (3.20). In particular, they are independent of $N \in \mathbb{N}$.

We now apply Theorem 3.14 to our particular setting.

THEOREM 3.16. *There exists $C' > 0$ and $R > 0$ such that for any $N \in \mathbb{N}$ and any $p > 2$,*

$$\left|\frac{\phi_N(z)}{z} - 1\right| < C' \cdot \omega_p(1/|z|) \quad \textit{for } |z| > R. \tag{3.26}$$

Proof. Let $N \in \mathbb{N}$. Let $\phi := \phi_N : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping such that $h_N \circ \phi^{-1}$ is holomorphic and $\phi(0) = 0$. Consider

$$\psi(z) := 1/\phi(1/z),$$

and define $I(r)$ as in equation (3.20). We wish to apply Theorem 3.14. To this end, we calculate

$$I(r) \leq \frac{k}{1-k^2}\iint_{(|z|<r)\cap \operatorname{supp}(\psi_{\bar z})} \frac{dx\,dy}{|z|^2} \leq \frac{k}{1-k^2}\sum_{j\geq j(r)} \iint_{G_j} \frac{dx\,dy}{|z|^2}, \tag{3.27}$$

where

$$j(r) \text{ is the smallest integer such that } 1/r < r_{j(r)} \cdot \exp(\pi/M_j) \tag{3.28}$$

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and

$$G_j := \{z \in \mathbb{C} : r_j^{-1} \cdot \exp(-\pi/M_j) \le |z| \le (r_j - 1)^{-1}\}. \tag{3.29}$$

As in the proof of [BL23, Theorem 4.8], we calculate

$$\begin{aligned}
\frac{k}{1-k^2} \sum_{j \ge j(r)} \iint_{G_j} \frac{dx\,dy}{|z|^2} &\lesssim \sum_{j \ge j(r)} \iint_{G_j} \frac{dx\,dy}{r_j^{-2} \exp(-2\pi/M_j)} \\
&= \sum_{j \ge j(r)} \frac{\pi((r_j-1)^{-2} - r_j^{-2} \exp(-2\pi/M_j))}{r_j^{-2} \exp(-2\pi/M_j)} \\
&\simeq \sum_{j \ge j(r)} \left(\left(\frac{r_j}{r_j - 1} \right)^2 \exp(2\pi/M_j) - 1 \right) \\
&\lesssim \sum_{j \ge j(r)} \left(\left(\frac{r_j}{r_j - 1} \right)^2 - 1 + \left(\frac{r_j}{r_j - 1} \right)^2 \frac{4\pi}{M_j} \right) \\
&\lesssim \sum_{j \ge j(r)} \left(\frac{2r_j - 1}{(r_j - 1)^2} + \frac{8\pi}{M_j} \right) \\
&\lesssim \sum_{j \ge j(r)} \left(\frac{1}{r_j} + \frac{1}{M_j} \right) \\
&\lesssim \left(\frac{1}{2} \right)^{j(r)},
\end{aligned} \tag{3.30}$$

where we have used the fact that $M_j = 2^j$, Corollary 3.5, and the inequality

$$\exp(x) \le 1 + 2x \quad \text{for all } x \le 1.$$

Next, we note that

$$2^{2^{(j+1)(j+2)/2}} > 2^{2^{1+\cdots+j} + \cdots + 2^j} > r_j \cdot \exp(\pi/M_j). \tag{3.31}$$

Thus, it is readily calculated that

$$r_j \cdot \exp(\pi/M_j) > 1/r \implies (j+1)(j+2) > \log\log r^{-1}, \tag{3.32}$$

and so since $(1/2)^j \simeq (1/2)^{\sqrt{(j+1)(j+2)}}$, it follows that

$$r_j \cdot \exp(\pi/M_j) > \exp(-\pi/M_{j(r)})/r > 1/(2r) \implies \left(\tfrac{1}{2}\right)^j \lesssim \left(\tfrac{1}{2}\right)^{\sqrt{\log\log r^{-1}}}. \tag{3.33}$$

Together, equations (3.27)–(3.33) imply that $I(r)$ is finite and has order $O(\omega_1(r))$ as $r \searrow 0$, as needed. Thus, we may apply Theorem 3.14 to deduce that there exist $c > 0$ and $r > 0$, so that

$$|\psi(z)/z - \psi'(0)| < c \cdot \omega_p(|z|) \quad \text{for } |z| < r. \tag{3.34}$$

By multiplying ψ by a complex constant, we may assume that $\psi'(0) = 1$. Since $1/\phi(1/z) = \psi(z)$, the inequality (3.26) follows by taking $R = 1/r$ and $C' > c$. By Proposition 3.12 and Remark 3.15, the constants C' and R do not depend on N since the above big-O estimates for $I(r)$ do not depend on N. □

For the rest of the paper, we fix $C', R > 0$ so that Theorem 3.16 holds.

THEOREM 3.17. *Let $\varepsilon > 0$. There exists $N_\varepsilon \in \mathbb{N}$ such that for $N > N_\varepsilon$, we have*

$$|\phi_N(z) - z| < \varepsilon \quad \text{for } |z| < R. \tag{3.35}$$

Proof. Let μ_N be the Beltrami coefficient of ϕ_N. As $N \to \infty$, we have $\mu_N \to 0$ pointwise. Thus, we have $|\phi_N(z) - z| \to 0$ uniformly on the compact set $|z| \le R$. □

We conclude by restating Proposition 3.6 that listed the critical points and values of f_N, but now adapted to account for the new behavior of the function f_N near 0.

LEMMA 3.18. *Let $\{\zeta_j\}_{j=1}^{2^N-1}$ denote the $2^N - 1$ many critical points of $g_N(z)$ contained in $B(0, r_N)$. Then, the only critical points of f_N are the simple critical points given by $\phi_N(\zeta_j)$ for $j = 1, \ldots, 2^N - 1$, and the simple critical points given by*

$$\phi_N\bigg(r_j \cdot \exp\bigg(i\frac{(2k_j - 1)\pi}{M_j}\bigg)\bigg), \quad j \ge N, 1 \le k_j \le M_j. \tag{3.36}$$

The only singular values of f_N are the critical values $(\pm c_j r_j^{M_j})_{j=N}^{\infty}$ and the critical values $(g_N(\zeta_j))_{j=1}^{2^N-1}$.

4. *Mapping behavior near ∞*

Having proven in §3 all the estimates on the 'correction' map ϕ that we will need, we can now begin describing the mapping behavior of the function $f := h \circ \phi^{-1}$. In §4, we will introduce the annuli A_k, V_k, B_k for $k \ge 1$: these regions will be central to the proof of Theorem 1.1, as discussed in §2. We will also prove in Lemmas 4.17 and 4.18 the fundamental relations:

$$f(B_k) \subset B_{k+1} \quad \text{and} \quad A_{k+1} \subset f(V_k) \quad \text{for all } k \ge 1.$$

In this section, and in the rest of the paper, we will consider the case $p = 2 \cdot \sqrt{2}$ for Definition 3.13. The following lemma gives us estimates for how $\omega_{2\cdot\sqrt{2}}(|z|^{-1})$ decays as $z \to \infty$.

LEMMA 4.1. *There exists $k_0 \in \mathbb{Z}$ so that if $k \ge k_0$ and $|z| \ge 1/20r_k$, then*

$$\omega_{2\sqrt{2}}\bigg(\frac{1}{|z|}\bigg) \le \bigg(\frac{1}{2}\bigg)^{\sqrt{k}/4}. \tag{4.1}$$

Proof. This is just a simple calculation using Definition 3.13 and Corollary 3.5. Indeed, for all $k \ge 10$, we have

$$\begin{aligned}
\log\log\bigg(\frac{r_k}{20}\bigg) &\ge \log\log\bigg(\frac{2^{M_{k-1}}}{20}\bigg) \\
&\ge \log\log(2^{M_{k-5}}) \\
&= \log(M_{k-5}\log 2) \\
&= \log(2^{k-5}\log 2) \\
&= (k-5)\log 2 + \log\log 2.
\end{aligned}$$

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Therefore, there exists a value k_0 so that for all $k \geq k_0$, we have $\log\log(r_k/20) \geq \frac{1}{2}k$. For all such k, we verify that when $|z| \geq (r_k/20)$, we have

$$\omega_{2\sqrt{2}}\left(\frac{1}{|z|}\right) = \left(\frac{1}{2}\right)^{(1/(2\sqrt{2}))\sqrt{\log\log|z|}} \leq \left(\frac{1}{2}\right)^{(1/(2\sqrt{2}))\sqrt{\log\log(r_k/20)}}$$
$$\leq \left(\frac{1}{2}\right)^{(1/(2\sqrt{2}))\sqrt{k/2}} = \left(\frac{1}{2}\right)^{\sqrt{k}/4}.$$

This yields equation (4.1) as desired. □

Remark 4.2. Note that by perhaps choosing k_0 larger, we may additionally assume that $r_k \geq (1/20)r_{k_0} \geq R$ for all $k \geq k_0$. In this case, Theorem 3.16 and Lemma 4.1 imply that for all $k \geq k_0$, if $|z| \geq (r_k/20)$, we have $(\phi_N(z)/z) \in B(1, C' \cdot 2^{-\sqrt{k}/4})$.

LEMMA 4.3. *Let k_0 be as in Remark 4.2. For all $k \geq k_0$, if $|z| \geq (r_k/20)$ and $N \geq 5$, we have*

$$\big(1 - C' \cdot (\tfrac{1}{2})^{\sqrt{k}/4}\big)|z| \leq |\phi_N(z)| \leq \big(1 + C' \cdot (\tfrac{1}{2})^{\sqrt{k}/4}\big)|z|. \tag{4.2}$$

Moreover, if $z \in \phi_N(\{w : |w| \geq (r_k/20)\})$, then

$$\frac{1}{\big(1 + C' \cdot (\tfrac{1}{2})^{\sqrt{k}/4}\big)}|z| \leq |\phi_N^{-1}(z)| \leq \frac{1}{\big(1 - C' \cdot (\tfrac{1}{2})^{\sqrt{k}/4}\big)}|z|. \tag{4.3}$$

Proof. Equation (4.2) is just a rearrangement of equation (3.26), but using the estimate (4.1). For the second equation, just note that if $z \in \phi_N(\{w : |w| \geq (r_k/20)\})$, then there exists w with $|w| \geq (r_k/20)$ so that $\phi_N(w) = z$. Then, equation (4.2) holds with $w = \phi_N^{-1}(z)$, so equation (4.3) is just a rearrangement of equation (4.2). □

Remark 4.4. We will always assume that the integer $N \in \mathbb{N}$ satisfies $N \geq k_0$.

Next, we will do some re-indexing of variables. This will make our notation easier to read and more consistent with [**Bis18**].

Definition 4.5. Given the parameters M_k, c_k, and r_k from Definition 3.1, and given any integer $N \geq 1$, we define

$$n_k := M_{k+N-1} = 2^{N+k-1}, \tag{4.4}$$

$$R_k := r_{k+N-1}, \tag{4.5}$$

and

$$C_k := c_{k+N-1}. \tag{4.6}$$

Next, for any given $N \geq 1$, we define

$$\alpha_k := \frac{1}{(1 - C' \cdot (1/2)^{(\sqrt{k+N-1})/4})} \tag{4.7}$$

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and

$$\beta_k := \frac{1}{(1 + C' \cdot (1/2)^{(\sqrt{k+N-1})/4})}. \tag{4.8}$$

Remark 4.6. As $k \to \infty$, the sequence (α_k) decreases monotonically to 1, and (β_k) increases monotonically to 1. We will always assume the integer $N \in \mathbb{N}$ is large enough so that for all $k \geq 1$, we have

$$\frac{99}{100} < \beta_k < 1 < \alpha_k < \frac{101}{100}. \tag{4.9}$$

The specific constants above are not important, we just need β_k and α_k to be sufficiently close to 1 for all large $k \geq 1$.

Remark 4.7. We emphasize that the parameters in Definition 4.5 depend on $N \in \mathbb{N}$; we omit this dependence in the notation for readability. We also remark that Definition 3.1 implies that we still have $R_{k+1} = C_k(R_k/2)^{n_k}$ for all $k \geq 1$. Finally, we have the equalities $R_1 = r_N$ and $n_1 = M_N$. We will occasionally switch between the two forms of notation.

The inequalities (3.7), (3.8), (3.9), and (3.10) that apply to $(r_j)_{j=1}^{\infty}$ can now be restated as follows by applying Definition 4.5.

LEMMA 4.8. *Fix an integer $N \geq 5$. Then, for all $k \geq 1$, we have*

$$R_k^{n_k} \cdot C_k \geq R_k^{n_{k-1}+1}, \tag{4.10}$$

$$R_{k+1} \geq 2^{-n_k} \cdot R_k^{n_{k-1}+1}, \tag{4.11}$$

$$R_k \geq 2^{2^{k+N-2}}, \quad \textit{and}, \tag{4.12}$$

$$R_{k+1} \geq 4R_k^2. \tag{4.13}$$

The next lemma describes some relationships among the n_k terms that we use freely throughout the paper.

LEMMA 4.9. *For all $k \geq 1$, we have:*

(1) $2n_k = n_{k+1}$;
(2) $2^N + \sum_{j=1}^{k} n_j = n_{k+1}$.

Proof. Part (1) is obvious. Part (2) is a simple calculation:

$$2^N + \sum_{j=1}^{k} n_j = 2^N \left(1 + \sum_{j=0}^{k-1} 2^j\right) = 2^N \cdot 2^k = 2^{N+k} = n_{k+1}.$$

This proves the claim. □

We denote open round annuli centered at the origin by $A(r, R) = \{z : r < |z| < R\}$. Next, we will define the following sequence of annuli. See Figures 2 and 3.

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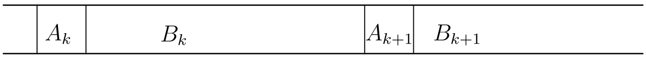

FIGURE 3. A visualization of A_k and B_k viewed on the cylinder. The annuli A_k have constant modulus, and the annulus B_k have very large and increasing moduli.

Definition 4.10. Given any $N \geq 1$, we define

$$A_k = A\big(\tfrac{1}{4}R_k, 4R_k\big), \quad B_k = \overline{A\big(4R_k, \tfrac{1}{4}R_{k+1}\big)}, \quad V_k = A\big(\tfrac{2}{5}R_k, \tfrac{3}{5}R_k\big). \tag{4.14}$$

We now begin to describe the mapping behavior of f in terms of the annuli in Definition 4.10.

PROPOSITION 4.11. *The zeros of f_N that satisfy $|z| \geq \frac{1}{4}R_1$ are contained in $\bigcup_{k=1}^{\infty} A_k$. In fact, each A_k contains exactly n_k many simple zeros, each located inside $A(\frac{3}{5}R_k, \frac{5}{4}R_k)$.*

Proof. This follows for f_N by combining equation (3.12) and Lemma 4.3. □

LEMMA 4.12. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, for all $k \geq 1$, and for all $z \in A(\frac{5}{4}R_k, \frac{3}{4}R_{k+1})$, we have*

$$f_N(z) = C_{k+1} \cdot (\phi_N^{-1}(z))^{n_{k+1}}. \tag{4.15}$$

Proof. If $z \in A(\exp(\pi/n_k) \cdot R_k, R_{k+1})$, then

$$f_N \circ \phi_N(z) = C_{k+1} \cdot z^{n_{k+1}}.$$

Therefore, if $z \in \phi_N(A(\exp(\pi/n_k) \cdot R_k, R_{k+1}))$, we must have

$$f_N(z) = C_{k+1} \cdot (\phi_N^{-1}(z))^{n_{k+1}}.$$

Therefore, it is sufficient to show that there is an M so that for all $N \geq M$, and for all $k \geq 1$, we have

$$A\left(\frac{5}{4}R_k, \frac{3}{4}R_{k+1}\right) \subset \phi_N\left(A\left(\exp\left(\frac{\pi}{n_k}\right) \cdot R_k, R_{k+1}\right)\right).$$

The existence of such an M is a simple calculation using Lemma 4.3. □

LEMMA 4.13. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, for all $k \geq 1$, and for all $z \in A(\frac{5}{4}R_k, \frac{3}{4}R_{k+1})$,*

$$\beta_k^{n_{k+1}} C_{k+1}|z|^{n_{k+1}} \leq |f_N(z)| \leq \alpha_k^{n_{k+1}} C_{k+1}|z|^{n_{k+1}}. \tag{4.16}$$

Proof. This follows immediately from Lemma 4.3, Definition 4.5, and Lemma 4.12. □

When estimating f near $|z| = R_1$, we will require in some situations the following lemma, which is similar to Lemma 4.13. Recall that by Definition 3.9 for all $z \in B(0, r_N - 1)$, we have $f_N(z) = q_N \circ \phi_N^{-1}(z)$, where $q_N(z) = c_N z^{M_N} + r_N z = C_1 z^{n_1} + R_1 z$.

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LEMMA 4.14. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, we have*

$$\frac{1}{2}c_N|z|^{M_N} \leq |q_N(z)| \leq 2c_N|z|^{M_N} \quad \text{for all } z \in A\left(\frac{1}{20}R_1, \frac{19}{20}R_1\right). \tag{4.17}$$

Proof. This is a simple but somewhat tedious application of Lemma 4.8. By the triangle inequality, we obtain for all $z \in A((1/20)R_1, (19/20)R_1)$ that

$$c_N|z|^{M_N}\left(1 - \frac{r_N}{c_N|z|^{M_N-1}}\right) \leq |q_N(z)| \leq c_N|z|^{M_N}\left(1 + \frac{r_N}{c_N|z|^{M_N-1}}\right).$$

On the one hand, we have by Lemma 3.5 that

$$\begin{aligned}
\max_{z\in A((1/20)R_1,(19/20)R_1)}\left(1 + \frac{r_N}{c_N|z|^{M_N-1}}\right) &= 1 + \frac{r_N}{c_N((1/20)r_N)^{M_N-1}} \\
&= 1 + \frac{r_N(1/20)r_N}{(1/10)^{M_N}c_N(r_N/2)^{M_N}} \\
&= 1 + \frac{10^{M_N}}{20}\cdot\frac{r_N^2}{r_{N+1}} \\
&\leq 1 + \frac{10^{M_N}}{20}\cdot\frac{r_N^2}{2^{-M_N}r_N^{M(N-1)}} \\
&\leq 1 + \frac{20^{M_N}}{20}\cdot\frac{1}{r_N^{M(N-2)}} \\
&= 1 + \frac{1}{20}\left(\frac{20}{r_N^{1/4}}\right)^{M_N}.
\end{aligned}$$

By Lemma 4.8, there exists M so that for all $N \geq M$, we have $r_N^{1/4} \geq 40$ and, therefore, we obtain

$$\max_{z\in A((1/20)R_1,(19/20)R_1)}\left(1 + \frac{r_N}{c_N|z|^{M_N-1}}\right) \leq 1 + \frac{1}{20}\cdot\left(\frac{1}{2}\right)^{M_N} < 2.$$

Therefore, we obtain for all $z \in z \in A((1/20)R_1, (19/20)R_1)$ that

$$|q_N(z)| \leq 2c_N|z|^{M_N}. \tag{4.18}$$

The proof of the other inequality is similar. □

LEMMA 4.15. *For all N sufficiently large, we have*

$$\frac{1}{2}\beta_1^{M_N}c_N|z|^{M_N} \leq |f_N(z)| \leq 2\alpha_1^{M_N}c_N|z|^{M_N} \quad \text{for all } z \in A\left(\frac{1}{10}R_1, \frac{9}{10}R_1\right). \tag{4.19}$$

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Proof. By Lemma 4.3, there exists $M \in \mathbb{N}$ so that for all $N \geq M$, we have

$$\phi_N^{-1}\left(A\left(\frac{1}{10}R_1, \frac{9}{10}R_1\right)\right) \subset A\left(\frac{1}{20}R_1, \frac{19}{20}R_1\right).$$

By perhaps choosing M larger, we have for all $N \geq M$ that $R_1 - 1 \geq (19/20)R_1$ as well. Then, for all $z \in A((1/10)R_1, (9/10)R_1)$, we have by Lemmas 4.3 and 4.14 that

$$\begin{aligned}\max_{z\in A((1/10)R_1,(9/10)R_1)} |f_N(z)| &= \max_{z\in A((1/10)R_1,(9/10)R_1)} |q_N(\phi_N^{-1}(z))| \\ &\leq 2c_N|\phi_N^{-1}(z)|^{M_N} \leq 2c_N\alpha_1^{M_N}|z|^{M_N}. \end{aligned} \tag{4.20}$$

Similarly, we obtain

$$\begin{aligned}\min_{z\in A((1/10)R_1,(9/10)R_1)} |f_N(z)| &= \min_{z\in A((1/10)R_1,(9/10)R_1)} |q_N(\phi_N^{-1}(z))| \\ &\geq \tfrac{1}{2}c_N|\phi_N^{-1}(z)|^{M_N} \geq \tfrac{1}{2}c_N\beta_1^{M_N}|z|^{M_N}. \end{aligned} \tag{4.21}$$

This proves the claim. □

We are now ready to prove some basic lemmas about the macroscopic mapping behavior of the function f_N. First, we will need the following basic lemma.

LEMMA 4.16. *Suppose that g is holomorphic on an annulus $W = A(a, b)$ and continuous up to the boundary of W. Let $U = A(c, d)$.*

(1) *If $|g(z)| \leq c$ on $|z| = a$ and $|g(z)| \geq d$ on $|z| = b$, then $U \subset g(W)$.*
(2) *Suppose g has no zeros in W and that $g(\partial W) \subset \overline{U}$. Then, $g(W) \subset \overline{U}$.*

Proof. The first part uses the fact that holomorphic maps are open. The second part is an application of the maximum principle. A detailed proof can be found in [**Bis18**, Lemma 11.1]. □

Next, we prove the following lemma about the mapping behavior on A_k, where we will see the dynamics of f_N are the most interesting.

LEMMA 4.17. *There exists $M \in \mathbb{N}$ such that for all $N \geq M$ and for all $k \geq 1$, we have*

$$A_{k+1} \subset f_N(V_k) \subset f_N(A_k). \tag{4.22}$$

Proof. First, we prove the case of $k \geq 2$. In this setting, by Lemma 4.13,

$$\max_{|z|=(2/5)R_k} |f_N(z)| \leq \max_{|z|=(2/5)R_k} \alpha_k^{n_k}C_k|z|^{n_k} \leq \alpha_k^{n_k}C_k\left(\tfrac{2}{5}R_k\right)^{n_k} \leq \alpha_k^{n_k}\left(\tfrac{4}{5}\right)^{n_k}\cdot C_k\cdot\left(\tfrac{1}{2}R_k\right)^{n_k}.$$

By equation (4.9), we have $\alpha_k \cdot \frac{4}{5} \leq \frac{7}{8}$. Therefore, if $N \geq 5$, since $R_{k+1} = C_k \cdot (R_k/2)^{n_k}$, we end up with

$$\max_{|z|=(2/5)R_k} |f_N(z)| \leq \left(\tfrac{7}{8}\right)^{n_k}R_{k+1} < \tfrac{1}{4}R_{k+1}. \tag{4.23}$$

Next, observe that by Lemma 4.13, we have

$$\min_{|z|=(3/5)R_k} |f_N(z)| \geq \min_{|z|=(3/5)R_k} \beta_k^{n_k}C_k|z|^{n_k} = \beta_k^{n_k}C_k\left(\tfrac{3}{5}R_k\right)^{n_k} = \beta_k^{n_k}\cdot\left(\tfrac{6}{5}\right)^{n_k}\cdot C_k\cdot\left(\tfrac{1}{2}R_k\right)^{n_k}.$$

By equation (4.9), we have $\beta_k \cdot \frac{6}{5} \geq \frac{9}{8}$. If $N \geq 5$, we end up with

$$\min_{|z|=(3/5)R_k} |f_N(z)| \geq \left(\tfrac{9}{8}\right)^{n_k} R_{k+1} > 4R_{k+1}. \tag{4.24}$$

The lemma for the case of $k \geq 2$ now follows from Lemma 4.16, part (1).

The case of $k = 1$ is almost exactly the same, except we now have to use Lemma 4.15. By following the exact same steps as above, we obtain since $N \geq 5$ that

$$\max_{|z|=(2/5)R_1} |f_N(z)| \leq 2\left(\tfrac{7}{8}\right)^{n_1} R_2 < \tfrac{1}{4}R_2 \tag{4.25}$$

and

$$\min_{|z|=(3/5)R_1} |f_N(z)| \geq \tfrac{1}{2}\left(\tfrac{9}{8}\right)^{n_1} R_2 > 4R_2. \tag{4.26}$$

The lemma for the case of $k = 1$ now follows from Lemma 4.16, part (1). □

LEMMA 4.18. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$ and for all $k \geq 1$, we have*

$$f_N(B_k) \subset B_{k+1}. \tag{4.27}$$

Proof. We will adopt a similar strategy to Lemma 4.17, using Lemma 4.16, part (2). First, we make the important observation that $C_{k+1}/C_k = R_k^{-n_k}$ for all $k \geq 1$. Next, observe that if $|z| = 4R_k$, then we have by Lemma 4.13,

$$\begin{aligned}
\max_{|z|=4R_k} |f_N(z)| &\leq \max_{|z|=4R_k} \alpha_k^{n_{k+1}} C_{k+1} |z|^{n_{k+1}} \\
&= \alpha_k^{n_{k+1}} C_{k+1} (4R_k)^{n_{k+1}} \\
&= \alpha_k^{n_{k+1}} 8^{n_{k+1}} \cdot C_k R_k^{-n_k} \cdot \left(\tfrac{1}{2}R_k\right)^{n_{k+1}} \\
&= \alpha_k^{n_{k+1}} 8^{n_{k+1}} C_k \left(\tfrac{1}{2}R_k\right)^{n_k} \cdot R_k^{-n_k} \left(\tfrac{1}{2}R_k\right)^{n_k} \\
&= \alpha_k^{n_{k+1}} 8^{n_{k+1}} 2^{-n_k} R_{k+1}. \\
&= (\alpha_k^2)^{n_k} 32^{n_k} R_{k+1}.
\end{aligned}$$

By equation (4.9), we have $\alpha_k^2 \leq 2$ for all $k \geq 1$. By Lemma 4.8, there exists M so that for all $N \geq M$, and for all $k \geq 1$, we have $R_k^{1/2} > 256$. Therefore,

$$\frac{64^{n_{k+1}} R_{k+1}}{R_{k+2}} \leq \frac{64^{n_{k+1}} R_{k+1}}{(1/2)^{n_{k+1}} R_{k+1}^{n_k+1}} = \left(\frac{128}{R_{k+1}^{1/2}}\right)^{n_{k+1}} \leq \left(\frac{1}{2}\right)^{n_{k+1}}. \tag{4.28}$$

By equation (4.28), and since $N \geq 5$, we have for all $k \geq 1$,

$$\max_{|z|=4R_k} |f_N(z)| \leq 64^{n_{k+1}} R_{k+1} \leq \left(\tfrac{1}{2}\right)^{n_{k+1}} R_{k+2} < \tfrac{1}{8} R_{k+2}. \tag{4.29}$$

Next, observe that by Lemma 4.13, we similarly have

$$\begin{aligned}
\min_{|z|=4R_k} |f_N(z)| &\geq \min_{|z|=4R_k} \beta_k^{n_{k+1}} C_{k+1} |z|^{n_{k+1}} \\
&= \beta_k^{n_{k+1}} C_k R_k^{-n_k} \cdot 2^{n_{k+1}} \left(\frac{R_k}{2}\right)^{n_k} \cdot \left(\frac{R_k}{2}\right)^{n_k} \\
&= (\beta_k^2)^{n_k} 2^{n_k} R_{k+1} = (2\beta_k^2)^{n_k} R_{k+1}.
\end{aligned}$$

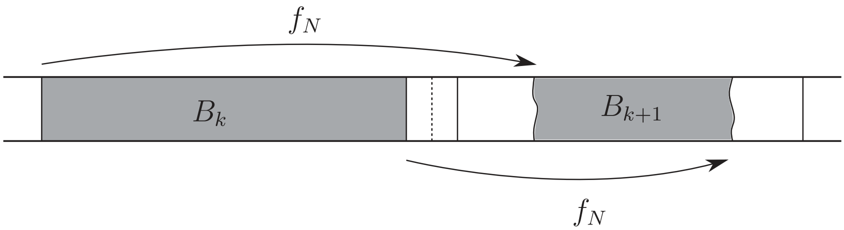

FIGURE 4. f_N maps the annulus B_k into the annulus B_{k+1}. The picture is not to scale; in reality, the modulus of B_{k+1} is much larger than the modulus of $f_N(B_k)$.

By equation (4.9), we have $\beta_k^2 \cdot 2 \geq \frac{3}{2}$. Therefore, since $N \geq 5$, we have for all $k \geq 1$ that

$$\min_{|z|=4R_k} |f_N(z)| \geq \left(\tfrac{3}{2}\right)^{n_k} R_{k+1} > 8R_{k+1}. \tag{4.30}$$

Therefore, by equations (4.29) and (4.30), for all $k \geq 1$, we have

$$f_N(|z| = 4R_k) \subset A\left(8R_k, \tfrac{1}{8}R_{k+1}\right) \subset B_{k+1}. \tag{4.31}$$

We can use similar techniques as above to analyze the behavior of f_N on the outermost boundary of B_k (see Figure 4). Indeed, we have

$$\max_{|z|=(1/4)R_{k+1}} |f_N(z)| < \tfrac{1}{8}R_{k+2} \tag{4.32}$$

and

$$\min_{|z|=(1/4)R_{k+1}} |f_N(z)| > 8R_{k+1}. \tag{4.33}$$

Therefore, by equations (4.33) and (4.32), we have

$$f_N\left(|z| = \tfrac{1}{4}R_{k+1}\right) \subset A\left(8R_k, \tfrac{1}{8}R_{k+1}\right) \subset B_{k+1}. \tag{4.34}$$

As we commented at the start, this proves the lemma. □

We conclude this section by recording the location of the critical points and values of f in relation to the annuli A_k, B_k.

PROPOSITION 4.19. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, all critical points z of f_N with $|z| > \frac{1}{4}R_1$ belong to $\bigcup_{k=1}^{\infty} A_k$. Moreover, if $z \in A_k$ is a critical point, then $f_N(z) \in B_{k+1}$.*

Proof. By Proposition 3.6 and Lemma 4.3, there exists M so that for all $N \geq M$, all critical points of f_N satisfying $|z| \geq \frac{1}{4}R_1$ belong to $\bigcup_{k=1}^{\infty} A_k$. If $z \in A_k$ is a critical point, then Proposition 3.6 also says that $|f_N(z)| = C_k R_k^{n_k}$. To see that $C_k R_k^{n_k} \in B_{k+1}$, first notice that we have the identity

$$C_k R_k^{n_k} = 2^{n_k} C_k \left(\tfrac{1}{2}R_k\right)^{n_k} = 2^{n_k} R_{k+1}. \tag{4.35}$$

It follows immediately from equation (4.35) that since $N \geq 5$, we have

$$C_k R_k^{n_k} = 2^{n_k} R_{k+1} > 8R_{k+1}. \tag{4.36}$$

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However, we have by Lemma 4.8 that

$$\frac{2^{n_k} R_{k+1}}{R_{k+2}} \leq \frac{8^{n_k}}{R_{k+1}^{n_k}} \leq \left(\frac{1}{2}\right)^{n_k}.$$

So since $N \geq 5$, we obtain from equation (4.35) that

$$C_k R_k^{n_k} \leq \left(\tfrac{1}{2}\right)^{n_k} R_{k+2} < \tfrac{1}{8} R_{k+2}. \tag{4.37}$$

Therefore, by equations (4.36) and (4.37), we have $f_N(z) \in B_{k+1}$. □

Remark 4.20. For the rest of the paper, we will always assume that N is large enough so that all of the statements and inequalities in this section are valid.

5. *Mapping behavior near* 0

Having studied the mapping behavior of f in the region $|z| > R_1/4$ in §4, we now study in §5 the mapping behavior of f in $|z| < R_1/4$. Recall that in $|z| < R_1/4$, the mapping f satisfies $f(z) = q_N \circ \phi_N^{-1}(z)$, where $q_N(z) = c_N z^{M_N} + r_N z$. This polynomial was chosen so as to have a Cantor Julia set of dimension $\ll 1$. Thus, when we consider f as a polynomial-like mapping by restricting the domain of f to a subdomain of $|z| < R_1/4$, we will see that the Julia set of this polynomial-like mapping has dimension $\ll 1$.

We begin with the following lemma about the polynomial $q_N(z)$.

LEMMA 5.1. *Let $q_N(z) = c_N z^{M_N} + r_N \cdot z$. Then, the derivative of $q_N(z)$ is*

$$q_N'(z) = c_N M_N z^{M_N - 1} + r_N. \tag{5.1}$$

The non-zeros of q_N are given by

$$z = \left(\frac{-r_N}{c_N}\right)^{1/M_N - 1}. \tag{5.2}$$

The critical points of q_N are given by

$$z = \left(\frac{-r_N}{c_N M_N}\right)^{1/M_N - 1}. \tag{5.3}$$

The critical values of q_N lie on the circle

$$|z| = \left(\frac{r_N}{c_N M_N}\right)^{1/M_N - 1} \cdot r_N \cdot \left(1 - \frac{1}{M_N}\right). \tag{5.4}$$

The value of $|q_N'(z)|$ when z is a zero of q_N satisfies

$$|q_N'(z)| = r_N(M_N - 1). \tag{5.5}$$

Proof. These are all simple calculations. We only verify equation (5.4). If z is a critical point of q_N, then we calculate that

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$$\begin{aligned} q_N(z) &= c_N\left(\frac{-r_N}{c_N M_N}\right)^{M_N/M_N-1} + r_N\left(\frac{-r_N}{c_N M_N}\right)^{1/M_N-1} \\ &= \left(\frac{-r_N}{c_N M_N}\right)^{1/M_N-1} \cdot \left(r_N - \frac{r_N}{M_N}\right) \\ &= \left(\frac{-r_N}{c_N M_N}\right)^{1/M_N-1} \cdot r_N \cdot \left(1 - \frac{1}{M_N}\right). \end{aligned}$$

The result follows upon taking the absolute value. □

We will now prove that the critical values of q_N map to B_1. This will be crucial in dimension estimates that require coverings that are built by considering the inverse f^{-1}. First, we need the following technical lemma.

LEMMA 5.2. *For all $N \geq 10$, we have*

$$2^{M_{N-7}} r_{N-1} \leq \left(\frac{r_N}{c_N M_N}\right)^{1/M_N-1} \leq \frac{1}{\sqrt{2}} r_{N-1}^2. \tag{5.6}$$

Proof. Recall first that $r_N = c_{N-1}(r_{N-1}/2)^{M_{N-1}}$ by definition. Therefore, we calculate

$$\frac{r_N}{c_N} = \frac{1}{2^{M_{(N-1)}}} \frac{c_{N-1} r_{N-1}^{M_{(N-1)}}}{c_N} = \frac{1}{2^{M_{(N-1)}}} \frac{r_{N-1}^{M_{(N-1)}}}{r_{N-1}^{-M_{(N-1)}}} = \frac{1}{2^{M_{(N-1)}}} r_{N-1}^{M_N}. \tag{5.7}$$

First, we prove the upper bound for equation (5.6). First, note that we have for all $N \geq 5$ that

$$2^{-M_{(N-1)}/M_N-1} \leq \frac{1}{\sqrt{2}}, \tag{5.8}$$

so that by combining equations (5.7) and (5.8), we obtain

$$\left(\frac{r_N}{c_N M_N}\right)^{1/M_N-1} = \left(\frac{1}{M_N}\right)^{1/M_N-1} \cdot 2^{-M_{(N-1)}/M_N-1} \cdot r_{N-1} \cdot r_{N-1}^{1/M_N-1} \leq \frac{1}{\sqrt{2}} r_{N-1}^2. \tag{5.9}$$

To prove the lower bound for equation (5.6), since $N \geq 5$, note that we have

$$\left(\frac{1}{M_N}\right)^{1/M_N-1} \geq \frac{1}{2} \quad \text{and} \quad 2^{-M_{N-1}/M_N-1} \geq \frac{1}{2}. \tag{5.10}$$

By two applications of Corollary 3.5, since $N \geq 10$, we obtain

$$r_{N-1}^{1/M_N-1} \geq 2^{-M_{(N-2)}/M_N-1} \cdot r_{N-2}^{M_{(N-3)}/M_N-1} \geq 2^{-1/2} r_{N-2}^{1/8} \geq 2^{-1/2} 2^{M_{N-3}/8} = 2^{M_{N-6}-1/2}. \tag{5.11}$$

Therefore, in a similar way to equation (5.9), except this time using equations (5.10) and (5.11), we have

$$\begin{aligned} \left(\frac{r_N}{c_N M_N}\right)^{1/M_N-1} &= 2^{-M_{(N-1)}/M_N-1} \cdot \left(\frac{1}{M_N}\right)^{1/M_N-1} \cdot r_{N-1} \cdot r_{N-1}^{1/M_N-1} \\ &\geq 2^{M_{N-6}-1/2-2} r_{N-1} \geq 2^{M_{N-7}} r_{N-1}. \end{aligned} \tag{5.12}$$

Equations (5.9) and (5.12) combine to prove equation (5.6). □

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LEMMA 5.3. *For $N \geq 10$, the critical values of q_N belong to B_1, and the critical points satisfy $|z| \leq r_{N-1}^2$.*

Proof. If z is a critical point, by Lemmas 5.1 and 5.2, we have

$$|z| = \left(\frac{r_N}{c_N M_N}\right)^{1/M_N - 1} \leq \frac{1}{\sqrt{2}} r_{N-1}^2. \tag{5.13}$$

Recall that by Corollary 3.5 that if $N \geq 10$, we have $r_{N-1}^2 \leq \frac{1}{4} r_N$ and $r_N^2 \leq \frac{1}{4} r_{N+1}$. So by Lemmas 5.1 and 5.2, if z is a critical point, then

$$|q_N(z)| = \left(\frac{r_N}{c_N M_N}\right)^{1/M_N - 1} r_N \left(1 - \frac{1}{M_N}\right) \leq \frac{1}{\sqrt{2}} r_{N-1}^2 r_N \leq \frac{1}{4\sqrt{2}} r_N^2 < \frac{1}{16\sqrt{2}} r_{N+1} \tag{5.14}$$

and

$$|q_N(z)| \geq 2^{M_{(N-7)}} r_{N-1} r_N \cdot \left(1 - \frac{1}{M_N}\right) > 8 r_N. \tag{5.15}$$

Therefore, by Definition 4.5, we have the critical values of q_N contained in B_1, as desired. □

Now we introduce a polynomial-like mapping by restricting f to a subdomain U, defined as follows.

Definition 5.4. For the rest of this section, we will use the following definitions.

(1) Define $D = B(0, \frac{1}{4} R_1)$.
(2) Define $r = 16 r_{N-1}^2 R_1$. This is the same as $r = 16 r_{N-1}^2 r_N$ by equation (4.5).
(3) Define $V = B(0, r)$ and $U' = q_N^{-1}(V)$.
(4) Define $U = \phi_N(U')$.

LEMMA 5.5. *For all $N \geq 10$, the triple $q_N : U' \to V$ is a degree 2^N polynomial-like mapping. Moreover, all $2^N - 1$ many critical points of q_N belong to $U' \subset D$.*

Proof. We first verify that $U' \subset D$. Note that if $|z| = \frac{1}{4} R_1 = \frac{1}{4} r_N$, then by Lemma 4.14, we have

$$\begin{aligned} |q_N(z)| &\geq \frac{1}{2} c_N \left(\frac{1}{4} r_N\right)^{M_N} \\ &= \left(\frac{1}{2}\right)^{M_N + 1} r_{N+1} \\ &\geq \left(\frac{1}{4}\right)^{M_N + 1/2} r_N^{M_{(N-1)}} r_N \quad \text{(Lemma 4.8)} \\ &= \frac{1}{2} \left(\frac{r_N^{1/4}}{4}\right)^{M_N} r_N^{M_{(N-2)}} r_N. \end{aligned} \tag{5.16}$$

Therefore, since $N \geq 10$, we have $r_N^{1/4}/4 > 8$ and we deduce that

$$|q_N(z)| \geq 16 r_N^3 > 16 r_{N-1}^2 r_N. \tag{5.17}$$

Therefore, we must have $U' \subset D$.

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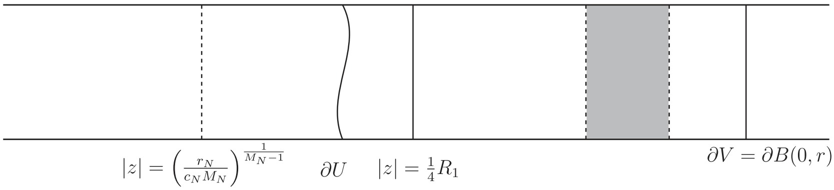

FIGURE 5. A schematic for Definition 5.4 and Lemma 5.5. The critical points for q_N lie on the circle $|z| = (r_N/c_N M_N)^{1/M_N - 1}$, and the associated critical values lie in an annulus contained in V, illustrated in gray. So, U contains the critical points of q_N, and it will also be verified that $U \subset D$.

By equation (5.14), the critical values of q_N satisfy $|z| \le r_{N-1}^2 r_N$. Therefore, V contains all $2^N - 1$ many critical values of q_N, so that U' contains the $2^N - 1$ many critical points of q_N. It now follows from Lemma A.11 that $q_N : U' \to V$ is a proper degree 2^N branched covering map, and it follows from Theorem A.13 that U' is a Jordan domain. Since U' is contained in D, which is compactly contained in V, $q_N : U' \to V$ is a degree 2^N polynomial-like mapping (see Figure 5). □

LEMMA 5.6. *Let U be as in Definition 5.4 and $N \ge 10$. Then, $U \subset D$ and the triple $f_N : U \to V$ is a degree 2^N polynomial-like mapping.*

Proof. By Lemma 4.3, we verify that $U = \phi_N(U') \subset D$. Therefore, by Lemma 3.18, $f_N = q_N \circ \phi_N^{-1} : U \to V$ is a proper, degree 2^N branched covering map, and is therefore a degree 2^N polynomial-like mapping. □

The rest of §5 is devoted to showing the filled Julia set of $f_N : U \to V$ has dimension $\ll 1$. We will do so by constructing a cover by pulling back $B(0, 4R_1)$ under appropriate branches of f^{-1}.

Remark 5.7. Suppose that $B(0, R)$ is the disk of radius R centered at the origin, where we take any $R \in [4R_1, 8R_1]$, so that $B(0, R) \subset V$. By Lemmas 5.5 and 5.6, $\overline{B(0, R)}$ contains the $2^N - 1$ many critical points of $f_N : U \to V$. However, by equation (5.15), $B(0, R)$ does not contain the $2^N - 1$ many critical values of $f_N : U \to V$. It follows then from Lemma A.15 that $f_N^{-1}(B) \subset U$ is the disjoint union of 2^N many Jordan domains B_i, $i = 1, \ldots, 2^N$ such that $f_N : B_i \to B$ is conformal.

Remark 5.7 motivates the following definition (see Figure 6).

Definition 5.8. Define $\gamma := \{z : |z| = 4R_1\}$. Then, γ is a circle that surrounds the critical points of f_N contained in D, but not the critical values associated to those critical points.

(1) Let $\Gamma_1 = f_N^{-1}(\gamma)$ be the disjoint union of the 2^N many Jordan curves contained in D that f_N maps to γ. We denote the elements of Γ_1 by γ_1.
(2) Let $\Gamma_n = f_N^{-n}(\gamma)$ be the disjoint union of 2^{Nn} many Jordan curves in D that get mapped by f_N^n to γ. We denote the elements of Γ_N by γ_n.
(3) Given $\gamma_n \in \Gamma_n$, we define $\widehat{\gamma_n}$ to be the bounded simply connected domain with boundary given by γ_n. We define $\widehat{\Gamma_n}$ to be the set of all $\widehat{\gamma_n}$.

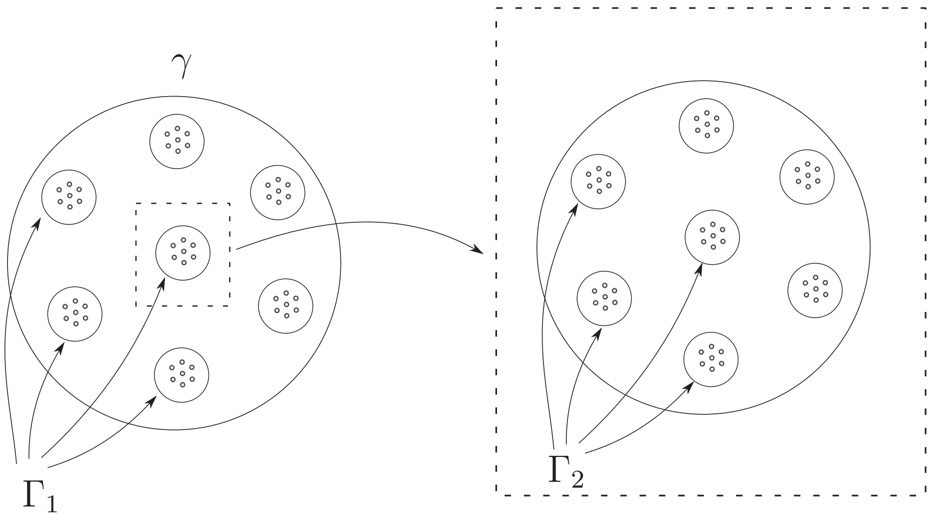

FIGURE 6. Illustration of Definition 5.8 of the families Γ_n.

Remark 5.9. An alternative definition for γ in Definition 5.8 would be $\sigma = \{z : |z| = 8R_1\}$, and we analogously could define Σ_n, elements $\sigma_n \subset \Sigma_n$, and $\widehat{\sigma_n}$. Then, for each $\sigma_n \in \Sigma_n$, $\widehat{\sigma_n}$ contains exactly one element $\gamma_n \in \Gamma_n$, and the modulus of $\widehat{\sigma_n} \setminus \overline{\widehat{\gamma_n}}$ is bounded below by $(2\pi)^{-1} \log 2 > 0$. For each $\sigma_n \in \Sigma_n$, there exists some element $\sigma_{n-1} \in \Sigma_{n-1}$ such that $f_N : \widehat{\sigma_n} \to \widehat{\sigma_{n-1}}$ is conformal. This means that the corresponding mapping $f_N : \widehat{\gamma_n} \to \widehat{\gamma_{n-1}}$ is conformal, and by Remark A.7, Corollaries A.4 and A.6 apply with constants that do not depend on the integers N or n.

We now estimate the diameters of our covering of Definition 5.8.

LEMMA 5.10. *Let γ, Γ_n, and $\widehat{\Gamma_n}$ be as in Definition 5.8. Then, there exists a value $M \in \mathbb{N}$ so that for all $N \geq M$, for all $n \geq 1$, and for every $\gamma_n \in \Gamma_n$, we have*

$$\text{diameter}(f_N(\gamma_n)) \geq R_1 \text{ diameter}(\gamma_n).$$

Proof. Choose some $\widehat{\gamma_1} \in \widehat{\Gamma_1}$, and let $z_0 \in \widehat{\gamma_1}$ be a zero for f_N. Such a z_0 exists since $f_N(\gamma_1) = \gamma$ surrounds the origin. Then, by Corollary A.6 and Remark 5.9 applied to the appropriate branch of the inverse $f_N^{-1} : B(0, 4R_1) \to \widehat{\gamma_1}$, there exists a constant $C > 0$ such that

$$\gamma_1 \subset B\left(z_0, \frac{C(4R_1)}{|f_N'(z_0)|}\right).$$

Therefore, by Theorem 3.17 and Lemma 5.1, there exists another constant $L > 0$ so that

$$\frac{\text{diameter}(f_N(\gamma_1))}{\text{diameter}(\gamma_1)} \geq \frac{1}{C} \cdot |f_N'(z_0)| = \frac{1}{C} \cdot R_1(M_N - 1) \cdot |(\phi_N^{-1})'(z_0)| \geq \frac{R_1}{C} \cdot L \cdot (M_N - 1).$$

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This proves the estimate for the case of $n = 1$. For $n > 1$, by Corollary A.4 and Remark 5.9, there exists a constant $c > 1$ such that

$$\frac{\text{diameter}(f_N(\gamma_n))}{\text{diameter}(\gamma_n)} \geq \frac{1}{c} \frac{\text{diameter}(f_N^{n-1}(f_N(\gamma_n)))}{\text{diameter}(f_N^{n-1}(\gamma_n))}$$
$$\geq \frac{1}{c} \frac{\text{diameter}(\gamma)}{\text{diameter}(\gamma_1)} \geq \frac{R_1}{c \cdot C} \cdot L \cdot (M_N - 1).$$

Therefore, there exists $M \in \mathbb{N}$ so that for all $N \geq M$ and for all $n \geq 1$, we have

$$\frac{\text{diameter}(f_N(\gamma_n))}{\text{diameter}(\gamma_n)} \geq R_1.$$

This is exactly what we wanted to show. □

Lemma 5.10 allows us to deduce the Hausdorff dimension of the Julia set of the polynomial-like mapping $f_N : U \to V$.

THEOREM 5.11. *Let $t > 0$ be given. Then, there exists an integer M so that for all $N \geq M$, the Hausdorff dimension of the filled Julia set of $f_N : U \to V$ is at most t.*

Proof. For each $n \geq 1$, $\widehat{\Gamma_n}$ is a covering of the Julia set of (f_N, U, V). Fix $t > 0$. If $\widehat{\gamma_n} \in \widehat{\Gamma_n}$, then $f_N(\widehat{\gamma_n}) =: \widehat{\gamma_{n-1}} \in \widehat{\Gamma_{n-1}}$. Therefore, by Lemma 5.10, we have

$$\sum_{\widehat{\gamma_n} \in \widehat{\Gamma_n}} \text{diameter}(\widehat{\gamma_n})^t \leq R_1^{-t} \cdot 2^N \cdot \sum_{\widehat{\gamma_{n-1}} \in \widehat{\Gamma_{n-1}}} \text{diameter}(\widehat{\gamma_{n-1}})^t. \tag{5.18}$$

It follows from equation (5.18) that

$$\sum_{n=1}^{\infty} \sum_{\widehat{\gamma_n} \in \widehat{\Gamma_n}} \text{diameter}(\widehat{\gamma_n})^t \leq \text{diameter}(\gamma)^t \cdot \sum_{n=1}^{\infty} 2^{Nn} R_1^{-tn}. \tag{5.19}$$

The sum in equation (5.19) converges if and only if

$$2^N R_1^{-t} < 1.$$

Therefore, we use Corollary 3.5 to see that

$$R_1^{-t} 2^N = r_N^{-t} 2^N \leq 2^{N - M_{N-1} t}. \tag{5.20}$$

Choose M so that for all $N \geq M$, we have $N - 2^{N-1} t < 0$, so that we obtain $2^N R_1^{-t} < 1$. For such a choice of N, equation (5.19) converges, and for any $\eta > 0$, there exists some value n so that

$$\sum_{\widehat{\gamma_n} \in \widehat{\Gamma_n}} \text{diameter}(\widehat{\gamma_n})^t < \eta.$$

Since $\widehat{\Gamma_n}$ is a covering of the filled Julia set of $f_N : U \to V$, it follows that its Hausdorff dimension is bounded above by t. □

We conclude §5 by showing that the critical values of f_N lying in $|z| < R_1/4$ map to B_1. This will be crucial in constructing covers of the Julia set of f_N by pulling back the annuli A_k under f_N.

LEMMA 5.12. *For all $N \geq 10$, if z is a critical point of f_N contained in D, then $f_N(z) \in B_1$.*

Proof. The only critical points of f_N contained inside of D are of the form $\phi_N(z)$, where z is a critical point of q_N. By Lemma 5.3, the critical values of q_N are contained in B_1. Therefore, the critical values of f_N associated to the critical points in D belong to B_1. □

6. *Location of the Julia set*

In this section, we refine our understanding of the behavior of f in the annuli A_k. Namely, we prove that unless z belongs to V_k or a collection of small balls (which we call *petals* in Definition 6.2 below), then $f(z) \in B_{k+1}$. This is crucial to our understanding of the structure of $\mathcal{J}(f)$, since it is readily observed (see Lemma 6.1 below) that the annuli B_k lie in $\mathcal{F}(f)$. We will also further describe the behavior of f in the aforementioned petals.

LEMMA 6.1. *There exists $M \in \mathbb{N}$ such that for all $N \geq M$, B_k belongs to the Fatou set of f_N for all $k \geq 1$.*

Proof. By Lemma 4.18, there exists $M \in \mathbb{N}$ such that for all $N \geq M$, we have $f_N(B_k) \subset B_{k+1}$. This implies that each point in B_k escapes locally uniformly to ∞. □

Therefore, when $|z| \geq \frac{1}{4}R_1$, the Julia set of f_N is contained in $\bigcup_{k=1}^{\infty} A_k$. For each $k \geq 1$, Proposition 4.11 says that f_N has n_k many zeros contained in A_k. For a given $k \geq 1$, let $\{w_j^k\}_{j=1}^{n_k}$ denote these zeros. Following the terminology in [**Bis18**], we introduce some notation for the balls containing the zeros of f_N inside of A_k (see Figure 7).

Definition 6.2. For $k \geq 1$, let $\mathcal{P}_k = \bigcup_{j=1}^{n_k} B(w_j^k, R_k/2^{n_k})$ be the *petals* of f_N inside of A_k. A connected component $P_k \subset \mathcal{P}_k$ will be called a *petal*.

As already mentioned, we will now prove several lemmas (Lemmas 6.3–6.8) detailing the mapping behavior of f within the annulus A_k, most crucially within the subannulus V_k and the petals P_k.

LEMMA 6.3. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, if $k \geq 1$, then*

$$f_N\big(A\big(\tfrac{5}{4}R_k, 4R_k\big)\big) \subset B_{k+1}.$$

Proof. By Proposition 4.11 and by Lemma 4.16, it is sufficient to verify that there exists $M \in \mathbb{N}$ so that for all $N \geq M$, we have $f_N(|z| = 4R_k) \subset B_{k+1}$ and $f_N(|z| = \frac{5}{4}R_k) \subset B_{k+1}$. The former has already been verified in equation (4.31), so we only need to prove the latter. By the maximum principle for holomorphic functions, we have

$$\max_{|z|=(5/4)R_k} |f_N(z)| \leq \max_{|z|=4R_k} |f_N(z)| < \tfrac{1}{8}R_{k+2}. \tag{6.1}$$

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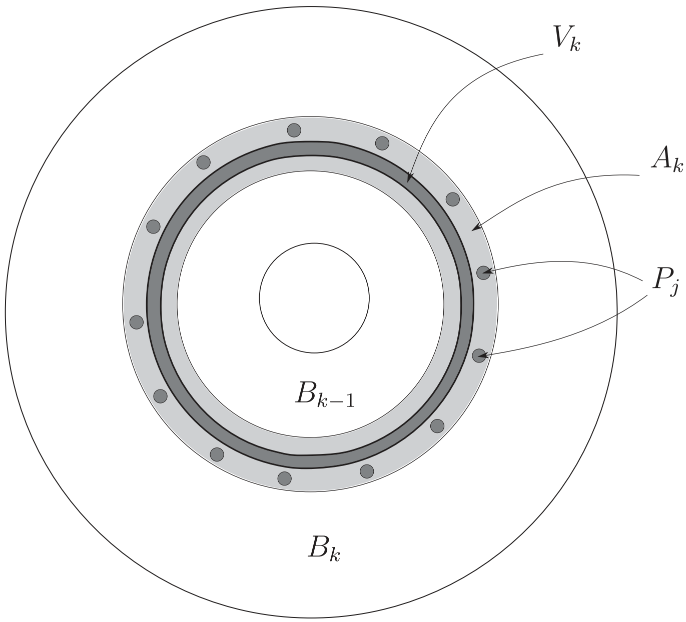

FIGURE 7. Illustration of Definition 6.2 of the petals P_k. The annuli B_{k-1}, B_k are in white, the annulus A_k is in light gray, and the annulus V_k and petals P_k are in dark gray.

By Lemma 4.13, we have

$$\begin{aligned}\min_{|z|=(5/4)R_k} |f_N(z)| &\geq \min_{|z|=(5/4)R_k} C_{k+1}\beta_k^{n_{k+1}}|z|^{n_{k+1}} = \beta_k^{n_{k+1}} C_k\left(\frac{R_k}{2}\right)^{n_k}\left(\frac{5}{2}\right)^{n_{k+1}}\left(\frac{1}{2}\right)^{n_k}\\ &\geq \beta_k^{n_{k+1}} R_{k+1}\left(\frac{5}{4}\right)^{n_{k+1}}.\end{aligned}$$

By equation (4.9), we have $\beta_k \cdot \frac{5}{4} \geq \frac{6}{5}$. Therefore, since $N \geq 5$, we have

$$\min_{|z|=(5/4)R_k} |f_N(z)| \geq \left(\tfrac{6}{5}\right)^{n_{k+1}} R_{k+1} > 8R_{k+1}. \tag{6.2}$$

It follows from equations (6.2) and (6.1) that

$$f_N\left(|z| = \tfrac{5}{4}R_k\right) \subset A\left(8R_{k+1}, \tfrac{1}{8}R_{k+2}\right) \subset B_{k+1}. \tag{6.3}$$

As discussed at the beginning, this proves the claim. □

LEMMA 6.4. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, if $k \geq 1$, then*

$$f_N\left(A\left(\tfrac{1}{4}R_k, \tfrac{2}{5}R_k\right)\right) \subset B_k.$$

Proof. We consider the cases of $k \geq 2$ and $k = 1$ separately.

When $k \geq 2$, equation (4.34) implies that $f_N(|z| = \frac{1}{4}R_k) \subset B_k$. By equation (4.23), we have $\max_{|z|=(2/5)R_k} |f_N(z)| \leq \frac{1}{4}R_{k+1}$. Next, we observe that by Lemma 4.8,

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$$\min_{|z|=(2/5)R_k} |f_N(z)| \geq \min_{|z|=(2/5)R_k} \beta_k^{n_k} C_k |z|^{n_k} = \beta_k^{n_k} C_k \big(\tfrac{4}{5}\big)^{n_k} \big(\tfrac{1}{2} R_k\big)^{n_k} = \beta_k^{n_k} R_{k+1} \big(\tfrac{4}{5}\big)^{n_k}$$
$$\geq \beta_k^{n_k} \big(\tfrac{4}{10}\big)^{n_k} R_k^{n_k - 1} R_k.$$

Since $N \geq 5$, by equation (4.9) and Lemma 4.8, we have

$$\beta_k \tfrac{4}{10} R_k^{1/2} \geq 4. \tag{6.4}$$

Therefore, we obtain

$$\min_{|z|=(2/5)R_k} |f_N(z)| \geq 4^{n_k} R_k > 8R_k. \tag{6.5}$$

It follows that

$$f_N\big(|z| = \tfrac{2}{5} R_k\big) \subset A\big(8R_k, \tfrac{1}{8} R_{k+1}\big) \subset B_k \tag{6.6}$$

whenever $k \geq 2$. Therefore, the case of $k \geq 2$ follows from part (2) of Lemma 4.16 and Proposition 4.11.

For the $k = 1$ case, we use slightly different estimates. First, notice that by following a similar argument as above, but applying Lemma 4.15, we obtain

$$\min_{|z|=(2/5)R_1} |f_N(z)| \geq \tfrac{1}{2} 4^{n_1} R_1 > 8R_1.$$

This, combined with equation (4.25), allows us to conclude that

$$f_N\big(|z| = \tfrac{2}{5} R_1\big) \subset B_1. \tag{6.7}$$

Next, by following similar reasoning as in equation (5.16), except this time applying Lemma 4.15, we have

$$\min_{|z|=(1/4)R_1} |f_N(z)| \geq \frac{1}{4} \beta_1^{M_N} \left(\frac{r_N^{1/2}}{4}\right)^{M_N} r_N^{M_{N-2}} r_N = \frac{1}{4} \left(\frac{\beta_1 r_N^{1/2}}{4}\right)^{M_N} r_N^{M_{N-2}} r_N.$$

By equation (4.9) and Lemma 4.8, along with similar reasoning as equation (5.17), we have

$$\min_{|z|=(1/4)R_1} |f_N(z)| > 16 r_{N-1}^2 r_N = 16 r_{N-1}^2 R_1^2 > 4R_1. \tag{6.8}$$

So by equation (6.8) and the maximum principle used with equation (4.25), we have

$$f_N\big(|z| = \tfrac{1}{4} R_1\big) \subset B_1. \tag{6.9}$$

The $k = 1$ case now follows from Proposition 4.11 and part (2) of Lemma 4.16. □

LEMMA 6.5. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, if $k \geq 1$, we have*

$$f_N\big(|z| = \tfrac{3}{5} R_k\big) \subset B_{k+1}. \tag{6.10}$$

Proof. We again must argue the $k = 1$ and $k \geq 2$ cases separately.

When $k \geq 2$, we have by Lemma 4.13 that

$$\max_{|z|=(3/5)R_k} |f_N(z)| \leq \alpha_k^{n_k} C_k \big(\tfrac{3}{5} R_k\big)^{n_k} = \alpha_k^{n_k} C_k \big(\tfrac{6}{5}\big)^{n_k} \big(\tfrac{1}{2} R_k\big)^{n_k} = \alpha_k^{n_k} \big(\tfrac{6}{5}\big)^{n_k} R_{k+1}.$$

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By equation (4.9), we have $\alpha_k \frac{6}{5} \leq \frac{7}{5}$. Therefore, since $N \geq 5$, we obtain

$$\frac{(\alpha_k(6/5))^{n_k} R_{k+1}}{R_{k+2}} \leq \frac{(7/5)^{n_k} R_{k+1}}{2^{-n_{k+1}} R_{k+1}^{n_k+1}} = \left(\frac{28}{5R_{k+1}}\right)^{n_k} < \left(\frac{1}{4}\right)^{n_k}.$$

It follows that

$$\max_{|z|=(3/5)R_k} |f_N(z)| \leq \left(\tfrac{1}{4}\right)^{n_k} R_{k+2} < \tfrac{1}{8} R_{k+2}. \tag{6.11}$$

Therefore, we have by equations (4.24) and (6.11) that

$$f_N\big(|z| = \tfrac{3}{5} R_k\big) \subset A\big(8R_{k+1}, \tfrac{1}{8} R_{k+2}\big) \subset B_{k+1}. \tag{6.12}$$

For the $k = 1$ case, the arguments are similar, and by using Lemma 4.15 and requiring $N \geq 5$, we obtain

$$\max_{|z|=(3/5)R_1} \leq 2\big(\tfrac{1}{4}\big)^{n_k} R_3 < \tfrac{1}{8} R_3. \tag{6.13}$$

Therefore, by equations (4.26) and (6.13), we obtain

$$f_N\big(|z| = \tfrac{3}{5} R_1\big) \subset A\big(8R_2, \tfrac{1}{8} R_3\big) \subset B_3. \tag{6.14}$$

The lemma now follows from equations (6.12) and (6.14). □

COROLLARY 6.6. *For all $k \geq 1$, we have $f_N(V_k) \subset B_k \cup A_{k+1} \cup B_{k+1}$.*

Proof. By equations (6.6) and (6.7), along with equations (6.12) and (6.14), we have $f_N(|z| = \frac{2}{5} R_k) \subset B_k$ and $f_N(|z| = \frac{3}{5} R_k) \subset B_{k+1}$ for all $k \geq 1$. Therefore, by part (2) of Lemma 4.16, we have $f_N(V_k) \subset B_k \cup A_{k+1} \cup B_{k+1}$, as desired. □

The following lemma asserts that f_N is conformal on every petal $P_k \subset \mathcal{P}_k$, with large expansion.

LEMMA 6.7. *There exists an $M \in \mathbb{N}$ and a constant $\lambda > 0$ so that for all $N \geq M$, for all $k \geq 1$, and for all zeros w_j^k in A_k, $j = 1, \ldots, n_k$, the mapping*

$$f_N : B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k) \to f_N(B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k)) \tag{6.15}$$

is conformal. Moreover, we have

$$B\left(w_j^k, \frac{R_k}{2^{n_k}}\right) \subset B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k) \tag{6.16}$$

and

$$B(0, 4R_{k+1}) \subset f_N\left(B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right). \tag{6.17}$$

Proof. Note that by Lemmas A.19 and 4.3, there exist constants $\lambda > 0$ and $\delta > 0$ so that

$$B(0, \delta C_k R_k^{n_k}) \subset f_N(B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k)) \subset B\big(0, \tfrac{1}{2} C_k R_k^{n_k}\big). \tag{6.18}$$

Moreover, Lemmas A.19 and 4.3 imply that the mapping in equation (6.15) is injective, and therefore conformal. Next, note that

$$\frac{2^{-n_k} R_k}{\lambda(\exp(\pi/n_k) - 1)R_k} \le \frac{2^{-n_k}}{\lambda\pi/n_k} = \frac{n_k}{2^{n_k}\pi\lambda} \xrightarrow{k\to\infty} 0. \tag{6.19}$$

Therefore, there exists M so that for all $N \ge M$, we have equation (6.15) and $B(w_j^k, R_k/2^{n_k}) \subset B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k)$. It remains to verify that equation (6.17) holds.

To see this, note that by Theorem A.1 and equation (6.18), we have

$$\begin{aligned} |(f_N)'(w_j^k)| &\ge \frac{1}{4}\frac{\operatorname{dist}(0, \partial B(0, \delta C_k R_k^{n_k}))}{\operatorname{dist}(w_j^k, \partial B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k))} \\ &= \frac{\delta}{4\lambda}\frac{C_k R_k^{n_k}}{(\exp(\pi/n_k) - 1)R_k} \\ &\ge \frac{\delta}{8\lambda\pi} n_k C_k R_k^{n_k - 1} \\ &= \frac{\delta}{8\lambda\pi} 2^{n_k} n_k R_{k+1} R_k^{-1}. \end{aligned}$$

Therefore, if w_j^k is a zero of f_N in A_k, we have

$$|(f_N)'(w_j^k)| \ge \frac{\delta}{8\lambda\pi} 2^{n_k} n_k R_{k+1} R_k^{-1}. \tag{6.20}$$

Next, consider the branch of the inverse $f_N^{-1} : B(0, \delta C_k R_k^{n_k}) \to D'$, where

$$D' \subset B(w_j^k, \lambda(\exp(\pi/n_k) - 1)R_k).$$

Since $\delta > 0$ is fixed, there exists a perhaps larger $M \in \mathbb{N}$ so that for all $N \ge M$, we have $\delta 2^{n_1} > 4$. Therefore,

$$\delta C_k R_k^{n_k} = \delta 2^{n_k} C_k \big(\tfrac{1}{2} R_k\big)^{n_k} = \delta 2^{n_k} R_{k+1} > 2^{n_k - 1} 4 R_{k+1} > 4R_{k+1}. \tag{6.21}$$

We can further deduce from equation (6.21) that the modulus of the annulus $B(0, \delta C_k R_k^{n_k}) \setminus \overline{B(0, 4R_{k+1})}$ is bounded below by some fixed constant independent of k. Denote $D = f_N^{-1}(B(0, 4R_{k+1})) \subset D'$. By applying Corollary A.6 to $f_N^{-1} : B(0, 4R_{k+1}) \to D$, we see that there exists a constant $L' > 0$ such that

$$\frac{4R_{k+1}}{L'}\frac{1}{|f'(w_j^k)|} \le r_{D,w_j^k} \le R_{D,w_j^k} \le L'\frac{4R_{k+1}}{|f'(w_j^k)|}. \tag{6.22}$$

Since the modulus of $B(0, \delta C_k R_k^{n_k}) \setminus \overline{B(0, 4R_{k+1})}$ is bounded below by some fixed constant independent of k, the constant L' is independent of k and w_j^k. By perhaps increasing M, we have that for all $N \ge M$,

$$4R_{k+1}L'|f'(w_j^k)|^{-1} \le 4R_{k+1}L'\frac{8\lambda\pi}{\delta n_k} 2^{-n_k} R_{k+1}^{-1} R_k \le \frac{R_k}{2^{n_k}}. \tag{6.23}$$

Therefore, by equations (6.22) and (6.23), and equation (A.6), we have

$$D \subset B\left(w_j^k, L' \frac{4R_{k+1}}{|f'(w_j^k)|}\right) \subset B\left(w_j^k, \frac{R_k}{2^{n_k}}\right).$$

This proves equation (6.17), which is exactly what we wanted to show. □

LEMMA 6.8. *For all* $k \geq 1$, $f_N(A(\frac{3}{5}R_k, \frac{5}{4}R_k) \setminus \overline{\bigcup_{j=1}^{n_k} B(w_j^k, R_k/2^{n_k})}) \subset B_{k+1}$.

Proof. First, observe that every connected component of the boundary of $A(\frac{3}{5}R_k, \frac{5}{4}R_k) \setminus \overline{\bigcup_{j=1}^{n_k} B(w_j^k, R_k/2^{n_k})}$ is mapped inside of B_{k+1} by f_N. Indeed, by Lemma 6.5, $f_N(|z| = \frac{3}{5}R_k) \subset B_{k+1}$, and by equation (6.3), $f_N(|z| = \frac{5}{4}R_k) \subset B_{k+1}$. The rest of the connected components of the boundary of $A(\frac{3}{5}R_k, \frac{5}{4}R_k) \setminus \overline{\bigcup_{j=1}^{n_k}(w_j^k, R_k/2^{n_k})}$ are the boundaries of the petals $P_k \subset \mathcal{P}_k$. By equation (6.17),

$$f_N\left(\partial B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right) \subset \{|z| \geq 4R_{k+1}\}. \tag{6.24}$$

By equations (4.37) and (6.18), we have

$$f_N\left(\partial B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right) \subset B\left(0, \frac{1}{2}C_k R_k^{n_k}\right) \subset B\left(0, \frac{1}{4}R_{k+2}\right). \tag{6.25}$$

Equations (6.24) and (6.25) imply

$$f_N\left(\partial B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right) \subset B_{k+1}. \tag{6.26}$$

By Proposition 4.11, f_N has no additional zeros in $A(\frac{3}{5}R_k, \frac{5}{4}R_k) \setminus \overline{\bigcup_{j=1}^{n_k} B(w_j^k, R_k/2^{n_k})}$. The result now follows from the maximum principle and minimum principle for non-zero holomorphic functions. □

With Lemmas 6.3–6.8 in hand, we can now deduce the following about the Julia set of f_N.

THEOREM 6.9. *There exists* $M \in \mathbb{N}$ *so that if* $N \geq M$, *then for all* $k \geq 1$,

$$(\mathcal{J}(f_N) \cap A_k) \subset \left(\bigcup_{j=1}^{n_k} B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right) \cup V_k.$$

Proof. We will show that all other points get mapped to B_k or B_{k+1}, so that they belong to the Fatou set of f_N by Lemma 6.1. Suppose that $z \in A_k \cap \mathcal{J}(f_N)$, but $z \notin \mathcal{P}_k$. Then, by Lemma 6.3, $z \notin A(\frac{5}{4}R_k, 4R_k)$, and by Lemma 6.4, $z \notin A(\frac{1}{4}R_k, \frac{2}{5}R_k)$. Similarly, Lemma 6.8 and the assumption that $z \notin \mathcal{P}_k$ shows that $z \notin A(\frac{3}{5}R_k, \frac{5}{4}R_k)$. Finally, observe that we cannot have $z \in \{|z| = \frac{2}{5}R_k\}$, $z \in \{|z| = \frac{3}{5}R_k\}$, or $z \in \{|z| = \frac{5}{4}R_k\}$ by equations (6.6) and (6.7), (6.10), and (6.3), respectively. Since $z \in A_k \cap J(f_N)$, but $z \notin \mathcal{P}_k$, it follows that $z \in A(\frac{2}{5}R_k, \frac{3}{5}R_k) = V_k$, which proves the theorem. □

The same argument of the proof of Theorem 6.9 yields the following useful result.

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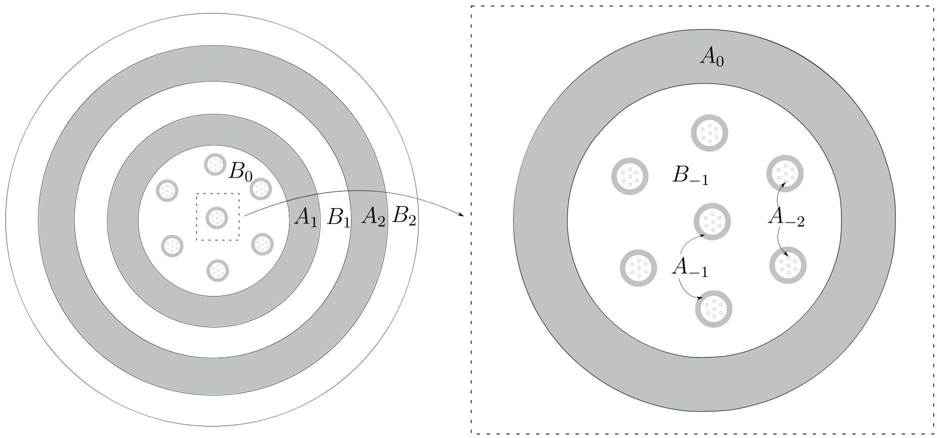

FIGURE 8. Illustration of Definition 7.1 of A_k, B_k for negative k. Each connected component of A_0 maps conformally onto A_1; therefore, each connected component of A_0 contains 2^N many connected components of A_{-1}. The picture repeats itself as we zoom in. The set B_0 is the region between the outer boundaries of the components of A_0 and the inner boundary of A_1.

LEMMA 6.10. *Suppose that $z \in A_k$ and $f_N(z) \in A_j$ for some $k \geq 1$ and some $j \in \mathbb{Z}$. Then,*

$$z \in V_k \cup \left(\bigcup_{j=1}^{n_k} B\left(w_j^k, \frac{R_k}{2^{n_k}}\right)\right).$$

7. *Conformal mapping behavior*

We begin §7 by defining A_k, B_k for negative indices k, simply by pulling back (by f) the definition of A_k, B_k for positive k (see Figure 8). With this definition, we will deduce that if $z \in \mathcal{J}(f)$, then either z maps to the filled Julia set of the polynomial like mapping (f, U, V), or all iterates of z lie in $\bigcup_{k\in\mathbb{Z}} A_k$. We have already estimated the dimension of the Julia set of (f, U, V) in §5, so in this section, we study the set of points that have orbits lying in $\bigcup_{k\in\mathbb{Z}} A_k$.

As in Definition 5.4(1), we will use the notation $D = B(0, \frac{1}{4}R_1)$ for the remainder of this section.

Definition 7.1. We define

$$A_0 = \{z : z \in D,\ f_N(z) \in A_1\}. \tag{7.1}$$

For integers $k \geq 1$, we define

$$A_{-k} = \{z : z, \ldots, f_N^k(z) \in D,\ f_N^{k+1}(z) \in A_1\}. \tag{7.2}$$

We define B_k and V_k for $k \leq 0$ in the exact same way.

Notation 7.2. We will use the following notation in this section.

(1) The polynomial-like mapping (f_N, U, V) will be the one defined as in Lemma 5.6.

(2) The filled Julia set of (f_N, U, V) will be denoted as E.

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LEMMA 7.3. *Let $z \in \mathbb{C}$. Then, exactly one of the following is true.*

(1) *We have $z \in E$.*
(2) *There exists $k \in \mathbb{Z}$ so that $z \in B_k$.*
(3) *There exists $k \in \mathbb{Z}$ so that $z \in A_k$.*

Proof. The filled Julia set E of (f_N, U, V) is forward invariant and contained in D. Therefore, if $z \in E$, it is impossible for $z \in A_k$ or $z \in B_k$ for any integer k by Definition 7.1. So we will suppose that $z \notin E$. By Definition 4.14, if $|z| > \frac{1}{4}R_1$, then there exists $k \geq 0$ so that z must belong to exactly one of B_k or A_k. Therefore, we only have to focus our attention on the case $|z| \leq \frac{1}{4}R_1$.

Suppose first that $z \in U \subset \overline{D}$, recalling that $U \subset \overline{D}$ by Lemma 5.6. Then, since $z \notin E$, there exists a smallest integer $l \geq 1$ so that $f_N^l(z) \notin U$. First, we consider the case that $|f_N^l(z)| > R_1/4$. Then, by equation (6.9), $f_N(\partial D) \subset B_1$, so by our choice of l and the maximum principle, we must have either $f_N^l(z) \in A_1$ or $f_N^l(z) \in B_1$. It follows from Definition 7.1 that either $z \in A_{1-l}$ or $z \in B_{1-l}$.

Next, we consider the case where $|f_N^l(z)| \leq \frac{1}{4}R_1$, so that $f^l(z) \in \overline{D} \setminus U$. Observe that by Definition 5.4, $f_N(\partial U) = \{z : |z| = 16r_{N-1}^2 R_1\} \subset B_1$. Indeed, we certainly have $16r_{N-1}^2 R_1 > 4R_1$, and we may argue using Lemma 4.8 that $16r_{N-1}^2 R_1 < R_2/4$. We also have $f_N(\partial D) \subset B_1$ by equation (6.9). Therefore, by Proposition 4.11, we must have $f^{l+1}(z) \in B_1$, so that $z \in B_{-l}$. □

LEMMA 7.4. *Suppose that $z \in f_N^{-1}(A_k)$ for some integer $k \in \mathbb{Z}$. Then, $z \in A_j$ for some integer j.*

Proof. By assumption, we have $f_N(z) \in A_k$ for some integer k. By Lemma 7.3, either $z \in E$, $z \in A_j$ for some integer j, or $z \in B_j$ for some integer j. We cannot have $z \in E$, because $f_N(E) \subset E$. If $z \in B_j$ and $j \leq 0$, then by Definition 7.1, we must have $f_N(z) \in B_{j+1}$. If $z \in B_j$ for some $j \geq 1$, then $f_N(z) \in B_{j+1}$ by Lemma 4.18. Therefore, we cannot have $z \in B_j$ for any integer j. The only remaining possibility is that we must have $z \in A_j$ for some integer j. □

Notation 7.5. For an open set $\Omega \subset \mathbb{C}$, we let $\widehat{\Omega}$ denote the union of Ω and its bounded complementary components.

Definition 7.6. A domain $A \subset \mathbb{C}$ is a topological annulus if the complement of A has two connected components. We say A is a Jordan annulus if the boundary of A consists of two Jordan curves.

Definition 7.7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. We define $CV(f)$ to be the set of all critical values of f, $AV(f)$ the set of asymptotic values of f, and $SV(f) = \overline{CV(f) \cup AV(f)}$ to be the set of all singular values of f. We define the postsingular set of f by

$$P(f) = \overline{\{f^n(z) : z \in SV(f),\ n \geq 0\}}. \tag{7.3}$$

We define the postcritical set of f in a similar way.

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LEMMA 7.8. *For all sufficiently large N, the postsingular set of f_N coincides with the postcritical set, and is a subset of $\bigcup_{k\geq 1} B_k$.*

Proof. By Lemma 3.18, f_N has no asymptotic values. By Proposition 4.19, for all sufficiently large N, all of the critical points of f_N with $|z| \geq \frac{1}{4}R_1$ are mapped into B_{k+1} for some $k \geq 1$. By Lemma 5.12, all of the critical points of f_N with $|z| \leq \frac{1}{4}R_1$ are mapped into B_1. Therefore, $SV(f_N) \subset \bigcup_{k\geq 1} B_k$. It follows from Lemma 4.18 that $P(f_N) \subset \bigcup_{k\geq 1} B_k$ as well. □

Next, we note some of the basic covering map behavior on the annuli A_k.

LEMMA 7.9. *For all $k \leq 1$, we let Z_k denote A_k or V_k.*

(1) *Let $k \leq 0$ and suppose that $\widehat{Z}'_k$ is a connected component of $\widehat{Z}_k$, and $\widehat{Z}'_{k+1}$ is a connected component of $\widehat{Z}_{k+1}$ so that $f_N(\widehat{Z}'_k) = \widehat{Z}'_{k+1}$. Then, $f_N : \widehat{Z}'_k \to \widehat{Z}'_{k+1}$ is conformal, and every connected component of $\widehat{Z}_k$ is a Jordan domain.*
(2) *Let $k \leq 0$ and suppose that Z'_k is a connected component of Z_k, and Z'_{k+1} is a connected component of Z_{k+1} so that $f_N(Z'_k) = Z'_{k+1}$. Then, $f_N : Z'_k \to Z'_{k+1}$ is conformal, and every connected component of Z_k is a Jordan annulus.*
(3) *For all $k \leq 1$, Z_k consists of exactly $2^{(-k+1)N}$ many connected components.*

Proof. To prove part (1), note that by Lemma 7.8, there are no critical values of f_N contained in $\overline{B(0, 4R_1)}$. Therefore, the claim follows by Lemma A.15, which further implies that each connected component of $\widehat{Z}_k$ for $k \leq 0$ is a Jordan domain.

Part (2) follows immediately from part (1).

To see part (3), note that Lemma A.15 implies that Z_0 consists of 2^N many connected components. Each connected component of $\widehat{Z}_0$ is mapped conformally onto $\widehat{Z}_1$ by f_N. Therefore, $f_N(\widehat{Z}_0)$ contains the 2^N many connected components of Z_0, and it follows that each connected component of $\widehat{Z}_0$ contains 2^N many connected components of Z_{-1}. Therefore, Z_{-1} consists of 2^{2N} many connected components. By proceeding similarly, we deduce that every connected component of $\widehat{Z}_{k+1}$ for $k \leq -1$ contains 2^N many connected components of Z_k, and Z_k therefore must have $2^{N(-k+1)}$ many connected components. □

LEMMA 7.10. *Let $k \geq 1$. Let $W = f_N^{-1}(A_{k+1}) \cap V_k$, which is non-empty by Lemma 4.17. Then, $f_N : W \to A_{k+1}$ is a degree n_k covering map.*

Proof. When $k \geq 2$, observe that on $\phi_N^{-1}(W)$, we have $f_N \circ \phi_N = C_k z^{n_k}$ and that the mapping $f_N \circ \phi_N : \phi_N^{-1}(W) \to A_{k+1}$ is a degree n_k covering map. Since ϕ_N is quasiconformal, it follows that $f_N : W \to A_k$ is a degree n_k covering map as well. The $k = 1$ case is similar, except this time, we use the fact that on $\phi_N^{-1}(W)$, $f_N \circ \phi_N(z) = q_N(z)$ is a degree n_1 covering map to conclude that $f_N \circ \phi_N : \phi_N^{-1}(W) \to A_2$ is a degree n_1 covering map. □

Recall that by Lemma 6.1, for $k \geq 1$, the annuli B_k belong to the Fatou set of f_N. This motivates the following definition.

Definition 7.11. For $k \geq 1$, we define Ω_k to be the Fatou component that contains B_k.

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Remark 7.12. It is readily verified that each Ω_k is multiply connected. By Definition 3.9, we have $f_N(0) = 0$, and 0 is in the Julia set of f_N since it is a repelling fixed point. Indeed, by Definition 3.9, we have

$$|f_N'(0)| = |q_N'(0)| \cdot |\phi_N'(0)|.$$

We verify from equation (5.1) that $q_N'(0) = r_N$, and by Theorem 3.17, for all N sufficiently large, we may assume that $\frac{1}{2} \leq |\phi_N'(0)| \leq \frac{3}{2}$. Therefore, for all N sufficiently large, by Lemma 4.8, we have

$$|f_N'(0)| \geq \tfrac{1}{2} r_N > 1.$$

Therefore, 0 is a repelling fixed point for f_N. Since $B_k \subset \Omega_k$ for all $k \geq 1$, it follows that Ω_k cannot be simply connected.

LEMMA 7.13. *Suppose that $j, k \geq 1$ and $j \neq k$. Then, $\Omega_j \neq \Omega_k$.*

Proof. By Lemma 4.18, we have $f_N(B_k) \subset B_{k+1}$ for all $k \geq 1$. Therefore, we must have $f_N(\Omega_k) \subset \Omega_{k+1}$ for all $k \geq 1$, and since f_N has no asymptotic values by Lemma 3.18, we actually have

$$f_N(\Omega_k) = \Omega_{k+1} \tag{7.4}$$

for all $k \geq 1$, (see [**Her98**, Corollary 2]).

Suppose for the sake of contradiction that we have $\Omega_j = \Omega_k$ for some $k > j \geq 1$. It follows from Definition 7.11 that $\Omega_j = \Omega_{j+1}$. This, combined with equation (7.4), implies that Ω_j is unbounded. This contradicts [**Bak75**, Theorem 1]. □

Definition 7.14. We define

$$A := \bigcup_{k \in \mathbb{Z}} A_k \tag{7.5}$$

and

$$X := \{z : f^n(z) \in A, n = 0, 1, \ldots\}. \tag{7.6}$$

LEMMA 7.15. *There exists $M \in \mathbb{N}$ so that for all $N \geq M$, we have $f_N^{-1}(A) \subset A$.*

Proof. By Lemma 7.4, we have $f_N^{-1}(A_k) \subset A$ for each integer k. Since

$$f_N^{-1}(A) = \bigcup_{k \in \mathbb{Z}} f_N^{-1}(A_k),$$

the conclusion follows immediately. □

Definition 7.16. Recall that the filled Julia set of the polynomial-like mapping (f_N, U, V) from Lemma 5.6 is denoted as E. Define

$$E' = \bigcup_{n=0}^{\infty} f_N^{-n}(E). \tag{7.7}$$

LEMMA 7.17. *The Hausdorff dimension of E and E' are the same.*

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Proof. This follows immediately from Definition 7.16 along with equations (A.11) and (A.12). □

The following lemmas give us some basic rules for how orbits of points in X behave.

LEMMA 7.18. *The Julia set of $f_N : \mathbb{C} \to \mathbb{C}$ is a subset of $E' \cup X$.*

Remark 7.19. In fact, we actually have $\mathcal{J}(f_N) = E' \cup X$, and this will become apparent in the later sections.

Proof. Since (f_N, U, V) is a polynomial-like mapping and E is its filled Julia set, E coincides with the closure of the repelling periodic cycles for (f_N, U, V). These are also repelling periodic cycles for f_N viewed as an entire function, so $E \subset \mathcal{J}(f_N)$. Since $\mathcal{J}(f_N)$ is backwards invariant, it follows immediately from Definition 7.16 that $E' \subset \mathcal{J}(f_N)$ as well.

Now, suppose that $z \in \mathcal{J}(f_N)$, but $z \notin E'$. The set E' is both forward and backward invariant. Therefore, for all $n \geq 0$, we have $f_N^n(z) \notin E'$ and, in particular, we have $f_N^n(z) \notin E$. By Lemma 6.1 and Definition 7.1, if $f_N^n(z) \in B_k$ for some $n \geq 0$ and $k \in \mathbb{Z}$, we must have z in the Fatou set of f_N. Therefore, by Lemma 7.3, we must have $f_N^n(z) \in A$ for all $n \geq 0$ so that $z \in X$, as desired. □

LEMMA 7.20. *Suppose that $z \in A_k$ and $f_N(z) \in A_j$ for some $j \in \mathbb{Z}$. Then, $j \leq k+1$.*

Proof. First, suppose that $z \in A_k$ for some $k \leq 0$. Then, by Definition 7.1, $f_N(z) \in A_{k+1}$, so that $j = k+1$.

Next, suppose that $z \in A_k$ for some $k \geq 1$. Then, by equation (4.31), we have $f_N(|z| = 4R_k) \subset B_{k+1}$. Therefore, by the maximum principle for holomorphic functions, we must have $f_N(B(0, 4R_k)) \subset \widehat{B}_{k+1}$. So if $f_N(z) \in A_j$, we must have $j \leq k+1$. □

LEMMA 7.21. *Suppose that $z \in A_k$ and $f_N(z) \in A_j$ for some $k \geq j$. Then, $k \geq 1$, and there exists a petal $P_k \subset \mathcal{P}_k$ such that $z \in P_k$.*

Proof. If $k \leq 0$, we have $f_N(z) \in A_{k+1}$ by Definition 7.1, so we must have $k \geq 1$.

By Lemma 6.10, we must have $z \in V_k$ or we must have $z \in P_k$ for some petal $P_k \subset \mathcal{P}_k$. If $z \in V_k$, then by Corollary 6.6, we must have $f_N(z) \in B_k \cup A_{k+1} \cup B_{k+1}$. Since $f_N(z) \in A_j$ and $j \leq k$, we must have $z \in P_k$ for some petal $P_k \subset \mathcal{P}_k$. □

LEMMA 7.22. *Suppose that $\Omega \subset A_{k+1}$ is a Jordan domain for some $k \geq 1$. Then, the number of connected components of $f_N^{-1}(\Omega)$ that are contained in A_k is n_{k+1}, and each component is a Jordan domain.*

Proof. By Lemma 7.8, there are no singular values of f_N in $\overline{U}$. Therefore, $f_N^{-1}(U)$ is the disjoint union of Jordan domains. By Lemma 7.10, there are exactly n_k many connected components of $f_N^{-1}(U)$ contained inside of V_k. By Lemma 6.7, there is exactly one connected component of $f_N^{-1}(U)$ contained in each petal $P_k \subset \mathcal{P}_k$, which

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yields n_k many more connected components of $f_N^{-1}(U)$. There are no other connected components contained inside of A_k by Lemma 6.10. Therefore, the total number of connected components contained inside of A_k is $n_k + n_k = n_{k+1}$. □

Definition 7.23. If a point $z \in X$, then for each $n \geq 0$, $f_N^n(z) \in A_{k(n,z)}$ for some integer $k(n, z)$. By Lemma 7.20,

$$k(z, n+1) \leq k(z, n) + 1. \tag{7.8}$$

We call the sequence $(k(z, n))_{n=0}^{\infty}$ the orbit sequence of z.

This inspires the following definition, which will be crucial to the proof of Theorem 1.1.

Definition 7.24. Suppose that $z \in X$. For $n \geq 1$, if $k(z, n) < k(z, n-1) + 1$, we will say that z *moves backwards* on the nth iterate. We will sometimes omit the iterate n and just say that z moves backwards. We let Y denote the set of all points in X that move backwards for infinitely many distinct iterates, and we let Z denote the set of all points in X that move backwards for only finitely many iterates.

Remark 7.25. Suppose that $z, f(z), \ldots, f^n(z) \in A$, but we perhaps do not have $f^{n+1}(z) \in A$. Then, we may still define the finite orbit sequence $(k(z, j))_{j=0}^{n}$. Therefore, we may still speak of a point z moving backwards for the iterates where its finite orbit sequence is defined.

Remark 7.26. Let W be connected with $W \subset A$ and suppose that $f_N^j(W) \subset A$ for $j = 1, \ldots, n$. Then, since W is connected, each point $z \in W$ has the same finite orbit sequence $(k(z, j))_{j=0}^{n}$. In these situations, we will say that the set W moves backwards whenever any of the points $z \in W$ move backwards.

Remark 7.27. It follows immediately from Definition 7.24 that

$$X = Y \cup Z.$$

8. *A first dimension estimate*

We deduced in §7 that $\mathcal{J}(f) \subset E' \cup X$. We have already estimated the dimension of E', and we now move on to estimating the dimension of $X = Y \sqcup Z$. This section will be devoted to estimating the dimension of Y (the set of points that move backwards infinitely often). In fact, we will show that the dimension of Y can be taken arbitrarily close to 0. Although the details are somewhat technical, the idea is simple: we build a sequence of coverings $\mathcal{C}_m$ of Y by pulling back the annuli A_k under appropriate branches of the inverse of f^m. The diameters of elements in $\mathcal{C}_m$ are estimated by standard distortion estimates for conformal mappings.

More precisely, our goal in this section is to show that for any $t > 0$, there exists some $M \in \mathbb{N}$ so that for all $N \geq M$, we have $\dim_H(Y) \leq t$. We will start by formally constructing a sequence of coverings $\mathcal{C}_m$ of $Y \cap A_1$, for $m \geq 0$, using the dynamics of f_N. Our initial covering $\mathcal{C}_0$ will have exactly one element, the annulus A_1. We first describe how to construct $\mathcal{C}_1$ from $\mathcal{C}_0$ (see Figure 9).

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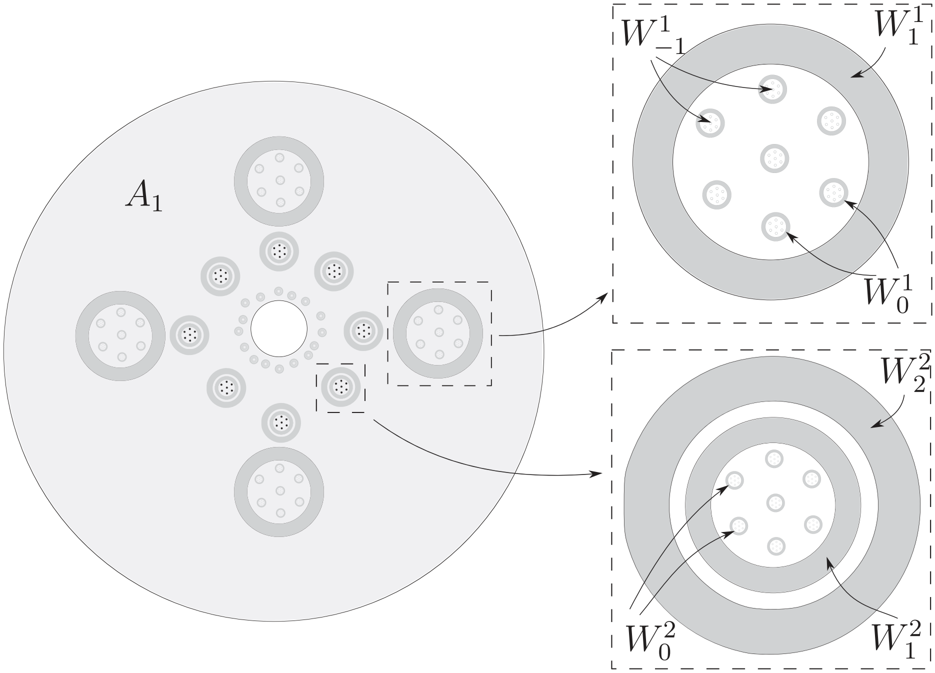

FIGURE 9. Illustration of the covering $\mathcal{C}_1$ and the notation W^n_k for elements of $\mathcal{C}_1$ (see Notation 8.3). The elements of $\mathcal{C}_m$ for $m > 1$ are obtained by essentially placing a scaled-down copy of $\mathcal{C}_{m-1}$ in each annulus in the covering $\mathcal{C}_{m-1}$.

LEMMA 8.1. *There exists a collection of sets $\mathcal{C}_1$ that has the following properties.*

(1) *Every element in $\mathcal{C}_1$ is a subset of an element in $\mathcal{C}_0$.*

(2) *$\mathcal{C}_1$ is a countable cover of $Y \cap A_1$.*

(3) *Let W be an element of $\mathcal{C}_1$. Then, there exists an integer $n \geq 1$ and an integer $k \in \mathbb{Z}$ such that $f_N^{n-1}(W) \subset A_n$, and $f_N^n(W)$ is a connected component of A_k for $k \leq n$.*

(4) *Every element of $\mathcal{C}_1$ moves backwards once.*

Proof. For each $z \in A_1 \cap Y$, by Definition 7.24 and equation (7.8), there is a smallest positive integer n so that $f_N^n(z) \in A_k$ for some $k \leq n$. We remark that it is possible that k is a non-positive integer. Let W denote the connected component of $f_N^{-n}(A_k)$ that contains z. The collection of all distinct components obtained by applying this procedure to all $z \in Y$ is denoted by $\mathcal{C}_1$. We check that the properties in the lemma hold.

(1) By Lemma 7.15, $W \subset A$, and since W is connected and contains $z \in A_1$, we have $W \subset A_1$.

(2) Any two elements of $\mathcal{C}_1$ are disjoint. Since A_1 is bounded and all elements of $\mathcal{C}_1$ are open, the collection $\mathcal{C}_1$ is countable.

(3) This follows from the construction of each W, since we chose the smallest positive integer n such that $f_N^n(z) \in A_k$ for $k \leq n$.

(4) By Lemma 7.15, $f_N^j(W) \subset A$ for $j = 0, \ldots, n$, so this claim follows since n is the smallest positive integer so that $f_N^n(z) \in A_k$ for $k \leq n$.

This proves the claim. □

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We now show how to construct $\mathcal{C}_m$ for $m > 1$. Our procedure is inductive and described in the following lemma.

LEMMA 8.2. *Let $m \geq 1$. Suppose there exists a collection of subsets $\mathcal{C}_m$ that satisfy the following properties.*

- *$\mathcal{C}_m$ is a countable cover of $Y \cap A_1$.*
- *Let W be an element of $\mathcal{C}_m$. Then, there exists an integer $n \geq 1$ and an integer $k \in \mathbb{Z}$ such that $f_N^n(W)$ is a connected component of A_k and $f_N^{n-1}(W) \subset A_j$ for some $j \geq k$. Moreover, W moves backwards for the mth time on the nth iterate.*

Then, there exists a collection of subsets $\mathcal{C}_{m+1}$ that satisfy the following properties.

(1) *$\mathcal{C}_{m+1}$ is a refinement of $\mathcal{C}_m$, i.e., every element in $\mathcal{C}_{m+1}$ is a subset of an element in $\mathcal{C}_m$.*
(2) *Each element of $\mathcal{C}_m$ contains countably many elements of $\mathcal{C}_{m+1}$, and $\mathcal{C}_{m+1}$ is a countable cover of $Y \cap A_1$.*
(3) *Every element of $\mathcal{C}_{m+1}$ moves backwards $m+1$ many times.*

Moreover, let W be an element of $\mathcal{C}_{m+1}$, and let $z \in Y$ satisfy $z \in W \subset W'$, where W' is an element of $\mathcal{C}_m$. Let $(k(z, j))_{j=0}^{\infty}$ denote the orbit sequence of z. Let n be the value such that $f_N^{n-1}(W) \subset f_N^{n-1}(W') \subset A_{k(z,n-1)}$, $f_N^n(W')$ is a connected component of $A_{k(z,n)}$, where $k(z, n) \leq k(z, n-1)$ and z moves backwards for the mth time on the nth iterate. Then, there exists a value $q \geq 1$ such that $f_N^{n+q-1}(W) \subset A_{k(z,n+q-1)}$, $f_N^{n+q}(W)$ is a connected component of $A_{k(z,n+q)}$, where $k(z, n+q) \leq k(z, n+q-1)$, and W moves backwards for the $m+1$st time on the $n+q$th iterate.

Proof. Choose $z \in Y \cap A_1$. Then, there exists an element W' of $\mathcal{C}_m$ containing z. Let n be the integer such that $f_N^n(W') = A_{k(z,n)}$, $k(z, n) \leq k(z, n-1)$, and z moves backwards for the mth time on the nth iterate. Then, since $z \in Y$, it must move backwards again. Therefore, there exists a smallest value $q \geq 1$ such that $f_N^{n+q}(z) \in A_{k(z,n+q)}$ and $k(z, n+q) \leq k(z, n+q-1)$. We let W denote the connected component of $f_N^{-(n+q)}(A_{k(z,n+q)})$ that contains z, and we let $\mathcal{C}_{m+1}$ denote the collection of all distinct components obtained by applying this procedure to all $z \in Y$. We now prove the desired properties.

(1) Let W be an element of $\mathcal{C}_{m+1}$. Then by construction, there exists some point $z \in Y \cap A_1$ contained inside of W. Let W' denote the element of $\mathcal{C}_m$ that contains z. Let n be the integer so that $f^n(W') = A_k$ for some k, where W' moves backwards for the mth time on the nth iterate. We must have $f^n(W) \subset A$, so since W is connected, we have $f^n(W) \subset A_k$ and it follows that $W \subset W'$.
(2) We already know $\mathcal{C}_m$ is countable and by the construction, $\mathcal{C}_{m+1}$ covers $Y \cap A_1$. Let W' be an element of $\mathcal{C}_m$, and let n be the integer such that W' moves backwards for the mth time on the nth iterate. Then, $f_N^n(W') = A_k$ for some integer k. If $k \geq 1$, then by equation (6.18), there exists countably many elements of $\mathcal{C}_{m+1}$ contained in W'. If $k \leq 0$ is negative, then $f_N^{n-k+1}(W') = A_1$ by Lemma 7.9, and we can apply the same reasoning for the $k \geq 1$ case to see that there exists countably many elements of $\mathcal{C}_{m+1}$ contained in W'. Since there are countably many elements W' in $\mathcal{C}_m$, the collection $\mathcal{C}_{m+1}$ is also countable.
(3) That every element of $\mathcal{C}_{m+1}$ moves backwards $m+1$ many times follows from the definition of n and the choice of q in the construction. □

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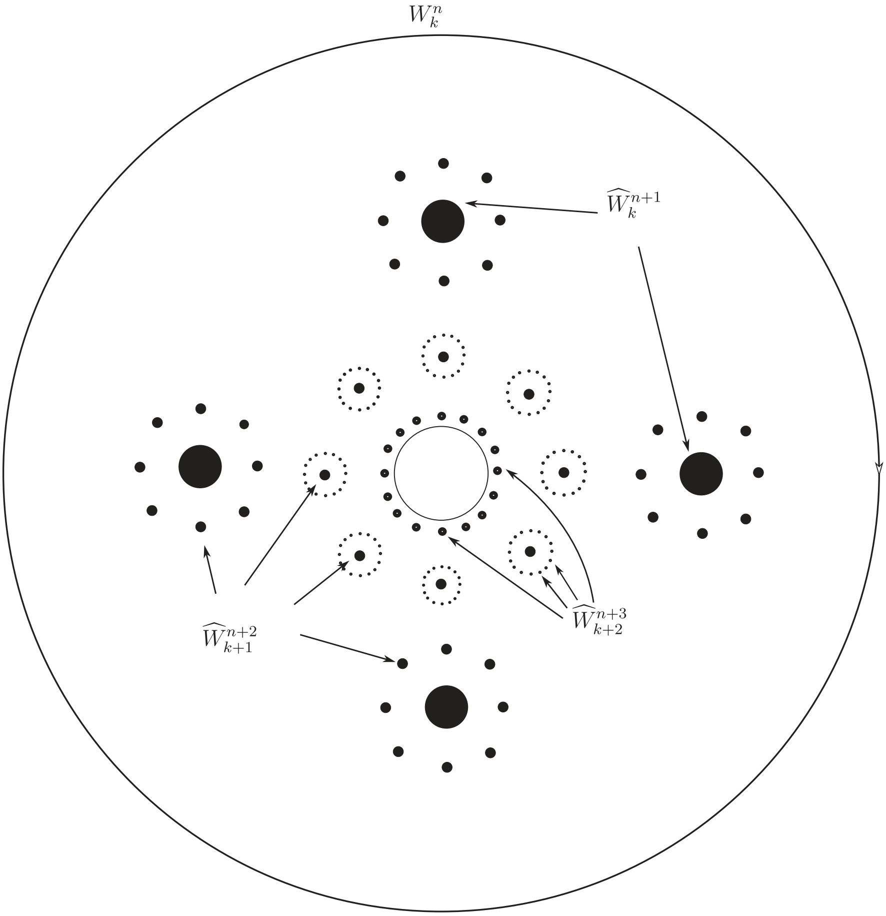

FIGURE 10. An illustration of components of the form $\widehat{W}_{k+q-1}^{n+q} \in \widehat{\mathcal{C}}_{m+1}$ contained inside of some $W_k^n \in \mathcal{C}_m$ for $q = 1, 2$, and 3. The component W_k^n is bounded by the innermost and outermost circle.

In the rest of our analysis, we will mostly be focused on understanding what happens when we refine from $\mathcal{C}_m$ to $\mathcal{C}_{m+1}$. Therefore, the following notation will be convenient (see Figures 9 and 10).

Notation 8.3. Let W' be an element of $\mathcal{C}_m$, and let $W \subset W'$ be an element of $\mathcal{C}_{m+1}$. We will denote W' as W_k^n, where $n \geq 1$ is the iterate where W' moves backwards for the mth time, and $f_N^n(W_k^n) = A_k$ for some integer k. We will say that W' is of the form W_k^n for $n \geq 1$ and $k \in \mathbb{Z}$. Likewise, we will denote W as W_j^{n+q}, where $n + q$ is the iterate where W moves backwards for the $m + 1$st time, and $f_N^{n+q}(W) = A_j$. We will say W is of the form W_j^{n+q} for $q \geq 1$.

The following lemma states that the mappings used to define the collections $\mathcal{C}_m$ are conformal with bounded distortion. We state it precisely below.

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LEMMA 8.4. *Let $m > 0$ and let W_k^n be an element of $\mathcal{C}_m$. Then, there exists a Jordan domain B containing W_k^n such that $f_N^n : B \to f_N^n(B)$ is conformal. Moreover:*

(1) *when $k \leq 0$, the modulus of $B \setminus \widehat{W}_k^n$ is bounded below by $(2\pi)^{-1}\log(2)$, and $f_N^n(B)$ is an element $\widehat{\sigma}_{1-k}$ from Remark 5.9;*

(2) *when $k \geq 1$, $f_N^n(B) = B(0, 4R_{k+1})$, and the modulus of $B \setminus \widehat{W}_k^n$ is bounded below by $(2\pi)^{-1}\log(2)$.*

Proof. Fix $m > 0$ and choose some $k \geq 1$. Let W_k^n be an element of $\mathcal{C}_m$. By following equation (6.21) and using the fact that $N \geq 5$, we deduce

$$\frac{\delta}{2} C_k R_k^{n_k} > 2^{n_{k-1}-1} 4R_{k+1} > 16R_{k+1}. \tag{8.1}$$

Therefore, we have

$$\widehat{A}_k = B(0, 4R_k) \subset B(0, 4R_{k+1}) \subset B\left(0, \frac{\delta}{2} C_k R_k^{n_k}\right) \subset B\left(0, \frac{\delta}{2} C_j R_j^{n_j}\right). \tag{8.2}$$

Let B denote the connected component of $f_N^{-n}(B(0, 4R_{k+1}))$ that contains W_k^n. By Lemma 7.21, $f_N^{n-1}(W_k^n)$ is the subset of a petal $P_j \subset A_j$ for some $j \geq k$. Since $f_N^n(\widehat{W}_k^n) = B(0, 4R_k)$, $f_N^{n-1}(\widehat{W}_k^n)$ contains a zero w of f_N and we can deduce by equation (8.2) that we have

$$f_N^{n-1}(B) \subset B(w, \lambda(\exp(\pi/n_j) - 1)R_j) \subset A_j.$$

Therefore, by equation (6.18), $f_N : f_N^{n-1}(B) \to B(0, 4R_{k+1})$ is conformal. Since $f_N^{n-1}(B) \subset A_j$, we have $f_N^l(B) \subset A$ for $l = 0, \ldots, n-1$ by Lemma 7.15. Therefore, by Lemma 7.8, $f_N^{n-1} : B \to f_N^{n-1}(B)$ is conformal. Therefore, the composition $f_N^n : B \to B(0, 4R_{k+1})$ is conformal and B is a Jordan domain. The modulus lower bound for $B \setminus \widehat{W}_k^n$ follows from equation (4.13).

Now, consider the case of $k \leq 0$. Let A_k' be the connected component of A_k such that $f_N^n(W_k^n) = A_k'$. Then, the boundary of $\widehat{A}_k'$ is one of the elements γ_{1-k} of Γ_{1-k} from Definition 5.8. Therefore, there exists an element σ_{1-k} from Remark 5.9 so that the modulus of $\widehat{\sigma}_{1-k} \setminus \widehat{\gamma}_{1-k}$ is bounded below by $(2\pi)^{-1}\log(2)$. Let B denote the connected component of $f_N^{-n}(\widehat{\sigma}_{k+1})$ that contains W_k^n. Then, $f_N^n : B \to \widehat{\sigma}_{k+1}$ is conformal by a similar argument to the $k \geq 1$ case. □

Remark 8.5. It follows immediately from Lemma 8.4 that for all $m \geq 0$, every element of $\mathcal{C}_m$ is a Jordan annulus.

Remark 8.6. Let $m \geq 0$ and let W_k^n be any element of $\mathcal{C}_m$, and let K be any compact subset of $\widehat{W}_k^n$. Let B be the Jordan domain from Lemma 8.4 containing W_k^n. By Remark A.7, Corollary A.4 applies with $D = B$ to $f_N^n : B \to f_N^n(B)$ with $U = W_k^n$ and constant $C = L'' > 0$ that does not depend on m, the element W_k^n, or the compact set K.

We now begin estimating the diameters of elements in $\mathcal{C}_m$ for the purpose of estimating the dimension of Y. The following lemma is the same as [**Bis18**, inequality (17.1)]. Our proof is similar, and we include all the details for the sake of the reader.

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LEMMA 8.7. *Fix some $t > 0$ and let $W_k^n \in \mathcal{C}_m$ be given. Then, there exists some $M \in \mathbb{N}$ so that for all $N \geq M$, we have*

$$\sum_{W_{k-1}^n \subset \widehat{W_k^n}} \text{diameter}(W_{k-1}^n)^t \leq \frac{1}{100} \text{diameter}(W_k^n)^t, \tag{8.3}$$

where the sum in equation (8.3) is taken over all components of the form W_{k-1}^n in $\mathcal{C}_m$ that are contained in $\widehat{W}_k^n$.

Remark 8.8. The specific constant $1/100$ is not particularly important. In fact, by increasing M, it can be replaced by any arbitrarily small positive constant.

Proof. The proof splits into two cases: the case of $k > 1$ and the case of $k \leq 1$.

Suppose that $k > 1$. Then, there is exactly one element of the form $W_{k-1}^n \subset \widehat{W}_k^n$. Indeed, the mapping $f_N^n : \widehat{W}_k^n \to \widehat{A}_k$ is a conformal bijection, and since there is only one $A_{k-1} \subset \widehat{A}_k$ when $k > 1$, there can only be one $W_{k-1}^n \subset \widehat{W}_k^n$. Thus, we have

$$\begin{aligned}\frac{\text{diameter}(W_{k-1}^n)^t}{\text{diameter}(W_k^n)^t} &\overset{\text{Remark 8.6}}{\leq} (L'')^t \frac{\text{diameter}(A_{k-1})^t}{\text{diameter}(A_k)^t} = (L'')^t \frac{R_{k-1}^t}{R_k^t} \\ &\overset{\text{Lemma 4.8}}{\leq} (L'')^t \left(\frac{1}{4R_{k-1}}\right)^t \leq \frac{(L'')^t}{R_1^t}.\end{aligned} \tag{8.4}$$

Suppose that $k \leq 1$. Then, there exists a connected component A_k' of A_k so that $f_N^n : \widehat{W}_k^n \to \widehat{A}_k'$ is conformal. By Lemma 7.9, we have the following composition of conformal bijections:

$$\widehat{W}_k^n \xrightarrow{f_N^n} \widehat{A}_k' \xrightarrow{f_N^{1-k}} \widehat{A}_1.$$

Therefore, the number of elements of the form $W_{k-1}^n \subset \widehat{W}_k^n$ is equal to the number of connected components of A_0, which is 2^N by part (3) of Lemma 7.9. Next, recall that the outermost boundary of each connected component of A_k and A_{k-1} coincides with an element γ_{-k+1} and an element γ_{-k+2} from Definition 5.8, respectively (here, we take the convention that γ_0 is $\{z : |z| = 4R_1\}$). Therefore, we obtain

$$\begin{aligned}\frac{\sum_{W_{k-1}^n \subset \widehat{W_k^n}} \text{diameter}(W_{k-1}^n)^t}{\text{diameter}(W_k^n)^t} &\overset{\text{Remark 8.6}}{\leq} (L'')^t \frac{\sum_{A_{k-1}' \subset A_k'} \text{diameter}(A_{k-1}')^t}{\text{diameter}(A_k')^t} \\ &\overset{\text{Lemma 5.10}}{\leq} (L'')^t \frac{2^N (\text{diameter}(A_k')^t / R_1^t)}{\text{diameter}(A_k')^t} = \frac{(L'')^t \cdot 2^N}{R_1^t}.\end{aligned} \tag{8.5}$$

By equations (8.4) and (8.5), the conclusion of the lemma now follows by choosing M large enough so that for all $N \geq M$, we have

$$\frac{(L'')^t \cdot 2^N}{R_1^t} \leq \frac{1}{100}.$$

Such an M exists by applying Lemma 4.8 and inequality (4.12); see equation (5.20). □

We move on to describing how to change the covering $\mathcal{C}_m$ of $Y \cap A_1$ by topological annuli into a simpler covering $\widehat{\mathcal{C}}_m$ by topological disks.

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Definition 8.9. We define $\widehat{\mathcal{C}}_0$ to be $\widehat{A}_1$ and we define $\widehat{\mathcal{C}}_1$ to be the collection

$$\{\widehat{W}_n^n : n \geq 1 \text{ and } W_n^n \in \mathcal{C}_1\}. \tag{8.6}$$

Remark 8.10. When $m = 1$, we note that the covering $\widehat{\mathcal{C}}_m$ satisfies:

- $\widehat{\mathcal{C}}_m$ is a covering of $\mathcal{C}_m$, and hence is a covering of $A_1 \cap Y$; and
- if $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$, then $k \geq 1$.

Definition 8.11. Let $m \geq 1$ and assume that $\widehat{\mathcal{C}}_m$ has been constructed and satisfies parts (1) and (2) of Remark 8.10. Let $\widehat{W}_k^n \in \mathcal{C}_m$. Then, $\widehat{W}_k^n$ contains a sequence of components $W_j^n \in \mathcal{C}_m$ for $j \leq k$. Fix W_j^n, and consider the elements of $\mathcal{C}_{m+1}$ contained inside of W_j^n of the form W_{j+q-1}^{n+q}. If $j \geq 1$, then all $q \geq 1$ occur. If $j \leq 0$, then q must satisfy $q \geq 2 - j$. Either way, for each valid choice of q, the elements of $\widehat{\mathcal{C}}_m$ that lie inside of W_j^n are defined to be the components $\widehat{W}_{j+q-1}^{n+q}$. Doing this for all $j \leq k$, we obtain all of the elements of $\widehat{\mathcal{C}}_{m+1}$ contained in $\widehat{W}_k^n$. The covering $\widehat{\mathcal{C}}_{m+1}$ is defined to be the collection of all such elements obtained in this way for each $\widehat{W}_k^n \in \mathcal{C}_m$.

Next, we need the following technical lemma. Recall that in Definition 4.5, we defined $n_k = 2^{N+k-1}$.

LEMMA 8.12. *Fix some $t > 0$. For $k \geq 1$, define $L_k = n_1 \cdots n_k$, and let $\varepsilon > 0$ be given. Then, there exists $M \in \mathbb{N}$ such that for all $N \geq M$, we have*

$$\sum_{k=1}^{\infty} 2^k L_k R_k^{-t} < \varepsilon. \tag{8.7}$$

Proof. We will use the ratio test. Let a_k denote the kth term of equation (8.7). Then, for any $k \geq 1$, we have, by applying Lemma 4.8, there exists M so that for all $N \geq M$,

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{2^{k+1} n_1 \cdots n_k n_{k+1} R_{k+1}^t}{2^k n_1 \cdots n_k R_k^t} \\
&= 2n_{k+1}\left(\frac{R_k}{R_{k+1}}\right)^t \\
&\leq 2n_{k+1}\left(\frac{1}{4R_k}\right)^t \\
&\leq \frac{2}{4^t} n_{k+1} 2^{-t2^{k+N-2}} \\
&\leq 8n_{k-1}\left(\frac{1}{2^t}\right)^{n_{k-1}}.
\end{aligned} \tag{8.8}$$

Since $nx^n \to 0$ as $n \to \infty$ whenever $x \in (0, 1)$, the series converges by the ratio test. In fact, a stronger statement is true. By perhaps choosing M larger, we may arrange for the ratio in equation (8.8) to be arbitrarily small for all $k \geq 1$. We may also arrange for the first term of equation (8.7), $2n_1 R_1^{-t}$, to be arbitrarily small. It follows that we can make equation (8.7) arbitrarily small. $\square$

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The proof of Lemma 8.14 is the same as the proof of [**Bis18**, equation (17.2)]. We include the details for the sake of the reader. First, we introduce the following convenient definition.

Definition 8.13. Let $m \geq 0$ and let $W_j^n \in \mathcal{C}_m$ be given. Fix some value $q \geq 1$. We define $W_j^n(q)$ to be the set of all elements $\widehat{W}_{j+q-1}^{n+q} \in \widehat{\mathcal{C}}_{m+1}$ that are a subset of W_j^n.

LEMMA 8.14. *Fix some $t > 0$. Let $W_j^n \in \mathcal{C}_m$ be given for $m \geq 0$. Then, there exists a $M \in \mathbb{N}$ so that for all $N \geq M$, we have*

$$\sum_{q=1}^{\infty} \sum_{W_j^n(q)} \operatorname{diameter}(\widehat{W}_{j+q-1}^{n+q})^t \leq \frac{1}{100} \operatorname{diameter}(W_j^n)^t. \tag{8.9}$$

Proof. Fix an arbitrary element $W_j^n \in \mathcal{C}_m$, and choose an arbitrary element of the form $\widehat{W}_{j+q-1}^{n+q} \in \widehat{\mathcal{C}}_{m+1}$ contained inside of W_j^n for some $q \geq \max\{1, 2-j\}$.

First, we observe that the mapping

$$f_N^n : W_j^n \to A_j$$

is conformal by Lemma 8.4. Therefore, by Lemma 8.4 and Corollary A.4, we obtain

$$\frac{\operatorname{diameter}(\widehat{W}_{j+q-1}^{n+q})^t}{\operatorname{diameter}(W_j^n)^t} \leq (L'')^t \frac{\operatorname{diameter}(f_N^n(\widehat{W}_{j+q-1}^{n+q}))^t}{\operatorname{diameter}(A_j)^t}. \tag{8.10}$$

Next, we observe that the mapping

$$f_N^q : f_N^n(\widehat{W}_{j+q-1}^{n+q}) \to A_{j+q-1}$$

is conformal. Noting that $j+q-1 \geq 1$, if B is the Jordan domain from Lemma 8.4 that contains $\widehat{W}_{j+q-1}^{n+q}$, then $f_N^n(B) \subset A_j$. Let B' be the connected component of $f_N^{-(n+q)}(B(0, 2R_{j+q}))$ so that $B' \subset B$ and the modulus of $B \setminus \overline{B'}$ is bounded above by $(2\pi)^{-1}\log(2)$. Therefore, by Corollary A.4 and Remark 8.6, we obtain

$$\frac{\operatorname{diameter}(f_N^n(\widehat{W}_{j+q-1}^{n+q}))^t}{\operatorname{diameter}(f_N^n(B'))^t} \leq (L'')^t \frac{\operatorname{diameter}(A_{j+q-1})^t}{\operatorname{diameter}(f_N^{n+q}(B'))^t} = (L'')^t \frac{\operatorname{diameter}(A_{j+q-1})^t}{\operatorname{diameter}(B(0, 2R_{j+q}))^t}. \tag{8.11}$$

Combining equations (8.10) and (8.11), we obtain

$$\begin{aligned}
&\operatorname{diameter}(\widehat{W}_{j+q-1}^{n+q})^t \\
&\leq (L'')^{2t} \frac{\operatorname{diameter}(f_N^n(B))^t}{\operatorname{diameter}(B(0, 2R_{j+q}))^t} \cdot \frac{\operatorname{diameter}(A_{j+q-1})^t}{\operatorname{diameter}(A_j)^t} \cdot \operatorname{diameter}(W_j^n)^t \\
&\leq (L'')^{2t} \frac{\operatorname{diameter}(A_j)^t}{\operatorname{diameter}(B(0, 2R_{j+q}))^t} \cdot \frac{\operatorname{diameter}(A_{j+q-1})^t}{\operatorname{diameter}(A_j)^t} \cdot \operatorname{diameter}(W_j^n)^t \\
&= (2(L'')^2)^t \left(\frac{R_{j+q-1}}{R_{j+q}}\right)^t \cdot \operatorname{diameter}(W_j^n)^t \\
&\leq \left(\frac{(L'')^2}{2}\right)^t \cdot \frac{1}{R_{j+q-1}^t} \operatorname{diameter}(W_j^n)^t.
\end{aligned}$$

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Next, for a fixed value $q \geq 1$, we want to count the total number of components of the form $\widehat{W}^{n+q}_{j+q-1} \in \widehat{\mathcal{C}}_{m+1}$ contained in W^n_j. Suppose that $j \geq 1$. First, the mapping $f^n_N : W^n_j \to A_j$ is a conformal mapping. Next, for each $i = 0, 1, \ldots, q-1$, we have $f^{n+i}_N(\widehat{W}^{n+q}_{j+q-1}) \subset A_{j+i}$, and $f^{n+q}_N(\widehat{W}^{n+q}_{j+q-1}) = A_{j+q-1}$.

Since there are n_{j+q-1} many petals in A_{j+q-1}, there are n_{j+q-1} many connected components of $f^{-1}_N(A_{j+q-1})$ contained inside of A_{j+q-1} by Lemma 7.21. For such a connected component U of $f^{-1}_N(A_{j+q-1})$, by repeatedly applying Lemma 7.22, there are exactly $n_{j+1} \cdot n_{j+2} \cdots n_{j+q-1} = 2^{q-1} n_j \cdots n_{j+q-2}$ many connected components contained in A_j that map conformally onto U. Therefore, the total number of components of the form $\widehat{W}^{n+q}_{j+q-1} \in \widehat{\mathcal{C}}_{m+1}$ contained in W^n_j is bounded above by

$$(2^{q-1} n_j \cdots n_{j+q-2}) \cdot n_{j+q-1} \leq 2^q n_j \cdots n_{j+q-1} \leq 2^{q+j-1} L_{q+j-1}. \tag{8.12}$$

When $j \leq 0$, for a fixed value $q \geq 1$, counting the total number of components of the form $\widehat{W}^{n+q}_{j+q-1} \in \mathcal{C}_{m+1}$ contained in W^n_j is similar to the $j \geq 1$ case, with just one additional complication. In this case, we must have $q \geq 2 - j$, and we have $f^n_N(W^{n+q}_{j+q-1}) \subset U$, where $U = f^{-(q-1)}_N(A_{j+q-1}) \cap A'_j$ and A'_j is the connected component of A_j that contains $f^n_N(W^{n+q}_{j+q-1})$. However, by Lemma 7.9, $f^{1-j}_N : U \to f^{1-j}_N(U)$ is conformal. So by similar reasoning as equation (8.12), the number of components of the form W^{n+q}_{j+q-1} contained inside of W^n_j is bounded above by

$$2^{j+q-2} n_1 \cdots n_{j+q-2} \cdot n_{j+q-1} \leq 2^{j+q-1} L_{j+q-1}, \tag{8.13}$$

for each $q \geq 2 - j$. Therefore,

$$\begin{aligned}
&\sum_{q=1}^{\infty} \sum_{W^n_j(q)} \operatorname{diameter}(\widehat{W}^{n+q}_{j+q-1})^t \\
&\quad \leq \operatorname{diameter}(W^n_j)^t \cdot \left(\frac{(L'')^2}{2}\right)^t \sum_{q \geq \max\{1, 2-j\}}^{\infty} 2^{j+q-1} L_{j+q-1} R^{-t}_{j+q-1} \\
&\quad \leq \operatorname{diameter}(W^n_j)^t \cdot \left(\frac{(L'')^2}{2}\right)^t \sum_{k=1}^{\infty} 2^k L_k R_k^{-t}.
\end{aligned}$$

The result now follows by choosing M so large that for all $N \geq M$,

$$\sum_{k=1}^{\infty} 2^k L_k R_k^{-t} < \frac{1}{100} \left(\frac{2}{(L'')^2}\right)^t.$$

Such an M exists by Lemma 8.12. □

We will now show that the sum of the diameters of every distinct element W of $\widehat{\mathcal{C}}_{m+1}$ is comparable to the sum of the diameters of every distinct element V of $\widehat{\mathcal{C}}_m$.

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LEMMA 8.15. *Fix some $t > 0$. Let $m \geq 1$ be given. Then,*

$$\sum_{W \in \widehat{\mathcal{C}}_{m+1}} \operatorname{diameter}(W)^t \leq \frac{1}{10} \sum_{V \in \widehat{\mathcal{C}}_m} \operatorname{diameter}(V)^t. \tag{8.14}$$

Proof. Let $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. Define

$$G := \{W \in \widehat{\mathcal{C}}_{m+1} : W \subset \widehat{W}_k^n\}.$$

If $W \in E$, then there exists $j \leq k$ so that $W \subset W_j^n \in \mathcal{C}_m$ and $W_j^n \subset \widehat{W}_k^n$. For fixed j, define

$$G_j := \{W \in G : W \subset W_j^n\}.$$

It follows from Lemma 8.14 that

$$\sum_{W \in G_j} \operatorname{diameter}(W)^t \leq \frac{1}{100} \operatorname{diameter}(W_j^n)^t.$$

Since $G = \bigcup_{j \leq k} G_j$, we obtain

$$\sum_{W \in G} \operatorname{diameter}(W)^t \leq \frac{1}{100} \sum_{W_j^n \in \mathcal{C}_m,\, W_j^n \subset \widehat{W}_k^n} \operatorname{diameter}(W_j^n)^t. \tag{8.15}$$

By repeatedly applying Lemma 8.7, we have for any fixed $j \leq k$ that

$$\sum_{W_j^n \subset \widehat{W}_k^n} \operatorname{diameter}(W_j^n)^t \leq \left(\frac{1}{100}\right)^{k-j} \operatorname{diameter}(W_k^n)^t. \tag{8.16}$$

By combining equations (8.15) and (8.16), we deduce

$$\sum_{W \in E} \operatorname{diameter}(W)^t \leq \sum_{j \leq k} \left(\frac{1}{100}\right)^{k-j+1} \operatorname{diameter}(W_k^n)^t \leq \frac{1}{10} \operatorname{diameter}(W_k^n)^t.$$

The claim now follows by summing over all $\widehat{W}_k^n \in \widehat{\mathcal{C}}_m$. □

THEOREM 8.16. *With the notation as above,*

$$\sum_{m=1}^{\infty} \sum_{W \in \widehat{\mathcal{C}}_m} \operatorname{diameter}(W)^t < \infty. \tag{8.17}$$

As a consequence, we have $\dim_H(Y \cap A_1) \leq t$.

Proof. We obtain a geometric sum by Lemma 8.15. Indeed,

$$\sum_{m=1}^{\infty} \sum_{W \in \widehat{\mathcal{C}}_m} \operatorname{diameter}(W)^t \leq \sum_{m=1}^{\infty} \left(\frac{1}{10}\right)^m \operatorname{diameter}(A_1)^t < \infty.$$

Therefore, for every $\varepsilon > 0$, there exists $m \geq 0$ so that

$$\sum_{W \in \widehat{\mathcal{C}}_m} \operatorname{diameter}(W)^t < \varepsilon.$$

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Since $\widehat{\mathcal{C}}_m$ covers $Y \cap A_1$ for each $m \geq 0$, by applying equation (A.10), we deduce that $H^t(Y \cap A_1) = 0$. It follows immediately that $\dim_H(Y \cap A_1) \leq t$. $\square$

COROLLARY 8.17. *We have* $\dim_H(Y) \leq t$.

Proof. First, we will observe that our arguments above apply to the case of $A_k \cap Y$ for $k > 1$, with simple modifications made to the definitions of $\mathcal{C}_m$. Therefore, $\dim_H(Y \cap A_k) \leq t$ for all $k \geq 1$. If $k \leq 0$, let A'_k be a connected component of A_k. Then, by repeatedly applying Lemma 7.9, we see that f_N^{k+1} maps A'_k onto A_1 conformally. It follows that $\dim_H(Y \cap A'_k) = \dim_H(Y \cap A_1)$, and we deduce that $\dim_H(Y \cap A_k) = \dim_H(Y \cap A_1)$ for all $k \leq 0$.

Since $Y \subset A$, we conclude by equation (A.11) that

$$\dim_H(Y) = \dim_H(Y \cap A) = \sup_{k \in \mathbb{Z}} \dim(A_k \cap Y) \leq t,$$

as desired. $\square$

9. *Jordan Fatou boundary components*

Recall that we have proven $\mathcal{J}(f) \subset E' \cup Y \cup Z$, and that E' and Y may be taken to have arbitrarily small (positive) dimension. We now move on to estimating the dimension of Z (those points whose orbits always stay in A, and eventually only move forward). It will be necessary to partition Z as follows.

Definition 9.1. Let

$$Z_1 := \left\{ z \in Z : \text{there exists } l \geq 0 \text{ such that for all } j \geq 0,\ f_N^{l+j}(z) \in \bigcup_{k \geq 1} V_k \right\} \tag{9.1}$$

and

$$Z_2 := Z \setminus Z_1. \tag{9.2}$$

Note that $Z = Z_1 \sqcup Z_2$.

Our primary objective over the next three sections is the proof of the following theorem.

THEOREM 9.2. *Z_1 is the disjoint union of countably many C^1 Jordan curves, and Z_2 has Hausdorff dimension* 0.

In this section, we will focus on proving that Z_1 consists of a disjoint union of Jordan curves, and in §10, we will prove that they are C^1. Lastly, in §11, we will study Z_2.

For the entirety of the next three sections, we choose M so large so that for all $N \geq M$ and $k \geq 1$, we have for all $z \in V_k$ that

$$\tfrac{1}{2} \leq |\phi'_N(z)| \leq 2. \tag{9.3}$$

The existence of such an M follows from the Cauchy estimate and Lemma 4.3.

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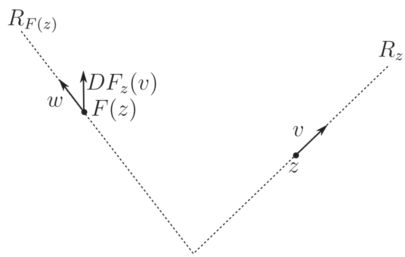

FIGURE 11. A schematic for the statement of Lemma 9.3.

Recall from Definition 7.11 that for $k \geq 1$, Ω_k is the Fatou component containing B_k. We first study the set $Z_1 \cap \overline{\Omega_k}$ for $k \geq 1$.

LEMMA 9.3. *Let $F : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $z \in \mathbb{C}$, and suppose that $F'(z) \neq 0$, $z \neq 0$, and $F(z) \neq 0$. Let R_z, $R_{F(z)}$ denote the rays starting at the origin and passing through z, $F(z)$, respectively. Let $v \in T_z\mathbb{C}$ denote the outward pointing tangent vector to R_z based at z and let $w \in T_{F(z)}\mathbb{C}$ be the image of the outward pointing tangent vector to $R_{F(z)}$ based at $F(z)$. Then, the angle between $DF_z(v)$ and w is given by* $\arg((z/F(z))F'(z))$ *(see Figure 11).*

Proof. First, we consider the case that $F(z) = z$. Then, letting $v \in T_z\mathbb{C}$ denote the unit tangent vector pointing in the direction of R_z, we have

$$DF_z(v) = F'(z)v = |F'(z)| \arg(F'(z))v \in T_z\mathbb{C}.$$

This proves the result in the case where $w = z$. When $F(z) \neq z$, the result follows from applying the above reasoning to the function $\zeta \mapsto (z/F(z)) \cdot F(\zeta)$. □

The following lemma follows a similar strategy to [**Bis18**, Lemma 18.1].

LEMMA 9.4. *Let $\varepsilon > 0$ and $n, k \in \mathbb{N}$ be given. Suppose that ϕ is a univalent function on the annulus $A(\frac{1}{4}R_k, \frac{3}{4}R_k)$ and suppose that $|\phi(z)/z - 1| < \varepsilon$ on $A(\frac{2}{5}R_k, \frac{3}{5}R_k)$. Define*

$$F(z) = (\phi(z))^n.$$

For any fixed $\tau \in [0, 2\pi)$, parameterize the segment $S(\tau) = \{re^{i\tau} : \frac{2}{5}R_k \leq r \leq \frac{3}{5}R_k\}$ as $\gamma_\tau(r) = re^{i\tau}$, $r \in [\frac{2}{5}R_k, \frac{3}{5}R_k]$. Suppose that $F \circ \gamma_\tau$ and γ_φ intersect at some point z. Then, the angle between the tangent vectors of $F \circ \gamma_\tau$ and γ_φ based at $F(z)$ is $O(\varepsilon)$ as $\varepsilon \to 0$.

Proof. Following Lemma 9.3, it is sufficient to estimate $\arg(zF'(z)/F(z))$. To that end, first observe that by the chain rule, we have

$$z\frac{F'(z)}{F(z)} = z\frac{n(\phi(z))^{n-1}\phi'(z)}{(\phi(z))^n} = \frac{z}{\phi(z)} \cdot n\phi'(z). \tag{9.4}$$

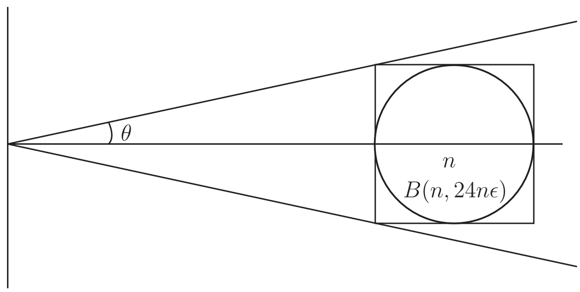

FIGURE 12. A schematic for the proof of Lemma 9.4.

Let $g(\zeta) = \phi(\zeta) - \zeta$. Then, $g'(\zeta) = \phi'(\zeta) - 1$. If $z \in A(\frac{2}{5}R_k, \frac{3}{5}R_k)$, then $B(z, \frac{1}{10}R_k) \subset A(\frac{1}{4}R_k, \frac{3}{4}R_k)$, so that Cauchy estimates say we must have

$$\begin{aligned}
|g'(z)| &\leq \frac{\max_{B(z,(1/10)R_k)} |g(\zeta)|}{(1/10)R_k} \\
&= \frac{10}{R_k} \cdot \max_{B(z,(1/10)R_k)} |\phi(\zeta) - \zeta| \\
&= \frac{10}{R_k} \cdot \max_{B(z,(1/10)R_k)} |\zeta| \cdot |\phi(\zeta)/\zeta - 1| \\
&\leq 10 \cdot \varepsilon.
\end{aligned}$$

It follows that for all $z \in A(\frac{2}{5}R_k, \frac{3}{5}R_k)$, we have $|\phi'(z) - 1| < 10\varepsilon$, so that $\phi'(z) \in B(1, 10\varepsilon)$. This means that $n\phi'(z) \in B(n, 10n\varepsilon)$. By assumption, we have $(\phi(z)/z) \in B(1, \varepsilon)$. Therefore, if $\varepsilon < \frac{1}{2}$, we have $(z/\phi(z)) \in B(1, 2\varepsilon)$ (see Figure 12).

Putting everything together, we have

$$\frac{z}{\phi(z)} \cdot n\phi'(z) \in B(n, 24n\varepsilon). \tag{9.5}$$

Indeed, if $a \in B(1, 2\varepsilon)$ and $b \in B(n, 10n\varepsilon)$, we have

$$\begin{aligned}
|ab - n| = |a(b - n) + an - n| &\leq |a| \cdot |b - n| + n \cdot |a - 1| \\
&\leq (1 + 2\varepsilon)10n\varepsilon + 2\varepsilon n \\
&= 12n\varepsilon + 20n\varepsilon^2 \\
&= 12n\varepsilon\big(1 + \tfrac{5}{3}\varepsilon\big) \\
&< 24n\varepsilon
\end{aligned}$$

whenever $\varepsilon < 3/5$. Therefore, for all ε sufficiently small, we have

$$\arg\left(z\frac{F'(z)}{F(z)}\right) \leq \arctan(24 \cdot 2\varepsilon) = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

This proves the claim; see Figure 12. $\square$

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Definition 9.5. Let Ω_k be given for some $k \geq 1$. Then, the outermost boundary component of Ω_k is contained in V_{k+1} by Theorem 6.9. Define

$$\Gamma_{k,n} := \{z \in V_{k+1} : f_N^j(z) \in V_{k+j+1},\ j = 0, 1, \ldots, n\}. \tag{9.6}$$

By Lemmas 4.17 and 7.10, for $n \geq 1$, each $\Gamma_{k,n}$ is a topological annulus compactly contained inside of $\Gamma_{k,n-1}$, and $\Gamma_{k,1}$ is compactly contained inside of $\Gamma_{k,0} = V_{k+1}$. We define

$$\Gamma_k := \bigcap_{n=1}^{\infty} \Gamma_{k,n}. \tag{9.7}$$

The remainder of this section will be devoted to showing that Γ_k is in fact a Jordan curve, and in the next section, we will show that it is C^1.

Definition 9.6. Fix some $k \geq 1$ and let $n \geq 0$ be arbitrary. Each V_{k+n+1} has a foliation of closed circles centered around the origin, including the inner and outer boundary of V_{k+n+1}. When $n = 0$, this is a foliation of $\Gamma_{k,0}$, which we denote by $\mathcal{U}_{k,0}$. When $n \geq 1$, by pulling this foliation back to $\Gamma_{k,n}$ by f_N^n, we obtain a foliation $\mathcal{U}_{k,n}$ of $\Gamma_{k,n}$ by analytic Jordan curves by Lemma 7.10.

Remark 9.7. Let $n \geq 1$. It is readily verified from equation (9.6) that $f_N(\Gamma_{k,n}) = \Gamma_{k+1,n-1}$. Similarly, we can verify using Definition 9.6 that if $\gamma \in \mathcal{U}_{k,n}$, then $f_N(\gamma) \in \mathcal{U}_{k+1,n-1}$.

LEMMA 9.8. *Let $k \geq 1$, and suppose that $\gamma_n \in \mathcal{U}_{k,n}$ and $\gamma_m \in \mathcal{U}_{k,m}$ for $m > n \geq 0$. Suppose that γ_n and γ_m intersect at some point z. Let $\tau_n(z)$ and $\tau_m(z)$ denote the counter-clockwise oriented unit tangent vectors of γ_n and γ_m at z. Likewise, let $\nu_n(z)$ and $\nu_m(z)$ denote the outward pointing normal vectors of γ_n and γ_m at z. Then,*

$$|\tau_n(z) - \tau_m(z)| = |\nu_n(z) - \nu_m(z)| \leq O\bigg(\sum_{l=n}^{m-1} 2^{-\sqrt{N+k+l}/4}\bigg). \tag{9.8}$$

Proof. We first consider the case $m = 1$ and $n = 0$. In this case, γ_0 is a circle, and $f_N(\gamma_1)$ is a circle in V_{k+2}. Let z be a point of intersection of γ_0 and γ_1, so that $\nu_0(z)$ and $\nu_1(z)$ are the corresponding outward pointing normal vectors based at z. Let R be the ray through the origin passing through z, let R' be the ray passing through $f_N(z)$. Since $f_N(\gamma_1)$ is a circle, $Df_N : T_z\mathbb{C} \to T_{f_N(z)}\mathbb{C}$ maps $\nu_1(z)$ to the outward pointing normal of a circle based at $f_N(z)$. Therefore, $\nu_0(z)$ coincides with the tangent vector to R based at z, $(Df_N)_z(\nu_0)$ coincides with the tangent vector to $f_N(R)$ based at $f_N(z)$, and $(Df_N)_z(\nu_1)$ coincides with the tangent vector to R' based at $f_N(z)$. Therefore, by Lemma 9.4, the angle between $(Df_N)_z(\nu_0)$ and $(Df_N)_z(\nu_1)$ is $O(2^{-\sqrt{k+N}})$. Since f_N is conformal at z, we deduce that

$$|\tau_1(z) - \tau_0(z)| = |\nu_1(z) - \nu_0(z)| \leq O(2^{-\sqrt{k+N}}).$$

Next, we consider the case of $m > 1$ and $n = m - 1$. By Lemma 7.10, the angle between γ_m and γ_{m-1} at z is the same as the angle between $f_N^{m-1}(\gamma_m)$ and $f_N^{m-1}(\gamma_{m-1})$ at the point

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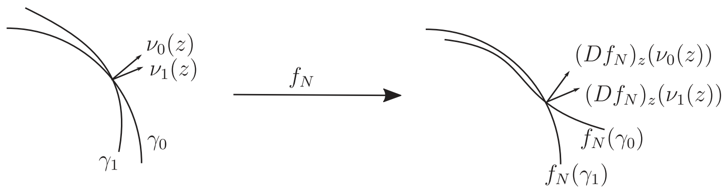

FIGURE 13. A schematic for the proof of Lemma 9.8 in the key step of $m = 1$ and $n = 0$. In this case, γ_0 and $f_N(\gamma_1)$ are circles, and $(Df_N) : T_z\mathbb{C} \to T_{f_N(z)}\mathbb{C}$ maps the outward pointing normal $\nu_1(z)$ to γ_1 to an outward pointing normal $(Df_N)_z(\nu_1(z))$ of the circle $f_N(\gamma_1)$. This allows us to apply Lemma 9.4 to estimate the angle between $(Df_N)_z(\nu_1(z))$ and $(Df_N)_z(\nu_0(z))$, which coincides with the angle between $\nu_1(z)$ and $\nu_0(z)$.

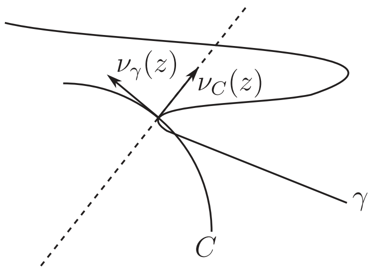

FIGURE 14. A schematic for the proof of Corollary 9.9. If some ray R passed through γ at more than one point, there is a ray R' tangent to γ at some other point z. If C is the circle centered at the origin passing through z, the normal vectors $\nu_\gamma(z)$ and $\nu_C(z)$ make an angle of $\pi/2$ with each other.

$f_N^{m-1}(z)$. We also have that $f_N^{m-1}(\gamma_m) \in \mathcal{U}_{k+m-1,1}$ and $f_N^{m-1}(\gamma_{m-1}) \in \mathcal{U}_{k+m-1,0}$, so that $f_N^{m-1}(\gamma_{m-1})$ is a circle. Therefore, we have by Lemma 9.4 that

$$|\tau_m(z) - \tau_{m-1}(z)| \le O(2^{-(\sqrt{N+k+m-1})/4}).$$

The lemma now follows by applying the triangle inequality. Let $m > n$. Then,

$$|\tau_m(z) - \tau_n(z)| \le \sum_{l=n}^{m-1} |\tau_l(z) - \tau_{l+1}(z)| \le O\bigg(\sum_{l=n}^{m-1} 2^{-(\sqrt{N+k+l})/4}\bigg).$$

This proves the claim (see Figure 13). □

COROLLARY 9.9. *Let $k \ge 1$, $n \ge 0$, and let γ be any element of $\mathcal{U}_{k,n}$. Then, there exists $M \in \mathbb{N}$ so that if $N \ge M$, $\gamma \cap \{re^{i\theta} : 0 < r < \infty\}$ is a single point for any θ.*

Proof. Suppose that some ray R through the origin intersected γ more than once. Since γ is an analytic Jordan curve, this implies that there exists a point z on γ and a ray R' such that R' passes through z and is tangent to γ. This, in turn, implies that the circle passing through z centered at the origin makes an angle of $\pi/2$ with γ. For all N sufficiently large, this is a contradiction to Lemma 9.8 (see Figure 14). □

LEMMA 9.10. *Suppose that $k \ge 2$, $z \in V_k$, and $f_N(z) \in V_{k+1}$. Then,*

$$\phi_N^{-1}(z) \in A\Big(\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{1/n_k} R_k, \tfrac{1}{2}\big(\tfrac{3}{4}\big)^{1/n_k} R_k\Big).$$

Proof. Recall that by Lemma 4.12, for $z \in V_k$, we have $f_N = h_N \circ \phi_N^{-1}(z) = C_k(\phi_N^{-1}(z))^{n_k}$. Suppose for the sake of a contradiction that $|\phi_N^{-1}(z)| \leq \frac{1}{2}(\frac{1}{4})^{1/n_k} R_k$. Then,

$$|f_N(z)| = C_k(|\phi_N^{-1}(z)|)^{n_k} \leq C_k\big(\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{1/n_k} R_k\big)^{n_k} = \tfrac{1}{4} R_{k+1} < \tfrac{2}{5} R_{k+1}.$$

Since we have $f_N(z) \in V_{k+1}$, we have a contradiction and deduce that we must have $|\phi_N^{-1}(z)| > \frac{1}{2}(\frac{1}{4})^{1/n_k} R_k$.

Similarly, suppose for the sake of a contradiction that $|\phi_N^{-1}(z)| \geq \frac{1}{2}(\frac{3}{4})^{1/n_k} R_k$. Then,

$$|f_N(z)| \geq \tfrac{3}{4} R_{k+1} > \tfrac{3}{5} R_{k+1}.$$

Since we have $f_N(z) \in V_{k+1}$, we have a contradiction and deduce that we must have $|\phi_N^{-1}(z)| < \frac{1}{2}(\frac{3}{4})^{1/n_k} R_k$. The claim follows. □

LEMMA 9.11. *Let $k, n \geq 1$ and suppose that $z \in \Gamma_{k,n}$. Then,*

$$|f_N'(z)| \geq \frac{1}{4} n_{k+1} \frac{R_{k+2}}{R_{k+1}}. \tag{9.9}$$

Proof. By the chain rule, we have

$$f_N'(z) = n_{k+1} C_{k+1} (\phi_N^{-1}(z))^{n_{k+1}-1} (\phi_N^{-1})'(z).$$

Since $z \in \Gamma_{k,n}$ and $k \geq 1$, we have $f_N(z) \in V_{k+2}$. Therefore, by Lemma 9.10, we must have

$$\phi_N^{-1}(z) \in A\big(\tfrac{1}{2}\big(\tfrac{1}{4}\big)^{1/n_{k+1}} R_{k+1}, \tfrac{1}{2}\big(\tfrac{3}{4}\big)^{1/n_{k+1}} R_{k+1}\big).$$

Therefore, by equation (9.3), we have

$$\begin{aligned}
|f_N'(\phi_N^{-1})(z)| &\geq n_{k+1} C_{k+1} (|\phi_N^{-1}|)^{n_{k+1}-1} |(\phi_N^{-1})'(z)| \\
&\geq \frac{1}{2} n_{k+1} C_{k+1} \left(\frac{1}{2}\left(\frac{1}{4}\right)^{1/n_{k+1}} R_{k+1}\right)^{n_{k+1}-1} \\
&= \frac{1}{2} n_{k+1} \left(\frac{1}{4}\right)^{(n_{k+1}-1)/n_{k+1}} \frac{1}{(1/2) R_{k+1}} R_{k+2} \\
&\geq \frac{2}{2 \cdot 4} n_{k+1} \frac{R_{k+2}}{R_{k+1}} \\
&= \frac{1}{4} n_{k+1} \frac{R_{k+2}}{R_{k+1}}.
\end{aligned}$$

This is precisely what we wanted to show. □

Remark 9.12. It follows from Corollary 9.9 that if Γ is an element of $\mathcal{U}_{k,n}$, then we can parameterize Γ as

$$\gamma : [0, 2\pi] \to \Gamma$$

$$\gamma(\theta) = r(\theta) \cdot e^{i\theta}$$

for some $\mathbb{R}^+$-valued function $r(\theta)$ on $[0, 2\pi]$. Equivalently, we have

$$\gamma(\theta) = (r(\theta)\cos(\theta), r(\theta)\sin(\theta)).$$

Definition 9.13. Fix $k \geq 1$ and $m \geq 1$. Let R_θ be the ray starting at the origin with angle θ. Define

$$w_{k,m}(\theta) = \text{length}(R_\theta \cap \Gamma_{k,m}) \tag{9.10}$$

and

$$w_{k,m} = \sup_{\theta\in[0,2\pi)} w_{k,m}(\theta). \tag{9.11}$$

LEMMA 9.14. *Fix $k \geq 1$. Then,*

$$w_{k,m} \leq \frac{8^{m-1}}{n_{k+1}\cdots n_{k+m}} R_{k+1}.$$

In particular, $w_{k,m} \to 0$ as $m \to \infty$.

Proof. Fix $k \geq 1$. Note that by Lemma 9.10, we have

$$w_{k,1} \leq \frac{1}{2}R_{k+1}\bigg(\bigg(\frac{3}{4}\bigg)^{1/n_{k+1}} - \bigg(\frac{1}{4}\bigg)^{1/n_{k+1}}\bigg) \leq \frac{1}{2}R_{k+1}\frac{2}{n_{k+1}} = \frac{R_{k+1}}{n_{k+1}},$$

where for the second inequality, we have used the easily verified fact that $(3/4)^x - (1/4)^x \leq 2x$ for all sufficiently small $x > 0$.

Therefore, for all $k \geq 1$, we have

$$w_{k,1} \leq \frac{R_{k+1}}{n_{k+1}}. \tag{9.12}$$

Next, fix some $m > 1$, and define $S_\theta = R_\theta \cap \Gamma_{k,m}$. Then, $f_N(S_\theta)$ is a curve in $\Gamma_{k+1,m-1}$ with one endpoint on the inner boundary of $\Gamma_{k+1,m-1}$ and the other endpoint on the outer boundary of $\Gamma_{k+1,m-1}$. Then, by Lemma 9.4, the angle between $f_N(S_\theta)$ and any radial segment it meets is $O(2^{-\sqrt{N+k}/4})$. We will now show that this implies that the length of $f_N(S_\theta)$ is bounded above by $2w_{k+1,m-1}$.

Indeed, first observe that by Lemma 9.4, $f_N(S_\theta)$ intersects any circle centered at 0 at most once. Thus, we may parameterize $f_N(S_\theta)$ as $\gamma(r) = r\exp(i\theta(r))$ for $r \in [r_1, r_2]$ with $r_2 - r_1 \leq w_{k+1,m-1}$ and some $[0, 2\pi)$-valued function $\theta(r)$. Suppose that the radial arc

$$\sigma(r) := r\exp(i\theta_0),\ r \in [r_1, r_2]$$

intersects $f_N(S_\theta)$ at some point $z_0 = r_0 e^{i\theta_0}$. Then, the angle φ between the tangent vectors of σ and $f_N(S_\theta)$ at the point z_0 is given by the usual dot-product formula,

$$\cos(\varphi) = \frac{\text{Re}(\sigma'(r_0))\text{Re}(\gamma'(r_0)) + \text{Im}(\sigma'(r_0))\text{Im}(\gamma'(r_0))}{\|\sigma'(r_0)\|\cdot\|\gamma'(r_0)\|} = \frac{1}{\sqrt{1 + \big(r_0\theta'(r_0)\big)^2}}.$$

Recall that $\varphi = O(2^{-\sqrt{N+k}})$ by Lemma 9.4. Thus, for all sufficiently large N, we have that $\cos(\varphi) \in [0.9, 1]$. It follows that $|\theta'(r_0)| \leq (1/r_0)$. The above reasoning holds for all $r_0 \in [r_1, r_2]$, and so it follows that

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$$\text{length}(f_N(S_\theta)) = \int_{r_1}^{r_2} \|\gamma'(r)\| dr = \int_{r_1}^{r_2} \sqrt{1+\big(r\theta'(r)\big)^2}\, dr \leq \sqrt{2}(r_2-r_1) < 2w_{k+1,m-1}.$$

However, we can establish a lower bound for the length of $f_N(S_\theta)$ using Lemma 9.11. Indeed, we have

$$\text{length}(f_N(S_\theta)) = \int_{S_\theta} |f_N'(z)||dz| \geq w_{k,m}(\theta)\cdot \frac{n_{k+1}}{4}\frac{R_{k+2}}{R_{k+1}}.$$

Therefore, we have

$$w_{k,m} \leq 2w_{k+1,m-1}\frac{4}{n_{k+1}}\frac{R_{k+1}}{R_{k+2}} = \frac{8w_{k+1,m-1}}{n_{k+1}}\frac{R_{k+1}}{R_{k+2}}.$$

Therefore, we have for all $k \geq 1$ and all $m \geq 1$,

$$\begin{aligned} w_{k,m} &\leq \frac{8^{m-1}}{n_{k+1}\cdots n_{k+m}} w_{k+m-1,1}\frac{R_{k+1}}{R_{k+m}} \\ &\leq \frac{8^{m-1}}{n_{k+1}\cdots n_{k+m-1}}\frac{R_{k+1}}{R_{k+m}}\frac{R_{k+m}}{n_{k+m}} = \frac{8^{m-1}}{n_{k+1}\cdots n_{k+m}} R_{k+1}. \end{aligned}$$

This is what we wanted to show, and it also follows that $w_{k,m} \to \infty$ as $m \to \infty$, as desired. $\square$

THEOREM 9.15. *For each $k \geq 1$, Γ_k is a Jordan curve. Furthermore, Γ_k intersects any ray $\{z : \arg(z) = \theta\}$ in exactly one point.*

Proof. Fix $k \geq 1$. By Corollary 9.9, there exist Jordan-curve parameterizations of the form

$$\gamma_n^{\text{in}}(\theta) = r_n^{\text{in}}(\theta)e^{i\theta}, \theta \in [0, 2\pi], \tag{9.13}$$

$$\gamma_n^{\text{out}}(\theta) = r_n^{\text{out}}(\theta)e^{i\theta}, \theta \in [0, 2\pi] \tag{9.14}$$

of the inner and outer boundaries (respectively) of $\Gamma_{k,n}$. Let $m \geq n$. Then, by Lemma 9.14 and since $\Gamma_{k,m} \subset \Gamma_{k,n}$, we have the estimate

$$|r_m^{\text{in}}(\theta) - r_n^{\text{in}}(\theta)| = r_m^{\text{in}}(\theta) - r_n^{\text{in}}(\theta) \leq r_n^{\text{out}}(\theta) - r_n^{\text{in}}(\theta) = w_{k,n} \xrightarrow{n\to\infty} 0. \tag{9.15}$$

By equation (9.15), we can conclude that γ_n^{in} has a continuous limit:

$$\gamma^{\text{in}}(\theta) := r^{\text{in}}(\theta)e^{i\theta}, \quad \theta \in [0, 2\pi].$$

Similar reasoning allows us to conclude that γ_n^{out} has a continuous limit:

$$\gamma^{\text{out}}(\theta) := r^{\text{out}}(\theta)e^{i\theta}, \quad \theta \in [0, 2\pi],$$

and moreover by equation (9.15),

$$\gamma^{\text{out}}([0, 2\pi]) = \gamma^{\text{in}}([0, 2\pi]). \tag{9.16}$$

Note that $\gamma^{\text{in}}([0, 2\pi])$ is a Jordan curve since r^{in} is continuous. Furthermore, $\gamma^{\text{in}}([0, 2\pi]) \subset \Gamma_k$ since $r^{\text{in}}(\theta) \geq r_n^{\text{in}}(\theta)$ for all n and θ. Moreover, we must have $\Gamma_k \subset \gamma^{\text{in}}([0, 2\pi])$ by equation (9.16). Thus, $\Gamma_k = \gamma^{\text{in}}([0, 2\pi])$ is a Jordan curve. $\square$

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10. *Smooth Fatou boundary components*

In this section, we continue our study of the set Z_1. We will first prove that each Jordan curve Γ_k is in fact a C^1 curve (see Theorem 10.2 below). Then, we will conclude that the set Z_1 is a disjoint union of C^1 curves and, in particular, has dimension 1. We begin with a precise definition of a C^1 curve.

Definition 10.1. We say that a Jordan curve Γ is C^1 if there exists a C^1 parameterization $\gamma : [0, 2\pi] \to \mathbb{C}$ of the curve Γ satisfying $\gamma'(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$.

THEOREM 10.2. *For every $k \geq 1$, Γ_k is C^1.*

Remark 10.3. Theorem 10.2 gives only a partial answer to the question: what is the regularity of the curves Γ_k? Are they C^2, smooth, analytic? This question is also asked in [**Bis18**, §21]. The authors were not able to prove that the curves Γ_k are C^2 with the approach in the current paper.

We will consider the case $k = 1$ to simplify notation, and we will fix a point $z_0 \in \Gamma_1$ throughout this section. We will sometimes omit the subscript N from f_N and ϕ_N, and simply write f or ϕ.

Definition 10.4. For $m \geq 1$, let s_m be such that the circle $|z| = s_m$ passes through $f^m(z_0) \in V_{2+m}$ (see Figure 15), and define

$$\gamma_m^m(\theta) := s_m \exp(i\theta) \quad \text{for } \theta \in [0, 2\pi]. \tag{10.1}$$

For $0 \leq k < m$, define

$$\gamma_k^m(\theta) := \phi\left(\sqrt[n_{k+2}]{\frac{\gamma_{k+1}^m(n_{k+2} \cdot \theta)}{C_{k+2}}}\right) \quad \text{for } \theta \in [0, 2\pi], \tag{10.2}$$

where the branch of $\sqrt[n_{k+2}]{\cdot}$ chosen depends on θ and is such that equation (10.2) defines a parameterization of a Jordan curve surrounding 0.

Remark 10.5. By precomposing γ_0^m with a translation of $[0, 2\pi]$ mod 2π, we may assume there is $\theta_0 \in [0, 2\pi]$ not depending on m with $z_0 = \gamma_0^m(\theta_0) \in \gamma_0^m([0, 2\pi])$.

LEMMA 10.6. *Let $\theta \in [0, 2\pi]$. Then,*

$$(\gamma_0^m)'(\theta) = i\phi(\gamma_0^m(\theta)) \cdot \prod_{k=1}^{m-1} \frac{\phi(f^k \circ \gamma_0^m(\theta))}{f^k \circ \gamma_0^m(\theta)} \cdot \prod_{k=0}^{m-1} \frac{1}{\phi'(f^k \circ \gamma_0^m(\theta))}. \tag{10.3}$$

Proof. By Lemmas 4.12 and 4.17, we have that

$$\gamma_k^m([0, 2\pi]) \subset V_{k+2} \quad \text{for } 0 \leq k \leq m. \tag{10.4}$$

Thus, the definition in equation (10.2) is such that:

$$f^m \circ \gamma_0^m(\theta) = s_m \exp(in_2 \cdots n_{m+1}\theta) \quad \text{for } \theta \in [0, 2\pi]. \tag{10.5}$$

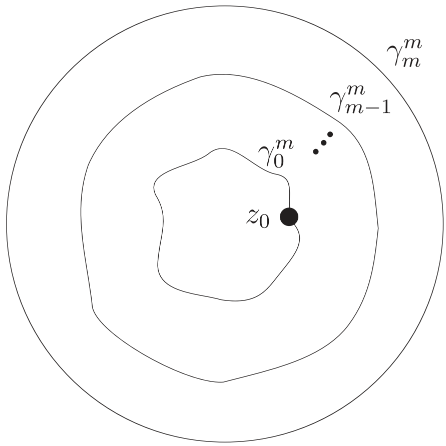

FIGURE 15. Illustration of a brief sketch of the curves γ_k^m for $0 \le k \le m$.

An application of the chain rule then yields:

$$f'(f^{m-1} \circ \gamma_0^m(\theta)) \cdot \cdots \cdot f'(\gamma_0^m(\theta)) \cdot (\gamma_0^m)'(\theta) = s_m i n_2 \cdots n_{m+1} \exp(i n_2 \cdots n_{m+1}\theta). \tag{10.6}$$

By Lemma 4.12 and equation (10.4), we have that

$$f'(z) = C_k n_k (\phi(z))^{n_k - 1} \phi'(z) = n_k f(z) \frac{\phi'(z)}{\phi(z)} \quad \text{for } z \in \gamma_{k-2}^m([0, 2\pi]). \tag{10.7}$$

Thus, equations (10.6) and (10.7) yield

$$\begin{aligned}(\gamma_0^m)'(\theta) \cdot \prod_{k=1}^{m} \left(n_{k+1} \cdot f^k(\gamma_0^m(\theta)) \cdot \frac{\phi'(f^{k-1} \circ \gamma_0^m(\theta))}{\phi(f^{k-1} \circ \gamma_0^m(\theta))} \right) \\ = s_m i n_2 \cdots n_{m+1} \exp(i n_2 \cdots n_{m+1}\theta). \end{aligned} \tag{10.8}$$

Thus, by using equation (10.5) and isolating for $(\gamma_0^m)'(\theta)$ in equation (10.8) yields equation (10.3). $\square$

Note that although the curve $\gamma_0^m([0, 2\pi])$ depends on m, the point $z_0 = \gamma_0^m(\theta_0)$ does not depend on m.

LEMMA 10.7. *The sequence $(\gamma_0^m)'(\theta_0)$ converges as $m \to \infty$.*

Proof. Let $k \ge 0$. Since $f^k \circ \gamma_0^m(\theta_0) \in V_{k+2}$ by equation (10.4), we have by Lemma 4.3 that

$$\left| \frac{\phi(f^k \circ \gamma_0^m(\theta))}{f^k \circ \gamma_0^m(\theta)} - 1 \right| \le C' \cdot 2^{(-\sqrt{k+N+2})/4}. \tag{10.9}$$

The standard Cauchy estimates then apply to show that

$$\left| \phi'(f^k \circ \gamma_0^m(\theta_0)) - 1 \right| \le C' \cdot 2^{-(\sqrt{k+N+2})/4}. \tag{10.10}$$

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Note that the right-hand sides of equations (10.9) and (10.10) are summable over k. Moreover, it is easily verified that if two complex sums $\sum_k |z_k - 1|$ and $\sum_k |w_k - 1|$ both converge, then so does $\prod_k z_k w_k$. Thus, we conclude from Lemma 10.6 and equations (10.9) and (10.10) that $(\gamma_0^m)'(\theta_0)$ converges. $\square$

To summarize, we have thus far defined the curves $\gamma_0^m([0, 2\pi])$ and their parameterizations, and we have shown that $(\gamma_0^m)'(\theta_0)$ converges as $m \to \infty$. Next, we will show that (γ_0^m) converges on the following dense subset of $[0, 2\pi]$.

Definition 10.8. Let $\Theta_m \in [0, 2\pi]$ be such that $f^m \circ \gamma_0^m(\theta_0) = \gamma_m^m(\Theta_m)$. By equation (10.2), there are $n_{m+1} \cdot \cdots \cdot n_2$ many points $\theta \in [0, 2\pi]$ such that

$$f^m(\gamma_0^m(\theta)) = \gamma_m^m(\Theta_m). \tag{10.11}$$

Denote the collection of θ satisfying equation (10.11) as $\mathcal{A}_m$, and define

$$\mathcal{A} := \bigcup_{m \geq 1} \mathcal{A}_m. \tag{10.12}$$

LEMMA 10.9. *Let $\theta \in \mathcal{A}$. Then, the sequence $(\gamma_0^k)'(\theta)$ converges as $k \to \infty$, uniformly over $\mathcal{A}$.*

Proof. Since $\theta \in \mathcal{A}$, we have that $\theta \in \mathcal{A}_m$ for some m. Consider γ_0^{m+1}. Since $z_0 \in \gamma_0^{m+1}([0, 2\pi])$, it follows from the definition of Θ_m that $\gamma_m^{m+1}([0, 2\pi])$ passes through the point $\gamma_m^m(\Theta_m)$. Moreover, by precomposing γ_m^{m+1} with a translation of $[0, 2\pi]$ mod 2π if necessary, we may assume that

$$\gamma_m^{m+1}(\Theta_m) = \gamma_m^m(\Theta_m). \tag{10.13}$$

Thus, we conclude that

$$\gamma_0^{m+1}(\theta) = \gamma_0^m(\theta) \quad \text{for } \theta \in \mathcal{A}_m, \tag{10.14}$$

and arguing recursively, we see that for all $k \geq 1$, we have

$$\gamma_0^{m+k}(\theta) = \gamma_0^m(\theta) \quad \text{for } \theta \in \mathcal{A}_m. \tag{10.15}$$

Thus, the sequence

$$(f^k \circ \gamma_0^m(\theta))_{k=1}^{\infty} \tag{10.16}$$

in fact does not depend on m. Thus, equation (10.3) and the same exact argument as for Lemma 10.7 show that in fact $(\gamma_0^k)'(\theta)$ converges as $k \to \infty$ for any $\theta \in \mathcal{A}_m$, with convergence that is uniform over m and the set $\mathcal{A}_m$. $\square$

To deduce convergence of $(\gamma_0^k)'$ on all of $[0, 2\pi]$, we will need the following proposition, whose proof is elementary and hence is omitted.

PROPOSITION 10.10. *Let X be a complete metric space, $f_n : X \to \mathbb{C}$ a sequence of uniformly continuous functions, and assume (f_n) converges uniformly on a dense subset of X. Then, the sequence f_n converges uniformly on all of X.*

LEMMA 10.11. *The functions $\gamma_k' : [0, 2\pi] \to \mathbb{C}$ converge uniformly.*

Proof. Since $\mathcal{A}$ is dense in $[0, 2\pi]$ by equation (10.2), Lemma 10.9 implies that the functions $(\gamma_0^k)'$ converge uniformly (as $m \to \infty$) on a dense subset of $[0, 2\pi]$. We conclude by Proposition 10.10 that the functions $(\gamma_0^k)'$ converge uniformly on $[0, 2\pi]$. □

To deduce that the functions γ_k converge, we will use the following elementary result (see [**Tao14**, Theorem 3.7.1]).

PROPOSITION 10.12. *Let $\gamma_k : [0, 2\pi] \to \mathbb{C}$ be a sequence of C^1 functions. Suppose that the functions γ_k' converge uniformly to a function g, and suppose furthermore that $\gamma_k(\theta_0)$ converges for some $\theta_0 \in [0, 2\pi]$. Then, the functions γ_k converge uniformly to a C^1 function $\gamma_\infty : [0, 2\pi] \to \mathbb{C}$, and $\gamma_\infty' = g$.*

LEMMA 10.13. *The functions $\gamma_k : [0, 2\pi] \to \mathbb{C}$ converge uniformly to a C^1 function γ_∞ : $[0, 2\pi] \to \mathbb{C}$.*

Proof. This is a direct application of Lemma 10.11 and Proposition 10.12. □

To prove that Γ_1 is a C^1 curve, it remains to show that γ_∞' does not vanish, and that $\gamma_\infty([0, 2\pi]) = \Gamma_1$.

LEMMA 10.14. *The curve γ_∞ satisfies $\gamma_\infty'(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$.*

Proof. Consider equation (10.3) for $\theta \in \mathcal{A}$. If we suppose by way of contradiction that $\gamma_\infty'(\theta) = 0$, then one of the infinite products in equation (10.3) must converge to 0, and so either

$$\sum_{k=1}^{\infty} \log\left(\frac{\phi(f^k \circ \gamma_0^m(\theta))}{f^k \circ \gamma_0^m(\theta)}\right) \tag{10.17}$$

or

$$\sum_{k=1}^{\infty} \log\left(\phi'(f^k \circ \gamma_0^m(\theta))\right) \tag{10.18}$$

must diverge. We will show that, in fact, both of the sums in equations (10.17), (10.18) converge. Indeed, we have

$$\left|\sum_{k=1}^{\infty} \log\left(\frac{\phi(f^k \circ \gamma_0^m(\theta))}{f^k \circ \gamma_0^m(\theta)}\right)\right| \lesssim \sum_{k=1}^{\infty} \left|\frac{\phi(f^k \circ \gamma_0^m(\theta))}{f^k \circ \gamma_0^m(\theta)} - 1\right|, \tag{10.19}$$

and the right-hand side of equation (10.19) converges by equation (10.9). Thus, equation (10.17) converges, and similarly we can use equation (10.10) to show that equation (10.18) converges. Moreover, we deduce that the sums in equations (10.17), (10.18) are bounded uniformly over $\theta \in \mathcal{A}$. Thus, we have proven that the sums in equations (10.17), (10.18) are bounded uniformly over a dense subset of $[0, 2\pi]$, and hence γ_∞' is bounded away from 0 uniformly over a dense subset of $[0, 2\pi]$. Hence, γ_∞' does not vanish on $[0, 2\pi]$. □

LEMMA 10.15. *The function γ_∞ parameterizes Γ_1, in other words, $\gamma_\infty([0, 2\pi]) = \Gamma_1$.*

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Proof. It is straightforward to see that $\gamma_\infty([0, 2\pi]) \subset \Gamma_1$. Indeed, since each $\theta \in \mathcal{A}$ satisfies $f^n(\gamma_\infty(\theta)) \in \bigcup_k V_k$ for all n, we have that $\gamma_\infty(\theta) \in \Gamma_1$ for $\theta \in \mathcal{A}$. Since $\mathcal{A}$ is dense in $[0, 2\pi]$ and Γ_1 is closed, it follows that $\gamma_\infty([0, 2\pi]) \subset \Gamma_1$. To show that $\gamma_\infty([0, 2\pi]) = \Gamma_1$, we will need to use the fact (proven in Theorem 9.15) that Γ_1 is a Jordan curve. Indeed, suppose by way of contradiction that $\gamma_\infty([0, 2\pi]) \subsetneq \Gamma_1$. Then, $\gamma_\infty([0, 2\pi])$ is a strict subset of Γ_1, and since $\gamma_\infty([0, 2\pi])$ is closed (as γ_∞ is continuous), it follows that there is an open interval $I \subset [0, 2\pi]$ such that $\gamma_\infty([0, 2\pi]) \subset \Gamma_1 \setminus \Gamma_1(I)$, where we use Γ_1 to also denote the parameterization of Γ_1. However, by Theorem 9.15, this means that $\gamma_\infty([0, 2\pi])$ has empty intersection with a sector of the form $\{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2\}$. However, then by uniform convergence, this would mean that for all sufficiently large m, we have that $\gamma_0^m([0, 2\pi])$ has empty intersection with $\{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2\}$, and this is a contradiction since each $\gamma_0^m([0, 2\pi])$ is a Jordan curve surrounding 0. $\square$

Thus, we have proven Theorem 10.2. We will deduce that Z_1 is one-dimensional, but first we need a few preliminary results.

LEMMA 10.16. *For each $k \geq 1$, Γ_k is a connected component of $\mathcal{J}(f_N)$. Moreover, the outer boundary of Ω_k is equal to the inner boundary of Ω_{k+1}, which is equal to Γ_k.*

Proof. We first show that $\Gamma_k \subset \mathcal{J}(f_N)$. If $z \in \Gamma_k$ and $\varepsilon > 0$, then for all sufficiently large n, there exists a petal $P \subset A_n$ such that $f_N^{-n}(P) \subset B(z, \varepsilon)$, where we use a branch of the inverse of $f_N : \Gamma_{k,n} \to V_n$ (see Figure 9). Since any petal contains a 0 of f_N, and $0 \in \mathcal{J}(f_N)$, it follows that $B(z, \varepsilon) \cap \mathcal{J}(f_N) \neq \emptyset$. Thus, as ε is arbitrary, we have proven $\Gamma_k \subset \mathcal{J}(f_N)$.

Next, we show that $\Gamma_k \subset \mathcal{J}(f_N)$ is indeed a component of $\mathcal{J}(f_N)$. Let K denote the component of $\mathcal{J}(f_N)$ that contains Γ_k. Since Γ_k is connected, we have $\Gamma_k \subset K$. Suppose by way of contradiction that $\Gamma_k \subsetneq K$. Note that

$$\Gamma_k := \bigcap_{n=1}^{\infty} \Gamma_{k,n} = \bigcap_{n=1}^{\infty} \{\zeta \in V_k : f_N^n(\zeta) \in V_{k+n} \text{ for all } n \geq 1\}.$$

Thus, the assumption $\Gamma_k \subsetneq K$ implies that there must be some point $\zeta \in K \setminus \Gamma_k$ and n such that $f_N^n(\zeta)$ is on the boundary of V_{k+n}. However, the boundary of V_{k+n} is mapped to the Fatou set, and this is a contradiction. Thus, $\Gamma_k \subset \mathcal{J}(f_N)$ is indeed a component of $\mathcal{J}(f_N)$.

Next, we show that Γ_k coincides with the inner boundary of Ω_{k+1}. Recall that Ω_{k+1} was defined to be the Fatou component containing B_{k+1}. Since we have proven that Γ_k is a Jordan curve component of $\mathcal{J}(f_N)$, it suffices to show that if $z \in \Gamma_k$ and $\varepsilon > 0$, then $B(z, \varepsilon) \cap \Omega_{k+1} \neq \emptyset$. Let $z \in \Gamma_k$ and $\varepsilon > 0$. As observed in the previous paragraph, the boundary of each V_k belongs to the Fatou set. Moreover, B_{k+1} and the outer boundary of V_k both belong to Ω_{k+1} by Theorem 6.9. By similar reasoning, the outer boundary of V_k and the outer boundary of $\Gamma_{k,1}$ belong to Ω_{k+1}, and recursively, we see that the outer boundary of $\Gamma_{k,n}$ belongs to Ω_{k+1} for all $n \geq 1$. Since the outer boundaries of $\Gamma_{k,n}$ limit on Γ_k by the proof of Theorem 9.15, we see that $B(z, \varepsilon) \cap \Omega_{k+1} \neq \emptyset$, as needed.

Lastly, it remains to show that Γ_k coincides with the outer boundary of Ω_k. It suffices to show that if $z \in \Gamma_k$ and $\varepsilon > 0$, then $B(z, \varepsilon) \cap \Omega_k \neq \emptyset$. Our reasoning is similar to that given in the previous paragraph. Namely, note that the outer boundary of B_k and the inner boundary of V_k both belong Ω_k by Theorem 6.9. Similarly, the inner boundary of V_k and the inner boundary of $\Gamma_{k,1}$ belong to the same Fatou component Ω_k. Recursively, we see that the inner boundaries of $\Gamma_{k,n}$ all belong to the same Fatou component Ω_k for all n. By the proof of Theorem 9.15, we see that $B(z, \varepsilon) \cap \Omega_k \neq \emptyset$ as needed. □

LEMMA 10.17. *For the set Z_1, we have $Z_1 \subset \mathcal{J}(f_N)$.*

Proof. Let $z \in Z_1$, so that by definition, there exists $l \geq 0$ so that for all $j \geq 0$, $f_N^l(z) \in \bigcup_{k\geq 1} V_k$. Since $z \in Z$, we may, by perhaps increasing l, further assume that $f_N^l(z)$ never moves backwards. Let $m \geq 1$ be such that $f_N^l(z) \in V_m$. Since $f_N^l(z)$ does not move backwards and $f_N^l(z) \in \bigcup_{k\geq 1} V_k$, we deduce that $f_N^{l+1}(z) \in V_{m+1}$. By similar reasoning, we see that in fact, $f_N^{l+j}(z) \in V_{m+j}$ for all $j \geq 0$. Thus, by Definition 9.5, $f_N^l(z) \in \Gamma_{m-1}$. By Lemma 10.16, $\Gamma_{m-1} \subset \mathcal{J}(f_N)$, and so $f_N^l(z) \in \mathcal{J}(f_N)$. □

LEMMA 10.18. *Let Γ be a component of Z_1. Then, there exist $p, n \geq 1$ and a Jordan domain B containing Γ such that $f_N^n|_B$ is conformal, and $f_N^n(\Gamma) = \Gamma_p$.*

Proof. It will be convenient to denote $V := \bigcup_{j\geq 1} V_j$. Let Γ be a component of Z_1, and let $z \in \Gamma$. Since $z \in Z$, there is a positive integer $m \geq 0$ so that z moves backwards precisely m times. Thus, there is an element $W_k^n \in \mathcal{C}_m$ containing z. Let us first assume $k \geq 1$. By Lemma 8.4, there exists a Jordan domain B containing W_k^n such that $f_N^n(B) \to B(0, 4R_{k+1})$ is conformal and $f_N^n(W_k^n) = A_k$. In particular, since $\partial A_k \subset \mathcal{F}(f_N)$, we deduce that $\Gamma \subset W_k^n$. In particular, we have that $f_N^n(\Gamma) \subset A_k$. By our choice of $W_k^n \in \mathcal{C}_m$, we have that $f_N^n(\Gamma)$ can only move forward. Moreover, by Lemma 6.9, we have that either $f_N^n(\Gamma) \subset V_k$ or $f_N^n(\Gamma)$ is a subset of a petal $P_k \subset \mathcal{P}_k$. Since $z \in Z_1$, there exists a smallest $l \geq 0$ and $p \geq 1$ such that $f_N^{n+l}(z) \in V_p$ and $f_N^{n+l+j}(z) \in V$ for all $j \geq 0$. Moreover, by Lemma 6.7, there exists a Jordan domain B' with $W_n^k \subset B' \subset B$ such that $f_N^l : f_N^n(B') \to f_N^{n+l}(B')$ is conformal. Thus, $f_N^{n+l}(\Gamma) \subset V_p$. Now, consider an arbitrary $z' \in \Gamma$. Since $f_N^{n+l}(z')$ only moves forward and $f_N^{n+l+j}(z') \in V$ for all $j \geq 0$, we have that $f_N^{n+l+j}(z') \in V_{p+j}$ for all $j \geq 0$. Thus, by Definition 9.5, $f_N^{n+l}(z') \in \Gamma_{p-1}$. Since $z' \in \Gamma$ was arbitrary, we have that $f_N^{n+l}(\Gamma) \subset \Gamma_{p-1}$. Lastly, since $f_N^{n+l} : B' \to f_N^{n+l}(B')$ is conformal, we have that $f_N^{n+l}(\Gamma) = \Gamma_{p-1}$, and hence the proof is finished in the case where $k \geq 1$. If $k < 1$, by Lemma 8.4, we have a Jordan domain B containing W_k^n such that $f_N^n|_B$ is conformal, and $f_N^n(W_k^n) = A_k$ is mapped conformally onto A_1, whence the above reasoning applies. □

COROLLARY 10.19. *The set Z_1 is a countable disjoint union of C^1 Jordan curves. In particular, the Hausdorff dimension of Z_1 is equal to* 1.

Proof. By Theorem 10.2 and Lemma 10.18, each component of Z_1 is a conformal image of a C^1 Jordan curve, and hence each component of Z_1 is a C^1 Jordan curve. Since any Jordan curve has non-empty interior, there can be at most countably many components of Z_1. Lastly, $\dim(Z_1) = 1$ follows from Lemma A.10. □

11. *Singleton boundary components*

In this last section, we analyze finally the set Z_2. Recall that we have proven that

$$\mathcal{J}(f) \subset E' \cup Y \cup Z_1 \cup Z_2,$$

and so to prove part (1) of Theorem 1.1, it only remains to estimate the dimension of Z_2. In this section, we will prove that, in fact, Z_2 has dimension 0 and consists of uncountably many singletons. We begin by constructing a sequence of covers for Z_2.

Recall from Definition 9.1 that

$$Z_2 = \left\{ z \in Z : \text{ there exist arbitrarily large } n \text{ such that } f_N^n(z) \notin \bigcup_{k \geq 1} V_k \right\}. \tag{11.1}$$

We first analyze Z_2 intersected with the closure of a Fatou component Ω_k.

LEMMA 11.1. *Suppose that $z \in Z_2 \cap \overline{\Omega_k}$ for some $k \geq 1$. Then, $z \in A_k$.*

Proof. Since $z \in Z_2 \subset X$, the orbit sequence of z is $(k(z, n))_{n=0}^{\infty}$, and we have

$$f_N^n(z) \in A_{k(z,n)}$$

for all $n \geq 0$.

Note that $\overline{\Omega_k} \subset A_k \cup B_k \cup A_{k+1}$. Since $z \in X$, we must have $z \in A_k$ or $z \in A_{k+1}$. Suppose for the sake of contradiction that we had $z \in A_{k+1}$. Since $z \in \overline{\Omega_k}$, we have $f_N^l(z) \in \overline{\Omega_{k+l}}$ for all $l \geq 0$. The outermost boundary component of Ω_k is Γ_k by Lemma 10.16, and $\Gamma_k \subset V_{k+1}$ by equation (9.7). Therefore, we must have

$$z \in A\left(\tfrac{1}{4} R_{k+1}, \tfrac{3}{5} R_{k+1}\right).$$

By Lemma 6.4, $f_N(A(\frac{1}{4} R_{k+1}, \frac{2}{5} R_{k+1})) \subset B_{k+1}$, so since $z \in X$, we must have $z \in V_{k+1}$. Since $f_N(V_{k+1}) \subset B_{k+1} \cup A_{k+2} \cup B_{k+2}$ by Corollary 6.6, we deduce that $f_N(z) \in A_{k+2} \cap \overline{\Omega_{k+1}}$.

By repeating the reasoning above, we deduce that $f_N^l(z) \in V_{k+l+1}$ for all $l \geq 0$. This contradicts the fact that $z \in Z_2$, so we must have $z \in A_k$. □

LEMMA 11.2. *Suppose that $z \in \overline{\Omega_k} \cap Z_2$ for some $k \geq 1$. Then, $z \in \partial\Omega_k$.*

Proof. Recall that

$$\{z \in \mathbb{C} : 4R_k \leq |z| \leq R_{k+1}/4\} \subset \Omega_k. \tag{11.2}$$

Suppose for the sake of contradiction that $z \in \Omega_k$. Then, there exists $\varepsilon > 0$ so that $B(z, \varepsilon) \subset \Omega_k$. By [**BRS13**, Theorem 1.2], there exists $m > 0$ and $\alpha > 0$ so that for all $n \geq m$, we have

$$A(|f_N^n(z)|^{1-\alpha}, |f_N^n(z)|^{1+\alpha}) \subset f_N^n(B(z, \varepsilon)) \subset \Omega_{k+n}. \tag{11.3}$$

By Lemma 11.1,

$$\tfrac{1}{4} R_{k+j} \leq |f_N^j(z)| \leq 4R_{k+j} \tag{11.4}$$

holds for all $j \geq 0$. Notice that by Lemma 4.8,

$$\frac{(1/4^{1+\alpha})R_{k+n}^{1+\alpha}}{4^{1-\alpha}R_{k+n}^{1-\alpha}} = \frac{1}{16}R_{k+n}^{2\alpha} \xrightarrow{n\to\infty} \infty.$$

Then, by perhaps increasing m, we have for all $n \geq m$ that

$$\frac{(1/4^{1+\alpha})R_{k+n}^{1+\alpha}}{4^{1-\alpha}R_{k+n}^{1-\alpha}} > 1.$$

Therefore, for all $n \geq m$, the annulus $A(4^{1-\alpha}R_{k+n}^{1-\alpha}, (1/4^{(1+\alpha)})R_{k+n}^{1+\alpha})$ is not empty and

$$A\left(4^{1-\alpha}R_{k+n}^{1-\alpha}, \frac{1}{4^{(1+\alpha)}}R_{k+n}^{1+\alpha}\right) \subset A(|f_N^n(z)|^{1-\alpha}, |f_N^n(z)|^{1+\alpha}) \subset \Omega_{k+n}. \tag{11.5}$$

By perhaps increasing m one last time, we can use Lemma 4.8 to deduce that for all $n \geq m$, we have

$$4^{1-\alpha}R_{k+n}R_{k+n}^{-\alpha} < \tfrac{1}{4}R_{k+n}. \tag{11.6}$$

By equations (11.3) and (11.6), we deduce that $f_N^n(B(z,\varepsilon))$ contains a point $w \in B_{k+n-1} \subset \Omega_{k+n-1}$. This is a contradiction to the fact that $f_N^n(B(z,\varepsilon)) \subset \Omega_{k+n}$ for all $n \geq m$. □

Recall that we introduced the petals $P_j \subset \mathcal{P}_j$ for all $j \geq 1$ in Definition 6.2.

LEMMA 11.3. *Suppose that $z \in Z_2 \cap A_k$ and suppose that the orbit sequence of z is $(k(z,n))_{n=0}^{\infty} = (k, k+1, k+2, \ldots)$. Then,*

$$z \in \bigcap_{l=1}^{\infty}\left(\bigcup_{j\geq l} f_N^{-j}(\mathcal{P}_{k+j}) \cap A_k\right). \tag{11.7}$$

Proof. Since $z \in A_k$ and $(k(z,n))_{n=0}^{\infty} = (k, k+1, k+2, \ldots)$, we have $f_N^l(z) \in A_{k+l}$ for all $l \geq 0$. Since $z \notin Z_1$, there exists infinitely many positive integers j so that $f_N^j(z) \notin V_{k+j}$. For those values of j, we still must have $f_N^{j+1}(z) \in A_{k+j+1}$, so Lemma 6.10 implies that $f_N^j(z) \in \mathcal{P}_{k+j}$. The inclusion in equation (11.7) follows immediately. □

LEMMA 11.4. *Suppose that $z \in \partial\Omega_k$ for some $k \geq 1$. Then, $z \in Z$ and the orbit sequence of z is either $(k(z,n))_{n=0}^{\infty} = (k, k+1, \ldots)$ or $(k(z,n))_{n=0}^{\infty} = (k+1, k+2, \ldots)$. In the latter case, we must have $z \in \Gamma_k \subset Z_1$.*

Proof. Let $z \in \partial\Omega_k$. First, recall that $\partial\Omega_k \subset A_k \cup A_{k+1}$. Since Ω_k is a bounded Fatou component, we have $f(\partial\Omega_k) = \partial\Omega_{k+1}$ (see [**BRS13**, paragraph above Theorem 3.2]). Therefore, we have $f_N^n(z) \in A_{k+n} \cup A_{k+n+1}$ for all $n \geq 0$.

Suppose that $z \in A_{k+1} \cap \partial\Omega_k$. We argue similarly to Lemma 11.1. By Lemma 10.16 and equation (9.7), the outermost boundary of Ω_k is a subset of V_{k+1}. Therefore, we must have

$$z \in A\left(\tfrac{1}{4}R_{k+1}, \tfrac{3}{5}R_{k+1}\right).$$

Since $f_N(A(\frac{1}{4}R_{k+1}, \frac{2}{5}R_{k+1})) \subset B_{k+1}$ and $z \in \mathcal{J}(f_N)$, by Lemma 6.1, we must have $z \in V_{k+1}$. Since $f_N(V_{k+1}) \subset B_{k+1} \cup A_{k+2} \cup B_{k+2}$ and $f_N(z) \in \partial\Omega_{k+1}$, we obtain $f_N(z) \in A_{k+2}$.

By iterating the reasoning above, we deduce that $f_N^l(z) \in V_{k+l+1}$ for all $l \geq 0$, so that $z \in \Gamma_k$ and $(k(z, n))_{n=0}^{\infty} = (k+1, k+2, \ldots)$.

The other possibility is that $z \in A_k \cap \partial\Omega_k$. Since $f_N(z) \in \partial\Omega_{k+1}$, we must have $f_N(z) \in A_{k+1} \cup A_{k+2}$. By Lemma 7.20, we must have $f_N(z) \in A_{k+1}$. By repeating this reasoning, we deduce that $f_N^l(z) \in A_{k+l}$ for all $l \geq 0$, and $(k(z, n))_{n=0}^{\infty} = (k, k+1, \ldots)$. □

COROLLARY 11.5. *For all $k \geq 1$, we have*

$$Z_2 \cap \partial\Omega_k \subset \bigcap_{l=1}^{\infty} \bigg(\bigcup_{j \geq l} f_N^{-j}(\mathcal{P}_{k+j}) \cap A_k \bigg). \tag{11.8}$$

Proof. Let $z \in Z_2 \cap \partial\Omega_k$. By Lemma 11.1, we have $z \in A_k$, and by Lemma 11.4, the orbit sequence of z must be $(k(z, n))_{n=0}^{\infty} = (k, k+1, k+2, \ldots)$. The result now follows from Lemma 11.3. □

To estimate the Hausdorff dimension of $Z_2 \cap \partial\Omega_k$, we will need the following estimates on the expansion of f_N on the petals $\mathcal{P}_k$ for all $k \geq 1$.

LEMMA 11.6. *There exists M so that for all $N \geq M$ and for all $k \geq 1$ and all $z \in \mathcal{P}_k$, we have*

$$|f_N'(z)| \geq \frac{1}{4} n_k \frac{R_{k+1}}{R_k}. \tag{11.9}$$

Proof. Let w be a zero of f_N contained inside of some connected component P_k of $\mathcal{P}_k$. First, note that there exists M so that for all $N \geq M$ and all $k \geq 1$, the modulus of $B(w, \lambda(\exp(\pi/n_k) - 1)R_k) \setminus \overline{B(w, R_k/2^{n_k})}$ is bounded below $(2\pi)^{-1} \log 2$. Therefore, by Theorem A.2 and Lemma 6.7, there exists a constant $P \geq 1$ that does not depend on k or N so that for all $z \in B(w, R_k/2^{n_k})$, we have

$$\frac{1}{P} \leq \frac{|f_N'(z)|}{|f_N'(w)|} \leq P. \tag{11.10}$$

Therefore, by equation (6.20), we have

$$|f_N'(z)| \geq \frac{1}{P}|f_N'(w)| \geq \frac{1}{P} \frac{\delta 2^{n_k}}{8\lambda\pi} n_k \frac{R_{k+1}}{R_k}.$$

By Lemma 4.8, there exists $M \in N$ so that for all $N \geq M$, we have

$$\frac{1}{P} \frac{\delta 2^{n_k}}{8\lambda\pi} \geq \frac{1}{4}. \tag{11.11}$$

Equation (11.9) follows immediately. □

COROLLARY 11.7. *Fix some $k \geq 2$. Suppose that $z \in f_N^{-j}(\mathcal{P}_{k+j}) \cap A_k$ for some $j \geq 1$. Then,*

$$|(f_N^j)'(z)| \geq \frac{1}{4^j} \frac{R_{k+j}}{R_k} \prod_{l=0}^{j-1} n_{l+k}. \tag{11.12}$$

Proof. By repeatedly applying Lemma 6.10, we see that for each $l = 1, \ldots, j$, we have either $f_N^{-l}(P_{k+j})$ belongs to $\mathcal{P}_{k+j-l}$ or V_{k+j-l}. Therefore, if $z \in f_N^{-j}(P_{k+j}) \cap A_k$, we have by Lemmas 9.11 and 11.6 and the chain rule that

$$|(f_N^j)'(z)| \geq \prod_{l=0}^{j-1} \frac{1}{4} n_{l+k} \frac{R_{l+k+1}}{R_{l+k}} = \frac{1}{4^j} \frac{R_{k+j}}{R_k} \cdot \prod_{l=0}^{j-1} n_{l+k}.$$

This is exactly what we wanted to show. □

THEOREM 11.8. *We have* $\dim_H(Z_2 \cap \partial\Omega_k) = 0$.

Proof. Let $j \geq 1$ and W be a connected component of $f_N^{-j}(P_{k+j}) \cap A_k$. If $z \in W$, then we have by Theorem A.3 that there exists a constant P' independent of N, k, and j such that

$$\text{diameter}(W) \leq \frac{P'}{|(f_N^j)'(z)|} \text{diameter}(P_{k+j}) \leq \frac{P' 4^j}{\prod_{l=0}^{j-1} n_{l+k}} \frac{R_k}{R_{k+j}} \frac{R_{k+j}}{2^{n_{k+j}}} \leq P' \frac{R_k}{2^{n_{k+j}}}. \tag{11.13}$$

Fix some $t > 0$. For $j \geq 1$, we define

$$G_j = \{W \ : W \subset A_k \text{ and } f_N^j(W) = P_{k+j} \text{ for some } P_{k+j} \subset \mathcal{P}_{k+j}\}.$$

For each petal P_{k+j}, there are $n_{k+1} \cdots n_{k+j}$ many connected components $W \in G_j$ by Lemma 7.22. Since there are n_{k+j} many connected components of $\mathcal{P}_{k+j}$, and recalling that $L_k = n_1 \cdots n_k$, we count the number of connected components of G_j as

$$n_{k+1} \cdots n_{k+j} \cdot n_{k+j} = 2^j n_k \cdots n_{k+j-1} \cdot n_{k+j} \leq 2^j L_{k+j}.$$

Therefore, we have

$$\sum_{W \in G_j} \text{diameter}(W)^t \leq 2^j L_{k+j} (P')^t \frac{R_k^t}{2^{t \cdot n_{k+j}}}. \tag{11.14}$$

Therefore, for any fixed $l \geq 1$,

$$\sum_{j \geq l} \sum_{W \in G_j} \text{diameter}(W)^t \leq (P')^t \cdot R_k^t \sum_{j \geq l} 2^j L_{k+j} \left(\frac{1}{2^t}\right)^{n_{k+j}}. \tag{11.15}$$

This series converges by the ratio test. Indeed,

$$\lim_{j \to \infty} \frac{2^{j+1}}{2^j} \frac{L_{k+j+1}}{L_{k+j}} \left(\frac{1}{2^t}\right)^{n_{k+j}} = \lim_{j \to \infty} 4 n_{k+j} \left(\frac{1}{2^t}\right)^{n_{k+j}} = 0. \tag{11.16}$$

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Since equation (11.15) converges, we have that for any $\varepsilon > 0$, if l is sufficiently large, then

$$\sum_{j \geq l} \sum_{W \in G_j} \mathrm{diameter}(W)^t < \varepsilon.$$

By Corollary 11.5, we conclude that $H^t(Z_2 \cap \partial\Omega_k) = 0$. Since $t > 0$ was arbitrary, we further conclude that $\dim_H(Z_2 \cap \partial\Omega_k) = 0$. □

Now that we know that $\dim_H(Z_2 \cap \partial\Omega_k) = 0$ for all $k \geq 0$, we move on to estimating $\dim_H(Z_2)$.

LEMMA 11.9. *Suppose that $z \in A_k$ for some $k \geq 1$. Suppose further that the orbit sequence of z is $k(z, n) = (k, k+1, k+2, \ldots)$. Then, $z \in \overline{\Omega_k}$.*

Proof. Since $z \in Z_2 \cap A_k$, by Lemma 11.3, we have

$$z \in \bigcap_{l=1}^{\infty} \bigg(\bigcup_{j \geq l} f_N^{-j}(\mathcal{P}_{k+j}) \cap A_k \bigg).$$

Therefore, there exists a sequence (l_j) of increasing integers so that $f_N^{l_j}(z) \in P_{k+l_j}$ for some petal $P_{k+l_j} \in \mathcal{P}_{k+l_j}$. Let W_{l_j} denote the connected component of $f_N^{-l_j}(P_{k+l_j})$ that contains z. By equation (11.13), $\mathrm{diameter}(W_{l_j}) \to 0$ as $j \to \infty$.

Let $\varepsilon > 0$ be given. Then, there exists l_j such that $W_{l_j} \subset B(z, \varepsilon)$. By equation (6.17), there exists a point $w \in P_{k+l_j}$ so that $|f_N(w)| = 3R_{k+l_j+1}$. By Lemma 6.3, $f_N^2(w) \in B_{k+l_j+2}$, so that $f_N^2(w) \in \Omega_{k+l_j+2}$. Therefore, the element of $f_N^{-l_j}(w)$ that belongs to W_{l_j} belongs to Ω_k. Therefore, $B(z, \varepsilon) \cap \Omega_k$ is not empty, and since $\varepsilon > 0$ was arbitrary, it follows that $z \in \overline{\Omega_k}$. □

COROLLARY 11.10. *We have*

$$Z_2 \subset \bigcup_{j \geq 0} f_N^{-j} \bigg(\bigcup_{k=1}^{\infty} Z_2 \cap \partial\Omega_k \bigg). \tag{11.17}$$

Moreover, we have $\dim_H(Z_2) = 0$.

Proof. Since $z \in Z_2$, there exists $m \geq 0$ and $k \geq 1$ so that $f_N^m(z) \in A_k$, and the orbit sequence of $f_N^m(z)$ is strictly increasing and given by $(k, k+1, \ldots)$. It follows that $f_N^m(z) \in \overline{\Omega_k}$ by Lemma 11.9. Thus, by Lemma 11.2, we have that $f_N^m(z) \in \partial\Omega_k$. Therefore, equation (11.17) holds.

By equations (A.12) and (A.11), it follows from equation (11.17) that $\dim_H(Z_2) = 0$. □

COROLLARY 11.11. *The set Z_2 is totally disconnected. In particular, every connected component of Z_2 is a point.*

Proof. The Hausdorff dimension of any non-singleton connected set is bounded below by 1. Since $\dim_H(Z_2) = 0 < 1$, Z_2 cannot have any non-singleton connected components. □

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COROLLARY 11.12. *Let $k \geq 1$. Then, $\partial\Omega_k$ consists of countably many C^1 smooth Jordan curves and uncountably many singleton components. The singleton components coincide with $\Omega_k \cap Z_2$ and the C^1 smooth components coincide with $\Omega_k \cap Z_1$.*

Proof. Since $\partial\Omega_k \subset Z$ by Lemma 11.4, we have

$$\partial\Omega_k = (\partial\Omega_k \cap Z_1) \sqcup (\partial\Omega_k \cap Z_2).$$

Every component of $\Omega_k \cap Z_1$ is a C^1 smooth Jordan curve by Corollary 10.19, and every component of $\Omega_k \cap Z_2$ is a singleton by Corollary 11.11. There are uncountably many such components by [**RS19**, Theorem 7.1]. □

COROLLARY 11.13. *Let Ω be a Fatou component of f_N. Then, $\partial\Omega$ consists of uncountably many singleton components and countably many C^1 smooth Jordan curves.*

Proof. Note that every Fatou component of f_N is bounded. Let Γ denote the connected component of the boundary of Ω that separates Ω from ∞. Since $\Gamma \subset \mathcal{J}(f_N)$, by Lemma 7.18, we have $\Gamma \subset E' \cup X = E' \cup Y \cup Z_1 \cup Z_2$. Since Γ is a non-singleton connected set, we must have $\Gamma \subset Z_1$. By Lemma 10.18, there exists $p, n \geq 1$ and a Jordan domain B containing Γ such that $f_N^n|_B$ is conformal and $f_N^n(\Gamma) = \Gamma_p$. Consequently, we have $f_N^n(\Omega) = \Omega_p$ and $f_N^n|_\Omega$ is conformal. The result now follows from Corollary 11.12. □

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A. *Appendix. Supplementary details*

In this appendix, we collect several classical theorems and definitions used throughout the paper, and we will briefly prove a technical result (needed in §6) on the behavior of the interpolating map of [**BL23**] near its zeros. We begin with the statements of some classical distortion theorems for conformal mappings.

THEOREM A.1. (Koebe 1/4 theorem) *Let $D \subset \mathbb{C}$ be a domain and let $z_0 \in D$, and suppose that $f : D \to f(D)$ is conformal. Then,*

$$\frac{1}{4}|f'(z_0)| \leq \frac{\operatorname{dist}(f(z_0), \partial(f(D)))}{\operatorname{dist}(z_0, \partial D)} \leq 4|f'(z_0)|. \tag{A.1}$$

THEOREM A.2. *Let D be a simply connected domain and let $f : D \to f(D)$ be a conformal mapping. Let U be a relatively compact subset of D. Then, there is a constant C that depends only on the modulus of $D \setminus \overline{U}$ such that*

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$$\frac{1}{C} \leq \sup_{z,w\in U} \frac{|f'(z)|}{|f'(w)|} \leq C. \tag{A.2}$$

Next, we need the following consequence of the Koebe distortion theorem. The statement below is [**McM94**, Theorem 2.9].

THEOREM A.3. *Let U and D be simply connected domains with U compactly contained in D. Let $f : D \to f(D)$ be conformal. Then, there exists a constant C that depends only on the modulus of $D \setminus \overline{U}$ such that for any $x, y, z \in U$, we have*

$$\frac{1}{C}|f'(x)| \leq \frac{|f(y) - f(z)|}{|y - z|} \leq C|f'(x)|. \tag{A.3}$$

Using the BiLipschitz estimate (A.3), we obtain the following corollary.

COROLLARY A.4. (Koebe distortion theorem) *Let D be simply connected, let U be open and compactly contained in D, and let K be a compact subset of $\overline{U}$. Suppose $f : D \to f(D)$ is conformal. Then, there is a constant C that depends only on the modulus of $D \setminus \overline{U}$ so that*

$$\frac{1}{C}\frac{\text{diameter}(K)}{\text{diameter}(U)} \leq \frac{\text{diameter}(f(K))}{\text{diameter}(f(U))} \leq C\frac{\text{diameter}(K)}{\text{diameter}(U)}. \tag{A.4}$$

We can also deduce the following corollary using equation (A.3), but we first need the following definitions.

Definition A.5. Let $f : D \to f(D)$ be a conformal mapping, and $B = B(z_0, r)$ be compactly contained in D. We define the *inner radius* of $f(B)$ by

$$r_{f(B),f(z_0)} := \sup\{t : B(f(z_0), t) \subset f(B)\}. \tag{A.5}$$

We similarly define the *outer radius* of $f(B)$ by

$$R_{f(B),f(z_0)} := \inf\{t : f(B) \subset B(f(z_0), t)\}. \tag{A.6}$$

COROLLARY A.6. *Let D be a simply connected domain and let $f : D \to f(D)$ be conformal. Let $B = B(z_0, r)$ be a disk compactly contained inside of D. Then, there exists a constant C that depends only on the modulus of $D \setminus \overline{B}$ so that*

$$C^{-1}|f'(z_0)|r \leq r_{f(B),f(z_0)} \leq R_{f(B),f(z_0)} \leq C|f'(z_0)|r. \tag{A.7}$$

Remark A.7. In this paper, we will frequently encounter the following situation. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let D_n be a sequence of simply connected domains in $\mathbb{C}$. Let U_n be open and relatively compact in D_n, and let K_n be a compact subset of U_n. Suppose that f, when restricted to D_n, is conformal, and suppose that the modulus of $D_n \setminus \overline{U_n}$ is bounded below by some fixed constant $\delta > 0$ that does not depend on n. Then, there exists a single constant C so that equation (A.4) holds for all pairs of domains U_n and K_n. A similar assertion is true for equation (A.7).

We now recall some basic facts about Hausdorff dimension, following [**Mat95**].

Definition A.8. Let $A \subset \mathbb{C}$ be a set. We define the *α-Hausdorff measure* of A to be the quantity

$$H^{\alpha}(A) := \lim_{\delta \to 0} H^{\alpha}_{\delta}(A)$$
$$:= \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} \text{diameter}(U_i)^{\alpha} : A \subset \bigcup_{i=1}^{\infty} U_i,\ \text{diameter}(U_i) < \delta \right\} \right), \quad \text{(A.8)}$$

where the infimum is taken over all countable covers of A by sets $\{U_i\}_{i=1}^{\infty}$.

One can verify by directly using Definition A.8 that if $H^t(A) < \infty$, then $H^s(A) = 0$ for all $s > t$, and similarly, if $H^t(A) > 0$, then $H^s(A) = \infty$ for all $s < t$. Therefore, the following definition is well defined.

Definition A.9. Let $A \subset \mathbb{C}$ be a set. The *Hausdorff dimension* of A is defined to be

$$\dim_H(A) := \sup\{t : \ H^t(A) = \infty\} = \inf\{t : H^t(A) = 0\}. \quad \text{(A.9)}$$

We also use the following well-known facts about Hausdorff dimension.

LEMMA A.10. *Let $A \subset \mathbb{C}$ be a set and let $s \geq 0$. Then:*

(1) *$H^s(A) = 0$ if and only if for all $\varepsilon > 0$, there exists sets $E_i \subset \mathbb{C}$, $i = 1, 2, \ldots$ such that $A \subset \bigcup_{i=1}^{\infty} E_i$ and*

$$\sum_{i=1}^{\infty} \text{diameter}(E_i)^s < \varepsilon; \quad \text{(A.10)}$$

(2) *suppose that $A = \bigcup_{i=1}^{\infty} A_i$ for some sets $A_i \subset \mathbb{C}$. Then,*

$$\dim_H(A) = \dim_H\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_{i \geq 1} \dim_H(A_i); \quad \text{(A.11)}$$

(3) *let $S \subset \mathbb{C}$ be a set and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then,*

$$\dim_H(S) = \dim_H(f(S)) = \dim_H(f^{-1}(S)). \quad \text{(A.12)}$$

We will now record some useful lemmas about branched coverings that are topological in nature. The following are [**RGS19**, Propositions 3.1 and 3.2].

LEMMA A.11. *Let $f : X \to Y$ be a branched covering map between two non-compact, simply connected Riemann surfaces. Suppose that $U \subset Y$ is a simply connected domain and let U' be a connected component of $f^{-1}(U)$ such that U' contains only finitely many critical points of f. Then, $f : U' \to U$ is a proper map and U' is simply connected. Additionally, if the boundary of U is a Jordan curve in Y that contains no critical values of f, then the boundary of U' is a Jordan curve in X.*

LEMMA A.12. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function and suppose that $U \subset \mathbb{C}$ is simply connected. Suppose that U contains no asymptotic values of f and that the critical*

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values of f inside of U form a discrete set. Then, f is a branched covering from each connected component of $f^{-1}(U)$ onto U.

The following version of the Riemann–Hurwitz formula is from [Ste93].

THEOREM A.13. *Let V and W be domains in $\mathbb{C}$, and suppose the connectivity (the number of complementary components) of V is m and the connectivity of W is n. Let $f : V \to W$ be a proper, branched covering map of degree k that has r many critical points. Then,*

$$m - 2 = k(n - 2) + r. \tag{A.13}$$

We also make use of polynomial-like mappings, see [DH85].

Definition A.14. Let Ω, $\Omega' \subset \mathbb{C}$ be Jordan domains and suppose that Ω is compactly contained inside of Ω'. A holomorphic mapping $f : \Omega \to \Omega'$ is called a *degree d polynomial-like mapping* if it is a proper, degree d, branched covering map. Given a polynomial-like mapping, we denote its *filled Julia set* by

$$K_f = \bigcap_{n=1}^{\infty} f^{-n}(\Omega).$$

We make frequent use out of the following lemma.

LEMMA A.15. *Suppose that $f : X \to Y$ is a degree d branched covering map between two simply connected planar domains with only finitely many critical points. Let $U \subset Y$ be a Jordan domain. Suppose that $\overline{U}$ does not contain any critical values of f. Then, there are d many connected components of $f^{-1}(U) \subset X$, each of which is a Jordan domain that is mapped conformally onto U by f.*

Proof. Since $f : X \to Y$ only has finitely many critical points in X, every connected component U' of $f^{-1}(U)$ is a Jordan domain and $f : U' \to U$ is proper, finitely branched covering map by Lemmas A.11 and A.12. Since $\overline{U}$ contains no critical values of $f : X \to Y$, the mapping $f : U' \to U$ has no critical points. Since U' and U are each Jordan domains, it follows from Theorem A.13 that $f : U' \to U$ is conformal. Since f is degree d, it follows that we must have d many connected components of $f^{-1}(U)$. $\square$

Next, we state the main result of [BL23], which is central to the proof of Theorem 1.1. We refer the reader to [BL23] for a detailed discussion and proof.

Definition A.16. Let $(M_j)_{j=1}^{\infty} \in \mathbb{N}$ be increasing and $(r_j)_{j=1}^{\infty} \in \mathbb{R}^+$. We say that $(M_j)_{j=1}^{\infty}$, $(r_j)_{j=1}^{\infty}$ are *permissible* if

$$r_{j+1} \geq \exp(\pi / M_j) \cdot r_j \quad \text{for all } j \in \mathbb{N}, r_j \xrightarrow{j\to\infty} \infty, \text{ and } \sup_j \frac{M_{j+1}}{M_j} < \infty. \tag{A.14}$$

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THEOREM A.17. *Let $(M_j)_{j=1}^{\infty}$, $(r_j)_{j=1}^{\infty}$ be permissible, $r_0 := 0$, and $c \in \mathbb{C}^\star := \mathbb{C} \setminus \{0\}$. Set*

$$c_1 := c \quad \text{and} \quad c_j := c_{j-1} \cdot r_{j-1}^{M_{j-1}-M_j} = c \cdot \prod_{k=2}^{j} r_{k-1}^{M_{k-1}-M_k} \quad \text{for } j \geq 2. \tag{A.15}$$

Then, there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ and a quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ such that

$$f \circ \phi(z) = c_j z^{M_j} \quad \text{for } r_{j-1} \cdot \exp(\pi/M_{j-1}) \leq |z| \leq r_j,\ j \in \mathbb{N}. \tag{A.16}$$

Moreover, if $\sum_{j=1}^{\infty} M_j^{-1} < \infty$, then $|\phi(z)/z - 1| \to 0$ as $z \to \infty$. The only singular values of f are the critical values $(\pm c_j r_j^{M_j})_{j=1}^{\infty}$.

Finally, we record the following important lemma that is used in §6.

LEMMA A.18. *Let $g_{n,2n}$ be the function in* [**BL23**, Proposition 3.13] *and let w be a zero of g contained inside $A(1, \exp(\pi/n))$. There exists constants $0 < \lambda < 1/8$ and $\delta > 0$, which do not depend on n, so that*

$$B(0, \delta) \subset g_{n,2n}(B(w, \lambda(\exp(\pi/n) - 1))) \subset B(0, 1/2). \tag{A.17}$$

Moreover, $g_{n,2n}$ is injective on $B(w, \lambda(\exp(\pi/n) - 1))$.

Proof. It is possible to prove this result directly from the definition of $g_{n,2n}$ [**BL23**, Definition 3.11], but it will be more straightforward if we use some general results about quasiconformal mappings. We will let $B_\lambda := B(w, \lambda(\exp(\pi/n) - 1))$ and denote the Jacobian of g by J_g.

Let w be a zero of g contained inside $A(1, \exp(\pi/n))$. It follows from [**BL23**, Definition 3.11] that g is injective (and hence quasiconformal) in

$$B\bigg(w, \frac{\exp(\pi/2n) - 1}{2}\bigg).$$

We first show that the first inclusion in equation (A.17) holds for small λ. To this end, we appeal to [**AG85**, Theorem 1.8], which implies that there is a constant c depending only on $K(g_{n,2n})$ (in particular, c does not depend on n or λ) so that

$$d(g(w), \partial g(B_\lambda)) \geq c \cdot \lambda(\exp(\pi/2n) - 1) \cdot \exp\bigg(\frac{1}{2m(B_\lambda)} \int_{B_\lambda} \log(J_g)\bigg). \tag{A.18}$$

It is readily calculated that there is a constant C independent of n and λ such that $J_g(z) \geq C \cdot n^2$ for $z \in B_\lambda$. Thus, from equation (A.18), we conclude that

$$d(g(w), \partial g(B_\lambda)) \geq c\lambda(\exp(\pi/2n) - 1)Cn \geq c\lambda \cdot \pi/2n \cdot Cn = c\lambda\pi C/2.$$

We conclude that the first inclusion in equation (A.17) holds for δ that depends only on λ.

Next, we show that for sufficiently small λ the second inclusion in equation (A.17) also holds. Indeed, since quasiconformal mappings are quasisymmetric (see

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[**AIM09**, Theorem 3.6.2]), there exists a constant η depending only on $K(g_{n,2n})$ (and, in particular, η does not depend on n) such that for all $\lambda \leq (\exp(\pi/2n) - 1)/4$, we have

$$\sup_{\theta \in [0,2\pi]} \frac{|g(w) - g(w + e^{i\theta}\lambda(\exp(\pi/n) - 1))|}{|g(w) - g(w + \lambda(\exp(\pi/n) - 1)))|} \leq \eta. \tag{A.19}$$

It is readily seen from the definition of $g_{n,2n}$ that

$$\sup_n |g(w) - g(w + \lambda(\exp(\pi/n) - 1)))| \xrightarrow{\lambda \to 0} 0,$$

so that by equation (A.19), we conclude that the second inclusion in equation (A.17) holds for all sufficiently small λ. □

LEMMA A.19. *Let* $g = g_{n,2n,x,c}$ *be the function in* [**BL23**, Proposition 3.19] *and let* w *be a zero of* g *contained inside* $A(x, \exp(\pi/n) \cdot x)$. *There exists constants* $0 < \lambda < 1/8$ *and* $\delta > 0$, *which do not depend on* n *or* x, *so that*

$$B(0, \delta \cdot cx^n) \subset g(B(w, \lambda(\exp(\pi/n) - 1)x)) \subset B(0, 1/2 \cdot cx^n). \tag{A.20}$$

Moreover, g *is injective on* $B(w, \lambda(\exp(\pi/n) - 1)x)$.

Proof. This follows immediately from the definition of $g_{n,2n,x}$ and Lemma A.18. Indeed, we have

$$g_{n,2n,x,c} = (z \mapsto cx^n z) \circ g_{n,2n} \circ \left(z \mapsto \frac{z}{x}\right).$$

The inclusions in equation (A.20) now follow from equation (A.17) □

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