



Hierarchically hyperbolic groups and uniform exponential growth

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Abstract

We give several sufficient conditions for uniform exponential growth in the setting of virtually torsion-free hierarchically hyperbolic groups. For example, any hierarchically hyperbolic group that is also acylindrically hyperbolic has uniform exponential growth. In addition, we provide a quasi-isometric characterization of hierarchically hyperbolic groups without uniform exponential growth. To achieve this, we gain new insights on the structure of certain classes of hierarchically hyperbolic groups. Our methods give a new unified proof of uniform exponential growth for several examples of groups with notions of non-positive curvature. In particular, we obtain the first proof of uniform exponential growth for certain groups that act geometrically on CAT(0) cubical spaces of dimension 3 or more. Under additional hypotheses, we show that a quantitative Tits alternative holds for hierarchically hyperbolic groups.

1 Introduction

A finitely generated group has (*uniform*) *exponential growth* if the number of elements that can be spelled with words of bounded length grows (uniformly) exponentially fast with respect to *any* finite generating set. Exponential growth rates and uniform exponential growth rates

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are of interest in a broad range of areas, including differential geometry, dynamical system theory, and the theory of unitary representations (see [28] and citations therein).

Gromov asked if every finitely generated group with exponential growth has uniform exponential growth. However, this is not the case: the first example of a group with exponential growth but not uniform exponential growth was constructed by Wilson [73], and additional counterexamples have since been constructed [5, 60, 72]. However, Gromov's question is still open for finitely presented groups.

Many classes of groups are known to either be virtually nilpotent or have uniform exponential growth. This form of growth gap was shown for linear groups by Eskin, Mozes, and Oh [31]; for hyperbolic groups by Koubi [53]; for fundamental groups of manifolds with pinched negative curvature by Besson, Coutois, and Gallot [7]; for finitely generated subgroups of the mapping class group by Mangahas [55] and, more generally, automorphism groups of one-ended hyperbolic groups by Kropholler, Lyman, and Ng [51]; for finitely generated special groups with trivial center by Kerr [49]; for linearly growing subgroups of $\text{Out}(F_n)$ by Bering [10]; and for groups acting without global fixed points on 2-dimensional $\text{CAT}(0)$ cube complexes, with some generalizations to higher dimensions, by work of Gupta, Jankiewicz, and Ng [36]. We note that the full mapping class group was known to have uniform exponential growth (via its action on homology) by Anderson, Aramayona, and Shackleton [1], and the torsion-free case of 2-dimensional cubical groups was shown by Kar and Sageev [54].

In the case of torsion-free cubical groups, it is remarkable that all known proofs rely heavily on the assumption that the cube complex has low dimension or that it has isolated flats, a strong form of relative hyperbolicity. Indeed, the authors are not aware of any general proof that works in dimensions higher than 2. Moreover, some of the more curious cubical groups do not act geometrically on $\text{CAT}(0)$ cube complexes with isolated flats. For example, the genus 2 handlebody group acts geometrically on a $\text{CAT}(0)$ cube complex [43], but Dehn twist flats can intersect along infinite subgroups, so this cube complex cannot have isolated flats.

Our main contribution, a combination of the main body of the paper and the appendix, is the following.

Theorem 1.1 *Let G be a group virtually acting freely and cocompactly on a locally finite, finite-dimensional $\text{CAT}(0)$ cube complex X , and assume that X has a factor system. Then exactly one of the following holds: G has uniform exponential growth or G is virtually abelian.*

Any group acting geometrically on a $\text{CAT}(0)$ cube complex with a factor system is a *hierarchically hyperbolic group* [44, Theorem A]. This is a starting point for results such as Theorem 1.1 and, for example, an alternative proof of rank-rigidity for $\text{CAT}(0)$ cube complexes with a factor system by Durham, Hagen, and Sisto [25, Corollary 9.24], a result originally proved by Caprace and Sageev [21]. Beyond hierarchical hyperbolicity, factor systems also appear in work of Incerti-Medici and Zalloum [47] and implicitly in work of Genevois [34]. Many cube complexes admitting a cocompact group action have factor systems; Hagen and Susse provide several sufficient conditions [44]. However, recently Shepherd constructed the first example of a cube complex without a factor system but admitting a proper cocompact group action [67]. It is unknown whether this group admits a different cubulation with a factor system, or whether it is hierarchically hyperbolic.

The main technical result of this paper is an analogue of Theorem 1.1 for the larger class of hierarchically hyperbolic groups. This is a broad class of groups introduced by Behrstock, Hagen, and Sisto [13] whose structure is similar to that of mapping class groups and $\text{CAT}(0)$ cubical groups. This class of groups includes hyperbolic groups, mapping class groups, many

CAT(0) cubical groups, fundamental groups of most 3-manifolds, and various combinations of the above groups, including direct products, certain quotients, and graph products [13, 17].

Hierarchically hyperbolic groups and, more generally, hierarchically hyperbolic spaces are defined axiomatically, generalizing the Masur–Minsky hierarchy machinery for mapping class groups [58]. Roughly speaking, a *hierarchically hyperbolic space* (HHS) consists of a metric space \mathcal{X} along with the following data: an index set \mathfrak{S} of *domains* with three relations (nesting, transversality, and orthogonality), δ -hyperbolic spaces \mathcal{CU} associated to each domain $U \in \mathfrak{S}$, and projection maps $\mathcal{X} \rightarrow \mathcal{CU}$ and $\mathcal{CU} \rightarrow \mathcal{CV}$ (defined for certain $U, V \in \mathfrak{S}$) satisfying certain conditions. We denote this entire package of information by $(\mathcal{X}, \mathfrak{S})$. In some sense, this set of hyperbolic spaces can be thought of as a set of coordinate spaces. We are used to understanding the space \mathbb{R}^n by associating to a point a n -tuple of elements of \mathbb{R} , which is a hyperbolic space. A simplistic but useful viewpoint on hierarchically hyperbolic space is to think of the hierarchical structure as nothing but a more complicated coordinate system on the metric space \mathcal{X} . We discuss this in more detail in Sect. 2.5. A *hierarchically hyperbolic group* (HHG) is essentially a group whose Cayley graph is an HHS such that the action of the group on the Cayley graph is compatible with the HHS structure; we use (G, \mathfrak{S}) to denote a HHG.

The following is a structure theorem for virtually torsion-free hierarchically hyperbolic groups providing a sufficient condition for uniform exponential growth.

Theorem 1.2 *Let (G, \mathfrak{S}) be an infinite virtually torsion-free hierarchically hyperbolic group. Then either G has uniform exponential growth, or there is a space E such that the Cayley graph of G is quasi-isometric to $\mathbb{Z} \times E$.*

We note that the two possible outcomes in the theorem are not mutually exclusive: a simple example is given by the group $\mathbb{Z} \times F_2$, where F_2 is a free group of rank two. Such a group is clearly a product of the form $\mathbb{Z} \times E$, but it has uniform exponential growth because it surjects onto F_2 .

Remark 1.3 The virtually torsion-free hypothesis can be replaced by the weaker condition that every generating set admits an infinite order element with uniformly short word length (see Theorem 4.1).

When G is a group virtually acting freely and cocompactly on a locally finite, finite-dimensional CAT(0) cube complex with a factor system and G is not directly decomposable, then Theorem 1.1 follows immediately from Theorem 1.2. In the case that G is directly decomposable, Theorem 1.1 follows from applying Theorem 1.2 and Theorem A.1 from the appendix.

1.1 HHG with uniform exponential growth

The first consequence of Theorem 1.2 is that if the Cayley graph of a hierarchically hyperbolic group G is not quasi-isometric to a (nontrivial) product, then G has uniform exponential growth. We state several corollaries giving conditions under which this is the case. There is significant overlap in the situations covered by these corollaries; our goal is simply to highlight a wide variety of conditions that imply uniform exponential growth.

A subset Y of a metric space X is *Morse* if every (λ, ε) -quasi-geodesic in X with endpoints on Y is contained in a uniform neighborhood (depending on λ, ε) of the subset Y .

Corollary 5.1 *Every non-virtually cyclic virtually torsion-free hierarchically hyperbolic group which has an asymptotic cone containing a cut-point has uniform exponential growth. In particular, if the Cayley graph of a virtually torsion-free hierarchically hyperbolic group G contains an unbounded Morse quasi-geodesic, then G has uniform exponential growth.*

One particularly nice class of hierarchically hyperbolic groups to which Corollary 5.1 can be applied is those which are *acylindrically hyperbolic*. The action of a group G on a metric space X is *acylindrical* if for all $\varepsilon > 0$ there exist constants $R, N \geq 0$ such that for all $x, y \in X$ with $d(x, y) \geq R$,

$$\#\{g \in G \mid d(x, gx) \leq \varepsilon \text{ and } d(y, gy) \leq \varepsilon\} \leq N.$$

A group is *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a hyperbolic space, that is, such that the limit set of the action contains at least three points.¹ It is unknown if all acylindrically hyperbolic groups have uniform exponential growth. However, it follows from Sisto [68] that every acylindrically hyperbolic group contains an infinite order Morse element, that is, an infinite order element g such that the quasi-geodesic $\langle g \rangle$ in the Cayley graph of G is Morse. Thus we immediately obtain the following result.

Corollary 1.4 *Virtually torsion-free hierarchically hyperbolic groups which are acylindrically hyperbolic have uniform exponential growth.*

The following gives another way of using Morse subspaces to determine that G is not quasi-isometric to a product with unbounded factors.

Corollary 5.2 *Every virtually torsion-free hierarchically hyperbolic group which is not virtually cyclic and contains an infinite Morse subgroup of infinite index has uniform exponential growth.*

For any hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, the index set \mathfrak{S} contains a domain which is largest under the nesting relation; we will always denote this domain S and its associated hyperbolic space CS . Given a hierarchically hyperbolic group, we can use the geometry of the hyperbolic space CS to determine that G is not quasi-isometric to a product with unbounded factors.

Corollary 5.3 *Let (G, \mathfrak{S}) be a virtually torsion-free hierarchically hyperbolic group such that CS is a non-elementary hyperbolic space. Then G has uniform exponential growth.*

Under the assumptions of Corollary 5.3, we actually obtain more information than what is stated in Theorem 1.2. We can additionally show that G satisfies a *quantitative Tits alternative*. We will make this precise in the next subsection.

Example 1.5 In addition to proving uniform exponential growth for a large class of cubical groups including right-Angled Artin and Coxeter groups and uniform lattices in products of hyperbolic cube complexes, Theorem 1.2 gives a single, unified proof that the following groups have uniform exponential growth.

- (1) Non-elementary virtually torsion-free hyperbolic groups. These groups are acylindrically hyperbolic [63], so we may apply Corollary 1.4. Uniform exponential growth was first shown for these groups by Gromov [37] and Delzant [22] (see [35, Theorem(vii)] for a precise statement) and generalized by Koubi [53] (without the torsion-free hypothesis).

¹ Equivalently, a group is acylindrically hyperbolic if it is not virtually cyclic and admits an acylindrical action on a hyperbolic space with unbounded orbits.

- (2) Non-exceptional mapping class groups. These groups are acylindrically hyperbolic [16, 57] and virtually torsion-free [46, Corollary 1.5]. Uniform exponential growth was first shown by Anderson, Aramayona, and Shackleton [1].
- (3) Many orientable 3-manifold groups. Specifically, if M is geometric then it suffices that M admits a complete metric locally isometric to \mathbb{H}^3 or $\mathbb{H}^2 \times \mathbb{R}$. In the non-geometric case, it suffices to have M be a flip graph 3-manifold or certain mixed 3-manifolds. These groups are torsion-free and acylindrically hyperbolic [59], so we may apply Corollary 1.4. The class of hierarchically hyperbolic 3-manifold groups to which our theorem applies is broader than stated here, but rather technical. For example, the manifold need not be prime, but cannot have any Nil or Sol components (see [13, Remark 10.2]). Uniform exponential growth is already known for 3-manifold groups (see for example [23] and references therein). For non-geometric 3-manifolds, this follows from the action on its JSJ-tree and work of Bucher and de la Harpe [8]. In the geometric case, this follows from work of Besson, Courtois, and Gallot [7] for hyperbolic 3-manifolds and from the fact that uniform exponential growth is inherited from quotients for Seifert fibered manifolds.
- (4) Graph products of virtually torsion-free hierarchically hyperbolic groups. Such groups are hierarchically hyperbolic by [17] and virtually torsion-free by [48, Corollary 3.4]. When the defining graph is not a join and $G \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$, the space \mathcal{CS} is non-elementary by work of Berlyne and Russell [18] extending work of Berlai and Robbio [17], and thus we may apply Corollary 5.3. This class includes free products and direct products of virtually torsion-free hierarchically hyperbolic groups. Uniform exponential growth for graph products of this form is known to follow from work of Bucher and de la Harpe [8], as long as the underlying graph is not complete, and from Antolín and Minasyan in the general case [4, Corollary 1.5].
- (5) A virtually torsion-free tree of hierarchically hyperbolic groups satisfying the conditions of [13, Corollary 8.24] or [17]. For instance, groups of the form $G_1 *_C G_2$, where G_i is hyperbolic and C is 2-ended, are hierarchically hyperbolic [65]. For the standard hierarchical structure on such groups, \mathcal{CS} is a tree (which is not a quasi-line), and so we may apply Corollary 5.3. Uniform exponential growth follows for such groups by Bucher and de la Harpe [8].

So far we have only provided conditions that are sufficient to guarantee that an HHG is not quasi-isometric to a non-trivial product, whereas Theorem 1.2 gives a more precise characterization of the product structure. Thus, Theorem 1.2 allows us to conclude that certain hierarchically hyperbolic groups which are quasi-isometric to a product still have uniform exponential growth. One example is the following.

Example 1.6 (Burger-Mozes) Consider the group G constructed by Burger and Mozes in [15] as the first example of a torsion-free simple group which acts cocompactly on the product of two trees. It is known that G is quasi-isometric to the product of two trees (which are not lines). Moreover, G was shown to be a hierarchically hyperbolic group by Behrstock, Hagen, and Sisto [12, Section 8]. However, there is no space E such that G is quasi-isometric to $\mathbb{Z} \times E$. Indeed, such a space E would have to be a quasi-tree by work of Fujiwara and Whyte [33, Theorem 0.1] together with bounds on the asymptotic (Assouad-Nagata) dimension [30, Theorem 4.3], [9, Theorem 2.4]. Such a quasi-isometry would induce a bi-Lipschitz homeomorphism on asymptotic cones, contradicting a result of Kapovich and Leeb on the nonexistence of certain bi-Lipschitz maps from products of two trees [50, Corollary 2.15]. By applying Theorem 1.2, we obtain a new proof that G has uniform exponential growth. This result also follows from the structure of G as an amalgamated free product of two free

groups and work of Bucher and de la Harpe [8]. Theorem A.1 in the appendix gives another proof that G has uniform exponential growth.

This example can be extended to give a new proof of uniform exponential growth for all *BMW-groups* (this terminology is introduced and described in [19]). This class of groups, generalizes the original construction of Burger and Mozes. A group G is a BMW-group if it acts by isometries on the product of two trees $T_1 \times T_2$ such that every element preserves the product decomposition and the action on the vertex set of $T_1 \times T_2$ is free and transitive.

1.2 A quantitative Tits alternative

Most known proofs of uniform exponential growth, including the proof of Theorem 1.2, demonstrate that one can produce a pair of elements with bounded word length with respect to *any* generating set that generate a free semigroup. In light of this, one can ask under what conditions one can find a pair of uniformly short elements which freely generate an actual subgroup. In groups which satisfy a Tits alternative, producing a free basis with bounded word length can be seen as a *quantitative* Tits alternative.

In our proof of Theorem 4.1, we use work of Breuillard and Fujiwara [11] to produce short elements that generate a free semigroup. Under the additional assumption of *hierarchical acylindricity*, discussed in Sect. 5, we can upgrade our argument using earlier work of Fujiwara [32] to produce a genuine free subgroup, showing the following quantitative Tits alternative holds for many hierarchically hyperbolic groups.

Proposition 5.5 *Let (G, \mathfrak{S}) be a virtually torsion-free hierarchically hyperbolic group such that G is not quasi-isometric to $\mathbb{Z} \times E$ for any metric space E . Suppose that either*

- (1) \mathcal{CS} is non-elementary, or
- (2) G is hierarchically acylindrical.

Then for any generating set X of G , there exists a free subgroup of G generated by two elements whose word length with respect to X is uniformly bounded.

We remark that existence of uniformly short free subgroups in Proposition 5.5 proves that G has *uniform non-amenability* (see, for example, [62]), which implies uniform exponential growth. Amenability can be formulated in terms of existence of Følner sets, which allows for averaging over the group. Heuristically, uniform non-amenability means that non-existence of Følner sets can be witnessed by balls of uniformly bounded radius in any Cayley graph.

1.3 HHGs without uniform exponential growth

We now turn our attention to the class of hierarchically hyperbolic groups that do not have uniform exponential growth. Since every finitely generated abelian group is hierarchically hyperbolic, this provides a large class of examples that lack even (non-uniform) exponential growth. On the other hand, HHGs are finitely presented and satisfy a Tits alternative: every finitely generated subgroup of a hierarchically hyperbolic group either contains a non-abelian free group or is virtually abelian [26]. In light of this, we ask the following question.

Question 1.7 *Does there exist a hierarchically hyperbolic group that is not virtually abelian and does not have uniform exponential growth?*

Either a positive or negative answer to this question would be of significant interest. A positive answer would prove that all hierarchically hyperbolic groups are either virtually

abelian or have uniform exponential growth, while a negative answer would provide an example of a finitely presented group which has exponential but not uniform exponential growth, answering a question of Gromov. Although our techniques do not allow us to answer Question 1.7, we obtain a structural classification of the cases where uniform exponential growth does not (or may not) hold. We obtain rather restrictive conditions on the hierarchical structure a group must satisfy in order to answer Question 1.7 in the affirmative.

Theorem 1.8 *Let G be a virtually torsion-free hierarchically hyperbolic group which is not virtually abelian and does not have uniform exponential growth. Then there exists a virtually G -invariant set of pairwise orthogonal domains \mathcal{W} such that for each $U \in \mathcal{W}$ the space CU is uniformly a quasi-line, and for each $V \notin \mathcal{W}$ either CV is uniformly bounded, or $V \perp U$ for all $U \in \mathcal{W}$.*

1.4 About the proof of Theorem 1.2

The proof of Theorem 1.2 has a similar structure to Mangahas's proof of uniform exponential growth for finitely generated subgroups of the mapping class group of a surface [55]. However, in this more general setting one needs to handle certain difficult behavior not present in the action of the mapping class group on the hierarchy of subsurface curve graphs. In particular, a general HHG does not contain a *pure* subgroup in the sense of Ivanov [46], that is, a finite index torsion-free subgroup such that for every domain U , elements that stabilize U act on the space CU either loxodromically or trivially. Indiscrete BMW-groups (see Example 1.6) give one class of examples of such phenomena. Indeed Caprace, Kropholler, Reid, and Wesolek [20, Corollary 32(i), (iv)] show that in these groups every finite index subgroup contains infinite order elements which are non-trivial elliptic isometries with respect to the action on one of the tree factors.

The proof of Theorem 1.2 splits into two cases. In the first case, we assume that there exist short words that act loxodromically on the hyperbolic spaces associated to two non-orthogonal domains. In this case we produce uniformly short powers that generate a free subgroup by playing ping-pong in the Cayley graph. If the first case doesn't hold, then we show that the action of (a finite index subgroup of) G on the set of domains must fix a collection \mathcal{W} of pairwise orthogonal domains pointwise. In this case, we show that either \mathcal{W} is a singleton or the top-level curve graph CS has bounded diameter. If \mathcal{W} is a singleton, we conclude that G has uniform exponential growth by finding uniformly short elements of G which are independent loxodromic isometries of CS , and thus have short powers generating a free subgroup. If CS has bounded diameter, we conclude that G is quasi-isometric to a product, and we next consider whether there are independent loxodromic isometries of CU for each $U \in \mathcal{W}$. If there are, then G has uniform exponential growth. Otherwise, we argue that each such CU is quasi-isometric to a line and use this to give a more explicit description of the product structure of G .

Organization: In Sect. 2 we review background material for uniform exponential growth, hierarchically hyperbolic groups, and tools to produce free (semi)groups. In Sect. 3, we give several structural results for when a hierarchically hyperbolic group contains invariant domains whose associated hyperbolic spaces are quasi-lines. This is followed by the proof of Theorem 1.2 in Sect. 4, where we also prove Theorem 1.8. In Sect. 5, we prove all of the corollaries and Proposition 5.5.

In the appendix, Gupta and Petyt prove Theorem A.1, a strengthening of Theorem 1.2 in the case of certain CAT(0) cubical groups which states that such groups either have uni-

form exponential growth or are virtually abelian. Together, Theorems 1.2 and A.1 prove Theorem 1.1.

2 Background and past results

We begin by recalling some preliminary notions about metric spaces, maps between them, and group actions. Given metric spaces X, Y , we use d_X, d_Y to denote the distance functions in X, Y , respectively. A map $f: X \rightarrow Y$ is:

- K -Lipschitz if there exists a constant $K \geq 1$ such that for every $x, y \in X$, $d_Y(f(x), f(y)) \leq Kd_X(x, y)$;
- (K, C) -coarsely Lipschitz if for every $x, y \in X$, $d_Y(f(x), f(y)) \leq Kd_X(x, y) + C$.
- a (K, C) -quasi-isometric embedding if there exist constants $K \geq 1$ and $C \geq 0$ such that for all $x, y \in X$,

$$\frac{1}{K}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + C,$$

- a (K, C) -quasi-isometry if it is a (K, C) -quasi-isometric embedding and, coarsely surjective, that is, Y is contained in the C -neighborhood of $f(X)$. In this case, we say X and Y are quasi-isometric.

For any interval $I \subseteq \mathbb{R}$, the image of an isometric embedding $I \rightarrow X$ is a geodesic and the image of a (K, C) -quasi-isometric embedding $I \rightarrow X$ is a (K, C) -quasigeodesic. A space X is a quasi-line if it is quasi-isometric to \mathbb{R} .

If any two points in X can be connected by a (K, C) -quasigeodesic, then we say X is a (K, C) -quasigeodesic space. If $K = C$, we may simply say that X is a K -quasigeodesic space. For all of the above notions, if the particular constants K, C are not important, we may drop them and simply say, for example, that a map is a quasi-isometry.

Throughout this paper, we will assume that all group actions are by isometries. The action of a group G on a metric space X is proper if the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite for every bounded subset $B \subseteq X$. The action is cobounded (respectively, cocompact) if there exists a bounded (respectively, compact) subset $B \subseteq X$ such that $X = \bigcup_{g \in G} gB$. If a group G acts on metric spaces X and Y , we say a map $f: X \rightarrow Y$ is G -equivariant if for every $x \in X$, $f(gx) = gf(x)$.

Given a metric space X and a subspace Y , we define the A -neighborhood of Y to be

$$N_A(Y) = \{x \in X \mid d_X(x, Y) \leq A\}.$$

Let X be a geodesic metric space and let $x, y, z \in X$. We denote by $[x, y]$ a geodesic segment between x and y . A geodesic triangle with vertices x, y, z is δ -slim if there is a constant $\delta \geq 0$ such that for any point $p \in [x, y]$, there is a point $m \in [y, z] \cup [x, z]$ such that $d_X(p, m) \leq \delta$. The space X is δ -hyperbolic if there is a constant $\delta \geq 0$ such that every geodesic triangle is δ -slim.

2.1 Uniform exponential growth

Given a finite collection of elements X containing the identity in a group, the growth function of X is

$$\beta_X(n) = |X^n|,$$

where $X^n = \{x_1 \dots x_n \mid x_i \in X\}$. This function $\beta_X(n)$ counts the number of elements that can be expressed as words in the alphabet X with length at most n . The *exponential growth rate* of a finite subset X of a group is

$$\lambda(X) := \lim_{n \rightarrow \infty} \frac{\log(\beta_X(n))}{n}.$$

Definition 2.1 ((Uniform) Exponential growth) A finitely generated group is said to have *exponential growth* if there is a finite generating set X such that

$$\lambda(X) > 0.$$

Such a group has *uniform exponential growth* if the infimum over all finite generating sets is bounded away from 0, that is,

$$\lambda_0 := \inf_{\substack{(X)=G \\ |X| < \infty}} \lambda(X) > 0.$$

Remark 2.2 One can also use the function

$$\omega(X) := \lim_{n \rightarrow \infty} \sqrt[n]{\beta_X(n)}$$

in place of $\lambda(X)$ to give an equivalent characterization of exponential growth rate. In this case, the growth is uniform if it is uniformly bounded above 1.

If $G = F_2$ is a free group of rank two and $X = \{1, a, b\}$ is a generating set, then it is easy to see that $|X^n| \geq 2^n$. Hence, $\lambda(X) \geq \log(2)$. In fact, since any generating set contains a pair of noncommuting elements and nonabelian subgroups of a free group are free, we have $\lambda(X') \geq \log(2)$ for any generating set X' . Therefore, $\lambda_0 \geq \log 2 > 0$, and so F_2 has uniform exponential growth. By the same reasoning, free semigroups have uniform exponential growth.

In light of this, most known proofs of uniform exponential growth make use of the following observation.

Observation 2.3 (Short free semigroups witness uniform exponential growth) If there is a constant N depending only on the group G such that for any generating set X there exists two elements with X -length at most N whose positive words generate a free semigroup, then G has uniform exponential growth with $\lambda_0 \geq \frac{\log(2)}{N}$.

The following result of Shalen and Wagreich gives bounds on the growth of a group given the growth of a finite index subgroup.

Lemma 2.4 ([70, Lemma 3.4]) *Let G be a group with finite generating set X , and let H be a finite index subgroup with $[G : H] = d$. Then there exists a generating set for H all of whose elements of have X -length at most $2d - 1$.*

This implies that if $[G : H] = d$ then

$$\lambda_0(G) \geq \frac{1}{2d - 1} \lambda_0(H),$$

thus, uniform exponential growth passes to finite index supergroups.

2.2 Finding free (semi)groups

In this section, we give multiple ways to find free (semi)groups given an action of a group on a hyperbolic metric space. We will assume all actions on metric spaces are by isometries. Together with Observation 2.3, these will be our key tools to show uniform exponential growth.

The first is a version of the standard ping-pong lemma.

Lemma 2.5 *Let G be a group acting on a set X , and let $a, b \in G$ have infinite order. Suppose there exist disjoint non-empty subsets $X_1, X_2 \subseteq X$ such that $a^n \cdot X_2 \subseteq X_1$ and $b^n \cdot X_1 \subseteq X_2$ for all $n \neq 0$. Then $\langle a, b \rangle$ is a free group of rank 2.*

Let G be a group acting on a hyperbolic metric space X with basepoint $x_0 \in X$, and let $g \in G$. The (stable) translation length of g is defined to be $\tau(g) = \lim_{n \rightarrow \infty} \frac{d(x_0, g^n x_0)}{n}$. If $\tau(g) > 0$, then g is a *loxodromic* isometry of X . Equivalently, g is loxodromic if it fixes exactly two points in the boundary ∂X of X . Such isometries act as translation along a quasi-geodesic axis in X . Two loxodromic isometries are *independent* if their fixed point sets in ∂X are disjoint.

The following result gives a method for producing free semigroups from an action on a hyperbolic space. While the statement is likely well-known, Breuillard and Fujiwara give an explicit proof in this context [11]. Their proof generalizes the analogous result for simplicial trees due to Bucher and de la Harpe [8]. When the hyperbolic space is a Hadamard manifold with $K \leq -1$, the result is due to Besson, Courtois, and Gallot [7].

Proposition 2.6 ([11, Proposition 11.1]) *For $\delta \geq 0$ let X be a δ -hyperbolic space, and $g, h \in \text{Isom}(X)$. Suppose g and h are loxodromic isometries whose fixed point sets in ∂X are not equal and $\tau(g), \tau(h) > 10000\delta$. Then some pair in $\{g^{\pm 1}, h^{\pm 1}\}$ generates a free semigroup.*

In particular, this result shows that given a pair of elements with stable translation length bounded from below, there are powers depending only on the displacement bound that generate a free semigroup. While it would be sufficient to use Proposition 2.6 to show uniform exponential growth, under the additional assumption that the action is acylindrical, we can construct genuine free subgroups generated by short conjugates of a single loxodromic.

Theorem 2.7 ([32, Proposition 2.3(2)]) *If G acts acylindrically on a δ -hyperbolic space containing elements $a, b \in G$ such that a acts loxodromically and $ba^n b^{-1} \neq a^{\pm n}$ for any $n \neq 0$ then there is a constant power p depending on δ and the acylindricity constants such that $\langle a^k, ba^k b^{-1} \rangle = \mathbb{F}_2 < G$ for all $k \geq p$.*

We note that the requirement that $ba^n b^{-1} \neq a^{\pm n}$ for any n ensures that a^k and $ba^k b^{-1}$ are independent loxodromic isometries.

2.3 Definition of a hierarchically hyperbolic group

We begin this subsection by recalling the definition of a hierarchically hyperbolic space (see [13], [25, Proposition 1.16]).

Definition 2.8 (Hierarchically hyperbolic space) The quasigeodesic space $(\mathcal{X}, d_{\mathcal{X}})$ is a *hierarchically hyperbolic space (HHS)* if there exists $\delta \geq 0$, an index set \mathfrak{S} , and a set $\{CW : W \in \mathfrak{S}\}$ of δ -hyperbolic spaces (CW, d_W) , such that the following conditions are satisfied:

- (1) **(Projections.)** There is a set $\{\pi_W : \mathcal{X} \rightarrow 2^{CW} \mid W \in \mathfrak{S}\}$ of *projections* sending points in \mathcal{X} to sets of diameter bounded by some $\xi \geq 0$ in the various $CW \in \mathfrak{S}$. Moreover, there exists K so that each π_W is (K, K) -coarsely Lipschitz and $N_K(\pi_W(\mathcal{X})) = CW$.
- (2) **(Nesting.)** \mathfrak{S} is equipped with a partial order \sqsubseteq , and either $\mathfrak{S} = \emptyset$ or \mathfrak{S} contains a unique \sqsubseteq -maximal element; when $V \sqsubseteq W$, we say V is *nested* in W . (We emphasize that $W \sqsubseteq W$ for all $W \in \mathfrak{S}$.) For each $W \in \mathfrak{S}$, we denote by \mathfrak{S}_W the set of $V \in \mathfrak{S}$ such that $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with V properly nested in W there is a specified subset $\rho_W^V \subset CW$ with $\text{diam}_{CW}(\rho_W^V) \leq \xi$. There is also a *projection* $\rho_V^W : CW \rightarrow 2^{CV}$.
- (3) **(Orthogonality.)** \mathfrak{S} has a symmetric and anti-reflexive relation called *orthogonality*: we write $V \perp W$ when V, W are orthogonal. Also, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. We require that for each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \emptyset$, there exists $W \in \mathfrak{S}_T - \{T\}$, so that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$. Finally, if $V \perp W$, then V, W are not \sqsubseteq -comparable.
- (4) **(Transversality and consistency.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$. There exists $\kappa_0 \geq 0$ such that if $V \pitchfork W$, then there are sets $\rho_W^V \subseteq CW$ and $\rho_V^W \subseteq CV$ each of diameter at most ξ and satisfying:

$$\min \left\{ d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W) \right\} \leq \kappa_0$$
 for all $x \in \mathcal{X}$. For $V, W \in \mathfrak{S}$ satisfying $V \sqsubseteq W$ and for all $x \in \mathcal{X}$, we have:

$$\min \left\{ d_W(\pi_W(x), \rho_W^V), \text{diam}_{CV}(\pi_V(x) \cup \rho_V^W(\pi_W(x))) \right\} \leq \kappa_0.$$

The preceding two inequalities are the *consistency inequalities* for points in \mathcal{X} .

Finally, if $U \sqsubseteq V$, then $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either that V is properly nested in W or that $V \pitchfork W$ and $W \not\sqsubseteq U$.

- (5) **(Finite complexity.)** There exists $n \geq 0$, the *complexity* of \mathcal{X} (with respect to \mathfrak{S}), so that any set of pairwise- \sqsubseteq -comparable elements has cardinality at most n .
- (6) **(Large links.)** There exist $\lambda \geq 1$ and $E \geq \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathfrak{S}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda d_W(\pi_W(x), \pi_W(x')) + \lambda$. Then there exists $\{T_i\}_{i=1, \dots, [N]} \subseteq \mathfrak{S}_W - \{W\}$ such that for all $T \in \mathfrak{S}_W - \{W\}$, either $T \in \mathfrak{S}_{T_i}$ for some i , or $d_T(\pi_T(x), \pi_T(x')) < E$. Also, $d_W(\pi_W(x), \rho_{T_i}^W) \leq N$ for each i .
- (7) **(Bounded geodesic image.)** There exists $E > 0$ such that for all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of CW , either $\text{diam}_{CV}(\rho_V^W(\gamma)) \leq E$ or $\gamma \cap N_E(\rho_V^W) \neq \emptyset$.
- (8) **(Partial Realization.)** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of \mathfrak{S} , and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq CV_j$. Then there exists $x \in \mathcal{X}$ so that:
 - $d_{V_j}(x, p_j) \leq \alpha$ for all j ,
 - for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubseteq V$, we have $d_V(x, \rho_V^{V_j}) \leq \alpha$, and
 - if $W \pitchfork V_j$ for some j , then $d_W(x, \rho_W^{V_j}) \leq \alpha$.

- (9) **(Uniqueness.)** For each $\kappa \geq 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d_{\mathcal{X}}(x, y) \geq \theta_u$, then there exists $V \in \mathfrak{S}$ such that $d_V(x, y) \geq \kappa$.

For ease of readability, given $U \in \mathfrak{S}$, we typically suppress the projection map π_U when writing distances in CU , that is, given $x, y \in \mathcal{X}$ and $p \in CU$ we write $d_U(x, y)$ for $d_U(\pi_U(x), \pi_U(y))$ and $d_U(x, p)$ for $d_U(\pi_U(x), p)$.

Heuristically, a hierarchically hyperbolic structure on a space \mathcal{X} is a means of organizing the space by the coarse geometry of the product regions in \mathcal{X} and their interactions. Nesting

gives a notion of sub-product regions and subspaces. Transversality gives a notion of separate or isolated subspaces. Orthogonality gives a notion of independent subspaces that together span a product region in \mathcal{X} . An excellent explanation of the intuition behind hierarchically hyperbolic spaces can be found in [69].

An important consequence of being an HHS is the existence of a distance formula, which relates distances in \mathcal{X} to distances in the hyperbolic spaces \mathcal{CU} . The notation $\{x\}_s$ means include x in the sum if and only if $x > s$.

Theorem 2.9 (*Distance formula; [13, Theorem 4.5]*) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then there exists s_0 such that for all $s \geq s_0$, there exist C, K so that for all $x, y \in \mathcal{X}$,*

$$d(x, y) \underset{K, C}{\asymp} \sum_{U \in \mathfrak{S}} \{d_U(x, y)\}_s.$$

We will now define the main object of this paper, hierarchically hyperbolic groups (HHG). Intuitively, a hierarchically hyperbolic group is a group whose Cayley graph is an HHS such that the action of the group on its Cayley graph is compatible with the HHS structure. The compatibility of the action is a key requirement, as can be seen by the fact that being an HHG is not a quasi-isometry invariant [64], though being an HHS is.

Definition 2.10 A group G said to be *hierarchically hyperbolic* if its Cayley graph X can be equipped with a hierarchically hyperbolic structure \mathfrak{S} such that the following hold.

- G acts cofinitely on \mathfrak{S} preserving the three relations.
- Given $g \in G$ and a domain $U \in \mathfrak{S}$, there is an isometry $g: \mathcal{CU} \rightarrow \mathcal{CgU}$, and these isometries are compatible with the group structure, in the sense that $g \circ h = gh$.
- For all $U \neq V$ satisfying either $V \sqsubseteq U$ or $V \pitchfork U$, and for all $g \in G$ and $x \in X$, the projection are equivariant with respect to the group action, meaning $g\pi_U(x) = \pi_{gU}(gx)$ and $g\rho_U^V = \rho_{gU}^{gV}$.

The last item differs from the original definition but can be assumed without loss of generality by [26, Section 2.1].

Remark 2.11 By the definition of a hierarchically hyperbolic group, there is finite set of domains U_1, \dots, U_k such that for every $W \in \mathfrak{S}$, there is some $i = 1, \dots, k$ such that \mathcal{CW} is isometric to \mathcal{CU}_i . It follows that for every $W \in \mathfrak{S}$, the diameter of \mathcal{CW} is either infinite or *uniformly* bounded.

In what follows we will consider an HHG (G, \mathfrak{S}) with respect to different finite generating sets. Let X and Y be two finite generating sets for a group G , and suppose that an HHG structure (G, \mathfrak{S}) is given, where distances in G are measured with d_X . Then the identity provides an equivariant quasi-isometry between (G, d_X) and (G, d_Y) . Note that this provides a hierarchically hyperbolic group structure on (G, d_Y) , where all the constants of the hierarchy axioms are the same, except the ones that involve distances in G . In particular, the only two such constants are the K of the projections of Axiom 1, and the constant θ_u of Axiom 9.

Remark 2.12 We say a constant k depends only on (G, \mathfrak{S}) when k depends only on the constants in the definition of the hierarchically hyperbolic structure on G which are independent of the generating set. Further, we will frequently refer to $D = \max\{\delta, \xi, \kappa_0, n, E\}$ as the *hierarchy constant*, which is also independent of the generating set.

Lemma 2.13 *Let $U, W, V \in \mathfrak{S}$ be such that U and W properly nest into V . If $d_V(\rho_V^U, \rho_V^W) > 2D$, then $U \pitchfork W$.*

Proof If $U \sqsubseteq W$ or $W \sqsubseteq U$, then $d_V(\rho_V^U, \rho_V^W) \leq D$ by the transversality and consistency axiom, which contradicts our assumption. If $U \perp W$, then there is a partial realization point $x \in \mathcal{X}$ such that $d_V(x, \rho_V^U) \leq D$ and $d_V(x, \rho_V^W) \leq E$. It follows that $d_V(\rho_V^U, \rho_V^W) \leq 2D$, which contradicts our assumption. Therefore $U \pitchfork W$. \square

2.4 Preliminaries on hierarchically hyperbolic groups

In this section, we recall the classification of hierarchical automorphisms from [25] and related results.

Definition 2.14 (Big set) For an HHG (G, \mathfrak{S}) , the *big set* of an element $g \in G$ is the collection of all domains such that the orbit map to the associated hyperbolic spaces is unbounded, that is, given a base point $x \in \mathcal{X}$ the *big set* is

$$\text{Big}(g) = \{U \in \mathfrak{S} \mid \text{diam}_{CU}(\langle g \rangle . x) \text{ is unbounded}\}.$$

Note that this collection is independent of base point.

Remark 2.15 There is a constant N , depending only on the hierarchy constant D such that any collection of pairwise non-transverse domains has cardinality at most N [14, Lemma 2.2]. Since the elements of $\text{Big}(g)$ must all be pairwise orthogonal (and hence pairwise non-transverse), it follows that $|\text{Big}(g)|$ is bounded by N for all $g \in G$. We fix this constant N for the remainder of the paper.

Definition 2.16 An element of a hierarchically hyperbolic group G is *elliptic* if it acts with bounded orbits on the Cayley graph X of G . It is *axial* if its orbit map induces a quasi-isometric embedding of a line in X .

Proposition 2.17 ([25, Lemma 6.3, Proposition 6.4, & Theorem 7.1]) *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. Then there exists a constant M between 0 and $N!$ depending only on D so that for all $g \in G$ the following hold.*

- (1) g is either elliptic or axial;
- (2) g is elliptic if and only if $\text{Big}(g) = \emptyset$;
- (3) for every $U \in \text{Big}(g)$, we have $g^M(U) = U$.

Remark 2.18 An element $g \in G$ is finite order if and only if $\text{Big}(g) = \emptyset$ [2, Lemma 1.7]. Therefore, if G is a torsion-free HHG, then every element of G has a non-empty big set.

If $g \in G$ has non-empty big set and fixes a domain $U \in \text{Big}(g)$, then g is a loxodromic isometry of CU . In particular, it follows from Proposition 2.17 (3) that g^M is a loxodromic isometry of CU for any domains $U \in \text{Big}(g)$.

Given an infinite order element $g \in G$ and a domain $U \in \mathfrak{S}$ such that g is loxodromic with respect to the action on CU , we let $\tau_U(g)$ denote the (stable) translation length of g in this action. In order for our proofs to produce words that are uniformly short with respect to any generating set of and HHG G , we would like to have a uniform lower bound on the translation lengths $\tau_U(g)$ for all $U \in \text{Big}(g)$. Unfortunately, this is not always the case [3, Theorem 1.4], but the following weaker condition does hold and is sufficient to prove uniform exponential growth.

Lemma 2.19 ([3]) *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. There exists a constant $\tau_0 > 0$ such that for every infinite order element $g \in G$ that fixes $\text{Big}(g)$ pointwise, there exists a domain $U \in \text{Big}(g)$ with $\tau_U(g) \geq \tau_0$.*

Throughout the paper, it will be important for us to pass to certain finite index subgroups while maintaining the hierarchical structure of the group. We do this with the following lemma.

Lemma 2.20 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group, and let H be a finite index subgroup of G . Then (H, \mathfrak{S}) is a hierarchically hyperbolic group with the same hierarchical structure as G .*

Proof Since H is a finite index subgroup of G , H is finitely generated. In particular, for each $U \in \mathfrak{S}$ the map $H \hookrightarrow G \xrightarrow{\pi_U} CU$ is a composition of two quasi-surjective, coarsely Lipschitz maps, and hence quasi-surjective coarsely Lipschitz itself. As the rest of the structure is unchanged, it is straightforward to verify that (H, \mathfrak{S}) is an HHS structure for H , possibly with different constants.

We will show that the three conditions from Definition 2.10 are satisfied. First, H acts on \mathfrak{S} because G does and $H \leq G$. The action of H on \mathfrak{S} still has finitely many orbits because H is finite index in G . As the other two conditions are concerned with single elements, they hold as every element of H is an element of G . \square

2.5 Hierarchical structures as coordinate systems

In this section, we will describe a product decomposition of G . More precisely, given an HHS $(\mathcal{X}, \mathfrak{S})$ there is a map $\pi : \mathcal{X} \rightarrow \prod_{W \in \mathfrak{S}} 2^{CW}$ defined by associating to each $x \in \mathcal{X}$ the tuple $\{\pi_W(x)\}_{W \in \mathfrak{S}}$. Our goal is to investigate the image of π into $\prod_{W \in \mathfrak{S}} 2^{CW}$, and determine when we can use the product structure of $\prod_{W \in \mathfrak{S}} 2^{CW}$ to induce a product structure on \mathcal{X} . We begin by recalling the definition of a κ -consistent tuple.

Definition 2.21 ([13, Definition 1.17]) Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space, fix $\kappa \geq 0$, and let $\vec{b} \in \prod_{W \in \mathfrak{S}} 2^{CW}$ be a tuple such that for each $W \in \mathfrak{S}$, the coordinate b_W is a subset of CW of diameter at most κ . The tuple \vec{b} is κ -consistent if:

- (1) $\min \{d_W(b_W, \rho_W^V), d_V(b_V, \rho_V^W)\} \leq \kappa$, whenever $V \pitchfork W$;
- (2) $\min \{d_W(b_W, \rho_W^V, \text{diam}_{CV}(b_V \cup \rho_V^W(b_W)))\} \leq \kappa$, whenever $V \sqsubseteq W$.

We denote the subset of $\prod_{W \in \mathfrak{S}} 2^{CW}$ consisting of κ -consistent tuples by Ω_κ .

Remark 2.22 Note that for κ large enough, the first condition holds automatically since π_W is surjective by Axiom 1.

We want to prove a sufficient condition on the index set \mathfrak{S} under which the group G quasi-isometrically decomposes as a product. This result can be deduced from discussions in [13, Sections 3 & 5]; we restate it here, along with its justification, for the sake of clarity and completeness.

Proposition 2.23 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $\tilde{\mathfrak{S}}$ consist of all $W \in \mathfrak{S}$ such that CW has infinite diameter. Suppose that $\tilde{\mathfrak{S}}$ can be partitioned as $\mathfrak{I}_1 \sqcup \cdots \sqcup \mathfrak{I}_n$ where $\mathfrak{I}_i \neq \emptyset$ for all i and every element of \mathfrak{I}_i is orthogonal to every element of \mathfrak{I}_j for $i \neq j$. Then there are infinite diameter metric spaces Y_i such that \mathcal{X} is quasi-isometric to $Y_1 \times \cdots \times Y_n$. Moreover, each Y_i can be equipped with an HHS structure.*

The main technical ingredient to prove the proposition is to establish a connection between G and the set of consistent tuples. As noted, there is a map $\pi: \mathcal{X} \rightarrow \prod_{W \in \mathfrak{S}} 2^{CW}$ defined by associating to each $x \in \mathcal{X}$ the tuple $\{\pi_W(x)\}_{W \in \mathfrak{S}}$. Thus, by setting $\kappa_1 = \max\{\kappa_0, \xi\}$, Axioms 1 and 4 of Definition 2.8 give that for each $\kappa \geq \kappa_1$, the map π has image in Ω_κ . The following theorem should be thought of as saying that the projection π has a quasi-inverse.

Theorem 2.24 ([13, Theorem 3.1]) *For each $\kappa \geq 1$ there exist $\theta_e, \theta_u \geq 0$ such that the following holds. Let $\vec{b} \in \Omega_\kappa$ be a κ -consistent tuple, and for each W let b_W denote the CW -coordinate of \vec{b} . Then the set $\Psi(\vec{b}) \subseteq \mathcal{X}$ defined as all $x \in \mathcal{X}$ so that $d_W(b_W, \pi_W(x)) \leq \theta_e$ for all $CW \in \mathfrak{S}$ is non empty and has diameter at most θ_u .*

The reason why " Ψ is a quasi-inverse of π " is not a precise statement is because we did not equip Ω_κ with a metric. The distance formula (Theorem 2.9) gives a constant s_0 such that for each $s \geq s_0$ there is a map $f_s: \Omega_\kappa \times \Omega_\kappa \rightarrow \mathbb{R}$ defined as

$$f_s(\vec{a}, \vec{b}) = \sum_{W \in \mathfrak{S}} \{d_W(a_W, b_W)\}_s,$$

such that for every $x, y \in \mathcal{X}$, the quantities $f_s(\pi(x), \pi(y))$ and $d_{\mathcal{X}}(x, y)$ are comparable. However, note that the map f_s is not a distance: it does not satisfy the triangle inequality and there exists $\vec{a} \neq \vec{b}$ such that $f_s(\vec{a}, \vec{b}) = 0$. To remedy this, we equip Ω_κ with the subspace metric coming from Ψ , which we denote by $d_{\mathcal{X}}$ with an abuse of notation.

The next ingredient in the proof of Proposition 2.23 is to show that one needs only focus on domains whose associated hyperbolic spaces have sufficiently large diameter. We first concern ourselves with subdividing \mathfrak{S} into blocks. Let $\mathfrak{S}' \subseteq \mathfrak{S}$ be any subset. It is straightforward to see that concept of a consistent tuple (Definition 2.21) can be generalized to $\prod_{W \in \mathfrak{S}'} 2^{CW}$. Let $\Omega_\kappa^{\mathfrak{S}'}$ be the set of κ -consistent tuples of $\prod_{W \in \mathfrak{S}'} 2^{CW}$.

Definition 2.25 Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and suppose that a basepoint $x \in \mathcal{X}$ is fixed. For $C < \kappa_0$ consider the set \mathfrak{S}_C consisting of all $W \in \mathfrak{S}$ such that $\text{diam}(CW) > C$. Given $\vec{a} \in \Omega_\kappa^{\mathfrak{S}_C}$ we define $\Psi_{\mathfrak{S}_C}(\vec{a}) = \Psi(\vec{b})$, where $\vec{b} \in \Omega_\kappa$ coincides with \vec{a} on \mathfrak{S}_C and $b_U := \pi_U(x)$ for $U \in \mathfrak{S} - \mathfrak{S}_C$.

Remark 2.26 The choice of basepoint is not very important: the distance formula shows that the Hausdorff distance between the images of $\Psi_{\mathfrak{S}_C}$ under different choices of basepoints is bounded in terms of C . For this reason, we will suppress the dependence.

Lemma 2.27 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then for each $0 \leq C < \kappa$ the spaces Ω_κ and $\Omega_\kappa^{\mathfrak{S}_C}$ equipped with the subspace metric are quasi-isometric.*

Proof Setting $s > C$, the coordinates associated to the elements of $\mathfrak{S} - \mathfrak{S}_C$ do not contribute to the distance formula. Thus the conclusion follows. \square

Lemma 2.27 is particularly useful when an HHS satisfies the *bounded domain dichotomy*, that is, when there exists C such that for each $U \in \mathfrak{S}$ either $\text{diam}(CU) \leq C$ or $\text{diam}(CU) = \infty$. Notably, Remark 2.11 states that all HHGs satisfy the bounded domain dichotomy. The following corollary is immediate.

Corollary 2.28 *Let \mathcal{X} be an HHS satisfying the bounded domain dichotomy, and let $\tilde{\mathfrak{S}}$ consist of all $W \in \mathfrak{S}$ such that CW has infinite diameter. Then there is a constant $\kappa > 0$ such that $\Psi_{\tilde{\mathfrak{S}}}: \Omega_\kappa^{\tilde{\mathfrak{S}}} \rightarrow \mathcal{X}$ is coarsely surjective, and so $\Omega_\kappa^{\tilde{\mathfrak{S}}}$ with the subspace metric is quasi-isometric to \mathcal{X} .*

We refer to $U \in \mathfrak{S}$ as being a(n) *(un)bounded domain* when its associated hyperbolic space \mathcal{CU} is (un)bounded. The last ingredient missing to Proposition 2.23 is a criterion to determine when a subspace of an HHS is itself an HHS.

Definition 2.29 Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. A subset $Y \in \mathcal{X}$ is *hierarchically quasi-convex* if:

- (1) the projection $\pi_U(Y)$ are uniformly quasi-convex for all $U \in \mathfrak{S}$; and
- (2) for each r there is an R such that if $x \in \mathcal{X}$ satisfies $d_U(x, \pi_U(Y)) \leq r$ for all U , then $d_{\mathcal{X}}(x, Y) \leq R$.

By [13, Proposition 5.6] every hierarchically quasi-convex subset of an HHS can be equipped with an HHS structure. We can now prove Proposition 2.23.

Proof of Proposition 2.23 By assumption, $\tilde{\mathfrak{S}}$ can be partitioned as $\mathfrak{T}_1 \sqcup \cdots \sqcup \mathfrak{T}_n$ where every element of \mathfrak{T}_i is orthogonal to every element of \mathfrak{T}_j when $i \neq j$. By consistency (see Definition 2.21), the set $\Omega_{\kappa}^{\tilde{\mathfrak{S}}}$ can be written as $\Omega_{\kappa}^{\mathfrak{T}_1} \times \cdots \times \Omega_{\kappa}^{\mathfrak{T}_n}$. Fix a basepoint $x \in \mathcal{X}$, and for each $\Omega_{\kappa}^{\mathfrak{T}_i}$ consider the map $\Psi_{\mathfrak{T}_i} : \Omega_{\kappa}^{\mathfrak{T}_i} \rightarrow \mathcal{X}$ defined by $\Psi_{\mathfrak{T}_i}(\vec{a}) = \Psi_{\tilde{\mathfrak{S}}}(\vec{b})$, where \vec{b} coincides with \vec{a} on \mathfrak{T}_i and is defined to be $\pi_U(x)$ otherwise. Let Y_i denote the resulting metric space. The distance formula yields that $Y_1 \times \cdots \times Y_n$ is quasi-isometric to $\Psi_{\tilde{\mathfrak{S}}}(\Omega_{\kappa}^{\tilde{\mathfrak{S}}})$. By Corollary 2.28, the latter coarsely coincides with G . We are left with proving that each Y_i is hierarchically quasi-convex. By definition, $\pi_W(Y_i)$ coarsely coincides with $\pi_W(\mathcal{X})$ for $W \in \mathfrak{T}_i$ and it coarsely coincides with $\pi_W(x)$ otherwise. This proves the first item of Definition 2.29. For the second, let $y \in \mathcal{X}$ be such that $d_W(y, x) \leq r$ for all $W \notin \mathfrak{T}_i$. Let $z \in Y_i$ be the realization point of the tuple defined as $\pi_W(y)$ for $W \in \mathfrak{T}_i$ and as $\pi_W(x)$ for $W \notin \mathfrak{T}_i$. By definition of \mathfrak{T}_i , such a tuple is consistent. By the distance formula, we can bound the distance between z and y in terms of r , which shows the second item of hierarchical quasi-convexity. \square

3 Structural results

In this section, we give several structural results which will be useful in the proof of Theorem 1.2.

Lemma 3.1 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group. Suppose \mathcal{U} is a G -invariant collection of pairwise orthogonal domains such that \mathcal{CU} has infinite diameter for each $U \in \mathcal{U}$. If there exists a domain $V \notin \mathcal{U}$ with $\text{diam}(\mathcal{CV}) = \infty$, then for any $U \in \mathcal{U}$, we have $U \not\sqsubseteq V$.*

Proof Suppose by way of contradiction that there exists a domain $U \in \mathcal{U}$ such that $U \sqsubseteq V$. For each $W \in \mathcal{U}$, fix any point $p_W \in \mathcal{CW}$, and let $p \in G$ be given by partial realization (Axiom 8 of Definition 2.8). Pick any $g \in G$, and consider the points $\pi_V(g)$ and $\pi_V(p)$. By the choice of p ,

$$d_V(p, \rho_V^U) \leq \alpha.$$

Now apply the isometry $pg^{-1} : \mathcal{CV} \rightarrow \mathcal{C}(pg^{-1}V)$. It follows that

$$d_{pg^{-1}V}(pg^{-1}\pi_V(p), pg^{-1}\rho_V^U) \leq \alpha.$$

Since $pg^{-1}\rho_V^U$ uniformly coarsely coincides with $\rho_{pg^{-1}V}^{pg^{-1}U}$, we have that $pg^{-1}\pi_V(p)$ uniformly coarsely coincides with $\rho_{pg^{-1}V}^{pg^{-1}U}$.

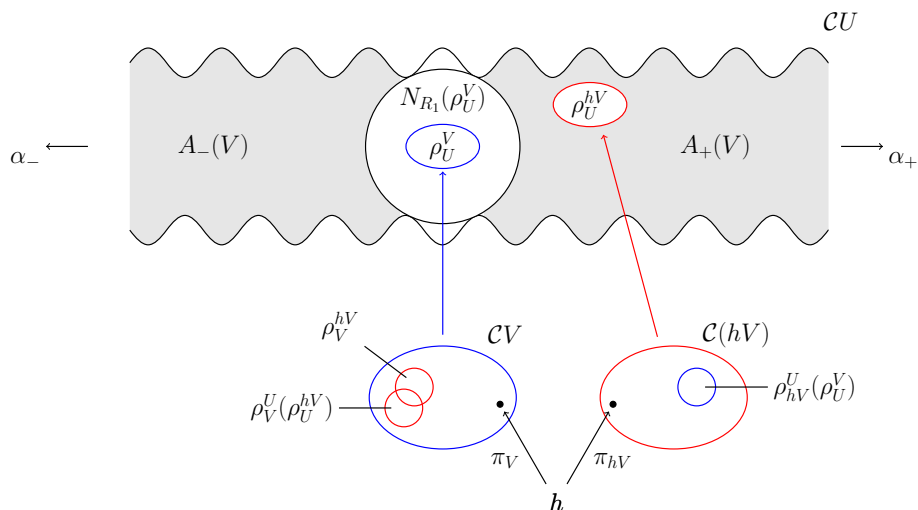


Fig. 1 A schematic of the spaces and projections in the proof of Proposition 3.2

As the action of G on \mathfrak{S} fixes \mathcal{U} setwise, it follows that $pg^{-1}U \in \mathcal{U}$. Moreover, $pg^{-1}U \subseteq pg^{-1}V$. Thus, by using partial realization as above, we have that $\pi_{pg^{-1}V}(p)$ uniformly coarsely coincides with $\rho_{pg^{-1}V}^{pg^{-1}U}$, and so $pg^{-1}\pi_V(p)$ uniformly coarsely coincides with $\pi_{pg^{-1}V}(p)$, as well. Moreover, $\pi_{pg^{-1}V}(p) = \pi_{pg^{-1}V}(pg^{-1}g)$, hence applying the inverse isometry gp^{-1} shows that the distance between $\pi_V(p)$ and $\pi_V(g)$ is uniformly bounded. Since g was arbitrary and π_V is coarsely surjective, it follows that CV has finite diameter, which contradicts our assumption on V . \square

The following proposition shows that any G -invariant domain whose associated hyperbolic space is a quasi-line that contains the axis of a loxodromic must be nest minimal.

Proposition 3.2 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group, and suppose there exists $U \in \mathfrak{S}$ such that $G.U = U$ and CU is Q -quasi-isometric to \mathbb{R} . If G contains an element acting by translation on CU , then for all $V \subsetneq U$, $\text{diam}(CV) < \infty$.*

Proof We remark that since we are solely concerned with understanding the spaces CW for $W \in \mathfrak{S}$, we can fix an arbitrary generating set to work with for the proof of this proposition. This assumption is only needed to prove Claim 2 below.

Let D be the hierarchy constant introduced in Remark 2.12, and let κ_1 be the constant from [13, Proposition 1.8]. Let $\partial CU = \{\alpha_+, \alpha_-\}$. For any domain V which properly nests into U , the nesting axiom (Axiom 2) gives that $\text{diam}_{CU}(\rho_U^V) \leq D$. The hyperbolic space CU is a Q -quasi-line for some constant Q ; we may assume without loss of generality that $Q > 1/\sqrt{2}$. It follows that there is a constant $R_1 > 2D + \kappa_1$ such that the neighborhood $N_{R_1}(\rho_U^V)$ disconnects CU . Let $A_+(V)$ and $A_-(V)$ be the two connected components of $CU \setminus N_{R_1}(\rho_U^V)$ containing α_+ and α_- , respectively, and let $A_{\pm}(V) = A_+(V) \cup A_-(V)$ denote their union. See Fig. 1. Since CU is a path connected Q -quasi-line by assumption, we have $\text{diam}_{CU}(CU \setminus (A_{\pm}(V))) \leq 2(Q^2 R_1 + Q^2 + Q)$. Let $R_2 = 2(Q^2 R_1 + Q^2 + Q)$, and note that $R_2 > D + \kappa_1$. The bounded geodesic image axiom (Axiom 7) states

that every geodesic segment in A_+ or A_- projects to CV with diameter at most D . Thus $\text{diam}_{CV}(\rho_V^U(A_+(V))) \leq 2D$ and $\text{diam}_{CV}(\rho_V^U(A_-(V))) \leq 2D$. Since the map ρ_U^V is G -equivariant, we have $A_\pm(V') = hA_\pm(V)$ whenever $V' = hV$.

The proof follows by contradiction using the following two claims, each relying on the assumption that there is a domain properly nested into U whose curve graph has infinite diameter.

Claim 1 If $V' \subsetneq U$ and CV' is unbounded, then for all $L > 0$ there is an unbounded domain $V \subsetneq U$ such that

$$d_V(1, \rho_V^U(A_\pm(V))) > L. \quad (1)$$

Claim 2 If $V \subsetneq U$ and CV is unbounded, then for all $L > 0$ there is an element $h \in G$ such that

$$d_V(h, \rho_V^U(A_\pm(V))) > L \quad \text{and} \quad d_U(\rho_U^{hV}, \rho_U^V) > L. \quad (2)$$

We complete the proof assuming the claims, which will be addressed later. Take $L = R_2$, and suppose there is a domain V that properly nests into U such that CV has unbounded diameter. By Claim 1, we may assume without loss of generality that V satisfies (1). Let $h \in G$ be the element from Claim 2. The second statement of (2) and Lemma 2.13 give that $V \triangleleft hV$. Since $G.U = U$, every element of G acts on CU by isometries; in particular, $hU = U$. We have $\text{diam}_{CU}(CU \setminus A_\pm(V)) = \text{diam}_{CU}(CU \setminus hA_\pm(V)) \leq R_2$, and consequently the second statement of Claim 2 implies that $\rho_U^{hV} \subset A_\pm(V)$ and $\rho_U^V \subset hA_\pm(V)$.

The second statement of Claim 2 and [13, Proposition 1.8] show that $\text{diam}_{CV}(\rho_V^{hV} \cup \rho_V^U(\rho_U^{hV})) \leq \kappa_1$. As $\rho_V^U(\rho_U^{hV}) \subseteq \rho_V^U(A_\pm(V))$, we thus have

$$d_V(h, \rho_V^{hV}) \geq d_V(h, \rho_V^U(A_\pm(V))) - \kappa_1 > R_2 - \kappa_1 \geq D, \quad (3)$$

where the second to last inequality follows from the first statement of Claim 2. Applying the fact that the projections ρ are G -equivariant to (1) yields

$$d_{hV}(h, \rho_{hV}^U(hA_\pm(V))) > L = R_2.$$

Since $\rho_U^V \subseteq hA_\pm(V)$ and $d_V(1, \rho_V^U(A_\pm(V))) = d_{hV}(h, \rho_{hV}^U(hA_\pm(V))) > R_2$, an analogous argument yields

$$d_{hV}(h, \rho_{hV}^V) \geq d_{hV}(h, \rho_{hV}^U(hA_\pm(V))) - \kappa_1 > R_2 - \kappa_1 \geq D. \quad (4)$$

However, the inequalities (3) and (4) contradict the transversality and consistency axiom (Axiom 4) applied to h projected to V and hV , which states that

$$\min \{d_V(h, \rho_V^{hV}), d_{hV}(h, \rho_{hV}^V)\} \leq D.$$

It remains to prove the two claims.

Proof of Claim 1 Let $L > 0$ be fixed, and consider $A_\pm(V')$. We have that $\rho_{V'}^U(A_\pm(V'))$ has bounded diameter. If $d_{V'}(1, \rho_{V'}^U(A_\pm(V'))) > L$ then we are done by taking $V = V'$. Otherwise, $d_{V'}(1, \rho_{V'}^U(A_\pm(V'))) \leq L$. Since $\rho_{V'}^U(A_\pm(V'))$ is bounded and $\pi_{V'}$ is D -coarsely surjective, there is an element $g^{-1} \in G$ so that $d_{V'}(g^{-1}, \rho_{V'}^U(A_\pm(V'))) > L + D$. By equivariance, we can apply g to obtain $d_{gV'}(1, \rho_{gV'}^U(gA_\pm(V'))) > L + D$. Taking $V = gV'$ completes the claim.

Proof of Claim 2 Let $L > D$ be fixed exceeding the hierarchy constant and $t \in G$ be an element acting by translation on \mathcal{CU} , which exists by assumption. Let γ be any isometry of \mathcal{CU} that fixes the endpoints and moves some point $x_0 \in \mathcal{CU}$ less than L . Then there is a constant $\tilde{L} \geq L$ depending only on the quasi-line constants of \mathcal{CU} (and not on the choice of γ) such that γ moves every point of \mathcal{CU} by at most \tilde{L} .

Let $\hat{G} \leq G$ be the index 2 subgroup of G that fixes $\partial\mathcal{CU}$ pointwise. Note that t acts as translation, and so $t \in \hat{G}$. Moreover, since G coarsely surjects onto \mathcal{CU} , so does \hat{G} . Pick $M > 0$ so that $M\tau_0 > 2\tilde{L} + D$, where τ_0 is as in Lemma 2.19.

As before, the coarse surjectivity of the K -coarsely Lipschitz map π_V guarantees the existence of an element $h' \in \hat{G}$ satisfying

$$d_V(h', \rho_V^U(A_{\pm}(V))) > \tilde{L} + KM|t| + K,$$

where $|t|$ is the word length of t in the fixed generating set. If $d_U(\rho_U^{h'V}, \rho_U^V) > L$, then we are done by taking $h = h'$, so assume $d_U(\rho_U^{h'V}, \rho_U^V) \leq L$.

Consider $h = h't^M$. Using the fact that π_V is Lipschitz and the triangle inequality, we have

$$\begin{aligned} d_V(h, \rho_V^U(A_{\pm}(V))) &\geq d_V(h', \rho_V^U(A_{\pm}(V))) - d_V(h't^M, h') \\ &\geq (\tilde{L} + KM|t| + K) - (KM|t| + K) \\ &\geq \tilde{L} \geq L. \end{aligned}$$

Thus the first statement of the claim holds. By the choice of \tilde{L} , we have that $d_U(x, h'x) \leq \tilde{L}$ for all $x \in \mathcal{CU}$. Thus

$$\begin{aligned} d_U(\rho_U^V, \rho_U^{h'V}) &\geq d_U(\rho_U^V, h't^M \rho_U^V) - D \\ &\geq d_U(\rho_U^V, t^M \rho_U^V) - d_U(t^M \rho_U^V, h't^M \rho_U^V) - D \\ &\geq (2\tilde{L} + D) - \tilde{L} - D \\ &\geq \tilde{L} \geq L, \end{aligned}$$

completing the proof of Claim 2. \square

Next, we give a sufficient condition for when a collection of pairwise orthogonal domains have associated hyperbolic spaces that are quasi-lines.

Proposition 3.3 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group and $U \in \mathfrak{S}$ a domain such that there is a pair of points $\alpha, \beta \in \partial\mathcal{CU}$ which are fixed pointwise by G . Then \mathcal{CU} is quasi-isometric to a line.*

Proof Let γ be a geodesic between the points $\alpha, \beta \in \partial\mathcal{CU}$, and let $h \in G$. We want to uniformly bound $d_U(h, \gamma)$. Since there exists $C = C(\mathfrak{S})$ such that π_U is C -coarsely surjective, this would prove the result. Let $g \in G$ be such that $d_U(g, \gamma) \leq C$, and consider $hg^{-1}\gamma$. Since all the generators fix $\alpha, \beta \in \partial\mathcal{CU}$, we have that $hg^{-1}\gamma$ is a geodesic of \mathcal{CU} with the same endpoints as γ . By the hyperbolicity of \mathcal{CU} , the Hausdorff distance between γ and $hg^{-1}\gamma$ is uniformly bounded. Moreover, by equivariance of the map π_U we have $d_U(h, hg^{-1}\gamma) = d_U(g, \gamma) \leq C$, which implies that $d_U(h, \gamma)$ is uniformly bounded, concluding the proof. \square

We end this section by describing domains which are transverse to a G -invariant domain whose associated hyperbolic space has infinite diameter.

Proposition 3.4 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group, and suppose there is a G -invariant domain $U \in \mathfrak{S}$ such that $\text{diam}(CU) = \infty$. For any $W \in \mathfrak{S}$ satisfying $W \pitchfork U$, the space CW has uniformly bounded diameter.*

Proof If there is no W transverse to U , the proposition is trivial. So assume it is not the case. Let $\Omega_\kappa \subset \prod_{W \in \mathfrak{S}} 2^{CW}$ and $\Phi: \Omega_\kappa \rightarrow 2^X$ be as in Sect. 2.5. Let $\kappa \geq \kappa_1$, and let Y be the subset of Ω_κ consisting of all tuples whose W -coordinate is ρ_W^U for each $W \pitchfork U$. Since CU has infinite diameter, $\Phi(Y)$ is an infinite diameter subset of G . Moreover, since U is G -invariant, so are Y and $\Phi(Y)$. Since G acts coboundedly on itself, we have that $\Phi(Y)$ coarsely coincides with G . Since Φ is a quasi-isometry, we conclude that Y coarsely coincides with Ω_κ . Thus, the spaces CW are uniformly bounded for every $W \pitchfork U$. \square

4 Proof of main theorem

Recall that for any generating set X and any $n \geq 1$, we denote by X^n the ball of radius n about the identity in the Cayley graph of G with respect to X .

Theorem 4.1 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group such that every generating set of G contains an infinite order element. There exists a constant $M > 0$ depending only on (G, \mathfrak{S}) such that one of the following occurs.*

- (a) G is virtually abelian.
- (b) For any generating set X , there are elements $u, w \in X^M$ which form a basis for a free sub-semigroup.
- (c) There is a G -invariant collection \mathcal{W} of pairwise orthogonal domains such that G is quasi-isometric to $\mathbb{Z}^{|\mathcal{W}|} \times E$, where E is a non-elementary space.

Before turning to the proof of Theorem 4.1, we show how it implies Theorem 1.2.

Proof of Theorem 1.2 Let X be an arbitrary finite generating set for G . Consider the finite-index torsion-free subgroup H of G . Then (H, \mathfrak{S}) is an HHG by Lemma 2.20, there is a generating set X' for H , all of whose elements have X -length at most $2d - 1$, where $d = [G : H]$ by Lemma 2.4. Moreover, since H is torsion-free, every generating set contains an infinite order element. By Theorem 4.1 and Observation 2.3, the subgroup H either has uniform exponential growth or is quasi-isometric to $\mathbb{Z} \times E$ for some space E . Since H is finite index, it follows that the theorem holds for G . \square

For the remainder of the section, fix a generating set X for G with the convention that X contains the identity. Recall that N is the maximal number of pairwise non-transverse domains in \mathfrak{S} , which depends only on the hierarchy constants. By hypothesis, there is an infinite order generator $s' \in X$. Let $s = (s')^{N!}$, so that s fixes $\text{Big}(s)$ pointwise. Let $\tau_0 > 0$ and $W \in \text{Big}(s)$ be the constant and domain provided by Lemma 2.19, so that $\tau_W(s) > \tau_0$. Let

$$\mathcal{W} = X^N \cdot W \tag{5}$$

be the set of images of W under words of length at most N . Since X is finite, \mathcal{W} is a finite set.

The proof of Theorem 4.1 will be divided into two main cases using the following proposition.

Proposition 4.2 *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group such that every generating set of G contains an infinite order element, and let X be a generating set for G as above containing both $1 \in X$ and an infinite order element $s \in X$. Then one of the following holds.*

- (1) *The set \mathcal{W} defined in (5) contains two transverse domains. In particular, there is an element $g \in X^{2N}$ such that $W \pitchfork gW$.*
- (2) *The set \mathcal{W} is a G -invariant collection of pairwise orthogonal domains.*

Moreover, if Item 2 holds, then there is a finite index subgroup $\hat{G} \leq G$ of index at most $N!$ fixing \mathcal{W} pointwise.

Proof First, suppose there exist $V_1, V_2 \in \mathcal{W}$ with $V_1 \pitchfork V_2$. Then for $i = 1, 2$ there are elements $g_i \in X^N$ such that $V_i = g_i W$. By premultiplying by g_2^{-1} , we may assume that $V_1 = W$ and $V_2 = gW$ for some $g \in X^{2N}$. Thus the second statement of Item 1 holds.

Now suppose Item 1 does not hold, so that every pair of domains in \mathcal{W} is non-transverse. Suppose that \mathcal{W} is not G -invariant. Then X does not fix $\mathcal{W} = X^N \cdot \mathcal{W}$ setwise, and thus X does not fix $X^k \cdot \mathcal{W}$ setwise for any $1 \leq k \leq N$. Hence for each $1 \leq k \leq N$,

$$X^k \cdot \mathcal{W} \neq X^{k+1} \cdot \mathcal{W}.$$

Since the identity is contained in X , we have

$$X^k \cdot \mathcal{W} \subsetneq X^{k+1} \cdot \mathcal{W}.$$

In particular, this implies that $|X^N \cdot \mathcal{W}| \geq N + 1$. However, this is a contradiction, as there can be at most N pairwise non-transverse elements, and so \mathcal{W} is G -invariant.

Let $U, V \in \mathcal{W}$. Since U and V are not transverse, either $U \subseteq V$ or $U \perp V$. Suppose $U \subseteq V$, and consider the subcollection $\mathcal{U} = \{gU \mid g \in G\} \subseteq \mathcal{W}$. This subcollection is G -invariant by construction, and since $gU \subseteq U$ would imply $gU = U$ for any $g \in G$, the set \mathcal{U} consists of pairwise orthogonal domains. However, Lemma 3.1 applied to \mathcal{U} and $V \notin \mathcal{U}$ implies that $U \not\subseteq V$, which is a contradiction. Therefore \mathcal{W} is a G -invariant collection of pairwise orthogonal domains, and Item 2 holds.

For the moreover statement, we have $|\mathcal{W}| \leq N$ by definition of N . This induces a map to the symmetric group $G \rightarrow \text{Sym}(N)$ whose kernel is a subgroup of G of index at most $N!$ fixing \mathcal{W} pointwise. \square

We address the two cases of Proposition 4.2 in separate subsections.

4.1 Case 1

Assume that Item 1 of Proposition 4.2 holds, that is, there is an element $g \in X^{2N}$ such that $W \pitchfork gW$. If $|\cdot|_X$ denotes the word length with respect to the finite generating set X , then we have $|s|_X \leq N!$ and

$$|gsg^{-1}|_X \leq |g|_X + |s|_X + |g^{-1}|_X \leq 4N + N!.$$

We will show that there are uniform powers of s and gsg^{-1} that generate a free subgroup of G .

Proposition 4.3 *Let (G, \mathfrak{S}) be any hierarchically hyperbolic group satisfying the hypotheses of Case 1. There exists a constant k_1 depending only on (G, \mathfrak{S}) such that $\langle s^{k_1}, (gsg^{-1})^{k_1} \rangle \cong F_2$.*

Proof Let $t = gsg^{-1}$ and $V = gW$. Notice that t is loxodromic with respect to the action on CV and $\tau_V(t) > \tau_0$, since s and t are conjugate and $\tau_W(s) > \tau_0$ by the choice of W .

Let κ_0 be the constant from Axiom 4 (Transversality) of Definition 2.8. We will apply the ping-pong lemma to the following subsets of G :

$$Y_s = \left\{ x \in G : d_W(\pi_W(x), \rho_W^V) > \kappa_0 \right\} \quad \text{and} \quad Y_t = \left\{ x \in G : d_V(\pi_V(x), \rho_V^W) > \kappa_0 \right\}.$$

Transversality and consistency imply that these sets are disjoint. Note that for all $Z, T \in \mathfrak{S}$, the projection map $\pi_Z : G \rightarrow CZ$ is coarsely surjective and ρ_Z^T is a bounded subset of CZ whenever $T \pitchfork Z$. Since CW and CV are infinite diameter, this implies that Y_s and Y_t are non-empty.

Fix a constant $k \geq 2\kappa_0\tau_0^{-1}$ and a point $x \in Y_s$. By transversality and consistency, we have $d_V(x, \rho_V^W) \leq \kappa_0$. Using this fact in addition to Lemma 2.19 and the triangle inequality, we have

$$\begin{aligned} d_V(\rho_V^W, t^{k(2N+1)!}x) &\geq d_V(x, t^{k(2N+1)!}x) - d_V(x, \rho_V^W) \\ &\geq \tau_0 |k| - d_V(x, \rho_V^W) \\ &\geq 2\kappa_0 - \kappa_0 \\ &= \kappa_0. \end{aligned}$$

Thus $t^{k(2N+1)!}x \in Y_t$, and so $t^{k(2N+1)!}(Y_s) \subseteq Y_t$. Observe that the only requirement for k was $k \geq 2\kappa_0\tau_0^{-1}$. In particular, the conclusion holds for multiples of k . The same argument works with negative powers, so $t^{-k(2N+1)!}(Y_s) \subseteq Y_t$. By a symmetric argument, it follows that $s^{\pm k(2N+1)!}(Y_t) \subseteq Y_s$. Thus $\langle s^{k(2N+1)!}, t^{k(2N+1)!} \rangle \cong F_2$ by the ping-pong lemma. Setting $k_1 = 2\kappa_0\tau_0^{-1}(2N+1)!$ completes the proof. \square

We note that in the previous proposition (and in many of the later results), if we allow s and t to have different exponents, then we can find smaller constants $k_{1,s}$ and $k_{1,t}$ such that $\langle s^{k_{1,s}}, t^{k_{1,t}} \rangle \cong F_2$. In particular, we may take $k_{1,s} = 2\kappa_0\tau_0^{-1}m_s$ and $k_{1,t} = 2\kappa_0\tau_0^{-1}m_t$, for some $m_s, m_t \leq N$. Also, the stabilization power $(2N+1)!$ is not optimal since it is given by the kernel of a map from a copy of \mathbb{Z} to a cyclic subgroup of $\text{Sym}(2N+1)$, which can have size at most $LCD(1, 2, \dots, 2N+1)$, which grows slower than factorial. For ease of notation, however, we choose to use the larger uniform exponent.

4.2 Case 2

We now assume that Item 2 of Proposition 4.2 holds, that is, $\mathcal{W} \subseteq \mathfrak{S}$ (as in 5) is a finite collection of pairwise orthogonal domains that is stabilized by the action of G on \mathfrak{S} and fixed pointwise by the action of $\bar{G} := \ker(G \rightarrow \text{Sym}(N))$ on \mathfrak{S} .

Proposition 4.4 *Let (G, \mathfrak{S}) be any hierarchically hyperbolic group satisfying the hypotheses of Case 2. Suppose CS has infinite diameter. Then either G is virtually cyclic or there exists a constant k_2 depending only on (G, \mathfrak{S}) and an element $t \in X$ such that $\langle s^{k_2}, ts^{k_2}t^{-1} \rangle \cong F_2$.*

Proof Let $U \in \mathcal{W}$. Then, by definition, there exists $h \in G$ with $|h|_X \leq N$, such that $U = h.W$. As CW has infinite diameter and h acts as an isometry on the associated hyperbolic spaces, CU must have infinite diameter, as well.

Since CS has infinite diameter by assumption and $U \sqsubseteq S$, it follows from Lemma 3.1 applied with $\mathcal{U} = \mathcal{W}$ that $S \in \mathcal{W}$. By definition, $S \in \mathcal{W}$ implies that $S = g.W$ for some

$g \in G$ with $|g|_X \leq N$. However, the map g preserves the \sqsubseteq -levels of elements of \mathfrak{S} , and S is the unique \sqsubseteq -maximal domain in \mathfrak{S} . Thus, $S = g.W$ if and only if W has the same level as S , and we conclude that $W = S$. This implies that $S \in \text{Big}(s)$. (In fact, this implies that $S = \text{Big}(s)$ by [25, Lemma 6.7], but we will not need this stronger statement.)

The action of G on \mathcal{CS} is cobounded and acylindrical by [12, Corollary 14.4]. Let $E(s)$ denote the stabilizer of the endpoints of the axis of s in $\partial\mathcal{CS}$. If for every generator $r \in X$ we have $r \in E(s)$, then G is virtually cyclic by [24, Lemma 6.5].

Otherwise, there exists a generator $t \in X \setminus \{s\}$ such that $t \notin E(s)$, and hence t does not stabilize the endpoints of the axis of s in $\partial\mathcal{CS}$. In particular, $|\partial\mathcal{CS}| \geq 3$, that is, \mathcal{CS} is a non-elementary hyperbolic space.

By [24, Corollary 6.6], $t \notin E(s)$ if and only if $ts^n t^{-1} \neq s^{\pm n}$ for any $n \neq 0$. Therefore, by the choice of t , Theorem 2.7 guarantees the existence of a constant k_2 such that $\langle s^{k_2}, ts^{k_2} t^{-1} \rangle \cong F_2$. \square

Notice that the proof of Proposition 4.4 shows that whenever Item 2 of Proposition 4.2 holds and \mathcal{CS} has infinite diameter, there exist two uniformly short elements which are independent loxodromic elements with respect to the action on \mathcal{CS} . Such elements are necessarily Morse [68]. On the other hand, when Item 1 of Proposition 4.2 holds, the uniformly short elements we find are not necessarily Morse. We ask if it is always possible to find uniformly short Morse elements in a hierarchically hyperbolic group which generate a free group.

4.3 Proof of Theorem 4.1

Let k_1 be the constant from Proposition 4.3, k_2 the constant from Proposition 4.4, and δ the hyperbolicity constant of \mathcal{CU} for any $U \in \mathfrak{S}$. Let X be an arbitrary generating set for G , let $s' \in X$ be an infinite order element, and let $s = (s')^{N!}$. Let $W \in \text{Big}(s)$ and τ_0 be the domain and constant from Lemma 2.19; note that τ_0 is independent of the choice of generating set X . Also let

$$k_3 = \lceil 10000\delta\tau_0^{-1} \rceil,$$

and

$$M \geq \max\{k_1, k_2 + 2, 3(k_3 + 2)(N + 1)!\}.$$

Our goal is to show that one of the following occurs:

- (a) G is virtually abelian;
- (b) there exist two words of length at most M that generate a free semigroup; or
- (c) G is quasi-isometric to a product $\mathbb{Z}^{|\mathcal{W}|} \times E$, where E has infinite diameter and is not quasi-isometric to \mathbb{Z}^n .

One of the two cases of Proposition 4.2 must hold. If Item 1 holds, then (b) holds by Proposition 4.3. So, suppose Item 2 holds and the set

$$\mathcal{W} = X^N.W$$

defined in (5) is fixed setwise by G .

If $S \in \mathcal{W}$, then \mathcal{CS} has infinite diameter, and so (a) or (b) holds by Proposition 4.4. If $S \notin \mathcal{W}$, then $\text{diam}(\mathcal{CS}) < \infty$ (in particular, it is uniformly bounded) by applying Lemma 3.1 with $\mathcal{U} = \mathcal{W}$.

By passing to a further finite index subgroup, we can assume that \mathcal{W} is fixed pointwise by G . Indeed, consider the subgroup $\hat{G} = \ker(G \rightarrow \text{Sym}(\mathcal{W}))$ of index at most $N!$ which

fixes \mathcal{W} pointwise. By Lemma 2.4, there is a generating set Y' for \hat{G} all of whose elements have X -length at most $2N! - 1$. This means that if we can prove the desired trichotomy for \hat{G} , it will follow for G . By definition, every domain $U \in \mathcal{W}$ supports the axis of at least one element in X^{2N+1} . Observe also that, by Proposition 2.17 there is a constant K between 0 and $N!$ such that $g^K \in \hat{G}$. Expand the generating set for \hat{G} to be

$$Y = Y' \cup \{g^K : g \in X^{2N+1}\}.$$

Elements of Y have X -length at most $(2N+1)N! < 3(N+1)!$. Since each domain of \mathcal{W} was in the big set of some element of X^{2N+1} , each domain is also in the big set of some element of Y .

For the rest of the proof, we restrict our attention to \hat{G} , which acts on \mathcal{CU} for each $U \in \mathcal{W}$. For each $gW = U \in \mathcal{W}$, there exists an element $s_U = gsg^{-1} \in G$ with $|s_U|_X \leq 2N+N!$ that acts loxodromically on \mathcal{CU} . Thus $s_U^K \in \hat{G}$ also acts loxodromically on \mathcal{CU} , and $|s_U^K|_Y = 1$, by the definition of Y . Let s_U^\pm be the fixed point of s_U^K on $\partial\mathcal{CU}$. We claim that either (b) holds or all the generators fix $\{s_U^+, s_U^-\}$ setwise. Indeed, if t is an element of Y that does not fix $\{s_U^+, s_U^-\}$, the conjugate $t^{-1}s_U^K t$ is an independent loxodromic with respect to the action on \mathcal{CU} . By Lemma 2.19, there is a uniform lower bound on the translation length of $s_U^K = gs^K g^{-1}$ (which is equal to the translation length of $t^{-1}s_U^K t$) with respect to the action on \mathcal{CU} . Therefore, Proposition 2.6 implies that for k_3 defined as above, some pair in $\{(s_U^K)^{\pm k_3}, t^{-1}(s_U^K)^{\pm k_3} t\}$ generates a free semigroup, and hence (b) holds.

Thus, we may assume that for each $U \in \mathcal{W}$, the set $\{s_U^+, s_U^-\}$ is \hat{G} -invariant. By Proposition 3.3, we conclude that \mathcal{CU} is a quasi-line for each $U \in \mathcal{W}$. Let $\mathcal{W} = \{W_1, \dots, W_n\}$ for some n , and let $\tilde{\mathcal{S}} = \{V \in \tilde{\mathcal{S}} \mid \text{diam}(\mathcal{CV}) = \infty\}$. We claim that $Z \perp W_i$ for each $Z \in \tilde{\mathcal{S}} - \mathcal{W}$ and for all i . To see this, suppose that \mathcal{CZ} is unbounded. Then Lemma 3.1 and Proposition 3.2 imply that for each i , either $Z \perp W_i$ or $Z \pitchfork W_i$. Since W_i is \hat{G} -invariant, by Proposition 3.4, we must have $Z \perp W_i$. Thus, we can partition $\tilde{\mathcal{S}}$ into pairwise orthogonal sets as follows:

$$\tilde{\mathcal{S}} = \{W_1\} \sqcup \dots \sqcup \{W_n\} \sqcup (\tilde{\mathcal{S}} - \mathcal{W}).$$

Let $\Omega_\kappa^{\tilde{\mathcal{S}}}$ be as in Sect. 2.5. By Proposition 2.23, we conclude that \hat{G} (and therefore G) is quasi-isometric to $\mathbb{Z}^{|\mathcal{W}|} \times \Omega_\kappa^{\tilde{\mathcal{S}} - \mathcal{W}}$. If $\Omega_\kappa^{\tilde{\mathcal{S}} - \mathcal{W}}$ is quasi-isometric to \mathbb{Z}^m for some m , then (a) holds. Otherwise, (c) holds with respect to the initial generating set, X , and $E = \Omega_\kappa^{\tilde{\mathcal{S}} - \mathcal{W}}$, completing the proof of Theorem 4.1.

Theorem 1.8 follows immediately from the proof of Theorem 4.1.

5 Applications

We begin by proving Corollaries 5.1, 5.2, and 5.3 from the introduction, whose statements we recall for the convenience of the reader.

Corollary 5.1 *Every non-virtually cyclic virtually torsion-free hierarchically hyperbolic group which has an asymptotic cone containing a cut-point has uniform exponential growth. In particular, if the Cayley graph of a virtually torsion-free hierarchically hyperbolic group G contains an unbounded Morse quasi-geodesic, then G has uniform exponential growth.*

Proof of Corollary 5.1 Let G be a non-virtually cyclic virtually torsion-free hierarchically hyperbolic group. It follows from [29, Proposition 1.1] that having a cut-point in an asymptotic cone of G is equivalent to G having super-linear divergence. However, this cannot occur if

G is quasi-isometric to a product with unbounded factors, and therefore G has uniform exponential growth by Theorem 1.2.

The second statement follows from [29, Proposition 3.24], which shows that if a geodesic metric space X has an unbounded Morse quasi-geodesic, then every asymptotic cone of X has a cut-point. \square

Corollary 5.2 *Every virtually torsion-free hierarchically hyperbolic group which is not virtually cyclic and contains an infinite Morse subgroup of infinite index has uniform exponential growth.*

Proof of Corollary 5.2 Let G be a non-virtually cyclic virtually torsion-free hierarchically hyperbolic group, and let $H \leq G$ be an infinite Morse subgroup of infinite index. If G is quasi-isometric to a product with unbounded factors, then either H is quasi-isometric to the Cayley graph of G or H has bounded diameter in the Cayley graph of G . In the first case, we reach a contradiction with the fact that H is infinite index, and in the second case we reach a contradiction with the fact that H is infinite. Thus G has uniform exponential growth by Theorem 1.2. \square

Corollary 5.3 *Let (G, \mathfrak{S}) be a virtually torsion-free hierarchically hyperbolic group such that CS is a non-elementary hyperbolic space. Then G has uniform exponential growth.*

Proof of Corollary 5.3 Let (G, \mathfrak{S}) be a virtually torsion-free hierarchically hyperbolic group such that CS is a non-elementary hyperbolic space. The result follows immediately from Proposition 4.4. \square

We now turn our attention to the quantitative Tits alternative described in Proposition 5.5. Under the additional assumption that (G, \mathfrak{S}) is hierarchically acylindrical, our proof of Theorem 4.1 can be adjusted to generate free subgroups rather than free semigroups. Hierarchical acylindricity was introduced by Durham, Hagen, and Sisto in [25] to generalize the following property of mapping class groups: for any non-annular subsurface $\Sigma \subseteq S$, the subgroup $MCG(\Sigma) \leq MCG(S)$ acts acylindrically on domains corresponding to Σ .

To make this precise in the HHG setting, let

$$\text{Stab}(U) = \{g \in G : gU = U\}.$$

By definition of HHG, $\text{Stab}(U)$ acts on CU . Let K_U be the kernel of the action, namely the subgroup $\{g \in \text{Stab}(U) : g.x = x \ \forall x \in CU\}$.

Definition 5.4 A hierarchically hyperbolic group is *hierarchically acylindrical* if $\text{Stab}(U) / K_U$ acts acylindrically on CU , for all $U \in \mathfrak{S}$.

For example, right-angled Artin groups are hierarchically acylindrical because parabolic subgroups act acylindrically on the contact graph corresponding to the associated subgraph of the defining graph. More generally, fundamental groups of compact special cube complexes are hierarchically acylindrical. However, not all hierarchically hyperbolic group structures are hierarchically acylindrical. An example is given by the group constructed by Burger–Mozes (see Example 1.6); for further discussion see [26] and [3].

Proposition 5.5 *Let (G, \mathfrak{S}) be a virtually torsion-free hierarchically hyperbolic group such that G is not quasi-isometric to $\mathbb{Z} \times E$ for any metric space E . Suppose that either*

(1) *CS is non-elementary, or*

(2) G is hierarchically acylindrical.

Then for any generating set X of G , there exists a free subgroup of G generated by two elements whose word length with respect to X is uniformly bounded.

Proof of Proposition 5.5 Fix constants as in the proof of Theorem 4.1. The only time that free semigroups are produced in the proof of Theorem 4.1 is when Item 2 of Proposition 4.2 holds and CS is an elementary hyperbolic space. Equivalently, this occurs when two elements have independent axes in an infinite diameter domain that properly nests into S . In this case, we pass to a subgroup \hat{G} with finite generating set Y which fixes \mathcal{W} pointwise, and find elements $s, t \in Y$ such that s and $t^{-1}st$ are independent loxodromic isometries of CU for some $U \subsetneq S$. By hierarchical acylindricity, \hat{G}/K_U acts nonelementarily and acylindrically on CU . Let \bar{s} and \bar{t} be the images of s and t in the quotient. Applying Theorem 2.7, there exists a constant k_4 such that $(\bar{s}^{k_4}, \bar{t}\bar{s}^{k_4}\bar{t}^{-1}) \cong \mathbb{F}_2$ in \hat{G}/K_U . Since free groups are Hopfian, this lifts to a free subgroup of \hat{G} . In particular, the constant M in Theorem 4.1 can be updated to be

$$M \geq \max\{k_1, k_2 + 2, 3(k_4 + 2)(N + 1)!\}.$$

□

Remark 5.6 The proof of Proposition 5.5 shows that the conclusion of Proposition 5.5 also holds in slightly more generality. In particular, it holds for any virtually torsion-free HHG in which Item 1 of Proposition 4.2 holds for every finite generating set X .

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Appendix A: Uniform exponential growth for cocompactly cubulated groups By Radhika Gupta and Harry Petyt

The aim of this appendix is to show that cubical groups that admit a *factor system* (see Sect. 1) either have uniform exponential growth or are virtually abelian. This extends the work of the main body of the paper, in which this is shown to hold when the cube complex does not split as a direct product (Theorem 1.2). The arguments here show that this irreducibility assumption can be dropped.

Theorem A.1 *Let G be a group virtually acting freely cocompactly on a locally finite, finite dimensional $CAT(0)$ cube complex X , and assume that X has a factor system. Then either G has uniform exponential growth or G is virtually abelian.*

Since the arguments of Sect. 4 require that the groups involved are (virtually) torsion-free, we consider free actions only, as any proper action of a torsion-free group on a $CAT(0)$ cube complex is free.

The class to which Theorem A.1 applies is very large [44], although Shepherd recently found examples of cubulations of groups that do not admit factor systems [67]. More specifically, the class includes: compact special groups [12]; the Burger-Mozes group, and more

generally BMW-groups in the sense of [19]; certain Artin groups [38]; and any graph product of these [18]. We get a new proof of uniform exponential growth for some of these groups:

- *Compact special groups.* These groups are linear [27, 45], and hence satisfy a UEG alternative by [31] and the Tits alternative [71].
- *The mapping class group of the genus-two handlebody.* It is a subgroup of the mapping class group of the closed surface of genus two, which is linear by [6, 52], and hence has UEG by [31]. Note that in this case, linearity didn't come from virtual specialness. This group acts geometrically on a CAT(0) cube complex with a factor system by [56].
- *Burger–Mozes group.* It acts freely on a 2-dimensional CAT(0) cube complex, so a UEG alternative follows by [54]. It also appears as Example 1.6, but rather more machinery is needed there. Our proof does not depend on the 2-dimensionality, and we need little technology.

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Preliminaries

Let us briefly record a few definitions and recall some lemmas.

Lemma A.2 *Let $G = G_1 \times G_2$ be a finitely generated group. If G_1 has uniform exponential growth, then G has uniform exponential growth.*

Proof This is a special case of the simple fact that if $G \rightarrow H$ is a surjective homomorphism and H has uniform exponential growth, then so does G . \square

CAT(0) cube complexes

We refer the reader to [66] for an introduction to CAT(0) cube complexes and groups acting on them. We highlight two points for later use; also see [12, §2]. Firstly, each hyperplane h has two associated combinatorial hyperplanes. These are parallel copies of h in the carrier of h that are as far apart as possible. In particular, they are subcomplexes, unlike h itself. However, like h , they are convex. Secondly, if Y is a convex subcomplex of a CAT(0) cube complex X and $x \in X^{(0)}$, then the *gate* of x in Y , denoted $\text{g}_Y(x)$, is the unique closest vertex of Y to x .

The *contact graph* of a CAT(0) cube complex X , denoted CX , is defined to be the graph whose 0-skeleton is the set of hyperplanes of X , with an edge (h, h') whenever the carriers of h and h' intersect. See [40, 41] for more information on contact graphs, including a proof that they are quasitrees, and, in particular, hyperbolic.

Lemma A.3 *Let h_1, h_2, h_3 be distinct hyperplanes of a CAT(0) cube complex X , such that h_2 separates h_1 from h_3 . Then any path P in CX from h_1 to h_3 passes through the 1-neighbourhood of h_2 .*

Proof The hyperplanes h_1 and h_3 lie in different components of $X \setminus h_2$, so P must contain a hyperplane whose carrier intersects the carrier of h_2 . \square

Note that an action on a CAT(0) cube complex induces an action on the contact graph. We refer to [21, §3] for the definition of an essential action.

Lemma A.4 (Double-skewering, [21, p.853], [42, Lem. 2.11, Cor. 4.5]) Suppose G acts essentially and cocompactly on a locally finite, finite dimensional CAT(0) cube complex X . If w and v are disjoint hyperplanes with halfspaces $\overleftarrow{w} \subsetneq \overleftarrow{v}$, then there is a hyperbolic isometry $g \in G$ such that $g\overleftarrow{v} \subsetneq \overleftarrow{w}$. Moreover, if $d_{CX}(w, v) > 2$, then g can be taken to act loxodromically on CX .

The contact graph comes with a coarsely Lipschitz projection map from X to CX that sends each point $x \in X$ to the diameter-1 subset of CX consisting of all hyperplanes whose carriers contain x . More generally, if Y is a convex subcomplex of X , then projection from X to CY is defined as the composition of the projection $Y \rightarrow CY$ with g_Y .

Factor systems

A *factor system* for a CAT(0) cube complex X is the data of a collection of convex subcomplexes satisfying certain conditions. Factor systems were introduced in [12], and any factor system for X gives it the geometry of a hierarchically hyperbolic space. In general, X may have many different factor systems, which allow for greater flexibility. However, if X has a factor system, then it always has a “simplest” factor system, called the *hyperclosure* [44]. Therefore, when X has a factor system we shall assume that it is the hyperclosure of X .

Definition A.5 (Hyperclosure) For a CAT(0) cube complex X , the hyperclosure of X is the intersection \mathcal{F} of all sets \mathcal{F}' of convex subcomplexes of X satisfying the following conditions.

- \mathcal{F}' contains X and every combinatorial hyperplane of X .
- If $Y, Y' \in \mathcal{F}'$, then $g_Y(Y') \in \mathcal{F}'$.
- If $Y \in \mathcal{F}'$ and Y' is parallel to Y , in the sense that any hyperplane crossing one crosses both, then $Y' \in \mathcal{F}'$.

If there exists an $N > 0$ such that for all $x \in X$, there are at most N subcomplexes $F \in \mathcal{F}$ with $x \in F$, then the hyperclosure \mathcal{F} is a factor system. Moreover, if \mathcal{F} is not a factor system, then X does not have any factor system [44, Rem. 1.15]. Note that the hyperclosure always contains the vertices of X .

Given a factor system for X , there is an associated hierarchically hyperbolic structure on X . This structure includes a hyperbolic space $\hat{C}Y$ for the parallelism class of each convex subcomplex Y in the factor system. The space $\hat{C}Y$ is obtained from the contact graph CY by coning off certain subgraphs, so any projection to CY induces a projection to $\hat{C}Y$. (Note that the notation here disagrees slightly with that of the main body of the paper, where the hyperbolic spaces associated to a hierarchically hyperbolic space are always denoted CY . For cube complexes it is standard to reserve that notation for contact graphs.) When the factor system is the hyperclosure of X , the hyperbolic space $\hat{C}X$ associated to X is quasi-isometric to the contact graph CX [12, Rem. 8.18].

If $X = \prod_{i=1}^n X_i$ is a direct product, then the hyperclosure \mathcal{F} of X can be concretely described in terms of the hyperclosures \mathcal{F}_i of the X_i . Indeed, a subcomplex Y of X lies in \mathcal{F} exactly when it decomposes as $Y = \prod_{i=1}^n Y_i$ for some $Y_i \in \mathcal{F}_i$. From this decomposition it follows that $\hat{C}Y$ is unbounded if and only if exactly one $\hat{C}Y_i$ is unbounded and all the other Y_i are vertices. Thus, in this case, Y is a subcomplex of some standard factor $v_1 \times \cdots \times v_{i-1} \times X_i \times v_{i+1} \times \cdots \times v_n$. By abuse of notation we say Y is a subcomplex of X_i .

Main result

In what remains we shall prove Theorem A.1. We begin with a specialisation of Theorem 4.1 to a certain kind of cocompactly cubulated group.

Proposition A.6 *Let $X = \prod_{i=1}^n X_i$ be a direct product of finite dimensional $CAT(0)$ cube complexes. If X has a factor system and the contact graph of each X_i is unbounded but not a quasiline, then any group acting freely cocompactly on X has uniform exponential growth.*

Proof As discussed in Sect. 1, any element of the hyperclosure \mathcal{F} of X whose associated hyperbolic space is unbounded is a subcomplex of some X_i . Moreover, $\hat{C}X_i$ is not a quasiline, as it is quasi-isometric to CX_i . Any group G acting properly cocompactly on X by cubical automorphisms is a hierarchically hyperbolic group, and the hierarchy structure of G is the same as that of X . Note that the set of parallelism classes $\{X_1, \dots, X_n\}$ is necessarily G -invariant (see [21, Prop. 2.6], for instance).

Let N be the constant from Remark 2.15. Let S be a generating set of G , let $s \in S^{N!}$ be an infinite order element that fixes its bigset pointwise, and let $W \in \text{Big}(s)$ be the domain provided by Lemma 2.19. Recall from Sect. 4 that $\mathcal{W} = S^N \cdot W$. This is a subset of parallelism classes of elements of \mathcal{F} because \mathcal{F} is G -invariant. By Propositions 4.2 and 4.3, either G has uniform exponential growth, or \mathcal{W} is a G -invariant set of subcomplexes such that $\prod_{B \in \mathcal{W}} B$ is a subcomplex of X , and each $\hat{C}B$ is unbounded. In this latter case, $\mathcal{W} \subset \{X_1, \dots, X_n\}$ by Lemma 3.1, so none of the $\hat{C}B$ are quasilines. This suffices, because the argument of Theorem 4.1 shows that if none of the $\hat{C}B$ are quasilines, then G has uniform exponential growth. \square

Lemma A.7 *Suppose that a group G acts essentially and cocompactly on a locally finite, finite dimensional, irreducible $CAT(0)$ cube complex Y . Then the contact graph CY of Y is unbounded, and if CY is a quasiline, then so is Y .*

Proof The contact graph CY is unbounded by [42, Cor. 4.7]. Suppose that CY is a quasiline. Let h and h' be two hyperplanes of Y that are at distance at least 3 in CY . The “moreover” statement of Lemma A.4 provides a corresponding hyperbolic isometry $g \in G$ that acts loxodromically on CY . Perhaps after subdividing Y , an application of [39, Thm 1.4] shows that g is combinatorially hyperbolic on Y , and is therefore rank-one by [42, Thm 4.1]. Let A be an axis of g in Y , and let C be the cubical convex hull of A . By [42, Lem. 4.8], C is at finite Hausdorff distance from A , and thus it is a quasiline. Moreover, since g is loxodromic on the quasiline CY , the projection of C to CY is coarsely onto.

It suffices to show that $C = Y$. Since C is convex, it is enough to prove that every hyperplane of Y crosses C . Suppose h_1 is a hyperplane of Y that does not cross C . By the “implication (3) \implies (4)” part of the proof of [42, Thm 4.1], the gate $g_C(h_1)$ of h_1 to C has uniformly bounded diameter. Since the projection map $Y \rightarrow CY$ is coarsely Lipschitz and since CY is a quasiline, there is a ball $B \subset CY$ of uniformly bounded diameter that contains the projection of $g_C(h_1)$ and disconnects CY .

Let h_2 and h_3 be hyperplanes of Y that cross C , have $d_{CY}(h_i, B) > 1 + \text{diam}(B)$, and lie on opposite sides of B inside CY . Since h_1 does not cross C , it cannot separate h_2 and h_3 . It follows from Lemma A.3 that neither h_2 nor h_3 separates the other from h_1 . Thus, the h_i form a facing triple that are pairwise at distance greater than 3 in CY . By Lemma A.4 (Double skewering), there exists an isometry g that acts loxodromically on CY and has the property that h_1 separates $g^n h_2$ from h_2 , and hence from h_3 , for all positive n . From Lemma A.3,

we see that CY must have at least three ends, a contradiction. Thus every hyperplane of Y crosses C , so $Y = C$. \square

We are now in a position to prove Theorem A.1. We restate it for convenience, minus the word “virtually”, as that generalisation is immediate from [70].

Theorem A.1 Let G be a group acting freely cocompactly on a locally finite, finite dimensional CAT(0) cube complex X , and assume that X has a factor system. Then either G has uniform exponential growth or G is virtually abelian

Proof By passing to the essential core, we may assume that X is essential (see [21, §3.2]), which implies that the action of G on X is essential, by cocompactness. Now, by [21, Prop. 2.6], there is a decomposition $X = X^1 \times X^2$, where $X^1 = \prod_{i=1}^n X_i$ is a product of irreducible non-euclidean complexes, and X^2 is euclidean. Since X is essential, if X^1 is bounded then it is trivial. In this case, G acts properly cocompactly on a flat, and so is virtually abelian.

Otherwise, [61, Cor.2.8] provides a finite index subgroup of G that splits as $G_1 \times G_2$, where G_j acts properly cocompactly on X^j . By Lemma A.2, it suffices to show that G_1 has uniform exponential growth, and by [70] we may pass to a finite index subgroup G'_1 of G_1 that fixes the factors of X^1 . According to Lemma A.7, the contact graph of each X_i is unbounded but not a quasiline. The conditions of Proposition A.6 are therefore met by X^1 , on which G'_1 is acting properly cocompactly. Thus G'_1 has uniform exponential growth, completing the proof. \square

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