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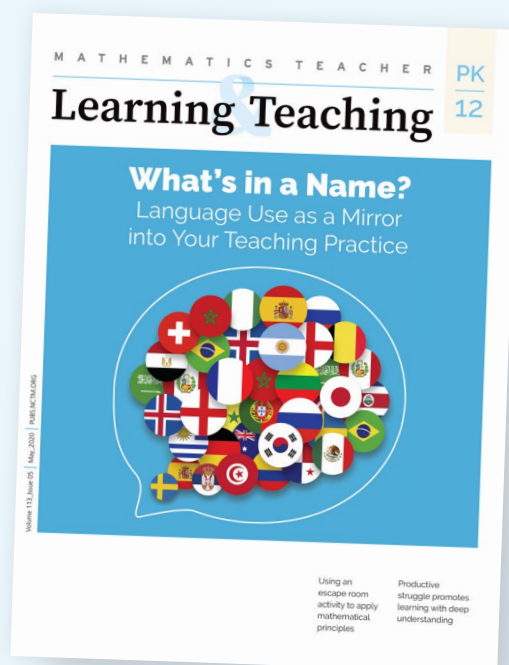
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Leveraging Early Algebraic Experiences

Understand how to consider your students' informal algebraic experiences from early childhood to support their formal learning of algebra.

Margaret Walton and Janet Walkoe

As a parent (Janet) and an aunt (Margaret), we have both spent time playing with young children. In these moments, our teacher and researcher “radar” has been activated as we watch children reason about tasks, games, and activities. We have both caught ourselves wondering, “How does kids’ thinking evolve and how do they build on early experiences?” Margaret recently considered this question as she played with her friend’s one-year-old daughter, Penny, and a set of blocks on the floor. Margaret and Penny established a routine during which Margaret handed Penny one block at a time.

Penny placed each block in a bucket, one on top of the other, until Margaret ran out of blocks. Penny then took the blocks out of the bucket one at a time and handed them back to Margaret. Once Penny returned all the blocks to Margaret, the process started again. Penny was the clear leader of this activity, as she expectantly looked for Margaret to hand her another block, when she began to reverse the process once the bucket was full, and when she showed no regard for whether Margaret’s knees would give out from kneeling so long! Researchers have started to consider how thinking like

Penny's could be beneficial for learning formal algebra later in life (Walkoe & Levin, 2020).

At first glance, Penny's thinking seems to have little to do with algebra. However, with Margaret's help, Penny repeatedly carried out a process and then "undid" it. The process here is similar to certain kinds of functional thinking, like moving back and forth between inputs and outputs. Walkoe and Levin (2020) considered the connection between children's experiences, like Penny's, and more formal algebraic ideas. They explored what they called Seeds of Algebraic Thinking—sub-conceptual resources (e.g., balance, grouping objects) children attain as they interact with their world—and how these seeds could be leveraged in formal algebraic thinking (Levin & Walkoe, 2022; Walkoe & Levin, 2020). In this article, we will summarize the theory behind Seeds of Algebraic Thinking, how they connect to formal algebraic reasoning, and how they can be leveraged during classroom instruction.

SEEDS OF ALGEBRAIC THINKING

Seeds of Algebraic Thinking (Levin & Walkoe, 2022; Walkoe & Levin, 2020) comes from the Knowledge in Pieces (KiP) perspective of learning (diSessa, 2018; Hammer, 2000). KiP is a systems approach to learning that stems from the constructivist idea that people learn by building on prior knowledge (Piaget 1964; Smith et al., 1993). As people experience the world, they acquire small, sub-conceptual knowledge elements. When people engage in a particular context, these sub-conceptual resources get elicited and organized in ways that are potentially productive in that situation. Activation of the resources is highly context-dependent. As the person becomes more familiar with the context, either through trial and error or formal direction, resources become foregrounded or backgrounded, some are added or dropped, and connections among

resources are strengthened or weakened, "fine tuning" the system to the particular situation.

Walkoe and Levin (2020) argue that Seeds of Algebraic Thinking are a class of resources available to children that get elicited as they learn algebra. For example, many children experience the idea of balance as they walk along a sidewalk curb or play on a seesaw. Such experiences might contribute to children developing a *balance* seed of algebra (Levin & Walkoe, 2022; Walkoe & Levin, 2020). Later in life, students in an algebra class activate the *balance* seed (along with others) as they learn to balance equations. If teachers recognize that various seeds can be activated while learning about equations, they might use these images (e.g., a seesaw) to make the connection explicit between students' physical experiences with balance and balancing an equation. Levin and Walkoe (2022) discuss this idea further and illustrate what seeds might be leveraged in particular situations.

An important characteristic of resources like Seeds of Algebraic Thinking from the KiP perspective is that they might be productive or unproductive for learning about a particular topic, but they are not inherently correct or incorrect. A well-used example in science education is that many people think the outside temperature is hotter in the summer because the Earth is closer to the Sun (Dove, 1998). This thinking likely relates to people's life experiences: as you approach a heat source, like a stove, the heat intensifies. This resource, *closer means stronger*, is reasonable, based on one's interactions with their environment (Hammer, 2000). However, it is not a productive resource when thinking about the relationship between the Earth, Sun, and seasons. As a teacher, then, it is important to understand from where children's ideas might originate (i.e., what seeds they might be activating) and whether these seeds are productive for the situation.

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It is also important for teachers to know how to help students refine their thinking, should their activated seeds be less productive (Walkoe et al., 2022). An example is when students add numerators and denominators when they add fractions. They are used to seeing a plus sign and adding the numbers on either side. The seed *plus sign means combine* is activated. Rather than focusing on the “incorrectness” of such an answer, a teacher could validate the student’s thinking by verbally recognizing where the idea came from and acknowledging that the idea is useful for adding the numerators. However, the student needs to ensure that the denominator is the same before adding the numerators. The teacher might leverage other seeds relating to parts and wholes to help the student understand why they need to get like-denominators before adding.

To better illustrate the seeds perspective, we provide an example using children’s game play (Video 1). This video was recorded by Janet when Janet’s daughter was playing a game with her friends.

During the video, three 6-year-olds play the card game Rat-a-Tat Cat (Gamewright, 2023). Each player receives two cards that are face up, and two that are face down, which other players never see. The cards are worth zero to nine points. The purpose of the game is to get the lowest score. A player picks a card from the deck in the middle and decides whether they want to exchange the card for one in their hand or discard it. Once a player thinks they have the lowest score of the group, they yell “Rat-a-Tat Cat!” Each player gets one additional turn and then players add their cards for their final score.

As the children in the video play the game, we noticed several Seeds of Algebraic Thinking that the children could activate later in life when they learn formal algebra. For example, the goal of the game is to get the lowest possible score. When children get closer and closer to the lowest score, possible seeds that are evoked are *closing in*, or the idea that some desired value falls between two closer and closer values, and *minimum* or lowest value.

During algebra instruction, students might activate these seeds as they learn about minima and maxima of polynomials. For instance, the minimum of a parabola on a graph could evoke experiences of aiming for a low score or shopping for the cheapest price. A teacher could leverage these resources by contextualizing graphs with situations like games or bargain hunting, or by asking students to create a context.

The children in the video also discuss “bad” and “not bad” cards. At one point, a child selects a card with a seven on it and decides to discard it. A different child remarks, “Sevens and up are bad, but six and down [are not].” The children exhibit a *belong or not* seed. *Belong or not* relates to understanding that a number, object, etc. may or may not belong to a set of numbers, objects, etc. The children in the video agree that the numbers 0–6 belong to a set of preferable cards, while 7–9 do not.

Another experience that could contribute to a *belong or not* seed is when children engage in Which One Does Not Belong tasks. A child might be given a picture of a group of objects to determine which object is unlike the others and why. In formal algebra learning, this seed might be activated as students make generalizations that apply to more than one number. Generalization could relate to algebra topics like inequalities, domain and range of functions, and others. In the case of finding domains and ranges of functions, the *belong or not* seed might be especially useful for thinking about how the domain and range relate when solving for domain algebraically. If a teacher knew to look for the *belong or not* seed in student thinking, they might ask students to check their answers for domain by plugging in the values they found to determine whether the output belongs in the range.

Another seed that the children in the video exhibit is *replacement* (Walkoe & Levin, 2020). The *replacement* seed might help children understand that certain numbers, items, etc. are an appropriate substitute for another item. For Rat-a-Tat Cat, the children understand that their cards are not fixed and that they have opportunities to switch out the cards in their

Video 1 Children Playing Rat-a-Tat Cat



 Watch the full video online.

hands. However, they know that their cards cannot take on any value: The cards are limited by the kinds of cards in the game. A different example is when young children play doctor with toys as patients. They understand that a stuffed bear or a doll can serve as a patient, but a pencil would be less appropriate. The *replacement* seed could be activated for students during algebra instruction as they learn about variables and start to understand that variables can assume different numerical values, depending on a problem's context. For more information about the trajectories of Seeds of Algebraic Thinking from early childhood to the algebra classroom, see Levin & Walkoe (2022).

REAL LIFE EXAMPLE: SEARCH HISTORY AND THE MANGO PROBLEM

As teacher educators, we have explored the usefulness of the seeds framework for teaching and learning algebra. We created a video library of children exhibiting seeds that we use with teachers in professional development. We present one clip (Video 2) as a clear example of middle school students using seeds that they likely gained from early childhood experiences. The task the students worked on is below (adapted from National Council of Teachers of Mathematics, 2023):

Bluebird Middle School offers a mid-afternoon “grab and go” snack that students can get in the cafeteria during their study hall. The best snack day is mango day! When Terrell got to the cafeteria, he found only one bowl full of mangoes left. He was hungry, so he took $\frac{1}{6}$ of

the mangoes in the bowl. A few minutes later, Hana walked into the cafeteria and took $\frac{1}{5}$ of the mangoes that Terrell had left in the bowl. Still later, Gabriela took $\frac{1}{4}$ of the remaining mangoes. Even later, Sahil took $\frac{1}{3}$ of what was left. Finally at the end of study hall, CJ took $\frac{1}{2}$ of the remaining mangoes. Only three mangoes remained. *How many mangoes were originally in the bowl?*

Many of the students solved this problem by “undoing” or working backwards using the three remaining mangoes and given fractions to determine the original number of mangoes. Undoing is an important algebraic concept that can be applied to, for example, using inverse operations to solve equations or factoring polynomials.

Two students in the video, Aiden and Felix, started with three mangoes and undid the problem by multiplying three mangoes by two, which is the denominator of the fraction of mangoes that CJ took. Aiden and Felix repeated this process at each step of the problem and got 2,160 mangoes (Figure 1). Felix and Aiden then explained their solution to another student, Nadia:

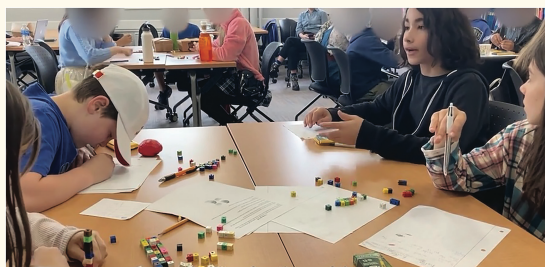
Felix: You look at the denominator and then multiply 3 by that denominator, so 3 times $\frac{1}{2}$, and the denominator is 2, so it's times 2, that's 6. And then it was $\frac{1}{3}$, the denominators 3, so you would multiply and get 18 and $\frac{1}{4}$. . .

Nadia: I know, but why do you multiply that 3 [meaning the three remaining mangoes] times the denominator?

Aiden: Think of it, oh, you know, like search history? Think of it as kind of like, going back through search history and then you're, you're like each time. . . basically, you're trying to find your way back.

While Aiden and Felix did not get the correct answer, they had a productive idea, possibly stemming from the *doing/undoing* seed. The *doing/undoing* seed is about understanding that many processes can be done in reverse. For Penny at the beginning of this article, she understood that the blocks could be stacked in the bucket and then unstacked in reverse order. With the Mango Problem, Felix and Aiden know that a series of steps led to three mangoes left. They understand that they can retrace those steps, which is a constructive way to view the problem. Aiden even

Video 2 Students Solving the Mango Problem



 Watch the full video online.

likened the process to the everyday task of retracing one's Internet search history. Aiden's search history example is applicable to his middle school life, but it has similar characteristics to Penny's block routine. In both situations, there is a process of moving forward (stacking the blocks in the bucket and performing an Internet search) and moving backward (taking the blocks out of the bucket and retracing one's search history). It is possible that this *doing/undoing* seed developed for Aiden and Felix early in life, perhaps through play, like Penny's blocks routine.

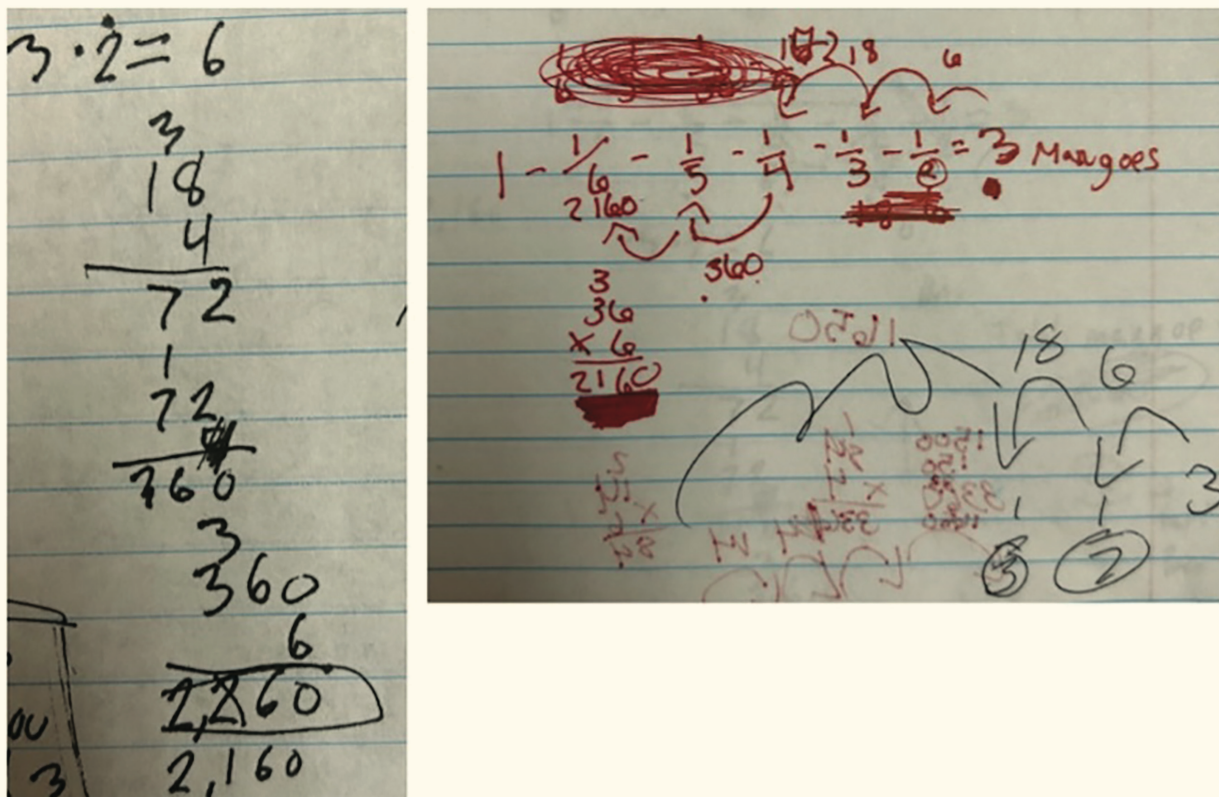
How might a teacher with knowledge of seeds respond to Aiden and Felix? The *doing/undoing* seed is potentially helpful, but the students here need guidance to use it productively. We have several ideas that leverage the *doing/undoing* seed. First, a teacher might ask the students to "do" or move forward through the problem starting with their answer, 2,160. The students could start by taking one-sixth of 2,160 and continue through the steps of the problem. The students would likely realize that they will not

end with three mangoes and must reassess their work.

Another possibility is to use manipulatives, like counting cubes, which were available to Felix and Aiden. A teacher might ask the students to model the first few steps of undoing the problem using the counting cubes for mangoes. For Felix and Aiden, their mistake might become apparent as they try to undo the second to last step of the problem. Based on their work, they would have six cubes (mangoes) from undoing the last step of the problem. A teacher could ask, "Now show me Sahil putting back his mangoes." According to Felix and Aiden, Sahil putting back $\frac{1}{3}$ of the mangoes should result in 18 cubes total. Felix and Aiden already have six cubes, so they would need to add 12 more. However, 12 is $\frac{2}{3}$, not $\frac{1}{3}$, of 18. Thus, 12 cannot be the number of mangoes that Sahil took. This exercise might again prompt Felix and Aiden to revise their method.

A teacher could also support the students to activate the *doing/undoing* seed and model the problem with counting blocks by asking questions like, "Could

Figure 1 Aiden's Work (Left) and Felix's Work (Right) on the Mango Problem



you show me what it would look like to take $\frac{1}{3}$ of the mangoes and be left with 6?” Students might build on the *doing* part of the seed by trial and error. They might test whether taking $\frac{1}{3}$ of different numbers of blocks leaves them with 6. Students might also utilize the *undoing* part of the seed by asking themselves, “6 is $\frac{2}{3}$ of what number?” They could use blocks to reason about the question or might even set up the equation $\frac{2}{3}x = 6$. In each case, the teacher’s questions might prompt the students to redirect their *doing/undoing* seed in a more productive way for solving the problem.

SUPPORTING EQUATION WRITING WITH SEEDS OF ALGEBRAIC THINKING

Once Aiden and Felix understood the mangoes problem through the *doing/undoing* seed, a teacher might employ other seeds to help the students think about writing an equation for the problem. Students often struggle with translating words into equations, including defining a variable and expressing operations, coefficients, and constants in terms of a situation (Copraro & Joffrion, 2006). The *replacement* seed might help. As seen in the Rat-a-Tat Cat example, children could begin to develop the *replacement* seed with card games that allow values of cards in one’s hand to change. Those changes are limited by the kinds of cards in the game. The mango problem also has limitations. One could try any number of mangoes, but only 18 will work, based on the problem narrative.

We have seen students who, if asked to solve the mango problem by writing one or more equations, do not know where to start. In response, we have often heard teachers ask, “What is the variable?” This question may draw students’ attention to the fact that they are looking for the number of mangoes, but it does not support them to conceptually understand an equation as a tool; for example, if one is seeking an unknown value(s) and knows the constraints of the problem, then they can often create one or more equations to help.

From a Seeds of Algebraic Thinking perspective, a teacher might instead ask Aiden and Felix questions like, “What are some potential values for the mangoes in the bowl? Can the answer be all the values you

listed? How will you know if the number of mangoes is any of the values you listed?” If the students identify the number of mangoes as the variable x , the teacher could follow up with something like, “What is acting on the variable, based on the words in the problem?” In response, Aiden and Felix might identify that Terrell took $\frac{1}{6}$ of the mangoes and write $x - \frac{1}{6}x$. From there, the students could be well on their way to translating the other actions in the mango problem into expressions that ultimately result in the equation $\frac{1}{6}x = 3$ (for the full solution, see this Illuminations Lesson [link online]).

The kinds of questions above could activate Aiden’s and Felix’s *replacement* seeds. The questions could reinforce the idea that if one has a situation with an unknown, x , there are a limited number of values (depending on the kind of equation) that can replace x . Writing an equation is a way to describe the parameters of the situation so that the value can be found. Said differently, like the cards in a Rat-a-Tat Cat hand, the number of mangoes cannot be replaced with anything; words in the problem limit it. Translating those words into an equation helps one find that number. Finally, if Aiden and Felix eventually got to the equation $\frac{1}{6}x = 3$, the teacher might question the students in a way that again activates the *doing/undoing* seed, pointing them toward inverse operations to solve.

CONCLUSION

Aiden’s and Felix’s idea of *doing/undoing* is a potentially productive seed for their math learning. This seed likely came from a myriad of life experiences, especially those from when they were young. Like Aiden and Felix, students come to school with plenty of resources for learning algebra (e.g., Walkoe & Levin, 2020). In this article, we argue that if teachers can recognize and identify these resources, they can work to connect algebraic ideas to children’s lived experiences. Looking for evidence of students using their informal experiences and leveraging those experiences during instruction is another way to expand the learning opportunities we provide to our students to understand algebra. —

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