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WELL-POSEDNESS OF A PSEUDO-PARABOLIC KWC SYSTEM IN MATERIALS SCIENCE*

HARBIR ANTIL†, DAIKI MIZUNO‡, AND KEN SHIRAKAWA§

Abstract. The original KWC system is widely used in materials science. It was proposed in [R. Kobayashi, J. A. Warren, and W. C. Carter, *Phys. D*, 140 (2000), pp. 141–150] and is based on the phase field model of planar grain boundary motion. This model suffers from two key challenges. First, it is difficult to establish its relation to physics, in particular a variational model. Second, it lacks uniqueness. The former has been recently studied within the realm of BV theory. The latter only holds under various simplifications. This article introduces a pseudo-parabolic version of the KWC system. A direct relationship with variational model (as gradient flow) and uniqueness are established without making any unrealistic simplifications. Namely, this is the first KWC system which is both physically and mathematically valid. The proposed model overcomes the well-known open issues.

Key words. KWC-type system, pseudo-parabolic nature, existence, uniqueness, regularity, continuous dependence

MSC codes. 35A15, 35G50, 35G61, 35K70, 74N20

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1. Introduction.

The Kobayashi–Warren–Carter (KWC) system consists of a set of nonsmooth parabolic PDEs and is widely used in materials science [23, 24]. It is based on the phase field model of planar grain boundary motion. This model suffers from two fundamental challenges: (i) *Physics:* It is difficult to establish the KWC system as the gradient flow of a variational model. (ii) *Mathematics:* The solutions to the KWC system are known to be unique only under special cases. Both of these challenges make it difficult to rigorously use this model in practice or carry our new material design via optimization [5, 6].

This article aims to overcome both of these challenges by introducing a pseudo-parabolic version of the KWC system. Well-posedness (both existence and uniqueness) of the resulting system (S), which arises from gradient flow based on the *KWC energy,*

$$
\begin{aligned}
\mathcal{F} : [\eta,\theta] \in [L^2(\Omega)]^2 \mapsto \mathcal{F}(\eta,\theta) \\
:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega G(\eta)\,dx + \int_\Omega \alpha(\eta)|D\theta| \\ \qquad \text{if } [\eta,\theta] \in [H^1(\Omega)\cap L^\infty(\Omega)] \times BV(\Omega), \\ \infty \quad \text{otherwise} \end{cases}
\end{aligned} \tag{1.1}
$$

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†Department of Mathematical Sciences and the Center for Mathematics and Artificial Intelligence (CMAI), George Mason University, Fairfax, VA 22030 USA (hantil@gmu.edu).
‡Division of Mathematics and Informatics, Graduate School of Science and Engineering, Chiba University, 1–33, Yayoi-cho, Inage-ku, 263–8522, Chiba, Japan (d-mizuno@chiba-u.jp).
§Department of Mathematics, Faculty of Education, Chiba University 1–33, Yayoi-cho, Inage-ku, 263–8522, Chiba, Japan (sirakawa@faculty.chiba-u.jp).

is established. In this context, $\Omega \subset \mathbb{R}^N$ is a bounded spatial domain of dimension $N \in \{1,2,3\}$. $|D\theta|$ denotes the total variation of the variational measure $D\theta$ of $\theta \in BV(\Omega)$ (cf. [4, section 3.1]), and $\int_\Omega \alpha(\eta)|D\theta|$ denotes the total variation of $\theta \in BV(\Omega)$ with the weight $\alpha(\eta) \in H^1(\Omega) \cap L^\infty(\Omega)$ given as follows (cf. [29, section 2.1]):

$$\begin{aligned}&\int_\Omega \alpha(\eta)|D\theta| \\ &\quad := \inf\left\{ \varliminf_{n\to\infty} \int_\Omega \alpha(\eta)|\nabla\varphi_n|\,dx \;\middle|\; \begin{array}{l}\{\varphi_n\}_{n=1}^\infty \subset W^{1,1}(\Omega) \text{ such that} \\ \varphi_n \to \theta \text{ in } L^1(\Omega) \text{ as } n\to\infty\end{array} \right\} \\ &\qquad \text{for } [\eta,\theta] \in [H^1(\Omega)\cap L^\infty(\Omega)] \times BV(\Omega).\end{aligned}$$

Namely, this article addresses open issues from previous works that deal with the KWC system (cf. [26, 29, 30, 31, 32, 35]) and its regularized versions (cf. [5, 6, 7, 22, 25, 38]).

Next, we describe the system (S). Let $0 < T < \infty$ be a fixed final time, and let $\Gamma := \partial\Omega$ be the boundary of Ω. Especially when $N > 1$, Γ is assumed to be sufficiently smooth, with the unit outer normal n_Γ. Besides, we let $Q := (0,T)\times\Omega$ and $\Sigma := (0,T)\times\Gamma$. Then the pseudo-parabolic system denoted by (S), with two constants $\mu > 0$ and $\nu > 0$, is given by

(S)

$$\begin{cases}\partial_t\eta - \Delta\big(\eta + \mu^2\partial_t\eta\big) + g(\eta) + \alpha'(\eta(t))|\nabla\theta| = u(t,x) \text{ for } (t,x)\in Q, \\ \nabla\big(\eta+\mu^2\partial_t\eta\big)\cdot n_\Gamma = 0 \text{ on } \Sigma, \\ \eta(0,x) = \eta_0(x) \text{ for } x\in\Omega,\end{cases}$$

$$\begin{cases}\alpha_0(\eta)\partial_t\theta - \operatorname{div}\left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu^2\nabla\partial_t\theta\right) = v(t,x) \text{ for } (t,x)\in Q, \\ \left(\alpha(\eta)\dfrac{D\theta}{|D\theta|} + \nu^2\nabla\partial_t\theta\right)\cdot n_\Gamma = 0 \text{ on } \Sigma, \\ \theta(0,x) = \theta_0(x) \text{ for } x\in\Omega.\end{cases}$$

Here, the unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are order parameters that indicate the *orientation order* and *orientation angle* of the polycrystal body, respectively. Besides, $\eta_0 = \eta_0(x)$ and $\theta_0 = \theta_0(x)$ is the *initial data.* Moreover, $u = u(t,x)$ and $v = v(t,x)$ are the forcing terms. Additionally, $\alpha_0 = \alpha_0(\eta)$ and $\alpha = \alpha(\eta)$ are fixed positive-valued functions to reproduce the mobilities of grain boundary motions. Finally, $g = g(\eta)$ is a perturbation for the orientation order η, having a nonnegative potential $G = G(\eta)$, i.e., $\frac{d}{d\eta}G(\eta) = g(\eta)$.

A generic form of the "KWC system" is given by the evolution equation (cf. [23, 24])

$$(1.2)\qquad \begin{aligned}&-\mathcal{A}_0\big(\eta(t)\big)\frac{d}{dt}\begin{bmatrix}\eta(t)\\ \theta(t)\end{bmatrix} = \frac{\delta}{\delta[\eta,\theta]}\mathcal{F}(\eta(t),\theta(t)) + \begin{bmatrix}u(t)\\ v(t)\end{bmatrix} \\ &\quad \text{in } [L^2(\Omega)]^2 \text{ for } t\in(0,T),\end{aligned}$$

which is motivated by the gradient flow of the free energy, namely the KWC energy (1.1), with a functional derivative $\frac{\delta}{\delta[\eta,\theta]}\mathcal{F}$, and an unknown-dependent monotone operator $\mathcal{A}_0(\eta) \subset [L^2(\Omega)]^2$. Here, the evolution equation (1.2) can be considered as the common root of the original KWC system (cf. [23, 24]) and our system (S). Indeed, our system (S) is derived from the evolution equation (1.2) in the case when

$$\mathcal{A}_0(\eta): \begin{bmatrix} \widetilde{\eta} \\ \widetilde{\theta} \end{bmatrix} \in [H^2(\Omega)]^2 \mapsto \mathcal{A}_0(\eta) \begin{bmatrix} \widetilde{\eta} \\ \widetilde{\theta} \end{bmatrix} := \begin{bmatrix} \widetilde{\eta} - \mu^2 \Delta \widetilde{\eta} \\ \alpha_0(\eta)\widetilde{\theta} - \nu^2 \Delta \widetilde{\theta} \end{bmatrix} \in [L^2(\Omega)]^2, \tag{1.3}$$

for each $\eta \in L^2(\Omega)$, subject to the zero-Neumann boundary condition,

while the original KWC system corresponds to the case when $\mu = \nu = 0$.

In recent years, the principal issue has been to clarify the variational structure (representation) of the functional derivative $\frac{\delta}{\delta[\eta,\theta]}\mathcal{F}$ of the nonsmooth and nonconvex energy $\mathcal{F}$ in (1.1). The positive answer to the issue was obtained in [29, 30, 35] by means of BV theory (cf. [2, 3, 4, 8, 13, 19]), and this work has provided a basis of the study of the KWC system, e.g., the existence and large-time behavior [29, 30], the observations under other boundary conditions [31, 32], the time-periodic solution [26], and so on.

However, the uniqueness of solutions has been a significant challenge, due to the velocity term $\alpha_0(\eta)\partial_t\theta$ and the singular diffusion flux $\alpha(\eta)\frac{D\theta}{|D\theta|}$, both of which depend on the unknown-dependent mobilities. Therefore, previous researchers have implemented the following modifications to the modeling framework (1.1) and (1.3):

- resetting α_0 to be a function which is independent of η (effectively a constant);
- modifying the free-energy functional to a more relaxed form:

$$\mathcal{F}_\varepsilon : [\eta,\theta] \in [L^2(\Omega)]^2 \mapsto \mathcal{F}_\varepsilon(\eta,\theta) = \begin{cases} \mathcal{F}(\eta,\theta) + \dfrac{\varepsilon^2}{2}\displaystyle\int_\Omega |\nabla\theta|^2\,dx \text{ if } \theta \in H^1(\Omega), \\ \infty \quad \text{otherwise} \end{cases}$$

with a small constant $\varepsilon > 0$.

These modifications have been pivotal in addressing the uniqueness challenges (cf. [22, 38]) and several advanced issues, such as the optimal control problems (see [5, 6, 7, 25]).

In light of this, we can expect that the pseudo-parabolic nature of our system will effectively address the uniqueness challenge. This is due to the positive constants μ and ν in (1.3), which are expected to bring a smoothing effect for the regularity of solution.

In fluid dynamics, the pseudo-parabolic regularization is known as Voigt regularization, which is a regularization method proposed by Voigt [37]. Indeed, some pseudo-parabolic models are named after Voigt, such as the Euler–Voigt equation (cf. [20, 21]) and the Navier–Stokes–Voigt equation (cf. [33, 34]). A number of previous works have succeeded in obtaining the well-posedness, including the existence, uniqueness, and continuous dependence of solution, with stronger regularity than that in the standard variational framework (cf. [9, 11, 27, 28, 33, 34]). This provides a strong motivation to try such regularizations for the KWC model. On the other hand, from the singular diffusion flux $\alpha(\eta)\frac{D\theta}{|D\theta|}$, it can be expected that the results of this paper will contribute to the development of research on Bingham-type fluids (cf. [14, 15]).

Consequently, we set the goal to clarify the similarities and differences between our pseudo-parabolic system and the original parabolic KWC system. To this end, we prove the following two Main Theorems, concerned with the well-posedness of our pseudo-parabolic system (S), i.e., the evolution equation (1.2) under (1.1) and (1.3).

Main Theorem 1. Existence and regularity of solution to (S).
Main Theorem 2. Uniqueness of solution to (S) and continuous dependence with respect to the initial data and forcings.

These Main Theorems will provide the positive answer to our earlier expectation regarding the effectiveness of the pseudo-parabolic nature of our system in resolving the uniqueness issue. Also, the Main Theorems will focus on two conflicting properties: the singularity in the diffusion flux $\alpha(\eta)\frac{D\theta}{|D\theta|}$, and the smoothing effect encouraged by the Laplacian in (1.3). This conflicting situation will be clarified by the differences in regularity between components: $\eta \in W^{1,2}(0,T;H^2(\Omega))$ and $\theta \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ in Main Theorem 1. Moreover, the results of this paper will form a fundamental part of the optimization problem in grain boundary motion, which will be explored in a forthcoming paper.

Finally, we conclude this section by mentioning that there are several works on the singular limit of the sequence of KWC-type energies $\{\mathcal{E}_\delta\}_{\delta>0}$, given as

$$\mathcal{E}_\delta(\eta,\theta) := \frac{\delta}{2}\int_\Omega |\nabla\eta|^2\,dx + \frac{1}{2\delta}\int_\Omega (\eta-1)^2\,dx + \int_\Omega \alpha(\eta)|D\theta|$$
$$\text{for } [\eta,\theta] \in [H^1(\Omega)\cap L^\infty(\Omega)] \times BV(\Omega) \text{ and } \delta > 0$$

as $\delta \downarrow 0$. In [18], the one-dimensional problem is discussed, while in [17] the multidimensional case is discussed. The singular limit of flow has recently been studied in [16].

Outline. Preliminaries are given in section 2, and on this basis, the Main Theorems are stated in section 3. For the proofs of the Main Theorems, we prepare section 4 to set up an approximation method for (S). Based on these, Main Theorems 1 and 2 are proved in sections 5 and 6, respectively, by means of the auxiliary results obtained in section 4.

2. Preliminaries.

We begin by prescribing the notations used throughout this paper.

<u>Notations in real analysis.</u> We define

$$r \vee s := \max\{r,s\} \text{ and } r \wedge s := \min\{r,s\} \text{ for all } r,s \in [-\infty,\infty],$$

and especially we write

$$[r]^+ := r \vee 0 \text{ and } [r]^- := -(r \wedge 0) \text{ for all } r \in [-\infty,\infty].$$

Additionally, for any $M > 0$, let $\mathcal{T}_M : \mathbb{R} \longrightarrow [-M,M]$ be the truncation operator, defined as

$$\mathcal{T}_M : r \in \mathbb{R} \mapsto (r \vee (-M)) \wedge M \in [-M,M].$$

Let $d \in \mathbb{N}$ be a fixed dimension. We denote by $|y|$ and $y \cdot z$ the Euclidean norm of $y \in \mathbb{R}^d$ and the scalar product of $y,z \in \mathbb{R}^d$, respectively, i.e.,

$$|y| := \sqrt{y_1^2 + \cdots + y_d^2} \quad \text{and} \quad y \cdot z := y_1 z_1 + \cdots + y_d z_d$$
$$\text{for all } y = [y_1,\ldots,y_d],\ z = [z_1,\ldots,z_d] \in \mathbb{R}^d.$$

Besides, we let

$$\mathbb{B}^d := \left\{ y \in \mathbb{R}^d \;\middle|\; |y| < 1 \right\} \text{ and } \mathbb{S}^{d-1} := \left\{ y \in \mathbb{R}^d \;\middle|\; |y| = 1 \right\}.$$

We denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure, and we denote by $\mathcal{H}^d$ the d-dimensional Hausdorff measure. In particular, the measure theoretical phrases, such as "a.e.", "dt", and "dx", and so on, are all with respect to the Lebesgue measure in each corresponding dimension. Also on a Lipschitz-surface S, the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension. In particular, if S is C^1-surface, then we simply denote by dS the area element of the integration on S.

For a Borel set $E \subset \mathbb{R}^d$, we denote by $\chi_E : \mathbb{R}^d \longrightarrow \{0,1\}$ the characteristic function of E. Additionally, for a distribution ζ on an open set in $\mathbb{R}^d$ and any $i \in \{1, \dots, d\}$, let $\partial_i \zeta$ be the distributional differential with respect to the ith variable of ζ. We also consider the differential operators, such as ∇, div, ∇^2, and so on, in the distributional sense.

***Abstract notations (cf.* [10, Chapter II]).** For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X and denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and its dual X^*. In particular, when X is a Hilbert space, we denote by $(\cdot,\cdot)_X$ the inner product of X.

For two Banach spaces X and Y, let $\mathscr{L}(X;Y)$ be the Banach space of bounded linear operators from X into Y.

For Banach spaces $X_1, \dots, X_d$ with $1 < d \in \mathbb{N}$, let $X_1 \times \cdots \times X_d$ be the product Banach space endowed with the norm $|\cdot|_{X_1 \times \cdots \times X_d} := |\cdot|_{X_1} + \cdots + |\cdot|_{X_d}$. However, when all $X_1, \dots, X_d$ are Hilbert spaces, $X_1 \times \cdots \times X_d$ denotes the product Hilbert space endowed with the inner product $(\cdot,\cdot)_{X_1 \times \cdots \times X_d} := (\cdot,\cdot)_{X_1} + \cdots + (\cdot,\cdot)_{X_d}$ and the norm $|\cdot|_{X_1 \times \cdots \times X_d} := \left(|\cdot|_{X_1}^2 + \cdots + |\cdot|_{X_d}^2 \right)^{\frac{1}{2}}$. In particular, when all $X_1, \dots, X_d$ coincide with a Banach space Y, the product space $X_1 \times \cdots \times X_d$ is simply denoted by $[Y]^d$.

Basic notations. Let $0 < T < \infty$ be a fixed constant of time, and let $N \in \{1,2,3\}$ be a fixed dimension. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a boundary $\Gamma := \partial\Omega$, and when $N > 1$, Γ has C^∞-regularity, with the unit outer normal n_Γ. Additionally, as notations of base spaces, we let

$$H := L^2(\Omega), \ V := H^1(\Omega), \ \mathscr{H} := L^2(0,T;H), \text{ and } \mathscr{V} := L^2(0,T;V).$$

Let $W_0 \subset H^2(\Omega)$ be the closed linear subspace of H, given by

$$W_0 := \{ z \in H^2(\Omega) \,|\, \nabla z \cdot n_\Gamma = 0 \text{ on } \Gamma \}.$$

Let A_N be a differential operator, defined as

$$A_N : z \in W_0 \subset H \mapsto A_N z := -\Delta z \in H.$$

It is well known that $A_N \subset H \times H$ is linear, positive, and self-adjoint, and the domain W_0 is a Hilbert space, endowed with the inner product

$$(z_1, z_2)_{W_0} := (z_1, z_2)_H + (A_N z_1, z_2)_H \ \big(= (z_1, z_2)_V\big) \text{ for } z_k \in W_0, \ k = 1, 2.$$

Moreover, there exists a positive constant C_0 such that

$$|z|_{H^2(\Omega)}^2 \le C_0 \big(|A_N z|_H^2 + |z|_H^2 \big) \text{ for all } z \in W_0. \tag{2.1}$$

Notations for the time discretization. Let $\tau > 0$ be a constant of the time step-size, and let $\{t_i\}_{i=0}^\infty \subset [0,\infty)$ be the time sequence, defined as

$$t_i := i\tau, \ i = 0, 1, 2, \dots.$$

Let X be a Banach space. Then, for any sequence $\{[t_i, z_i]\}_{i=0}^\infty \subset [0,\infty) \times X$, we define the *forward time-interpolation* $[\overline{z}]_\tau \in L^\infty_{\mathrm{loc}}([0,\infty); X)$, the *backward time-interpolation*

$[\underline{z}]_\tau \in L^\infty_{\mathrm{loc}}([0,\infty);X)$, and the *linear time-interpolation* $[z]_\tau \in W^{1,2}_{\mathrm{loc}}([0,\infty);X)$ by letting

$$\begin{cases} [\overline{z}]_\tau(t) := \chi_{(-\infty,0]} z_0 + \displaystyle\sum_{i=1}^\infty \chi_{(t_{i-1},t_i]}(t) z_i, \\ [\underline{z}]_\tau(t) := \displaystyle\sum_{i=0}^\infty \chi_{(t_i,t_{i+1}]}(t) z_i \qquad\qquad\qquad\qquad \text{in } X \text{ for } t \geq 0, \\ [z]_\tau(t) := \displaystyle\sum_{i=1}^\infty \chi_{[t_{i-1},t_i)}(t) \left(\frac{t-t_{i-1}}{\tau} z_i + \frac{t_i - t}{\tau} z_{i-1} \right), \end{cases}$$

respectively.

In the meantime, for any $q \in [1,\infty)$ and any $\zeta \in L^q_{\mathrm{loc}}([0,\infty);X)$, we denote by $\{\zeta_i\}_{i=0}^\infty \subset X$ the sequence of time-discretization data of ζ, defined as

$$\text{(2.2a)} \qquad \zeta_0 := 0 \text{ in } X \text{ and } \zeta_i := \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \zeta(\varsigma)\, d\varsigma \text{ in } X \text{ for } i = 1,2,3,\ldots.$$

As is easily checked, the time-interpolations $[\overline{\zeta}]_\tau, [\underline{\zeta}]_\tau \in L^q_{\mathrm{loc}}([0,\infty);X)$ for the above $\{\zeta_i\}_{i=0}^\infty$ fulfill

$$\text{(2.2b)} \qquad [\overline{\zeta}]_\tau \to \zeta \text{ and } [\underline{\zeta}]_\tau \to \zeta \text{ in } L^q_{\mathrm{loc}}([0,\infty);X) \text{ as } \tau \downarrow 0.$$

3. Main results.

In this paper, the main assertions are discussed under the following assumptions:

(A1) $\mu > 0$ and $\nu > 0$ are fixed constants.

(A2) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz continuous function with a nonnegative primitive $G \in C^1(\mathbb{R})$. Moreover, g satisfies the following condition:

$$\liminf_{\xi \downarrow -\infty} g(\xi) = -\infty, \ \limsup_{\xi \uparrow \infty} g(\xi) = \infty.$$

(A3) $\alpha_0 : \mathbb{R} \longrightarrow (0,\infty)$ is a locally Lipschitz continuous function, and $\alpha : \mathbb{R} \longrightarrow [0,\infty)$ is a C^1-class convex function, such that

$$\alpha'(0) = 0 \quad \text{and} \quad \delta_\alpha := \inf \alpha_0(\mathbb{R}) > 0.$$

(A4) $u, v \in L^2_{\mathrm{loc}}([0,\infty);H)$, and $u \in L^\infty(Q)$.

(A5) The initial data $[\eta_0, \theta_0]$ belong to the class $[W_0]^2 \subset [H^2(\Omega)]^2$.

Now, the main results are stated as follows.

Main Theorem 1 (existence and regularity). *Under assumptions* (A1)–(A5)*, the system* (S) *admits a solution* $[\eta,\theta] \in [\mathscr{H}]^2$ *in the following sense:*

(S0) $[\eta,\theta] \in \left[W^{1,2}(0,T;W_0) \cap L^\infty(Q)\right] \times \left[W^{1,2}(0,T;V) \times L^\infty(0,T;W_0)\right]$.

(S1) η *solves the following variational identity:*

$$\begin{gathered}(\partial_t \eta(t), \varphi)_H + (\nabla(\eta + \mu^2 \partial_t \eta)(t), \nabla\varphi)_{[H]^N} + (g(\eta(t)), \varphi)_H \\ + (\alpha'(\eta(t))|\nabla\theta(t)|, \varphi)_H = (u(t), \varphi)_H \\ \text{for any } \varphi \in V \text{ and a.e. } t \in (0,T).\end{gathered}$$

(S2) θ *solves the following variational inequality:*

$$\begin{gathered}\big((\alpha_0(\eta)\partial_t\theta)(t), \theta(t) - \psi\big)_H + \nu^2 (\nabla\partial_t\theta(t), \nabla(\theta(t) - \psi))_{[H]^N} \\ + \int_\Omega \alpha(\eta(t))|\nabla\theta(t)|\, dx \leq \int_\Omega \alpha(\eta(t))|\nabla\psi|\, dx + (v(t), \theta(t) - \psi)_H \\ \text{for any } \psi \in V \text{ and a.e. } t \in (0,T).\end{gathered}$$

(S3) $[\eta(0), \theta(0)] = [\eta_0, \theta_0]$ *in* $[H]^2$.

Main Theorem 2 (uniqueness and continuous dependence). *Under assumptions* (A1)–(A5)*, let* $[\eta^k, \theta^k]$, $k = 1, 2$, *be two solutions to* (S) *with two initial values* $[\eta_0^k, \theta_0^k]$ *and two forcings* $[u^k, v^k]$, $k = 1, 2$. *Then, there exists a constant* $C_1 = C_1(\nu) > 0$, *depending on* ν, *such that*

$$J(t) \le C_1(\nu)\big(J(0) + |u^1 - u^2|^2_{\mathscr{H}} + |v^1 - v^2|^2_{\mathscr{H}}\big)$$
$$\text{for any } t \in [0, T] \text{ and any } T > 0,$$

where

$$J(t) := |(\eta^1 - \eta^2)(t)|_H^2 + \mu^2 |\nabla(\eta^1 - \eta^2)(t)|^2_{[H]^N} + |\sqrt{\alpha_0(\eta^1)}(\theta^1 - \theta^2)(t)|_H^2$$
$$+ \nu^2 |\nabla(\theta^1 - \theta^2)(t)|^2_{[H]^N} \quad \text{for any } t \ge 0.$$

Remark 3.1 (cf. [1]). In Main Theorem 1, we note that the variational inequality in (S2) has an equivalent form as an evolution equation. Indeed, referring to [1, Main Theorem 2], this variational inequality can be reformulated as follows:

$$\text{(3.1)} \qquad \begin{cases} \alpha_0(\eta(t))\partial_t\theta(t) - \operatorname{div}\big(\alpha(\eta(t)\boldsymbol{\omega}^*(t)) + \nu^2\nabla\partial_t\theta(t)\big) = v(t) \text{ in } H, \\ \big(\alpha(\eta(t)\boldsymbol{\omega}^*(t)) + \nu^2\nabla\partial_t\theta(t)\big)\cdot n_\Gamma = 0 \text{ in } H^{-\frac{1}{2}}(\Gamma), \end{cases}$$

with a vectorial function $\boldsymbol{\omega}^* \in L^2(0, T; [H]^N)$ satisfying

$$\text{(3.2)} \qquad \boldsymbol{\omega}^*(t) \in \operatorname{Sgn}(\nabla\theta(t)) \text{ a.e. in } \Omega,$$

where $\operatorname{Sgn} \subset \mathbb{R}^N \times \mathbb{R}^N$ denotes the subdifferential of the Euclidean norm of $\mathbb{R}^N$. In mathematics, the condition (S2) would be more useful for the efficient proofs of the Main Theorems than (3.1) with (3.2).

Remark 3.2. In the proof of Main Theorem 1, the restriction of spatial dimension $N \in \{1, 2, 3\}$ will be important to guarantee the embedding $H^2(\Omega) \subset L^\infty(\Omega)$, and it will be essential to obtain the regularity as in (S1) (see Remark 5.1 in section 5). Additionally, the positivity of the both constants μ and ν associated with the pseudo-parabolicity of the system (S) will play a key role, too.

Meanwhile, in the proof of Main Theorem 2, it can be said that the assumptions $N \in \{1, 2, 3\}$, $\mu > 0$, and $\nu > 0$ are just sufficient conditions. For instance, it is possible to prove Main Theorem 2 by using only the embedding $H^1(\Omega) \subset L^4(\Omega)$ that immediately follows from $N \in \{1, 2, 3\}$. Furthermore, for the pseudo-parabolicity, we note that the constant $C_1(\nu)$ for the estimate will actually be obtained independently of the constant $\mu > 0$, i.e., the pseudo-parabolicity of η. Hence, in our proof of the uniqueness, it can be expected that the essence would be only the pseudo-parabolicity of θ, and our method could be extended to the case of $\mu = 0$ under the standard parabolic regularity of η.

4. Approximating method. In the Main Theorems, the solution to (S) will be obtained by means of the time-discretization method. In this light, let $\tau \in (0, 1)$ be a constant of the time-step size, and let $\varepsilon \in (0, 1)$ be a relaxation constant. Based on this, we adopt the following time-discretization scheme $(\mathrm{AP})_\tau^\varepsilon$ as our approximating problem of (S):

$(\mathrm{AP})_\tau^\varepsilon$: To find $\{[\eta_i, \theta_i]\}_{i=0}^\infty \subset [W_0]^2$ satisfying

$$\text{(4.1)} \qquad \frac{\eta_i^\varepsilon - \eta_{i-1}^\varepsilon}{\tau} + A_N\left(\eta_i^\varepsilon + \frac{\mu^2}{\tau}(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)\right) + g(\mathcal{T}_M\eta_i^\varepsilon)$$
$$+ \alpha'(\mathcal{T}_M\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon) = u_i \text{ in } H,$$

$$\begin{aligned}(4.2)\qquad &\frac{\alpha_0(\mathcal{T}_M\eta^\varepsilon_{i-1})}{\tau}(\theta^\varepsilon_i-\theta^\varepsilon_{i-1})-\mathrm{div}\big(\widetilde{\alpha_M}(\eta^\varepsilon_{i-1})\nabla\gamma_\varepsilon(\nabla\theta^\varepsilon_i)\big)\\ &\quad+\frac{\nu^2}{\tau}A_N(\theta^\varepsilon_i-\theta^\varepsilon_{i-1})=v_i \text{ in } H\\ &\text{for } i=1,2,3,\dots, \text{ subject to } [\eta^\varepsilon_0,\theta^\varepsilon_0]=[\eta_0,\theta_0] \text{ in } [H]^2.\end{aligned}$$

In this context, $\gamma_\varepsilon\in C^{0,1}(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)$ is a smooth approximation of the Euclidean norm $|\cdot|\in C^{0,1}(\mathbb{R}^N)$, defined as

$$\gamma_\varepsilon: y\in\mathbb{R}^N\mapsto\gamma_\varepsilon(y):=\sqrt{\varepsilon^2+|y|^2}\in[0,\infty).$$

Also, we define an approximating free-energy $\widetilde{\mathcal{F}}_\varepsilon$ on $[H]^2$ by setting

$$\begin{aligned}&\widetilde{\mathcal{F}}_\varepsilon:[\eta,\theta]\in[H]^2\mapsto\widetilde{\mathcal{F}}_\varepsilon(\eta,\theta)\\ (4.3)\qquad &:=\begin{cases}\dfrac12\displaystyle\int_\Omega|\nabla\eta|^2\,dx+\int_\Omega\widetilde{G}_M(\eta)+\int_\Omega\widetilde{\alpha}_M(\eta)\gamma_\varepsilon(D\theta)\\ \qquad\qquad \text{if } [\eta,\theta]\in H^1(\Omega)\times BV(\Omega),\\ \infty\quad\text{otherwise}\end{cases}\\ &\text{for any } 0<\varepsilon<1,\end{aligned}$$

where $\widetilde{\alpha}_M\in C^{1,1}(\mathbb{R})$ and $\widetilde{G}_M\in C^{1,1}(\mathbb{R})$ are nonnegative primitives of $\alpha'\circ\mathcal{T}_M\in W^{1,\infty}(\mathbb{R})$ and $g\circ\mathcal{T}_M\in W^{1,\infty}(\mathbb{R})$, respectively, and

$$\begin{aligned}&\int_\Omega\widetilde{\alpha}_M(\eta)\gamma_\varepsilon(D\theta)\\ &:=\inf\left\{\varliminf_{n\to\infty}\int_\Omega\widetilde{\alpha}_M(\eta)\gamma_\varepsilon(\nabla\varphi_n)\,dx\;\middle|\;\begin{array}{l}\{\varphi_n\}_{n=1}^\infty\subset W^{1,1}(\Omega)\\ \text{such that } \varphi_n\to\theta \text{ in}\\ L^1(\Omega) \text{ as } n\to\infty\end{array}\right\}\\ &\text{for } [\eta,\theta]\in H^1(\Omega)\times BV(\Omega), \text{ and } 0<\varepsilon<1.\end{aligned}$$

Finally, u_i,v_i are given as in (2.2).

The solution to $(\mathrm{AP})^\varepsilon_\tau$ is given as follows.

Definition 4.1. *The sequence of functions* $\{[\eta^\varepsilon_i,\theta^\varepsilon_i]\}_{i=0}^\infty$ *is called a solution to* $(\mathrm{AP})^\varepsilon_\tau$ *iff* $\{[\eta^\varepsilon_i,\theta^\varepsilon_i]\}_{i=0}^\infty\subset[W_0]^2$, *and* $[\eta^\varepsilon_i,\theta^\varepsilon_i]$ *fulfills* (4.1) *and* (4.2) *for any* $i=1,2,3,\dots$.

In this paper, the following theorem will play an important role for the proof of the Main Theorems.

Theorem 4.2 (solvability of the approximating problem). *There exists a sufficiently small constant* $\tau_0\in(0,1)$ *such that for any* $\tau\in(0,\tau_0)$ *and* $\varepsilon\in(0,1)$, $(\mathrm{AP})^\varepsilon_\tau$ *admits a unique solution* $\{[\eta^\varepsilon_i,\theta^\varepsilon_i]\}_{i=0}^\infty$. *Additionally, the following energy inequality holds:*

$$\begin{aligned}(4.4)\qquad &\frac{1}{4\tau}|\eta^\varepsilon_i-\eta^\varepsilon_{i-1}|^2_H+\frac{\mu^2}{\tau}|\nabla(\eta^\varepsilon_i-\eta^\varepsilon_{i-1})|^2_{[H]^N}+\frac{\delta_\alpha}{2\tau}|\theta^\varepsilon_i-\theta^\varepsilon_{i-1}|^2_H\\ &\quad+\frac{\nu^2}{\tau}|\nabla(\theta^\varepsilon_i-\theta^\varepsilon_{i-1})|^2_{[H]^N}+\widetilde{\mathcal{F}}_\varepsilon(\eta^\varepsilon_i,\theta^\varepsilon_i)\\ &\le\widetilde{\mathcal{F}}_\varepsilon(\eta^\varepsilon_{i-1},\theta^\varepsilon_{i-1})+\frac{\tau}{2}|u_i|^2_H+\frac{\tau}{2\delta_\alpha}|v_i|^2_H \quad \textit{for any } i=1,2,3,\dots.\end{aligned}$$

Theorem 4.2 is proved through several lemmas.

LEMMA 4.3. *For arbitrary $\widetilde{\theta} \in V$, $\widetilde{\eta}_0 \in W_0$, and $\widetilde{u} \in H$, we consider the following elliptic problem:*

$$\frac{1}{\tau}(\eta - \widetilde{\eta}_0) + A_N\left(\eta + \frac{\mu^2}{\tau}(\eta - \widetilde{\eta}_0)\right) + g(\mathcal{T}_M\eta) + \alpha'(\mathcal{T}_M\eta)\gamma_\varepsilon(\nabla\widetilde{\theta}) = \widetilde{u} \text{ in } H. \tag{4.5}$$

Then, there exists a small constant $\tau_1 \in (0,1)$, depending only on $|g'|_{L^\infty(-M,M)}$, and for any $0 < \tau < \tau_1$, the elliptic problem (4.5) *admits a unique solution $\eta \in W_0$.*

Proof. First, for any $\eta^\dagger \in V$, we define a functional $\Upsilon : H \longrightarrow (-\infty, \infty]$ as follows:

$$\Upsilon : z \in H \mapsto \Upsilon(z) := \begin{cases} \dfrac{1}{2\tau}\displaystyle\int_\Omega |z - \widetilde{\eta}_0|^2\,dx + \frac{1}{2}\int_\Omega |\nabla z|^2\,dx + \frac{\mu^2}{2\tau}\int_\Omega |\nabla(z - \widetilde{\eta}_0)|^2\,dx \\ \quad + \displaystyle\int_\Omega g(\mathcal{T}_M\eta^\dagger)z\,dx + \int_\Omega \widetilde{\alpha}_M(z)\gamma_\varepsilon(\nabla\widetilde{\theta})\,dx \\ \quad - \displaystyle\int_\Omega \widetilde{u}z\,dx \quad \text{if } z \in V, \\ \infty \text{ otherwise.} \end{cases}$$

As is easily checked, Υ is proper, l.s.c., strictly convex, and coercive, and its unique minimizer solves the following elliptic equation:

$$\frac{1}{\tau}(\eta - \widetilde{\eta}_0) + A_N\left(\eta + \frac{\mu^2}{\tau}(\eta - \widetilde{\eta}_0)\right) + g(\mathcal{T}_M\eta^\dagger) + \alpha'(\mathcal{T}_M\eta)\gamma_\varepsilon(\nabla\widetilde{\theta}) = \widetilde{u} \text{ in } H. \tag{4.6}$$

Now, we define an operator $S_\tau : V \longrightarrow H^2(\Omega)$ which maps any $\eta^\dagger \in V$ to the unique solution to (4.6) and consider the smallness condition of τ for S to be contractive. Here, let $\eta_k := S_\tau\eta_k^\dagger \in H^2(\Omega)$, $k = 1, 2$. By taking differences of (4.6), multiplying both sides by $\eta_1 - \eta_2$, and applying Young's inequality, we see from (A1) and (A2) that

$$\frac{1 \wedge \mu^2}{2\tau}|\eta_1 - \eta_2|_V^2 \leq \frac{\tau|g'|^2_{L^\infty(-M,M)}}{2}|\eta_1^\dagger - \eta_2^\dagger|_H^2.$$

Therefore, if we assume that

$$0 < \tau < \tau_1 := \left(\frac{1 \wedge \mu^2}{|g|^2_{L^\infty(-M,M)}}\right)^{\frac{1}{2}}, \tag{4.7}$$

then the mapping S_τ becomes a contraction mapping from V into itself. Therefore, applying Banach's fixed point theorem, we find a unique fixed point $\widetilde{\eta} \in V$ of S_τ under the condition (4.7). The identity $S_\tau\widetilde{\eta} = \widetilde{\eta}$ implies that $\widetilde{\eta}$ is the unique solution to (4.5). ∎

LEMMA 4.4. *For arbitrary $\widetilde{\eta} \in H^2(\Omega)$, $\widetilde{\theta}_0 \in W_0$, and $\widetilde{v} \in H$, we consider the following elliptic equation:*

$$\begin{cases} \alpha_0(\mathcal{T}_M\widetilde{\eta})\dfrac{\theta - \widetilde{\theta}_0}{\tau} - \operatorname{div}\left(\widetilde{\alpha}_M(\widetilde{\eta})\nabla\gamma_\varepsilon(\nabla\theta) + \dfrac{\nu^2}{\tau}\nabla(\theta - \widetilde{\theta}_0)\right) = \widetilde{v} \text{ a.e. in } \Omega, \\ \nabla\theta \cdot n_\Gamma = 0 \text{ a.e. on } \Gamma. \end{cases} \tag{4.8}$$

Then, for any $0 < \tau < 1$, (4.8) *admits a unique solution $\theta \in W_0$.*

Proof. Let us consider a proper, l.s.c., strictly convex, and coercive function Υ_*, defined as follows:

$$\Upsilon_* : z \in H \mapsto \Upsilon_*(z) := \begin{cases} \dfrac{1}{2\tau}\displaystyle\int_\Omega \alpha_0(\mathcal{T}_M\widetilde{\eta})|z-\widetilde{\theta}_0|^2\,dx + \int_\Omega \widetilde{\alpha}_M(\widetilde{\eta})\gamma_\varepsilon(\nabla z)\,dx \\ \quad + \dfrac{\nu^2}{2\tau}\displaystyle\int_\Omega |\nabla(z-\widetilde{\theta}_0)|^2\,dx - \int_\Omega \widetilde{v}z\,dx \quad \text{if } z \in V, \\ \infty \text{ otherwise.} \end{cases}$$

As is discussed in [1, Theorem 1], the unique minimizer $\widetilde{\theta}$ of Υ_* solves (4.8), and $\widetilde{\theta}$ belongs to W_0. ∎

Proof of Theorem 4.2. Let us fix any $\tau \in (0,\tau_1)$ and any $\varepsilon \in (0,1)$. Then, for any $i \in \mathbb{N}$, we can obtain $\theta_i^\varepsilon \in W_0$ by applying Lemma 4.4 in the case that

$$\widetilde{\eta} = \eta_{i-1}^\varepsilon,\ \widetilde{\theta}_0 = \theta_{i-1}^\varepsilon, \text{ and } \widetilde{v} = v_i \text{ in } H.$$

Moreover, for any $i \in \mathbb{N}$, the component $\eta_i^\varepsilon \in W_0$ can be obtained by applying Lemma 4.3 in the case that

$$\widetilde{\theta} = \theta_i^\varepsilon,\ \widetilde{\eta}_0 := \eta_{i-1}^\varepsilon, \text{ and } \widetilde{u} = u_i \text{ in } H.$$

Thus, we can find the unique solution $\{[\eta_i^\varepsilon, \theta_i^\varepsilon]\}_{i=0}^\infty \subset [W_0]^2$ to $(\mathrm{AP})_\tau^\varepsilon$.

Next, we verify the inequality (4.4). Multiplying both sides of (4.1) with $\eta_i^\varepsilon - \eta_{i-1}^\varepsilon$, we see that

$$\begin{aligned} (4.9)\qquad & \frac{1}{2\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2 + \frac{\mu^2}{\tau}|\nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|_{[H]^N}^2 + \frac{1}{2}|\nabla\eta_i^\varepsilon|_{[H]^N}^2 - \frac{1}{2}|\nabla\eta_{i-1}^\varepsilon|_{[H]^N}^2 \\ &+ \big(g(\mathcal{T}_M\eta_i^\varepsilon), \eta_i^\varepsilon-\eta_{i-1}^\varepsilon\big)_H + \big(\alpha'(\mathcal{T}_M\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon), \eta_i^\varepsilon-\eta_{i-1}^\varepsilon\big)_H \le \frac{\tau}{2}|u_i|_H^2 \\ &\qquad \text{for } i = 1,2,3,\dots \end{aligned}$$

via the following computations:

$$(\nabla\eta_i^\varepsilon, \nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon))_{[H]^N} \ge \frac{1}{2}\big(|\nabla\eta_i^\varepsilon|_{[H]^N}^2 - |\nabla\eta_{i-1}^\varepsilon|_{[H]^N}^2\big),$$

$$\frac{\mu^2}{\tau}(\nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon), \nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon))_{[H]^N} = \frac{\mu^2}{\tau}|\nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|_{[H]^N}^2,$$

and

$$(u_i, \eta_i^\varepsilon-\eta_{i-1}^\varepsilon)_H \le \frac{1}{2\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2 + \frac{\tau}{2}|u_i|_H^2.$$

In addition, by using (A2), it is obtained that

$$\begin{aligned} (4.10)\qquad & \big(g(\mathcal{T}_M\eta_i^\varepsilon), \eta_i^\varepsilon-\eta_{i-1}^\varepsilon\big)_H \ge \int_\Omega \widetilde{G}_M(\eta_i^\varepsilon)\,dx - \int_\Omega \widetilde{G}_M(\eta_{i-1}^\varepsilon)\,dx \\ &+ \big(g(\mathcal{T}_M\eta_i^\varepsilon) - g(\mathcal{T}_M\eta_{i-1}^\varepsilon), \eta_i^\varepsilon-\eta_{i-1}^\varepsilon\big)_H - \frac{1}{2}|g'|_{L^\infty(-M,M)}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2 \\ &\ge \int_\Omega \widetilde{G}_M(\eta_i^\varepsilon)\,dx - \int_\Omega \widetilde{G}_M(\eta_{i-1}^\varepsilon)\,dx - \frac{3}{2}|g'|_{L^\infty(-M,M)}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2 \\ &\text{for } i = 1,2,3,\dots, \end{aligned}$$

and by the convexity of $\widetilde{\alpha}_M$,

$$\begin{aligned} (4.11)\qquad & \big(\alpha'(\mathcal{T}_M\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon), \eta_i^\varepsilon-\eta_{i-1}^\varepsilon\big)_H \ge \int_\Omega \widetilde{\alpha}_M(\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx \\ &\qquad - \int_\Omega \widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx \text{ for } i = 1,2,3,\dots. \end{aligned}$$

On account of (4.9)–(4.11), it is inferred that

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{3\tau}{2}|g'|_{L^\infty(-M,M)}\right)\frac{1}{\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2+\frac{\mu^2}{\tau}|\nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|_{[H]^N}^2\\
&\qquad+\frac{1}{2}|\nabla\eta_i^\varepsilon|_{[H]^N}^2+\int_\Omega \widetilde{G}_M(\eta_i^\varepsilon)\,dx+\int_\Omega\widetilde{\alpha}_M(\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx\\
&\le\frac{1}{2}|\nabla\eta_{i-1}^\varepsilon|_{[H]^N}^2+\int_\Omega \widetilde{G}_M(\eta_{i-1}^\varepsilon)\,dx+\int_\Omega\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx+\frac{\tau}{2}|u_i|_H^2\\
&\qquad\qquad\text{for } i=1,2,3,\ldots.
\end{aligned}
\tag{4.12}
$$

On the other hand, by multiplying both sides of (4.2) by $\theta_i^\varepsilon-\theta_{i-1}^\varepsilon$, and using (A3) and the convexity of γ_ε, we have

$$
\begin{aligned}
&\frac{\delta_\alpha}{2\tau}|\theta_i^\varepsilon-\theta_{i-1}^\varepsilon|_H^2+\frac{\nu^2}{\tau}|\nabla(\theta_i^\varepsilon-\theta_{i-1}^\varepsilon)|_{[H]^N}^2+\int_\Omega\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx\\
&\qquad\le\int_\Omega\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_{i-1}^\varepsilon)\,dx+\frac{\tau}{2\delta_\alpha}|v_i|_H^2\quad\text{for } i=1,2,3,\ldots
\end{aligned}
\tag{4.13}
$$

via the following computation:

$$
\frac{1}{\tau}(\alpha_0(\mathcal{T}_M\eta_{i-1})(\theta_i^\varepsilon-\theta_{i-1}^\varepsilon),\theta_i^\varepsilon-\theta_{i-1}^\varepsilon)_H\ge\frac{\delta_\alpha}{\tau}|\theta_i^\varepsilon-\theta_{i-1}^\varepsilon|_H^2,
$$

$$
\begin{aligned}
&(\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\nabla\gamma_\varepsilon(\nabla\theta_i^\varepsilon),\nabla(\theta_i^\varepsilon-\theta_{i-1}^\varepsilon))_{[H]^N}\\
&\qquad\ge\int_\Omega\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon)\,dx-\int_\Omega\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\gamma_\varepsilon(\nabla\theta_{i-1}^\varepsilon)\,dx,
\end{aligned}
$$

and

$$
(u_i,\theta_i^\varepsilon-\theta_{i-1}^\varepsilon)_H\le\frac{\delta_\alpha}{2\tau}|\theta_i^\varepsilon-\theta_{i-1}^\varepsilon|_H^2+\frac{\tau}{2\delta_\alpha}|u_i|_H^2.
$$

Now, let us set τ_0 as

$$
\tau_0:=\min\left\{\tau_1,\frac{1}{6|g'|_{L^\infty(-M,M)}}\right\},\ \text{with the constant } \tau_1 \text{ as in Lemma 4.3.}
$$

Then, from (4.12) and (4.13), we obtain that

$$
\begin{aligned}
&\frac{1}{4\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_H^2+\frac{\mu^2}{\tau}|\nabla(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|_{[H]^N}^2+\frac{\delta_\alpha}{2\tau}|\theta_i^\varepsilon-\theta_{i-1}^\varepsilon|_H^2\\
&\quad+\frac{\nu^2}{\tau}|\nabla(\theta_i^\varepsilon-\theta_{i-1}^\varepsilon)|_{[H]^N}^2+\widetilde{\mathcal{F}}_\varepsilon(\eta_i^\varepsilon,\theta_i^\varepsilon)\\
&\quad\le\widetilde{\mathcal{F}}_\varepsilon(\eta_{i-1}^\varepsilon,\theta_{i-1}^\varepsilon)+\frac{\tau}{2}|u_i|_H^2+\frac{\tau}{2\delta_\alpha}|v_i|_H^2\quad\text{for } i=1,2,3,\ldots.
\end{aligned}
$$

Thus, we conclude the proof of Theorem 4.2. ∎

5. Proof of Main Theorem 1.

Main Theorem 1 is proved by verifying several claims, which are divided in the following subsections.

Subsection 5.1. Boundedness of $\{[\eta^\varepsilon]_\tau\}$ in $W^{1,2}(0,T;W_0)$.
Subsection 5.2. Boundedness of $\{[\theta^\varepsilon]_\tau\}$ in $L^\infty(0,T;W_0)$.

Subsection 5.3. Existence of a limit $[\eta, \theta]$ for a subsequence of $\{[[\eta^\varepsilon]_\tau, [\theta^\varepsilon]_\tau]\}$.
Subsection 5.4. Verification of (S0)–(S3) for $[\eta, \theta]$.

Remark 5.1. We note that the assumptions $N \in \{1,2,3\}$, $\mu > 0$, and $\nu > 0$, mentioned in Remark 3.2, will be essential to deriving the boundedness of $\{[\theta^\varepsilon]_\tau\}$.

To prepare the proof, we set

$$n_\tau := \min\{n \in \mathbb{N} \,|\, n\tau \geq T\}. \tag{5.1}$$

Additionally, under the notations as in Theorem 4.2, we invoke (2.2a) and (2.2b), and take a small constant $\tau_* \in (0, \tau_0)$, such that

$$\tau \sum_{i=1}^{n_\tau} \bigl(|u_i|_H^2 + |v_i|_H^2\bigr) \leq |u|_{\mathscr{H}}^2 + |v|_{\mathscr{H}}^2 + 1 \text{ whenever } \tau \in (0, \tau_*).$$

Also, with (A1), (A3), (A4), and (A5) in mind, we set the constant $M > 0$ of truncation, so large to satisfy that

$$M \geq |\eta_0|_{L^\infty(\Omega)}, \; g(M) \geq |u|_{L^\infty(Q)}, \text{ and } g(-M) \leq -|u|_{L^\infty(Q)}. \tag{5.2}$$

Then, it immediately follows that

$$\widetilde{G}_M(\eta_0) = G(\eta_0), \text{ and } \widetilde{\alpha}_M(\eta_0) = \alpha(\eta_0). \tag{5.3}$$

Additionally, we prepare the following lemmas.

LEMMA 5.2. *Let us fix $\varepsilon > 0$, $w \in W_0$, and $\alpha^\circ \in L^\infty(\Omega) \cap V$. Then, for any $L \geq |\alpha^\circ|_{L^\infty(\Omega)}$, there exists a constant $C_4(L) > 0$, depending only on L, and being independent of ε and w, such that*

$$\bigl(\mathrm{div}(\alpha^\circ \nabla \gamma_\varepsilon(\nabla w)), \Delta w\bigr)_H \geq -|\nabla^2 w|_{[H]^{N\times N}}^2 - C_4(L)(|\alpha^\circ|_V^2 + 1)(|w|_V^2 + 1).$$

Proof. This lemma is immediately obtained as a straightforward consequence of [1, Lemma 3.2]. ∎

LEMMA 5.3 (comparison principle). *We assume that $\eta^1, \eta^2 \in W^{1,2}(0,T;W_0)$, $\eta_0^1, \eta_0^2 \in W_0$, $\widetilde{\theta} \in \mathscr{V}$, $\widetilde{u} \in \mathscr{H}$, and*

$$\begin{cases} (-1)^{i-1}\Bigl(\partial_t \eta^i - \Delta(\eta^i + \mu^2 \partial_t \eta^i) + g(\mathcal{T}_M \eta^i) + \alpha'(\mathcal{T}_M \eta^i)|\nabla \widetilde{\theta}|\Bigr) \\ \qquad \leq (-1)^{i-1}\widetilde{u} \text{ a.e. in } Q, \\ \eta^i(0) = \eta_0^i \text{ a.e. in } \Omega. \end{cases} \tag{5.4}$$

Then, there exists a constant $C_9 > 0$ such that

$$\bigl|[\eta^1 - \eta^2]^+(t)\bigr|_V^2 \leq C_9 \bigl|[\eta_0^1 - \eta_0^2]^+\bigr|_V^2 \textit{ for any } t \in [0,T].$$

Proof. Taking the difference of two inequality (5.4) for η^i, $i = 1, 2$, and multiplying both sides by $[\eta^1 - \eta^2]^+(t)$, we see that

$$
\begin{aligned}
(5.5)\qquad & \frac{1}{2}\frac{d}{dt}\big|[\eta^1-\eta^2]^+(t)\big|_H^2+\frac{\mu^2}{2}\frac{d}{dt}\big|\nabla[\eta^1-\eta^2]^+(t)\big|_{[H]^N}^2 \\
&\quad +\big|\nabla[\eta^1-\eta^2]^+(t)\big|_{[H]^N}^2+(g(\mathcal{T}_M\eta^1(t))-g(\mathcal{T}_M\eta^2(t)),[\eta^1-\eta^2]^+(t))_H \\
&\quad +\int_\Omega\big(\alpha'(\mathcal{T}_M\eta^1(t))-\alpha'(\mathcal{T}_M\eta^2(t))\big)|\nabla\tilde{\theta}(t)|,[\eta^1-\eta^2]^+(t)\,dx \\
&\le 0 \quad \text{for a.e. } t\in(0,T).
\end{aligned}
$$

Here, from assumption (A1), it is deduced that

$$
\begin{aligned}
&(g(\mathcal{T}_M\eta^1(t))-g(\mathcal{T}_M\eta^2(t)),[\eta^1-\eta^2]^+(t))_H \\
(5.6)\qquad &\quad \ge -|g'|_{L^\infty(-M,M)}\big|[\eta^1-\eta^2]^+(t)\big|_H^2 \quad \text{for a.e. } t\in(0,T).
\end{aligned}
$$

Also, by the monotonicity of $\alpha'\circ\mathcal{T}_M$, we can say that

$$
\big(\alpha'(\mathcal{T}_M\eta^1)-\alpha'(\mathcal{T}_M\eta^2)\big)[\eta^1-\eta^2]^+\ge 0 \quad \text{a.e. in } Q.
$$

Hence one can see that

$$
\begin{aligned}
&\int_\Omega\big(\alpha'(\mathcal{T}_M\eta^1(t))-\alpha'(\mathcal{T}_M\eta^2(t))\big)|\nabla\tilde{\theta}(t)|,[\eta^1-\eta^2]^+(t)\,dx \\
(5.7)\qquad &\quad \ge 0 \quad \text{for a.e. } t\in(0,T).
\end{aligned}
$$

Now, in light of (5.5)–(5.7), it is deduced that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\big|[\eta^1-\eta^2]^+(t)\big|_H^2+\mu^2\big|\nabla[\eta^1-\eta^2]^+(t)\big|_{[H]^N}^2\Big) \\
&\quad \le 2|g'|_{L^\infty(-M,M)}\big|[\eta^1-\eta^2]^+(t)\big|_H^2 \quad \text{for a.e. } t\in(0,T).
\end{aligned}
$$

Applying Gronwall's inequality, we arrive at

$$
\begin{aligned}
(5.8)\qquad & \big|[\eta^1-\eta^2]^+(t)\big|_H^2+\mu^2\big|\nabla[\eta^1-\eta^2]^+(t)\big|_{[H]^N}^2 \\
&\quad \le e^{2T|g'|_{L^\infty(-M,M)}}\Big(\big|[\eta_0^1-\eta_0^2]^+\big|_H^2+\mu^2\big|\nabla[\eta_0^1-\eta_0^2]^+\big|_{[H]^N}^2\Big) \\
&\qquad\qquad \text{for any } t\in[0,T].
\end{aligned}
$$

Inequality (5.8) finishes the proof of Lemma 5.3 with the constant

$$
C_9:=\frac{1+\mu^2}{1\wedge\mu^2}e^{2T|g'|_{L^\infty(-M,M)}}. \qquad \square
$$

Now, the claims of subsections 5.1–5.4 are verified as follows.

Subsection 5.1. Boundedness of $\{[\eta^\varepsilon]_\tau\}$ in $W^{1,2}(0,T;W_0)$. In this subsection, we prove the following lemma, which provides the uniform estimate of $\{[\eta^\varepsilon]_\tau\}$ in $W^{1,2}(0,T;W_0)$.

LEMMA 5.4. *Let $\tau\in(0,\tau_*)$. Then, there exists a constant $C_2>0$, independent of ε and τ, such that*

$$
(5.9)\qquad \frac{1}{\tau}\sum_{i=1}^{n_\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|_{H^2(\Omega)}^2\le C_2\big(|\eta_0|_{H^2(\Omega)}^2+|\theta_0|_V^2+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\big).
$$

Proof. First, from the definition of $\widetilde{\mathcal{F}}_\varepsilon$ (4.3), (5.3), and the embedding $W_0 \subset L^\infty(\Omega)$ under $N \le 3$, it is seen that

$$\begin{aligned}\widetilde{\mathcal{F}}_\varepsilon(\eta_0,\theta_0) &= \frac{1}{2}\int_\Omega |\nabla\eta_0|^2\,dx + \int_\Omega \widetilde{G}_M(\eta_0)\,dx + \int_\Omega \widetilde{\alpha}_M(\eta_0)\gamma_\varepsilon(\nabla\theta_0)\,dx \\ &= \frac{1}{2}\int_\Omega |\nabla\eta_0|^2\,dx + \int_\Omega G(\eta_0)\,dx + \int_\Omega \alpha(\eta_0)\gamma_\varepsilon(\nabla\theta_0)\,dx \\ &\le \frac{1}{2}|\eta_0|_V^2 + |G(\eta_0)|_{L^1(\Omega)} + |\alpha(\eta_0)|_{L^2(\Omega)}^2 + \mathcal{L}^N(\Omega) + |\theta_0|_V^2.\end{aligned}$$

Hence,

$$C_F := \sup_{\varepsilon\in(0,1)} \widetilde{\mathcal{F}}_\varepsilon(\eta_0,\theta_0) < \infty.$$

Also, from (4.4), (5.1), Theorem 4.2, and Hölder's inequality, it is observed that

$$\begin{aligned}(5.10)\quad |\theta_i^\varepsilon|_V^2 &\le 2|\theta_0|_V^2 + 2\left(\sum_{i=1}^{n_\tau}|\theta_i^\varepsilon - \theta_{i-1}^\varepsilon|_V\right)^2 \\ &\le 2|\theta_0|_V^2 + 2(T+1)\sum_{i=1}^{n_\tau}\frac{1}{\tau}|\theta_i^\varepsilon - \theta_{i-1}^\varepsilon|_V^2 \\ &\le 2|\theta_0|_V^2 + \frac{4(T+1)}{\delta_\alpha \wedge \nu^2}\left(\widetilde{\mathcal{F}}_\varepsilon(\eta_0,\theta_0) + \frac{1}{2(1\wedge\delta_\alpha)}\big(|u|_{\mathscr{H}}^2 + |v|_{\mathscr{H}}^2 + 1\big)\right) \\ &\le \frac{4(C_F+1)(T+1)}{1\wedge\delta_\alpha^2\wedge\nu^4}\big(|\theta_0|_V^2 + |u|_{\mathscr{H}}^2 + |v|_{\mathscr{H}}^2 + 1\big) \quad \text{for any } i = 1,2,3,\ldots,n_\tau.\end{aligned}$$

Next, we verify the estimate (5.9). Multiplying both sides of (4.1) by $-\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)$ and applying Young's inequality, it can be seen that

$$\begin{aligned}(5.11)\quad \frac{3\mu^2}{4\tau}|\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)|_H^2 &\le \frac{1}{2}\big(|\Delta\eta_{i-1}^\varepsilon|_H^2 - |\Delta\eta_i^\varepsilon|_H^2\big) \\ &\quad + (g(\mathcal{T}_M\eta_i^\varepsilon), \Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon))_H + (\alpha'(\mathcal{T}_M\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta_i^\varepsilon), \Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon))_H \\ &\quad + \frac{\tau}{\mu^2}|u_i|_H^2 \quad \text{for } i = 1,2,3,\ldots,n_\tau\end{aligned}$$

via the following calculations:

$$(-\Delta\eta_i^\varepsilon, -\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon))_H \ge \frac{1}{2}\big(|\Delta\eta_i^\varepsilon|_H^2 - |\Delta\eta_{i-1}^\varepsilon|_H^2\big)$$

and

$$(u_i, -\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon))_H \le \frac{\mu^2}{4\tau}|\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)|_H^2 + \frac{\tau}{\mu^2}|u_i|_H^2.$$

In addition, we compute that

$$\begin{aligned}(5.12)\quad &(g(\mathcal{T}_M\eta_i^\varepsilon), \Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon))_H \\ &\quad \le \frac{\tau}{\mu^2}|g|_{L^\infty(-M,M)}^2\mathcal{L}^N(\Omega) + \frac{\mu^2}{4\tau}|\Delta(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)|_H^2\end{aligned}$$

and

$$
\begin{aligned}
(5.13)\qquad &(\alpha'(\mathcal{T}_M\eta_i^\varepsilon)\gamma_\varepsilon(\nabla\theta), \varDelta(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon))_H \\
&\leq \frac{\tau}{\mu^2}|\alpha'|^2_{L^\infty(-M,M)}\int_\Omega \big(\varepsilon^2+|\nabla\theta_i^\varepsilon|^2\big)\,dx+\frac{\mu^2}{4\tau}|\varDelta(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|^2_H \\
&\leq \frac{\tau}{\mu^2}|\alpha'|^2_{L^\infty(-M,M)}\big(\mathcal{L}^N(\Omega)+|\theta_i^\varepsilon|_V^2\big)+\frac{\mu^2}{4\tau}|\varDelta(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|^2_H \\
&\qquad \text{for } i=1,2,3,\ldots,n_\tau.
\end{aligned}
$$

On account of (5.10)–(5.13), we infer that

$$
\begin{aligned}
(5.14)\qquad &\frac{\mu^2}{4\tau}|\varDelta(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|^2_H \\
&\leq \frac{1}{2}\big(|\varDelta\eta_{i-1}^\varepsilon|^2_H-|\varDelta\eta_i^\varepsilon|^2_H\big)+\frac{\tau\mathcal{L}^N(\Omega)}{\mu^2}\big(|g|^2_{L^\infty(-M,M)}+|\alpha'|_{L^\infty(-M,M)}\big) \\
&\quad+\frac{4\tau(C_F+1)(T+1)|\alpha'|^2_{L^\infty(-M,M)}}{\mu^2(1\wedge\delta_\alpha^2\wedge\nu^4)}\big(|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big) \\
&\leq \frac{1}{2}\big(|\varDelta\eta_{i-1}^\varepsilon|^2_H-|\varDelta\eta_i^\varepsilon|^2_H\big)+\tau\widetilde{C}_3\big(|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big) \\
&\qquad \text{for } i=1,2,3,\ldots,n_\tau,
\end{aligned}
$$

with

$$
\widetilde{C}_3:=\frac{4(C_F+\mathcal{L}^N(\Omega)+1)(T+1)\big(|g|^2_{L^\infty(-M,M)}+|\alpha'|^2_{L^\infty(-M,M)}+1\big)}{\mu^2(1\wedge\delta_\alpha^2\wedge\nu^4)}.
$$

Hence, taking the sum of (5.14) with respect to $i=1,2,3,\ldots,n_\tau$, one can deduce from (2.1), (4.4), and (5.14) that

$$
\begin{aligned}
&\frac{1}{\tau}\sum_{i=1}^{n_\tau}|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|^2_{H^2(\Omega)}\leq\frac{C_0}{\tau}\sum_{i=1}^{n_\tau}\big(|A_N(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|^2_H+|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|^2_H\big) \\
&=\frac{C_0}{\tau}\sum_{i=1}^{n_\tau}\big(|\varDelta(\eta_i^\varepsilon-\eta_{i-1}^\varepsilon)|^2_H+|\eta_i^\varepsilon-\eta_{i-1}^\varepsilon|^2_H\big) \\
&\leq\frac{2C_0}{\mu^2}|\varDelta\eta_0|^2_H+\frac{4C_0\widetilde{C}_3(T+1)}{\mu^2}\big(|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big) \\
&\quad+4C_0\left(\widetilde{\mathcal{F}}_\varepsilon(\eta_0,\theta_0)+\frac{1}{2(1\wedge\delta_\alpha)}\big(|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big)\right) \\
&\leq\frac{2NC_0}{\mu^2}|\eta_0|^2_{H^2(\Omega)}+\frac{4C_0\widetilde{C}_3(T+1)}{\mu^2}\big(|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big) \\
&\quad+\frac{4C_0(C_F+1)}{1\wedge\delta_\alpha}\big(|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big) \\
&\leq C_2\big(|\eta_0|^2_{H^2(\Omega)}+|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big),
\end{aligned}
$$

where

$$
C_2:=\frac{4NC_0(T+1)(\widetilde{C}_3+C_F+1)}{1\wedge\mu^2\wedge\delta_\alpha}.
$$

Thus, we conclude the proof of Lemma 5.4. ∎

Subsection 5.2. Boundedness of $\{[\theta^\varepsilon]_\tau\}$ in $L^\infty(0,T;W_0)$. The objective of this subsection is formalized as establishing the following lemma.

LEMMA 5.5. *There exist a small time-step size $\tau_{**} \in (0, \tau_*)$ and a constant $C_5 > 0$ such that for any $\tau \in (0, \tau_{**})$, the following estimate holds:*

$$|\theta_i^\varepsilon|_{H^2(\Omega)}^2 \le C_5\big(|\eta_0|_{H^2(\Omega)}^2 + |\theta_0|_{H^2(\Omega)}^2 + |u|_{\mathscr{H}}^2 + |v|_{\mathscr{H}}^2 + 1\big)^2 \quad \text{for } i = 1, 2, 3, \dots, n_\tau. \tag{5.15}$$

Proof. First, we note that Lemma 5.4 leads to the boundedness of $\{\eta_i^\varepsilon\}_{i=0}^{n_\tau}$ in $H^2(\Omega)$, with the following estimate:

$$\begin{aligned}
|\eta_i^\varepsilon|_{H^2(\Omega)}^2 &\le 2|\eta_0|_{H^2(\Omega)}^2 + 2\left(\sum_{i=1}^{n_\tau} |\eta_i^\varepsilon - \eta_{i-1}^\varepsilon|_{H^2(\Omega)}\right)^2 \\
&\le 2|\eta_0|_{H^2(\Omega)}^2 + 2(T+1)\sum_{i=1}^{n_\tau} \frac{1}{\tau}|\eta_i^\varepsilon - \eta_{i-1}^\varepsilon|_{H^2(\Omega)}^2 \\
&\le 2(T+1)(C_2+1)\big(|\eta_0|_{H^2(\Omega)}^2 + |\theta_0|_V^2 + |u|_{\mathscr{H}}^2 + |v|_{\mathscr{H}}^2 + 1\big) \\
&\qquad \text{for } i = 1, 2, 3, \dots, n_\tau.
\end{aligned} \tag{5.16}$$

Moreover, by (5.16) and continuous embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$ under $N \le 3$, we see that $\widetilde{\alpha}_M(\eta_i^\varepsilon) \in L^\infty(\Omega) \cap V$ for any $i = 1, 2, 3, \dots, n_\tau$, with the following estimates holding:

$$\begin{aligned}
|\widetilde{\alpha}_M(\eta_i^\varepsilon)|_{L^\infty(\Omega)}^2 &\le 2\alpha(0)^2 + 2|\alpha'|_{L^\infty(-M,M)}^2 |\eta_i^\varepsilon|_{L^\infty(\Omega)}^2 \\
&\le 2\alpha(0)^2 + 2(C_{H^2}^{L^\infty})^2 |\alpha'|_{L^\infty(-M,M)}^2 |\eta_i^\varepsilon|_{H^2(\Omega)}^2
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
|\widetilde{\alpha}_M(\eta_i^\varepsilon)|_V^2 &\le \mathcal{L}^N(\Omega)|\widetilde{\alpha}_M(\eta_i^\varepsilon)|_{L^\infty(\Omega)}^2 + |\alpha'|_{L^\infty(-M,M)}^2 |\nabla \eta_i^\varepsilon|_{[H]^N}^2 \\
&\le 2\mathcal{L}^N(\Omega)\big(\alpha(0)^2 + (C_{H^2}^{L^\infty})^2 |\alpha'|_{L^\infty(-M,M)}^2 |\eta_i^\varepsilon|_{H^2(\Omega)}^2\big) + |\alpha'|_{L^\infty(-M,M)}^2 |\eta_i^\varepsilon|_{H^2(\Omega)}^2 \\
&\le \widetilde{C}_6\big(|\eta_i^\varepsilon|_{H^2(\Omega)}^2 + 1\big) \quad \text{for } i = 1, 2, 3, \dots, n_\tau,
\end{aligned} \tag{5.18}$$

where $C_{H^2}^{L^\infty}$ is a constant of the embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$, and

$$\widetilde{C}_6 := 2\big(\mathcal{L}^N(\Omega)\alpha(0)^2 + \mathcal{L}^N(\Omega)(C_{H^2}^{L^\infty})^2 + 1\big)\big(|\alpha'|_{L^\infty(-M,M)}^2 + 1\big).$$

Next, we verify the inequality (5.15). Let us consider multiplying both sides of (4.2) by $-\Delta\theta_i^\varepsilon$.

By applying Young's inequality, we have

$$\frac{\nu^2}{\tau}\big(-\Delta(\theta_i^\varepsilon - \theta_{i-1}^\varepsilon), -\Delta\theta_i^\varepsilon\big)_H \ge \frac{\nu^2}{2\tau}\big(|\Delta\theta_i^\varepsilon|_H^2 - |\Delta\theta_{i-1}^\varepsilon|_H^2\big) \tag{5.19}$$

and

$$(v_i, -\Delta\theta_i^\varepsilon)_H \le \frac{1}{2}|\Delta\theta_i^\varepsilon|_H^2 + \frac{1}{2}|v_i|_H^2 \quad \text{for } i = 1, 2, 3, \dots, n_\tau. \tag{5.20}$$

Moreover, from (4.4) and (A3), we see that

$$
\begin{aligned}
(5.21)\qquad & \frac{1}{\tau}\big(\alpha_0(\mathcal{T}_M\eta_{i-1}^\varepsilon)(\theta_i^\varepsilon-\theta_{i-1}^\varepsilon),-\Delta\theta_i^\varepsilon\big)_H \\
&\geq -\frac{1}{2\tau^2}|\alpha_0|^2_{L^\infty(-M,M)}|\theta_i^\varepsilon-\theta_{i-1}^\varepsilon|_H^2-\frac{1}{2}|\Delta\theta_i^\varepsilon|_H^2 \\
&\geq -\frac{|\alpha_0|^2_{L^\infty(-M,M)}}{\delta_\alpha}\cdot\frac{1}{\tau}\big(\widetilde{\mathcal{F}}_\varepsilon(\eta_i^\varepsilon,\theta_i^\varepsilon)-\widetilde{\mathcal{F}}_\varepsilon(\eta_{i-1}^\varepsilon,\theta_{i-1}^\varepsilon)\big) \\
&\quad -\frac{|\alpha_0|_{L^\infty(0,M,M)}}{2\delta_\alpha}\Big(|u_i|_H^2+\frac{1}{\delta_\alpha}|v_i|_H^2\Big)-\frac{1}{2}|\Delta\theta_i^\varepsilon|_H^2 \\
&\qquad \text{for } i=1,2,3,\dots,n_\tau.
\end{aligned}
$$

Using (2.1), (5.16)–(5.18), and applying Lemma 5.2 to the case that

$$\alpha^\circ=\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon) \text{ and } w=\theta_i^\varepsilon \text{ for each } i\in\{1,2,\dots,n_\tau\}$$

and

$$
\begin{aligned}
L=\;&2\alpha(0)^2+2(C_{H^2}^{L^\infty})^2|\alpha'|^2_{L^\infty(-M,M)} \\
&\cdot 2(T+1)(C_2+1)\big(|\eta_0|^2_{H^2(\Omega)}+|\theta_0|_V^2+|u|^2_{\mathscr{H}}+|v|^2_{\mathscr{H}}+1\big),
\end{aligned}
$$

it is observed that

$$
\begin{aligned}
(5.22)\qquad & \big(\mathrm{div}(\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)\nabla\gamma_\varepsilon(\nabla\theta_i^\varepsilon)),\Delta\theta_i^\varepsilon\big)_H \\
&\geq -|\nabla^2\theta_i^\varepsilon|^2_{[H]^{N\times N}}-C_4(L)(|\widetilde{\alpha}_M(\eta_{i-1}^\varepsilon)|_V^2+1)(|\theta_i^\varepsilon|_V^2+1) \\
&\geq -C_0\big(|A_N\theta_i^\varepsilon|_H^2+|\theta_i^\varepsilon|_H^2\big) \\
&\quad -C_4(L)\big(\widetilde{C}_6\big(|\eta_{i-1}^\varepsilon|^2_{H^2(\Omega)}+1\big)+1\big)\big(|\theta_i^\varepsilon|_V^2+1\big) \\
&\geq -C_0|\Delta\theta_i^\varepsilon|_H^2-(C_4(L)+C_0)(\widetilde{C}_6+1)\big(|\theta_i^\varepsilon|_V^2+1\big)\big(|\eta_{i-1}^\varepsilon|^2_{H^2(\Omega)}+1\big) \\
&\qquad \text{for } i=1,2,3,\dots,n_\tau.
\end{aligned}
$$

Now, by using (5.19)–(5.22), we will obtain that

$$(5.23)\qquad \frac{1}{\tau}(X_i-X_{i-1})\leq\widetilde{C}_7(X_i+F_i)\ \text{ for } i=1,2,3,\dots,n_\tau,$$

with

$$
\begin{cases}
X_i:=\nu^2|\Delta\theta_i^\varepsilon|_H^2+\dfrac{|\alpha_0|^2_{L^\infty(-M,M)}}{\delta_\alpha}\widetilde{\mathcal{F}}_\varepsilon(\eta_i^\varepsilon,\theta_i^\varepsilon) \\
F_i:=\big(|\theta_i^\varepsilon|_V^2+|u_i|_H^2+|v_i|_H^2+1\big)\big(|\eta_{i-1}^\varepsilon|^2_{H^2(\Omega)}+1\big)
\end{cases}
\quad \text{for } i=1,2,3,\dots,n_\tau,
$$

and

$$\widetilde{C}_7:=\frac{2(C_4(L)+C_0+1)(\widetilde{C}_6+1)(|\alpha_0|^2_{L^\infty(-M,M)}+1)}{1\wedge\delta_\alpha^2\wedge\nu^2}\geq 2.$$

Here, let us take $\tau_{**}\in(0,\tau_*)$ satisfying

$$\tau_{**}<\min\left\{\tau_*,\frac{1}{2\widetilde{C}_7}\right\},\ \text{and in particular, } 1-\tau_{**}\widetilde{C}_7>\frac{1}{2}.$$

Then, applying the discrete version of Gronwall's lemma (cf. [12, section 3.1]) to (5.23), one can see from (5.10), (5.16), and (5.23) that

$$
\begin{aligned}
(5.24)\qquad X_i &\le \widetilde{C}_7 e^{2\widetilde{C}_7(T+1)}\left(X_0+\tau\sum_{i=1}^{n_\tau}F_i\right)\\
&\le \widetilde{C}_7 e^{2\widetilde{C}_7(T+1)}\left(\nu^2|\Delta\theta_0|_H^2+\frac{|\alpha_0|^2_{L^\infty(-M,M)}C_F}{\delta_\alpha}\right)\\
&\quad+\frac{4\widetilde{C}_7 e^{2\widetilde{C}_7(T+1)}(C_F+3)(T+1)^2}{1\wedge\delta_\alpha^2\wedge\nu^4}\left(|\theta_0|_V^2+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\right)\\
&\qquad\cdot 2(T+1)(C_2+2)\left(|\eta_0|^2_{H^2(\Omega)}+|\theta_0|_V^2+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\right)\\
&\le \widetilde{C}_8\left(|\eta_0|^2_{H^2(\Omega)}+|\theta_0|^2_{H^2(\Omega)}+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\right)^2 \quad \text{for } i=1,2,3,\dots,n_\tau,
\end{aligned}
$$

where

$$
\widetilde{C}_8 := \frac{8\widetilde{C}_7 e^{2\widetilde{C}_7(T+1)}(C_F+3)(T+1)^3(C_2+N\nu^2+|\alpha_0|_{L^\infty(-M,M)}+2)}{1\wedge\delta_\alpha^2\wedge\nu^4}.
$$

In light of (2.1), (5.10), and (5.24), we arrive at

$$
\begin{aligned}
|\theta_i^\varepsilon|^2_{H^2(\Omega)} &\le C_0\left(|A_N\theta_i^\varepsilon|_H^2+|\theta_i^\varepsilon|_H^2\right)=C_0\left(|\Delta\theta_i^\varepsilon|_H^2+|\theta_i^\varepsilon|_H^2\right)\\
&\le \frac{C_0}{\nu^2}X_i+\frac{4C_0(C_F+1)(T+1)}{1\wedge\delta_\alpha^2\wedge\nu^4}\left(|\theta_0|_V^2+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\right)\\
&\le \frac{4C_0(C_F+\widetilde{C}_8+1)(T+1)}{1\wedge\delta_\alpha^2\wedge\nu^4}\left(|\eta_0|^2_{H^2(\Omega)}+|\theta_0|^2_{H^2(\Omega)}+|u|_{\mathscr{H}}^2+|v|_{\mathscr{H}}^2+1\right)^2\\
&\qquad\qquad \text{for } i=1,2,3,\dots,n_\tau.
\end{aligned}
$$

Thus, we conclude the proof of Lemma 5.5 with the constant

$$
C_5=\frac{4C_0(C_F+\widetilde{C}_8+1)(T+1)}{1\wedge\delta_\alpha^2\wedge\nu^4}.
$$

∎

Subsection 5.3. Existence of a limit $[\boldsymbol{\eta},\boldsymbol{\theta}]$ for a subsequence of $\{[[\boldsymbol{\eta}^\varepsilon]_\tau, [\boldsymbol{\theta}^\varepsilon]_\tau]\}$. As a consequence of Lemmas 5.4 and 5.5 and Theorem 4.2, the following boundednesses are derived:

- $\{[\eta^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$ is bounded in $W^{1,2}(0,T;W_0)$;
- $\{[\overline{\eta}^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$, $\{[\underline{\eta}^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$ is bounded in $L^\infty(0,T;W_0)$;
- $\{[\theta^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$ is bounded in $L^\infty(0,T;W_0)$ and in $W^{1,2}(0,T;V)$;
- $\{[\overline{\theta}^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$, $\{[\underline{\theta}^\varepsilon]_\tau \,|\, \varepsilon\in(0,1),\tau\in(0,\tau_{**})\}$ is bounded in $L^\infty(0,T;W_0)$.

Therefore, by applying Aubin's type compactness theory (cf. [36, Corollary 4]), we can find sequences $\{\varepsilon_n\}_{n=1}^\infty\subset(0,1)$, $\{\tau_n\}_{n=1}^\infty\subset(0,\tau_{**})$ and a pair of functions $[\eta,\theta]\in[\mathscr{H}]^2$ such that $\varepsilon_n\searrow 0$ and $\tau_n\searrow 0$ as $n\to\infty$, and we obtain the following convergences as $n\to\infty$:

$$
(5.25)\qquad \begin{cases}\eta_n:=[\eta^{\varepsilon_n}]_{\tau_n}\to\eta \text{ in } C([0,T];V) \text{ and weakly in } W^{1,2}(0,T;W_0),\\ \theta_n:=[\theta^{\varepsilon_n}]_{\tau_n}\to\theta \text{ in } C([0,T];V) \text{ and weakly in } W^{1,2}(0,T;V)\\ \qquad \text{and weakly-}* \text{ in } L^\infty(0,T;W_0).\end{cases}
$$

Besides, having in mind:

$$\begin{cases} \max\left\{|[\overline{\eta}^{\varepsilon_n}]_{\tau_n} - \eta_n|_V\,,\,|[\underline{\eta}^{\varepsilon_n}]_{\tau_n} - \eta_n|_V\right\} \leq \displaystyle\int_{\Delta_{i,\tau_n}} |\partial_t \eta_n|_V\,dt < \infty, \\ \max\left\{|[\overline{\theta}^{\varepsilon_n}]_{\tau_n} - \theta_n|_V\,,\,|[\underline{\theta}^{\varepsilon_n}]_{\tau_n} - \theta_n|_V\right\} \leq \displaystyle\int_{\Delta_{i,\tau_n}} |\partial_t \theta_n|_V\,dt < \infty, \end{cases}$$
$$\text{where } \Delta_{i,\tau} := [t_{i-1}, t_i) \cap (0,T),$$

we can derive that

$$\text{(5.26)} \qquad \begin{cases} \overline{\eta}_n := [\overline{\eta}^{\varepsilon_n}]_{\tau_n} \to \eta \text{ and } \underline{\eta}_n := [\underline{\eta}^{\varepsilon_n}]_{\tau_n} \to \eta \text{ in } L^\infty(0,T;V) \text{ and} \\ \text{weakly-}* \text{ in } L^\infty(0,T;W_0), \\ \overline{\theta}_n := [\overline{\theta}^{\varepsilon_n}]_{\tau_n} \to \theta \text{ and } \underline{\theta}_n := [\underline{\theta}^{\varepsilon_n}]_{\tau_n} \to \theta \text{ in } L^\infty(0,T;V) \text{ and} \\ \text{weakly-}* \text{ in } L^\infty(0,T;W_0). \end{cases}$$

Thus, we have verified the claim of subsection 5.3.

Subsection 5.4. Verification of (S0)–(S3) for $[\boldsymbol{\eta},\boldsymbol{\theta}]$. Now, we verify that the limiting pair $[\eta,\theta]$ satisfies (S0)–(S3). The initial condition (S3) can be easily confirmed as follows:

$$\eta(0) = \lim_{n\to\infty} \eta_n(0) = \eta_0 \text{ and } \theta(0) = \lim_{n\to\infty} \theta_n(0) = \theta_0 \text{ in } H.$$

Let us take an arbitrary open interval $I \subset (0,T)$. Then, in light of (4.1), (4.2), and the convexity of γ_ε, the sequences as in (5.25) and (5.26) should fulfill the following two variational forms:

$$\text{(5.27)} \quad \begin{aligned} &\int_I (\partial_t \eta_n(t), \varphi)\,dt + \int_I (\nabla(\overline{\eta}_n + \mu^2 \partial_t \eta_n)(t), \nabla\varphi)_{[H]^N}\,dt \\ &\quad + \int_I (g(\mathcal{T}_M\,\overline{\eta}_n(t)), +\alpha'(\mathcal{T}_M\,\overline{\eta}_n(t))\gamma_\varepsilon(\nabla\overline{\theta}_n(t)), \varphi)_H\,dt = \int_I ([\overline{u}]_{\tau_n}(t), \varphi)_H\,dt \\ &\qquad\qquad \text{for all } \varphi \in V \text{ and } n = 1,2,3,\dots,n_\tau \end{aligned}$$

and

$$\text{(5.28)} \quad \begin{aligned} &\int_I \big((\alpha_0(\mathcal{T}_M \underline{\eta}_n)\partial_t\theta_n)(t), \overline{\theta}_n(t) - \psi\big)_H\,dt + \int_I\int_\Omega \alpha(\underline{\eta}_n(t))\gamma_\varepsilon(\nabla\overline{\theta}_n(t))\,dxdt \\ &\quad + \nu^2 \int_I (\nabla\partial_t\theta_n(t), \nabla(\overline{\theta}_n(t) - \psi))_{[H]^N}\,dt \\ &\leq \int_I\int_\Omega \alpha(\underline{\eta}_n(t))\gamma_\varepsilon(\nabla\psi)\,dxdt + \int_I ([\overline{v}]_{\tau_n}(t), \overline{\theta}_n(t) - \psi)_H\,dt \\ &\qquad\qquad \text{for all } \psi \in V \text{ and } n = 1,2,3,\dots,n_\tau. \end{aligned}$$

On this basis, having in mind (5.25), (A1), (A2), and the fact that

$$\gamma_\varepsilon \to |\cdot| \text{ uniformly on } \mathbb{R}^N \text{ as } \varepsilon \to 0,$$

letting $n \to \infty$ in (5.27) and (5.28) yields that

$$\begin{aligned} &\int_I (\partial_t \eta(t), \varphi)\,dt + \int_I (\nabla(\eta + \mu^2\partial_t\eta)(t), \nabla\varphi)_{[H]^N}\,dt \\ &\quad + \int_I (g(\mathcal{T}_M\,\eta(t)) + \alpha'(\mathcal{T}_M\,\eta(t))|\nabla\theta(t)|, \varphi)_H\,dt = \int_I (u(t), \varphi)_H\,dt \\ &\qquad\qquad \text{for any } \varphi \in V \end{aligned}$$

and

$$\begin{aligned}
&\int_I \big((\alpha_0(\mathcal{T}_M\eta)\partial_t\theta)(t), \theta(t)-\psi\big)_H\,dt + \int_I\int_\Omega \alpha(\eta(t))|\nabla\theta(t)|\,dxdt\\
&\quad + \nu^2\int_I (\nabla\partial_t\theta(t), \nabla(\theta(t)-\psi))_{[H]^N}\,dt\\
&\quad \leq \int_I\int_\Omega \alpha(\eta(t))|\nabla\psi|\,dxdt + \int_I (v(t), \theta(t)-\psi)_H\,dt \quad \text{for any } \psi\in V,
\end{aligned}$$

respectively. Since $I\subset(0,T)$ is arbitrary, $[\eta,\theta]$ should satisfy (S1) and (S2) if $|\eta|_{L^\infty(Q)}\leq M$.

Finally, we verify $\eta\in L^\infty(Q)$. By (5.2), the following inequalities can be obtained:

$$\begin{cases} \partial_t M - \mathit{\Delta}(M+\mu^2\partial_t M) + g(M) + \alpha'(M)|\nabla\theta(t)| \geq u,\\ \partial_t(-M) - \mathit{\Delta}((-M)+\mu^2\partial_t(-M)) + g(-M) + \alpha'(-M)|\nabla\theta(t)| \leq u \end{cases}$$
$$\text{a.e. in } Q.$$

Hence, applying Lemma 5.3 to the case when

$$[\eta^1,\eta^2,\widetilde{\theta},\widetilde{u}] = [\eta, M, \theta, u],\ \ [\eta_0^1,\eta_0^2] = [\eta_0, M]$$

and

$$[\eta^1,\eta^2,\widetilde{\theta},\widetilde{u}] = [-M, \eta, \theta, u],\ \ [\eta_0^1,\eta_0^2] = [-M, \eta_0],$$

we arrive at

$$\begin{cases} \big|[\eta-M]^+(t)\big|_V^2 \leq C_9\big|[\eta_0-M]^+\big|_V^2 = 0\\ \big|[-M-\eta]^+(t)\big|_V^2 \leq C_9\big|[-M-\eta_0]^+\big|_V^2 = 0, \end{cases} \text{for any } t\in[0,T], \tag{5.29}$$

respectively. This implies that

$$|\eta(t)|_{L^\infty(\Omega)} \leq M \quad \text{for any } t\in[0,T].$$

Thus, we complete the verifications of (S0)–(S3) and conclude that $[\eta,\theta]$ is a solution to (S).

6. Proof of Main Theorem 2.

Main Theorem 2 will be obtained by means of a Gronwall-type inequality.

Let $[\eta^k,\theta^k]$, $k=1,2$, be the solutions to (S) corresponding to initial values η_0^k, θ_0^k and forcings u^k, v^k, $k=1,2$. Let us set

$$M_0 := |\eta^1|_{L^\infty(Q)} \vee |\eta^2|_{L^\infty(Q)} \text{ and } \delta_*(\nu) := 1\wedge\delta_\alpha\wedge\nu^2 \tag{6.1}$$

and take the difference between the variational formulas for η^k, $k=1,2$, and put $\varphi := (\eta^1-\eta^2)(t)$. Then, by using (A1), the monotonicity of α', and Young's inequality, we see that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(|(\eta^1-\eta^2)(t)|_H^2+\mu^2|\nabla(\eta^1-\eta^2)(t)|_{[H]^N}^2\big)\\
&\qquad -|g|_{L^\infty(-M_0,M_0)}|(\eta^1-\eta^2)(t)|_H^2\\
&\leq \int_\Omega \alpha'(\eta^1(t))(\eta^2-\eta^1)(t)|\nabla\theta^1(t)|\,dx\\
&\qquad+\int_\Omega \alpha'(\eta^2(t))(\eta^1-\eta^2)(t)|\nabla\theta^2(t)|\,dx\\
&\qquad+\big((\eta^1-\eta^2)(t),(u^1-u^2)(t)\big)_H\\
&\leq \int_\Omega |\alpha(\eta^1(t))-\alpha(\eta^2(t))||\nabla(\theta^1-\theta^2)(t)|\,dx\\
&\qquad+|(\eta^1-\eta^2)(t)|_H|(u^1-u^2)(t)|_H\\
&\leq \frac{|\alpha'|_{L^\infty(-M_0,M_0)}}{2}\Big(|(\eta^1-\eta^2)(t)|_H^2+|\nabla(\theta^1-\theta^2)(t)|_{[H]^N}^2\Big)\\
&\qquad+\frac{1}{2}|(\eta^1-\eta^2)(t)|_H^2+\frac{1}{2}|(u^1-u^2)(t)|_H^2 \quad \text{for a.e. } t>0.
\end{aligned}
\tag{6.2}
$$

On the other hand, by putting $\psi=\theta^2$ in the variational inequality for θ^1, and $\psi=\theta^1$ in the one for θ^2, and by taking the sum of two inequalities, we have

$$
\begin{aligned}
&\big(\alpha_0(\eta^1)\partial_t\theta^1(t)-\alpha_0(\eta^2)\partial_t\theta^2(t),(\theta^1-\theta^2)(t)\big)_H\\
&\qquad+\frac{1}{2}\frac{d}{dt}\big(\nu^2|\nabla(\theta^1-\theta^2)(t)|_{[H]^N}^2\big)\\
&\leq \int_\Omega \alpha(\eta^1(t))\big(|\nabla\theta^2(t)|-|\nabla\theta^1(t)|\big)\,dx\\
&\qquad+\int_\Omega \alpha(\eta^2(t))\big(|\nabla\theta^1(t)|-|\nabla\theta^2(t)|\big)\,dx\\
&\qquad+\big((\theta^1-\theta^2)(t),(v^1-v^2)(t)\big)_H\\
&\leq \int_\Omega |\alpha(\eta^1(t))-\alpha(\eta^2(t))||\nabla(\theta^1-\theta^2)(t)|\,dx\\
&\qquad+|(\theta^1-\theta^2)(t)|_H|(v^1-v^2)(t)|_H\\
&\leq \frac{|\alpha'|_{L^\infty(-M_0,M_0)}}{2}\Big(|(\eta^1-\eta^2)(t)|_H^2+|\nabla(\theta^1-\theta^2)(t)|_{[H]^N}^2\Big)\\
&\qquad+\frac{1}{2}|(\theta^1-\theta^2)(t)|_H^2+\frac{1}{2}|(v^1-v^2)(t)|_H^2 \quad \text{a.e. } t>0.
\end{aligned}
\tag{6.3}
$$

Here, we can compute the first term in (6.3) as follows:

$$
\begin{aligned}
&\big(\alpha_0(\eta^1(t))\partial_t\theta^1(t)-\alpha_0(\eta^2(t))\partial_t\theta^2(t),(\theta^1-\theta^2)(t)\big)_H\\
&\quad=\frac{1}{2}\frac{d}{dt}\Big|\sqrt{\alpha_0(\eta^1(t))}(\theta^1-\theta^2)(t)\Big|_H^2-\frac{1}{2}\int_\Omega \alpha_0'(\eta^1(t))\partial_t\eta^1(t)|(\theta^1-\theta^2)(t)|^2\,dx\\
&\qquad+\int_\Omega (\alpha_0(\eta^1(t))-\alpha_0(\eta^2(t)))\partial_t\theta^2(t)(\theta^1-\theta^2)(t)\,dx.
\end{aligned}
\tag{6.4}
$$

Also, by using (6.1), the continuous embedding from $H^1(\Omega)$ to $L^4(\Omega)$ under $N\leq 3$, and Young's inequality, one can see that

$$
\begin{aligned}
(6.5) \qquad & -\int_\Omega \alpha_0'(\eta^1(t))\partial_t\eta^1(t)|(\theta^1-\theta^2)(t)|^2\,dx \\
& \geq -|\alpha_0'|_{L^\infty(-M_0,M_0)}|\partial_t\eta^1(t)|_H|(\theta_1-\theta^2)(t)|^2_{L^4(\Omega)} \\
& \geq -(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}|\partial_t\eta^1(t)|_H|(\theta_1-\theta^2)(t)|^2_V \\
& \geq -\frac{(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}}{\delta_*(\nu)}|\partial_t\eta^1(t)|_H \\
& \quad \cdot \left(|\sqrt{\alpha_0(\eta^1(t))}(\theta^1-\theta^2)(t)|^2_H+\nu^2|\nabla(\theta^1-\theta^2)(t)|^2_{[H]^N}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.6) \qquad & \int_\Omega(\alpha_0(\eta^1(t))-\alpha_0(\eta^2(t)))\partial_t\theta^2(t)(\theta^1-\theta^2)(t)\,dx \\
& \geq -|\alpha_0'|_{L^\infty(-M_0,M_0)}|\partial_t\theta^2(t)|_{L^4(\Omega)}|(\eta^1-\eta^2)(t)|_H|(\theta^1-\theta^2)(t)|_{L^4(\Omega)} \\
& \geq -(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}|\partial_t\theta^2(t)|_V|(\eta^1-\eta^2)(t)|_H|(\theta^1-\theta^2)(t)|_V \\
& \geq -\frac{(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}}{2}|\partial_t\theta^2(t)|_V\left(|(\eta^1-\eta^2)(t)|^2_H+|(\theta^1-\theta^2)(t)|^2_V\right) \\
& \geq -\frac{(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}}{2\delta_*(\nu)}|\partial_t\theta^2(t)|_V \\
& \quad \cdot\left(|(\eta^1-\eta^2)(t)|^2_H+|\sqrt{\alpha_0(\eta^1(t))}(\theta^1-\theta^2)(t)|^2_H+\nu^2|\nabla(\theta^1-\theta^2)(t)|^2_{[H]^N}\right) \\
& \qquad \text{for a.e. } t>0,
\end{aligned}
$$

where $C_{H^1}^{L^4}$ is a constant of the continuous embedding from $H^1(\Omega)$ to $L^4(\Omega)$.

Therefore, putting

$$J_1(t):=|(u^1-u^2)(t)|^2_H+|(v^1-v^2)(t)|^2_H \quad \text{for } t\geq 0$$

and

$$\widetilde{C}_{10}:=2\left(|\alpha'|_{L^\infty(-M_0,M_0)}+|g'|_{L^\infty(-M_0,M_0)}+(C_{H^1}^{L^4})^2|\alpha_0'|_{L^\infty(-M_0,M_0)}+1\right),$$

it is deduced from (6.2)–(6.6) that

$$\frac{d}{dt}J(t)\leq\frac{\widetilde{C}_{10}}{\delta_*(\nu)}\left(|\partial_t\eta^1(t)|_H+|\partial_t\theta^2(t)|_V+1\right)J(t)+J_1(t) \quad \text{a.e. } t>0, \tag{6.7}$$

Applying Gronwall's lemma in (6.7), it can be obtained that for any $T>0$,

$$J(t)\leq C_1(\nu)\left(J(0)+\int_0^T J_1(s)\,ds\right) \quad \text{for any } t\in[0,T],$$

with

$$C_1(\nu):=\exp\left(\frac{\widetilde{C}_{10}\sqrt{T}}{\delta_*(\nu)}\left(|\partial_t\eta^1|_{\mathscr{H}}+|\partial_t\theta^2|_{\mathscr{V}}+\sqrt{T}\right)\right).$$

Thus, we finish the proof of Main Theorem 2.

REFERENCES

[1] T. Aiki, D. Mizuno, and K. Shirakawa, *A class of initial-boundary value problems governed by pseudo-parabolic weighted total variation flows*, Adv. Math. Sci. Appl., 32 (2023), pp. 311–341, https://mcm-www.jwu.ac.jp/∼aikit/AMSA/pdf/abstract/2023/Top_2023_015.pdf.

[2] M. Amar and G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11 (1994), pp. 91–133, https://doi.org/10.1016/S0294-1449(16)30197-4.

[3] M. Amar, V. De Cicco, and N. Fusco, *A relaxation result in BV for integral functionals with discontinuous integrands*, ESAIM Control Optim. Calc. Var., 13 (2007), pp. 396–412, https://doi.org/10.1051/cocv:2007015.

[4] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, New York, 2000.

[5] H. Antil, S. Kubota, K. Shirakawa, and N. Yamazaki, *Optimal control problems governed by 1-D Kobayashi-Warren-Carter type systems*, Math. Control Relat. Fields, 11 (2021), pp. 253–289, https://doi.org/10.3934/mcrf.2020036.

[6] H. Antil, S. Kubota, K. Shirakawa, and N. Yamazaki, *Constrained optimization problems governed by PDE models of grain boundary motions*, Adv. Nonlinear Anal., 11 (2022), pp. 1249–1286, https://doi.org/10.1515/anona-2022-0242.

[7] H. Antil, K. Shirakawa, and N. Yamazaki, *A class of parabolic systems associated with optimal controls of grain boundary motions*, Adv. Math. Sci. Appl., 27 (2018), pp. 299–336.

[8] H. Attouch, G. Buttazzo, and G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, 2nd ed., MOS-SIAM Ser. Optim. 17, SIAM, Philadelphia, Mathematical Optimization Society, Philadelphia, 2014, https://doi.org/10.1137/1.9781611973488.

[9] L. C. Berselli and L. Bisconti, *On the structural stability of the Euler-Voigt and Navier-Stokes-Voigt models*, Nonlinear Anal., 75 (2012), pp. 117–130, https://doi.org/10.1016/j.na.2011.08.011.

[10] H. Brézis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Math. Stud. 5, North-Holland, Amsterdam, London; Elsevier, New York, 1973.

[11] Y. Cao, E. M. Lunasin, and E. S. Titi, *Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models*, Commun. Math. Sci., 4 (2006), pp. 823–848, http://projecteuclid.org/euclid.cms/1175797613.

[12] E. Emmrich, *Discrete Versions of Gronwall's Lemma and their Application to the Numerical Analysis of Parabolic Problems*, Technical report 637, Institute of Mathematics, Technische Universität, Berlin, http://www3.math.tu-berlin.de/preprints/files/Preprint-637-1999.pdf, 1999.

[13] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, revised ed., Textb. Math., CRC Press, Boca Raton, FL, 2015.

[14] T. Fukao, *On a variational inequality of Bingham and Navier-Stokes type in three dimensional space*, in Nonlinear Analysis in Interdisciplinary Sciences—Modellings, Theory and Simulations, GAKUTO Internat. Ser. Math. Sci. Appl. 36, Gakkotosho, Tokyo, 2013, pp. 57–71.

[15] T. Fukao and T. Kashiwabara, *H^2-Regularity for Stationary and Non-stationary Bingham Problems with Perfect Slip Boundary Condition*, preprint, https://arxiv.org/abs/2404.18333, 2024.

[16] Y. Giga, A. Kubo, H. Kuroda, J. Okamoto, K. Sakakibara, and M. Uesaka, *Fractional Time Differential Equations as a Singular Limit of the Kobayashi–Warren–Carter System*, preprint, https://arxiv.org/abs/2306.15235, 2023.

[17] Y. Giga, J. Okamoto, K. Sakakibara, and M. Uesaka, *On a Singular Limit of the Kobayashi–Warren–Carter Energy*, preprint, https://arxiv.org/abs/2205.14314, 2022.

[18] Y. Giga, J. Okamoto, and M. Uesaka, *A finer singular limit of a single-well Modica-Mortola functional and its applications to the Kobayashi-Warren-Carter energy*, Adv. Calc. Var., 16 (2023), pp. 163–182, https://doi.org/10.1515/acv-2020-0113.

[19] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monogr. Math. 80, Birkhäuser Verlag, Basel, 1984, https://doi.org/10.1007/978-1-4684-9486-0.

[20] D. D. Holm, J. E. Marsden, and T. S. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math., 137 (1998), pp. 1–81, https://doi.org/10.1006/aima.1998.1721.

[21] D. D. Holm, J. E. Marsden, and T. S. Ratiu, *Euler-Poincaré models of ideal fluids with nonlinear dispersion*, Phys. Rev. Lett., 80 (1998), pp. 4173–4176, https://doi.org/10.1103/PhysRevLett.80.4173.

[22] A. Ito, N. Kenmochi, and N. Yamazaki, *A phase-field model of grain boundary motion*, Appl. Math., 53 (2008), pp. 433–454, https://doi.org/10.1007/s10492-008-0035-8.

[23] R. Kobayashi, J. A. Warren, and W. C. Carter, *A continuum model of grain boundaries*, Phys. D, 140 (2000), pp. 141–150, https://doi.org/10.1016/S0167-2789(00)00023-3.

[24] R. Kobayashi, J. A. Warren, and W. C. Carter, *Grain boundary model and singular diffusivity*, in Free Boundary Problems: Theory and Applications, II (Chiba, 1999), GAKUTO Internat. Ser. Math. Sci. Appl. 14, Gakkōtosho, Tokyo, 2000, pp. 283–294.

[25] S. Kubota, R. Nakayashiki, and K. Shirakawa, *Optimal control problems for 1D parabolic state-systems of KWC types with dynamic boundary conditions*, Adv. Math. Sci. Appl., 29 (2020), pp. 583–637.

[26] S. Kubota and K. Shirakawa, *Periodic solutions to Kobayashi–Warren–Carter systems*, Adv. Math. Sci. Appl., 32 (2023), pp. 511–553, https://mcm-www.jwu.ac.jp/~aikit/AMSA/pdf/abstract/2023/Top_2023_022.pdf.

[27] A. Larios, Y. Pei, and L. Rebholz, *Global well-posedness of the velocity-vorticity-Voigt model of the 3D Navier-Stokes equations*, J. Differential Equations, 266 (2019), pp. 2435–2465, https://doi.org/10.1016/j.jde.2018.08.033.

[28] A. Larios and E. S. Titi, *On the higher-order global regularity of the inviscid Voigt-regularization of three-dimensional hydrodynamic models*, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), pp. 603–627, https://doi.org/10.3934/dcdsb.2010.14.603.

[29] S. Moll and K. Shirakawa, *Existence of solutions to the Kobayashi–Warren–Carter system*, Calc. Var. Partial Differential Equations, 51 (2014), pp. 621–656, https://doi.org/10.1007/s00526-013-0689-2.

[30] S. Moll, K. Shirakawa, and H. Watanabe, *Energy dissipative solutions to the Kobayashi–Warren–Carter system*, Nonlinearity, 30 (2017), pp. 2752–2784, https://doi.org/10.1088/1361-6544/aa6eb4.

[31] S. Moll, K. Shirakawa, and H. Watanabe, *Kobayashi-Warren-Carter type systems with nonhomogeneous Dirichlet boundary data for crystalline orientation*, Nonlinear Anal., 217 (2022), 112722, https://doi.org/10.1016/j.na.2021.112722.

[32] R. Nakayashiki and K. Shirakawa, *Kobayashi-Warren-Carter system of singular type under dynamic boundary condition*, Discrete Contin. Dyn. Syst. Ser. S, 16 (2023), pp. 3746–3783, https://doi.org/10.3934/dcdss.2023162.

[33] A. P. Oskolkov, *The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 38 (1973), pp. 98–136.

[34] A. P. Oskolkov, *On the theory of unsteady flows of Kelvin-Voigt fluids*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 115 (1982), pp. 191–202, 310.

[35] K. Shirakawa, H. Watanabe, and N. Yamazaki, *Solvability of one-dimensional phase field systems associated with grain boundary motion*, Math. Ann., 356 (2013), pp. 301–330, https://doi.org/10.1007/s00208-012-0849-2.

[36] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96, https://doi.org/10.1007/BF01762360.

[37] W. Voigt, *Ueber innere reibung fester körper, insbesondere der metalle*, Annalen der Physik, 283 (1892), pp. 671–693, https://doi.org/10.1002/andp.18922831210.

[38] N. Yamazaki, *Global attractors for non-autonomous phase-field systems of grain boundary motion with constraint*, Adv. Math. Sci. Appl., 23 (2013), pp. 267–296.