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Largest hyperbolic action of 3-manifold groups

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Abstract

The set of equivalence classes of cobounded actions of a group G on different hyperbolic metric spaces carries a natural partial order. Following Abbott–Balasubramanya–Osin, the group G is $\mathcal{H}$*-accessible* if the resulting poset has a largest element. In this paper, we prove that every nongeometric 3-manifold has a finite cover with $\mathcal{H}$-inaccessible fundamental group and give conditions under which the fundamental group of the original manifold is $\mathcal{H}$-inaccessible. We also prove that every Croke–Kleiner admissible group (a class of graphs of groups that generalizes fundamental groups of three-dimensional graph manifolds) has a finite index subgroup that is $\mathcal{H}$-inaccessible.

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1 | INTRODUCTION

A fixed group G will admit many different cobounded actions on different hyperbolic metric spaces. Abbott, Balasubramanya, and Osin in [1] show that the set of equivalence classes of cobounded hyperbolic actions of a group G carries a natural partial order; see Section 2.1. The resulting poset is called the *poset of hyperbolic structures* on G, denoted as $\mathcal{H}(G)$. Roughly speaking, one action is larger than another if the smaller space can be formed by equivariantly collapsing some subspaces of the larger. The motivation is that the larger an action is in this partial order, the more information about the geometry of the group it should provide.

The posets $\mathcal{H}(G)$ remain mysterious, especially for groups with features of nonpositive curvature, which tend to have uncountable posets of hyperbolic structures [1]. First steps toward understanding these posets were made in [4, 8] and [5]. While [8] and [5] give complete descriptions of $\mathcal{H}(G)$ for the groups in question, it appears essentially impossible to do this, given current technology, for groups with strong features of nonpositive curvature. One measure of how complicated the poset $\mathcal{H}(G)$ is the (non)-existence of a *largest* element of $\mathcal{H}(G)$, that is, an element that is greater than or equal to every other element of the poset. When a largest element exists, we say that the group G is *$\mathcal{H}$-accessible*; otherwise, the group is *$\mathcal{H}$-inaccessible*. The first and third authors show that many groups with features of nonpositive curvature are $\mathcal{H}$-inaccessible, including Anosov mapping tori [5]. In this paper, we extend these results to a large class of 3-manifold groups: up to possibly taking a twofold cover of the manifold, we show that all 3-manifold groups are $\mathcal{H}$-inaccessible. This is further evidence that the poset $\mathcal{H}(G)$ is quite complicated, even for reasonably nice groups; see Section 1.1 for further discussion.

We first consider fundamental groups of nongeometric 3-manifolds with empty or toroidal boundary. If M is a compact, orientable, irreducible nongeometric 3-manifold, then there exists a nonempty minimal union $\mathcal{T}$ of disjoint essential tori in M such that each connected component of $M \setminus \mathcal{T}$ is Seifert-fibered or hyperbolic. This is called the *torus decomposition of* M, and the connected components of $M \setminus \mathcal{T}$ are called *pieces*. Our first result shows that if M contains certain types of pieces, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible. To describe these pieces, we introduce the class of *nonelementary* Seifert-fibered manifolds, which are those whose base orbifolds are orientable and hyperbolic. Nonelementary graph manifolds are those whose Seifert pieces are nonelementary, and a mixed manifold is nonelementary if all of its Seifert-fibered pieces and maximal graph manifold components are nonelementary. Any graph manifold or Seifert-fibered manifold is finitely covered by a nonelementary one. See Section 4 for a more in-depth discussion.

Theorem 1.1. *Let M be a nongeometric 3-manifold with empty or toroidal boundary. If the torus decomposition of M contains any of the following, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible:*

(1) *a hyperbolic piece that contains a boundary torus of M;*
(2) *two hyperbolic pieces glued along a torus;*
(3) *an isolated nonelementary Seifert-fibered piece; or*
(4) *a nonelementary maximal graph manifold component.*

We note a straightforward corollary of Theorem 1.1.

Corollary 1.2. *If M is one of the following types of 3-manifolds, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible:*

(a) *a finite nontrivial connected sum of finite volume cusped hyperbolic 3-manifolds;*
(b) *a nonelementary mixed manifold.*

Proof. Part (a) follows from Theorem 1.1(1). For part (b), observe that a mixed manifold must contain either: two hyperbolic pieces glued along a torus, an isolated nonelementary Seifert-fibered piece, or a nonelementary graph manifold component with boundary. In these cases, we apply Theorem 1.1 (2), (3), or (4), respectively. □

The first and third authors give a straightforward criterion to show that a group is $\mathcal{H}$-inaccessible [4, Lemma 1.4]; see Lemma 2.3. In this paper, we show in Lemma 4.3 that if a peripheral subgroup of a relatively hyperbolic group satisfies this criterion, then the group itself also satisfies the criterion. There is a natural relatively hyperbolic structure on $\pi_1(M)$ in which the peripheral subgroups are fundamental groups of certain tori, Seifert-fibered pieces, and graph manifolds. The proof Theorem 1.1 proceeds by showing that, in each case, there is a peripheral subgroup that is $\mathcal{H}$-inaccessible because it satisfies Lemma 2.3, and then applying Lemma 4.3. The key step is the following theorem.

Theorem 1.3. *If M is a nonelementary graph manifold or a nonelementary Seifert-fibered manifold, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible.*

Theorem 1.3 generalizes [4, Theorem 1.1], in which the first and third authors show that flip graph manifold groups, a special class of graph manifold groups, are $\mathcal{H}$-inaccessible.

The class of nonelementary graph manifold groups belongs to a larger class of graphs of groups called *Croke–Kleiner admissible groups* that were introduced by Croke and Kleiner in [15]. Roughly speaking, Croke–Kleiner admissible groups are modeled on the JSJ structure of graph manifolds where the (fundamental groups of the) Seifert-fibered pieces are replaced by central extensions G of general hyperbolic groups H:

$$1 \to Z(G) = \mathbb{Z} \to G \to H \to 1.$$

In some sense, Croke–Kleiner admissible groups are the simplest interesting groups constructed algebraically from a finite number of hyperbolic groups. The $\mathcal{H}$-inaccessibility of nonelementary graph manifold groups follows from the following result for Croke–Kleiner admissible groups.

Theorem 1.4. *Every Croke–Kleiner admissible group has a subgroup of index at most 2 which is $\mathcal{H}$-inaccessible.*

A nongeometric 3-manifold M always has a double cover in which all Seifert-fibered pieces are nonelementary, and hence passing to a further finite cover if necessary, we obtain a finite cover $M' \to M$ such that M' is either a nonelementary graph manifold or a nonelementary mixed manifold. Combining Theorems 1.1 and 1.3 yields the following corollary.

Corollary 1.5. *If M is a nongeometric 3-manifold, then $\pi_1(M)$ has a finite index subgroup H such that every finite index subgroup $K \leqslant H$ is $\mathcal{H}$-inaccessible.*

So far, we have only discussed nongeometric 3-manifolds. In some cases, we can also understand the $\mathcal{H}$-accessibility of (finite-index subgroups of) geometric 3-manifold groups.

Proposition 1.6. *Every 3-manifold with Nil or Sol geometry has a finite cover whose fundamental group is $\mathcal{H}$-inaccessible. The fundamental group of a closed hyperbolic 3-manifold is $\mathcal{H}$-accessible, while the fundamental group of a finite-volume cusped hyperbolic 3-manifold is $\mathcal{H}$-inaccessible.*

Proof. If a 3-manifold M has the geometry of Sol, then M is a torus bundle over a one-dimensional orbifold (an interval with reflection boundary points or a circle), and thus, M has a double cover that is a torus bundle with Anosov monodromy. The $\mathcal{H}$-inaccessibility of the fundamental group of this bundle then follows from work of the first and third authors [4].

If the geometry of M is Nil, M is a Seifert-fibered 3-manifold, and M is finitely covered by a torus bundle M' with unipotent monodromy, and the only possible hyperbolic actions are lineal and elliptic. On the other hand, the abelianization of $\pi_1(M')$ is virtually $\mathbb{Z}^2$, so this yields infinitely many homomorphisms $\mathbb{Z}^2 \to \mathbb{R}$ modulo scaling, and infinitely many inequivalent actions on

$\mathbb{R}$ by translations. Since such lineal actions are incomparable by [1, Theorem 2.3], $\pi_1(M')$ is $\mathcal{H}$-inaccessible.

The fundamental group of a closed hyperbolic 3-manifold is a hyperbolic group, and so is $\mathcal{H}$-accessible. The result for a finite-volume cusped hyperbolic 3-manifold is Corollary 4.4. □

In Section 4.4, we consider general finitely generated 3-manifold groups. This includes fundamental groups of reducible 3-manifolds, certain geometric 3-manifolds, and 3-manifolds with nontoroidal boundary. In Proposition 4.10, we characterize the H-accessibility of many such 3-manifold groups. In particular, any finitely generated fundamental group of a hyperbolic 3-manifold without rank-1 cusps is $\mathcal{H}$-accessible.

Many basic questions about posets of hyperbolic structures are still open. Surprisingly, it is still unknown whether $\mathcal{H}$-inaccessibility of a finite index normal subgroup of G passes to $\mathcal{H}$-inaccessibility of the ambient group G. In the setting of nongeometric 3-manifolds, the only cases in which we are unable to determine the $\mathcal{H}$-(in)accessibility of the fundamental group are a manifold all of whose Seifert-fibered pieces are elementary and a graph manifold whose underlying graph contains a single vertex.

Question 1.7. Let M be a graph manifold whose underlying graph contains only one vertex. Is $\pi_1(M)$ $\mathcal{H}$-inaccessible?

We suspect that the answer to this question is yes, but the techniques in this paper do not apply. See Section 4.5 for a more thorough discussion of these manifolds.

1.1 | Implications for the poset $\mathcal{H}(G)$

There is no known example of a finitely generated nonhyperbolic group with a (nontrivial) largest hyperbolic action. This paper is further evidence that if such an example exists, it is unlikely to be straightforward. In particular, it is unlikely to be found among hierarchically hyperbolic groups, a generalization of mapping class groups that includes many 3-manifold groups.

Question 1.8. Does there exist a finitely generated nonhyperbolic group whose poset of hyperbolic structures is nontrivial and has a largest element?

This is in stark contrast to the subposet $\mathcal{AH}(G) \subseteq \mathcal{H}(G)$ of *acylindrindrically hyperbolic structures*, consisting of the equivalence classes of all cobounded *acylindrical* actions on a hyperbolic space. There are many examples of nonhyperbolic groups that have a largest element in $\mathcal{AH}(G)$, including all hierarchically hyperbolic groups and hence many 3-manifold groups [1, 2]. This paper, along with [4, 5, 8], shows that the collection of cobounded hyperbolic actions, without any additional assumptions, is perhaps too complicated to hope for a single action to capture all the negative curvature of the group. Indeed, even all of the negative curvature of the (very non-hyperbolic) group $\mathbb{Z}^2$, which can been seen via its many actions on lines, cannot be captured in a single hyperbolic space. Our main tool for proving $\mathcal{H}$-inaccessibility shows this same phenomenon holds for more complicated groups: having a pair of commuting elements with opposite dynamical behavior in two hyperbolic actions is enough to obstruct the existence of a largest action.

However, one could still ask about whether there are other nice structural properties of the poset $\mathcal{H}(G)$ for certain groups. For example, does the poset have any symmetries? What is the structure of meets and joins in $\mathcal{H}(G)$? While containing a largest element is not possible for most 3-manifold groups G, one could instead ask about the set $\mathcal{M}$ of *maximal elements* of the poset. For example, does every chain in $\mathcal{H}(G)$ have a maximal element? If so, then collectively, the set $\mathcal{M}$ should capture all of the information about the hyperbolicity in G.

2 | PRELIMINARIES

2.1 | $\mathcal{H}$-accessibility

In this section, we review the partial order on cobounded group actions introduced in [1]. Fix a group G. If G acts coboundedly on two metric spaces X and Y, we say $G \curvearrowright X$ *is dominated by* $G \curvearrowright Y$, written $G \curvearrowright X \preceq G \curvearrowright Y$, if there exists a coarsely G-equivariant coarsely Lipschitz map $Y \to X$. The preorder $\preceq$ induces an equivalence relation $G \curvearrowright X \sim G \curvearrowright Y$ if and only if $G \curvearrowright X \preceq G \curvearrowright Y$ and $G \curvearrowright Y \preceq G \curvearrowright X$. It descends to a partial order $\preccurlyeq$ on the set of equivalence classes. We denote the equivalence class of an action by $[G \curvearrowright X]$.

Definition 2.1. Given a group G, the *poset of hyperbolic structures on* G is defined to be

$$\mathcal{H}(G) := \{[G \curvearrowright X] \mid G \curvearrowright X \text{ is cobounded and } X \text{ is hyperbolic}\},$$

equipped with the partial order $\preccurlyeq$.

By [1, Proposition 3.12], this is equivalent to the original definition of $\mathcal{H}(G)$ in terms of generating sets. We say that an element of a poset is *largest* when it is greater than or equal to every other element of the poset. Such an element is unique.

Definition 2.2. A group G is *$\mathcal{H}$-accessible* if the poset $\mathcal{H}(G)$ has a largest element. Otherwise, it is *$\mathcal{H}$-inaccessible*.

The following lemma gives a simple criterion to check if a group is $\mathcal{H}$-inaccessible.

Lemma 2.3 [4, Lemma 1.4]. *Let G be a group. Suppose that there are commuting elements $a, b \in G$ and hyperbolic actions $G \curvearrowright X$ and $G \curvearrowright Y$ such that*

(1) *a acts loxodromically and b acts elliptically in the action $G \curvearrowright X$; and*
(2) *b acts loxodromically in the action $G \curvearrowright Y$.*

Then, there does not exist a hyperbolic action $G \curvearrowright Z$ such that $G \curvearrowright X \preceq G \curvearrowright Z$ and $G \curvearrowright Y \preceq G \curvearrowright Z$.

We will typically apply this lemma to spaces X and Y that are quasi-isometric to lines. These actions on quasi-lines will be constructed using quasi-morphisms, which, in turn, will be constructed using hyperbolic actions of G with very weak proper discontinuity properties (see [9]).

Definition 2.4. Let $G \curvearrowright X$ be a hyperbolic action and $g \in G$ be loxodromic with fixed points $\{g^{\pm}\} \subset \partial X$. The element g is *WWPD* if the orbit of the pair (g^+, g^-) is discrete in the space $\partial X \times \partial X \setminus \Delta$, where $\Delta = \{(x, x) \mid x \in \partial X\}$ is the diagonal subset. A WWPD element g is *WWPD*$^+$ if any $h \in G$ that stabilizes $\{g^{\pm}\}$ as a *set* also fixes $\{g^{\pm}\}$ pointwise.

The following proposition follows from [10, Corollary 3.2].

Proposition 2.5. *Suppose that $G \curvearrowright X$ is a hyperbolic action with a* WWPD$^+$ *element $g \in G$. There is a homogeneous quasi-morphism $q : G \to \mathbb{R}$ such that $q(g) \neq 0$ and $q(h) = 0$ for any element $h \in G$ that acts elliptically on X.*

Homogeneous quasi-morphisms, in turn, give rise to actions on quasi-lines.

Proposition 2.6 [1, Lemma 4.15]. *Let $q : G \to \mathbb{R}$ be a nonzero homogeneous quasi-morphism. Then there is an action of G on a quasi-line $\mathcal{L}$ such that $q(g) \neq 0$ if and only if g acts loxodromically on $\mathcal{L}$.*

2.2 | Projection axioms

We will use the Bestvina–Bromberg–Fujiwara projection complex machinery developed in [9] to obtain actions on quasi-trees. In this section, we review this machinery.

Definition 2.7. Let $\mathbb{Y}$ be a collection of geodesic spaces equipped with projection maps

$$\{\pi_Y : \mathbb{Y} \setminus \{Y\} \to Y\}_{Y \in \mathbb{Y}}.$$

Let $d_Y(X, Z) = \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for $X \neq Y \neq Z \in \mathbb{Y}$. The pair $(\mathbb{Y}, \{\pi_Y\}_{Y \in \mathbb{Y}})$ satisfies the *projection axioms* for a *projection constant* $\xi \geqslant 0$ if

(1) $\operatorname{diam}(\pi_Y(X)) \leqslant \xi$ whenever $X \neq Y$;
(2) if X, Y, Z are distinct and $d_Y(X, Z) > \xi$, then $d_X(Y, Z) \leqslant \xi$; and
(3) for $X \neq Z$, the set $\{Y \in \mathbb{Y} \mid d_Y(X, Z) > \xi\}$ is finite.

For a fixed $K > 0$ and a pair $(\mathbb{Y}, \pi_Y)$ satisfying the projection axioms for some constant ξ, Bestvina, Bromberg, and Fujiwara construct a *quasi-tree of spaces* $C_K(\mathbb{Y})$ in [9]. If $\mathbb{Y}$ admits an action of the group G so that $\pi_{gY}(gX) = g\pi_Y(X)$ for any $g \in G$ and $X, Y \in \mathbb{Y}$, then G acts by isometries on $C_K(\mathbb{Y})$; see [9, Section 4.4]. Moreover, they show that if $K > 4\xi$ and $\mathbb{Y}$ is a collection of metric spaces uniformly quasi-isometric to $\mathbb{R}$, then $C_K(\mathbb{Y})$ is an unbounded quasi-tree [9, Theorem 4.14].

The following is a useful example to keep in mind throughout the paper and is discussed in detail in the introduction of [9].

Example 2.8. Let G be a discrete group of isometries of $\mathbb{H}^2$ and $g_1, \ldots, g_k \in G$ a finite collection of independent loxodromic elements with axes $\gamma_1, \ldots, \gamma_k$, respectively. Let $\mathbb{Y}$ be the set of all G-translates of $\gamma_1, \ldots, \gamma_k$, and given $Y \in \mathbb{Y}$, define π_Y to be closest point projection in $\mathbb{H}^2$. Since all translates of each γ_i are convex, this is a well-defined 1-Lipschitz map, and it follows from hyperbolicity that the projection of one translate of an axis onto another has uniformly bounded diameter. In [9], it is verified that the pair $(\mathbb{Y}, \pi_Y)$ satisfies the projection axioms for some constant ξ.

Given a pair $(\mathbb{Y}, \{\pi_Y\}_{Y\in\mathbb{Y}})$ satisfying the projection axioms and three domains $X, Y, Z \in \mathbb{Y}$, there is a notion of distance between a point of X and a point of Z from the point of view of Y, which we now describe. Let $x \in X$ and $z \in Z$. If X, Y, Z are all distinct, then define $d_Y(x, z) := d_Y(X, Z)$. If $Y = X$ and $Y \neq Z$, then define $d_Y(x, z) := \mathrm{diam}(\{x\} \cup \pi_Y(z))$, where the diameter is measured in Y. Finally, if $X = Y = Z$, then let $d_Y(x, z)$ be the distance in Y between x and z. The spaces $Y \in \mathbb{Y}$ naturally embed into $\mathcal{C}_K(\mathbb{Y})$, so the distance $d_{\mathcal{C}_K(\mathbb{Y})}(x, z)$ is also defined.

We have the following upper bound on distance in $\mathcal{C}_K(\mathbb{Y})$. Set $[t]_K = t$ if $t \geqslant K$ and $[t]_K = 0$ if $t < K$.

Proposition 2.9 [9, Lemma 4.4]. *Let $(\mathbb{Y}, \pi_{Y\in\mathbb{Y}})$ satisfy the projection axioms with constant ξ. For any K sufficiently large,*

$$d_{\mathcal{C}_K(\mathbb{Y})}(x, z) \leqslant 6K + 4 \sum_{Y\in\mathbb{Y}} [d_Y(x, z)]_K.$$

This distance formula was originally stated with a modified distance function. In [9], the distance defined above was denoted as d_Y^π, and the modified distance was denoted as d_Y. However, since $d_Y(x, z) \leqslant d_Y^\pi(x, z)$ for all point x, z, the inequality holds with the distance d_Y^π as well. As we will not need the modified distance function in this paper, we use the simpler notation d_Y for d_Y^π and choose to state the proposition with this distance.

2.3 | Croke–Kleiner-admissible groups

In this section, we review Croke–Kleiner admissible groups [15] and the associated Bass–Serre space, a notion defined in [17].

Definition 2.10. A *graph of groups* $\mathcal{G} = (\Gamma, \{G_\mu\}, \{G_\alpha\}, \{\tau_\alpha\})$ is a connected graph Γ together with a group G_σ for each $\sigma \in V(\Gamma) \cup E(\Gamma)$ (here $V(\Gamma)$ and $E(\Gamma)$ denote vertices and edges), and an injective homomorphism $\tau_\alpha : G_\alpha \to G_\mu$ for each oriented edge α, where μ denotes the terminal vertex of α.

Definition 2.11. A graph of groups $\mathcal{G} = (\Gamma, \{G_\mu\}, \{G_\alpha\}, \{\tau_\alpha\})$ is called *admissible* if the following hold.

(1) $\mathcal{G}$ is a finite graph with at least one edge.
(2) Each vertex group G_μ has center $Z_\mu := Z(G_\mu) \cong \mathbb{Z}$, $H_\mu := G_\mu/Z_\mu$ is a nonelementary hyperbolic group, and every edge group G_α is isomorphic to $\mathbb{Z}^2$.
(3) Let α_1 and α_2 be distinct edges oriented toward a vertex μ, and for $i = 1, 2$, let $K_i \subset G_\mu$ be the image of the edge homomorphism $G_{\alpha_i} \to G_\mu$. Then for every $g \in G_\mu$, gK_1g^{-1} is not commensurable with K_2, and for every $g \in G_\mu \setminus K_i$, gK_ig^{-1} is not commensurable with K_i.
(4) For every edge group G_α with $\alpha = [\alpha^-, \alpha^+]$ (oriented from α^- to α^+), the subgroup of G_α generated by $\tau_\alpha^{-1}(Z_{\alpha^+})$ and $\tau_{\bar{\alpha}}^{-1}(Z_{\alpha^-})$ has finite index in G_α.

A group G is *admissible* if it is the fundamental group of an admissible graph of groups. Such groups are often called *Croke–Kleiner admissible groups*.

Lemma 2.12 [17, Lemma 4.2]. *Let $\mathcal{G} = (\Gamma, \{G_\mu\}, \{G_\alpha\}, \{\tau_\alpha\})$ be a Croke–Kleiner admissible group. For each edge $\alpha = [\alpha^-, \alpha^+]$ of $\mathcal{G}$, denote*

$$C_\alpha = \tau_\alpha\big(\tau_{\bar{\alpha}}^{-1}(Z_{\alpha^-})\big),$$

which is a subgroup of G_{α^+}. Each vertex group G_μ has an infinite generating set S_μ so that the following holds.

(1) $\mathrm{Cay}(G_\mu, S_\mu)$ *is quasi-isometric to a line.*
(2) *The inclusion map $Z_\mu \to \mathrm{Cay}(G_\mu, S_\mu)$ is a Z_μ-equivariant quasi-isometry.*
(3) *For each edge α with $\alpha^+ = \mu$, we have that C_α is uniformly bounded in $\mathrm{Cay}(G_\mu, S_\mu)$.*

Remark 2.13. The quasi-line $\mathrm{Cay}(G_\mu, S_\mu)$ satisfies the following.

- The center Z_μ of G_μ acts loxodromically on $\mathrm{Cay}(G_\mu, S_\mu)$.
- If ω is an adjacent vertex to μ in Γ, then each cyclic subgroup of G_μ conjugate to Z_ω acts elliptically on $\mathrm{Cay}(G_\mu, S_\mu)$.

Let $\mathcal{G}$ be a graph of finitely generated groups, and let $G \curvearrowright T$ be the action of $G = \pi_1(\mathcal{G})$ on the associated Bass–Serre tree of $\mathcal{G}$ (we refer the reader to [15, Section 2.5] for a brief discussion). For each vertex v of the Bass–Serre tree T, let $\check{v}$ denote the vertex μ of Γ so that v represents gG_μ for some g in G. For each vertex group G_μ and edge group G_α, fix once and for all finite symmetric generating sets J_μ and J_α respectively, such that $J_\alpha = J_{\bar{\alpha}}$ and $\tau_\alpha(J_\alpha) \subseteq J_{\alpha^+}$.

We briefly sketch the description of the Bass–Serre space X for the graph of groups $\mathcal{G}$ and refer the reader to [17, Definition 2.10] for a full description of the space. Given a vertex v of T, the associated vertex space X_v of X is a graph isometric to $\mathrm{Cay}(G_{\check{v}}, J_{\check{v}})$. If e is a (directed) edge in T, then the associated edge space X_e is isometric to $\mathrm{Cay}(G_{\check{e}}, J_{\check{e}})$. Edges are added between the vertex and edge spaces so that the maps $\tau_{\check{e}}$ induce isometric embeddings of the edge spaces into the vertex spaces, which we denote by $\tau_e : X_e \to X_{e^+}$ and $\tau_{\bar{e}} : X_e \to X_{e^-}$.

Suppose that $\mathcal{G}$ is an admissible graph of groups with Bass–Serre tree T and Bass–Serre space X. For each vertex μ of Γ, let S_μ be given by Lemma 2.12. Without loss of generality, we can assume that J_μ is contained in S_μ, where J_μ is the fixed generating set of G_μ.

Definition 2.14 (Subspaces L_v and $\mathcal{H}_v$). Suppose that the vertex $v \in T$ represents $gG_{\check{v}}$. Let L_v be the graph with vertex set $gG_{\check{v}}$ and with an edge connecting $x, y \in gG_{\check{v}}$ if $x^{-1}y \in S_{\check{v}}$. In particular, L_v is isometric to $\mathrm{Cay}(G_{\check{e}}, S_{\check{e}})$, which is a quasi-line by Lemma 2.12.

Let $\mathcal{H}_v$ be the graph with vertex set $gG_{\check{v}}$ and with an edge connecting $x, y \in gG_{\check{v}}$ if $x^{-1}y \in J_{\check{v}} \cup Z_{\check{v}}$. It is isometric to $\mathrm{Cay}(G_{\check{v}}, J_{\check{v}} \cup Z_{\check{v}})$.

Since L_v and $\mathcal{H}_v$ are each obtained from X_v by adding extra edges, there are distance nonincreasing maps $p_v : X_v \to L_v$ and $i_v : X_v \to \mathcal{H}_v$ that are the identity on vertices. The space $\mathcal{H}_v$ is constructed to represent the geometry of $H_{\check{v}} = G_{\check{v}}/Z_{\check{v}}$ and is relatively hyperbolic:

Lemma 2.15 [17, Lemma 2.15]. *$\mathcal{H}_v$ is hyperbolic relative to the collection*

$$\mathcal{P}_v = \{\ell_e := i_v(\tau_e(X_e)) \mid e \in E(T) \text{ such that } e^+ = v\}.$$

It follows from [30] that there is a coarse closest point projection map

$$\mathrm{proj}_{\ell_e} : \mathcal{H}_v \to \ell_e,$$

which is coarsely Lipschitz with constants independent of e and v.

Remark 2.16. As peripheral subsets in a relatively hyperbolic space, the sets $\{\ell_f \mid f \in E(T)$ and $f^+ = v\}$ together with the maps proj_{ℓ_f} satisfy the projection axioms for a constant ξ_0.

We now show that if e is an edge from u to v, the various maps defined above can be composed to form a quasi-isometry between the quasi-line $\ell_e \in \mathcal{P}_u$ and the quasi-line L_v. Let $\psi_e : \ell_{\bar{e}} \to L_v$ be the map from [17, Lemnma 6.16] that is defined as the restriction to $\ell_{\bar{e}}$ of the composition

$$p_v \circ \tau_e \circ \tau_{\bar{e}}^{-1} \circ i_u^{-1}. \tag{1}$$

In [17, Lemma 6.16], the authors prove that ψ_e is coarsely Lipschitz and note that ψ_e is, in fact, a quasi-isometry. Here, we provide details for why it follows that ψ_e is a quasi-isometry. First, we prove a more general result.

Lemma 2.17. *Let ℓ_1, ℓ_2 be two quasi-lines, and suppose that a group G acts coboundedly on both ℓ_1 and ℓ_2. Any G-equivariant coarsely Lipschitz map $\psi : \ell_1 \to \ell_2$ is a quasi-isometry.*

Proof. Since there is a G-equivariant coarsely Lipschitz map from ℓ_1 to ℓ_2, we have $[G \curvearrowright \ell_1] \succcurlyeq [G \curvearrowright \ell_2]$ in the poset $\mathcal{H}(G)$. However, since $G \curvearrowright \ell_1$ and $G \curvearrowright \ell_2$ are both lineal, [1, Theorem 4.22] implies that these actions must be equivalent. Thus, there is a coarsely G-equivariant quasi-isometry $\Phi : \ell_1 \to \ell_2$. We will show that Φ and ψ differ by a uniformly bounded amount, which will then show that ψ is also a quasi-isometry.

Fix a basepoint $x_0 \in \ell_1$. Since $G \curvearrowright \ell_1$ is cobounded, there is a constant B such that for any $x \in \ell_1$, there is some $g \in G$ such that $d_{\ell_1}(x, gx_0) \leqslant B$. Since ψ is coarsely Lipschitz and Φ is a quasi-isometry, there is a constant A, depending on B and the coarse Lipschitz constants for Φ and ψ, such that $d_{\ell_2}(\Phi(x), \Phi(gx_0)) \leqslant A$ and $d_{\ell_2}(\psi(x), \psi(gx_0)) \leqslant A$. Moreover, since Φ is coarsely G-equivariant, there is a constant C such that $d_{\ell_2}(\Phi(gx_0), g\Phi(x_0)) \leqslant C$. Let $D = d_{\ell_2}(\Phi(x_0), \psi(x_0))$.

By the triangle inequality and G-equivariance of ψ, we have

$$\begin{aligned} d_{\ell_2}(\Phi(x), \psi(x)) &\leqslant d_{\ell_2}(\Phi(x), g\Phi(x_0)) + d_{\ell_2}(g\Phi(x_0), g\psi(x_0)) + d_{\ell_2}(\psi(gx_0), \psi(x)) \\ &\leqslant (A + C) + D + A, \end{aligned}$$

completing the proof. $\square$

We now complete the proof that ψ_e is a quasi-isometry.

Lemma 2.18. *There are constants $\lambda \geqslant 1$ and $c \geqslant 0$ depending only on $\mathcal{G}$ such that the following holds. For any oriented edge e in the Bass–Serre tree T of $\mathcal{G}$, the map $\psi_e : \ell_{\bar{e}} \to L_v$ is a (λ, c)-quasi-isometry.*

Proof. In [17, Lemma 6.16], the authors prove that the map ψ_e is coarsely Lipschitz. Moreover, from the definitions of $\ell_{\bar{e}}$ and L_v as Cayley graphs with respect to infinite generating sets, $G_{\check{e}}$ acts by isometries on both, and ψ_e is $G_{\check{e}}$-equivariant. Therefore, ψ_e is a quasi-isometry by Lemma 2.17. As there are only finitely many G-orbits of edges in T, we can choose the constants of these quasi-isometries to be independent of the edge e. □

3 | CROKE–KLEINER ADMISSIBLE GROUPS AND $\mathcal{H}$-INACCESSIBILITY

In this section, we prove Theorem 1.4: every Croke–Kleiner admissible group has a finite index subgroup that is $\mathcal{H}$-inaccessible.

Fix a Croke–Kleiner admissible group $\mathcal{G} = (\Gamma, \{G_\mu\}, \{G_\alpha\}, \{\tau_\alpha\})$. We partition the vertex set T^0 of the Bass–Serre tree into two disjoint collections of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ such that if v and v' are in $\mathcal{V}_i$, then $d_T(v, v')$ is even. Since any automorphism of T either preserves $\mathcal{V}_1$ and $\mathcal{V}_2$ setwise or interchanges them, we have the following.

Lemma 3.1 [23, Lemma 4.6]. *Let $\mathcal{G} = (\Gamma, \{G_\mu\}, \{G_\alpha\}, \{\tau_\alpha\})$ be a Croke–Kleiner admissible group. There exists a subgroup $G' \leqslant G = \pi_1(\mathcal{G})$ of index at most 2 in G so that G' preserves $\mathcal{V}_1$ and $\mathcal{V}_2$ and G' is also a Croke–Kleiner admissible group.*

Let G' be the finite index subgroup of G given by Lemma 3.1. In light of Lemma 2.3, to show that G' is $\mathcal{H}$-inaccessible, it suffices to construct commuting $a_i \in G'$ and actions $G' \curvearrowright X_i$ for $i = 1, 2$ such that a_i is elliptic with respect to the action $G' \curvearrowright X_{3-i}$ and loxodromic with respect to the action $G' \curvearrowright X_i$. Our spaces X_i will be quasi-trees of metric spaces.

3.1 | Construction of group actions

For notational simplicity, we replace G by its index $\leqslant 2$ subgroup G'. For each vertex v in the Bass–Serre tree T, let L_v be the quasi-line from Definition 2.14. Recall that $gL_v = L_{gv}$ for any group element g in G.

Let $\mathbb{L}_1$ be the collection of quasi-lines $\{L_v\}_{v \in \mathcal{V}_1}$ and $\mathbb{L}_2$ be the collection of quasi-lines $\{L_v\}_{v \in \mathcal{V}_2}$. We define a projection of L_v to $L_{v'}$ in $\mathbb{L}_i$ as follows.

Definition 3.2 (Projection maps in $\mathbb{L}_i$). For any two distinct vertices $v, v' \in \mathcal{V}_i$, let $e' = [w, u]$ and $e = [u, v]$ denote the last two (oriented) edges in $[v', v]$. The *projection* from $L_{v'}$ into L_v is

$$\Pi_{L_v}(L_{v'}) := \psi_e(\mathrm{proj}_{\ell_{\bar{e}}}(\ell_{e'})),$$

where $\psi_e : \ell_{\bar{e}} \to L_v$ and $\mathrm{proj}_{\ell_{\bar{e}}} : \mathcal{H}_u \to \ell_{\bar{e}}$ are the maps introduced in Section 2.

The fact that $d(v, v')$ is even is not necessary for Definition 3.2, only that $d(v, v') \geqslant 2$.

We will verify that the $\mathbb{L}_i$ with these projection maps satisfy the projection axioms (see Definition 2.7) for $i = 1, 2$. Let $d_{L_a}(L_b, L_c)$ be the projection distance $\operatorname{diam}\big(\Pi_{L_a}(L_b) \cup \Pi_{L_a}(L_b)\big)$.

Lemma 3.3. *There exists a constant $\lambda > 0$ such that $\operatorname{diam}(\Pi_{L_v}(L_{v'})) \leqslant \lambda$ for any distinct $v, v' \in \mathcal{V}_i$ for $i = 1, 2$. Moreover, if $a, b, c \in \mathcal{V}_i$ are distinct vertices with $d_T(a, [b, c]) \geqslant 2$, then $\Pi_{L_a}(L_c) = \Pi_{L_a}(L_b)$.*

Proof. By Remark 2.16, there is a uniform bound on the diameter of $\mathrm{proj}_{\ell_{\bar{e}}}(\ell_{e'})$. Combined with the fact that ψ_e is uniformly coarsely Lipschitz, this gives the constant λ. Considering the convex hull of $\{a\} \cup [b, c]$, we see that, orienting $[c, a]$ and $[b, a]$ toward a, the last two edges of $[c, a]$ are the same as the last two edges of $[b, a]$. Hence, by definition, $\Pi_{L_a}(L_c) = \Pi_{L_a}(L_b)$. □

Let v be a vertex of the Bass–Serre tree T. By Remark 2.16, the collection $\{\ell_f = i_v(\tau_f(X_f)) \mid f \in E(T)$ such that $f^+ = v\}$ satisfies the projection axioms with a constant ξ_0. Let d_ℓ denote the projection distances with respect to proj_ℓ. The following lemma follows immediately from Lemma 2.18 and the definitions of d_{ℓ_e} and d_{L_v}.

Lemma 3.4. *There exists a constant $\lambda > 0$ such that the following holds. Let u, v, w be distinct vertices in $\mathcal{V}_1$ contained in* $\mathrm{Lk}(o)$ *for some vertex o in $\mathcal{V}_2$. Let $e = [w, o]$, $e_1 = [u, o]$, and $e_2 = [v, o]$. Then*

$$\frac{1}{\lambda} d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) - \lambda \leqslant d_{L_w}(L_u, L_v) \leqslant \lambda d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) + \lambda.$$

We are now ready to verify the projection axioms.

Proposition 3.5. *There exists $\xi > 0$ such that for each $i \in \{1, 2\}$, $\mathbb{L}_i$ together with the projection maps* proj_ℓ *satisfies the projection axioms.*

Proof. We verify the projection axioms for $\mathbb{L}_1$. The case for $\mathbb{L}_2$ is identical. The constant ξ will be defined explicitly during the proof.

Axiom 1: This follows from Lemma 3.3.

Axiom 2: Let u, v, w be distinct vertices in $\mathcal{V}_1$. In the course of the proof, we will compute a constant $\xi > 0$ such that if $d_{L_w}(L_u, L_v) > \xi$, then $d_{L_u}(L_w, L_v) \leqslant \xi$.

Since $d_{L_w}(L_u, L_v) > 0$, it follows from Lemma 3.3 that either w lies on $[u, v]$ or $d_T(w, [u, v]) = 1$. If w lies on $[u, v]$, then since $u, w, v \in \mathcal{V}_1$, we have $d_T(u, [w, v]) \geqslant 2$ and $d_T(v, [u, w]) \geqslant 2$. Axiom 2 thus follows from Lemma 3.3.

On the other hand, suppose that $d(w, [u, v]) = 1$. Let $o \in [u, v]$ be adjacent to w and consider the vertices $u', v' \in \mathrm{Lk}(o) \cap [u, v]$ that lie in $[u, o]$ and $[o, v]$, respectively. If $u \neq u'$, then $d_{L_u}(L_w, L_v) = 0$, and so, we may assume without loss of generality that $u = u'$. Furthermore, $\pi_{L_u}(L_v) = \pi_{L_u}(L_{v'})$ by definition. Thus, to prove the upper bound on $d_{L_u}(L_w, L_v)$, it suffices to assume that $v = v'$, in which case u, v, w all lie in $\mathrm{Lk}(o)$, where $o \in \mathcal{V}_2$.

Let $e = [w, o]$, $e_1 = [u, o]$, and $e_2 = [v, o]$. It follows from Lemma 3.4 that

$$\frac{1}{\lambda} d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) - \lambda \leqslant d_{L_w}(L_u, L_v) \leqslant \lambda d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) + \lambda.$$

Again applying Lemma 3.4 with the roles of u, v, w exchanged, we have that

$$\frac{1}{\lambda} d_{\ell_{e_1}}(\ell_e, \ell_{e_2}) - \lambda \leqslant d_{L_u}(L_w, L_v) \leqslant \lambda d_{\ell_{e_1}}(\ell_e, \ell_{e_2}) + \lambda.$$

Since $\{\ell_f \mid f \in E(T)$ and $f^+ = o\}$ satisfies the projection axioms with constant ξ_0, it follows that $d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) > \xi_0$ implies that $d_{\ell_{e_1}}(\ell_e, \ell_{e_2}) \leqslant \xi_0$. Since there are finitely many choices for o up to the action G', the constant ξ_0 may be chosen independently of o. Thus, setting $\xi = \lambda\xi_0 + \lambda$, the above inequalities show that $d_{L_W}(L_u, L_v) > \xi$ implies that $d_{L_u}(L_w, L_v) \leqslant \xi$. This verifies Axiom 2.

Axiom 3: For distinct $u, v \in \mathcal{V}_1$, we will prove the set

$$\{w \in \mathcal{V}_1 \mid d_{L_w}(L_u, L_v) > \xi\}$$

is finite. By Lemma 3.3, any such vertex w is either contained in the interior of $[u, v]$ or satisfies $d(w, [u, v]) = 1$. The first case yields at most $d(u, v) - 1$ choices for w.

Suppose $d(w, [u, v]) = 1$. As in the proof of Axiom 2, we can assume that u, v, w lie in $\mathrm{Lk}(o)$ for some vertex o in $\mathcal{V}_2$. Let $e_1 = [u, o], e_2 = [v, o]$, and $e = [w, o]$. By Lemma 3.4, we have $d_{L_w}(L_u, L_v) \leqslant \lambda d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) + \lambda$. Since $\xi = \lambda\xi_0 + \lambda$, it follows that

$$\{w \in \mathrm{Lk}(o) \,|\, d_{L_w}(L_u, L_v) > \xi\} \subset \{w \in \mathrm{Lk}(o) \,|\, d_{\ell_e}(\ell_{e_1}, \ell_{e_2}) > \xi_0\}.$$

The projection axioms for $\{\ell_f \mid f \in E(T) \text{ and } f^+ = v\}$ imply that the latter set is finite, and so, the former set must also be finite. Since there are finitely many possibilities for o, this verifies Axiom 3. □

Lemma 3.6. *For each $i = 1, 2$, the action of $G = \pi_1(\mathcal{G})$ on the collection $\mathbb{L}_i = \{L_v \mid v \in \mathcal{V}_i\}$ satisfies*

$$\Pi_{gL_v}(gL_u) = g\Pi_{L_v}(L_u)$$

for any $v \in \mathcal{V}_i$ and any $g \in G$.

Proof. This follows immediately from the definition of Π and the fact that the maps proj and ψ are G-equivariant in the sense that $\mathrm{proj}_{\overline{ge}}(\ell_{gf}) = g \cdot \mathrm{proj}_{\bar{e}}(\ell_f)$ and $\psi_{ge}(gx) = g\psi_e(x)$. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let G' be the finite index subgroup of G given by Lemma 3.1, which is also a Croke–Kleiner admissible group. Without loss of generality, we replace G by G' for the rest of the proof.

By Proposition 3.5, the collection of quasi-lines $\mathbb{L}_i = \{L_v \mid v \in \mathcal{V}_i\}$ satisfies the projection axioms with a constant ξ for $i = 1, 2$. Fix $K > 4\xi$. The unbounded quasi-trees of metric spaces $\mathcal{C}_K(\mathbb{L}_1)$ and $\mathcal{C}_K(\mathbb{L}_2)$ are themselves quasi-trees, and they admit unbounded isometric actions $G \curvearrowright \mathcal{C}_K(\mathbb{L}_1)$ and $G \curvearrowright \mathcal{C}_K(\mathbb{L}_2)$.

Since the underlying graph of G is bipartite, we can choose an edge $\check{e}$ in Γ which is not a loop. Choosing the orientation of $\check{e}$ correctly, $\mu = \check{e}^-$ and $\omega = \check{e}^+$ have lifts in T belonging to $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. By construction, elements of Z_μ and Z_ω are loxodromic and elliptic in the action on $\mathcal{C}_K(\mathbb{L}_1)$, respectively, and elliptic and loxodromic in the action on $\mathcal{C}_K(\mathbb{L}_2)$, respectively. By Lemma 2.3, we conclude that the group G is $\mathcal{H}$-inaccessible. □

Every graph manifold has a finite cover that is a graph manifold N containing at least two Seifert-fibered spaces such that each Seifert-fibered piece has orientable, hyperbolic base orbifold. We call such a graph manifold *nonelementary* in Section 4. Since $\pi_1(N)$ is a Croke–Kleiner admissible group, the following corollary is immediate from Theorem 1.4.

Corollary 3.7. *Every graph manifold has a finite cover whose fundamental group is $\mathcal{H}$-inaccessible.*

It is still unknown whether $\mathcal{H}$-inaccessibility of a finite index normal subgroup of G passes to $\mathcal{H}$-inaccessibility of the ambient group G. Thus, it is natural to ask whether the "finite cover" condition in Corollary 3.7 can be removed. We will address this question in the following section.

4 | $\mathcal{H}$-ACCESSIBILITY OF 3-MANIFOLD GROUPS

The goal of this section is to prove Theorem 1.1, which gives conditions under which the fundamental group of a nongeometric 3-manifold is $\mathcal{H}$-inaccessible.

We begin by recalling some definitions and facts about 3-manifolds. Let M be a compact, connected, orientable, irreducible 3-manifold with empty or toroidal boundary. By the geometrization theorem for 3-manifolds of Perelman [24–26] and Thurston, either

(1) the manifold M is *geometric*, in the sense that its interior admits one of the following geometries: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$, Nil, and Sol; or
(2) the manifold M is *nongeometric*. In this case, the torus decomposition of 3-manifolds yields a nonempty minimal union $\mathcal{T} \subset M$ of disjoint essential tori, unique up to isotopy, such that each component of $M \backslash \mathcal{T}$ is either a Seifert-fibered piece or a hyperbolic piece.

We refer the reader to [29] for background on geometric structures on 3-manifolds. A Seifer-fibered piece is called *nonelementary* if its base orbifold is orientable and hyperbolic, and it is called *isolated* if it is not glued to any other Seifert-fibered piece.

The manifold M is called a *graph manifold* if all the pieces of $M \backslash \mathcal{T}$ are Seifert-fibered. A graph manifold is *nonelementary* if it contains at least two pieces and all pieces are nonelementary. In other words, a nonelementary graph manifold is obtained by gluing at least two and at most finitely many nonelementary Seifer-fibered manifolds, where the gluing maps between the Seifert components do not identify (unoriented) Seifert fibers up to homotopy.

We will call a nongeometric manifold M a *mixed manifold* if it is not a graph manifold. If there is a subcollection $\mathcal{T}'$ of $\mathcal{T}$ and a connected component of $M \backslash \mathcal{T}'$ that is a graph manifold, then this connected component is called a *graph manifold component* of the mixed manifold M. A graph manifold component is *maximal* if it is not properly contained in another graph manifold component. A mixed manifold is *nonelementary* if all maximal graph manifold components and Seifert-fibered pieces are nonelementary.

Remark 4.1. Every graph (respectively, mixed) manifold is finitely covered by a nonelementary graph (respectively, mixed) manifold (see, e.g., [27, Lemma 3.1], [19, Lemma 2.1]).

Our starting point for proving Theorem 1.1 is the following lemma, which describes when $\pi_1(M)$ is relatively hyperbolic.

Lemma 4.2 [12, 16]. *Let $M_1, \dots, M_k$ be the maximal graph manifold components and Seifert-fibered pieces of the torus decomposition of M. Let $S_1, \dots, S_\ell$ be the tori in the boundary of M that bound a hyperbolic piece, and let $T_1, \dots, T_m$ be the tori in the torus decomposition of M that separate two* *hyperbolic components. Then $\pi_1(M)$ is hyperbolic relative to*

$$\mathbb{P} = \{\pi_1(M_p)\}_{p=1}^{k} \cup \{\pi_1(S_q)\}_{q=1}^{\ell} \cup \{\pi_1(T_r)\}_{r=1}^{m}.$$

This relatively hyperbolic structure on $\pi_1(M)$ is useful because of the following result, which gives a criterion for relatively hyperbolic groups to be $\mathcal{H}$-inaccessible.

Lemma 4.3. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group. If there is a peripheral subgroup $P \in \mathbb{P}$ that satisfies the hypotheses of Lemma 2.3, then G is $\mathcal{H}$-inaccessible.*

Before proving the lemma, we state an immediate corollary, which gives a different proof of [4, Theorem 6.2].

Corollary 4.4 [4, Theorem 6.2]. *The fundamental group of a finite-volume cusped hyperbolic 3-manifold is $\mathcal{H}$-inaccessible.*

We now turn to the proof of Lemma 4.3.

Proof of Lemma 4.3. To see that $\mathcal{H}(G)$ does not contain a largest element, we will construct two actions of G on hyperbolic spaces with commuting elements $a, b \in G$ that satisfy the hypotheses of Lemma 2.3. To do this, we will apply the machinery of *induced actions* from [3].

Since P satisfies the hypotheses of Lemma 2.3 by assumption, there are commuting elements $a, b \in P$ and isometric actions $P \curvearrowright X$ and $P \curvearrowright Y$ on hyperbolic spaces such that a and b act loxodromically and elliptically, respectively, in the action $P \curvearrowright X$, and b acts loxodromically in the action $P \curvearrowright Y$. For all $Q \in \mathbb{P} \setminus \{P\}$, fix the trivial action of Q on a point. By [3, Corollary 4.11(a)], there exist hyperbolic spaces Z_X, Z_Y on which G acts by isometries, associated to the collection of actions $\{Q \curvearrowright * \mid Q \in \mathbb{P} \setminus \{P\}\} \cup \{P \curvearrowright X\}$ and the collection of actions $\{Q \curvearrowright * \mid Q \in \mathbb{P} \setminus \{P\}\} \cup \{P \curvearrowright Y\}$, respectively. Moreover, there are coarsely P-equivariant quasi-isometric embeddings $X \to Z_X$ and $Y \to Z_Y$. Therefore, a acts loxodromically and b acts elliptically in the action $G \curvearrowright Z_X$, while b acts loxodromically in the action $G \curvearrowright Z_Y$. This completes the proof. $\square$

In light of Lemmas 4.2 and 4.3, to prove the $\mathcal{H}$-inaccessibility of $\pi_1(M)$, it suffices to understand its peripheral subgroups. In Section 4.1, we analyze the fundamental groups of the Seifert-fibered pieces. The more difficult subgroups to understand are the fundamental groups of the maximal graph manifold components. We consider these in Section 4.2 and give conditions under which they satisfy Lemma 2.3; see Proposition 4.8. In Section 4.3, we put these results together and prove Theorem 1.1. Up to this point, we have been assuming that M has empty or toroidal boundary. Finally, in Section 4.4, we consider 3-manifolds with higher genus boundary components.

4.1 | Seifert-fibered manifolds

In this section, we analyze Seifert-fibered pieces.

Lemma 4.5. *Let $1 \to \mathbb{Z} \xrightarrow{i} G \xrightarrow{\pi} H \to 1$ be a short exact sequence where $\mathbb{Z}$ is central in G and H is a nonelementary hyperbolic group. Then, G is $\mathcal{H}$-inaccessible.*

Proof. Choose a finite generating set J of H and consider the hyperbolic action $G \curvearrowright \mathrm{Cay}(H, J)$. Let a be a generator of the group $\mathbb{Z}$, and let b be an element of G such that $\pi(b)$ is loxodromic in $H \curvearrowright \mathrm{Cay}(H, J)$. The element b is thus loxodromic in the action $G \curvearrowright \mathrm{Cay}(H, J)$, as well, while the element a is elliptic (in fact, trivial) in this action.

Since every integral cohomology class of a hyperbolic group is bounded (see [22]), the central extension $\mathbb{Z} \to G \to H$ corresponds to a bounded element of $H^2(H, \mathbb{Z})$. Hence, [17, Lemma 4.1] provides a quasi-morphism $\phi : G \to \mathbb{Z}$ that is unbounded on $i(\mathbb{Z})$. By [1, Lemma 4.15], there exists a generating set S for G such that $L := \mathrm{Cay}(G, S)$ is a quasi-line and the inclusion $\mathbb{Z} \to L$ induced by i is a $\mathbb{Z}$-equivariant quasi-isometry. We thus obtain a hyperbolic action $G \curvearrowright L$ for which a is loxodromic. Since $a \in Z(G)$, the elements a and b commute. By Lemma 2.3, G is $\mathcal{H}$-inaccessible. □

Corollary 4.6. *Let M be a nonelementary Seifert-fibered manifold. Then, $\pi_1(M)$ is $\mathcal{H}$-inaccessible.*

Proof. Let $\varphi : M \to \Sigma$ be a Seifert fibration. Since $S^1 \to M \to \Sigma$ is a circle bundle over Σ, there is a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(\Sigma) \to 1,$$

where $\mathbb{Z}$ is the normal cyclic subgroup of $\pi_1(M)$ generated by a fiber. The group $\mathbb{Z}$ is central in $\pi_1(M)$ since Σ is orientable (see, e.g., [20, Proposition 10.4.4]). By Lemma 4.5, the group $\pi_1(M)$ is $\mathcal{H}$-inaccessible. □

4.2 | $\mathcal{H}$-accessibility of nonelementary graph manifolds

Let M be a three-dimensional nonelementary graph manifold with Seifert-fibered pieces $M_1, \ldots, M_k$ in its torus decomposition. There is an induced graph-of-groups structure $\mathcal{G}$ on $\pi_1(M)$ with underlying graph Γ as follows. There is a vertex of Γ for each M_i, with vertex group $\pi_1(M_i)$. Each edge group is $\mathbb{Z}^2$, the fundamental group of a torus in the decomposition. The edge monomorphisms come from the two different gluings of the torus into the two adjacent Seifert-fibered components. With this graph of groups structure, $\pi_1(M)$ is a Croke–Kleiner admissible group.

The universal cover $\widetilde{M}$ of M is tiled by a countable collection of copies of the universal covers $\widetilde{M}_1, \ldots, \widetilde{M}_k$. We call these subsets *vertex spaces*. We refer to boundary components of vertex spaces as *edge spaces*. Two vertex spaces are either disjoint or intersect along an edge space. Let T be the Bass–Serre tree of $\mathcal{G}$.

Applying Theorem 1.4 to the Croke–Kleiner admissible group $\pi_1(M)$, we obtain a cover $M' \to M$ of degree 2 such that $\pi_1(M')$ is not $\mathcal{H}$-accessible. However, this is not enough to conclude $\mathcal{H}$-inaccessibility of $\pi_1(M)$, as it is unknown whether $\mathcal{H}$-inaccessibility of a finite index subgroup passes to the ambient group. In this section, we will show that $\pi_1(M)$ itself is $\mathcal{H}$-inaccessible; see Proposition 4.8.

We begin with a lemma. Let $\pi_\alpha(\beta)$ be the closest point projection of a line β to a line α in a hyperbolic space. Let $d_\alpha(\cdot, \cdot)$ be in the resulting projection distances.

Lemma 4.7. *Let F be a two-dimensional hyperbolic orbifold with nonempty boundary and universal cover $\widetilde{F}$, and let $\mathbb{L}$ be the collection of boundary lines of $\widetilde{F}$. For any $\alpha \in \mathbb{L}$ and any loxodromic*

$\gamma \in \pi_1(F)$ whose axis in $\widetilde{F}$ is also a line in $\mathbb{L}$, the following holds. There exists a constant $\lambda > 0$ such that for any $n \in \mathbb{Z}$ and any line $\beta \in \mathbb{L} \setminus \{\alpha, \gamma^n(\alpha)\}$, we have

$$d_\beta(\alpha, \gamma^n(\alpha)) \leqslant \lambda.$$

The proof of this lemma is very similar to that of [4, Lemma 5.6]. We refer the reader to that paper for some figures that may be helpful; see, in particular, [4, Figure 8].

Proof of Lemma 4.7. Since $\mathbb{L}$ is a $\pi_1(F)$-invariant collection of axes in the hyperbolic plane $\mathbb{H}^2$ with disjoint limit sets, it follows from Example 2.8 that $(\mathbb{L}, \pi_\ell)$ satisfies the projection axioms for some constant ξ. In particular, there exists a constant $\xi > 1$ such that $\operatorname{diam}(\pi_\ell(\ell')) \leqslant \xi$ for distinct elements ℓ and ℓ' in $\mathbb{L}$. Let

$$\lambda = \max\{\xi, d(\alpha, \gamma(\alpha)) + 2\xi, d(\alpha, \gamma^2(\alpha)) + 2\xi\}.$$

Let $\ell \in \mathbb{L}$ denote the axis of γ in $\widetilde{F}$. If $\ell = \alpha$, then

$$d_\beta(\alpha, \gamma^n(\alpha)) = d_\beta(\alpha, \alpha) \leqslant \xi \leqslant \lambda,$$

and the result holds. For the remainder of the proof, we assume $\ell \neq \alpha$ and consider two cases.

Case 1: $\beta \notin \{\gamma^k(\alpha) \,|\, k \in \mathbb{Z}\}$

In this case, there exists a unique $k_0 \in \mathbb{Z}$ such that β lies between $\gamma^{k_0}(\alpha)$ and $\gamma^{k_0+1}(\alpha)$. That is, $\partial\mathbb{H}^2$ minus the endpoints of $\gamma^{k_0}(\alpha)$ and $\gamma^{k_0+1}(\alpha)$ consists of four intervals, one containing the endpoints of β, one containing the endpoints of all $\gamma^i(\alpha)$ for $i \notin \{k_0, k_0+1\}$, and the other two disjoint from all endpoints of lines in $\mathbb{L}$. Fixing an appropriate orientation on β, we partially order subintervals $I = [x, y]$ and $J = [z, w]$ of β (with $x \leqslant y$ and $z \leqslant w$ in the orientation) by $I \leqslant J$ if $x \leqslant z$ and $y \leqslant w$. Then, the projections of the lines $\gamma^k(\alpha)$ onto β occur in the following order:

$$\pi_\beta\big(\gamma^{k_0}(\alpha)\big) < \pi_\beta\big(\gamma^{k_0-1}(\alpha)\big) < \pi_\beta\big(\gamma^{k_0-2}(\alpha)\big) < \ldots < \pi_\beta(\ell)$$

and

$$\pi_\beta(\ell) < \ldots < \pi_\beta\big(\gamma^{k_0+3}(\alpha)\big) < \pi_\beta\big(\gamma^{k_0+2}(\alpha)\big) < \pi_\beta\big(\gamma^{k_0+1}(\alpha)\big).$$

Thus,

$$d_\beta(\alpha, \gamma^n(\alpha)) \leqslant d_\beta\big(\gamma^{k_0}(\alpha), \gamma^{k_0+1}(\alpha)\big) \leqslant d\big(\gamma^{k_0}(\alpha), \gamma^{k_0+1}(\alpha)\big) + 2\xi,$$

where $d(\gamma^{k_0}(\alpha), \gamma^{k_0+1}(\alpha))$ denotes the distance between $\gamma^{k_0}(\alpha)$ and $\gamma^{k_0+1}(\alpha)$ in the hyperbolic plane. The final inequality follows from the fact that the nearest point projection is a 1-Lipschitz map and that $\pi_\ell(\ell')$ has diameter at most ξ for any distinct lines $\ell, \ell' \in \mathbb{L}$. Since $d(\gamma^{k_0}(\alpha), \gamma^{k_0+1}(\alpha))) = d(\alpha, \gamma(\alpha))$, it follows that

$$d_\beta(\alpha, \gamma^n(\alpha)) \leqslant d(\alpha, \gamma(\alpha)) + 2\xi \leqslant \lambda.$$

Case 2: $\beta = \gamma^k(\alpha)$ for some integer $k \neq 0, n$. Using an analogous argument to Case 1, we see that $d_\beta(\alpha, \gamma^n(\alpha))$ is bounded above by

$$d_\beta(\gamma^{k-1}(\alpha), \gamma^{k+1}(\alpha)) \leqslant d(\alpha, \gamma^2(\alpha)) + 2\xi \leqslant \lambda. \qquad \square$$

Proposition 4.8. *The fundamental group of a nonelementary graph manifold is $\mathcal{H}$-inaccessible.*

Proof. Let $\mathcal{G}$ be the graph-of-groups structure on $\pi_1(M)$ with underlying graph Γ described at the beginning of this section. The assumption that the graph manifold M is nonelementary ensures that there are at least two vertices in the graph Γ. We divide the proof into two cases, depending on the location of loops in Γ.

Fix an edge α in Γ that is not a loop, and label the vertex α^- by μ and the vertex α^+ by ω. Let T_α be the torus in M associated to the edge α. Let v and w be two adjacent vertices in the tree T such that $\widetilde{M}_v$ and $\widetilde{M}_w$ are the universal covers of the Seifert pieces M_μ and M_ω, respectively. Let z_μ and z_ω be the generators of Z_μ and Z_ω, respectively.

Case 1: Suppose that there is no loop in Γ based at the vertex μ. Let

$$\mathbb{W}_\mu := \left\{ L_v \,\middle|\, \widetilde{M}_v \text{ is a lift of the Seifert-fibered piece } M_\mu \right\}$$

If L_v and $L_{v'}$ are two distinct elements in $\mathbb{W}_\mu$, then $d(v, v') \geqslant 2$ (though they are not necessarily an even distance apart). In this case, the techniques in Section 3 apply to define projection maps between L_v and $L_{v'}$. Note that the assumption $d(v, v') \geqslant 2$ is necessary in order to make such a definition. The proof of Proposition 3.5 applies to show that the projection axioms are satisfied for $\mathbb{W}_\mu$. This yields a cobounded action $\pi_1(M) \curvearrowright \mathcal{C}_K(\mathbb{W}_\mu)$ such that z_μ is loxodromic and z_ω is elliptic.

Case 2: Suppose that there is a loop in Γ based at the vertex μ.

As in Section 3, we partition the vertex set T^0 into two disjoint collections of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ such that if z and z' both lie in $\mathcal{V}_i$, then $d(z, z')$ is even. Applying Theorem 1.4 to the Croke–Kleiner admissible group $\pi_1(M)$, we obtain a degree 2 cover $M' \to M$ such that $\pi_1(M')$ is $\mathcal{H}$-inaccessible and $\pi_1(M')$ preserves $\mathcal{V}_1$ and $\mathcal{V}_2$.

Assume without loss of generality that v is in $\mathcal{V}_1$ and w is in $\mathcal{V}_2$. Let

$$\mathbb{Q}_\mu := \left\{ L_u \,\middle|\, u \in \mathcal{V}_1 \text{ and } \widetilde{M}_u \text{ is a lift of } M_\mu \right\}.$$

As in the previous case, the techniques of Section 3 suffice to define projection maps for $\mathbb{Q}_\mu$ and the proof of Proposition 3.5 shows that the projection axioms are satisfied by $\mathbb{Q}_\mu$, and thus, we obtain quasi-trees of spaces $\mathcal{C}_K(\mathbb{Q}_\mu)$ for sufficiently large K. Since $\pi_1(M')$ preserves $\mathbb{Q}_\mu$, we obtain an action $\pi_1(M') \curvearrowright \mathcal{C}_K(\mathbb{Q}_\mu)$ as in Section 3.

Passing to a power of two if necessary, we may assume that $z_\mu, z_\omega \in \pi_1(M')$. As shown in the proof of Theorem 1.4, the element z_μ acts loxodromically on $\mathcal{C}_K(\mathbb{Q}_\mu)$, while z_ω acts elliptically on $\mathcal{C}_K(\mathbb{Q}_\mu)$. By the construction of $\mathcal{C}_K(\mathbb{Q}_\mu)$, and since vertex groups are central extensions of $\mathbb{Z}$, the element z_μ is a WWPD$^+$ element in the action $\pi_1(M') \curvearrowright \mathcal{C}_K(\mathbb{Q}_\mu)$. Hence, Proposition 2.5 provides a homogeneous quasi-morphism $q_K : \pi_1(M') \to \mathbb{R}$ satisfying $q_K(z_\omega) = 0$ and $q_K(z_\mu) \neq 0$.

Our goal is to extend q_K to a homogeneous quasi-morphism $\pi_1(M) \to \mathbb{R}$ while ensuring that z_ω and z_μ still have trivial and nontrivial image, respectively. Let $h \in \pi_1(M)$ be a representative of the nontrivial coset of $\pi_1(M')$ in $\pi_1(M)$. Define a function $q'_K : \pi_1(M') \to \mathbb{R}$ by

$$q'_K(x) := q_K(x) + q_K(hxh^{-1}).$$

Note that q'_K is constant on conjugacy classes of $\pi_1(M)$, that is, $q'_K(yxy^{-1}) = q'_K(x)$ for any $y \in \pi_1(M)$ and $x \in \pi_1(M')$). Hence, it follows from the proof of [11, Lemma 7.2] that q'_K extends to a homogeneous quasi-morphism $\rho_K : \pi_1(M) \to \mathbb{R}$ defined by $\rho_K(x) := q'_K(x^2)/2$ for each $x \in \pi_1(M)$.

Lemma 4.9. *Suppose that there is a loop in Γ based at μ. For K large enough, we have $\rho_K(z_\omega) = 0$ and $\rho_K(z_\mu) \neq 0$.*

We defer the proof of the lemma for the moment and assume this result to complete the proof of Proposition 4.8. Since $\rho_K : \pi_1(M) \to \mathbb{R}$ is a nonzero homogeneous quasi-morphism, we obtain from Proposition 2.6 an action $\pi_1(M) \curvearrowright \mathcal{L}$ on a quasi-line. Moreover, since $\rho_K(z_\mu) \neq 0$ and $\rho_K(z_\omega) = 0$, the element z_μ is loxodromic, while z_ω is elliptic in this action.

Now, consider the other endpoint ω of α. Suppose first there is not a loop in Γ based at ω. Interchanging the roles of μ and ω in Case 1 above produces an action $\pi_1(M) \curvearrowright \mathcal{C}_K(\mathbb{W}_\omega)$ such that z_μ is elliptic and z_ω is loxodromic. On the other hand, if there is a loop in Γ based at ω, then interchanging the roles of μ and ω in Case 2 above produces an action $\pi_1(M) \curvearrowright \mathcal{L}'$ on a quasi-line in which (after possibly passing to a power of 2) z_μ is elliptic and z_ω is loxodromic.

Regardless of which combination of cases holds for the vertices μ and ω, we have produced two actions on hyperbolic spaces and two commuting elements z_μ and z_ω that satisfy the conditions of Lemma 2.3, which concludes the proof. □

We now prove Lemma 4.9.

Proof of Lemma 4.9. Recall that $v \in \mathcal{V}_1$. As $h \in \pi_1(M)$ is a representative of the nontrivial coset of $\pi_1(M')$ in $\pi_1(M)$, we have $hv \in \mathcal{V}_2$. Note that $\widetilde{M}_{hv}$ is also a lift of M_μ, even though hv is not in $\mathcal{V}_1$. Fix a vertex v_0 adjacent to hv such that $\widetilde{M}_{v_0}$ is a lift of M_μ in $\widetilde{M}$. This ensures that L_{v_0} is in $\mathbb{Q}_1$. Let $l \in \mathbb{L}_{hv}$ be the boundary line of $\widetilde{F}_{hv}$ corresponding to the edge $[v_0, hv]$.

We will first show that $\rho_K(z_\omega) = 0$. Since q_K is a homogeneous quasi-morphism and $z_\omega \in \pi_1(M')$, we have that

$$\rho_K(z_\omega) = q'_K(z_\omega) = q_K(z_\omega) + q_K(hz_\omega h^{-1}) = 0 + q_K(hz_\omega h^{-1}).$$

By Proposition 2.5, to show $\rho_K(z_\omega) = q_K(hz_\omega h^{-1}) = 0$, it suffices to show that $hz_\omega h^{-1}$ is elliptic in the action $\pi_1(M') \curvearrowright \mathcal{C}_K(\mathbb{Q}_1)$. Let $\xi > 0$ be the projection constant of the projection complexes $\mathbb{Q}_1$ and $\mathbb{Q}_2$. Since M_{hv} is a Seifert-fibered piece, we have $\widetilde{M}_{hv} = \widetilde{F}_{hv} \times \mathbb{R}$, where F_{hv} is the base orbifold of M_{hv}. Applying Lemma 4.7 to the space F_{hv}, the collection of boundary lines of $\widetilde{F}_{hv}$, the fixed boundary line l, and the chosen element $\gamma = hz_\omega h^{-1}$, we obtain a constant $\lambda > 0$. We further enlarge λ so that it satisfies Lemma 3.4.

Choose $K > 4\xi + 4 + 2\lambda + \lambda^2$ large enough to apply Proposition 2.9, and let y_0 be a point in the projection of $L_{hz_\omega h^{-1}v_0}$ to L_{v_0}. We will show that $d_{\mathcal{C}_K(\mathbb{Q}_1)}(y_0, \gamma^n(y_0)) \leqslant 6K$ for all $n \in \mathbb{Z}$, which will imply that γ is elliptic in the action $\pi_1(M') \curvearrowright \mathcal{C}_K(\mathbb{Q}_1)$, as desired.

Fix $n \in \mathbb{Z}$. By Proposition 2.9, we have

$$d_{C_K(\mathbb{Q}_1)}(y_0, \gamma^n(y_0)) \leqslant 4 \sum_{\substack{u \in \mathcal{V}_1 \\ L_u \in \mathbb{Q}_1}} [d_{L_u}(y_0, \gamma^n(y_0))]_K + 6K. \tag{2}$$

Thus, it suffices to show that $d_{L_u}(y_0, \gamma^n(y_0)) < K$ for all $u \in \mathcal{V}_1$ such that $L_u \in \mathbb{Q}_1$. Since $L_u \in \mathbb{Q}_1$, $\widetilde{M}_u$ is a lift of M_μ.

We divide the proof into several cases, depending on the location of the vertex u.

Case 1: $u \in \{v_0, \gamma^n(v_0)\}$. We assume that $u = v_0$ as the case $u = \gamma^n(v_0)$ is proved similarly.

By assumption, $y_0 \in L_{v_0}$, and so $\gamma^n(y_0) \in L_{\gamma^n(y_0)}$. By definition,

$$d_{L_{v_0}}(y_0, \gamma^n(y_0)) = \operatorname{diam}(\{y_0\} \cup \Pi_{L_{v_0}}(L_{\gamma^n(v_0)}))$$

and

$$d_{L_{v_0}}(L_{\gamma(v_0)}, L_{\gamma^n(v_0)}) = \operatorname{diam}\big(\Pi_{L_{v_0}}(L_{\gamma(v_0)}) \cup \Pi_{L_{v_0}}(L_{\gamma^n(v_0)})\big).$$

As $y_0 \in \Pi_{L_{v_0}}(L_{\gamma(v_0)})$ and the diameter of $\Pi_{L_{v_0}}(L_{\gamma(v_0)})$ is no more than ξ, it follows that

$$\left|d_{L_{v_0}}(y_0, \gamma^n(y_0)) - d_{L_{v_0}}(L_{\gamma(v_0)}, L_{\gamma^n(v_0)})\right| \leqslant 2\xi. \tag{3}$$

The line l is the boundary line of $\widetilde{F}_{hv}$ associated to the edge $[v_0, hv]$. Recall that z_ω is an element of the edge group $G_{[v,w]}$, and so, it fixes the vertex v. Thus, $\gamma(hv) = hz_\omega h^{-1}(hv) = hz_\omega(v) = hv$, and so, the lines $\gamma(l)$ and $\gamma^n(l)$ are the boundary lines in $\widetilde{F}_{hv}$ associated to the edges $[hv, \gamma(v_0)]$ and $[hv, \gamma^n(v_0)]$, respectively.

Combining (3) with Lemmas 3.4 and 4.7 implies that

$$\begin{aligned} d_{L_{v_0}}(y_0, \gamma^n(y_0)) &\leqslant d_{L_{v_0}}(L_{\gamma(v_0)}, L_{\gamma^n(v_0)}) + 2\xi \\ &\leqslant \lambda d_l(\gamma(l), \gamma^n(l)) + \lambda + 2\xi \\ &= \lambda\, d_{\gamma^{-1}(l)}(l, \gamma^{n-1}(l)) + \lambda + 2\xi \leqslant \lambda^2 + \lambda + 2\xi < K. \end{aligned}$$

Case 2: $u \in \operatorname{Lk}(hv)$ *but* $u \notin \{v_0, \gamma^n(v_0)\}$. Let b be the boundary line of $\widetilde{F}_{hv}$ corresponding to the edge $[u, hv]$, so that $b \notin \{l, \gamma^n(l)\}$. By Lemma 4.7, we have that $d_b(l, \gamma^n(l)) \leqslant \lambda$. It follows from Lemma 3.4 that

$$\begin{aligned} d_{L_u}(y_0, \gamma^n(y_0)) &= d_{L_u}(L_{v_0}, L_{\gamma^n(v_0)}) \\ &\leqslant \lambda\, d_b(l, \gamma^n(l)) + \lambda \\ &\leqslant \lambda^2 + \lambda < K. \end{aligned}$$

Case 3: $u \notin \operatorname{Lk}(hv)$. In this case, $d(u, [v_0, \gamma^n(v_0)]) \geqslant 2$, and so,

$$d_{L_u}(y_0, \gamma^n y_0) = d_{L_u}(L_{v_0}, L_{\gamma^n(v_0)}) \leqslant \lambda < K.$$

We have shown that $d_{L_u}(y_0, \gamma^n(y_0)) < K$ for all $u \in \mathcal{V}_1$ such that $L_u \in \mathbb{Q}_1$. Therefore, (2) shows that $d_{C_K(\mathbb{Q}_1)}(y_0, \gamma^n(y_0)) \leqslant 6K$ for all n. It follows that γ is elliptic in the action $\pi_1(M') \curvearrowright C_K(\mathbb{Q}_1)$, and so, $q(z_\omega) = 0$.

To complete the proof, we need to verify that

$$\rho_K(z_\mu) = q'_K(z_\mu) = q_K(z_\mu) + q_K(hz_\mu h^{-1}) \neq 0.$$

Since $hz_\mu h^{-1}$ is a central element in $G_{hv} = \mathrm{Stab}_G(hv)$, it follows from Remark 2.13 that $hz_\mu h^{-1}$ acts elliptically on L_{v_0}, and thus, also on $C_K(\mathbb{Q}_1)$. By Proposition 2.5, we have $q_K(hz_\mu h^{-1}) = 0$. Since $q_K(z_\mu) \neq 0$, it follows that $\rho_K(z_\mu) \neq 0$. □

Theorem 1.3 now follows immediately from Corollary 4.6 and Proposition 4.8.

4.3 | Theorem 1.1

In this section, we put together the above results and prove Theorem 1.1, whose statement we recall for the convenience of the reader.

Theorem 1.1. *Let M be a nongeometric 3-manifold with empty or toroidal boundary. If the torus decomposition of M contains any of the following, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible:*

(1) *a hyperbolic piece that contains a boundary torus of M;*
(2) *two hyperbolic pieces glued along a torus;*
(3) *an isolated nonelementary Seifert-fibered piece; or*
(4) *a nonelementary maximal graph manifold component.*

Proof. Let $M_1, \dots, M_k$ be the maximal graph manifold components and isolated Seifert-fibered pieces of the torus decomposition of M. Let $S_1, \dots, S_\ell$ be the tori in the boundary of M that bound a hyperbolic piece, and let $T_1, \dots, T_m$ be the tori in the torus decomposition of M that separate two hyperbolic components of the torus decomposition. By Lemma 4.2, $\pi_1(M)$ is hyperbolic relative to

$$\mathbb{P} = \{\pi_1(M_p)\}_{p=1}^{k} \cup \{\pi_1(S_q)\}_{q=1}^{\ell} \cup \{\pi_1(T_r)\}_{r=1}^{m}.$$

In all of the cases (1)–(4), the collection $\mathbb{P}$ is nonempty.

In case (1), the collection $\{S_1, \dots, S_\ell\} \neq \emptyset$, while in case (2), $\{T_1, \dots, T_m\} \neq \emptyset$. Both of these collections consist of tori. Note that $\mathbb{Z}^2$ is $\mathcal{H}$-inaccessible: the projections of $\mathbb{Z}^2$ onto each factor yield two actions on lines to which Lemma 2.3 applies. Thus, if $\{\pi_1(S_q)\} \cup \{\pi_1(T_r)\}$ is nonempty, then $\pi_1(M)$ is $\mathcal{H}$-inaccessible by Lemma 4.3, proving the theorem in cases (1) and (2).

Next suppose that (3) holds, so that there is an isolated nonelementary Seifert-fibered piece M_p. By the proof of Corollary 4.6, we see that $\pi_1(M_p)$ has two actions to which Lemma 2.3 applies. By Lemma 4.3, $\pi_1(M)$ is $\mathcal{H}$-inaccessible.

Finally, suppose that (4) holds, so that there is a nonelementary maximal graph manifold component M_p. By the proof of Proposition 4.8, there are two commuting elements $a, b \in \pi_1(M_p)$ and two actions on hyperbolic spaces (in fact, quasi-trees) $\pi_1(M_p) \curvearrowright X$ and $\pi_1(M_p) \curvearrowright Y$ such that a and b are elliptic and loxodromic, respectively, in $\pi_1(M_p) \curvearrowright X$ and a is loxodromic in

$\pi_1(M_p) \curvearrowright Y$. Applying Lemma 4.3 to $P = \pi_1(M_p)$, we conclude that $\mathcal{H}(\pi_1(M))$ contains no largest element. □

4.4 | $\mathcal{H}$-accessibility of finitely generated 3-manifold groups

In this section, we explain how one might reduce the study of $\mathcal{H}$-accessibility of all finitely generated 3-manifold groups to the case of compact, orientable, irreducible, ∂-irreducible 3-manifold groups. In particular, we show that for any hyperbolic 3-manifold M without rank-1 cusps, if $\pi_1(M)$ is finitely generated then it is $\mathcal{H}$-accessible.

Let M be an orientable 3-manifold with finitely generated fundamental group. It follows from Scott's Core Theorem that M contains a compact codimension zero submanifold whose inclusion map is a homotopy equivalence [28], and thus, also an isomorphism on fundamental groups. We thus can assume that our 3-manifolds are compact.

The sphere-disk decomposition provides a decomposition of a compact, orientable 3-manifold M into irreducible, ∂-irreducible pieces $M_1, \ldots, M_k$. In particular, $\pi_1(M)$ is a free product $\pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k)$. Let $G_i := \pi_1(M_i)$. Note that $\pi_1(M)$ is hyperbolic relative to the collection $\mathbb{P} = \{G_1, \ldots, G_k\}$. In light of Lemma 4.3, the $\mathcal{H}$-inaccessibility of $\pi_1(M)$ follows whenever some G_i satisfies the conditions of Lemma 2.3. Hence, it suffices to investigate the $\mathcal{H}$-accessibility of the groups G_i.

If M has empty or toroidal boundary, then $\mathcal{H}$-accessibility of $\pi_1(M)$ is understood, except for a few sporadic cases, by Theorem 1.1. The following proposition addresses certain manifolds with higher genus boundary.

Proposition 4.10. *Let M be a compact, orientable, irreducible, ∂-irreducible 3-manifold that has at least one boundary component of genus at least 2. Then, $\pi_1(M)$ is $\mathcal{H}$-inaccessible under either of the following hypotheses:*

(1) *M has trivial torus decomposition and at least one torus boundary component; or*
(2) *M has nontrivial torus decomposition.*

On the other hand, if M has trivial torus decomposition and all boundary components have genus at least 2, then $\pi_1(M)$ is $\mathcal{H}$-accessible.

Proof. As in [31, Section 6.3], we can paste compact hyperbolic 3-manifolds with totally geodesic boundaries to the higher genus boundary components of M to obtain a finite volume hyperbolic manifold N (in case M has trivial torus decomposition) or a mixed 3-manifold (in case M has nontrivial torus decomposition).

If (1) holds, then the manifold N has toroidal boundary, and, by assumption, there is a boundary torus T for N that is also a boundary torus of M.

The subgroup $P := \pi_1(T) \cong \mathbb{Z}^2$ satisfies Lemma 2.3 and is a peripheral subgroup in the relatively hyperbolic structure on $\pi_1(N)$. The proof of Lemma 4.3 shows that there are commuting elements $a, b \in P$ and hyperbolic actions $\pi_1(N) \curvearrowright Z_X$ and $\pi_1(N) \curvearrowright Z_Y$ such that a and b act loxodromically and elliptically, respectively, in the action $\pi_1(N) \curvearrowright Z_X$, and b acts loxodromically in the action $G \curvearrowright Z_Y$. As $\pi_1(M)$ is a subgroup of $\pi_1(N)$, we obtain induced actions $\pi_1(M) \curvearrowright Z_X$ and $\pi_1(M) \curvearrowright Z_Y$. Since $a, b \in \pi_1(M)$, we see that $\pi_1(M)$ is $\mathcal{H}$-inaccessible by Lemma 2.3.

If (2) holds, then N has either empty or toroidal boundary and has the following properties:

(i) M is a submanifold of N with incompressible toroidal boundary;
(ii) cutting N along the tori in the torus decomposition of M yields the torus decomposition of N; and
(iii) each piece of M with a boundary component of genus at least 2 is contained in a hyperbolic piece of N.

In particular, it follows from (ii) and (iii) that N is a mixed 3-manifold, and hence, $\pi_1(N)$ is $\mathcal{H}$-inaccessible by Theorem 1.1.

In the proof of Theorem 1.1, we prove the $\mathcal{H}$-inaccessibility of $\pi_1(N)$ by showing that there are two commuting elements $a, b \in \pi_1(T)$ for some torus T in the torus decomposition of N and isometric actions $\pi_1(N) \curvearrowright Z_X$ and $\pi_1(N) \curvearrowright Z_Y$ on hyperbolic spaces, and then applying Lemma 2.3.

By (ii), T is also a torus in the torus decomposition of M. Thus, the induced actions $\pi_1(M) \curvearrowright Z_X$ and $\pi_1(M) \curvearrowright Z_Y$ satisfy the hypotheses of Lemma 2.3, and so, $\pi_1(M)$ is $\mathcal{H}$-inaccessible.

We now turn our attention to the final statement of the theorem. In this case, the manifold N is closed. A finitely generated subgroup H of N is a *virtual surface fiber subgroup* if N admits a finite cover $N' \to N$ such that H is a subgroup of $\pi_1(N')$ and H is a surface fiber subgroup of $\pi_1(N')$. Any finitely generated subgroup H of $\pi_1(N)$ is either a geometrically finite Kleinian group or a virtual surface fiber subgroup in $\pi_1(N)$ by the Covering Theorem (see [13]) and the Subgroup Tameness Theorem (see [6, 14] or [7, Theorem 4.1.2] for a statement). In particular, $\pi_1(M)$ is either a virtual surface fiber subgroup, in which case it is hyperbolic, or it is geometrically finite in $\pi_1(N)$. In the latter case, $\pi_1(M)$ is undistorted in $\pi_1(N)$ [18, Corollary 1.6], and we again conclude that $\pi_1(M)$ is hyperbolic, since undistorted subgroups of hyperbolic groups are hyperbolic. As a result, in either case, $\pi_1(M)$ is $\mathcal{H}$-accessible. □

4.5 | Graph manifolds with one vertex

As mentioned in the introduction, our methods do not apply to graph manifolds whose underlying graphs contain only one vertex. Our main tool is Lemma 4.7, a criterion for the nonexistence of a largest action. Intuitively, the idea is to find a $\mathbb{Z}^2$-subgroup of $\pi_1(M)$ and extend two incompatible actions of $\mathbb{Z}^2$ on lines to two incompatible actions of $\pi_1(M)$ on hyperbolic spaces. In general, extending actions from subgroups to groups is difficult (and not always possible) [3]. When the underlying graph of the graph manifold has more than one vertex, we apply the Bestvina–Bromberg–Fujiwara projection complex machinery developed in [9] and a certain construction of quasi-morphisms to overcome this difficulty in Section 4.2. These methods require that the graph has more than one vertex in an essential way: when the graph has a single vertex, we are not able to define projections between adjacent lifts of Seifert-fibered pieces to the universal cover.

Despite this, we expect that, in general, all graph manifold groups are $\mathcal{H}$-inaccessible. In one of the simplest examples where the underlying graph has only one vertex, we can show that this is the case.

Example 4.11. Let M be the 3-manifold that is the mapping torus over a punctured torus defined by a Dehn twist. This is a graph manifold consisting of a single Seifert-fibered space with three boundary components, two of which are identified; see [21]. As described in [21, Lemma 2.2], the fundamental group of M has presentation

$$\pi_1(M) \cong \langle a, b, t \mid [a, b] = 1, a^t = b \rangle.$$

We will show that $\pi_1(M)$ is $\mathcal{H}$-inaccessible.

First, notice that the abelianization of $\pi_1(M)$ is $\pi_1(M)^{Ab} \cong \langle a, t \mid [a, t] = 1 \rangle \cong \mathbb{Z}^2$. Thus, by first composing with the quotient map $\pi_1(M) \to \pi_1(M)^{Ab}$, we obtain uncountably many incomparable actions of $\pi_1(M)$ on lines; see [1, Example 4.23]. The elements a and b have the same image in $\pi_1(M)^{Ab}$, and so, in each of these lineal actions, they are either both loxodromic or both elliptic. In all but one of these actions, a and b are both loxodromic. Fix one such action $\pi_1(M) \curvearrowright \ell$.

Now, suppose that $\pi_1(M) \curvearrowright Z$ is a cobounded action on a hyperbolic space, and suppose $\pi_1(M) \curvearrowright Z \succeq \pi_1(M) \curvearrowright \ell$. Since a and b are both loxodromic with respect to the action on ℓ, they must both be loxodromic with respect to the action on Z, as well. Since they commute and are conjugate in $\pi_1(M)$, they must have the same fixed points in the boundary ∂Z and the same translation length. Moreover, the element t must also stabilize (setwise) the two fixed points of a in ∂Z, as t conjugates a to b. Thus, all of $\pi_1(M)$ stabilizes these two points in ∂Z, and so, $G \curvearrowright Z$ is a lineal action. Since lineal actions are always minimal [1, Corollary 4.12], we must have $\pi_1(M) \curvearrowright Z$ is equivalent to $\pi_1(M) \curvearrowright \ell$ in $\mathcal{H}(\pi_1(M))$. In particular, no element of $\mathcal{H}(\pi_1(M))$ can dominate all of the lineal actions, and so, $\pi_1(M)$ is $\mathcal{H}$-inaccessible.

Our proof that this 3-manifold group is $\mathcal{H}$-inaccessible depended on properties of a particular presentation of the group. It is not clear that the methods used in this example will generalize to all graph manifold groups with one vertex in the underlying graph. It is possible that a collection of ad hoc methods could be used to show that these remaining 3-manifold groups are $\mathcal{H}$-inaccessible.

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