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Dot products in $\mathbb{F}_q^3$ and the Vapnik-Chervonenkis dimension [☆]

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This paper is dedicated to the Ukrainian people who are bravely facing a brutal aggression.

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ABSTRACT

Given a set $E \subset \mathbb{F}_q^3$, where $\mathbb{F}_q$ is the field with q elements. Consider a set of "classifiers" $\mathcal{H}_t^3(E) = \{h_y : y \in E\}$, where $h_y(x) = 1$ if $x \cdot y = t$, $x \in E$, and 0 otherwise. We are going to prove that if $|E| \geq Cq^{\frac{11}{4}}$, with a sufficiently large constant $C > 0$, then the Vapnik-Chervonenkis dimension of $\mathcal{H}_t^3(E)$ is equal to 3. In particular, this means that for sufficiently large subsets of $\mathbb{F}_q^3$, the Vapnik-Chervonenkis dimension of $\mathcal{H}_t^3(E)$ is the same as the Vapnik-Chervonenkis dimension of $\mathcal{H}_t^3(\mathbb{F}_q^3)$. In some sense the proof leads us to consider the most complicated possible configuration that can always be embedded in subsets of $\mathbb{F}_q^3$ of size $\geq Cq^{\frac{11}{4}}$.

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1. Introduction

The purpose of this paper is to study the Vapnik-Chervonenkis dimension in the context of a naturally arising family of functions on subsets of the three-dimensional vector space over the finite field with q elements, denoted by $\mathbb{F}_q^3$. Let us begin by recalling some definitions and basic results (see e.g. [9], Chapter 6).

Definition 1.1. Let X be a set and $\mathcal{H}$ a collection of functions from X to $\{0, 1\}$. We say that $\mathcal{H}$ shatters a finite set $C \subset X$ if the restriction of $\mathcal{H}$ to C yields every possible function from C to $\{0, 1\}$.

Definition 1.2. Let X and $\mathcal{H}$ be as above. We say that a non-negative integer n is the VC-dimension of $\mathcal{H}$ if there exists a set $C \subset X$ of size n that is shattered by $\mathcal{H}$, and no subset of X of size $n+1$ is shattered by $\mathcal{H}$.

We are going to work with a class of functions $\mathcal{H}_t^d$, where $t \neq 0$. Let $X = \mathbb{F}_q^d$, and define

$$\mathcal{H}_t^d = \{h_y : y \in \mathbb{F}_q^d\}, \tag{1.1}$$

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where $y \in \mathbb{F}_q^d$, and $h_y(x) = 1$ if $x \cdot y = t$, and 0 otherwise. Let $\mathcal{H}_t^d(E)$ be defined the same way, but with respect to a set $E \subset \mathbb{F}_q^d$ i.e.

$$\mathcal{H}_t^d(E) = \{h_y : y \in E\},$$

where $h_y(x) = 1$ if $x \cdot y = t$ $(x \in E)$, and 0 otherwise.

Our main result is the following.

Theorem 1.3. *Let $\mathcal{H}_t^3(E)$ be defined as above with respect to $E \subset \mathbb{F}_q^3$, $t \neq 0$. If $|E| \geq Cq^{\frac{11}{4}}$, for some large enough constant C, then the VC-dimension of $\mathcal{H}_t^3(E)$ is equal to 3.*

Remark 1.4. Since $|\mathcal{H}_t^3(E)| = |E|$, it is clear that the VC-dimension of $\mathcal{H}_t^3(E)$ is at most $\log_2(|E|)$, so 3 is a clear improvement over this general estimate. It is not difficult to see that the VC-dimension is < 4 since three points determine a plane in $\mathbb{F}_q^3$, so the real challenge is to establish that some set of 3 points shatters. Moreover, our result says that in this sense, the learning complexity of subsets of $\mathbb{F}_q^3$ of size $> Cq^{\frac{11}{4}}$ is the same as that of the whole vector space $\mathbb{F}_q^3$.

Remark 1.5. In the case when $d = 2$ and the dot product $x \cdot y$ is replaced by $||x - y|| = (x_1 - y_1)^2 + (x_2 - y_2)^2$, the corresponding result, with the threshold $|E| \geq Cq^{\frac{15}{8}}$ was established by D. Fitzpatrick, E. Wyman and the first two listed authors of this paper ([2]). The techniques used to prove Theorem 1.3 are quite a bit different. On one hand, we have more room to roam in three dimensions. On the other, the non-translation invariant nature of the dot product requires special care.

In the case $d \geq 4$, an attempt to prove that the VC dimension is equal to d with a non-trivial threshold on the size of E runs into the difficulties that are quite reminiscent of those the authors ran into in the process of trying to extend the result mentioned in Remark 1.5 to higher dimensions. The biggest difficulty is posed by degenerate configurations. However, we can prove in higher dimensions that the VC dimension is at least 3 with a non-trivial threshold on the size of E. We shall address these issues in a sequel.

Remark 1.6. The case $d = 2$ of Theorem 1.3 is quite simple because two points determine a line. Shattering two points can be accomplished by a method quite similar to that used to prove Theorem 1.9 below.

Remark 1.7. As the reader shall see, the proof of Theorem 1.3 involves a construction of a reasonably complicated point configuration in E. For a general theory of such configurations in the context of dot products, see e.g. [3] and [8].

Remark 1.8. The concept of the VC-dimension plays an important role in many combinatorial problems. See, for example, [1], [4], and the references contained therein.

We can also prove that the VC-dimension is ≥ 2 under a much weaker assumption. More precisely, we have the following result.

Theorem 1.9. *Let $\mathcal{H}_t^3(E)$ be defined as above with respect to $E \subset \mathbb{F}_q^3$, $t \neq 0$. If $|E| > cq^{\frac{5}{2}}$ for an arbitrary c, then the VC-dimension of $\mathcal{H}_t^3(E)$ is ≥ 2.*

Remark 1.10. We do not know to what extent the exponent $\frac{11}{4}$ in Theorem 1.3 and the exponent $\frac{5}{2}$ are sharp, but we know that neither exponent can fall below 2 in view of sharpness examples for the distance problem in odd dimensions in [3].

From the point of view of learning theory, it is interesting to ask what the "learning task" is in the situation at hand. It can be described as follows. We are asked to construct a function $f : E \to \{0, 1\}$, $E \subset \mathbb{F}_q^3$, that is equal to 1 when $x \cdot y^* = t$, but we do not know the value of y^*. The fundamental theorem of statistical learning tells us that if the VC-dimension of $\mathcal{H}_t^3(E)$ is finite, we can find an arbitrarily accurate hypothesis (element of $\mathcal{H}_t^3(E)$ with arbitrarily high probability if we consider a randomly chosen sampling training set of sufficiently large size. The necessary sample size to learn effectively grows with the VC-dimension. For a precise, quantitative treatment of this idea, see [9].

2. Proof of Theorem 1.9

We prove Theorem 1.9 first because some of the ideas in the proof will be needed in the proof of Theorem 1.3. It is sufficient to prove that there exist $x_1, x_2, y_1, y_2, y_{12}, y^* \in E$ such that

- i) $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$,
- ii) $x_1 \cdot y_1 = t, x_2 \cdot y_1 \neq t$,

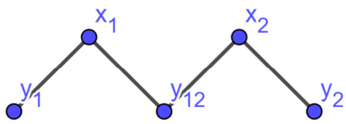

Fig. 1. Configuration for Theorem 1.9.

- iii) $x_2 \cdot y_2 = t, x_1 \cdot y_2 \neq t$,
- iv) $x_1 \cdot y^*, x_2 \cdot y^* \neq t$

The reason the above suffices to establish Theorem 1.9 is because it demonstrates that a set of size two can be shattered by $\mathcal{H}_t^3(E)$.

It suffices to find such a tuple $(x_1, x_2, y_{12}, y_1, y_2)$ under the additional assumption that for each $u \in E$, there are at most $C\frac{|E|}{q}$ vectors $v \in E$ such that $u \cdot v = t$. The following lemma allows us to reduce to this case.

Lemma 2.1. *Let $E \subseteq \mathbb{F}_q^3$ such that $|E| \geq Cq^{\frac{5}{2}}$ with C sufficiently large. Then there is a subset $E' \subseteq E$ with $|E'| \geq \frac{1}{2}|E|$, and for any $u \in E'$,*

$$\sum_{v \in E'} D_t(u, v) \leq \frac{22|E'|}{5q},$$

where $D_t(u, v) = 1$ when $u \cdot v = t$, and 0 otherwise.

Clearly if $\mathcal{H}_t^3(E')$ shatters some set of 3 points, then $\mathcal{H}_t^3(E)$ shatters the same set of 3 points. Moreover, if E satisfies the hypotheses of Theorem 1.3 or Theorem 1.9, then so does E'.

Proof. It follows immediately from the proof of the main result in [5] that

$$\sum_{u, v \in E} D_t(u, v) = \frac{|E|^2}{q} + R(t),$$

where $|R(t)| \leq |E|q < \frac{|E|^2}{10q}$. In particular,

$$\sum_{u \in E} \sum_{v \in E} D_t(u, v) \leq \frac{11|E|^2}{10q}.$$

This implies that at most $\frac{|E|}{2}$ distinct points $u \in E$ satisfy

$$\sum_{v \in E} D_t(u, v) \geq \frac{11|E|}{5q}.$$

Thus, for

$$E' := \left\{ u \in E : \sum_{v \in E} D_t(u, v) \leq \frac{11|E|}{5q} \right\},$$

we see that E' satisfies the conditions of the lemma. $\square$

With this lemma, we may assume without loss of generality that there are at most $C\frac{|E|}{q}$ vectors $v \in E$ with $u \cdot v = t$. By Theorem 2.2 in [6], there exist a set P_5 of ordered quintuples $(x_1, x_2, y_{12}, y_1, y_2)$ such that

$$y_1 \cdot x_1 = x_1 \cdot y_{12} = y_{12} \cdot x_2 = x_2 \cdot y_2 = t,$$

and

$$\left| |P_5| - \frac{|E|^5}{q^4} \right| \leq \frac{4}{\log 2} q^2 \frac{|E|^4}{q^4} \leq \frac{1}{2} \frac{|E|^5}{q^4}.$$

In particular, $|P_5| \geq \frac{1}{2}\frac{|E|^5}{q^4}$. Such a quintuple is represented in Fig. 1 as a graph with the vectors as vertices and edges between them if their dot product is t.

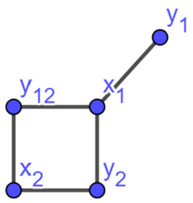

Fig. 2. Degeneracy case in which $x_1 \cdot y_2 = t$.

It remains to show that such a quintuple exists with $x_1 \cdot y_2 \neq t$ and $x_2 \cdot y_1 \neq t$. We first count the number of quintuples $(x_1, x_2, y_{12}, y_1, y_2)$ in P_5 with $x_1 \cdot y_2 = t$. This case of degeneracy is displayed in Fig. 2 above.

We have,

$$\begin{aligned}&\sum_{x_1,x_2,y_{12},y_1,y_2\in E} D_t(y_1,x_1)D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2)D_t(y_2,x_1)\\ &\quad= \sum_{x_1,x_2,y_{12},y_2\in E} D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2)D_t(y_2,x_1)\sum_{y_1\in E} D_t(y_1,x_1)\\ &\quad\leq \frac{22|E|}{5q}\sum_{x_1,x_2,y_{12},y_2\in E} D_t(y_1,x_1)D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2)\end{aligned}$$

This sum over x_1, x_2, y_{12}, y_2 is the number of 4-cycles in the dot-product graph on E, denoted C_4^{prod} in the notation of [6]. By Theorem 1.2 in [6],

$$\begin{aligned}&\left|\sum_{x_1,x_2,y_{12},y_2\in E} D_t(y_1,x_1)D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2) - \frac{|E|^4}{q^4}\right|\\ &\leq \frac{|E|^4}{q^4}\left(12q^{-\frac{1}{2}} + 8\frac{q^5}{|E|^2} + 28\frac{q^2}{|E|}\right)\\ &\leq \frac{|E|^4}{q^4}\left(12q^{-\frac{1}{2}} + \frac{8}{c^2} + \frac{28}{c}q^{-\frac{1}{2}}\right)\\ &\leq \frac{9|E|^4}{c^2q^4}.\end{aligned}$$

Thus,

$$\sum_{x_1,x_2,y_{12},y_2\in E} D_t(y_1,x_1)D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2) \leq \left(\frac{9}{c^2}+1\right)\frac{|E|^4}{q^4}$$

and

$$\begin{aligned}&\sum_{x_1,x_2,y_{12},y_1,y_2\in E} D_t(y_1,x_1)D_t(x_1,y_{12})D_t(y_{12},x_2)D_t(x_2,y_2)D_t(y_2,x_1)\\ &\quad\leq \frac{22|E|}{5q}\left(\frac{9}{c^2}+1\right)\frac{|E|^4}{q^4}\\ &\quad= \frac{22}{5}\left(\frac{9}{c^2}+1\right)\frac{|E|^5}{q^5} < \frac{|E|^5}{10q^4}.\end{aligned}$$

That is, the number of quintuples $(x_1, x_2, y_{12}, y_1, y_2)$ in P_5 with $x_1 \cdot y_2 = t$ is less than $\frac{|E|^5}{10q^4}$. Analogously, the number of quintuples $(x_1, x_2, y_{12}, y_1, y_2)$ in P_5 with $x_2 \cdot y_1 = t$ is less than $\frac{|E|^5}{10q^4}$. It follows that there exists a quintuple $(x_1, x_2, y_{12}, y_1, y_2)$ in P_5 with $x_1 \cdot y_2, x_2 \cdot y_1 \neq t$.

It only remains to construct $y^* \in E$ such that $x_1 \cdot y^* \neq t$ and $x_2 \cdot y^* \neq t$. Observe that

$$|\{x \in E : x \cdot x_1 = t\}| = q^2,$$

and

$$|\{x \in E : x \cdot x_2 = t\}| = q^2,$$

so since $|E| > 2q^2$, there exists y^* with the desired properties. This completes the proof of Theorem 1.9.

3. Proof of Theorem 1.3

Analogously with the proof of Theorem it suffices to find a set $\{x_1, x_2, x_3\}$ of 3 distinct points which is shattered by $\mathcal{H}_t^3(E)$. This is equivalent to finding $x_1, x_2, x_3, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}, y^* \in E$ with x_1, x_2, x_3 distinct such that

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = x_3 \cdot y_{123} = t$
- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$, $x_3 \cdot y_{12} \neq t$, and similarly for y_{13} and y_{23}
- $x_1 \cdot y_1 = t$, $x_2 \cdot y_1, x_3 \cdot y_1 \neq t$, and similarly for y_1 and y_2
- $x_1 \cdot y^*, x_2 \cdot y^*, x_3 \cdot y^* \neq t$

As in the previous section, we may assume without loss of generality that for all $u \in E$,

$$\sum_{v \in E} D_t(u, v) \leq \frac{22|E|}{5q}.$$

This time we will reduce to the case where the sum is bounded below as well, which follows analogously via a counterpart to Lemma 2.1.

Lemma 3.1. *For a set E satisfying the hypotheses of Theorem 1.3, there is a subset $E_0 \subseteq E$ with $|E_0| \geq \frac{1}{6}|E|$, and for any $u \in E_0$,*

$$\sum_{v \in E} D_t(u, v) \geq \frac{|E|}{5q}$$

Proof. Let

$$E_0 := \left\{ u \in E : \sum_{v \in E} D_t(u, v) \geq \frac{|E|}{5q} \right\},$$

so that we need only show that $|E_0| \geq \frac{1}{6}|E|$.

$$\begin{aligned}
\sum_{u, v \in E} D_t(u, v) &= \sum_{u \in E_0} \sum_{v \in E} D_t(u, v) + \sum_{u \notin E_0} \sum_{v \in E} D_t(u, v) \\
&\leq |E_0| \frac{22|E|}{5q} + (|E| - |E_0|) \frac{|E|}{5q} \\
&= \frac{|E|^2}{5q} + |E_0| \frac{21|E|}{5q}.
\end{aligned}$$

We know from the previous section that for E satisfying the hypotheses of Theorem 1.3,

$$\left| \sum_{u, v \in E} D_t(u, v) - \frac{|E|^2}{q} \right| < \frac{|E|^2}{10q}.$$

Thus,

$$\frac{9|E|^2}{10q} \leq \sum_{u, v \in E} D_t(u, v) \leq \frac{|E|^2}{5q} + |E_0| \frac{21|E|}{5q},$$

and so

$$|E_0| \geq \frac{|E|}{6}. \quad \square$$

With Lemma 2.1 and Lemma 3.1, we may assume that for all $u \in E$,

$$\frac{|E|}{5q} \leq \sum_{v \in E} D_t(u, v) \leq \frac{22|E|}{5q}.$$

As we noted in the beginning of this section, in order to conclude that $\mathcal{H}_t^3(E)$ has VC-dimension 3, we need to find $x_1, x_2, x_3, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}, y^* \in E$ with x_1, x_2, x_3 distinct such that

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = x_3 \cdot y_{123} = t$

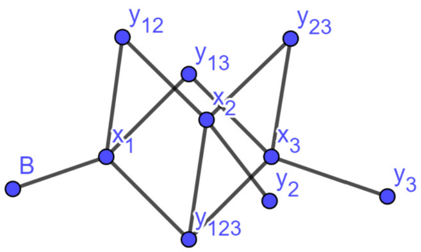

Fig. 3. Configuration for shattering a set of three points.

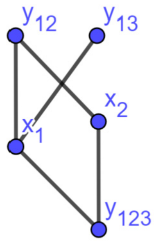

Fig. 4. Initial configuration used to build up to full shattering configuration.

- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$, $x_3 \cdot y_{12} \neq t$, and similarly for y_{13} and y_{23}
- $x_1 \cdot y_1 = t$, $x_2 \cdot y_1, x_3 \cdot y_1 \neq t$, and similarly for y_1 and y_2
- $x_1 \cdot y^*, x_2 \cdot y^*, x_3 \cdot y^* \neq t$

This configuration is displayed above in Fig. 3.

Let

$$A = \left\{(x, y, z, u, v) \in E^5 : x \cdot y = y \cdot z = z \cdot u = u \cdot x = v \cdot u = t\right\} \tag{3.1}$$

For $(x, y, z, u, v) \in A$, by identifying $x = y_{12}$, $y = x_2$, $z = y_{123}$, $u = x_1$, and $v = y_{13}$, this configuration corresponds to the graph shown in Fig. 4 above, which is a subgraph of the graph shown in Fig. 3. Our strategy is to use the symmetry of the larger configuration, in the sense that by removing y_1, y_2, y_3 and then identifying $x_1 = x_3$ and $y_{12} = y_{23}$, we obtain the smaller configuration.

We need the following result follows from [7], Chapter 2).

Lemma 3.2. *Let $E \subset \mathbb{F}_q^3$ with $|E| \geq Cq^{\frac{5}{2}}$, C sufficiently large. Then for A as in equation (3.1),*

$$\frac{1}{2}|E|^5 q^{-5} \leq |A| \leq 2|E|^5 q^{-5}.$$

Since the definition of the set A did not require that $x \neq z$, $y \neq u$, and $y \cdot v \neq t$, all of which will be necessary for our construction, we will find an upper bound for the number of elements of A which do not have these properties. The following lemma implies that these make up a small proportion of A.

Lemma 3.3. *Let $E \subset \mathbb{F}_q^3$ with $|E| \geq Cq^{\frac{5}{2}}$, C sufficiently large. Then*

$$|\{(x, y, z, v, u) \in A : y \cdot v = t, \text{ or } x = z, \text{ or } y = u\}| \leq 5|E|^2 q^3.$$

Proof. There are at most $|E|^2$ ways to produce a pair of distinct points u, y in E. Then, the intersection of the planes defined by $a \cdot u = t$ and $a \cdot y = t$ is at most a line since these planes are distinct. There are at most q^3 ways to choose 3 points on that line, $x, z, u \in E$. So, there are at most $|E|^2 q^3$ quintuples $(x, y, z, v, u) \in A$ with $y \cdot v = t$. For the case when $y = u$, by Corollary 4.5 in [6] there are at most $2\frac{|E|^4}{q^3} \leq 2|E|^2 q^3$ such quadruples of points (x, y, z, v) such that $x \cdot y = z \cdot y = v \cdot y = t$. For the case when $x = z$, by Theorem 2.2 in [6], there are at most $2\frac{|E|^4}{q^3}$ such quadruples of points (x, y, u, v) with $x \cdot y = x \cdot u = u \cdot v = t$. The conclusion follows. □

Let

$$A' = \{(x, y, z, v, u) \in A : y \cdot v \neq t, \ x \neq z, \ y \neq u\}.$$

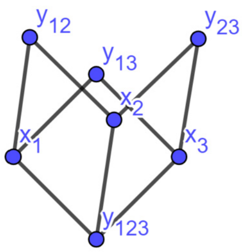

Fig. 5. Result of using Cauchy Schwarz.

If $|E| \geq Cq^{\frac{11}{4}}$, then

$$|A \setminus A'| \leq 5|E|^2 q^3 \leq \frac{5}{C^3}\frac{|E|^5}{q^5},$$

and thus $|A'| \geq \left(\frac{1}{2} - \frac{5}{C^3}\right)\frac{|E|^5}{q^5}$, in particular $|A'| \geq \frac{|E|^5}{4q^5}$.

Remark 3.4. We are now ready to take advantage of the symmetry of the configuration in Fig. 3. Ignoring y_1, y_2, y_3 for now, we can realize the rest of the configuration by taking a pair of quintuples $(x, y, z, u, v), (x', y, z, u', v) \in A'$ sharing the points y, z, and v.

For ease of notation we let A' denote both the set and its indicator function. Let

$$f(y, z, v) = \sum_{x,u \in E} A'(x, y, z, u, v).$$

Then

$$\frac{|E|^{10}}{16q^{10}} \leq |A'|^2 = \left(\sum_{y,z,v \in E} f(y, z, v)\right)^2$$

By Cauchy-Schwarz, and noting that $D_t(y, z) = 1$ whenever $f(y, z, v) \neq 0$, this is bounded by

$$\left(\sum_{y,z,v} f(y, z, v)^2\right)\left(\sum_{y,z,v} D_t(y, z)\right)$$

But for $y \in E$, $\sum_z D_t(y, z) \leq \frac{5|E|}{q}$, so

$$\frac{|E|^7}{80q^9} \leq \sum_{y,z,v} f(y, z, v)^2 = \sum_{y,z,v,} \left(\sum_{x,u} A(x, y, z, u, v)\right)^2$$
$$= \sum_{x,x',y,z,u,u',v} A(x, y, z, u, v)A(x', y, z, u', v).$$

By the one-to-one correspondence noted in Remark 3.4, the number of ordered tuples of vectors $(x_1, x_2, x_3, y_{12}, y_{13}, y_{23}, y_{123}) \in E^7$ such that

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = x_3 \cdot y_{123} = t$
- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$
- $x_1 \cdot y_{13} = x_3 \cdot y_{13} = t$
- $x_2 \cdot y_{23} = x_3 \cdot y_{23} = t$
- $x_2 \cdot y_{13} \neq t$
- $x_1 \neq x_2$, $x_3 \neq x_2$
- $y_{123} \neq y_{12}$, $y_{123} \neq y_{23}$

is at least $\frac{|E|^7}{80q^9}$. Fig. 5 above represents such a tuple.

We give a lower bound for the number of these tuples where $x_1 \neq x_3$. Suppose $x_1 = x_3$. Then we have six points $x_1, x_2, y_{12}, y_{13}, y_{23}, y_{123} \in E$ where

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = t$
- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$
- $x_1 \cdot y_{23} = x_2 \cdot y_{23} = t$
- $x_1 \cdot y_{13} = t$
- $x_2 \cdot y_{13} \neq t$
- $x_1 \neq x_2$
- $y_{123} \neq y_{12}, \ y_{123} \neq y_{23}$

We count the number of such tuples, summing first in y_{13} and then handling the remaining sum with Lemma 3.3. In the notation of Lemma 3.3, the sum in the second line of the following calculation corresponds to the case when $y \cdot v = t$.

$$\begin{aligned}
&\sum_{x_1,x_2,y_{12},y_{13},y_{23},y_{123} \in E} D_t(x_1, y_{123}) D_t(x_1, y_{12}) D_t(x_1, y_{23}) D_t(x_2, y_{123}) D_t(x_2, y_{12}) D_t(x_2, y_{23}) \\
&\qquad D_t(x_1, y_{13}) \\
&\leq \frac{5|E|}{q} \sum_{x_1,x_2,y_{12},y_{23},y_{123} \in E} D_t(x_1, y_{123}) D_t(x_1, y_{12}) D_t(x_1, y_{23}) D_t(x_2, y_{123}) D_t(x_2, y_{12}) D_t(x_2, y_{23}) \\
&\leq \frac{5|E|}{q} \cdot |E|^2 q^3 = 5|E|^3 q^2 \leq \frac{|E|^7}{800q^9}.
\end{aligned}$$

It follows that there exist at least

$$\frac{|E|^7}{80q^9} - \frac{|E|^7}{800q^9} \geq \frac{9|E|^7}{800q^9}$$

distinct tuples of vectors $(x_1, x_2, x_3, y_{12}, y_{13}, y_{23}, y_{123}) \in E^7$ such that

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = x_3 \cdot y_{123} = t$
- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$
- $x_1 \cdot y_{13} = x_3 \cdot y_{13} = t$
- $x_2 \cdot y_{23} = x_3 \cdot y_{23} = t$
- $x_2 \cdot y_{13} \neq t$
- $x_1 \neq x_2, \ x_3 \neq x_2, \ x_1 \neq x_3$
- $y_{123} \neq y_{12}, \ y_{123} \neq y_{23}$

Furthermore, for any such tuple, $y_{12} \cdot x_3 \neq t$. To see why, suppose otherwise. Then, both y_{12} and y_{123} lie on the intersection of the planes defined by $x_1 \cdot y = t$, $x_2 \cdot y = t$, and $x_3 \cdot y = t$. The intersection of two of these planes is either a line or the null set, since they are distinct. So, the intersection of all three is either a line, point, or the null set. Since two distinct points lie on the intersection, it must be a line. Furthermore, it must be the same line that is the intersection of any two of these planes. That is, if $y_{13} \cdot x_1 = t$ and $y_{13} \cdot x_3 = t$, then $y_{13} \cdot x_2 = t$ as well, a contradiction. By analogous reasoning $y_{23} \cdot x_1 \neq t$.

Now, fix one such tuple and observe that there are at least $\frac{|E|}{5q}$ vectors $y_1 \in E$ such that $x_1 \cdot y_1 = t$. However, there are at most q such y_1 where $x_2 \cdot y_1 = t$, since the intersection of the planes corresponding to x_1 and x_2 is at most a line. Likewise, there are at most q such y_1 where $x_3 \cdot y_1 = t$. Since $\frac{|E|}{5q} > 2q$, there exist a y_1 with $x_1 \cdot y_1 = t$, $x_2 \cdot y_1 \neq t$, and $x_3 \cdot y_1 \neq t$. We can also produce y_2 and y_3 in E with analogous properties. Since there are at most $3y^2$ vectors $y \in E$ such that $y \cdot x_i = t$ for some $i = 1, 2, 3$, we can also obtain a $y^* \in E$ where $y^* \cdot x_i \neq t$ for all $i = 1, 2, 3$.

We have obtained a sequence of vectors in E, $\{x_1, x_2, x_3, y_1, y_2, y_3, y_{12}, y_{13}, y_{23}, y_{123}, y^*\}$ such that

- $x_1 \cdot y_{123} = x_2 \cdot y_{123} = x_3 \cdot y_{123} = t$
- $x_1 \cdot y_{12} = x_2 \cdot y_{12} = t$, $x_3 \cdot y_{12} \neq t$, and similarly for y_{13} and y_{23}
- $x_1 \cdot y_1 = t$, $x_2 \cdot y_1, x_3 \cdot y_1 \neq t$, and similarly for y_1 and y_2
- $x_1 \cdot y^*, x_2 \cdot y^*, x_3 \cdot y^* \neq t$,

as desired.

Declaration of competing interest

We have no conflict of interest.

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