

Summing Gravitational Effects from Loops of Inflationary Scalars

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ABSTRACT

We develop a procedure for re-summing the large logarithms induced in gravity by loops of inflationary scalars. We first show how the scalar can be integrated out of the field equations in the presence of constant graviton field. We then extend this result to a fully conserved form which explains the need for a finite renormalization of the cosmological constant which was previously inferred from explicit computation. A variant of the renormalization group turns out to explain the large logarithmic corrections revealed by explicit computation in the electric field strength of gravitational radiation and in the potentials which characterize the response to a point mass. The implications for graviton loops are discussed.

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1 Prologue

The geometry of cosmology can be characterized by a scale factor $a(t)$, Hubble parameter $H(t)$ and first slow roll parameter $\epsilon(t)$,

$$ds^2 = -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \quad \Longrightarrow \quad H(t) \equiv \frac{\dot{a}}{a} \quad , \quad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} . \quad (1)$$

The accelerated expansion ($H > 0$ with $0 \leq \epsilon < 1$) of primordial inflation rips virtual particles out of the vacuum [1]. The phenomenon is largest for particles such as massless, minimally coupled (MMC) scalars and gravitons, which are both massless and not conformally invariant [2, 3]. This is what is thought to have caused the primordial spectra of gravitons [4] and scalars [5].

Because more and more quanta are created as inflation progresses, correlators which involve interacting MMC scalars and gravitons often show secular growth in the form of powers of $\ln[a(t)]$ [6–15]. For example, the 2-loop dimensionally regulated and fully renormalized expectation value of the stress tensor of an MMC scalar with a $\frac{\lambda}{4!}\phi^4$ self-interaction, on de Sitter background ($\epsilon = 0$) takes the perfect fluid form $\langle T_{\mu\nu} \rangle = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$ with energy density and pressure [6, 16, 17],

$$\rho(t) = \frac{\lambda H^4}{2^7 \pi^4} \times \ln^2(a) + O(\lambda^2) , \quad (2)$$

$$p(t) = \frac{\lambda H^4}{2^7 \pi^4} \left\{ -\ln^2(a) - \frac{2}{3} \ln(a) \right\} + O(\lambda^2) . \quad (3)$$

In the correlators of this theory each factor of λ can be associated with as many as two factors of $\ln(a)$. Contributions which saturate this bound are known as *leading logarithm*, whereas those which have fewer factors of $\ln(a)$ are known as *subleading*. In expression (3) the factor of $-\ln^2(a)$ is a leading logarithm whereas the factor of $-\frac{2}{3} \ln(a)$ is subleading.

During a prolonged period of inflation factors of $\ln[a(t)]$ can grow so large that they overwhelm even the smallest coupling constant. Developing a technique to sum up the series of leading logarithms may eventually be as important for cosmology as the renormalization group summation of leading momentum logarithms was to flat space quantum field theory. The late Alexei Starobinsky solved this problem for scalar potential models [18],

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - \frac{\lambda}{4!} \phi^4 \sqrt{-g} . \quad (4)$$

Starobinsky's technique can be proven to reproduce each order's leading logarithms [19] and, when $V(\phi)$ is bounded from below, it can be summed up to give the late time limits of cosmological correlators [20]. However, straightforward application of the method fails for more general theories [8, 21, 22].

To understand the problem with more general theories it is necessary to distinguish between “Active” fields which induce large logarithms, such as MMC scalars and gravitons, and “Passive” fields which do not, such as conformally coupled scalars, fermions, photons and even differentiated Active fields. One can sum the leading logarithms of theories which contain Passive fields without derivative interactions by integrating out the Passive fields, assuming the Active fields are constant in space and time. This results in a scalar potential model which can then be treated using Starobinsky's formalism [8, 21].

The large logarithms of scalar potential models derive from what DeWitt and Brehme termed the “tail” part [23] of the MMC scalar propagator,¹

$$D = 4 \implies i\Delta(x; x') = \frac{1}{4\pi^2} \frac{1}{aa'(x-x')^2} - \frac{H^2}{8\pi^2} \ln \left[\frac{1}{4} H^2 (x-x')^2 \right]. \quad (5)$$

The situation when derivative interactions are present is more complicated because renormalization provides a new mechanism for generating large logarithms. When using dimensional regularization it turns out that D -dependent factors of $a(t)$ do not occur in divergent primitive contributions, which are incompletely canceled by counterterms,

$$\begin{aligned} \left(\text{Primitive} : \frac{(2H)^{D-4}}{D-4} \right) - \left(\text{Counterterm} : \frac{(\mu a)^{D-4}}{D-4} \right) \\ = -\ln \left(\frac{\mu a}{2H} \right) + O(D-4). \quad (6) \end{aligned}$$

These sorts of large logarithms are correctly described by a variant of the renormalization group [27].

¹In this de Sitter background example (5), the tail is the logarithmic term. The original work on this subject was done by Hadamard [24] who considered the Cauchy problem for a class of linear partial differential equations which include those of Riemannian geometries. What DeWitt and Brehme did was to cast the same asymptotic form, Hadamard's “elementary solution”, in the geometrical language of bitensors and biscalars. The bitensor formalism had previously been introduced by Ruse [25] and Synge [26], but not applied to Hadamard's elementary solution.

Derivative interactions are also more complicated in the way one integrates out Passive fields in the presence of a constant Active background. When derivative interactions are absent the coupling to a constant Active field is typically a mass term, as in Yukawa [8] or scalar quantum electrodynamics in Lorenz gauge [21]. This leads to a conventional Coleman-Weinberg potential [28], of course modified by functions of the dimensionless ratio of $\frac{\phi}{H}$. Derivative interactions lead to a new type of effective potential in which the field strength of a Passive field depends upon constant Active fields [27]. In this paper we consider a third type of effective potential in which constant Active fields shift the Hubble parameter on which the Passive field propagators depend.²

The model we shall study is how a loop of MMC scalars on de Sitter background induce changes in gravity,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} + \frac{[R-(D-2)\Lambda]\sqrt{-g}}{16\pi G}, \quad (7)$$

where G is Newton's constant and $\Lambda = (D-1)H^2$ is the cosmological constant. The scalar loop contribution to the graviton self-energy was made about a decade ago [30], and used to solve the effective field equations for gravitational radiation [31, 32] and for the gravitational response to a point mass [33]. Unfortunately, the original computation fails to be conserved because it lacks a finite renormalization of the cosmological constant, the necessity for which was only realized recently [34]. This problem was compounded by representing the non-conserved graviton self-energy using a sum of structure functions which are automatically conserved [32]. When the computation was re-done, including the finite renormalization, and making no assumptions about structure functions, a somewhat different result was found for the $\ln[a(t)]$ correction to the potentials [35]. The effects on gravitational radiation were also computed to enough accuracy to reveal a logarithmic change in the electric component of the Weyl tensor. The purpose of this paper is to explain both logarithmic corrections using a variant of the renormalization group. We will also explain how to integrate the scalar out of the gravitational field equation for constant graviton background, and show that the induced stress tensor implies precisely the finite renormalization of the cosmological constant which is needed to make the graviton self-energy conserved [34].

²Hubble-induced effective potentials can also occur in theories for which matter fields assume a non-trivial background [29].

This paper consists of six sections. In Section 2 we review non-linear sigma models, which manifest large logarithms from renormalization (6) and also stochastic logarithms from effective potentials derived from Active fields modifying the field strengths of Passive fields. Section 3 shows that integrating the scalar out of (7) in the presence of a spacetime constant graviton background induces a new type of effective stress tensor by modifying the Hubble constant on which the scalar propagator depends. We show that this result explains the finite renormalization required in the exact calculation. Because the induced stress tensor is only valid at leading logarithm order, it is not conserved when the graviton field is allowed to depend on space and time. In Section 4 we show how the stress tensor can be extended to give a fully conserved form. A variant of the renormalization group is used in Section 5 to explain the large logarithms found for gravitational radiation and for the response to a point mass. Section 6 gives our conclusions and also discusses the prospects for extending this analysis to loops of gravitons.

2 Non-linear Sigma Models

Non-linear sigma models have the same $h\partial h\partial h$ derivative interactions as gravity, and induce the same factors of $\ln[a(t)]$ on de Sitter background, but without the complex index structure or the gauge issue. They have therefore received much attention as a simple venue for sorting out the complexities of derivative interactions [19, 27, 36–40]. A simple example consists of two scalar fields $A(x)$ and $B(x)$ with the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} - \frac{1}{2}\left(1 + \frac{1}{2}\lambda A\right)^2 \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} . \quad (8)$$

The first variations of its action provide the equations of motion,

$$\frac{\delta S[A, B]}{\delta A(x)} = \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu A \right] - \frac{1}{2}\lambda \left(1 + \frac{1}{2}\lambda A\right) \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} = 0 , \quad (9)$$

$$\frac{\delta S[A, B]}{\delta B(x)} = \partial_\mu \left[\left(1 + \frac{1}{2}\lambda A\right)^2 \sqrt{-g} g^{\mu\nu} \partial_\nu B \right] = 0 . \quad (10)$$

The above theory is not renormalizable and hence requires an ever-increasing number of counterterms.

** Perturbative Results*

The quantities computed were the expectation values of A , B and their squares, the mode functions $u_A(\eta, k)$ & $u_B(\eta, k)$, and the exchange potentials $P_A(\eta, r)$ & $P_B(\eta, k)$. The latter two are obtained from the scalar self-mass $-iM^2(x; x')$ which supplies the quantum corrections to the linearized effective field equation for a scalar $\Phi(x)$,

$$\mathcal{D}\Phi(x) - \int d^4x' M^2(x; x')\Phi(x') = J(x) , \quad (11)$$

where $\mathcal{D} \equiv \partial^\mu a^2 \partial_\mu$ is the kinetic operator and $J(x)$ the source. The quantum corrections to the propagation of scalar radiation are imprinted in the mode function correction which is obtained when $J(x) = 0$. The choice $J(x) = a(\eta)\delta^3(\mathbf{x})$ gives the scalar exchange potential.

The above perturbative calculations of self-masses and VEV's were performed in the “*in-in*” formalism which is causal and which allows only real self-masses [41–49]. The regulation technique is dimensional regularization which preserves coordinate invariance. As a result the Lagrangian (8) requires two counterterms per scalar field to renormalize the self-masses at 1-loop,

$$\begin{aligned} \Delta\mathcal{L}_{-iM^2} = & -\frac{1}{2}C_{A1} \square A \square A \sqrt{-g} - \frac{1}{2}C_{A2} R \partial_\mu A \partial_\nu A g^{\mu\nu} \sqrt{-g} \\ & -\frac{1}{2}C_{B1} \square B \square B \sqrt{-g} - \frac{1}{2}C_{B2} R \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} . \end{aligned} \quad (12)$$

Moreover, the VEV's of the squares of the scalar fields require composite operator renormalization which at 1-loop and 2-loop orders implies the following counterterms,³

$$A_{\text{ren}}^2 = A^2 + K_{A1}R + K_{A2}R A^2 + K_{A3}R^2 + O(\lambda^4) , \quad (13)$$

$$B_{\text{ren}}^2 = B^2 + K_{B1}R + K_{B2}R B^2 + K_{B3}R^2 + O(\lambda^4) . \quad (14)$$

Finally, to renormalize the 3-point vertex of the theory at 1-loop order, the counterterm Lagrangian is,

$$\begin{aligned} \Delta\mathcal{L}_{ABB} = & -\frac{1}{2}C_{ABB1} \square A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} - \frac{1}{2}C_{ABB2} \partial_\mu A \partial_\nu B \square B g^{\mu\nu} \sqrt{-g} \\ & -\frac{1}{2}C_{ABB3} A \square B \square B \sqrt{-g} - \frac{1}{2}C_{ABB4} R A \partial_\mu B \partial_\nu B g^{\mu\nu} \sqrt{-g} . \end{aligned} \quad (15)$$

³No renormalizations are needed for the VEV's of $A(x)$ which is ultraviolet finite to this order and the VEV of $B(x)$ vanishes to all orders by virtue of the shift symmetry of (8).

In (15) the last term is the curvature-dependent coupling constant renormalization. It turns out that at 1-loop there is no such renormalization and the associated 1-loop β -function vanishes [27, 39],

$$\delta\lambda \equiv \lambda_{\text{ren}} - \lambda = C_{AB B^4} \times R + O(\lambda^5) = 0 \lambda^3 + O(\lambda^5) \Rightarrow \quad (16)$$

$$\beta \equiv \frac{\partial \delta\lambda}{\partial \ln(\mu)} = 0 \lambda^3 + O(\lambda^5) . \quad (17)$$

All the coefficients in (12-15) are determined so that they absorb the primitive divergences of the perturbative diagrams. The leading logarithm renormalized results are most conveniently presented in the form of Table 1 [27, 39]:

Quantity	Leading Logarithms
$u_A(\eta, k)$	$\left\{ 1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4) \right\} \times \frac{H}{\sqrt{2k^3}}$
$u_B(\eta, k)$	$\left\{ 1 + 0 + O(\lambda^4) \right\} \times \frac{H}{\sqrt{2k^3}}$
$P_A(\eta, r)$	$\left\{ 1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(a) + \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4) \right\} \times \frac{KH}{4\pi} \ln(Hr)$
$P_B(\eta, r)$	$\left\{ 1 - \frac{\lambda^2 H^2}{32\pi^2} \ln(Hr) + O(\lambda^4) \right\} \times \frac{KH}{4\pi} \ln(Hr)$
$\langle \Omega A(x) \Omega \rangle$	$\left\{ 1 + \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4) \right\} \times \frac{\lambda H^2}{16\pi^2} \ln(a)$
$\langle \Omega A^2(x) \Omega \rangle_{\text{ren}}$	$\left\{ 1 - \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4) \right\} \times \frac{H^2}{4\pi^2} \ln(a)$
$\langle \Omega B(x) \Omega \rangle$	0
$\langle \Omega B^2(x) \Omega \rangle_{\text{ren}}$	$\left\{ 1 + \frac{3\lambda^2 H^2}{32\pi^2} \ln(a) + O(\lambda^4) \right\} \times \frac{H^2}{4\pi^2} \ln(a)$

Table 1: Leading logarithm renormalized perturbative results for the AB model to leading logarithm. **Red** denotes leading logarithms explained by the stochastic formalism while **green** denotes those explained by the renormalization group.

** Re-summation Techniques: Stochastic*

For the infrared secular contributions due to the ever-increasing number of degrees of freedom with super-horizon wavelengths $k < a(t)H$, the stochastic method can re-sum the leading contributions and provide the late time evolution; this was explicitly displayed by Starobinsky and Yokoyama [20] for a scalar $\Phi(x)$ in de Sitter spacetime with a non-derivative self-interacting

potential $V(\Phi)$ and a static late time evolution limit. The basic idea is to replace $\Phi(x)$ with a stochastic field $\varphi(x)$ which commutes with itself $[\varphi(x), \varphi(x')] = 0$, and whose correlators are completely free of ultraviolet divergences. The stochastic field $\varphi(x)$ is constructed from the same free creation and annihilation operators that appear in $\Phi(x)$ in such a way that the two fields produce the same leading logarithms at each order in perturbation theory. The requirement to concentrate on these infrared modes, simplifies the equation of motion to a Langevin equation obeyed by the stochastic field $\varphi(x)$,⁴

$$\dot{\varphi}(x) - \dot{\varphi}_o(x) = -\frac{1}{3H}V'(\varphi) \ , \quad (18)$$

where φ_o is the stochastic jitter given by the infrared truncated free field mode sum emanating from $\Phi(x)$.

- The extension of the method to scalars with derivative self-interactions can be seen by noting, for instance in the equation of motion (9) for $A(x)$, that a constant $A(x) = \bar{A}$ field is a field strength renormalization of $B(x)$. It follows that we can quantify the effect of the undifferentiated $A(x)$ to all orders by simply integrating out the differentiated $B(x)$ fields in (9) for constant $A(x) = \bar{A}$. Taking the VEV of (9) gives the first order equation for the 1-loop effective potential,

$$-V'_{\text{eff}}(\bar{A}) a^D = -\frac{1}{2}\lambda \left(1 + \frac{1}{2}\lambda \bar{A}\right) a^{D-2} \left\langle \Omega \left| \partial^\mu B(x) \partial_\mu B(x) \right| \Omega \right\rangle_{\bar{A}} \ , \quad (19)$$

$$= + \frac{\frac{1}{2}\lambda(D-1)kH^2 a^D}{1 + \frac{1}{2}\lambda \bar{A}} \ , \quad (20)$$

which upon taking the unregulated $D = 4$ limit and integrating gives the 1-loop effective potential,

$$V_{\text{eff}}(A) = -\frac{3H^4}{8\pi^2} \ln \left| 1 + \frac{1}{2}\lambda A \right| \ . \quad (21)$$

By integrating the derivative interactions out of the field equations in the presence of a constant scalar background, we obtained a curvature-dependent effective potential V_{eff} for which the standard stochastic procedure applies: one merely replaces V in the non-derivative re-summation rule with V_{eff} .

⁴For the purposes of this analysis we have converted from conformal time η to co-moving time t .

- When the generic field equation (9) of $A(x)$ is restricted to de Sitter spacetime we get,⁵

$$-\frac{d}{dt}(a^3\dot{A}) - V'_{\text{eff}}(A)a^3 = 0 \quad \implies \quad \dot{A} = \frac{\lambda H^3}{16\pi^2} \frac{1}{1 + \frac{1}{2}\lambda A} . \quad (22)$$

Equation (22) can be solved exactly, and for the initial condition $A_{\text{in}} = 0$ the solution is,

$$A_{\text{cl}} = \frac{2}{\lambda} \left[\sqrt{1 + \frac{\lambda^2 H^2}{16\pi^2} \ln(a)} - 1 \right] . \quad (23)$$

The Langevin equation (18) for the associated stochastic field $\mathcal{A}(x)$,

$$\dot{\mathcal{A}} - \dot{\mathcal{A}}_o = \frac{\lambda H^3}{16\pi^2} \frac{1}{1 + \frac{1}{2}\lambda \mathcal{A}} , \quad (24)$$

introduces the stochastic jitter coming from the infrared truncated free field $\mathcal{A}_o$. The general solution consists of the $\mathbb{C}$ -number solution (23) plus a series in powers of $\mathcal{A}_o$ coming from iterating (24),

$$\mathcal{A} = \mathcal{A}_{\text{cl}} + \mathcal{A}_o - \frac{\lambda^2 H^3}{32\pi^2} \int_0^t dt' \mathcal{A}_o + \frac{\lambda^3 H^3}{64\pi^2} \int_0^t dt' \mathcal{A}_o^2 + O(\lambda^4) . \quad (25)$$

* *Re-summation Techniques: Renormalization Group*

For the leading logarithms coming from the ultraviolet sector, the standard QFT re-summation methodology of the renormalization group needs, as well, an extension due to the presence of a curved spacetime: counterterms can be regarded as curvature-dependent renormalizations of the bare parameters present in the original theory. Consider, for instance, the renormalization of the composite operator $A^2(x)$. Of the general set of counterterms (13), the $K_{A^2} R A^2$ is a field strength renormalization,

$$A^2 = \sqrt{Z_{A^2}} \times A_{\text{ren}}^2 \quad , \quad Z_{A^2} = 1 - 2K_{A^2} \times R + O(\lambda^4) , \quad (26)$$

which is curvature dependent.

⁵Since the scale factor varies much faster than the field during inflation, it is preferable to have derivatives act on the scale factor instead of the field. Thus, the single time derivative of the Hubble friction term dominates over the second time and space derivatives terms.

- The re-summation of the leading logarithms from the renormalization group is well known in QFT: the 1-loop approximation to the running coupling re-sums the leading logarithmic behaviour of all the loop diagrams.⁶ Recall that the N -point function $G_N(x_1, \dots, x_N; \lambda; \mu)$ at a scale μ can be expressed in terms of its value at a scale μ_0 via the running coupling constant $\bar{\lambda}(\mu)$,

$$G_N(x_1, \dots, x_N; \lambda; \mu) = G_N(x_1, \dots, x_N; \bar{\lambda}(\mu); \mu_0) \times \exp \left[-N \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(\bar{\lambda}(\mu')) \right], \quad (27)$$

where $\gamma(\bar{\lambda}(\mu))$ is the γ -function of the associated field.

Since for the AB model the 1-loop β -function vanishes, so that $\bar{\lambda}(\mu) = \lambda$, the leading $\ln(\mu)$ re-summation becomes a power law,

$$G_N(x_1, \dots, x_N; \lambda; \mu) = G_N(x_1, \dots, x_N; \lambda; \mu_0) \times \left[\frac{\mu_0}{\mu} \right]^{N \gamma(\lambda)}. \quad (28)$$

* *Agreement: Stochastic Analysis*

To substantiate that the above re-summations do indeed produce the leading logarithms, we must compare their predictions against the perturbative results of Table 1.

- Starting from the stochastic re-summation, we expand the stochastic solution (25) in powers of λ and take its VEV,⁷

$$\langle \Omega | \mathcal{A}(t, \vec{x}) | \Omega \rangle = \frac{\lambda H^2}{16\pi^2} \ln(a) \left\{ 1 + \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4) \right\}, \quad (29)$$

as well as the VEV of its square,

$$\langle \Omega | \mathcal{A}^2(t, \vec{x}) | \Omega \rangle = \frac{H^2}{4\pi^2} \ln(a) \left\{ 1 - \frac{\lambda^2 H^2}{64\pi^2} \ln(a) + O(\lambda^4) \right\}. \quad (30)$$

Comparing (29) and (30) with their perturbative counterparts $\langle \Omega | A(x) | \Omega \rangle$ and $\langle \Omega | A^2(x) | \Omega \rangle$ in Table 1 shows perfect agreement, a highly non-trivial

⁶Furthermore, the running coupling obtained by solving the renormalization group equations with the β -function approximated with its ℓ -loop expression re-sums not only the leading logarithm arising at any order, but also the first $\ell-1$ subleading logarithms.

⁷The infrared truncated free field $\mathcal{A}_o$ has the canonically commuting creation and annihilation operators of A so that: $\langle \Omega | \mathcal{A}_o(t, \vec{x}) | \Omega \rangle = 0$ and $\langle \Omega | \mathcal{A}_o^2(t, \vec{x}) | \Omega \rangle = \frac{H^2 \ln(a)}{4\pi^2}$.

achievement indeed.

** Agreement: Renormalization Group*

A similar non-trivial agreement is maintained when we consider the renormalization group analysis and its re-summation of the leading logarithms. For instance, the Callan-Symanzik equation for the composite operator $A^2(x)$ has the form [27],

$$\left[a \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \lambda} + \gamma_{A^2} \right] \langle \Omega | A^2(x) | \Omega \rangle_{\text{ren}} = 0 . \quad (31)$$

The β and γ functions are,

$$\beta \equiv \frac{\partial \delta \lambda}{\partial \ln(\mu)} = 0 + O(\lambda^5) \quad , \quad \gamma_{A^2} \equiv \frac{\partial \ln(Z_{A^2})}{\partial \ln(\mu^2)} = 0 + O(\lambda^4) . \quad (32)$$

Therefore, there should no leading logarithms coming from this analysis to this order (2-loop) and indeed the corresponding perturbative result seen in Table 1 for $\langle \Omega | A^2(x) | \Omega \rangle_{\text{ren}}$ confirm this; both leading logarithms are stochastic effects as described above.⁸

- The curvature-dependent field strength renormalization of the composite operator $B^2(x)$ is [27],

$$Z_{B^2} = 1 - 2K_{B^2} \times R + O(\lambda^4) \quad \implies \quad \gamma_{B^2} = -\frac{3\lambda^2 H^2}{16\pi^2} + O(\lambda^4 H^4) , \quad (33)$$

so that its associated Callan-Symanzik equation predicts the leading logarithm contribution to this order to be $+\frac{3\lambda^2 H^2}{16\pi^2}$ which is identical to what is seen in Table 1 for $\langle \Omega | B^2(x) | \Omega \rangle_{\text{ren}}$.

** Comments*

- It turns out that all entries in Table 1 are reproduced by one *or* the other of the re-summation techniques. Yet another example, is the λ^0 order term $\frac{H^2}{4\pi^2} \ln(a)$ that appears in the VEV's of Table 1: it comes from the coincidence limit of the scalar propagator and is a stochastic effect; it could not be explained by the renormalization group since it does not obey the Callan-Symanzik equation.

- Among the physical conclusions from the above analysis, we could highlight the behaviour of the scalar field $A(x)$. It provides an example of a field

⁸The same is true for the renormalized VEV of $A(x)$.

which, (i) rolls down its potential at a faster rate due to its stochastic jitter, (ii) develops a positive mass from a stochastically induced effective potential, and (iii) has a time evolution that does not approach a static limit but grows and persists to arbitrarily late times.

3 Constant Graviton Induced Stress Tensor

On $D = 4$ dimensional de Sitter the cosmological constant is $\Lambda = 3H^2$. The gravitational field equation associated with our model (7) is,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + 3H^2 g_{\mu\nu} = 8\pi G \left[\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2}g_{\mu\nu} g^{\rho\sigma} \right] \partial_\rho \phi \partial_\sigma \phi. \quad (34)$$

We seek to integrate out the scalar fields on the right hand side of (34). If this could be done for an arbitrary metric field $g_{\mu\nu}(x)$ it would give us the exact 1-loop effective field equation, however, this is not possible. What we can do instead is to integrate out the scalars for constant graviton field, which should suffice at leading logarithm order. As we shall see, this constant background is just de Sitter spacetime with a different cosmological constant.

- Consider the general class of conformally rescaled backgrounds with constant H and arbitrary $\tilde{g}_{\mu\nu}(x)$,

$$g_{\mu\nu}(x) \equiv a^2 \tilde{g}_{\mu\nu}(x) \equiv a^2 \left[\eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \right], \quad a = -\frac{H\eta}{\kappa}. \quad (35)$$

Here and henceforth $\kappa^2 \equiv 16\pi G$ is the loop-counting parameter of quantum gravity. For geometries (35) we find,⁹

$$\Gamma^\rho_{\mu\nu} = aH \left(\delta^\rho_\mu \delta^0_\nu + \delta^\rho_\nu \delta^0_\mu - \tilde{g}^{0\rho} \tilde{g}_{\mu\nu} \right) + \tilde{\Gamma}^\rho_{\mu\nu}, \quad (36)$$

$$\begin{aligned} R^\rho_{\sigma\mu\nu} = & -H^2 \tilde{g}^{00} \left(\delta^\rho_\mu g_{\sigma\nu} - \delta^\rho_\nu g_{\sigma\mu} \right) \\ & + aH \left[\delta^\rho_\mu \tilde{\Gamma}^0_{\nu\sigma} - \delta^\rho_\nu \tilde{\Gamma}^0_{\mu\sigma} + \tilde{g}^{\rho\alpha} \left(\tilde{\Gamma}^0_{\mu\alpha} \tilde{g}_{\nu\sigma} - \tilde{\Gamma}^0_{\nu\alpha} \tilde{g}_{\mu\sigma} \right) \right] + \tilde{R}^\rho_{\sigma\mu\nu}, \end{aligned} \quad (37)$$

$$R_{\mu\nu} = -3H^2 \tilde{g}^{00} g_{\mu\nu} + aH \left[2\tilde{\Gamma}^0_{\mu\nu} + \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \tilde{\Gamma}^0_{\alpha\beta} \right] + \tilde{R}_{\mu\nu}, \quad (38)$$

$$R = -12H^2 \tilde{g}^{00} + 6\frac{H}{a} \tilde{g}^{\alpha\beta} \tilde{\Gamma}^0_{\alpha\beta} + \frac{1}{a^2} \tilde{R}. \quad (39)$$

⁹We define: $\tilde{\Gamma}^\rho_{\mu\nu} \equiv \frac{1}{2}\tilde{g}^{\rho\sigma} \left(\tilde{g}_{\sigma\mu,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma} \right)$.

Thus, the left hand side of (34) becomes,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + 3H^2 g_{\mu\nu} = 3H^2 \left(1 + \tilde{g}^{00}\right) g_{\mu\nu} + 2aH \left[\tilde{\Gamma}^0_{\mu\nu} - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \tilde{\Gamma}^0_{\alpha\beta} \right] + \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} . \quad (40)$$

- When we focus on geometries with $h_{\mu\nu}(x)$ constant (which implies constant $\tilde{g}_{\mu\nu}(x)$), so that only terms without derivatives acting on the field are selected, the curvature tensor (37) simplifies,

$$R^\rho_{\sigma\mu\nu} \Big|_{\tilde{g}_{\mu\nu}=c} = -H^2 \tilde{g}^{00} \left(\delta^\rho_\mu g_{\sigma\nu} - \delta^\rho_\nu g_{\sigma\mu} \right) , \quad (41)$$

and we recognize a de Sitter geometry albeit with a different cosmological constant,

$$\tilde{g}_{\mu\nu,\rho} = 0 \implies H^2 \longrightarrow -\tilde{g}^{00} H^2 . \quad (42)$$

- To integrate out the differentiated scalars present in the right hand side of (34),

$$\partial_\mu \phi(x) \partial_\nu \phi(x) \implies \left\langle \Omega \left| \partial_\mu \phi(x) \partial_\nu \phi(x) \right| \Omega \right\rangle , \quad (43)$$

the above expectation value is to be evaluated in the scalar vacuum for the background (35) with $\tilde{g}_{\mu\nu}(x)$ a spacetime constant.

What is needed for the computation of (43) is the coincidence limit of the mixed derivative of the scalar propagator in de Sitter spacetime, a quantity that has been evaluated using dimensional regularization (and the old Hubble parameter H) [6, 16],

$$\begin{aligned} \lim_{x' \rightarrow x} \partial_\mu \partial'_\nu i\Delta_{dS}(x; x') &= -\frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{2\Gamma(\frac{D}{2}+1)} \times g_{\mu\nu}(x) \Big|_{\tilde{g}_{\mu\nu}=c} \\ &\longrightarrow -\frac{3H^4}{32\pi^2} \times g_{\mu\nu}(x) \Big|_{\tilde{g}_{\mu\nu}=c} . \end{aligned} \quad (44)$$

The finite $D=4$ remainder in (44) emerges after the automatic subtraction of the quartic and quadratic divergences of the doubly differentiated propagator.

By effecting the replacement (42) in the evaluation (44), the right hand side (43) equals,

$$\left\langle \Omega \left| \partial_\mu \phi(x) \partial_\nu \phi(x) \right| \Omega \right\rangle_{g_{\mu\nu}=a^2\tilde{g}_{\mu\nu}} = -\frac{3[-\tilde{g}^{00}H^2]^2}{32\pi^2} \times g_{\mu\nu}(x) , \quad (45)$$

so that,

$$8\pi G \left\langle \Omega \left| \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right| \Omega \right\rangle_{g_{\mu\nu}=a^2 \tilde{g}_{\mu\nu}} = \frac{3\kappa^2 [-\tilde{g}^{00} H^2]^2}{64\pi^2} \times g_{\mu\nu}(x). \quad (46)$$

Note that (46) represents a negative contribution to the cosmological constant which can become arbitrarily large if $-\tilde{g}^{00}$ achieves arbitrarily large values.

- Substituting (40) and (46) into the gravitational field equations (34) we arrive at the leading logarithm form,

$$\begin{aligned} \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + 2aH \left[\tilde{\Gamma}^0_{\mu\nu} - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \tilde{\Gamma}^0_{\alpha\beta} \right] \\ = -3H^2 \left[1 + \tilde{g}^{00} - \frac{\kappa^2 H^2}{64\pi^2} (\tilde{g}^{00})^2 \right] g_{\mu\nu}, \end{aligned} \quad (47)$$

displaying the scalar-induced gravitational stress-energy tensor $T_{\mu\nu}^{\text{ind}}$,

$$T_{\mu\nu}^{\text{ind}} = \frac{3}{32\pi^2} \left[-\tilde{g}^{00} H^2 \right]^2 g_{\mu\nu}, \quad (48)$$

always with the understanding that $\tilde{g}_{\mu\nu}(x) = \text{constant}$.

For $\tilde{g}^{00} = -1$ the induced stress tensor (48) takes the form of a negative addition to the cosmological constant of,

$$\Delta\Lambda = -\frac{3\kappa^2 H^4}{64\pi^2}. \quad (49)$$

If we want the quantity “ H ” to represent the true Hubble parameter, at least initially, (49) must be absorbed by making a finite renormalization of the cosmological constant. This is precisely the finite renormalization which was previously recognized as being necessary to make the scalar loop contribution to the graviton self-energy conserved — see equations (33) and (38) of [34].

4 Generally Conserved Extension

The induced stress tensor (48) is not conserved when $\tilde{g}_{\mu\nu}$ is allowed to depend on space and time. It is important to understand that this is not an error but rather the inevitable consequence of working at leading logarithm order.

Consider, for example, the $\lambda\phi^4$ energy density and pressure (2-3). Under the symmetries of homogeneity and isotropy conservation reduces to $\dot{\rho} = -3H(\rho + p)$, and relations (2-3) do obey this,

$$\dot{\rho} = \frac{\lambda H^4}{2^7 \pi^2} \times 2H \ln(a) + O(\lambda^2), \quad (50)$$

$$-3H(\rho + p) = -3H \left\{ \frac{\lambda H^4}{2^7 \pi^2} \left[+ \ln^2(a) - \ln^2(a) - \frac{2}{3} \ln(a) \right] + O(\lambda^2) \right\}. \quad (51)$$

However, the right hand side (51) only agrees with the left (50) by virtue of the sub-leading factor of $-\frac{2}{3} \ln(a)$ in the pressure (3). With just the $\pm \ln^2(a)$ leading logarithm contributions, conservation is violated.

Although the violation of conservation of the induced stress tensor (48) has a good explanation, it does present us with a problem. The gravitational field equations (34) consist of 10 relations, for a general metric, of which 4 are automatically implied by conservation. If one adds the non-conserved stress tensor (48) this will no longer be true. All 10 of the equations cannot be used, because the 0th order equations are still conserved. So which 6 of the 10 equations should we solve, and why? Note that this sort of issue could not arise with the very similar effective potentials which are induced in non-linear sigma models; it is specific to gravity and it will have to be confronted as well when graviton loops are considered.

We believe that the answer is to extend the leading logarithm stress tensor (48) to a generally conserved form. The variation of any invariant would give such a conserved form, and it is worthwhile first considering the two 1-loop counterterms,

$$\Delta\mathcal{L} = c_1 R^2 \sqrt{-g} + c_2 C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \sqrt{-g}. \quad (52)$$

The variation of the Weyl counterterm will still contain a Weyl tensor, which vanishes for de Sitter, so this counterterm has no impact. The variation of the R^2 counterterm contributes to the stress tensor as,

$$\Delta T_{\mu\nu}^1 = -\frac{2}{\sqrt{-g}} \frac{\delta \Delta S_1}{\delta g^{\mu\nu}} = c_1 \left[g_{\mu\nu} R^2 - 4 \left(R_{\mu\nu} + g_{\mu\nu} \square - D_\mu D_\nu \right) R \right]. \quad (53)$$

For D -dimensional de Sitter with arbitrary Hubble parameter $\overline{H}$, the Ricci tensor is $R_{\mu\nu} = (D-1)\overline{H}^2 g_{\mu\nu}$. The covariant derivatives all vanish so we have,

$$g_{\mu\nu} R^2 - 4 \left(R_{\mu\nu} + g_{\mu\nu} \square - D_\mu D_\nu \right) R = D(D-1)^2 (D-4) \overline{H}^4 g_{\mu\nu}. \quad (54)$$

The factor of $(D - 4)$ means that only the divergent part of the counterterm c_1 can make a nonzero contribution to expression (53),

$$c_1 \longrightarrow \frac{\mu^{D-4}}{D-4} \times \frac{1}{2^7 \cdot 3^3 \cdot \pi^2} \quad (55)$$

The result does not quite agree with (48),

$$\Delta T_{\mu\nu}^1 = \frac{\overline{H}^4}{32\pi^2} g_{\mu\nu} . \quad (56)$$

Although the R^2 counterterm induces a stress tensor (56) which fails to agree with the leading logarithm result (48), it is close enough that we are motivated to consider more general functions of the Ricci scalar. Expressions (38) and (39) for the Ricci tensor and scalar suggest that we could regard $[-\tilde{g}^{00}H^2]$ as $\frac{R}{12}$ and try to extend the stress-energy tensor (48) so that it emerges from a more general Lagrangian,

$$\mathcal{L}_T = -f(R)\sqrt{-g} , \quad (57)$$

which gives the conserved source,

$$-\frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_T[g]}{\delta g^{\mu\nu}(x)} = -\frac{1}{2}\kappa^2 g_{\mu\nu} f(R) + \kappa^2 [R_{\mu\nu} + g_{\mu\nu} \square - D_\mu D_\nu] f'(R) , \quad (58)$$

where $\square \equiv \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu]$ is the scalar covariant d'Alembertian and D_μ is the covariant derivative operator. By substituting (38) and (39) into (58), by discarding the sub-leading terms involving $\tilde{\Gamma}_{\alpha\beta}^0$ and $\tilde{R}$, and by defining $X \equiv -12\tilde{g}^{00}H^2$, equation (58) reduces to a 1st order differential equation for the function $f(X)$,

$$\frac{X^2}{2^9 \cdot 3 \cdot \pi^2} = -f(X) + \frac{1}{2}X f'(X) = \frac{X^3}{2} \left(\frac{f(X)}{X^2} \right)' , \quad (59)$$

which has the solution,

$$f(X) = \frac{X^2 \ln(\frac{X}{12H^2})}{2^8 \cdot 3 \cdot \pi^2} , \quad (60)$$

whose 1st and 2nd derivatives equal,

$$f'(X) = \frac{2X \ln(\frac{X}{12H^2}) + X}{2^8 \cdot 3 \cdot \pi^2} , \quad f''(X) = \frac{2 \ln(\frac{X}{12H^2}) + 3}{2^8 \cdot 3 \cdot \pi^2} . \quad (61)$$

- Furthermore, the two derivative terms present in (58) are,

$$g_{\mu\nu}\square = \tilde{g}_{\mu\nu}\left[\tilde{g}^{\rho\sigma}\partial_\rho\partial_\sigma + 2aH\tilde{g}^{0\rho}\partial_\rho - \tilde{g}^{\rho\sigma}\tilde{\Gamma}^\alpha_{\rho\sigma}\partial_\alpha\right], \quad (62)$$

$$-D_\mu D_\nu = -\partial_\mu\partial_\nu + aH\left[\delta^0_\mu\partial_\nu + \delta^0_\nu\partial_\mu - \tilde{g}_{\mu\nu}\tilde{g}^{0\rho}\partial_\rho\right] + \tilde{\Gamma}^\rho_{\mu\nu}\partial_\rho. \quad (63)$$

It will turn out to be rather useful to extract the metric from the Ricci tensor (38),

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{4}g_{\mu\nu}R + 2aH\left[\tilde{\Gamma}^0_{\mu\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\tilde{\Gamma}^0_{\alpha\beta}\right] + \left[\tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{R}\right] \\ &\equiv \frac{1}{4}g_{\mu\nu}R + \Delta R_{\mu\nu}, \end{aligned} \quad (64)$$

and to define the dimensionless Ricci scalar $\mathcal{R}$ from (39),

$$R = 12H^2\left[-\tilde{g}^{00} + \frac{\tilde{g}^{\alpha\beta}\tilde{\Gamma}^0_{\alpha\beta}}{2aH} + \frac{\tilde{R}}{12a^2H^2}\right] \equiv 12H^2 \times \mathcal{R}. \quad (65)$$

In view of (64), (65) and (60), (61) the three terms on the right hand side of (58) become,

$$-\frac{1}{2}\kappa^2 g_{\mu\nu} f(R) = -6\epsilon H^2 \mathcal{R}^2 \ln(\mathcal{R}) g_{\mu\nu}, \quad (66)$$

$$\kappa^2 R_{\mu\nu} f'(R) = 3\epsilon H^2 \mathcal{R}^2 \left[2\ln(\mathcal{R}) + 1\right] g_{\mu\nu} + \epsilon \mathcal{R} \left[2\ln(\mathcal{R}) + 1\right] \Delta R_{\mu\nu}, \quad (67)$$

$$\kappa^2 \left[g_{\mu\nu}\square - D_\mu D_\nu\right] f'(R) = \epsilon \left[g_{\mu\nu}\square - D_\mu D_\nu\right] \left[2\mathcal{R} \ln(\mathcal{R}) + \mathcal{R}\right], \quad (68)$$

where $\epsilon \equiv \frac{\kappa^2 H^2}{64\pi^2}$.

Summing (66-68) gives,

$$\begin{aligned} \frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_T[g]}{\delta g^{\mu\nu}(x)} &= 3\epsilon H^2 \mathcal{R}^2 g_{\mu\nu} + \epsilon \mathcal{R} \left[2\ln(\mathcal{R}) + 1\right] \Delta R_{\mu\nu} \\ &\quad + \epsilon \left[g_{\mu\nu}\square - D_\mu D_\nu\right] \left[2\mathcal{R} \ln(\mathcal{R}) + \mathcal{R}\right]. \end{aligned} \quad (69)$$

The spacetime geometries of cosmological interest are homogeneous,

$$\tilde{g}_{00} = -\frac{1}{x(t)} \quad , \quad \tilde{g}_{0i} = 0 \quad , \quad \tilde{g}_{ij} = \delta_{ij}. \quad (70)$$

The quantities of interest have the following form,

$$\tilde{\Gamma}_{00}^0 = -\frac{a\dot{x}}{2x} \quad \Longrightarrow \quad \tilde{g}^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}^0 = \frac{1}{2}a\dot{x} , \quad (71)$$

$$\tilde{R}_{\mu\nu} = 0 \quad , \quad \Delta R_{\mu\nu} = -a^2 H \dot{x} \left[\frac{1}{x} \delta^0_\mu \delta^0_\nu + \frac{1}{4} \tilde{g}_{\mu\nu} \right] , \quad (72)$$

$$\mathcal{R} = x + \frac{\dot{x}}{4H} , \quad (73)$$

so that the relevant derivative operators become,

$$g_{\mu\nu} \square - D_\mu D_\nu = -g_{\mu\nu} \left[x \partial_t + 2xH + \frac{1}{2} \dot{x} \right] \partial_t - \frac{a^2}{x} \delta^0_\mu \delta^0_\nu \left[x \partial_t - xH + \frac{1}{2} \dot{x} \right] \partial_t . \quad (74)$$

Consequently, the two non-zero components in this background are,

$$\frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_T[g]}{\delta g^{00}(x)} = \epsilon g_{00} \times A , \quad (75)$$

$$A \equiv 3H^2 \mathcal{R}^2 + 3H \left(\frac{1}{4} \dot{x} - x \partial_t \right) \left[2\mathcal{R} \ln(\mathcal{R}) + \mathcal{R} \right] . \quad (76)$$

$$\frac{\kappa^2}{\sqrt{-g}} \frac{\delta S_T[g]}{\delta g^{ij}(x)} = \epsilon g_{ij} \times B , \quad (77)$$

$$B \equiv 3H^2 \mathcal{R}^2 - \left(\frac{1}{4} H \dot{x} + 2Hx \partial_t + \frac{1}{2} \dot{x} \partial_t + x \partial_t^2 \right) \left[2\mathcal{R} \ln(\mathcal{R}) + \mathcal{R} \right] , \quad (78)$$

and the conservation equation is satisfied,

$$\dot{A} = 3H(-A + B) . \quad (79)$$

Moreover, the extended source (69) has the proper correspondence limit (48) because when $\tilde{g}_{\mu\nu}$ is constant only the first term in the right hand side of (69) survives. It should also be noted that the extension we analyzed is by no means unique.

5 Renormalization Group Analysis

Analysis of the exact 1-loop effective field equations produced three results [35]. First, the electric component of the Weyl tensor for plane wave gravitational radiation ($h_{ij}(t, \vec{x}) = \epsilon_{ij}(\vec{k}, \lambda) u(t, k) e^{i\vec{k} \cdot \vec{x}}$) receives a logarithmic correction,

$$C_{0i0j}(t, \vec{x}) = C_{0i0j}^{(0)}(t, \vec{x}) \left\{ 1 - \frac{3GH^2}{10\pi} \times \ln(a) + O(G^2) \right\} . \quad (80)$$

A similar logarithmic correction affects the gravitational potentials,

$$ds^2 = -\left[1 - 2\Psi(t, r)\right]dt^2 + a^2(t)\left[1 - 2\Phi(t, r)\right]d\vec{x}\cdot d\vec{x}, \quad (81)$$

generated in response to a static point mass M ,

$$\Psi(t, r) = \frac{GM}{ar} \left\{ 1 + \frac{G}{20\pi a^2 r^2} - \frac{3GH^2}{10\pi} \times \ln(aHr) + O(G^2) \right\}, \quad (82)$$

$$\Phi(t, r) = -\frac{GM}{ar} \left\{ 1 - \frac{G}{60\pi a^2 r^2} - \frac{3GH^2}{10\pi} \times [\ln(aHr) + 1] + O(G^2) \right\}. \quad (83)$$

The factors of $\frac{G}{a^2 r^2}$ in (82-83) represent de Sitter versions of corrections which have long been known on flat space background [50–52]. They are not leading logarithm contributions and have no interest for us.

The three leading logarithm corrections (80) and (82-83) are not explained by the induced stress tensor we considered in the previous section. They closely resemble the logarithmic corrections previously found to the electric components of the field strength tensor for plane wave electromagnetic radiation [13], and to the Coulomb potential for a static point charge [12]. Those two results can be explained by a variant of the renormalization group [53], and it turns out that a similar explanation works for the gravitational logarithms (80) and (82-83).

The model (7) was the first quantum gravitational system ever studied using dimensional regularization [54]. It has long been known that this model requires two counterterms at 1-loop order,

$$\Delta\mathcal{L} = c_1 R^2 \sqrt{-g} + c_2 C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \sqrt{-g}. \quad (84)$$

The coefficients c_1 and c_2 which were used to derive the 1-loop results (80) and (82-83) are [30, 35],

$$c_1 = \frac{\mu^{D-4} \Gamma(\frac{D}{2})}{2^8 \pi^{\frac{D}{2}}} \frac{(D-2)}{(D-1)^2 (D-3)(D-4)}, \quad (85)$$

$$c_2 = \frac{\mu^{D-4} \Gamma(\frac{D}{2})}{2^8 \pi^{\frac{D}{2}}} \frac{2}{(D+1)(D-1)(D-3)^2 (D-4)}, \quad (86)$$

where μ is the scale of dimensional regularization.

Like the non-linear sigma models considered in Section 3, our model (7) is nonrenormalizable. Recall from Section 3 that it was necessary to sort the resulting BPHZ (Bogoliubov, Parasiuk [55], Hepp [56] and Zimmermann [57, 58])

counterterms into those which can be viewed as renormalizing parameters of the bare theory and those that provide irrelevant higher derivative contributions which do not induce large logarithms. Experience with the large electromagnetic logarithms induced by graviton loops [53] shows that this sometimes requires dissecting the original counterterms. For example, renormalizing the 1-loop graviton contribution to the vacuum polarization on de Sitter background requires three counterterms [59, 60],¹⁰

$$\begin{aligned} \Delta\mathcal{L} = & \Delta C H^2 F_{ij} F_{kl} g^{ik} g^{jl} \sqrt{-g} + \overline{C} H^2 F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} \\ & + C_4 D_\alpha F_{\mu\nu} D_\beta F_{\rho\sigma} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} . \end{aligned} \quad (87)$$

However, the curvature-dependent field strength renormalization which explains the large electromagnetic logarithms is [53],

$$\delta Z = -4 \left[\overline{C} - (3D-8)C_4 \right] H^2 . \quad (88)$$

The analogous decomposition of the gravitational counterterms (84) begins by writing the Eddington (R^2) counterterm as the sum of three terms,

$$R^2 = \left(R - D\Lambda \right)^2 + 2D\Lambda \left[R - (D-2)\Lambda \right] + D(D-4)\Lambda^2 . \quad (89)$$

(i) The first term in (89) involves products of second derivatives of the graviton field, and explicit computation shows that it engenders no large logarithms [35].

(ii) However, the second term can be viewed either as a renormalization of Newton's constant in the original model (7), or else as a renormalization of the graviton field strength. Because we have no information about higher order couplings it is convenient to adopt the latter interpretation.

(iii) The third term is finite.

The interpretation adopted for the second term in (89) applies as well to the special term arising, at quadratic order, from the term proportional to $\partial_0^2 h_{\nu\sigma}$ in the Weyl counterterm of (84),

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = -\frac{2\kappa}{a^4} \partial_\mu \partial_\rho h_{\nu\sigma} \tilde{C}^{\mu\nu\rho\sigma} + O(\kappa^3 h^3) \longrightarrow -\frac{2\kappa}{a^4} \partial_0^2 h_{ij} \tilde{C}_{0i0j} , \quad (90)$$

where $\tilde{C}^{\mu\nu\rho\sigma}$ is the conformally rescaled Weyl tensor. Direct calculation [35] shows that the two time derivatives can act on scale factors to eventually contribute to the large logarithms (80) and (82-83).

¹⁰The non-covariant ΔC counterterm arises from the unavoidable breaking of de Sitter invariance in the graviton propagator.

Hence the total field strength renormalization is,

$$\delta Z = D \left[2(D-1)c_1 - c_2 \right] \kappa^2 H^2 \implies \gamma \equiv \frac{\partial \ln(1+\delta Z)}{\partial \ln(\mu^2)} = \frac{3GH^2}{20\pi} . \quad (91)$$

The Callan-Symanzik equation for a 2-point Green's function reads,

$$\left[\frac{\partial}{\partial \ln(\mu)} + \beta_G \frac{\partial}{\partial G} + 2\gamma \right] G^{(2)} = 0 . \quad (92)$$

The β -function for G vanishes at the order we are working. From expression (6) we see that factors of the renormalization scale μ come in the form $\ln(\mu a)$, so that we can replace the derivative with respect to $\ln(\mu)$ by a derivative with respect to $\ln(a)$. The Weyl tensor (80) can be considered as the 2-point correlator between the operator C_{0i0j} and the single particle creation operator; similarly, the potentials (82-83) can be considered as 2-point correlators. It follows that the three large logarithms in (80) and (82-83) can all be explained using the renormalization group. If we ignore possible running of Newton's constant it is even possible to sum up the effects to conclude,

$$C_{0i0j}(t, \vec{x}) \longrightarrow C_{0i0j}^{(0)}(t, \vec{x}) \times \left[a(t) \right]^{-\frac{3GH^2}{10\pi}} , \quad (93)$$

$$\Psi(t, r) \longrightarrow \frac{GM}{a(t)r} \times \left[a(t)Hr \right]^{-\frac{3GH^2}{10\pi}} , \quad (94)$$

$$\Phi(t, r) \longrightarrow -\frac{GM}{a(t)r} \times \left[a(t)Hr \right]^{-\frac{3GH^2}{10\pi}} . \quad (95)$$

6 Epilogue

The continuous production of MMC scalars and gravitons during inflation causes loop corrections which would be constant in flat space to acquire secular growth factors in the form of powers of the logarithm of the scale factor [6–15]. Over a prolonged period of inflation these factors must overwhelm even the smallest coupling constant. At this point perturbation theory breaks down and one must employ some non-perturbative re-summation. Starobinsky's stochastic formalism [18, 20] sums the series of leading logarithms of scalar potential models (4) but it must be extended to correctly describe the leading logarithms of more general models. In these models one must

distinguish between Active fields which engender secular growth factors and Passive fields which do not. One form of extending Starobinsky's formalism is to integrate out Passive fields in the presence of a spacetime constant Active field background. The result is an effective potential which can then be treated using Starobinsky's formalism. With the work done in this paper we now know of three ways in which these effective potentials can be induced:

- Through a constant Active field giving rise to a Passive field mass, as in Yukawa [8];
- Through a constant Active field changing a Passive field strength, as in nonlinear sigma models [27]; and
- Through a constant Active field changing the Hubble parameter on which a Passive field propagator depends, as occurs in our model (7).

In Section 3 we considered an MMC scalar coupled to gravity (7) and integrated the scalar out of the gravitational field equation in the presence of a constant graviton background. It turns out that this is equivalent to merely changing the de Sitter Hubble constant according to the rule (42). The resulting stress tensor (48) precisely explains the finite renormalization of the cosmological constant which was previously noted as being necessary to make the graviton self-energy conserved [34].

In Section 4 we noted that the induced stress tensor (48) is not conserved when one goes beyond leading logarithm order to permit to graviton field to vary in space and time. This is not an error, but rather the inevitable consequence of working at leading logarithm order. However, it does pose a problem in solving the gravitational field equations. We showed that (48) can be extended to a form (69) which is generally conserved.

For theories which possess derivative interactions, such as non-linear sigma models and gravity, a second extension must be made of Starobinsky's formalism to include the large logarithms induced by the incomplete cancellation between primitive divergences and counterterms (6). These logarithms are described by a variant of the renormalization group in which some portion of the BPHZ counterterms can be regarded as renormalizing couplings or field strengths of the original theory. That procedure was described in Section 2 for non-linear sigma models, and we applied it in Section 5 to the large logarithms (80) and (82-83) which occur in our model (7). The two BPHZ counterterms (84) can be regarded as giving rise to a graviton

field strength renormalization (91) which explains all three of the large logarithms. If we ignore possible running of Newton's constant, it is even possible to give fully resummed results (93-95). We remark that it should be possible to determine any running of G by computing the 1PI 3-point function, the same as has recently been done for non-linear sigma models [39].

The point of making this study was to facilitate the development of a leading logarithm re-summation technique for graviton loops, so it is worth commenting on what will generalize and what may differ. First, the procedure of integrating out differentiated graviton fields for constant $\tilde{g}_{\mu\nu}$ should involve the same replacement (42) as for our scalar model (7). However, the more complicated form of the graviton propagator [61, 62] makes it likely that the induced stress tensor will contain a term proportional to $\delta_\mu^0 \delta_\nu^0$ in addition to one proportional to $g_{\mu\nu}$. Like the scalar-induced stress tensor (48), it is inevitable that the graviton-induced stress tensor will not be conserved when one permits the graviton field to depend on space and time. We plan to seek a fully conserved extension, but we do not expect that a local one such as (69) can be found. We also expect that 1-loop corrections to the graviton mode function and to the response to a point mass will involve stochastic fluctuations in these extended, effective field equations, in addition to renormalization group effects. Finally, we expect the need for more counterterms than just the two covariant possibilities (84), owing to the same de Sitter breaking which produced the noncovariant ΔC counterterm (87) for electromagnetism.

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