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EXTERIOR NONLOCAL VARIATIONAL INEQUALITIES ASSOCIATED WITH THE FRACTIONAL LAPLACE OPERATOR

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Abstract. This paper introduces a new class of variational inequalities where the obstacle is placed in the exterior domain that is disjoint from the observation domain. This is carried out with the help of nonlocal fractional operators. The need for such novel variational inequalities stems from the fact that the classical approach only allows placing the obstacle either inside the observation domain or on the boundary. An analysis of the continuous problem is provided. Additionally, penalization arguments to approximate the problem are discussed.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with a Lipschitz continuous boundary $\partial\Omega$. Moreover, let Σ_1, Σ_2 be nonempty open subsets of $\mathbb{R}^N \setminus \overline{\Omega}$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\overline{\Sigma}_1 \cup \overline{\Sigma}_2 = \mathbb{R}^N \setminus \Omega$. In this paper we introduce and study the following variational problem: Given $f \in W^{-s,2}(\mathbb{R}^N) \subset W^{-s,2}(\Omega, \Sigma_1)$, $z \in W_0^{s,2}(\Sigma_1)$, an obstacle $\varphi \in W^{s,2}(\Sigma_2)$, we want to solve the following minimization problem (in the sense of Definition 3.2 below):

$$\min_{u \in \mathcal{K}} J(u) \tag{1.1}$$

with the functional J given by

$$J(u) := \frac{C_{N,s}}{4} \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy - \langle f, u\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}, \tag{1.2}$$

where $0 < s < 1$ is a real number, $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$, and the set of constraints is given by

$$\mathcal{K} := \{u \in W^{s,2}(\mathbb{R}^N) : u = z \text{ in } \Sigma_1,\ u \leq \varphi \text{ in } \Sigma_2\}.$$

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Notice that $\mathcal{K}$ is a closed and convex subset of $W^{s,2}(\mathbb{R}^N)$. In Section 3 we shall give the assumptions on the sets Σ_1 and Σ_2 that show that the set $\mathcal{K}$ is in addition nonempty. We shall see that $W^{s,2}(\Omega,\Sigma_1)$ is a closed subspace of $W^{s,2}(\mathbb{R}^N)$ so that $W^{-s,2}(\mathbb{R}^N)$ (the dual of $W^{s,2}(\mathbb{R}^N)$) can be identified with a subspace of $W^{-s,2}(\Omega,\Sigma_1)$ (the dual of $W^{s,2}(\Omega,\Sigma_1)$). The precise definition of the Sobolev spaces involved will be given in Section 2.

Obstacle and equilibrium problems, in general, have a rich history. They can capture many applications from phase changes to friction. Though in all cases either the obstacle is placed in the interior of Ω or on the boundary $\partial\Omega$. We refer to the monographs [4, 19, 20] for more information and examples.

The main novelty of this paper is the introduction of the model (1.1) which due to the presence of the nonlocal (fractional) Laplacian, enables placement of the obstacle φ in the exterior of Ω and possibly disjoint from the boundary $\partial\Omega$. After establishing existence of solutions (using standard arguments) to (1.1), our first main result is given in Proposition 3.4 which shows the equivalence between the variational problem (1.1) and three other characterizations:

(i) variational inequality;
(ii) slack variable (Lagrange multiplier) formulation, and;
(iii) weak formulation in the distributional sense.

Notice that similar results in the classical setting are well-known. Due to the nonlocal nature of the fractional Laplacian, the existing results do not directly extend to the fractional setting. Indeed, for instance, one has to carefully account for the nonlocal normal derivative. Tools from convex analysis such as tangent cone, convex indicator function are also employed to establish these results. Some of these arguments may appear to be standard, but the details are delicate due to the problem being nonlocal.

Our second main result corresponds to penalization of the constraints in the set $\mathcal{K}$, firstly in L^2-sense $\epsilon^{-2}\|(u-\varphi)^+\|^2_{L^2(\Sigma_2)}$ (cf. (4.1)) and secondly in the Sobolev sense $\xi^{-1}\|(u-\varphi)^+\|^2_{W^{s,2}(\Sigma_2)}$ (cf. (4.13)). Here $v^+=\max\{v,0\}$. The former penalization is associated to the so-called Moreau-Yosida regularization. The latter has the distinct advantage of being able to provide a direct relationship between the slack variable for the original problem as formulated in the above item (ii) and its penalized version. After establishing convergence using Mosco convergence arguments in Proposition 4.2, we establish a convergence rate in ϵ in Theorem 4.4. We show that the penalized solution converges linearly in ϵ in $W^{s,2}(\Omega,\Sigma_1)$-norm. Moreover the constraint violation converges quadratically in ϵ. Theorem 4.5 provides linear convergence in ξ for both the solution u and the slack variable.

Going forward, several of the techniques developed here will be helpful for the local problems and also in deriving finite element approximations. Notice that a popular way to numerically tackle these problems is to solve the penalized problems. Then the final approximation errors are governed by ϵ (or ξ) and the discretization errors. This will be part of a future investigation.

For completeness, we also mention that fractional obstacle problems where the obstacle is in the interior have also received a significant attention recently. See for instance [3, 8, 21] for more information and details. However, as pointed out above, this is the first work that proposes to tackle the exterior obstacle problem.

The paper is organized as follows. In Section 2 we first introduce some notations and state some preliminary results. Our main work starts in Section 3, where we establish several equivalent formulations to the variational problem (1.1). In Section 4 we consider two penalty approaches. In the first case we consider L^2-penalty and in the second case we consider a penalty in a Sobolev norm. Convergence and the precise rates of convergence with respect to the penalty parameters are established.

2. Notation and preliminaries.

We begin this section by introducing some notations and give some preliminary results as they are needed throughout the paper. Some of the results given in this section are well-known, in particular we follow the notations from our previous works [2, 24].

2.1. Fractional order Sobloev spaces and the fractional Laplacian.

For $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ an arbitrary nonempty open set and $0 < s < 1$, we first define the classical Sobolev-Slobodecki space

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\},$$

and we endow it with the norm given by

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int_\Omega |u|^2\, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}.$$

We also let

$$W_0^{s,2}(\Omega) := \{u \in W^{s,2}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Recall that Σ_1, Σ_2 are nonempty open subsets of $\mathbb{R}^N \setminus \overline{\Omega}$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\overline{\Sigma}_1 \cup \overline{\Sigma}_2 = \mathbb{R}^N \setminus \Omega$. This implies that Σ_1 and Σ_2 have positive Lebesgue measures.

Now, throughout the remainder of the paper we assume that the open set $\Omega \subset \mathbb{R}^N$ is bounded and has a Lipschitz continuous boundary. We define the space

$$W^{s,2}(\Omega, \Sigma_1) := \left\{ u \in W^{s,2}(\mathbb{R}^N) \ : \ u = 0 \text{ in } \Sigma_1 \right\}$$

which is a Hilbert space endowed with the norm induced by $W^{s,2}(\mathbb{R}^N)$. We observe that in the definition of the space $W^{s,2}(\Omega, \Sigma_1)$ the set Ω is a priori not involved. But this is consistent with the literature where this space has been defined. Additionally, in Proposition 2.1 below it will be clear why Ω has been introduced in the definition.

Next, for $u \in W^{s,2}(\Omega, \Sigma_1)$, we let

$$\|u\|_{W^{s,2}(\Omega,\Sigma_1)} = \left(\int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{1/2} \tag{2.1}$$

where we recall that

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

The following result is contained in [1, Proposition 5].

Proposition 2.1. *The norm $\|\cdot\|_{W^{s,2}(\Omega,\Sigma_1)}$ given in (2.1) is equivalent to the one induced by $W^{s,2}(\mathbb{R}^N)$. As a consequence, $(W^{s,2}(\Omega, \Sigma_1), \|\cdot\|_{W^{s,2}(\Omega,\Sigma_1)})$ is a Hilbert space with the scalar product*

$$(u, v)_{W^{s,2}(\Omega,\Sigma_1)} = \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy.$$

The norm given in (2.1) is the motivation for introducing the set Ω in the definition of the space $W^{s,2}(\Omega,\Sigma_1)$. In addition, we observe that the norm does not depend on Σ_1.

We denote the dual spaces of $W^{s,2}(\mathbb{R}^N)$ and $W^{s,2}(\Omega,\Sigma_1)$ by $W^{-s,2}(\mathbb{R}^N)$ and $W^{-s,2}(\Omega,\Sigma_1)$, respectively. Moreover, we will use $\langle\cdot,\cdot\rangle$, to denote their duality pairing whenever it is clear from the context.

Next, we introduce the fractional Laplace operator. For $0<s<1$ we set

$$\mathbb{L}^1_s(\mathbb{R}^N) := \left\{ u:\mathbb{R}^N\to\mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N}\frac{|u(x)|}{(1+|x|)^{N+2s}}\,dx<\infty\right\}.$$

For $u\in\mathbb{L}^1_s(\mathbb{R}^N)$ and $\varepsilon>0$, we let

$$(-\Delta)^s_\varepsilon u(x) := C_{N,s}\int_{\{y\in\mathbb{R}^N,|y-x|>\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,\quad x\in\mathbb{R}^N,$$

where the normalized constant $C_{N,s}$ is given by

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{2.2}$$

and Γ is the usual Euler Gamma function (see, e.g. [5, 9, 7, 8, 10, 22, 23, 24]). Then, the fractional Laplacian $(-\Delta)^s$ is defined for $u\in\mathbb{L}^1_s(\mathbb{R}^N)$ by the formula

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy = \lim_{\varepsilon\downarrow 0}(-\Delta)^s_\varepsilon u(x),\quad x\in\mathbb{R}^N, \tag{2.3}$$

provided that the limit exists for a.e. $x\in\mathbb{R}^N$. We refer to [10] and the references therein for the class of functions for which the limit in (2.3) exists.

It has been shown in [6, Proposition 2.2] that for $u,v\in\mathcal{D}(\Omega)$ (the space of all continuously infinitely differentiable functions with compact support in Ω), we have that

$$\lim_{s\uparrow 1}\int_{\mathbb{R}^N} v(-\Delta)^s u\,dx = -\int_{\mathbb{R}^N} v\Delta u\,dx = -\int_\Omega v\Delta u\,dx = \int_\Omega \nabla u\cdot\nabla v\,dx.$$

This is where the constant $C_{N,s}$ given in (2.2) plays a crucial role.

Next, we introduce the realization in $L^2(\Omega)$ of the operator $(-\Delta)^s$ with the mixed zero Dirichlet exterior condition in Σ_1 and zero nonlocal Neumann exterior condition in Σ_2. For this, consider the continuous, closed and coercive bilinear form $\mathcal{E}: W^{s,2}(\Omega,\Sigma_1)\times W^{s,2}(\Omega,\Sigma_1)\to\mathbb{R}$ given for every $u,v\in W^{s,2}(\Omega,\Sigma_1)$ by

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy. \tag{2.4}$$

Let $(-\Delta)^s_{\Sigma_1}$ be the self-adjoint operator in $L^2(\Omega)$ associated with $\mathcal{E}$ in the following sense:

$$\begin{cases} D((-\Delta)^s_{\Sigma_1}) := \big\{u\in W^{s,2}(\Omega,\Sigma_1),\ \exists f\in L^2(\Omega):\\ \qquad\qquad \mathcal{E}(u,v) = (f,v)_{L^2(\Omega)}\ \forall v\in W^{s,2}(\Omega,\Sigma_1)\big\},\\ (-\Delta)^s_{\Sigma_1}u = f \ \text{ in } \ \Omega. \end{cases} \tag{2.5}$$

Remark 2.2. Since Ω is assumed to have a Lipschitz continuous boundary, assuming that Σ_1 has a continuous boundary, and as functions in $W^{s,2}(\Omega,\Sigma_1)$ are zeros on Σ_1, and $C_c^\infty(\mathbb{R}^N\setminus\Sigma_1)\subset D((-\Delta)^s_{\Sigma_1})$, one can use known results on density of continuous functions in fractional order Sobolev spaces (see e.g. [15, Chapter 1] and [13]) to show that $D((-\Delta)^s_{\Sigma_1})$ is dense in $L^2(\Omega)$ and in $W^{s,2}(\Omega,\Sigma_1)$.

In the forthcoming discussion, we also make use of the local fractional order Sobolev space

$$W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega) := \left\{u \in L^2_{\text{loc}}(\mathbb{R}^N \setminus \Omega) \ : \ u\varphi \in W^{s,2}(\mathbb{R}^N \setminus \Omega), \ \forall\, \varphi \in \mathcal{D}(\mathbb{R}^N \setminus \Omega)\right\}. \tag{2.6}$$

Furthermore, for $u \in W^{s,2}(\mathbb{R}^N)$, using the terminology from [11], we define the nonlocal normal derivative (or interaction operator) $\mathcal{N}_s$ as follows:

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{2.7}$$

Clearly $\mathcal{N}_s$ is a nonlocal operator and it is well defined on $W^{s,2}(\mathbb{R}^N)$ as we discuss next.

Lemma 2.3. *By* [14, Lemma 3.2]*, the interaction operator $\mathcal{N}_s$ maps continuously $W^{s,2}(\mathbb{R}^N)$ into $W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$. As a consequence we have that if $u \in W^{s,2}(\mathbb{R}^N)$, then $\mathcal{N}_s u \in L^2_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$.*

Despite the fact that $\mathcal{N}_s$ is defined on $\mathbb{R}^N \setminus \overline{\Omega}$, it is still known as the "normal" derivative. This is due to its similarity with the classical normal derivative, that is, it plays the same role for $(-\Delta)^s$ that the normal derivative does for the negative Laplace operator $-\Delta$ (see e.g. [2, Proposition 2.2]). Following the terminology from [2], we shall call $\mathcal{N}_s$ the *interaction operator* since it allows interaction between Ω and the exterior domain $\mathbb{R}^N \setminus \overline{\Omega}$.

We conclude this subsection by stating the integration by parts formulas for the fractional Laplacian, see [11, Lemma 3.3] for smooth functions and [2, Proposition 2.2] for functions in Sobolev spaces (by using some density arguments).

Proposition 2.4 (**The integration by parts formula I**)**.** *Let $u \in W^{s,2}(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$. Then, for every $v \in W^{s,2}(\mathbb{R}^N)$ we have that*

$$\begin{aligned}&\frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\, dxdy \\ &= \int_\Omega v(-\Delta)^s u\, dx + \int_{\mathbb{R}^N \setminus \Omega} v\mathcal{N}_s u\, dx.\end{aligned} \tag{2.8}$$

We observe the following.

Remark 2.5. Let $(-\Delta)^s_{\Sigma_1}$ be the operator defined in (2.5). Using the integration by parts formula (2.8) we can deduce that

$$\begin{cases} D((-\Delta)^s_{\Sigma_1}) = \left\{u \in W^{s,2}(\Omega, \Sigma_1) : \ \mathcal{N}_s u = 0 \ \text{ in } \Sigma_2, \ (-\Delta)^s|_\Omega \in L^2(\Omega)\right\}, \\ (-\Delta)^s_{\Sigma_1} u = (-\Delta)^s u \ \text{ in } \ \Omega. \end{cases} \tag{2.9}$$

The version of the integration-by-parts formula we will frequently use in this paper is the following (see e.g. [1, Proposition 6]).

Proposition 2.6 (**The integration by parts formula II**)**.** *Let $u, v \in W^{s,2}(\Omega, \Sigma_1)$. Then,*

$$\begin{aligned}&\frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\, dxdy \\ &= \langle (-\Delta)^s u, v\rangle_{W^{-s,2}(\Omega,\Sigma_1), W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2} v\mathcal{N}_s u\, dx.\end{aligned} \tag{2.10}$$

Proof. We notice that (2.10) has been stated differently in [1, Proposition 6] without providing a proof. They have replaced the duality map $\langle\cdot,\cdot\rangle$ with the scalar product $(\cdot,\cdot)_{L^2(\Omega)}$. We think the right formulation is as given in (2.10). For that reason we include the proof. We proceed in two steps.

Step 1: Firstly, we observe that $(-\Delta)^s_{\Sigma_1}$ defined in (2.5) can be viewed as a bounded operator from $W^{s,2}(\Omega,\Sigma_1)$ into its dual $W^{-s,2}(\Omega,\Sigma_1)$ given for $u,v\in W^{s,2}(\Omega,\Sigma_1)$ by

$$\begin{aligned}&\langle(-\Delta)^s_{\Sigma_1}u,v\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)}\\&=\frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy.\end{aligned}$$

Step 2: Let $u_n\in D((-\Delta)^s_{\Sigma_1})$ be a sequence that converges to u in $W^{s,2}(\Omega,\Sigma_1)$, as $n\to\infty$. Existence of u_n follows from Remark 2.2. It follows from Proposition 2.4 that for every $n\in\mathbb{N}$ and $v\in W^{s,2}(\Omega,\Sigma_1)$, we have that

$$\begin{aligned}&\frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy\\&=\int_\Omega v(-\Delta)^s u_n\,dx+\int_{\mathbb{R}^N\setminus\Omega}v\mathcal{N}_s u_n\,dx\\&=\langle(-\Delta)^s u_n,v\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)}+\int_{\Sigma_2}v\mathcal{N}_s u_n\,dx,\end{aligned}\tag{2.11}$$

where we have also used that $v=0$ in Σ_1. Since u_n converges to u in $W^{s,2}(\Omega,\Sigma_1)$, as $n\to\infty$, it follows from Step 1 that $(-\Delta)^s u_n$ converges to $(-\Delta)^s u$ in $W^{-s,2}(\Omega,\Sigma_1)$, as $n\to\infty$. It also follows from Lemma 2.3 (the continuity of the operator $\mathcal{N}_s$) that $\mathcal{N}_s u_n$ converges to $\mathcal{N}_s u$ in $L^2(\Sigma_2)$, as $n\to\infty$. Finally, using all the above convergences and taking the limit of both sides of (2.11), as $n\to\infty$, we get (2.10) and the proof is finished. □

2.2. **Useful results from convex analysis.** We will additionally require the following fundamental concepts from Convex Analysis. Consider a general problem of the form

$$\min_{w\in W} f(w)\quad\text{subject to } G(w)\in\mathcal{K}_G,\ w\in\mathcal{C}\tag{2.12}$$

where W and V are Banach spaces and $f:W\to\mathbb{R}$, $G:W\to V$ are continuously Fréchet differentiable. Further, suppose that $\mathcal{C}\subset W$ is a nonempty, closed set and convex and $\mathcal{K}_G\subset V$ is a closed, convex cone.

The feasible set is defined by

$$F:=\{w\in W: G(w)\in\mathcal{K}_G, w\in\mathcal{C}\}.\tag{2.13}$$

Then, when $F\subset W$ is nonempty, we define the tangent cone of F at $w\in F$ by

$$\begin{aligned}T(F;w):=\{\tau\in W:\ &\text{for each } k\in\mathbb{N},\exists r_k>0\ w_k\in F:\lim_{k\to\infty}w_k=w,\\&\lim_{k\to\infty}r_k(w_k-w)=\tau\}\end{aligned}\tag{2.14}$$

and the linearization cone at a point $w\in F$ by

$$L(F;w):=\{rh: r>0,\ h\in W,\ G(w)+G'(w)h\in\mathcal{K}_G,\ w+h\in\mathcal{C}\}.\tag{2.15}$$

For an optimal solution $\overline{w}$ of (2.12), it can be shown that the existence of Lagrange multipliers and construction of first order optimality conditions is dependent upon the linearization cone at $\overline{w} \in F$ to be contained in the tangent cone of F at $\overline{w} \in F$. That is,

$$L(F;\overline{w}) \subset T(F;\overline{w}).$$

This is sometimes referred to as the Abadie Constraint Qualification or just Constraint Qualification. A more detailed discussion of constraint qualifications has been addressed in [16, 17, 25].

When formulating the Lagrangian in Section 3 it is necessary to introduce a few additional definitions. We refer to [12] for more details. As before, suppose that W is a real Banach space and let W^* denote its topological dual with duality pairing $\langle \cdot,\cdot\rangle_{W^*,W}$. Given $f : W \to \mathbb{R}\cup\{+\infty\}$, its Fenchel conjugate is defined by $f^* : W^* \to \mathbb{R}\cup\{+\infty\}$, where

$$f^*(\lambda) := \sup_{w\in W}\{\langle \lambda, w\rangle_{W^*,W} - f(w)\}. \tag{2.16}$$

Additionally, for a convex set $\mathcal{W} \subset W$ we define the indicator functional of $\mathcal{W}$ by $I_{\mathcal{W}} : W \to \mathbb{R}\cup\{+\infty\}$ given by

$$I_{\mathcal{W}}(u) = \begin{cases} 0 & \text{if } u \in \mathcal{W} \\ +\infty & \text{if } u \notin \mathcal{W}. \end{cases} \tag{2.17}$$

In light of (2.16) and (2.17) we have the following result.

Lemma 2.7. *Consider the sets $\mathcal{W}^- := \{w \in W : w \le 0\}$ and $\mathcal{W}^+ := \{\eta \in W^* : \eta \ge 0\}$. If W is reflexive, then $I^*_{\mathcal{W}^-}(\lambda) = I_{\mathcal{W}^+}(\lambda)$ and $I^*_{\mathcal{W}^+}(u) = I_{\mathcal{W}^-}(u)$.*

Proof. Notice that the Fenchel conjugate of $I_{\mathcal{W}^-}$ is given by $I^*_{\mathcal{W}^-} : W^* \to \mathbb{R}\cup\{+\infty\}$ where,

$$I^*_{\mathcal{W}^-}(\lambda) = \sup_{w\in W}\{\langle \lambda, w\rangle_{W^*,W} - I_{\mathcal{W}^-}(w)\} = \sup_{v\in\mathcal{W}^-}\langle \lambda, v\rangle_{W^*,W} = I_{\mathcal{W}^+}(\lambda)\,.$$

The second equality in the lemma follows in a similar fashion. The proof is finished. □

3. **Well-posedness of the variational inequality.** Throughout the rest of the paper Ω, Σ_1, Σ_2 are as in the previous sections. We also assume the following regularity on Σ_1 and Σ_2.

Assumption 3.1. *We assume that Σ_1 has a continuous boundary and Σ_2 has the extension property in the sense that for every $\varphi \in W^{s,2}(\Sigma_2)$, there exists a function $\Phi \in W^{s,2}(\mathbb{R}^N)$ such that $\Phi|_{\Sigma_2} = \varphi$.*

The assumption that Σ_1 has a continuous boundary implies that $\overline{\mathcal{D}(\Sigma_1)}^{W^{s,2}(\Sigma_1)} = W^{s,2}_0(\Sigma_1)$ (see e.g. [13]) and the assumption that Σ_2 has the extension property shows that there is a constant $C > 0$ such that for every $\varphi \in W^{s,2}(\Sigma_2)$ we have the following estimate:

$$\|\Phi\|_{W^{s,2}(\mathbb{R}^N)} \le C\|\varphi\|_{W^{s,2}(\Sigma_2)},$$

where Φ is the extension of φ.

Next, we introduce the notion of solutions to the minimization problem (1.1). Before doing that we recall that given $z \in W^{s,2}_0(\Sigma_1)$ and $\varphi \in W^{s,,2}(\Sigma_2)$ we have let

$$\mathcal{K} := \{u \in W^{s,2}(\mathbb{R}^N) \,:\, u = z \text{ in } \Sigma_1,\ u \le \varphi \text{ in } \Sigma_2\}.$$

We also set

$$\mathcal{K}_0 := \{w \in W^{s,2}(\mathbb{R}^N) \,:\, w = 0 \text{ in } \Sigma_1,\ w \le \varphi \text{ in } \Sigma_2\}.$$

Now, assume in addition that Ω, Σ_1 and Σ_2 also have Lipschitz continuous boundaries so that Assumption 3.1 holds. Let $z \in W_0^{s,2}(\Sigma_1)$ and $\varphi \in W^{s,,2}(\Sigma_2)$. Define the function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$u := \begin{cases} z & \text{in } \Sigma_1 \\ \Phi & \text{in } \mathbb{R}^N \setminus \Sigma_1, \end{cases}$$

where $\Phi \in W^{s,2}(\mathbb{R}^N)$ is the extension of φ in the sense of Assumption 3.1. A simple computation shows that $u \in W^{s,2}(\mathbb{R}^N)$ and this function can be used to show that both sets $\mathcal{K}$ and $\mathcal{K}_0$ are nonempty as we explain below.

Firstly, we remark that in the classical setting with $H_0^1(\Omega)$-space and $u \leq \varphi$ a.e. in Ω, we need a compatibility condition on the obstacle φ. More precisely, we need $\varphi \geq 0$ on $\partial\Omega$. Indeed, if $\varphi < 0$ on $\partial\Omega$, then $u \notin H_0^1(\Omega)$. Secondly, one possible way to translate this condition to the current setting is that we need $\varphi \geq 0$ on $\partial\Sigma_1 \cap \partial\Sigma_2$. Otherwise, $u \neq 0$ on $\partial\Sigma_1 \cap \partial\Sigma_2$, which is especially required for $\frac{1}{2} < s < 1$ because $z = 0$ on $\partial\Sigma_1 \cap \partial\Sigma_2$. All the boundary values are understood in the trace sense. Under these assumptions, we have that both sets $\mathcal{K}$ and $\mathcal{K}_0$ are nonempty. The assumption that $\frac{1}{2} < s < 1$ is that, if $0 < s \leq \frac{1}{2}$ and $\mathcal{O} \subset \mathbb{R}^N$ is an open set with a Lipschitz continuous boundary, then $W_0^{s,2}(\mathcal{O}) = W^{s,2}(\mathcal{O})$ with equivalent norms, so that functions in $W^{s,2}(\mathcal{O})$ do not have well defined traces on $\partial\mathcal{O}$. (see e.g. [15, Chapter 1] for more details and [23] for more general assumptions on $\mathcal{O}$).

Next, we give our notion of solutions to (1.1).

Definition 3.2. Given $f \in W^{-s,2}(\mathbb{R}^N)$, $z \in W_0^{s,2}(\Sigma_1)$ and an obstacle $\varphi \in W^{s,2}(\Sigma_2)$, we say that $u \in \mathcal{K}$ solves (1.1), if $w := (u - z) \in \mathcal{K}_0$ solves the minimization problem

$$\min_{w \in \mathcal{K}_0} J(w), \tag{3.1}$$

where we recall that the functional J is given by (1.2).

Notice that $\mathcal{K}$ and $\mathcal{K}_0$ only differ by the fact that functions in $\mathcal{K}_0$ are zero in Σ_1. The next result states the well-posedness of the minimization problem (1.1) according to Definition 3.2.

Theorem 3.3. *Let $f \in W^{-s,2}(\mathbb{R}^N)$, $z \in W_0^{s,2}(\Sigma_1)$ and $\varphi \in W^{s,2}(\Sigma_2)$. Then, there exists a unique solution $u \in \mathcal{K}$ to the minimization problem (1.1) according to Definition 3.2.*

Proof. Let $u \in \mathcal{K}$ and set $w := u - z$. Notice that $w|_{\Sigma_1} = 0$ so that $w \in W^{s,2}(\Omega, \Sigma_1)$. Then, using the definition of the functional J, the minimization problem (3.1) can be rewritten as follows:

$$\min_{w \in \mathcal{K}_0} J(w) := \frac{C_{N,s}}{4} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}}\, dxdy - \langle f, w \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)}, \tag{3.2}$$

where we have used that $W^{s,2}(\Omega, \Sigma_1) \hookrightarrow W^{s,2}(\mathbb{R}^N)$ and that $f \in W^{-s,2}(\mathbb{R}^N)$. We recall from Proposition 2.1 that the norm

$$\|w\|_{W^{s,2}(\Omega,\Sigma_1)} = \left(\int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}$$

is equivalent to the one induced by the space $W^{s,2}(\mathbb{R}^N)$. Since $\mathcal{K}_0$ is nonempty, closed and convex, the existence of solutions to the minimization problem (3.2) follows from the direct method of the calculus of variations. Uniqueness is due to the fact that J is strictly convex. □

Throughout the remainder of the paper $\mathcal{E}$ denotes the bilinear form defined in (2.4) with domain $D(\mathcal{E}) = W^{s,2}(\Omega,\Sigma_1)$.

We have the following important result on various equivalent formulations to the minimization problem (1.1), hence to (3.1).

Proposition 3.4. *Let $f \in W^{-s,2}(\mathbb{R}^N)$, $z \in W_0^{s,2}(\Sigma_1)$ and $\varphi \in W^{s,2}(\Sigma_2)$. Then, the following assertions hold.*

(a) A function $u \in \mathcal{K}$ solves the minimization problem (1.1) in the sense of Definition 3.2 if and only if $w := (u - z) \in \mathcal{K}_0$ satisfies the variational inequality

$$\mathcal{E}(w, v - w) - \langle f, v - w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \geq 0,\ \forall v \in \mathcal{K}_0. \tag{3.3}$$

(b) The variational inequality (3.3) is equivalent to the following. There exists a non negative functional $\lambda \in W^{-s,2}(\Sigma_2)$ such that,

$$\begin{cases} \mathcal{E}(w,v) + \langle\lambda, v\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = \langle f, v\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} & \forall v \in W^{s,2}(\Omega,\Sigma_1)\\ w \leq \varphi & \text{in } \Sigma_2\\ \langle\lambda, \tilde{v}\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} \leq 0 & \forall\ \tilde{v} \in \mathbb{K}^-\\ \langle\lambda, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = 0, \end{cases} \tag{3.4}$$

where

$$\mathbb{K}^- := \{\tilde{v} \in W^{s,2}(\Sigma_2) : \tilde{v} \leq 0\}. \tag{3.5}$$

(c) Additionally, if the obstacle $\varphi \in W^{s,2}(\Sigma_2)\cap C(\Sigma_2)$ and the variational inequality (3.3) has a solution $w \in W^{s,2}(\Omega,\Sigma_1)\cap C(\Sigma_2)$, then (3.3) is also equivalent to the Euler-Lagrange equations

$$\begin{cases} (-\Delta)^s w = f & \text{in } \mathcal{D}(\Omega)',\\ (-\Delta)^s w = 0 & \text{in } \mathcal{D}(\Sigma_2)',\\ \mathcal{N}_s w \leq 0 & \text{in } \Sigma_2,\\ \mathcal{N}_s w = 0 & \text{in } \Sigma_2 \cap \{u < \varphi\},\\ u \leq \varphi & \text{in } \Sigma_2. \end{cases} \tag{3.6}$$

The last two conditions of (3.6) are also equivalent to the complementarity condition

$$(u - \varphi)\mathcal{N}_s w = 0 \quad \text{in } \Sigma_2. \tag{3.7}$$

Proof. Let $f \in W^{-s,2}(\mathbb{R}^N)$, $z \in W_0^{s,2}(\Sigma_1)$ and $\varphi \in W^{s,2}(\Sigma_2)$. We proceed in three steps.

Step 1: Let $u \in \mathcal{K}$ solve the minimization problem (1.1) in the sense of Definition 3.2. Then, by Definition 3.2 $w := (u - z) \in \mathcal{K}_0$ solves (3.2). From the convexity of $\mathcal{K}_0$, for all $v \in \mathcal{K}_0$ and all $t \in [0, 1]$, we know that $w + t(v - w) \in \mathcal{K}_0$. As a result, for every $t \in (0, 1]$, we have that

$$\begin{aligned} 0 &\leq \frac{1}{t}\Big(J(w + t(v - w)) - J(w)\Big)\\ &= \frac{C_{N,s}}{4t}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|(w + t(v - w))(x) - (w + t(v - w))(y)|^2}{|x - y|^{N+2s}}\,dxdy \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{t}\langle f, w + t(v-w)\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \\
&- \frac{C_{N,s}}{4t}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy + \frac{1}{t}\langle f, w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}.
\end{aligned}
$$

Calculating we obtain that for every $t \in (0,1]$,

$$
\begin{aligned}
0 \leq & \frac{1}{t}\Big(J(w+t(v-w)) - J(w)\Big) \\
= & \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(w(x)-w(y))((v-w)(x)-(v-w)(y)}{|x-y|^{N+2s}}\,dxdy \\
& - \langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \\
& + \frac{C_{N,s}t}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|(v-w)(x)-(v-w)(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
= & \mathcal{E}(w, v-w) - \langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \\
& + \frac{C_{N,s}t}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|(v-w)(x)-(v-w)(y)|^2}{|x-y|^{N+2s}}\,dxdy,
\end{aligned}
$$

so that taking the limit, as $t \downarrow 0$, we get (3.3).

Conversely, if $u - z =: w \in \mathcal{K}_0$ satisfies (3.3), then for any $v \in \mathcal{K}_0$, after a simple calculation we obtain that

$$
\begin{aligned}
J(v) = & J(w + (v-w)) \\
= & \frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy - \langle f, w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \\
& + \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(w(x)-w(y))((v-w)(x)-(v-w)(y))}{|x-y|^{N+2s}}\,dxdy \\
& - \langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \\
& + \frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|(v-w)(x)-(v-w)(y)|^2}{|x-y|^{N+2s}}\,dxdy. \qquad (3.8)
\end{aligned}
$$

Since by (3.3),

$$
\begin{aligned}
& \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(w(x)-w(y))((v-w)(x)-(v-w)(y))}{|x-y|^{N+2s}}\,dxdy \\
& \quad - \langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \geq 0,
\end{aligned}
$$

it follows from (3.8) that

$$
J(v) = J(w + (v-w)) \geq J(w).
$$

Hence, $w \in \mathcal{K}_0$ is a minimizer of J. By definition, this shows that u solves (1.1) in the sense of Definition 3.2. The proof of Part (a) is complete.

Step 2: First, we observe that $W^{s,2}(\Omega, \Sigma_1)$ is a closed subspace of $W^{s,2}(\mathbb{R}^N)$ and under Assumption 3.1 we have that $W^{s,2}(\mathbb{R}^N)$ is continuously embedded into $W^{s,2}(\Sigma_2)$ so that we have the following continuous embeddings:

$$
W^{s,2}(\Omega, \Sigma_1) \hookrightarrow W^{s,2}(\mathbb{R}^N) \hookrightarrow W^{s,2}(\Sigma_2).
$$

Now, let us consider another characterization of the solution to (3.2). In particular, we are able to include the constraint $w \in \mathcal{K}_0$ into the minimization problem if we instead consider the functional $\tilde{J}$ given by

$$\begin{aligned}\tilde{J}(w) =&\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+I_{\mathbb{K}^-}(w-\varphi),\end{aligned}\tag{3.9}$$

where $\mathbb{K}^-$ is given in (3.5) and $I_{\mathbb{K}^-}$ is the indicator function defined in (2.17). It follows from Lemma 2.7 that $I^*_{\mathbb{K}^+}(w-\varphi)=I_{\mathbb{K}^-}(w-\varphi)$, where $\mathbb{K}^+=\{\eta\in W^{-s,2}(\Sigma_2):\eta\geq 0\}$. In the definition of $\mathbb{K}^+$, by $\eta\geq 0$ we mean that $\langle\eta,v\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\geq 0$ for all $v\in W^{s,2}(\Sigma_2)$ with $v\geq 0$ a.e. in Σ_2. Therefore,

$$\begin{aligned}\inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\tilde{J}(w) =& \inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\Bigg(\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+I_{\mathbb{K}^-}(w-\varphi)\Bigg)\\=&\inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\Bigg(\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+I^*_{\mathbb{K}^+}(w-\varphi)\Bigg).\end{aligned}\tag{3.10}$$

Applying the definition of $I^*_{\mathbb{K}^+}$ further shows that the right hand side of (3.10) becomes

$$\begin{aligned}&\inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\Bigg(\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+\sup_{\lambda\in W^{-s,2}(\Sigma_2)}\{\langle\lambda,w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}-I_{\mathbb{K}^+}(\lambda)\}\Bigg).\end{aligned}\tag{3.11}$$

Since the supremum in (3.11) can only be reached when $\lambda\in\mathbb{K}^+$, we can deduce that

$$\begin{aligned}&\inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\Bigg(\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+\sup_{\lambda\in\mathbb{K}^+}\langle\lambda,w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\Bigg)\\=&\inf_{w\in W^{s,2}(\Omega,\Sigma_1)}\sup_{\lambda\in\mathbb{K}^+}\Bigg(\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+\langle\lambda,w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\Bigg).\end{aligned}\tag{3.12}$$

The above identities motivate the introduction of an associated Lagrangian $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L}(w,\eta):=&\frac{C_{N,s}}{4}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}\setminus\Omega)^2}\frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,dxdy\\&-\langle f,w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}+\langle\eta,w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}.\end{aligned}\tag{3.13}$$

Next, we define the tangent cone of $\mathcal{K}_0$ at $w\in\mathcal{K}_0$ by

$$\begin{aligned}T(\mathcal{K}_0;u):=&\Big\{\kappa\in W^{s,2}(\Omega,\Sigma_1):\text{ for each } k\in\mathbb{N},\ \exists r_k>0, w_k\in\mathcal{K}_0\\&:\lim_{k\to\infty}w_k=w,\ \lim_{k\to\infty}r_k(w_k-w)=\kappa\Big\}\end{aligned}$$

and the linearization cone at $w \in \mathcal{K}_0$ by

$$L(\mathcal{K}_0; w) := \Big\{ rh : r > 0, h \in W^{s,2}(\Omega, \Sigma_1), w + h - \varphi \in \mathbb{K}^- \Big\}.$$

Notice that, whenever $w \in \mathcal{K}_0$ solves (3.2) we have that $L(\mathcal{K}_0; w) \subset T(\mathcal{K}_0; w)$. Indeed, suppose that $\kappa \in L(\mathcal{K}_0; w)$. Then, for some $r > 0$ and $h \in W^{s,2}(\Omega, \Sigma_1)$ we have that $\kappa = rh$ where $w + h \leq \varphi$ in Σ_2. It follows from the convexity of $\mathcal{K}_0$ that $w + \frac{1}{k}h \in \mathcal{K}_0$ for any $k \in \mathbb{N}$. Then, choosing $w_k = w + \frac{1}{k}h$ and $r_k = kr$, we have that $\lim_{k\to\infty} w_k = w$ and

$$\lim_{k\to\infty} r_k(w_k - w) = \lim_{k\to\infty} kr(w + \frac{1}{k}h - w) = \kappa$$

so that $\kappa \in T(\mathcal{K}_0; w)$.

It is well-known (see e.g. [17]) that there exists a Lagrange multiplier $\lambda \in W^{-s,2}(\Sigma_2)$ so that (w, λ) satisfies the KKT conditions (3.4). It follows from the third condition in (3.4) that λ is non-negative in the sense that $\langle \lambda, \tilde{v} \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} \geq 0$ for every $\tilde{v} \in W^{s,2}(\Sigma_2)$ with $\tilde{v} \geq 0$ a. e. in Σ_2.

Conversely, taking $v := v - w$ in the first identity in (3.4) with $v \in \mathcal{K}_0$, we get that for every $v \in \mathcal{K}_0$,

$$\begin{aligned} &\mathcal{E}(w, v - w) - \langle f, v - w \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)} \\ &= -\langle \lambda, v - w \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} \\ &= -\langle \lambda, v - \varphi + \varphi - w \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} \\ &= -\langle \lambda, v - \varphi \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} + \langle \lambda, w - \varphi \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)}. \end{aligned} \tag{3.14}$$

It follows from the last identity in (3.4) that

$$\langle \lambda, w - \varphi \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} = 0. \tag{3.15}$$

Since $(v - \varphi) \in \mathbb{K}^-$, it follows from the third inequality in (3.4) that

$$-\langle \lambda, v - \varphi \rangle_{W^{-s,2}(\Sigma_2), W^{s,2}(\Sigma_2)} \geq 0. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16) we can deduce that

$$\mathcal{E}(w, v - w) - \langle f, v - w \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)} \geq 0,$$

for every $v \in \mathcal{K}_0$, and we have shown (3.3).

Step 3: It remains to show Part (c). Suppose that $w := (u - z) \in \mathcal{K}_0$ solves (3.3). Applying the integration by parts formula given in (2.10), we can rewrite the variational inequality (3.3) as follows: For all $v \in \mathcal{K}_0$, we have that

$$\begin{aligned} &\langle (-\Delta)^s w, v - w \rangle_{W^{-s,2}(\Omega,\Sigma_1), W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2} (v - w) \mathcal{N}_s w \, dx \\ &= \langle (-\Delta)^s w, v - w \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)} + \int_{\Sigma_2} (v - w) \mathcal{N}_s w \, dx \\ &\geq \langle f, v - w \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)}. \end{aligned} \tag{3.17}$$

Let $\zeta \in \mathcal{D}(\Omega)$ be arbitrary. It is clear that $(w + \zeta) \in \mathcal{K}_0$, so letting $v := w + \zeta$ in (3.17) yields

$$\langle (-\Delta)^s w - f, \zeta \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)} \geq 0.$$

Since this is also true for $-\zeta$, we can deduce that

$$\langle (-\Delta)^s w - f, \zeta \rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)} = 0 \quad \text{for all } \zeta \in \mathcal{D}(\Omega). \tag{3.18}$$

That is, $(-\Delta)^s w = f$ in $\mathcal{D}(\Omega)'$.

Now, suppose that $\psi \in \mathcal{D}(\Sigma_2)$. Then,

$$\begin{aligned}\mathcal{E}(w,\psi) &= \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy\\ &= \frac{C_{N,s}}{2}\Bigg(-\int_{\Sigma_2}\int_{\Omega}\frac{\psi(y)(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy\\ &\quad+\int_{\Omega}\int_{\Sigma_2}\frac{\psi(x)(w(x)-w(y))}{|x-y|^{N+2s}}\,dxdy\Bigg)\\ &= C_{N,s}\int_{\Sigma_2}\psi(x)\Bigg(\int_{\Omega}\frac{w(x)-w(y)}{|x-y|^{N+2s}}\,dy\Bigg)\,dx\\ &= \int_{\Sigma_2}\psi(x)\mathcal{N}_s w(x)\,dx.\end{aligned}$$

Combining this with (2.10) shows that

$$\langle(-\Delta)^s w,\psi\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)} = 0 \qquad \text{for all } \psi\in\mathcal{D}(\Sigma_2) \tag{3.19}$$

so that $(-\Delta)^s w = 0$ in $\mathcal{D}(\Sigma_2)'$.

Now, let $\varphi \in W^{s,2}(\Sigma_2)\cap C(\Sigma_2)$, assume that the solution w of (3.3) also belongs to $C(\Sigma_2)$, and consider the set $E := \{x\in\Sigma_2 : w(x)<\varphi(x)\}$. Since $w,\varphi\in C(\Sigma_2)$, we have that the set E is open. Let $\psi\in\mathcal{D}(E)$, the space of test functions in E. For sufficiently small $\epsilon>0$ we have that $v := w+\epsilon\psi\in\mathcal{K}_0$. Our variational inequality, along with (3.19) yields

$$\epsilon\int_{\Sigma_2}\psi\mathcal{N}_s w\,dx \geq \epsilon\langle f,\psi\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s.2}(\mathbb{R}^N)}.$$

Since this is also true for $-\psi$, we have that

$$\int_{\Sigma_2}\psi\mathcal{N}_s w\,dx = \langle f,\psi\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s.2}(\mathbb{R}^N)} \qquad \text{for all } \psi\in\mathcal{D}(E). \tag{3.20}$$

Since $f\in W^{-s,2}(\mathbb{R}^N)\hookrightarrow W^{-s,2}(\Omega,\Sigma_1)$, it follows from the Riesz representation theorem there exists a unique $\tilde f\in W^{s,2}(\Omega,\Sigma_1)\hookrightarrow W^{s,2}(\mathbb{R}^N)$ such that for every $\psi\in\mathcal{D}(E)$,

$$\begin{aligned}\langle f,\psi\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} &= (\tilde f,\psi)_{W^{s,2}(\Omega,\Sigma_1)}\\ &= \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(\tilde f(x)-\tilde f(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy.\end{aligned} \tag{3.21}$$

Since $\psi=0$ outside of Σ_2 and $\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2 = (\Omega\times\Omega)\cup(\Omega\times\mathbb{R}^N\setminus\Omega))\cup((\mathbb{R}^N\setminus\Omega)\times\Omega)$, we have that the identity (3.21) reduces to the following:

$$\begin{aligned}&\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(\tilde f(x)-\tilde f(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,dxdy\\ &= \int_{\Omega}\int_{\Sigma_2}\frac{\psi(x)(\tilde f(x)-\tilde f(y))}{|x-y|^{N+2s}}\,dxdy + \int_{\Sigma_2}\int_{\Omega}\frac{-\psi(y)(\tilde f(x)-\tilde f(y))}{|x-y|^{N+2s}}\,dxdy\\ &= \int_{\Sigma_2}\int_{\Omega}\frac{\psi(x)(\tilde f(x)-\tilde f(y))}{|x-y|^{N+2s}}\,dydx + \int_{\Sigma_2}\int_{\Omega}\frac{-\psi(x)(\tilde f(y)-\tilde f(x))}{|x-y|^{N+2s}}\,dydx\\ &= 2\int_{\Sigma_2}\int_{\Omega}\frac{\psi(x)(\tilde f(x)-\tilde f(y))}{|x-y|^{N+2s}}\,dydx\end{aligned}$$

$$= 2\int_{\Sigma_2} \psi(x)\mathcal{N}_s(\tilde{f})(x)\;dx. \tag{3.22}$$

It follows from (3.20), (3.21) and (3.22) that for all $\psi \in \mathcal{D}(E)$ we have that

$$\int_{\Sigma_2} \psi(x)\Big(\mathcal{N}_s w(x) - 2\mathcal{N}_s(\tilde{f})(x)\Big)dx = 0. \tag{3.23}$$

We can deduce from the fundamental lemma of the calculus of variations that

$$\mathcal{N}_s w = 2\mathcal{N}_s(\tilde{f}) \;\text{ in } \Sigma_2, \tag{3.24}$$

whenever $w < \varphi$ in Σ_2.

Next, since $W^{s,2}(\Omega,\Sigma_1)$ is a linear subspace of $L^2(\Omega)$, it follows from the Hahn Banach Theorem that, there exists a linear functional $\hat{f} \in (L^2(\Omega))^\star = L^2(\Omega)$ such that for every $\psi \in \mathcal{D}(E)$,

$$\langle f,\psi\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} = (\hat{f},\psi)_{L^2(\Omega)}. \tag{3.25}$$

It follows from (3.21), (3.22) and (3.25) that for every $\psi \in \mathcal{D}(E)$,

$$2\int_{\Sigma_2} \psi(x)\mathcal{N}_s(\tilde{f})(x)\;dx = \int_\Omega \psi(x)\hat{f}(x)\;dx = 0,$$

where we have also used that $\operatorname{supp}[\psi] \subset \Sigma_2$. It follows from the fundamental lemma of the calculus of variations that

$$\mathcal{N}_s(\tilde{f}) = 0 \quad \text{in } \Sigma_2.$$

This fact together with (3.24) implies that

$$\mathcal{N}_s w = 0 \quad \text{in } \{x \in \Sigma_2 : u(x) < \varphi(x)\}.$$

Now, suppose that $\psi \in \mathcal{D}(\Sigma_2)$ with $\psi \geq 0$. Substituting $v := w - \psi$ into (3.3), we can see that

$$\int_{\Sigma_2} \psi\mathcal{N}_s w\;dx \leq \langle f,\psi\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \tag{3.26}$$

for all non-negative test functions ψ defined in Σ_2. Since $\psi \in \mathcal{D}(\Sigma_2)$, the right-hand-side in (3.26) vanishes. Therefore,

$$\int_{\Sigma_2} \psi\mathcal{N}_s w\;dx \leq 0$$

for all $\psi \in \mathcal{D}(\Sigma_2)$ with $\psi \geq 0$. As a result,

$$\mathcal{N}_s w \leq 0 \quad \text{in } \Sigma_2.$$

Conversely, suppose that w satisfies (3.6). Then, for all $v \in \mathcal{K}_0$ we have that

$$\begin{aligned}
\mathcal{E}(w,v-w) =&\langle(-\Delta)^s w, v-w\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2}(v-w)\mathcal{N}_s w\;dx\\
=&\langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} + \int_{\{x\in\Sigma_2 : w(x)<\varphi(x)\}}(v-w)\mathcal{N}_s w\;dx\\
&+\int_{\{x\in\Sigma_2 : w(x)=\varphi(x)\}}(v-\varphi)\mathcal{N}_s w\;dx\\
\geq&\langle f, v-w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)}.
\end{aligned}$$

It remains to show the last assertion of the proposition. Indeed, since $\mathcal{N}_s w = 0$ in $\Sigma_2 \cap \{u < \varphi\}$ and $u \leq \varphi$ in Σ_2, it follows that $(u-\varphi)\mathcal{N}_s w = 0$ in Σ_2. We have shown that the last two conditions in (3.6) implies (3.7). Now, assume that (3.7) holds. This implies that $u - \varphi = 0$ in Σ_2 or $\mathcal{N}_s w = 0$ in Σ_2. This trivially implies that $\mathcal{N}_s w = 0$ in $\Sigma_2 \cap \{u < \varphi\}$ and $u \le \varphi$ in Σ_2. The proof is finished. □

4. **Penalization.** We now consider a variety of penalty formulations, whose purpose is to incorporate the constraint into our minimization problem and approximate our original formulation by a sequence of Fréchet differentiable functionals. We begin this section by analyzing a Moreau-Yosida type penalty formulation in $L^2(\Sigma_2)$, namely

$$\min_{W^{s,2}(\Omega,\Sigma_1)} J_\epsilon(w) \tag{4.1}$$

with

$$J_\epsilon(w) := \frac{1}{2}\mathcal{E}(w,w) - \langle f, w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} + \frac{\epsilon^{-2}}{2}\int_{\Sigma_2} [(w-\varphi)^+]^2\,dx,$$

where $\epsilon > 0$ is the penalty parameter and, for $\varphi \in W^{s,2}(\Sigma_2)$, we denote the positive part of φ as $\varphi^+ := \max\{\varphi, 0\}$. Further, we denote the negative part of φ to be $\varphi^- := \min\{\varphi, 0\}$ and notice that $\varphi = \varphi^+ + \varphi^-$.

As before, the direct method of the calculus of variations ensures a unique minimizer to J_ϵ, denoted by w_ϵ. Before stating our first result in this section, we recall the convergence in the sense of Mosco.

Definition 4.1. Let B be a Banach space. A sequence $F_n : B \to \mathbb{R}$, $n \in \mathbb{N}$, is said to converge to $F : B \to \mathbb{R}$ in the sense of Mosco, writing $F_n \xrightarrow{M} F$, as $n \to \infty$, if for every $v \in B$, the following two conditions are satisfied.

(a) There is a sequence $(v_n)_n$ that converges to v in B, as $n \to \infty$, and
$$\limsup_{n\to\infty} F_n(v_n) \le F(v).$$
(b) For every sequence $(v_n)_n$ that converges weakly to v in B, as $n \to \infty$, we have that
$$\liminf_{n\to\infty} F_n(v_n) \ge F(v).$$

We have the following convergence result where in abuse of language we use the terminology sequences instead of nets.

Proposition 4.2. *For every $\epsilon > 0$, there exists a unique solution w_ϵ to (4.1). Additionally, there exists a sub-sequence, that we still denote by $(w_\epsilon)_\epsilon$, of solutions that converges weakly to $w \in \mathcal{K}_0$, as $\epsilon \downarrow 0$, so that $J_\epsilon \xrightarrow{M} J$ (in the sense of Mosco), as $\epsilon \downarrow 0$.*

Proof. For each $\epsilon > 0$ and the minimization problem corresponding to J_ϵ, consider the resulting sequence of solutions $(w_\epsilon)_{\epsilon>0}$. From the coercivity of J, there exists a constant $C > 0$ independent of ϵ such that $\|w_\epsilon\|_{W^{s,2}(\Omega,\Sigma_1)} \le C$. Then, there is a subsequence that we still denote by $(w_\epsilon)_\epsilon$ that converges weakly to $w \in W^{s,2}(\Omega,\Sigma_1)$, as $\epsilon \downarrow 0$. From the weak lower semi-continuity of J, we have that

$$\begin{aligned} J(w) \le \liminf_{\epsilon\downarrow 0} J(w_\epsilon) &\le \liminf_{\epsilon\downarrow 0}\left(J(w_\epsilon) + \frac{\epsilon^{-2}}{2}\int_{\Sigma_2}[(w_\epsilon-\varphi)^+]^2\,dx\right) \\ &= \liminf_{\epsilon\downarrow 0} J_\epsilon(w_\epsilon). \end{aligned} \tag{4.2}$$

Further, we claim that the weak limit, w, belongs to $\mathcal{K}_0$. Indeed, let $v \in \mathcal{K}_0$ be fixed but arbitrary. Then, from the optimality of w_ϵ it follows that

$$J(w_\epsilon) + \frac{\epsilon^{-2}}{2}\int_{\Sigma_2} [(w_\epsilon - \varphi)^+]^2 \, dx \le J(v) + \frac{\epsilon^{-2}}{2}\int_{\Sigma_2} [(v - \varphi)^+]^2 \, dx = J(v), \tag{4.3}$$

where we have used the fact that $(v-\varphi)^+ = 0$ in Σ_2. It follows from (4.3) that

$$\int_{\Sigma_2} [(w_\epsilon - \varphi)^+]^2 \, dx \le 2\epsilon^2 (J(v) - J(w_\epsilon)) \le C\epsilon^2.$$

As a consequence of the weak lower semi-continuity, we have that

$$\int_{\Sigma_2} [(w-\varphi)^+]^2 \le \liminf_{\epsilon \downarrow 0} \int_{\Sigma_2} [(w_\epsilon - \varphi)^+]^2 = 0$$

so that $w \in \mathcal{K}_0$, and the claim is proved.

Further, from (4.2) we have that

$$J(w) \le \liminf_{\epsilon\downarrow 0} J_\epsilon(w_\epsilon).$$

Now, if $w \in W^{s,2}(\Omega, \Sigma_1)$, choosing the constant sequence $(w)_{\epsilon>0}$ gives us that

$$\limsup_{\epsilon\downarrow 0} J_\epsilon(w) = J(w).$$

We have shown that the conditions (a) and (b) in Definition 4.1 are satisfied. Hence, we have the convergence in the sense of Mosco. The proof is finished. □

Remark 4.3. We observe that generally, $w_\epsilon \in W^{s,2}(\Omega, \Sigma_1)$ will fail to satisfy $w_\epsilon \le \varphi$ in Σ_2. Therefore, it is necessary to estimate the error created by this penalization. Notice that Proposition 4.2 establishes convergence, but not the rate of convergence.

The next result shows a rate of convergence with respect to ϵ.

Theorem 4.4. *Assume that the complementary condition (3.7) holds. Then, the unique minimizer $w_\epsilon \in W^{s,2}(\Omega, \Sigma_1)$ of the penalized functional J_ϵ satisfies*

$$C_{N,s}\|w - w_\epsilon\|^2_{W^{s,2}(\Omega,\Sigma_1)} + \frac{\epsilon^{-2}}{2}\|(w_\epsilon - \varphi)^+\|^2_{L^2(\Sigma_2)} \le \epsilon^2 \|\mathcal{N}_s w\|^2_{L^2(\Sigma_2)}, \tag{4.4}$$

where we recall that the operator $\mathcal{N}_s$ is given in (2.7).

Proof. For any specified $\epsilon > 0$, the minimizer w_ϵ of (4.1) satisfies for all $v \in W^{s,2}(\Omega, \Sigma_1)$,

$$\mathcal{E}(w_\epsilon, v) + \epsilon^{-2}\int_{\Sigma_2} (w_\epsilon - \varphi)^+ v \, dx = \langle f, v\rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)}.$$

Applying the integration by parts formula given in (2.10), we obtain that

$$\begin{aligned}&\langle(-\Delta)^s w_\epsilon, v\rangle_{W^{-s,2}(\Omega,\Sigma_1), W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2} v\mathcal{N}_s w_\epsilon \, dx + \epsilon^{-2}\int_{\Sigma_2}(w_\epsilon - \varphi)^+ v \, dx \\ &= \langle f, v\rangle_{W^{-s,2}(\mathbb{R}^N), W^{s,2}(\mathbb{R}^N)}\end{aligned} \tag{4.5}$$

for all $v \in W^{s,2}(\Omega, \Sigma_1)$.

Further, from (2.10) and (4.5), taking $v := w - w_\epsilon$ as a test function, we obtain that,

$$\frac{C_{N,s}}{2}\|w - w_\epsilon\|^2_{W^{s,2}(\Omega,\Sigma_1)}$$

$$
\begin{aligned}
&= \langle (-\Delta)^s (w-w_\epsilon), w-w_\epsilon\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2} (w-w_\epsilon)\mathcal{N}_s(w-w_\epsilon)\ dx\\
&= \langle (-\Delta)^s w, w-w_\epsilon\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)} + \int_{\Sigma_2} (w-w_\epsilon)\mathcal{N}_s w\ dx\\
&\quad + \epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w-w_\epsilon)\ dx - \langle f, w-w_\epsilon\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N}. \qquad (4.6)
\end{aligned}
$$

The first two identities in (3.6) imply that

$$
\langle (-\Delta)^s w, w-w_\epsilon\rangle_{W^{-s,2}(\Omega,\Sigma_1),W^{s,2}(\Omega,\Sigma_1)} - \langle f, w-w_\epsilon\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} = 0. \quad (4.7)
$$

By (3.7), we have that

$$
\int_{\Sigma_2} (w-\varphi)\mathcal{N}_s w\ dx = \int_{\Sigma_2} (u-\varphi)\mathcal{N}_s w\ dx = 0.
$$

Then, using (4.7) we see that (4.6) becomes

$$
\frac{C_{N,s}}{2}\|w-w_\epsilon\|^2_{W^{s,2}(\Omega,\Sigma_1)} = -\int_{\Sigma_2} (w_\epsilon-\varphi)\mathcal{N}_s w\ dx + \epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w-w_\epsilon)\ dx. \quad (4.8)
$$

Since $w \le \varphi$ in Σ_2, we have that

$$
\begin{aligned}
&\epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w-w_\epsilon)\ dx\\
&= -\epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w_\epsilon-\varphi)\ dx - \epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(\varphi-w)\ dx\\
&\le -\epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w_\epsilon-\varphi)\ dx. \qquad (4.9)
\end{aligned}
$$

Since $(w_\epsilon-\varphi)^+(w_\epsilon-\varphi)^- = 0$, the right-hand-side in (4.9) becomes

$$
\begin{aligned}
&-\epsilon^{-2}\int_{\Sigma_2} [(w_\epsilon-\varphi)^+]^2\ dx - \epsilon^{-2}\int_{\Sigma_2} (w_\epsilon-\varphi)^+(w_\epsilon-\varphi)^-\ dx\\
&= -\epsilon^{-2}\int_{\Sigma_2} [(w_\epsilon-\varphi)^+]^2\ dx. \qquad (4.10)
\end{aligned}
$$

Combining (4.6)-(4.10) and recalling that $\mathcal{N}_s w \le 0$ in Σ_2, we get

$$
\begin{aligned}
\frac{C_{N,s}}{2}\|w-w_\epsilon\|^2_{W^{s,2}(\Omega,\Sigma_1)} &\le -\int_{\Sigma_2} (w_\epsilon-\varphi)\mathcal{N}_s w\ dx - \epsilon^{-2}\int_{\Sigma_2} [(w_\epsilon-\varphi)^+]^2\ dx\\
&= -\int_{\Sigma_2} (w_\epsilon-\varphi)^+\mathcal{N}_s w\ dx - \int_{\Sigma_2} (w_\epsilon-\varphi)^-\mathcal{N}_s w\ dx\\
&\quad - \epsilon^{-2}\int_{\Sigma_2} [(w_\epsilon-\varphi)^+]^2\ dx\\
&\le -\int_{\Sigma_2} (w_\epsilon-\varphi)^+\mathcal{N}_s w\ dx - \epsilon^{-2}\int_{\Sigma_2} [(w_\epsilon-\varphi)^+]^2\ dx. \qquad (4.11)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{C_{N,s}}{2}\|w-w_\epsilon\|^2_{W^{s,2}(\Omega,\Sigma_1)} + \epsilon^{-2}\|(w_\epsilon-\varphi)^+\|^2_{L^2(\Sigma_2)}\ dx\\
&\le -\int_{\Sigma_2} (w_\epsilon-\varphi)^+\mathcal{N}_s w\ dx
\end{aligned}
$$

$$\leq \frac{\epsilon^{-2}}{2}\|(w_\epsilon - \varphi)^+\|^2_{L^2(\Sigma_2)} + \frac{\epsilon^2}{2}\|\mathcal{N}_s w\|^2_{L^2(\Sigma_2)}, \tag{4.12}$$

where in the last step we have used Hölder's inequality and Young's inequality. We have shown (4.4) and the proof is finished. □

In the above penalized formulation (cf. (4.1)) and subsequently in Theorem 4.4, we have considered a penalization in the $L^2(\Sigma_2)$-norm. Next, we instead consider a penalization in the $W^{s,2}(\Sigma_2)$-norm. The advantage being that the optimality conditions for the penalized problem can be directly related to the original optimality system (3.4).

Motivated by Kikuchi and Oden [18, Chapter 3], we look for a penalty functional of the form

$$J_\xi(w) = \frac{1}{2}\mathcal{E}(w,w) - \langle f, w\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} + \frac{\xi^{-1}}{2}\|(w-\varphi)^+\|^2_{W^{s,2}(\Sigma_2)}. \tag{4.13}$$

From the strict convexity of J_ξ as well as the direct method of the calculus of variations we know that J_ξ has a unique minimizer, which we denote by w_ξ. Furthermore, w_ξ satisfies the optimality conditions

$$\mathcal{E}(w_\xi, v) + \frac{1}{\xi}((w_\xi - \varphi)^+, v)_{W^{s,2}(\Sigma_2)} = \langle f, v\rangle_{W^{-s,2}(\mathbb{R}^N),W^{s,2}(\mathbb{R}^N)} \tag{4.14}$$

for all $v \in W^{s,2}(\Omega,\Sigma_1)$. As we will see, considering such a penalty functional gives us a method to relate our penalized problem back to the optimality system in (3.4). More precisely, we have the following result.

Theorem 4.5. *Suppose that $w \in \mathcal{K}_0$ and $\lambda \in W^{-s,2}(\Sigma_2) \subset W^{-s,2}(\Omega,\Sigma_1)$ satisfy (3.4). Additionally, suppose that w_ξ minimizes J_ξ for a given $\xi > 0$. Let λ_ξ be the unique element in $W^{-s,2}(\Sigma_2)$ satisfying (by the Riesz representation theorem)*

$$\langle \lambda_\xi, v\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = \left(\frac{1}{\xi}(w_\xi - \varphi)^+, v\right)_{W^{s,2}(\Sigma_2)} \quad \text{for all } v \in W^{s,2}(\Sigma_2). \tag{4.15}$$

Then, there is a constant $C = C(N,s,\Omega) > 0$ independent of ξ such that

$$\|w_\xi - w\|_{W^{s,2}(\Omega,\Sigma_1)} \leq C\xi\|\lambda\|_{W^{-s,2}(\Sigma_2)} \tag{4.16}$$

and

$$\|\lambda_\xi - \lambda\|_{W^{-s,2}(\Sigma_2)} \leq C\xi\|\lambda\|_{W^{-s,2}(\Sigma_2)}. \tag{4.17}$$

In particular, we have that $w_\xi \to w$ in $W^{s,2}(\Omega,\Sigma_1)$ and $\lambda_\xi \to \lambda$ in $W^{-s,2}(\Sigma_2)$, as $\xi \downarrow 0$.

In addition, there is a constant $C > 0$ independent of ξ such that

$$\|(w_\xi - \varphi)^+\|_{W^{s,2}(\Sigma_2)} \leq C\xi. \tag{4.18}$$

Proof. It follows from (3.4), (4.14) and (4.15) that

$$\mathcal{E}(w_\xi - w, v) = \langle \lambda_\xi - \lambda, v\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} \tag{4.19}$$

for all $v \in W^{s,2}(\Omega,\Sigma_1)$. Taking $v := w_\xi - w$ in (4.19) yields

$$\mathcal{E}(w_\xi - w, w_\xi - w) = \langle \lambda_\xi - \lambda, w_\xi - w\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}. \tag{4.20}$$

Since by (3.4)

$$\langle \lambda, w - \varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = 0,$$

it follows that

$$\langle \lambda - \lambda_\xi, w - w_\xi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = \langle \lambda - \lambda_\xi, w - \varphi + \varphi - w_\xi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}$$

$$=\langle -\lambda_\xi, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} \\ - \langle \lambda-\lambda_\xi, w_\xi-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}. \quad (4.21)$$

It follows from the abstract result given in [18, Section 3.3], that for $h\in W^{s,2}(\Sigma_2)$, h^+ is characterized by the variational inequality

$$(h^+-h, g-h^+)_{W^{s,2}(\Sigma_2)}\ge 0, \quad (4.22)$$

for all $g\in\mathbb{K}^+ := \{g\in W^{s,2}(\Sigma_2): g\ge 0\}$. In addition, since $h^+\in\mathbb{K}^+$ we can easily deduce that

$$(h^-,h^+)_{W^{s,2}(\Sigma_2)}=0,\ (g^-,h^+)_{W^{s,2}(\Sigma_2)}\le 0,\ \ (h^+,h^+)_{W^{s,2}(\Sigma_2)}\ge 0,$$

for all $h,\ g\in W^{s,2}(\Sigma_2)$. Notice also that from the definition of λ_ξ, we have that

$$\langle \lambda_\xi-\lambda, \xi^{-1}(w_\xi-\varphi)^+\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} = (\lambda_\xi-\lambda,\lambda_\xi)_{W^{-s,2}(\Sigma_2)}.$$

Using all these facts and (4.15), we get from (4.21) that

$$\begin{aligned}\langle \lambda-\lambda_\xi, w-w_\xi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} =&\langle \lambda-\lambda_\xi, w-\varphi+\varphi-w_\xi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ =&-\langle \lambda-\lambda_\xi, w_\xi-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ &+\langle \lambda-\lambda_\xi, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ =&-\langle \lambda-\lambda_\xi, w_\xi-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ &+\langle \lambda, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ &-\langle \lambda_\xi, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ =&-\langle \lambda-\lambda_\xi, w_\xi-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ &+\langle \lambda, w-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ &-\left(\frac{1}{\xi}(w_\xi-\varphi)^+, w-\varphi\right)_{W^{s,2}(\Sigma_2)}.\end{aligned}$$

Since, $(w-\varphi)^+=0$ in Σ_2, we have that

$$\begin{aligned}&\left(\frac{1}{\xi}(w_\xi-\varphi)^+, w-\varphi\right)_{W^{s,2}(\Sigma_2)}\\ &=\left(\frac{1}{\xi}(w_\xi-\varphi)^+, (w-\varphi)^+\right)_{W^{s,2}(\Sigma_2)}+\left(\frac{1}{\xi}(w_\xi-\varphi)^+, (w-\varphi)^-\right)_{W^{s,2}(\Sigma_2)}\\ &=\left(\frac{1}{\xi}(w_\xi-\varphi)^+, (w-\varphi)^-\right)_{W^{s,2}(\Sigma_2)}\\ &\le 0.\end{aligned}$$

Notice also that

$$\langle \lambda_\xi, (w_\xi-\varphi)^-\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}=0$$

and

$$\langle \lambda, (w_\xi-\varphi)^-\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\le 0.$$

Using all the above facts we can deduce that

$$\begin{aligned}\langle \lambda-\lambda_\xi, w-w_\xi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} \le&\langle \lambda_\xi-\lambda, w_\xi-\varphi\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ \le&\langle \lambda_\xi-\lambda, (w_\xi-\varphi)^+\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)}\\ =&-\xi\,(\lambda_\xi-\lambda,\lambda_\xi)_{W^{-s,2}(\Sigma_2)}\\ =&-\xi\,(\lambda_\xi-\lambda,\lambda_\xi-\lambda+\lambda)_{W^{-s,2}(\Sigma_2)}\end{aligned}$$

$$
\begin{aligned}
&= -\xi\|\lambda_\xi - \lambda\|^2_{W^{-s,2}(\Sigma_2)} - \xi\,(\lambda_\xi - \lambda, \lambda)_{W^{-s,2}(\Sigma_2)} \\
&\leq \xi\|\lambda - \lambda_\xi\|_{W^{-s,2}(\Sigma_2)}\|\lambda\|_{W^{-s,2}(\Sigma_2)}.
\end{aligned}
$$

This latter estimate together with (4.20) and the coercivity of the bilinear form $\mathcal{E}$ yield that there is a constant $C = C(N, s, \Omega) > 0$ independent of ξ such that

$$\|w_\xi - w\|^2_{W^{s,2}(\Omega,\Sigma_1)} \leq C\xi\|\lambda - \lambda_\xi\|_{W^{-s,2}(\Sigma_2)}\|\lambda\|_{W^{-s,2}(\Sigma_2)}. \tag{4.23}$$

On the other hand, it follows from (4.19) that there is a constant $C > 0$ independent of ξ such that

$$\|\lambda - \lambda_\xi\|_{W^{-s,2}(\Sigma_2)} \leq C\|w_\xi - w\|_{W^{s,2}(\Omega,\Sigma_1)}. \tag{4.24}$$

Combining (4.23)-(4.24) we obtain that

$$\|w_\xi - w\|_{W^{s,2}(\Omega,\Sigma_1)} \leq C\xi\|\lambda\|_{W^{-s,2}(\Sigma_2)}$$

and we have shown (4.16). Combining (4.16) and (4.24) we get (4.17). Finally, the last assertion follows from (4.24).

It remains to prove (4.18). Indeed, taking $v := (w_\xi - \varphi)^+ \in W^{s,2}(\Sigma_2)$ as a test function in (4.15) we obtain the following estimate:

$$
\begin{aligned}
\|(w_\xi - \varphi)^+\|^2_{W^{s,2}(\Sigma_2)} &= \xi\langle\lambda_\xi, (w_\xi - \varphi)^+\rangle_{W^{-s,2}(\Sigma_2),W^{s,2}(\Sigma_2)} \\
&\leq \xi\|\lambda_\xi\|_{W^{-s,2}(\Sigma_2)}\|(w_\xi - \varphi)^+\|_{W^{s,2}(\Sigma_2)}.
\end{aligned}
$$

Since $\lambda_\xi \to \lambda$ in $W^{-s,2}(\Sigma_2)$, as $\xi \to 0$, it follows that the sequence $(\lambda_\xi)_\xi$ is bounded in $W^{-s,2}(\Sigma_2)$. This fact together with the previous estimate gives (4.18). ☐

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