Fluid-poroviscoelastic structure interaction problem with nonlinear coupling

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Abstract

We prove the existence of a weak solution to a fluid-structure interaction (FSI) problem between the flow of an incompressible, viscous fluid modeled by the Navier-Stokes equations, and a poroviscoelastic medium modeled by the Biot equations. The two are nonlinearly coupled over an interface with mass and elastic energy, modeled by a reticular plate equation, which is transparent to fluid flow. The existence proof is constructive, consisting of two steps. First, the existence of a weak solution to a regularized problem is shown. Next, a weak-classical consistency result is obtained, showing that the weak solution to the regularized problem converges, as the regularization parameter approaches zero, to a classical solution to the original problem, when such a classical solution exists. While the assumptions in the first step only require the Biot medium to be poroelastic, the second step requires additional regularity, namely, that the Biot medium is poroviscoelastic. This is the first weak solution existence result for an FSI problem with nonlinear coupling involving a Biot model for poro(visco)elastic media.

1 Introduction and motivation

In this paper we study a time-dependent nonlinearly coupled fluid-structure interaction problem between the flow of an incompressible, viscous fluid, modeled by the Navier-Stokes equations, and bulk poroviscoelasticity modeled by the Biot equations. Bulk poroviscoelasticity means that the dimensions of the "free fluid flow" domain and the poroviscoelastic medium domain are the same. In particular, in this manuscript we consider the 2D case, see Fig. 1, which captures the main mathematical difficulties of such coupling. The free fluid flow and the Biot poro(visco)elastic medium are coupled across the current location of the interface, which has inertia and elastic energy, modeled by the reticular plate equation. A reticular plate is a lattice-type structure characterized by two properties: periodicity and small thickness, where periodicity refers to periodic cells (holes) distributed in all directions [25]. The reticular plate interface is transparent to fluid flow. We are interested in the existence of finite energy weak solutions (of the Leray-Hopf type).

The problem we study here arises in many applications. In particular, we mention encapsulation of bioartificial organs [65] and blood flow in arteries which are modeled as poro(visco)elastic media to study drug transport through the vascular walls [3, 16, 17]. The reticular plate can be used to capture the elastodynamics behavior of the intima/elastic laminae layer of arterial walls which is in direct contact with the blood flow on one side, and a poroelastic medium consisting of the arterial media/adventitia complex on the other side.

From the mathematical point of view the primary difficulties in studying 2D or 3D Navier-Stokes equations nonlinearly coupled to the 2D or 3D bulk poro(visco)elasticity arise from the fact that the finite energy solutions do not posses sufficient regularity to (1) define the moving domain and the corresponding traces, and (2) guarantee that all the integrals in the weak formulation of the

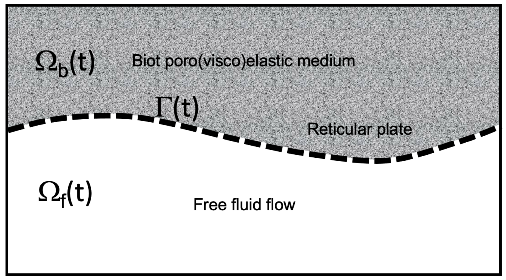

Figure 1: *A sketch of the fluid-poroelastic structure interaction domain.*

problem are well-defined. The first issue is related to the difficulties associated with 3D-3D fluid-structure coupling. The second issue is a consequence of the geometric nonlinearities associated with moving domain problems. These are the main reasons why to this day there have been no works on the existence of weak solutions for the Biot-Navier-Stokes coupled problems in which the coupling is assumed over a moving interface.

To get around these difficulties, we take the following approaches. First, the reticular plate at the interface associates mass and elastic energy to the interface, and regularizes the boundary of the fluid domain. This is, however, not sufficient to deal with the fact that in the nonlinearly coupled problem, the Biot domain is also moving, and as a result certain integrals in the weak formulation over the moving Biot domain are not well defined. This is why we introduce a "consistent **regularized** weak formulation" of the coupled problem by introducing a suitably constructed convolution in spatial variables and regularizing only the problematic terms in the weak formulation of the coupled problem. We then prove the existence of a weak solution to the regularized problem and show that as the regularization parameter tends to zero, this solution converges to the solution of the original nonregularized problem in the case when the original problem has a classical solution and the Biot poroelastic matrix is viscoelastic, where a classical solution is a solution that is smooth both temporally and spatially that hence satisfies the original system of PDEs for the original FPSI problem pointwise. We call this type of result a **weak-classical consistency** result. Namely, we prove that if a classical solution for the fluid-poroviscoelastic structure interaction (FPSI) problem without regularization exists on time-interval [0,T], then a sequence of weak solutions to the regularized FPSI problem constructed here, converges to the classical solution on $[0, T]$ as the regularization parameter converges to 0.

We mention that the existence of a weak solution to the regularized problem was considered by the authors of this manuscript in [40], where only the main steps in the proof were outlined. Here we present details of that proof, and show the weak-classical consistency result. Therefore, in this manuscript *we prove the existence of a weak solution to a nonlinearly coupled fluid-structure interaction problem between the flow of an incompressible, viscous fluid modeled by the Navier-Stokes equations, and a structure consisting of two solids – a thick poroviscoelastic medium modeled by the Biot equations, and a thin interface with mass modeled by a reticular plate equation.* We mention that no viscoelasticity is needed for the proof of the existence of a weak solution to the regularized problem. The existence of a weak solution to the regularized problem holds in the purely poroelastic case (and in the viscoelstic case). Viscoelasticity of the Biot poroelastic matrix is needed only in the proof of weak-classical consistency.

One of the interesting features of this work is that the proof of the existence of a weak solution is constructive. The main steps of the proof can be used to construct a numerical scheme to capture the physical solution to the problem [56]. The main idea of the proof is based on semidiscretizing the regularized FPSI problem in time by subdividing the time interval into N

subintervals of width Δt. At each time step we split the reticular plate subproblem from the regularized fluid-Biot subproblem using a Lie operator splitting strategy [31]. To deal with the moving domains we use the Lagrangian map for the Biot domain, and an Arbitrary Lagrangian-Eulerian mapping for the fluid domain, which maps a fixed, reference domain onto the current, physical domain. We switch between the reference domain formulation and moving domain formulation in the proof as needed. For each Δt, approximate solutions are constructed by "solving" the sequence of semidiscretized (linearized) problems defined on the current (approximate) moving domain for each $t_n = n\Delta t, n = 1, \ldots, N$. For each fixed Δt, we obtain energy estimates uniform in Δt, which allow us to deduce the existence of weakly and weakly* convergent subsequences. Since the problem is highly nonlinear, these are not sufficient to pass to the limit in the weak formulations of the approximate problems. Strong convergence of approximate sequences is then obtained by using several compactness results: the classical Aubin-Lions compactness lemma [52] for the Biot displacement, Arzela-Ascoli for the plate displacement, Dreher and Jüngel's compactness result [28] for the Biot and plate velocity and pore pressure, and a recent generalized Aubin-Lions-Simon compactness result by Muha and Čanić [47], to deal with the most involved part, which is the free fluid velocity defined on different time-dependent fluid domains. Once strongly convergent subsequences are obtained from the compactness results, one would like to pass to the limit in the weak formulation to show that the limits of the subsequences are weak solutions to the regularized fluid-poroelastic structure interaction problem. However, this cannot be done yet, since the velocity test functions are also defined on moving domains and we need to construct "appropriate" test functions which can be compared for different domains, and for which we can show converge to a test function of the limiting, continuous problem. Luckily, in contrast with the classical fluid-elastic structure interaction problems, in our case the fluid test functions decouple from the structure problem, and so it is a bit easier to construct appropriate test functions for which one can show uniform pointwise convergence to a test function for the continuous problem. With this final step, we can pass to the limit in the weak formulations of approximate problems and show that the limits of approximate subsequences satisfy the continuous weak formulation of the regularized problem. This existence result is local in time because we can guarantee the nondegeneracy of the fluid domains both for the free fluid flow and the filtrating flow through the poroelastic medium only locally in time. However, using the approaches presented in [20, Section 5] the time of existence can be extended to the maximal time until either (1) the moving fluid domain or Biot domain degenerates (e.g., the interface touches the bottom of the fluid domain or the top of the Biot domain), (2) the pores in the poroelastic matrix denegerate in the sense that the Lagrangian mapping stops being injective, or (3) $T = \infty$.

We finish this manuscript by addressing the weak-classical consistency of the regularized problem, namely, we prove, using a Gronwall-type estimate, that the energy of the difference between the weak solution of the regularized problem and the classical (temporally and spatially smooth) solution to the original, nonregularized problem with viscoelastic Biot poroelastic matrix, converges to zero as the regularization parameter tends to zero. While the main idea is simple, the estimates are quite nontrivial due to the fact that we need to work with the integrals over regularized Biot domains and compare them with the integrals over the nonregularized moving domains. Details are presented in Section 10.

2 Literature review

There is extensive past work on fluid-structure interaction studying fully coupled systems involving incompressible, Newtonian fluids interacting with elastic structures. In many fluid-structure interaction models considered in the literature, the solid structure, which is elastic or deformable, is modeled by equations of elasticity. The models first considered in the literature are linearly coupled fluid-structure interaction models [4, 5, 42], which pose the fluid equations on a fixed reference fluid domain, as a linearization that approximates real-life dynamics well when structure displacements and deformations are small. In cases when displacements and deformations of the structure are large, they can significantly affect the dynamics of the fluid in which case time-dependent moving fluid domains that depend on the displacement itself must be taken into account. There has been extensive work on studying such nonlinearly coupled models [8,20,22,23,26,27,32–36,41,43,44,47–51,55], in which the time-dependent and a priori unknown fluid domain evolves according to the displacement of the structure, giving rise to a fully coupled problem with two-way coupling between the fluid and structure that has significant geometric nonlinearities arising from the moving boundary. As a result of past work in fluid-structure interaction, significant progress has been made within the past 20 years in the mathematical analysis of fluid-structure systems involving incompressible fluids and elastic structures.

However, many elastic materials, such as biological tissues and sediments that interact with fluids are not impermeable and can admit fluid flow through their pores, in which case poroelasticity of the material needs to be taken into account. The study of poroelasticity was initiated in studies by Biot modeling soil consolidation [9, 10], but has since been extended to broader applications. Such porous media and poroelastic materials are found in applications to geoscience, including the study of fractures in porous and poroelastic materials [30, 45] and more recently, applications to biomedical science, including the study of the ocular poroelastic tissue known as the lamina cribrosa, which is related to understanding the onset of glaucoma [18], and the modeling of intestinal walls by equations of poroelasticity [66]. In addition to modeling, the mathematical study of poroelastic materials and the Biot equations in terms of well-posedness has also been an active area of research [6, 7, 11, 13–15, 54, 59, 60, 63].

More recently, there has been a need in applications to understand not just poroelastic materials by themselves, but the interaction between poroelastic materials and fluids. Mathematically, such systems are described by coupling fluid equations (e.g. the Navier-Stokes or Stokes system) with poroelasticity. These coupled problems are referred to as fluid-poroelastic structure interaction (FPSI) problems, and have been analyzed, for instance, in [2, 19, 61], where linear coupling between the free fluid equations and poroelastic medium was assumed. Recent progress in the design of bioartificial organs, see e.g., [65], sparked the need to study FPSI problems in which the fluid-structure interface has mass and elastic or poroelastic properties itself. Namely, in the recent work on the design of a bioartificial pancreas [65], the bioartifical pancreas consists of an *encapsulated* poroelastic agarose gel containing transplanted pancreatic cells, where the capsule containing the poroelastic medium is itself poroelastic, and it is designed to protect the transplanted cells within the poroelastic agarose gel from the patient's own immune cells, while allowing the passage of oxygen and nutrients to the cells for long time viability. This capsule is a thin poroelastic membrane/plate which sits at the interface between the poroelastic gel containing the transplanted cells, and the flow of blood carrying oxygen and nutrients to the bioartificial organ. The resulting mathematical problem in [65] is a fluid-poroelastic structure interaction problem in which the structure consists of two layers: a thin poroelastic plate located at the interface between the free fluid flow and a thick poroelastic medium modeled by the linear and nonlinear Biot equations, coupled over a fixed interface (linearized coupling). The well-posedness for this problem was studied in [12] for both the linear and nonlinear Biot equations, where the nonlinearity refers to the dependence of the permeability tensor in the Biot equations on the fluid content. In this work the fluid-structure interface with mass serves as a regularizing mechanism and provides sufficient information about the regularity of the interface and the free fluid domain to allow, for the first time, the proof of the existence of a finite energy weak solution.

None of the works that address weak solutions to fluid-structure interaction problems between the flow of an incompressible, viscous fluid and a poroelastic solid have taken into account nonlinear coupling over the moving interface. Such problems, however, are of importance in many applications, including the flow of blood in coronary arteries sitting on the surface of the heart, and contracting and relaxing with each heart beat [17,64]. To understand large displacements that occur due to the contractions of the heart muscle, and capture the flow of drugs through the vascular wall, nonlinear coupling between the blood flow and vascular walls, modeled as poro(visco)elastic media, needs to be taken into account. The goal of the current manuscript is to develop a well-posedness theory for a nonlinearly coupled (moving boundary) fluid-poroelastic structure interaction problem by constructing new tools for dealing with the equations of poroelasticity defined on a priori unknown and time-dependent domains.

3 Description of the main problem

We study fluid-structure interaction between the flow of an incompressible, viscous fluid and a multilayered poro(visco)elastic structure consisting of two layers: a thick poro(visco)elastic layer modeled by the Biot equations, and a thin elastic layer modeled by the reticular plate equation. The problem is set on a two dimensional domain, which embodies all the main mathematical difficulties associated with the analysis of this problem. The entire two dimensional domain $\hat{\Omega}$ is a union of the reference domain for the fluid subproblem $\hat{\Omega}_f$, the reference domain for the Biot poroviscoelastic material $\hat{\Omega}_b$, and the reference domain $\hat{\Gamma}$ of the elastic reticular plate which serves as the interface separating the free fluid flow and the Biot medium:

$$\hat{\Omega} = \hat{\Omega}_b \cup \hat{\Omega}_f \cup \hat{\Gamma}, \text{ where } \hat{\Omega}_b = (0, L) \times (0, R),\ \hat{\Gamma} = (0, L) \times \{0\},\ \hat{\Omega}_f = (0, L) \times (-R, 0).$$

These domains will evolve in time, giving rise to the time-dependent $\Omega(t) = \Omega_b(t) \cup \Omega_f(t) \cup \Gamma(t)$. We will be using the hat notation to denote objects associated with the reference domain. On each subdomain we will consider the following mathematical models.

3.1 The Biot equations on a moving domain

The Biot system consists of the elastodynamics equation, which in this work will be defined on the Lagrangian domain $\hat{\Omega}_b$, and the fluid equation, which in this work will be defined on the Eulerian domain $\Omega_b(t)$. Let $\hat{\boldsymbol{\eta}} : [0, T] \times \hat{\Omega}_b \to \mathbb{R}^2$ denote the displacement of the Biot poroviscoelastic matrix from its reference configuration, and let $\hat{p} : \hat{\Omega}_b \to \mathbb{R}$ denote the fluid pore pressure. To specify the fluid equation given in terms of the fluid pore pressure in Eulerian formulation, we introduce the **Lagrangian map** by

$$\hat{\boldsymbol{\Phi}}_b^\eta(t, \cdot) = \text{Id} + \hat{\boldsymbol{\eta}}(t, \cdot) : \hat{\Omega}_b \to \Omega_b(t), \tag{1}$$

with $(\boldsymbol{\Phi}_b^\eta)^{-1}(t, \cdot) : \Omega_b(t) \to \hat{\Omega}_b$ denoting its inverse. The Biot equations are then given by:

$$\rho_b \partial_{tt} \hat{\boldsymbol{\eta}} = \hat{\nabla} \cdot \hat{S}_b(\hat{\nabla} \hat{\boldsymbol{\eta}}, \hat{p}) \qquad \text{in } \hat{\Omega}_b, \tag{2}$$

$$\frac{c_0}{[\det(\hat{\nabla} \hat{\boldsymbol{\Phi}}_b^\eta)] \circ (\boldsymbol{\Phi}_b^\eta)^{-1}} \frac{D}{Dt} p + \alpha \nabla \cdot \frac{D}{Dt} \boldsymbol{\eta} = \nabla \cdot (\kappa \nabla p) \qquad \text{in } \Omega_b(t), \tag{3}$$

where $\frac{D}{Dt} = \frac{d}{dt} + \left(\left(\partial_t \boldsymbol{\eta}(t, \cdot) \circ (\boldsymbol{\Phi}_b^\eta)^{-1}(t, \cdot) \right) \cdot \nabla \right)$ is the material derivative. The first equation describes the elastodynamics of the poroelastic solid matrix, while the second equation models the conservation of mass principle of the filtrating fluid. See, e.g., [58, 67]. To recover the filtration fluid velocity $\boldsymbol{q}$, Darcy's law is used:

$$\boldsymbol{q} = -\kappa \nabla p \qquad \text{on } \Omega_b(t), \tag{4}$$

where κ is a positive permeability constant.

In this work, we will consider both the viscoelastic and the purely elastic consitutive models for the Biot poroelastic matrix with the Piola-Kirchhoff stress tensor for the viscoelastic case given by

$$\hat{S}_b(\nabla\boldsymbol{\eta}, p) = 2\mu_e \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) + \lambda_e(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}})\boldsymbol{I} + 2\mu_v \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}_t) + \lambda_v(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}_t)\boldsymbol{I} - \alpha \det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^\eta)\hat{p}(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^\eta)^{-t}, \tag{5}$$

where superscript t denotes matrix transposition and $A^{-t} = (A^{-1})^t$. The purely elastic case has the coefficients λ_v and μ_v equal to zero. Here, $\boldsymbol{D}$ denotes the symmetrized gradient, μ_e and λ_e are the Lamé parameters related to the elastic stress, μ_v and λ_v are the corresponding parameters related to the viscoelastic stress, and $\hat{\boldsymbol{\Phi}}_b^\eta$ is the Lagrangian map defined above. In equation (3) the Biot material displacement $\boldsymbol{\eta}$ and the pore pressure p are defined on the physical domain $\Omega_b(t)$ as

$$\boldsymbol{\eta}(t,\cdot) = \hat{\boldsymbol{\eta}}(t, (\boldsymbol{\Phi}_b^\eta)^{-1}(t,\cdot)), \qquad p(t,\cdot) = \hat{p}(t, (\boldsymbol{\Phi}_b^\eta)^{-1}(t,\cdot)), \text{ where } \Omega_b(t) = \hat{\boldsymbol{\Phi}}_b^\eta(t, \hat{\Omega}_b).$$

We remark that in the last term of the Piola-Kirchhoff stress tensor (5), we have used the Piola transform, which is a transformation that maps tensors in Lagrangian coordinates to corresponding tensors in Eulerian coordinates in such a way that divergence-free tensors in Lagrangian coordinates remain divergence free in Eulerian coordinates [24].

We note that *a priori* the notion of $\Omega_b(t)$ is not entirely clear, unless $\hat{\boldsymbol{\eta}}$ is sufficiently regular, and furthermore, the formulation of this problem makes sense only if the map $\hat{\boldsymbol{\Phi}}_b^\eta = \text{Id} + \hat{\boldsymbol{\eta}}$ is an injective map from $\hat{\Omega}_b$ to $\Omega_b(t)$. We address these important issues later.

3.2 The reticular plate equation

The elastodynamics of reticular plates, studied in [25] using homogenization, are governed by a plate-type equation, defined on the equilibrium middle surface of the homogenized plate or shell $\hat{\Gamma}$. The homogenized equation is given in terms of transverse displacement $\hat{\boldsymbol{\omega}} = \hat{\omega}\boldsymbol{e}_y$ from the reference configuration:

$$\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2\hat{\omega} = \hat{F}_p, \qquad \text{on } \hat{\Gamma}, \tag{6}$$

where ρ_p is the plate density coefficient and $\hat{F}_p$ is the external forcing on the plate in y direction, to be specified later in the coupling conditions. The constant ρ_p is the "average" plate density, which depends on the periodic structure. The in-plane bi-Laplacian $\hat{\Delta}^2$ (Laplace-Beltrami operator for curved $\hat{\Gamma}$'s) is associated with the elastic energy of the plate. Typically, there is a coefficient $\tilde{D}$ in front of the bi-Laplacian, which contains information about the periodicity of the structure and its stiffness properties [25]. In the present work, we will assume that it is equal to 1. The source term $\hat{F}_p$ corresponds to the loading of the poroelastic plate, which will come from the jump in the normal stress (traction) between the free fluid on one side and the thick Biot poroelastic structure on the other, see (7) below.

In our problem, the reticular plate separates the regions of free fluid flow and the Biot poroviscoelastic medium, and is transparent to the flow between the two. The time-dependent configuration of the plate

$$\Gamma(t) = \{(x, y) : 0 < x < L,\ y = \hat{\omega}(t, x)\},$$

forms the bottom boundary of the moving Biot domain $\Omega_b(t)$, and the remaining left, top, and right boundaries of the moving Biot domain $\Omega_b(t)$ are fixed in time. Hence, we impose $\boldsymbol{\eta} = 0$ on the left, top, and right boundaries of $\Omega_b(t)$. See Fig. 1. Hence, we can describe the moving domain $\Omega_b(t)$ as

$$\Omega_b(t) = \{(x,y) : 0 < x < L,\ \hat{\omega}(t,x) < y < R\}.$$

3.3 The Navier-Stokes equations on a moving domain

The free flow of an incompressible, viscous fluid will be modeled by the Navier-Stokes equations

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &=& \nabla\cdot\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi) \\ \nabla\cdot\boldsymbol{u} &=& 0 \end{array}\right\} \quad \text{in } \Omega_f(t), \tag{7}$$

where $\boldsymbol{u}$ is the fluid velocity and π is the fluid pressure. The Cauchy stress tensor is given by

$$\boldsymbol{\sigma}_f(\nabla\boldsymbol{u},\pi) = 2\nu\boldsymbol{D}(\boldsymbol{u}) - \pi\boldsymbol{I},$$

where π is the fluid pressure and ν is kinematic viscosity coefficient. Notice that the fluid problem is defined on a moving domain, which is not known *a priori*. The moving fluid domain $\Omega_f(t)$ is a function of time and it is determined by the plate displacement $\hat{\omega}$, as follows:

$$\Omega_f(t) = \{(x,y) : 0 < x < L, -R < y < \hat{\omega}(t,x)\}.$$

The fact that the free fluid domain depends on one of the unknowns in the problem presents a geometric nonlinearity that is difficult to deal with. We will be using the following **Arbitrary Lagrangian Eulerian (ALE) mapping $\hat{\boldsymbol{\Phi}}_f^{\omega}$** $: \hat{\Omega}_f \to \Omega_f(t)$ to map the fixed reference domain $\hat{\Omega}_f$ onto the current, physical domain $\Omega_f(t)$:

$$\hat{\boldsymbol{\Phi}}_f^{\omega}(\hat{x},\hat{y}) = \left(\hat{x}, \hat{y} + \left(1 + \frac{\hat{y}}{R}\right)\hat{\omega}\right), \qquad (\hat{x},\hat{y}) \in \hat{\Omega}_f. \tag{8}$$

In our analysis, we will use this ALE mapping to will switch between the fixed and moving boundary formulations of the coupled problem as needed.

3.4 The coupling conditions

The Navier-Stokes equations (7), the Biot equations (2), (3), and the reticular plate equation (6) are coupled across the moving reticular plate interface $\Gamma(t)$ via two sets of coupling conditions: the kinematic and dynamic coupling conditions. To state these conditions, we introduce the following notation:

- The Biot Cauchy stress tensor, defined on the physical domain, is obtained by applying the Piola transform to the Biot Cauchy stress tensor $\hat{S}_b(\nabla\boldsymbol{\eta}, p)$ on the reference domain:

$$\begin{aligned} S_b(\nabla\boldsymbol{\eta},p) &= [\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^{-1}\hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}},\hat{p})(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^t] \circ (\boldsymbol{\Phi}_b^{\eta})^{-1} \\ &= \left(\frac{1}{\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})}\left[2\mu_e\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) + \lambda_e(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}) + 2\mu_v\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}_t) + \lambda_v(\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}_t)\right](\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^{\eta})^t\right) \circ (\boldsymbol{\Phi}_b^{\eta})^{-1} - \alpha p\boldsymbol{I}. \end{aligned} \tag{9}$$

- The Eulerian structure velocity of the Biot poroviscoelastic matrix is given at each point of the physical domain $\Omega_b(t)$ by

$$\boldsymbol{\xi}(t,\cdot) = \partial_t\hat{\boldsymbol{\eta}}\left(t, (\boldsymbol{\Phi}_b^{\eta})^{-1}(t,\cdot)\right). \tag{10}$$

- The normal unit vector to the moving interface $\Gamma(t)$ will be denoted by $\boldsymbol{n}(t)$, and the normal unit vector to the reference configuration of the interface $\hat{\Gamma}$ will be denoted by $\hat{\boldsymbol{n}}$. Note that $\hat{\boldsymbol{n}} = \boldsymbol{e}_y$. The vectors $\boldsymbol{n}(t)$ and $\hat{\boldsymbol{n}}$ point outward from $\Omega_f(t)$ and Ω_f, and inward towards $\Omega_b(t)$ and Ω_b.

The following two sets of coupling conditions give rise to a well-defined bounded energy of the coupled problem:

(I) Kinematic coupling conditions:

- Continuity of normal components of velocity (conservation of mass of the fluid):

$$\boldsymbol{u} \cdot \boldsymbol{n}(t) = (\boldsymbol{q} + \boldsymbol{\xi}) \cdot \boldsymbol{n}(t), \qquad \text{on } (0,T) \times \Gamma(t). \tag{11}$$

- Slip in the tangential component of free fluid velocity, known as the Beavers-Joseph-Saffman condition [38,39]:

$$\beta(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{\tau}(t) = \boldsymbol{\sigma}_f \boldsymbol{n}(t) \cdot \boldsymbol{\tau}(t), \qquad \text{on } (0,T) \times \Gamma(t), \tag{12}$$

  where $\beta \geqslant 0$ is a constant and $\boldsymbol{\tau}(t)$ is the rightward pointing unit tangent vector to $\Gamma(t)$.

- Continuity of displacements:

$$\hat{\boldsymbol{\eta}} = \hat{\omega} \boldsymbol{e}_y, \qquad \text{on } (0,T) \times \hat{\Gamma}. \tag{13}$$

(II) Dynamic coupling conditions:

- Balance of forces describing the body forcing on the plate as the difference between the normal components of normal stress coming from the Biot medium on one side, and free fluid flow on the other:

$$\hat{F}_p = -\det(\nabla \hat{\boldsymbol{\Phi}}_f^{\omega})[\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) \circ \hat{\boldsymbol{\Phi}}_f^{\omega}](\nabla \hat{\boldsymbol{\Phi}}_f^{\omega})^{-t} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} + \hat{S}_b(\hat{\nabla} \hat{\boldsymbol{\eta}}, \hat{p}) \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}|_{\hat{\Gamma}}, \qquad \text{on } \hat{\Gamma}, \tag{14}$$

  where $\hat{\boldsymbol{\Phi}}_f^{\omega}$ is the Arbitrary Lagrangian-Eulerian (ALE) mapping defined in (16).

- Balance of pressure at the interface:

$$-\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) \boldsymbol{n}(t) \cdot \boldsymbol{n}(t) + \frac{1}{2}|\boldsymbol{u}|^2 = p, \qquad \text{on } (0,T) \times \Gamma(t). \tag{15}$$

3.5 The initial and boundary conditions

For the fluid, we will assume rigid walls on $\partial\Omega_f(t) \backslash \Gamma(t)$ and impose a no-slip condition

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega_f(t) \backslash \Gamma(t).$$

Similarly, we will assume that the boundaries of the Biot poroviscoelastic medium, excluding the interface $\Gamma(t)$, are rigid and impose

$$\hat{\boldsymbol{\eta}} = 0 \quad \text{and} \quad \hat{p} = 0, \qquad \text{on } \partial\hat{\Omega}_b \backslash \hat{\Gamma}.$$

Finally, we prescribe the following initial conditions:

$$\begin{aligned} \boldsymbol{u}(0) = \boldsymbol{u}_0 &\quad \text{in } \Omega_f(0), \\ \hat{\boldsymbol{\eta}}(0) = \hat{\boldsymbol{\eta}}_0,\ \partial_t \hat{\boldsymbol{\eta}}(0) = \hat{\boldsymbol{\xi}}_0 &\quad \text{in } \hat{\Omega}_b, \\ \hat{\omega}(0) = \hat{\omega}_0,\ \partial_t \hat{\omega}(0) = \hat{\zeta}_0 &\quad \text{in } \hat{\Gamma}, \\ \hat{p}(0) = \hat{p}_0 &\quad \text{in } \hat{\Omega}_b. \end{aligned}$$

3.6 Preview of the main results

Our first main result is the existence of a weak solution to a regularized FPSI problem, introduced in Sec. 5. The existence result holds for both elastic and viscoelastic Biot material. Here we state the theorem informally and refer the reader to Theorem 5.1 for the precise statement.

Theorem 3.1 (Existence of a weak solution to the regularized problem)**.** Let $\rho_b, \mu_e, \lambda_e, \alpha, \rho_p, \nu > 0$ and $\mu_v, \lambda_v \geqslant 0$. Moreover, assume that initial data are in the finite energy class and that initially, the interface does not touch the bottom boundary of the fluid domain and the top boundary of the Biot domain, and assume that certain compatibility conditions are satisfied. Then for every regularization parameter $\delta > 0$, there exists $T > 0$ (potentially depending on $\delta > 0$) such that there is a weak solution on $[0, T]$ to the regularized problem with regularization parameter δ. Furthermore, the weak solution to the regularized problem exists on a maximal time interval $[0, T]$, where either (1) $T = \infty$ or (2) T is finite and is the time at which either:

- the fluid or Biot domain degenerates so that the moving interface collides with the bottom boundary of $\hat{\Omega}_b$ or the top boundary of $\hat{\Omega}_b$) or
- the (regularized) Lagrangian mapping $\hat{\mathbf{\Phi}}_b^{\eta^\delta}$ for the Biot domain is no longer injective.

Our second main result is a weak-classical consistency result. Namely, in order to justify our regularization procedure and the corresponding definition of weak solutions to the regularized problem, we prove that weak solutions to the regularized problem indeed converge to the solution of the original (non-regularized) FPSI problem. More precisely, we prove the following result, made precise in Theorem 10.1.

Theorem 3.2 (Weak-classical consistency)**.** Assume that a classical (smooth) solution to the FPSI problem with a Biot poroviscoelastic medium exists on time-interval [0,T] for the case for which the viscoelasticity parameters $\mu_v, \lambda_v > 0$. Then every sequence of weak solutions to the regularized problem with regularization parameter $\delta > 0$ converges to the classical solution on $[0, T]$ as the regularization parameter δ converges to 0. In particular, the time interval of existence for the weak solutions to the regularized problem is uniform in regularization parameter and solutions to the regularized problem exists on the same time interval where the classical solution exists.

The heart of the proof of this theorem is a bootstrap argument presented in Section 10.4. Namely, the main issue is that geometric quantities, such as the determinant of the displacement, cannot be estimated by the energy, and thus are not uniformly bounded in the regularization parameter δ. We derive appropriate bounds by using a bootstrap argument in combination with optimal convergence rate estimates for the convolution regularization. The main technical issue in comparing the classical solution with weak solutions to the regularized problem is the fact that they are defined on different domains. Therefore, we use a change of variables that transfers fluid velocities as vector fields and preserves the divergence-free condition. This transformation was introduced by [37] and was used in proving weak-strong type of results in the context of FSI in [21, 53, 57]. The corresponding estimates are carried out in Section 10.3.

4 Definition of a weak solution

Because the problem under consideration is nonlinearly coupled, the fluid domain $\Omega_f(t)$ and the Biot poroviscoelastic domain $\Omega_b(t)$ in physical space are time-dependent and not known apriori. To handle the moving domains, it is useful to introduce the mappings that map the reference domains $\hat{\Omega}_b$, $\hat{\Gamma}$, and $\hat{\Omega}_f$ onto the moving domains that depend on time and on the solution itself.

4.1 Mappings between reference and physical domains

Let

$$\hat{\mathbf{\Phi}}^{\eta}_b(t,\cdot):\hat{\Omega}_b \to \Omega_b(t), \qquad \hat{\mathbf{\Phi}}^{\omega}_{\Gamma}(t,\cdot):\hat{\Gamma}\to\Gamma(t), \qquad \hat{\mathbf{\Phi}}^{\omega}_f(t,\cdot):\hat{\Omega}_f \to \Omega_f(t),$$

be such that

$$\begin{array}{ll} \hat{\mathbf{\Phi}}^{\eta}_b = \mathrm{Id} + \hat{\boldsymbol{\eta}}(\hat{x},\hat{y}), & (\hat{x},\hat{y})\in\hat{\Omega}_b \\ \hat{\mathbf{\Phi}}^{\omega}_{\Gamma}(\hat{x},0) = (\hat{x},\hat{\omega}(\hat{x})), & \hat{x}\in\hat{\Gamma} \\ \hat{\mathbf{\Phi}}^{\omega}_f(\hat{x},\hat{y}) = \left(\hat{x},\hat{y}+\left(1+\frac{\hat{y}}{R}\right)\hat{\omega}(\hat{x})\right), & (\hat{x},\hat{y})\in\hat{\Omega}_f, \end{array} \tag{16}$$

with the inverse

$$(\mathbf{\Phi}^{\omega}_f)^{-1}(x,y) = \left(x, -R + \frac{R}{R+\hat{\omega}}(R+y)\right). \tag{17}$$

We are using $(\hat{x},\hat{y})$ to denote the coordinates on the reference domain and (x,y) the coordinates on the physical domain. Note that these mapings are *time-dependent*, even though in the rest of this manuscript we will not explicitly notate this time dependence for ease of notation.

The **Jacobians of the transformations** are given by:

$$\hat{\mathcal{J}}^{\omega}_f = 1+\tfrac{\hat{\omega}}{R}, \qquad \hat{\mathcal{J}}^{\eta}_b = \det(\boldsymbol{I}+\hat{\nabla}\hat{\boldsymbol{\eta}}), \qquad \hat{\mathcal{J}}^{\omega}_{\Gamma} = \sqrt{1+|\partial_{\hat{x}}\hat{\omega}|^2}, \tag{18}$$

where $\hat{\mathcal{J}}^{\omega}_{\Gamma}$ measures the arc length difference of between the reference and deformed configuration of the plate. Notice that in the Jacobian $\hat{\mathcal{J}}^{\omega}_f$ we dropped the absolute value sign since our results will hold up until the time of domain degeneracy when $|\hat{\omega}| \geqslant R$.

Under these mappings the functions are transformed as follows.

Tranformations under $\mathbf{\Phi}^{\omega}_f$. The fluid velocity $\boldsymbol{u}$ defined on $\Omega_f(t)$ is transferred to the fixed reference domain $\hat{\Omega}_f$ by

$$\hat{\boldsymbol{u}}(t,\hat{x},\hat{y}) = \boldsymbol{u}\circ\hat{\mathbf{\Phi}}_f, \qquad \text{for } (\hat{x},\hat{y})\in\hat{\Omega}_f.$$

Recall that on the moving domain $\Omega_f(t)$, the fluid velocity $\boldsymbol{u}$ is divergence free, i.e., $\nabla\cdot\boldsymbol{u} = 0$. However, when we pull the fluid velocity back to the reference domain, $\hat{\boldsymbol{u}}$ is not necessarily divergence free on $\hat{\Omega}_f$. Hence, we want to reformulate the divergence free condition on the fixed reference domain.

The divergence free condition. Let g be a function defined on $\Omega_f(t)$, then

$$\nabla g = \nabla\left(\hat{g}\circ(\mathbf{\Phi}^{\omega}_f)^{-1}\right) = (\hat{\nabla}^{\omega}_f\hat{g})\circ(\mathbf{\Phi}^{\omega}_f)^{-1},$$

where $\hat{\nabla}^{\omega}_f$ is the *transformed gradient operator*:

$$\hat{\nabla}^{\omega}_f = \begin{pmatrix} \partial_{\hat{x}} - (R+y)\partial_{\hat{x}}\hat{\omega}\frac{R}{(R+\hat{\omega})^2}\partial_{\hat{y}} \\ \frac{R}{R+\hat{\omega}}\partial_{\hat{y}} \end{pmatrix} \quad \text{where} \quad y = \hat{y}+\left(1+\frac{\hat{y}}{R}\right)\hat{\omega}. \tag{19}$$

Therefore, the divergence free condition and the symmetrized gradient on the fixed reference domain $\hat{\Omega}_f$ are:

$$\hat{\nabla}^{\omega}_f\cdot\hat{\boldsymbol{u}} = 0, \qquad \hat{\boldsymbol{D}}^{\omega}_f(\hat{\boldsymbol{u}}) = \frac{1}{2}\left(\hat{\nabla}^{\omega}_f\hat{\boldsymbol{u}} + (\hat{\nabla}^{\omega}_f\hat{\boldsymbol{u}})^t\right).$$

Time derivatives. The time derivative transforms under the map $\hat{\mathbf{\Phi}}^{\omega}_f$ as follows:

$$\partial_t\boldsymbol{u} = \partial_t\hat{\boldsymbol{u}} - (\hat{\boldsymbol{w}}\cdot\hat{\nabla}^{\omega}_f)\hat{\boldsymbol{u}} \quad \text{where} \quad \hat{\boldsymbol{w}} = \frac{R+\hat{y}}{R}\partial_t\hat{\omega}\boldsymbol{e}_y. \tag{20}$$

Tranformations under $\mathbf{\Phi}_b^\omega$. Given a scalar function g defined on $\Omega_b(t)$ the pull back of g to the reference domain $\hat{\Omega}_b$ is given by

$$\hat{g} = g \circ \hat{\mathbf{\Phi}}_b^\eta.$$

We claim that for some differential operator $\hat{\nabla}_b^\eta$, which we will determine below,

$$\nabla g = \nabla \left(\hat{g} \circ (\mathbf{\Phi}_b^\eta)^{-1}\right) = (\hat{\nabla}_b^\eta \hat{g}) \circ (\mathbf{\Phi}_b^\eta)^{-1},$$

where ∇ is a gradient on the physical domain, $\hat{\nabla}$ is a gradient on the reference domain, and $\hat{\nabla}_b^\eta$ is a differential operator (different from $\hat{\nabla}$) on the reference domain. For any function g defined on the physical domain, we have that

$$\hat{\nabla}\left(g \circ \hat{\mathbf{\Phi}}_b^\eta\right) = [(\nabla g) \circ \hat{\mathbf{\Phi}}_b^\eta] \cdot (\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}).$$

Hence, for

$$\hat{\nabla}_b^\eta \hat{g} = (\nabla g) \circ \hat{\mathbf{\Phi}}_b^\eta,$$

we get the following explicit formula for the *transformed gradient operator* $\hat{\nabla}_b^\eta$ on $\hat{\Omega}_b$:

$$\hat{\nabla}_b^\eta \hat{g} = \left(\frac{\partial \hat{g}}{\partial \hat{x}}, \frac{\partial \hat{g}}{\partial \hat{y}}\right) \cdot (\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}})^{-1}. \tag{21}$$

Notice that the invertibility of the matrix $\boldsymbol{I} + \hat{\nabla}\hat{\boldsymbol{\eta}}$ will be related to whether the map $(\hat{x}, \hat{y}) \to (\hat{x}, \hat{y}) + \hat{\boldsymbol{\eta}}(\hat{x}, \hat{y})$ is a bijection between $\hat{\Omega}_b$ and $\Omega_b(t)$.

4.2 Weak solution

We now derive the definition of a weak solution to the given FPSI problem, by means of the following formal calculation. We start with the fluid equations and multiply by a test function $\boldsymbol{v}$. Recall the definition of the Eulerian structure velocity $\boldsymbol{\xi}$ from (10). For the inertia term of the Navier-Stokes equations, using the Reynold's transport theorem and integration by parts, we obtain:

$$\begin{aligned}
\int_{\Omega_f(t)} (\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} &= \frac{d}{dt}\int_{\Omega_f(t)} \boldsymbol{u} \cdot \boldsymbol{v} - \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} - \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\
&\quad + \frac{1}{2}\int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\
= \frac{d}{dt}\int_{\Omega_f(t)} \boldsymbol{u} \cdot \boldsymbol{v} - \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} &+ \frac{1}{2}\int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v}.
\end{aligned}$$

For the diffusive term of the Navier Stokes equations, we integrate by parts to obtain

$$-\int_{\Omega_f(t)} (\nabla \cdot \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)) \cdot \boldsymbol{v} = 2\nu \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) - \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{v},$$

where we used the fact that the test function $\boldsymbol{v}$ is divergence free to eliminate the pressure, and we use that the test function satisfies $\boldsymbol{v} = 0$ on $\partial\Omega_f(t)\backslash\Gamma(t)$ due to the boundary conditions for $\boldsymbol{u}$.

Next, we multiply the structure equation by a test function $\hat{\boldsymbol{\psi}}$ to obtain

$$\int_{\hat{\Omega}_b} (\rho_b \partial_{tt}\hat{\boldsymbol{\eta}} - \hat{\nabla} \cdot \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p})) \cdot \hat{\boldsymbol{\psi}} = \rho_b \left(\frac{d}{dt} \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{\psi}} - \int_{\Omega_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} \right)$$
$$+ \int_{\hat{\Omega}_b} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}) : \hat{\nabla}\hat{\boldsymbol{\psi}} + \int_{\hat{\Gamma}} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}) \boldsymbol{e}_y \cdot \hat{\boldsymbol{\psi}} = \rho_b \left(\frac{d}{dt} \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\boldsymbol{\psi}} - \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} \right)$$
$$+ \int_{\hat{\Omega}_b} (2\mu_e \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_e (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}))$$
$$- \alpha \int_{\Omega_b(t)} p(\nabla \cdot \boldsymbol{\psi}) + \int_{\hat{\Gamma}} \hat{S}_b(\nabla \hat{\boldsymbol{\eta}}, \hat{p}) \boldsymbol{e}_r \cdot \hat{\boldsymbol{\psi}}.$$

Except on $\hat{\Gamma}$, there are no boundary terms, because $\hat{\boldsymbol{\eta}} = 0$ on the left, top, and right boundaries of $\hat{\Omega}_b$, and hence the same condition holds for the corresponding test function $\hat{\boldsymbol{\psi}}$. Note that in the integral over $\Omega_b(t)$, $\boldsymbol{\psi} := \hat{\boldsymbol{\psi}} \circ (\boldsymbol{\Phi}_b^\eta)^{-1}$.

Finally, we test the second equation corresponding to the evolution of the pore pressure for the Biot poroviscoelastic medium with a test function r, and recall the definition of the Darcy velocity $\boldsymbol{q}$ from (4), keeping in mind that $\boldsymbol{n}$ is the *inward* normal vector to $\Omega_b(t)$:

$$\int_{\Omega_b(t)} \left(\frac{c_0}{[\det(\hat{\nabla}\hat{\boldsymbol{\Phi}}_b^\eta)] \circ (\boldsymbol{\Phi}_b^\eta)^{-1}} \frac{D}{Dt} p + \alpha \nabla \cdot \frac{D}{Dt} \boldsymbol{\eta} - \nabla \cdot (\kappa \nabla p) \right) r$$
$$= \int_{\hat{\Omega}_b} c_0 \partial_t \hat{p} \cdot \hat{r} + \int_{\Omega_b(t)} \alpha \left(\nabla \cdot \frac{D}{Dt} \boldsymbol{\eta} \right) r + \int_{\Omega_b(t)} \kappa \nabla p \cdot \nabla r - \int_{\Gamma(t)} (\boldsymbol{q} \cdot \boldsymbol{n}) r$$
$$= \frac{d}{dt} \int_{\hat{\Omega}_b} c_0 \hat{p} \cdot \hat{r} - \int_{\hat{\Omega}_b} c_0 \hat{p} \cdot \partial_t \hat{r} - \int_{\Omega_b(t)} \alpha \frac{D}{Dt} \boldsymbol{\eta} \cdot \nabla r - \alpha \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n}) r + \int_{\Omega_b(t)} \kappa \nabla p \cdot \nabla r - \int_{\Gamma(t)} (\boldsymbol{q} \cdot \boldsymbol{n}) r.$$

There are no boundary terms except on $\Gamma(t)$ from the integration by parts in the integral involving α and in the integral involving κ because of the Dirichlet boundary condition $r = 0$ (since $p = 0$) on the left, top, and right boundaries of $\hat{\Omega}_b$.

After adding the two stress terms, and recalling the definition of $\hat{\boldsymbol{\Phi}}_\Gamma^\omega$ in (16) and $\hat{\mathcal{J}}_\Gamma^\omega$ in (18) we obtain:

$$- \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) \boldsymbol{n} \cdot \boldsymbol{v} + \int_{\hat{\Gamma}} \hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}) \boldsymbol{e}_y \cdot \hat{\boldsymbol{\psi}}$$
$$= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) \boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} (\hat{S}_b(\hat{\nabla}\hat{\boldsymbol{\eta}}, \hat{p}) \boldsymbol{e}_y - \hat{\mathcal{J}}_\Gamma^\omega \cdot (\boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi) \boldsymbol{n}|_{\Gamma(t)} \circ \boldsymbol{\Phi}_\Gamma^\omega) \cdot \hat{\boldsymbol{\psi}}.$$

Since the displacement of the plate is only in the y direction so that $\hat{\boldsymbol{\eta}} = \hat{\omega} \boldsymbol{e}_y$ on $\hat{\Gamma}$, the test function $\hat{\boldsymbol{\psi}}$ points in the y direction on $\hat{\Gamma}$ as well. We will denote by $\hat{\varphi}$ the magnitude of $\hat{\boldsymbol{\psi}}|_{\hat{\Gamma}}$ so that $\hat{\boldsymbol{\psi}} = \hat{\varphi} \boldsymbol{e}_y$

on $\hat{\Gamma}$. By the dynamic coupling condition (14), we have that the previous expression is equal to

$$
\begin{aligned}
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} \hat{F}_p \cdot \hat{\varphi} = \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot (\boldsymbol{\psi} - \boldsymbol{v}) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{n}(\psi_n - v_n) + \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \boldsymbol{\sigma}_f(\nabla \boldsymbol{u}, \pi)\boldsymbol{n} \cdot \boldsymbol{n}(\psi_n - v_n) + \int_{\Gamma(t)} \beta(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) + \int_{\hat{\Gamma}} (\rho_p \partial_{tt}\hat{\omega} + \hat{\Delta}^2 \hat{\omega})\hat{\varphi} \\
&= \int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \int_{\Gamma(t)} \beta(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\psi_\tau - v_\tau) \\
&+ \frac{d}{dt}\left(\int_{\hat{\Gamma}} \rho_p \partial_t \hat{\omega} \cdot \hat{\varphi}\right) - \int_{\hat{\Gamma}} \rho_p \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega} \cdot \hat{\Delta}\hat{\varphi},
\end{aligned}
$$

where we used the coupling conditions (12) and (15) in the last step.

The weak formulation then follows by summing everything together.

Definition 4.1. The ordered four-tuple $(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, p)$ satisfies the weak formulation to the nonlinearly coupled FPSI problem if for every test function $(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, r)$ that is C_c^1 in time on $[0, T]$ taking values in the test space, satisfying $\hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y$ on $\hat{\Gamma}$, we have that

$$
\begin{aligned}
&- \int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^T \int_{\Omega_f(t)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_0^T \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\zeta \boldsymbol{e}_y \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\
&+ 2\nu \int_0^T \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^T \int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \beta \int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y - \boldsymbol{u}) \cdot \boldsymbol{\tau}(\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{\tau} \\
&- \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega} \cdot \hat{\Delta}\hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&+ \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) \\
&- \alpha \int_0^T \int_{\Omega_b(t)} p \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p} \cdot \partial_t \hat{r} - \alpha \int_0^T \int_{\Omega_b(t)} \frac{D}{Dt}\boldsymbol{\eta} \cdot \nabla r - \alpha \int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}) r \\
&+ \kappa \int_0^T \int_{\Omega_b(t)} \nabla p \cdot \nabla r - \int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta \boldsymbol{e}_y) \cdot \boldsymbol{n}) r \\
&= \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot \boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t \hat{\omega}(0) \cdot \hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}}(0) \cdot \hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0) \cdot \hat{r}(0). \qquad (22)
\end{aligned}
$$

Remark 4.1. It is immediate to see that a classical (temporally and spatially smooth) solution to the FPSI problem satisfies the weak formulation stated above. However, when considering less regular solutions (in particular, weak solutions in the class of finite-energy solutions), the above weak formulation is *is inadequate for the regularity of finite-energy solutions* for the following reason. By the energy estimates (see Section 5.2), the regularity of the structure displacement $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$ is $L^\infty(0, T, H^1(\hat{\Omega}_b))$, which is not enough regularity to interpret the term

$$\alpha \int_{\Omega_b(t)} p \nabla \cdot \boldsymbol{\psi},$$

since the test function has regularity $\hat{\boldsymbol{\psi}} \in H^1(\hat{\Omega}_b)$ on the *fixed reference domain*, due to the corresponding finite energy regularity of $\hat{\boldsymbol{\eta}}$. Hence, after changing variables, which adds an extra factor of $\det(\boldsymbol{I}+\hat{\nabla}\hat{\boldsymbol{\eta}})$ arising from the Jacobian, which is only in $L^\infty(0,T;L^1(\hat{\Omega}_b))$ in two dimensions, there is not enough regularity to guarantee that this integral is finite. Therefore, we cannot interpret the above notion of weak solution properly in the space of finite energy solutions, as the finite energy space does not have enough regularity to make sense of certain integrals in the weak formulation, involving the deformed domain $\Omega_b(t)$.

This is why we introduce a regularized problem, which is consistent with the original problem in the sense that weak solutions to the regularized problem converge, as the regularization parameter tends to zero, to a smooth solution of the original, nonregularized problem, when a smooth solution exists. This weak-classical consistency result will be shown in Sec. 10.

5 Regularized weak solution and statement of existence result

Since all the mathematical challenges related to the inability to properly interpret all of the terms in the weak solution arise fundamentally from the *lack of regularity* of $\hat{\boldsymbol{\eta}}$ on $\hat{\Omega}_b$, we will regularize $\hat{\boldsymbol{\eta}}$ via a convolution with a smooth, compactly supported kernel, and introduce an appropriate *regularized weak formulation* of the original FPSI problem. Because we are working on a bounded domain $\hat{\Omega}_b$, we must be careful to introduce the convolution in a way that preserves the Dirichlet condition on the left, top, and right boundaries of $\hat{\Omega}_b = (0,L)\times(0,R)$.

This is why we define an *extended domain* $\tilde{\Omega}_b$:

$$\tilde{\Omega}_b = [-L,2L]\times[-R,2R],$$

so that for $\delta < \min(L,R)$ the convolution of a function on $\tilde{\Omega}_b$ with a smooth function of compact support in the closed ball of radius δ gives a function defined on $\hat{\Omega}_b$. We then introduce an odd extension along the lines $\hat{x}=0$, $\hat{x}=L$, $\hat{y}=0$ and $\hat{y}=R$ as follows.

Definition 5.1. Given $\hat{\boldsymbol{\eta}}$ defined on $\hat{\Omega}_b$ satisfying $\hat{\boldsymbol{\eta}}=0$ on $\hat{x}=0$, $\hat{x}=L$, and $\hat{y}=R$ and $\hat{\boldsymbol{\eta}}=\hat{\omega}\boldsymbol{e}_y$ on $\hat{y}=0$, define the **odd extension of $\hat{\boldsymbol{\eta}}$ to $\tilde{\Omega}_b$** by keeping $\hat{\boldsymbol{\eta}}$ the same on $\hat{\Omega}_b=[0,L]\times[0,R]$ and defining $\hat{\boldsymbol{\eta}}$ outside of the closure of $\hat{\Omega}_b$ as follows:

1. On $[0,L]\times[-R,0]$, set $\hat{\boldsymbol{\eta}}(\hat{x},\hat{y}) = \hat{\omega}(\hat{x})\boldsymbol{e}_y + (\hat{\omega}(\hat{x})\boldsymbol{e}_y - \hat{\boldsymbol{\eta}}(\hat{x},-\hat{y}))$.
2. On $[0,L]\times[R,2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x},\hat{y}) = -\hat{\boldsymbol{\eta}}(\hat{x},2R-\hat{y})$.
3. On $[-L,0]\times[-R,2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x},\hat{y}) = -\hat{\boldsymbol{\eta}}(-\hat{x},\hat{y})$.
4. On $[L,2L]\times[-R,2R]$, set $\hat{\boldsymbol{\eta}}(\hat{x},\hat{y}) = -\hat{\boldsymbol{\eta}}(2L-\hat{x},\hat{y})$.

Let σ be a radially symmetric function on $\mathbb{R}^2$ with compact support in the closed ball of radius one such that $\displaystyle\int_{\mathbb{R}^2}\sigma = 1$, and define

$$\sigma_\delta = \delta^{-2}\sigma(\delta^{-1}\boldsymbol{x}), \qquad \text{on } \mathbb{R}^2.$$

Definition 5.2. We define the following **regularized functions** which are spatially smooth on $\hat{\Omega}_b$:

- The regularized Biot displacement obtained by extending $\hat{\boldsymbol{\eta}}$ to $\tilde{\Omega}_b$ by odd extension and defining:

$$\hat{\boldsymbol{\eta}}^\delta = \hat{\boldsymbol{\eta}} * \sigma_\delta, \qquad \text{on } \hat{\Omega}_b, \tag{23}$$

- The regularized Lagrangian mapping:

$$\hat{\mathbf{\Phi}}_b^{\eta^\delta}(t,\cdot) = \text{Id} + \hat{\boldsymbol{\eta}}^\delta(t,\cdot), \tag{24}$$

- The regularized moving Biot domain:

$$\Omega_b^\delta(t) = \hat{\mathbf{\Phi}}_b^{\eta^\delta}(t,\hat{\Omega}_b). \tag{25}$$

  Note that even though the kinematic coupling condition holds for $\hat{\boldsymbol{\eta}}$ in the sense that $\hat{\boldsymbol{\eta}}|_{\hat{\Gamma}} = \hat{\omega}\boldsymbol{e}_y$, it is *not necessarily true* that $\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}} = \hat{\omega}\boldsymbol{e}_y$. Therefore, we will also define:

- The regularized moving interface:

$$\Gamma^\delta(t) = \hat{\mathbf{\Phi}}_b^{\eta^\delta}(t,\hat{\Gamma}).$$

  Alternatively, $\hat{\Gamma}^\delta$ is the plate interface if it were displaced from the reference configuration $\hat{\Gamma}$ in the direction $\hat{\boldsymbol{\eta}}^\delta|_{\hat{\Gamma}}$, which is a purely transverse y displacement, as one can verify.

Note that by the way we extended $\hat{\boldsymbol{\eta}}$ to the larger domain $\tilde{\Omega}_b$ we have that

$$\hat{\boldsymbol{\eta}}^\delta = 0 \quad \text{on} \quad \partial\hat{\Omega}_b \backslash \hat{\Gamma}.$$

With these regularized versions of the Biot structure displacement and velocity, we can now define the notion of a *weak solution to the regularized weak FPSI problem with the regularization parameter δ*. We start by defining the solution and test space, which are motivated by the energy estimates in Section 5.2, and then we state the regularized weak formulation in the moving domain framework and in the fixed reference domain framework.

5.1 Functional spaces and definition of weak solutions

Definition 5.3. (Solution and test spaces for the regularized problem)

- *Fluid function space (moving domain/Eulerian formulation).*

$$V_f(t) = \{\boldsymbol{u} = (u_x, u_y) \in H^1(\Omega_f(t)) : \nabla\cdot\boldsymbol{u} = 0, \text{ and } \boldsymbol{u} = 0 \text{ when } x = 0, x = L, y = -R\}, \tag{26}$$

$$\mathcal{V}_f = L^\infty(0,T; L^2(\Omega_f(t))) \cap L^2(0,T; V_f(t)). \tag{27}$$

- *Fluid function space (fixed domain/Lagrangian formulation).*

$$V_f^\omega = \{\hat{\boldsymbol{u}} = (\hat{u}_x, \hat{u}_y) \in H^1(\hat{\Omega}_f) : \hat{\nabla}_f^\omega \cdot \hat{\boldsymbol{u}} = 0, \text{ and } \hat{\boldsymbol{u}} = 0 \text{ when } \hat{x} = 0, \hat{x} = L, \hat{y} = -R\}, \tag{28}$$

$$\mathcal{V}_f^\omega = L^\infty(0,T; L^2(\hat{\Omega}_f)) \cap L^2(0,T; V_f^\omega). \tag{29}$$

- *Plate function space.*

$$\mathcal{V}_\omega = W^{1,\infty}(0,T; L^2(\hat{\Gamma})) \cap L^\infty(0,T; H_0^2(\hat{\Gamma})). \tag{30}$$

- *Biot displacement function space.*

$$V_d = \{\hat{\boldsymbol{\eta}} = (\hat{\eta}_x, \hat{\eta}_y) \in H^1(\hat{\Omega}_b) : \hat{\boldsymbol{\eta}} = 0 \text{ for } \hat{x} = 0, \hat{x} = L, \hat{y} = R, \text{ and } \hat{\eta}_x = 0 \text{ on } \hat{\Gamma}\}, \tag{31}$$

$$\mathcal{V}_b = W^{1,\infty}(0,T; L^2(\hat{\Omega}_b)) \cap L^\infty(0,T; V_d) \cap H^1(0,T; V_d). \tag{32}$$

- *Biot pore pressure function space.*

$$V_p = \{\hat{p} \in H^1(\hat{\Omega}_b) : \hat{p} = 0 \text{ for } \hat{x} = 0, \hat{x} = L, \hat{y} = R\}, \tag{33}$$

$$\mathcal{Q}_b = L^\infty(0,T;L^2(\hat{\Omega}_b)) \cap L^2(0,T;V_p). \tag{34}$$

- *Weak solution space (moving domain).*

$$\mathcal{V}_{\text{sol}} = \{(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}_f \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{35}$$

- *Weak solution space (fixed domain).*

$$\mathcal{V}^\omega_{\text{sol}} = \{(\hat{\boldsymbol{u}}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}^\omega_f \times \mathcal{V}_\omega \times \mathcal{V}_b \times \mathcal{Q}_b : \hat{\boldsymbol{\eta}} = \hat{\omega}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{36}$$

- *Test space (moving domain).*

$$\mathcal{V}_{\text{test}} = \{(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in C^1_c([0,T); V_f(t) \times H^2_0(\hat{\Gamma}) \times V_d \times V_p) : \hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{37}$$

- *Test space (fixed domain).*

$$\mathcal{V}^\omega_{\text{test}} = \{(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in C^1_c([0,T); V^\omega_f \times H^2_0(\hat{\Gamma}) \times V_d \times V_p) : \hat{\boldsymbol{\psi}} = \hat{\varphi}\boldsymbol{e}_y \text{ on } \hat{\Gamma}\}. \tag{38}$$

Remark 5.1. Because $\hat{\Gamma}$ is one dimensional, for plate displacements $\hat{\omega} \in \mathcal{V}_\omega$, we have that $\hat{\omega} \in C(0,T;C^1(\hat{\Gamma}))$ and hence, there is a one-to-one correspondence between functions in $\mathcal{V}_{\text{sol}}$ and $\mathcal{V}^\omega_{\text{sol}}$ and functions in $\mathcal{V}_{\text{test}}$ and $\mathcal{V}^\omega_{\text{test}}$, given by composition with the ALE mapping (16).

Before we state the definition of a weak solution to the regularized problem, we introduce the following notation. Define the **transverse velocity** of the plate by the variable $\hat{\zeta}$, so that

$$\partial_t \hat{\omega} = \hat{\zeta}, \tag{39}$$

and let $\zeta = \hat{\zeta} \circ (\boldsymbol{\Phi}^\omega_\Gamma)^{-1}$.

Definition 5.4. (Weak solution to the regularized problem, **moving fluid domain formulation**) An ordered four-tuple $(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, p) \in \mathcal{V}_{\text{sol}}$ is a *weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter* δ if for every test function $(\boldsymbol{v}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in \mathcal{V}_{\text{test}}$,

$$\begin{aligned}
&-\int_0^T \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^T \int_{\Omega_f(t)} [((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v} - ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{u}] + \frac{1}{2}\int_0^T \int_{\Gamma(t)} (\boldsymbol{u}\cdot\boldsymbol{n} - 2\zeta\boldsymbol{e}_y\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{v} \\
&+ 2\nu\int_0^T \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^T \int_{\Gamma(t)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \beta\int_0^T \int_{\Gamma(t)} (\zeta\boldsymbol{e}_y - \boldsymbol{u})\cdot\boldsymbol{\tau}(\boldsymbol{\psi} - \boldsymbol{v})\cdot\boldsymbol{\tau} \\
&- \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t\hat{\omega}\cdot\partial_t\hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta}\hat{\omega}\cdot\hat{\Delta}\hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t\hat{\boldsymbol{\eta}}\cdot\partial_t\hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&+ \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla}\cdot\hat{\boldsymbol{\eta}})(\hat{\nabla}\cdot\hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla}\cdot\partial_t\hat{\boldsymbol{\eta}})(\hat{\nabla}\cdot\hat{\boldsymbol{\psi}}) \\
&- \alpha \int_0^T \int_{\Omega^\delta_b(t)} p\nabla\cdot\boldsymbol{\psi} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p}\cdot\partial_t\hat{r} - \alpha \int_0^T \int_{\Omega^\delta_b(t)} \frac{D^\delta}{Dt}\boldsymbol{\eta}\cdot\nabla r - \alpha \int_0^T \int_{\Gamma^\delta(t)} (\zeta\boldsymbol{e}_y\cdot\boldsymbol{n}^\delta) r \\
&+ \kappa \int_0^T \int_{\Omega^\delta_b(t)} \nabla p\cdot\nabla r - \int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta\boldsymbol{e}_y)\cdot\boldsymbol{n}) r \\
&= \int_{\Omega_f(0)} \boldsymbol{u}(0)\cdot\boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t\hat{\omega}(0)\cdot\hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t\hat{\boldsymbol{\eta}}(0)\cdot\hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0)\cdot\hat{r}(0),
\end{aligned} \tag{40}$$

where $\frac{D^\delta}{Dt} = \frac{d}{dt} + (\boldsymbol{\xi}^\delta \cdot \nabla)$ with $\boldsymbol{\xi}^\delta(t,\cdot) = \partial_t \hat{\boldsymbol{\eta}}^\delta(t, (\boldsymbol{\Phi}_b^{\eta^\delta})^{-1}(t,\cdot))$ is the material derivative with respect to the regularized displacement, $\boldsymbol{n}$ denotes the upward pointing normal vector to $\Gamma(t)$, and $\boldsymbol{n}^\delta$ denotes the upward pointing normal vector to $\Gamma^\delta(t)$.

Notice that only four terms contain regularization via convolution with parameter δ. While there are many different ways to write the regularized weak formulation, the regularization presented above is a regularization that deviates from the original, nonregularized problem, in the smallest possible number of terms, and is still consistent with the original, nonregularized problem, as we show later.

Remark 5.2. While the solution to the regularized problem above depends on the regularization parameter δ implicitly, to simplify notation we will drop the δ notation whenever it is clear from the context that we are working with the solution to the regularized problem.

Remark 5.3. We simplify notation by omitting the explicit compositions with the maps $\hat{\boldsymbol{\Phi}}_f^\omega$, $\hat{\boldsymbol{\Phi}}_\Gamma^\omega$, $\hat{\boldsymbol{\Phi}}_b^\eta$, and $\hat{\boldsymbol{\Phi}}_b^{\eta^\delta}$, and their inverses. The necessary compositions with such mappings will be clear from the context. For example,

$$-\alpha \int_0^T \int_{\Omega_b^\delta(t)} p \nabla \cdot \boldsymbol{\psi} \quad \text{means} \quad -\alpha \int_0^T \int_{\Omega_b^\delta(t)} \left(\hat{p} \circ (\boldsymbol{\Phi}_b^{\eta^\delta})^{-1}\right) \nabla \cdot \left(\hat{\boldsymbol{\psi}} \circ (\boldsymbol{\Phi}_b^{\eta^\delta})^{-1}\right),$$

and

$$-\int_0^T \int_{\Gamma(t)} ((\boldsymbol{u} - \zeta \boldsymbol{e}_y) \cdot \boldsymbol{n}) r \quad \text{means} \quad -\int_0^T \int_{\Gamma(t)} \left((\boldsymbol{u} - (\zeta \circ (\boldsymbol{\Phi}_\Gamma^\omega)^{-1})\boldsymbol{e}_y) \cdot \boldsymbol{n}\right) \left(\hat{r} \circ (\boldsymbol{\Phi}_b^\eta)^{-1}\right).$$

Next, we reformulate the definition of a regularized weak solution on the **fixed reference domain**. Recall that the Jacobians $\hat{\mathcal{J}}_f^\omega$, $\hat{\mathcal{J}}_b^\eta$, and $\hat{\mathcal{J}}_\Gamma^\omega$ in (18) will appear upon using a change of variables to map the problem onto the reference domain. To transform the first term in the weak formulation (40) above, we use (20) to transform the time derivatives and assume that $|\hat{\omega}| < R$ so that there is no domain degeneracy. After using (20) and (19) we get

$$\begin{aligned}
\int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} &= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot [(\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega)\hat{\boldsymbol{v}}] \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{R} \int_{\hat{\Omega}_f} \hat{\boldsymbol{u}} \cdot [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{v}}] \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{2R} \int_{\hat{\Omega}_f} \hat{\boldsymbol{u}} \cdot [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{v}}] + \frac{1}{2R} \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} \\
&\quad + \frac{1}{2R} \int_{\hat{\Omega}_f} [(R + \hat{y}) \partial_t \hat{\omega} \partial_{\hat{y}} \hat{\boldsymbol{u}}] \cdot \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega} \\
&= \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) [((\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega)\hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{u}} - ((\hat{\boldsymbol{w}} \cdot \hat{\nabla}_f^\omega)\hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{v}}] \\
&\quad + \frac{1}{2R} \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} - \frac{1}{2} \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega},
\end{aligned} \tag{41}$$

where we integrated by parts in the $\hat{y}$ direction. Note that the final term in (41) will combine with the following term in (40):

$$\int_0^T \int_{\Gamma(t)} (\zeta \boldsymbol{e}_y \cdot \boldsymbol{n}) \boldsymbol{u} \cdot \boldsymbol{v} = \int_0^T \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}}) \partial_t \hat{\omega}, \tag{42}$$

where we used $\boldsymbol{n} = (-\partial_{\hat{x}}\hat{\omega}, 1)/\hat{\mathcal{J}}_{\Gamma}^{\omega}$ for the normal vector to the interface and $\zeta \boldsymbol{e}_y|_{\Gamma(t)} = \partial_t \hat{\omega} \boldsymbol{e}_y$. Because the transformation from $\Gamma(t)$ to $\hat{\Gamma}$ cancels out the factor of $\hat{\mathcal{J}}_{\Gamma}^{\omega}$ in the unit normal vector, it is useful to define the following renormalized normal and tangent vectors:

$$\hat{\boldsymbol{n}}^{\omega} = (-\partial_{\hat{x}}\hat{\omega}, 1), \qquad \hat{\boldsymbol{\tau}}^{\omega} = (1, \partial_{\hat{x}}\hat{\omega}). \tag{43}$$

We similarly define

$$\hat{\boldsymbol{n}}^{\omega^{\delta}} = (-\partial_{\hat{x}}(\hat{\boldsymbol{\eta}}^{\delta}|_{\hat{\Gamma}}), 1). \tag{44}$$

We are now ready to state the definition of a weak solution to the regularized problem on the fixed reference domain.

Definition 5.5. (Weak solution to the regularized problem, **fixed fluid domain formulation**) An ordered four-tuple $(\hat{\boldsymbol{u}}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p}) \in \mathcal{V}_{\text{sol}}^{\omega}$ is a *weak solution to the regularized nonlinearly coupled FPSI problem with regularization parameter* δ if for all test functions $(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) \in \mathcal{V}_{\text{test}}^{\omega}$, the following equality holds:

$$\begin{aligned}
&- \int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{u}} \cdot \partial_t \hat{\boldsymbol{v}} + \frac{1}{2} \int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) [((\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) \cdot \hat{\nabla}_f^{\omega} \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{v}} - ((\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) \cdot \hat{\nabla}_f^{\omega} \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{u}}] \\
&- \frac{1}{2R} \int_0^T \int_{\hat{\Omega}_f} (\partial_t \hat{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} + \frac{1}{2} \int_0^T \int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}^{\omega} - \hat{\zeta} \boldsymbol{e}_y \cdot \hat{\boldsymbol{n}}^{\omega}) \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} + 2\nu \int_0^T \int_{\hat{\Omega}_f} \left(1 + \frac{\hat{\omega}}{R}\right) \hat{\boldsymbol{D}}(\hat{\boldsymbol{u}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{v}}) \\
&\qquad + \int_0^T \int_{\hat{\Gamma}} \left(\frac{1}{2} |\hat{\boldsymbol{u}}|^2 - \hat{p}\right) (\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}}^{\omega} + \frac{\beta}{\hat{\mathcal{J}}_{\Gamma}^{\omega}} \int_0^T \int_{\hat{\Gamma}} (\hat{\zeta} \boldsymbol{e}_y - \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{\tau}}^{\omega} (\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}) \cdot \hat{\boldsymbol{\tau}}^{\omega} \\
&\quad - \rho_p \int_0^T \int_{\hat{\Gamma}} \partial_t \hat{\omega} \cdot \partial_t \hat{\varphi} + \int_0^T \int_{\hat{\Gamma}} \hat{\Delta} \hat{\omega} \cdot \hat{\Delta} \hat{\varphi} - \rho_b \int_0^T \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}} \cdot \partial_t \hat{\boldsymbol{\psi}} + 2\mu_e \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) \\
&\quad + \lambda_e \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} \hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}}) : \hat{\boldsymbol{D}}(\hat{\boldsymbol{\psi}}) + \lambda_v \int_0^T \int_{\hat{\Omega}_b} (\hat{\nabla} \cdot \partial_t \hat{\boldsymbol{\eta}})(\hat{\nabla} \cdot \hat{\boldsymbol{\psi}}) \\
&\qquad - \alpha \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^{\delta}} \hat{p} \hat{\nabla}_b^{\eta^{\delta}} \cdot \hat{\boldsymbol{\psi}} - c_0 \int_0^T \int_{\hat{\Omega}_b} \hat{p} \cdot \partial_t \hat{r} - \alpha \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^{\delta}} \partial_t \hat{\boldsymbol{\eta}} \cdot \hat{\nabla}_b^{\eta^{\delta}} \hat{r} \\
&\qquad - \alpha \int_0^T \int_{\hat{\Gamma}} (\hat{\zeta} \boldsymbol{e}_y \cdot \hat{\boldsymbol{n}}^{\omega^{\delta}}) \hat{r} + \kappa \int_0^T \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^{\delta}} \hat{\nabla}_b^{\eta^{\delta}} \hat{p} \cdot \hat{\nabla}_b^{\eta^{\delta}} \hat{r} - \int_0^T \int_{\hat{\Gamma}} ((\hat{\boldsymbol{u}} - \hat{\zeta} \boldsymbol{e}_y) \cdot \hat{\boldsymbol{n}}^{\omega}) \hat{r} \\
&\qquad = \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot \boldsymbol{v}(0) + \rho_p \int_{\hat{\Gamma}} \partial_t \hat{\omega}(0) \cdot \hat{\varphi}(0) + \rho_b \int_{\hat{\Omega}_b} \partial_t \hat{\boldsymbol{\eta}}(0) \cdot \hat{\boldsymbol{\psi}}(0) + c_0 \int_{\hat{\Omega}_b} \hat{p}(0) \cdot \hat{r}(0),
\end{aligned} \tag{45}$$

where $\hat{\mathcal{J}}_b^{\eta^{\delta}}$ and $\hat{\mathcal{J}}_{\Gamma}^{\omega}$ are defined in (18), $\hat{\boldsymbol{w}}$ is defined in (20), $\hat{\nabla}_f^{\omega}$ in (19), $\hat{\nabla}_b^{\eta^{\delta}} \hat{g}$ in (21), and $\hat{\zeta}$ in (50).

5.2 Formal energy inequality

Here we show that the regularized problem is defined in a way that preserves the variational structure of the problem. More precisely, we formally prove that a weak solution to the regularized problem satisfies the following energy equality.

Lemma 5.1. Assuming that a weak solution exists, the following energy equality holds:

$$E^K(T) + E^E(T) + \int_0^T \left(D_f^V(t) + D_b^V(t) + D_{f_b}^V(t) + D_{\beta}^V(t)\right) dt = E^K(0) + E^E(0) \tag{46}$$

where

$$E^K(t) = \frac{1}{2}\int_{\Omega_f(t)} |\boldsymbol{u}(t)|^2 + \frac{1}{2}\rho_b \int_{\hat{\Omega}_b} |\partial_t \hat{\boldsymbol{\eta}}(t)|^2 + \frac{1}{2}c_0 \int_{\hat{\Omega}_b} |\hat{p}(t)|^2 + \frac{1}{2}\rho_p \int_{\hat{\Gamma}} |\partial_t \hat{\omega}(t)|^2$$

is the sum of the kinetic energy of the fluid, the kinetic energy of the Biot poroviscoelastic matrix motion, the kinetic energy of the filtrating fluid flow in the Biot medium, and the kinetic energy of the plate motion, $E^E(t)$ is defined by

$$E^E(t) = 2\mu_e \int_{\hat{\Omega}_b} |\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}})(t)|^2 + 2\lambda_e \int_{\hat{\Omega}_b} |\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}(t)|^2 + \int_{\hat{\Gamma}} |\hat{\Delta}\hat{\omega}(t)|^2,$$

which corresponds to the elastic energy of the Biot poroviscoelastic matrix and the elastic energy of the plate, and

$$D_f^V(t) = 2\nu \int_{\Omega_f(t)} |\boldsymbol{D}(\boldsymbol{u})|^2, \quad D_b^V(t) = 2\mu_v \int_{\hat{\Omega}_b} |\hat{\boldsymbol{D}}(\partial_t \hat{\boldsymbol{\eta}})|^2 + \lambda_v \int_{\hat{\Omega}_b} |\hat{\nabla}\cdot\partial_t \hat{\boldsymbol{\eta}}|^2,$$

$$D_{f_b}^V(t) = \kappa \int_{\Omega_b^\delta(t)} |\nabla p|^2, \quad D_\beta^V(t) = \beta \int_{\Gamma(t)} |(\boldsymbol{\xi}-\boldsymbol{u})\cdot\boldsymbol{\tau}|^2$$

correspond to dissipation due to fluid viscosity, viscosity of the Biot poroviscoelastic matrix, dissipation due to permeability effects, and dissipation due to friction in the Beavers-Joseph-Saffman slip condition.

Proof. To derive this energy equality we start by substituting $(\hat{\boldsymbol{v}}, \hat{\varphi}, \hat{\boldsymbol{\psi}}, \hat{r}) = (\hat{\boldsymbol{u}}, \hat{\zeta}, \partial_t \hat{\boldsymbol{\eta}}, \hat{p})$ into the regularized weak formulation (45) defined on the fixed reference domain and calculate

$$\frac{1}{2}\int_{\hat{\Gamma}} (\hat{\boldsymbol{u}} - \hat{\zeta}\boldsymbol{e}_y)\cdot\hat{\boldsymbol{n}}^\omega |\hat{\boldsymbol{u}}|^2 + \int_{\hat{\Gamma}} \left(\frac{1}{2}|\hat{\boldsymbol{u}}|^2 - \hat{p}\right)(\hat{\zeta}\boldsymbol{e}_y - \hat{\boldsymbol{u}})\cdot\hat{\boldsymbol{n}}^\omega - \int_{\hat{\Gamma}} ((\hat{\boldsymbol{u}} - \hat{\zeta}\boldsymbol{e}_y)\cdot\hat{\boldsymbol{n}}^\omega)\hat{p} = 0.$$

Furthermore, using integration by parts one obtains

$$\begin{aligned}
&\alpha\left(\int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^\delta} \hat{p} \hat{\nabla}_b^{\eta^\delta}\cdot\partial_t\hat{\boldsymbol{\eta}} + \int_{\hat{\Omega}_b} \hat{\mathcal{J}}_b^{\eta^\delta} \partial_t \hat{\boldsymbol{\eta}}\cdot\hat{\nabla}_b^{\eta^\delta}\hat{p} + \int_{\hat{\Gamma}} (\hat{\zeta}\boldsymbol{e}_y\cdot\hat{\boldsymbol{n}}^{\omega^\delta})\hat{p}\right) \\
&= \alpha\left(\int_{\Omega_b^\delta(t)} p\nabla\cdot\boldsymbol{\xi} + \int_{\Omega_b^\delta(t)} \boldsymbol{\xi}\cdot\nabla p + \int_{\Gamma^\delta(t)} (\zeta\boldsymbol{e}_y\cdot\boldsymbol{n}^\delta)p\right) = 0,
\end{aligned}$$

where $\boldsymbol{n}^\delta$ is the upward pointing unit normal vector to $\Gamma^\delta(t)$. Finally, by the Reynold's transport theorem

$$\int_0^T \int_{\Omega_f(t)} \boldsymbol{u}\cdot\partial_t\boldsymbol{u} + \frac{1}{2}\int_0^T \int_{\Gamma(t)} (\zeta\boldsymbol{e}_y\cdot\boldsymbol{n})|\boldsymbol{u}|^2 = \frac{1}{2}\int_{\Omega_f(T)} |\boldsymbol{u}|^2 - \frac{1}{2}\int_{\Omega_f(0)} |\boldsymbol{u}|^2.$$

By combining these calculations one obtains the final energy estimate:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_f(T)} |\boldsymbol{u}(T)|^2 + 2\nu\int_0^T \int_{\Omega_f(t)} |\boldsymbol{D}(\boldsymbol{u})|^2 + \beta\int_0^T \int_{\Gamma(t)} |(\boldsymbol{\xi}-\boldsymbol{u})\cdot\boldsymbol{\tau}|^2 + \frac{1}{2}\rho_p \int_{\hat{\Gamma}} |\partial_t\hat{\omega}(T)|^2 + \int_{\hat{\Gamma}} |\hat{\Delta}\hat{\omega}(T)|^2 \\
&\quad + \frac{1}{2}\rho_b \int_{\hat{\Omega}_b} |\partial_t\hat{\boldsymbol{\eta}}(T)|^2 + 2\mu_e \int_{\hat{\Omega}_b} |\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}})(T)|^2 + 2\lambda_e \int_{\hat{\Omega}_b} |\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}(T)|^2 + 2\mu_v \int_0^T \int_{\hat{\Omega}_b} |\hat{\boldsymbol{D}}(\partial_t\hat{\boldsymbol{\eta}})|^2 \\
&+ \lambda_v \int_0^T \int_{\hat{\Omega}_b} |\hat{\nabla}\cdot\partial_t\hat{\boldsymbol{\eta}}|^2 + \frac{1}{2}c_0 \int_{\hat{\Omega}_b} |\hat{p}(T)|^2 + \kappa\int_0^T \int_{\Omega_b^\delta(t)} |\nabla p|^2 = \frac{1}{2}\int_{\Omega_f(0)} |\boldsymbol{u}(0)|^2 + \frac{1}{2}\rho_p \int_{\hat{\Gamma}} |\partial_t\hat{\omega}(0)|^2 \\
&\quad + \int_{\hat{\Gamma}} |\hat{\Delta}\hat{\omega}(0)|^2 + \frac{1}{2}\rho_b \int_{\hat{\Omega}_b} |\partial_t\hat{\boldsymbol{\eta}}(0)|^2 + 2\mu_e \int_{\hat{\Omega}_b} |\hat{\boldsymbol{D}}(\hat{\boldsymbol{\eta}})(0)|^2 + 2\lambda_e \int_{\hat{\Omega}_b} |\hat{\nabla}\cdot\hat{\boldsymbol{\eta}}(0)|^2 + \frac{1}{2}c_0 \int_{\hat{\Omega}_b} |\hat{p}(0)|^2.
\end{aligned}$$

□

5.3 Statement of the main existence result for the regularized problem

We now state the main result on the existence of a weak solution to the regularized problem.

Theorem 5.1. Let $\rho_b, \mu_e, \lambda_e, \alpha, \rho_p, \nu > 0$ and $\mu_v, \lambda_v \geqslant 0$. Consider initial data for the plate displacement $\hat{\omega}_0 \in H_0^2(\hat{\Gamma})$, plate velocity $\hat{\zeta}_0 \in L^2(\hat{\Gamma})$, Biot displacement $\hat{\boldsymbol{\eta}}_0 \in H^1(\hat{\Omega}_b)$, Biot velocity $\hat{\boldsymbol{\xi}}_0 \in H^1(\hat{\Omega}_b)$ in the case of a viscoelastic Biot medium $\mu_v, \lambda_v > 0$ and $\hat{\boldsymbol{\xi}}_0 \in L^2(\hat{\Omega}_b)$ otherwise for the case of a purely elastic Biot medium, Biot pore pressure $\hat{p}_0 \in L^2(\hat{\Omega}_b)$, and fluid velocity $\boldsymbol{u}_0 \in H^1(\Omega_f(0))$ which is divergence-free. Suppose further that $|\hat{\omega}_0| \leqslant R_0 < R$ for some R_0, $\hat{\boldsymbol{\eta}}_0|_\Gamma = \hat{\omega}_0 \boldsymbol{e}_y$, and $\hat{\boldsymbol{\xi}}_0|_\Gamma = \hat{\zeta}_0 \boldsymbol{e}_y$, and for some arbitrary but fixed regularization parameter $\delta > 0$, suppose that $\mathrm{Id} + \hat{\boldsymbol{\eta}}_0^\delta$ is an invertible map with $\det(\boldsymbol{I} + \nabla \hat{\boldsymbol{\eta}}_0^\delta) > 0$. Then, there exists a weak solution $(\boldsymbol{u}, \hat{\omega}, \hat{\boldsymbol{\eta}}, \hat{p})$ to the regularized FPSI problem with regularization parameter δ on some time interval $[0, T]$, for some $T > 0$.

While T in general depends on δ, we will show that if there exists a smooth solution to the nonregularized FPSI problem, then this time T for the regularized problem is independent of δ. This will allow us to pass to the limit as $\delta \to 0$ and show that weak solutions to the regularized FPSI problems constructed in this manuscript, converge to a smooth solution of the original, nonregularized problem, when a smooth solution to the nonregularized problem exists.

Remark 5.4. The result above is a local result, since it holds up to some time $T > 0$, which needs to be sufficiently small. However, it is easy to show that this $T > 0$ can be made maximal, in the sense that it holds until the time for which $\mathrm{Id} + \hat{\boldsymbol{\eta}}^\delta$ fails to be invertible or $|\hat{\omega}(\cdot, x)| = R$ for some $x \in \hat{\Gamma}$ when the reticular plate collides with the boundary. This can be shown using a standard method, see e.g., pg. 397-398 of [20], or the proof of Theorem 7.1 in [47].

An important notational convention. For notational simplicity, we will no longer use the "hat" notation to distinguish between functions and domains in the physical or reference configuration: for example, we will denote both the pore pressure p on $\Omega_b(t)$ and $\hat{p}$ on $\hat{\Omega}_b$ by p, as the distinction between these two will be clear from context. In addition, we will remove the "hat" convention from the reference domains, and for example, we will denote the reference domain $\hat{\Omega}_b$ for the Biot medium by Ω_b. We will follow this notational convention for the rest of the manuscript.

The proof of Theorem 5.1 is constructive, and based on an operator splitting scheme. This is an approach that has been used in constructive existence proofs of weak solutions for a variety of FSI problems, see for example [47].

6 The splitting scheme

The splitting scheme is defined as follows. First, semidiscretize the problem in time by introducing the time step $\Delta t = T/N$, and subdivide the time interval $[0, T]$ into N subintervals, each of width Δt. The approximations of the fluid velocity, plate displacement and velocity, and Biot poroviscoelastic material displacement and pressure will be denoted by

$$(\boldsymbol{u}_N^{n+\frac{i}{2}}, \omega_N^{n+\frac{i}{2}}, \zeta_N^{n+\frac{i}{2}}, \boldsymbol{\eta}_N^{n+\frac{i}{2}}, p_N^{n+\frac{i}{2}}), \qquad \text{for } n = 0, 1, ..., N \text{ and } i = 0, 1.$$

For the splitting scheme we will work on the fixed reference domain and hence, we will semidiscretize the regularized weak formulation (45) on the fixed reference domain. Backwards Euler discretization will be used to approximate time derivatives, with the following shorthand notation:

$$\dot{f}_N^{n+\frac{i}{2}} = \frac{f_N^{n+\frac{i}{2}} - f_N^{n+\frac{i}{2}-1}}{\Delta t}.$$

6.1 The plate subproblem

Only the plate displacement and velocity $\omega_N^{n+\frac{1}{2}}$ and $\zeta_N^{n+\frac{1}{2}}$ are updated in this subproblem, leaving the remaining variables unchanged:

$$\boldsymbol{u}_N^{n+\frac{1}{2}} = \boldsymbol{u}_N^n, \qquad \boldsymbol{\eta}_N^{n+\frac{1}{2}} = \boldsymbol{\eta}_N^n, \qquad p_N^{n+\frac{1}{2}} = p_N^n.$$

The new plate displacement and velocity are calculated from the following weak formulation of the plate subproblems: find $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ and $\zeta_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$, such that

$$\int_\Gamma \left(\frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t}\right) \cdot \phi = \int_\Gamma \zeta_N^{n+\frac{1}{2}} \cdot \phi, \qquad \text{for all } \phi \in L^2(\Gamma), \tag{47}$$

$$\rho_p \int_\Gamma \left(\frac{\zeta_N^{n+\frac{1}{2}} - \zeta_N^n}{\Delta t}\right) \cdot \varphi + \int_\Gamma \Delta\omega_N^{n+\frac{1}{2}} \cdot \Delta\varphi = 0, \qquad \text{for all } \varphi \in H_0^2(\Gamma). \tag{48}$$

When $n = 0$, we set $\omega_N^{-\frac{1}{2}} = \omega(0)$ and $\zeta_N^0 = \zeta(0)$. In particular, $\omega(0)\boldsymbol{e}_y = \boldsymbol{\eta}(0)|_\Gamma$ and $\zeta(0)\boldsymbol{e}_y = \boldsymbol{\xi}(0)$.

Lemma 6.1. Problem (47), (48) has a unique solution which satisfies the following energy equality:

$$\begin{aligned} &\frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+\frac{1}{2}}|^2 + \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n+\frac{1}{2}} - \zeta_N^n|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_N^{n+\frac{1}{2}}|^2 + \frac{1}{2}\int_\Gamma |\Delta(\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}})|^2 \\ &= \frac{1}{2}\rho_p \int_\Gamma |\zeta_N^{n-\frac{1}{2}}|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_N^{n-\frac{1}{2}}|^2. \end{aligned} \tag{49}$$

Proof. To prove this, we first notice that

$$\zeta_N^{n+\frac{1}{2}} = \frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t} \tag{50}$$

so that $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ above satisfies the following weak formulation:

$$\rho_p \int_\Gamma \omega_N^{n+\frac{1}{2}} \cdot \varphi + (\Delta t)^2 \int_\Gamma \Delta\omega_N^{n+\frac{1}{2}} \cdot \Delta\varphi = \rho_p \int_\Gamma (\omega_N^{n-\frac{1}{2}} + (\Delta t)\zeta_N^n) \cdot \varphi, \qquad \text{for all } \varphi \in H_0^2(\Gamma).$$

The bilinear form

$$B[\omega, \varphi] = \rho_p \int_\Gamma \omega \cdot \varphi + (\Delta t)^2 \int_\Gamma \Delta\omega \cdot \Delta\varphi$$

is coercive on $H_0^2(\Gamma)$, and

$$\varphi \to \rho_p \int_\Gamma \left(\omega_N^{n-\frac{1}{2}} + (\Delta t)\zeta_N^n\right) \cdot \varphi$$

is a continuous linear functional on $H_0^2(\Gamma)$, since we will have $\omega_N^{n-\frac{1}{2}} \in H_0^2(\Gamma)$ and $\zeta_N^n \in L^2(\Gamma)$ by the way our splitting scheme is defined. Thus, by the Lax-Milgram lemma, there exists a unique solution $\omega_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$, from which we also recover $\zeta_N^{n+\frac{1}{2}} \in H_0^2(\Gamma)$ using (50) above.

The energy equality above follows by substituting $\varphi = \zeta_N^{n+\frac{1}{2}} = \frac{\omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}}}{\Delta t} \in H_0^2(\Gamma)$ into the weak formulation and using the identity

$$(a - b) \cdot a = \frac{1}{2}(|a|^2 + |a - b|^2 - |b|^2).$$

$\square$

6.2 The fluid and Biot subproblem

For the fluid and Biot subproblem, we update the quantities related to the fluid and the Biot medium. Due to the kinematic coupling between the Biot medium displacement and the plate displacement, we must also update the plate velocity, as the dynamics of the Biot medium affect the kinematics of the plate. In this step, only the plate displacement remains unchanged:

$$\omega_N^{n+1} = \omega_N^{n+\frac{1}{2}}.$$

To state the weak formulation of the fluid and Biot subproblem, we define the solution and test spaces, respectively:

$$\mathcal{V}_N^{n+1} = \{(\boldsymbol{u}, \zeta, \boldsymbol{\eta}, p) \in \mathcal{V}_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p\}, \tag{51}$$

$$\mathcal{Q}_N^{n+1} = \{(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in V_f^{\omega_N^n} \times H_0^2(\Gamma) \times V_d \times V_p : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma\}, \tag{52}$$

where V_f^ω, V_d, and V_p are defined in (28), (31), and (33).

The weak formulation now reads: find $(\boldsymbol{u}_N^{n+1}, \zeta_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_N^{n+1}$ defined on the reference domain, such that for all test functions $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{Q}_N^{n+1}$ defined on the reference domain, the following holds:

$$\begin{aligned}
&\int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \dot{\boldsymbol{u}}_N^{n+1} \cdot \boldsymbol{v} + 2\nu \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1}) : \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + \int_\Gamma \left(\frac{1}{2}\boldsymbol{u}_N^{n+1} \cdot \boldsymbol{u}_N^n - p_N^{n+1}\right)(\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{n}^{\omega_N^n} \\
&+ \frac{1}{2}\int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left[\left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y\right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{u}_N^{n+1}\right) \cdot \boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R} \boldsymbol{e}_y\right) \cdot \nabla_f^{\omega_N^n} \boldsymbol{v}\right) \cdot \boldsymbol{u}_N^{n+1}\right] \\
&+ \frac{1}{2R}\int_{\Omega_f} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u}_N^{n+1} \cdot \boldsymbol{v} + \frac{1}{2}\int_\Gamma (\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n}^{\omega_N^n} (\boldsymbol{u}_N^n \cdot \boldsymbol{v}) \\
&+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} \int_\Gamma (\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1}) \cdot \boldsymbol{\tau}^{\omega_N^n} (\boldsymbol{\psi} - \boldsymbol{v}) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} \left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n}{\Delta t}\right) \cdot \boldsymbol{\psi} \\
&+ \rho_p \int_\Gamma \left(\frac{\zeta_N^{n+1} - \zeta_N^{n+\frac{1}{2}}}{\Delta t}\right) \varphi + 2\mu_e \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_e \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) \\
&+ 2\mu_v \int_{\Omega_b} \boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_{\Omega_b} (\nabla \cdot \dot{\boldsymbol{\eta}}_N^{n+1})(\nabla \cdot \boldsymbol{\psi}) - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p_N^{n+1} \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} \\
&+ c_0 \int_{\Omega_b} \frac{p_N^{n+1} - p_N^n}{\Delta t} r - \alpha \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \dot{\boldsymbol{\eta}}_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha \int_\Gamma (\dot{\boldsymbol{\eta}}_N^{n+1} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r \\
&+ \kappa \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot \nabla_b^{(\eta_N^n)^\delta} r - \int_\Gamma [(\boldsymbol{u}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^{n+1}) \cdot \boldsymbol{n}^{\omega_N^n}] r = 0,
\end{aligned} \tag{53}$$

and

$$\int_\Gamma \left(\frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\right) \cdot \boldsymbol{\phi} = \int_\Gamma \zeta_N^{n+1} \boldsymbol{e}_y \cdot \boldsymbol{\phi}, \qquad \text{for all } \boldsymbol{\phi} \in L^2(\Gamma). \tag{54}$$

Lemma 6.2. Problem (53), (54) has a unique solution provided that the following assumptions hold:

1. Assumption 1A: *Boundedness of the plate displacement away from R.* There exists a positive constant R_{max} such that

$$|\omega_N^{k+\frac{i}{2}}| \leqslant R_{max} < R, \qquad \text{for all } k = 0, 1, ..., n \text{ and } i = 0, 1. \tag{55}$$

2. Assumption 2A: *Invertibility of the map from fixed to moving Biot domain.* The map

$$\text{Id} + (\boldsymbol{\eta}_N^n)^\delta : \Omega_b \to (\Omega_b)_N^{n,\delta} \qquad \text{is invertible,} \tag{56}$$

where we define $(\Omega_b)_N^{n,\delta}$ to be the image of Ω_b under the map $\text{Id} + (\boldsymbol{\eta}_N^n)^\delta$.

Additionally, the weak solution satisfies the following energy equality:

$$\frac{1}{2}\int_{\Omega_f}\left(1 + \frac{\omega_N^{n+1}}{R}\right)|\boldsymbol{u}_N^{n+1}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+1}|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1})|^2 + \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^{n+1}|^2$$
$$+ \frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+1}|^2 + 2\mu_v(\Delta t)\int_{\Omega_b}|\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1})|^2 + \lambda_v(\Delta t)\int_{\Omega_b}|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \kappa(\Delta t)\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}|\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}|^2$$
$$+ \frac{\beta(\Delta t)}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma|(\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+1} - p_N^n|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2$$
$$+ \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot(\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2 = \frac{1}{2}\int_{\Omega_f}\left(1 + \frac{\omega_N^n}{R}\right)|\boldsymbol{u}_N^n|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^n|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^n)|^2$$
$$+ \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^n|^2 + \frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+\frac{1}{2}}|^2.$$

The proof is based on using the Lax-Milgram Lemma. However, in this case the proof is more involved for two reasons. First, the bilinear form associated with problem (53) and (54) is not coercive on the Hilbert space $\mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$, because of a mismatch between the hyperbolic and parabolic scaling in the problem. The second reason is that it is not *a priori* clear that Korn's inequality, which is needed in the proof of the existence, holds for the Biot domain. To deal with the first difficulty and recover the coercive structure of the problem, the test functions can be rescaled so that

$$\boldsymbol{v} \to (\Delta t)\boldsymbol{v}, \qquad r \to (\Delta t)r. \tag{57}$$

This scaling of the test functions is valid because if $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, v) \in \mathcal{Q}_N^{n+1}$, then the rescaled test function satisfies $((\Delta t)^{-1}\boldsymbol{v}, \varphi, \boldsymbol{\psi}, (\Delta t)^{-1}r) \in \mathcal{Q}_N^{n+1}$ also. To deal with the second difficulty, one can show by explicit calculation that the following Korn's inequality holds for this problem:

Proposition 6.1. Korn's inequality for the Biot poroviscoelastic domain. For all $\boldsymbol{\eta} \in V_d$,

$$\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta})|^2 \geqslant \frac{1}{2}\int_{\Omega_b}|\nabla\boldsymbol{\eta}|^2.$$

Proof. By a standard approximation argument, it suffices to assume that $\boldsymbol{\eta}$ is smooth. Because $\eta_x = 0$ on Γ and because $\boldsymbol{\eta} = 0$ on the left, top, and right boundaries of Ω_b, we have from integration by parts, that

$$\int_{\Omega_b}\frac{\partial\eta_x}{\partial y}\frac{\partial\eta_y}{\partial x} = -\int_{\Omega_b}\eta_x\frac{\partial^2\eta_y}{\partial x\partial y} = \int_{\Omega_b}\frac{\partial\eta_x}{\partial x}\frac{\partial\eta_y}{\partial y}.$$

Therefore, by using the inequality $a^2 + 2ab + b^2 \geqslant 0$, we obtain

$$\begin{aligned}
\int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 &= \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial \eta_x}{\partial y} + \frac{\partial \eta_y}{\partial x}\right)^2 \\
&= \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial \eta_x}{\partial y}\right)^2 + \left(\frac{\partial \eta_y}{\partial x}\right)^2\right] + \frac{\partial \eta_x}{\partial y}\frac{\partial \eta_y}{\partial x} \\
&= \int_{\Omega_b} \left(\frac{\partial \eta_x}{\partial x}\right)^2 + \frac{\partial \eta_x}{\partial x}\frac{\partial \eta_y}{\partial y} + \left(\frac{\partial \eta_y}{\partial y}\right)^2 + \frac{1}{2}\left[\left(\frac{\partial \eta_x}{\partial y}\right)^2 + \left(\frac{\partial \eta_y}{\partial x}\right)^2\right] \geqslant \frac{1}{2}\int_{\Omega_b} |\nabla \boldsymbol{\eta}|^2.
\end{aligned}$$

□

Proof. **Proof of Lemma 6.2.** Rewrite the weak formulation (53) and (54) so that all of the functions at the $(n+1)$st time step are on the left hand side while all other quantities are on the right hand side. In addition, we rewrite ζ_N^{n+1} in terms of $\boldsymbol{\eta}_N^n$ and $\boldsymbol{\eta}_N^{n+1}$ by using (54):

$$\zeta_N^{n+1}\boldsymbol{e}_y = \frac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\Big|_\Gamma.$$

After using the rescaling (57) of the test functions, the weak formulation involves the following coercive and continuous bilinear form:

$$\begin{aligned}
&B[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\psi}, p, r] := (\Delta t)^2 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{u} \cdot \boldsymbol{v} \\
&+ \frac{1}{2}(\Delta t)^3 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left[\left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R}\boldsymbol{e}_y\right) \cdot \nabla^{\omega_N^n} \boldsymbol{u}\right) \cdot \boldsymbol{v} - \left(\left(\boldsymbol{u}_N^n - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R}\boldsymbol{e}_y\right) \cdot \nabla^{\omega_N^n} \boldsymbol{v}\right) \cdot \boldsymbol{u}\right] \\
&+ (\Delta t)^3 \cdot \frac{1}{2R} \int_{\Omega_f} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u} \cdot \boldsymbol{v} + \frac{1}{2}(\Delta t)^3 \int_\Gamma (\boldsymbol{u} - (\Delta t)^{-1}\boldsymbol{\eta}) \cdot \boldsymbol{n}^{\omega_N^n} (\boldsymbol{u}_N^n \cdot \boldsymbol{v}) \\
&+ 2\nu(\Delta t)^3 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}) : \boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v}) + (\Delta t)^2 \int_\Gamma \left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}_N^n - p\right) (\boldsymbol{\psi} - (\Delta t)\boldsymbol{v}) \cdot \boldsymbol{n}^{\omega_N^n} \\
&+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}} (\Delta t)^2 \int_\Gamma [(\Delta t)^{-1}\boldsymbol{\eta} - \boldsymbol{u}] \cdot \boldsymbol{\tau}^{\omega_N^n} (\boldsymbol{\psi} - (\Delta t)\boldsymbol{v}) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} \boldsymbol{\eta} \cdot \boldsymbol{\psi} + \rho_p \int_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{\psi} \\
&+ (2\mu_e(\Delta t)^2 + 2\mu_v(\Delta t)) \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}) : \boldsymbol{D}(\boldsymbol{\psi}) + (\lambda_e(\Delta t)^2 + \lambda_v(\Delta t)) \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\psi}) \\
&- \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\psi} + c_0(\Delta t)^2 \int_{\Omega_b} pr - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta} \cdot \nabla_b^{(\eta_N^n)^\delta} r \\
&- \alpha(\Delta t)^2 \int_\Gamma (\boldsymbol{\eta} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r + \kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \nabla_b^{(\eta_N^n)^\delta} p \cdot \nabla_b^{(\eta_N^n)^\delta} r \\
&- (\Delta t)^3 \int_\Gamma [(\boldsymbol{u} - (\Delta t)^{-1}\boldsymbol{\eta}) \cdot \boldsymbol{n}^{\omega_N^n}] r.
\end{aligned}$$

With this notation, the weak formulation reads: find $(\boldsymbol{u}_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$ such that for all test functions $(\boldsymbol{v}, \boldsymbol{\psi}, r) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$,

$$B[\boldsymbol{u}_N^{n+1}, \boldsymbol{v}, \boldsymbol{\eta}_N^{n+1}, \boldsymbol{\psi}, p_N^{n+1}, r] = (\Delta t)^2 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \boldsymbol{u}_N^n \cdot \boldsymbol{v} - \frac{1}{2}(\Delta t)^2 \int_\Gamma \boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{\omega_N^n}(\boldsymbol{u}_N^n \cdot \boldsymbol{v})$$
$$+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}(\Delta t) \int_\Gamma \boldsymbol{\eta}_N^n \cdot \boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi} - (\Delta t)\boldsymbol{v}) \cdot \boldsymbol{\tau}^{\omega_N^n} + \rho_b \int_{\Omega_b} (2\boldsymbol{\eta}_N^n - \boldsymbol{\eta}_N^{n-1}) \cdot \boldsymbol{\psi} + \rho_p \int_\Gamma (\boldsymbol{\eta}_N^n + (\Delta t)\zeta_N^{n+\frac{1}{2}} \boldsymbol{e}_y) \cdot \boldsymbol{\psi}$$
$$+ 2\mu_v(\Delta t) \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N^n) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v(\Delta t) \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N^n)(\nabla \cdot \boldsymbol{\psi})$$
$$+ c_0(\Delta t)^2 \int_{\Omega_b} p_N^n r - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta}_N^n \cdot \nabla_b^{(\eta_N^n)^\delta} r - \alpha(\Delta t)^2 \int_\Gamma (\boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}) r + (\Delta t)^2 \int_\Gamma (\boldsymbol{\eta}_N^n \cdot \boldsymbol{n}^{\omega_N^n}) r. \tag{58}$$

We now show that the bilinear form $B[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\psi}, p, r]$ is coercive and continuous as a bilinear form on the Hilbert space $\mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$, with the inner product given by

$$\langle (\boldsymbol{u}, \boldsymbol{\eta}, p), (\boldsymbol{v}, \boldsymbol{\psi}, r) \rangle = \int_{\Omega_f} (\boldsymbol{u} \cdot \boldsymbol{v} + \nabla \boldsymbol{u} : \nabla \boldsymbol{v}) + \int_{\Omega_b} (\boldsymbol{\eta} \cdot \boldsymbol{\psi} + \nabla \boldsymbol{\eta} : \nabla \boldsymbol{\psi}) + \int_{\Omega_b} (p \cdot r + \nabla p \cdot \nabla r).$$

We focus on establishing coercivity, since continuity follows by standard arguments. To show coercivity we calculate $B[\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\eta}, p, p]$. In this calculation we note that after integration by parts, the sum of the following terms becomes zero:

$$-\alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\eta} - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta} \cdot \nabla_b^{(\eta_N^n)^\delta} p - \alpha(\Delta t)^2 \int_\Gamma \left(\boldsymbol{\eta} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}\right) p = 0.$$

Indeed, to see this, we bring the integrals back to the time-dependent physical domain, **which we can do as long as $(\boldsymbol{\eta}_N^n)^\delta$ is a bijection from Ω_b to $(\Omega_b)_N^{n,\delta}$**, which is provided by Assumption 2A (56), and perform the following computation:

$$- \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} p \nabla_b^{(\eta_N^n)^\delta} \cdot \boldsymbol{\eta} - \alpha(\Delta t)^2 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} \boldsymbol{\eta} \cdot \nabla_b^{(\eta_N^n)^\delta} p - \alpha(\Delta t)^2 \int_\Gamma \left(\boldsymbol{\eta} \cdot \boldsymbol{n}^{(\omega_N^n)^\delta}\right) p$$
$$= -\alpha(\Delta t)^2 \left(\int_{(\Omega_b)_N^{n,\delta}} p \nabla \cdot \boldsymbol{\eta} + \int_{(\Omega_b)_N^{n,\delta}} \boldsymbol{\eta} \cdot \nabla p + \int_{\Gamma_N^{n,\delta}} (\boldsymbol{\eta} \cdot \boldsymbol{n}) p \right) = 0,$$

where we used integration by parts, the fact that $\boldsymbol{n}$ points outwards from Ω_f and hence inwards towards Ω_b, and also use that $\boldsymbol{\eta} = 0$ on the left, right, and top boundaries of Ω_b. Combining this with the fact that $(\Delta t)\zeta_N^{n+\frac{1}{2}} = \omega_N^{n+\frac{1}{2}} - \omega_N^{n-\frac{1}{2}} = \omega_N^{n+1} - \omega_N^n$, we obtain

$$B[\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}, \boldsymbol{\eta}, p, p] := (\Delta t)^2 \int_{\Omega_f} \left(1 + \frac{\omega_N^n + \omega_N^{n+1}}{2R}\right) |\boldsymbol{u}|^2 + 2\nu(\Delta t)^3 \int_{\Omega_f} \left(1 + \frac{\omega_N^n}{R}\right) \left|\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u})\right|^2$$
$$+ \frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}(\Delta t) \int_\Gamma \left|(\boldsymbol{\eta} - (\Delta t)\boldsymbol{u}) \cdot \boldsymbol{\tau}^{\omega_N^n}\right|^2 + \rho_b \int_{\Omega_b} |\boldsymbol{\eta}|^2 + \rho_p \int_\Gamma |\boldsymbol{\eta}|^2 + (2\mu_e(\Delta t)^2 + 2\mu_v(\Delta t)) \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2$$
$$+ (\lambda_e(\Delta t)^2 + \lambda_v(\Delta t)) \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}|^2 + c_0(\Delta t)^2 \int_{\Omega_b} |p|^2 + \kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2.$$

Coercivity of this form follows from the fact that $|\omega_N^{k+\frac{i}{2}}| < R$, see Assumption 1A in (55), and Korn inequality, see Proposition 6.1, once we handle the last term and show that

$$\kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2 \geqslant c \int_{\Omega_b} |\nabla p|^2,$$

for some positive constant $c > 0$. To show this, we first recall the definitions

$$\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta), \qquad \nabla_b^{(\eta_N^n)^\delta} p = \nabla p \cdot (\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}.$$

Then, letting $|\cdot|$ denote the matrix norm, we have

$$\kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p|^2 \geqslant \kappa(\Delta t)^3 \int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^{-2} |\nabla p|^2. \tag{59}$$

Assumption 2A (56) implies that $\boldsymbol{I} + (\boldsymbol{\eta}_N^n)^\delta$ is an invertible map from Ω_b to $(\Omega_b)_N^{n,\delta}$, and we further note that $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|$ is continuous on $\overline{\Omega_b}$ and hence is bounded from above. Thus, $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^{-2} \geqslant c_0 > 0$ for some positive constant c_0. The assumption that $\boldsymbol{I} + (\boldsymbol{\eta}_N^n)^\delta$ is invertible implies that $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) > 0$. However, since this determinant is a continuous function on the compact set $\overline{\Omega_b}$, we conclude that there exists a positive constant $c_1 > 0$ such that $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) \geqslant c_1 > 0$. This establishes coercivity.

Existence of a unique weak solution $(\boldsymbol{u}_N^{n+1}, \boldsymbol{\eta}_N^{n+1}, p_N^{n+1}) \in \mathcal{V}_f^{\omega_N^n} \times V_d \times V_p$ now follows from the Lax-Milgram lemma. From here, we recover ζ_N^{n+1}, by using $\zeta_N^{n+1}\boldsymbol{e}_y = \left.\dfrac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\right|_\Gamma$. Note that $\left.\dfrac{\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n}{\Delta t}\right|_\Gamma$ points in the y direction because the trace of any function $\boldsymbol{\eta} \in V_d$ on Γ points in the y direction by definition, see (31).

Energy equality: We substitute $\boldsymbol{v} = \boldsymbol{u}_N^{n+1}$, $\varphi = \zeta_N^{n+1}$, $\boldsymbol{\psi} = \dot{\boldsymbol{\eta}}_N^{n+1}$, and $r = p_N^{n+1}$ into (53), and use the identity

$$(a - b) \cdot a = \frac{1}{2}(|a|^2 + |a - b|^2 - |b|^2).$$

Since $\omega_N^{n+1} = \omega_N^{n+\frac{1}{2}}$ and $(\Delta t)\zeta_N^{n+\frac{1}{2}} = \omega_N^{n+\frac{1}{2}} - \omega_N^n$, we obtain the following energy equality:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_f}\left(1 + \frac{\omega_N^{n+1}}{R}\right)|\boldsymbol{u}_N^{n+1}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+1}|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1})|^2 + \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^{n+1}|^2 \\
&+ \frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+1}|^2 + 2\mu_v(\Delta t)\int_{\Omega_b}|\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1})|^2 + \lambda_v(\Delta t)\int_{\Omega_b}|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}|^2 + \kappa(\Delta t)\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}|\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}|^2 \\
&+ \frac{\beta(\Delta t)}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma|(\dot{\boldsymbol{\eta}}_N^{n+1} - \boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+1} - \dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+1} - p_N^n|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2 \\
&+ \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot(\boldsymbol{\eta}_N^{n+1} - \boldsymbol{\eta}_N^n)|^2 = \frac{1}{2}\int_{\Omega_f}\left(1 + \frac{\omega_N^n}{R}\right)|\boldsymbol{u}_N^n|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^n|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_N^n|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^n)|^2 \\
&+ \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^n|^2 + \frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+\frac{1}{2}}|^2,
\end{aligned}$$

where the terms containing parameter α cancel out after bringing the integrals back to the time-dependent domain, integrating by parts, and recalling that the normal vector points inward towards the Biot domain:

$$\begin{aligned}
-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\dot{\boldsymbol{\eta}}_N^{n+1} - \alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1} - \alpha\int_\Gamma\left(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta}\right)p_N^{n+1} \\
= -\alpha\int_{(\Omega_b)_N^{n,\delta}}p_N^{n+1}(\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}) - \alpha\int_{(\Omega_b)_N^{n,\delta}}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla p_N^{n+1} - \alpha\int_{\Gamma_N^{n,\delta}}(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n})p_N^{n+1} = 0.
\end{aligned}$$

This completes the proof of the Lemma. $\square$

6.3 The coupled semi-discrete problem: weak formulation and energy

To obtain uniform energy estimates for approximate solutions of our semidiscretized scheme it is useful to present the scheme in monolithic form:

$$
\begin{aligned}
&\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\dot{\boldsymbol{u}}_N^{n+1}\cdot\boldsymbol{v}+2\nu\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1}):\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{v})+\int_\Gamma\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{u}_N^n-p_N^{n+1}\right)(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{n}^{\omega_N^n}\\
&+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left[\left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{v}-\left(\left(\boldsymbol{u}_N^n-\zeta_N^{n+\frac{1}{2}}\frac{R+y}{R}\boldsymbol{e}_y\right)\cdot\nabla_f^{\omega_N^n}\boldsymbol{v}\right)\cdot\boldsymbol{u}_N^{n+1}\right]\\
&+\frac{1}{2R}\int_{\Omega_f}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{v}+\frac{1}{2}\int_\Gamma(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}^{\omega_N^n}(\boldsymbol{u}_N^n\cdot\boldsymbol{v})+\frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi}-\boldsymbol{v})\cdot\boldsymbol{\tau}^{\omega_N^n}\\
&+\rho_b\int_{\Omega_b}\left(\frac{\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n}{\Delta t}\right)\cdot\boldsymbol{\psi}+\rho_p\int_\Gamma\left(\frac{\zeta_N^{n+1}-\zeta_N^n}{\Delta t}\right)\varphi+2\mu_e\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi})+\lambda_e\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta}_N^{n+1})(\nabla\cdot\boldsymbol{\psi})\\
&+2\mu_v\int_{\Omega_b}\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi})+\lambda_v\int_{\Omega_b}(\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1})(\nabla\cdot\boldsymbol{\psi})-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}+c_0\int_{\Omega_b}\frac{p_N^{n+1}-p_N^n}{\Delta t}r\\
&-\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r-\alpha\int_\Gamma(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta})r+\kappa\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta}r\\
&-\int_\Gamma[(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}^{\omega_N^n}]r+\int_\Gamma\Delta\omega_N^{n+\frac{1}{2}}\cdot\Delta\varphi=0,\ \forall(\boldsymbol{v},\varphi,\boldsymbol{\psi},r)\in\mathcal{Q}_N^{n+1},
\end{aligned}
\tag{60}
$$

$$
\int_\Gamma\left(\frac{\omega_N^{n+\frac{1}{2}}-\omega_N^{n-\frac{1}{2}}}{\Delta t}\right)\phi=\int_\Gamma\zeta_N^{n+\frac{1}{2}}\phi,\qquad\int_\Gamma\left(\frac{\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n}{\Delta t}\right)\cdot\boldsymbol{\phi}=\int_\Gamma\zeta_N^{n+1}\boldsymbol{e}_y\cdot\boldsymbol{\phi},\ \forall\phi,\boldsymbol{\phi}\in L^2(\Gamma).\tag{61}
$$

This formulation implies uniform energy estimates for the following discrete energy and discrete dissipation:

$$
\begin{aligned}
E_N^{n+\frac{i}{2}}&=\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)|\boldsymbol{u}_N^{n+\frac{i}{2}}|^2+\frac{1}{2}\rho_b\int_{\Omega_b}|\dot{\boldsymbol{\eta}}_N^{n+\frac{i}{2}}|^2+\frac{1}{2}c_0\int_{\Omega_b}|p_N^{n+\frac{i}{2}}|^2+\mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+\frac{i}{2}})|^2,.\\
&\quad+\frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_N^{n+\frac{i}{2}}|^2+\frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+\frac{i}{2}}|^2+\frac{1}{2}\int_\Gamma|\Delta\omega_N^{n+\frac{i}{2}}|^2,\ i=0,1.\\
D_N^{n+1}&=2\nu(\Delta t)\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{D}_f^{\omega_N^n}(\boldsymbol{u}_N^{n+1})\right|^2+2\mu_v(\Delta t)\int_{\Omega_b}|\boldsymbol{D}(\dot{\boldsymbol{\eta}}_N^{n+1})|^2+\lambda_v(\Delta t)\int_{\Omega_b}|\nabla\cdot\dot{\boldsymbol{\eta}}_N^{n+1}|^2\\
&\quad+\kappa(\Delta t)\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}|\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}|^2+\frac{\beta(\Delta t)}{\mathcal{J}_\Gamma^{\omega_N^n}}\int_\Gamma\left|(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}\right|^2.
\end{aligned}
\tag{62}
$$

Lemma 6.3. The following **discrete energy equalities** hold for the semi-discretized formulation (60), (61):

$$
E_N^{n+\frac{1}{2}}+\frac{1}{2}\rho_p\int_\Gamma\left|\zeta_N^{n+\frac{1}{2}}-\zeta_N^n\right|^2+\frac{1}{2}\int_\Gamma\left|\Delta(\omega_N^{n+\frac{1}{2}}-\omega_N^{n-\frac{1}{2}})\right|^2=E_N^n\tag{63}
$$

$$
\begin{aligned}
&E_N^{n+1}+D_N^{n+1}+\frac{1}{2}\int_{\Omega_f}\left(1+\frac{\omega_N^n}{R}\right)\left|\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n\right|^2+\frac{1}{2}\rho_b\int_{\Omega_b}\left|\dot{\boldsymbol{\eta}}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^n\right|^2+\frac{1}{2}c_0\int_{\Omega_b}\left|p_N^{n+1}-p_N^n\right|^2\\
&+\mu_e\int_{\Omega_b}\left|\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n)\right|^2+\frac{1}{2}\lambda_e\int_{\Omega_b}\left|\nabla\cdot(\boldsymbol{\eta}_N^{n+1}-\boldsymbol{\eta}_N^n)\right|^2+\frac{1}{2}\rho_p\int_\Gamma|\zeta_N^{n+1}-\zeta_N^{n+\frac{1}{2}}|^2=E_N^{n+\frac{1}{2}}.
\end{aligned}
\tag{64}
$$

We remark that the terms not included in the definition of $E_N^{n+\frac{i}{2}}$ and D_N^{n+1}, appearing in (63) and (64), are **numerical dissipation terms**.

These energy identities immediately imply that $E_N^{n+\frac{i}{2}}$ and $\sum_{n=1}^{N} D_N^n$ are **uniformly bounded** by a constant C independent of n and N.

The semidiscretized splitting scheme defines semidiscretized approximations of the solution to the regularized problem at discrete time points. To work with approximate functions and show that they converge to the solution of the continuous problem, we need to extend the semidiscrete approximations to the entire time interval and investigate uniform boundedness of those approximate solution functions. This is done next.

7 Approximate solutions

Now that we have defined the numerical solutions at each time step, we collect the solutions into approximate solutions defined on the whole time interval $[0, T]$, for which we will obtain uniform estimates from our previous energy estimates.

We define the following two extensions of the approximate functions to the entire interval $[0, T]$:

- Piecewise constant approximate solutions, for $(n-1)\Delta t < t \leqslant n\Delta t$:

$$\boldsymbol{u}_N(t) = \boldsymbol{u}_N^n, \quad \boldsymbol{\eta}_N(t) = \boldsymbol{\eta}_N^n, \quad p_N(t) = p_N^n, \quad \omega_N(t) = \omega_N^{n-\frac{1}{2}}, \quad \zeta_N(t) = \zeta_N^{n-\frac{1}{2}}, \quad \zeta_N^*(t) = \zeta_N^n;$$

- Linear interpolations:

$$\overline{\boldsymbol{\eta}}_N(n\Delta t) = \boldsymbol{\eta}_N^n, \qquad \overline{p}_N(n\Delta t) = p_N^n, \qquad \overline{\omega}_N(n\Delta t) = \omega_N^{n-\frac{1}{2}}, \qquad \text{for } n = 0, 1, ..., N,$$

  where we formally set $\omega_N^{-\frac{1}{2}} = \omega_0$.

Note that by construction, we have that

$$\partial_t \overline{\omega}_N = \zeta_N, \qquad \partial_t \overline{\boldsymbol{\eta}}_N|_\Gamma = \zeta_N^* \boldsymbol{e}_y.$$

From the preceding energy estimates, we have the following lemma on uniform boundedness.

Lemma 7.1. Uniform boundedness of approximate solutions. Assume:

1. Assumption 1B: Uniform boundedness of plate displacements. There exists a positive constant R_{max} such that for all N,

$$|\omega_N^{n-\frac{1}{2}}| \leqslant R_{max} < R, \qquad \text{for all } n = 0, 1, ..., N, \tag{65}$$

$$\left|(\boldsymbol{\eta}_N^n)^\delta|_\Gamma\right| \leqslant R_{max} < R, \qquad \text{for all } n = 0, 1, ..., N. \tag{66}$$

2. Assumption 2B: Uniform invertibility of the ALE mapping (Jacobian). There exists a positive constant c_0 such that for all N,

$$\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta) \geqslant c_0 > 0, \qquad \text{for all } n = 0, 1, ..., N. \tag{67}$$

3. Assumption 2C: Uniform boundedness of the ALE mapping (matrix norm). There exists positive constants c_1 and c_2 such that for all N,

$$|(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}| \leqslant c_1, \qquad |\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta| \leqslant c_2, \qquad \text{for all } n=0,1,...,N. \tag{68}$$

Then for all N:

- $\boldsymbol{u}_N$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega_f))$ and $L^2(0,T;H^1(\Omega_f))$.
- $\boldsymbol{\eta}_N$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega_b))$.
- p_N is uniformly bounded in $L^\infty(0,T;L^2(\Omega_b))$ and $L^2(0,T;H^1(\Omega_b))$.
- ω_N is uniformly bounded in $L^\infty(0,T;H_0^2(\Gamma))$.

In addition, we have the following estimates on the linear interpolations.

- $\overline{\boldsymbol{\eta}}_N$ is uniformly bounded in $W^{1,\infty}(0,T;L^2(\Omega_b))$.
- $\overline{\boldsymbol{\omega}}_N$ is uniformly bounded in $W^{1,\infty}(0,T;L^2(\Gamma))$.

Remark 7.1. A crucial remark about invertibility. At first, it would appear that to show the uniform boundedness results above, we also need to have a fourth assumption, which is Assumption 2A (56) from before, that the map $\text{Id}+(\boldsymbol{\eta}_N^n)^\delta : \Omega_b \to \mathbb{R}^2$ is injective (and is hence a bijection onto its image), for each $n=0,1,...,N$ and for all N. However, this is implied by an injectivity theorem, see Ciarlet [24] Theorem 5-5-2. Note also that Assumption 1A (55) from before is automatically satisfied once we verify Assumption 1B (65), (66). In particular, this injectivity theorem is as follows. Since $\det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta) > 0$ by Assumption 2B (67), it suffices to show that $\text{Id}+(\boldsymbol{\eta}_N^n)^\delta = \boldsymbol{\varphi}_0$ on $\partial\Omega_b$, for some injective mapping $\boldsymbol{\varphi}_0 : \overline{\Omega_b} \to \mathbb{R}^2$, for example a standard ALE mapping $\boldsymbol{\varphi}_0(x,y) = \left(x, y+\left(1-\frac{y}{R}\right)\omega\right)$ can be used. This implies the very useful fact that $(\text{Id}+(\boldsymbol{\eta}_N^n)^\delta)(\overline{\Omega_b}) = \boldsymbol{\varphi}_0(\overline{\Omega_b})$, which means that the *deformed configuration is fully determined by the behavior on the boundary.*

Proof. The uniform boundedness of approximate solutions follows from the uniform energy estimates. More precisely, the uniform boundedness of $\boldsymbol{u}_N$ in $L^\infty(0,T;L^2(\Omega_f))$ follows from Assumption 1B (65). The uniform boundedness of $\boldsymbol{u}_N$ in $L^2(0,T;H^1(\Omega_f))$ follows from Korn's inequality on the fluid domain. The uniform boundedness of $\boldsymbol{\eta}_N$ in $L^\infty(0,T;H^1(\Omega_b))$ follows from combining the uniform energy estimates with Korn's inequality, stated in Proposition 6.1. To establish the uniform boundedness of p_N in $L^2(0,T;H^1(\Omega_b))$, we recall that by the uniform dissipation estimate,

$$\sum_{n=1}^{N} \kappa(\Delta t)\int_{\Omega_b} \mathcal{J}_b^{(\eta_N^n)^\delta} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C,$$

for some constant C uniform in N, where $\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)$, and $\nabla_b^{(\eta_N^n)^\delta} r = \nabla r \cdot (\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}$ on Ω_b. By Assumption 2B (67), we conclude that

$$(\Delta t)\sum_{n=1}^{N} \int_{\Omega_b} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C.$$

Since on Ω_b, we have that $\nabla p_N^{n+1} = \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} \cdot (\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta)$, we use Assumption 2C (68), which implies $|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta| \leqslant c_2$, and obtain the estimate

$$(\Delta t) \sum_{n=1}^{N} \int_{\Omega_b} |\nabla p_N^{n+1}|^2 \leqslant |\boldsymbol{I} + \nabla(\boldsymbol{\eta}_N^n)^\delta|^2 \cdot (\Delta t) \sum_{n=1}^{N} \int_{\Omega_b} |\nabla_b^{(\eta_N^n)^\delta} p_N^{n+1}|^2 \leqslant C,$$

for a constant C independent of N. Thus, p_N is uniformly bounded in $L^2(0,T;H^1(\Omega_b))$.

□

The above uniform boundedness result implies the following weak convergence results.

Proposition 7.1. Assume that the three assumptions listed in Lemma 7.1 hold. Then, there exists a subsequence such that the following weak convergence results hold:

- $\boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ weakly* in $L^\infty(0,T;L^2(\Omega_f))$, $\quad \boldsymbol{u}_N \rightharpoonup \boldsymbol{u}$ weakly in $L^2(0,T;H^1(\Omega_f))$,
- $\boldsymbol{\eta}_N \rightharpoonup \boldsymbol{\eta}$ weakly* in $L^\infty(0,T;H^1(\Omega_b))$, $\quad \overline{\boldsymbol{\eta}}_N \rightharpoonup \overline{\boldsymbol{\eta}}$ weakly* in $W^{1,\infty}(0,T;L^2(\Omega_b))$,
- $p_N \rightharpoonup p$ weakly* in $L^\infty(0,T;L^2(\Omega_b))$, $\quad p_N \rightharpoonup p$ weakly in $L^2(0,T;H^1(\Omega_b))$,
- $\omega_N \rightharpoonup \omega$ weakly* in $L^\infty(0,T;H_0^2(\Gamma))$, $\quad \overline{\omega}_N \rightharpoonup \overline{\omega}$ weakly* in $W^{1,\infty}(0,T;L^2(\Gamma))$.

Furthermore, $\boldsymbol{\eta} = \overline{\boldsymbol{\eta}}$ and $\omega = \overline{\omega}$.

To use these results and to be able to construct approximate solutions, it is essential to show that the assumptions from Lemma 7.1 hold. This is given by the following lemma.

Lemma 7.2. Suppose that the initial data satisfies $|\omega_0| \leqslant R_0 < R$ for some R_0, and suppose that $\boldsymbol{\eta}_0$ has the property that $\text{Id} + (\boldsymbol{\eta}_0)^\delta$ is invertible with $\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) \geqslant c_0 > 0$ on Ω_b for some positive constant c_0. Then, **there exists a sufficiently small time** $T > 0$ such that for all N, all three assumptions in Lemma 7.1 hold and the splitting scheme is well defined until time T.

Proof. First, notice that the assumptions on the initial data immediately imply that the three assumptions from Lemma 7.1 hold for the initial data, i.e., for $n = 0$. In particular, there exist constants α_0, α_1, and α_2 such that

$$\det(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta) \geqslant \alpha_0 > 0, \tag{69}$$

$$|\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta| \geqslant \alpha_1 > 0, \qquad |(\boldsymbol{I} + \nabla(\boldsymbol{\eta}_0)^\delta)^{-1}| \geqslant \alpha_2 > 0. \tag{70}$$

This is because $\det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_0)^\delta)$, $|\boldsymbol{I}+\nabla(\boldsymbol{\eta}_0)^\delta|$, and $|(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_0)^\delta)^{-1}|$ are positive continuous functions on the compact set $\overline{\Omega_b}$.

Next, we want to define an appropriate time $T > 0$ such that the three assumptions hold uniformly for all N and $n\Delta t$ up to time T. To do this, we use the energy estimates. Define the initial energy determined by the initial data by E_0. Then, by the uniform energy estimates, we have that

$$E_N^{k+\frac{1}{2}} \leqslant E_0, \qquad E_N^{k+1} \leqslant E_0, \qquad \text{for all } k = 0, 1, ..., N-1.$$

Therefore, after completing both subproblems of the scheme on the time step $[k\Delta t, (k+1)\Delta t]$, we obtain that

$$||\dot{\boldsymbol{\eta}}_N^n||_{L^2(\Omega_b)} \leqslant C, \qquad \text{for } n = 0, 1, ..., k+1, \tag{71}$$

$$||\omega_N^{n+\frac{1}{2}}||_{H_0^2(\Gamma)} \leqslant C, \qquad \text{for } n = 0, 1, ..., k, \tag{72}$$

$$||\zeta_N^{n+\frac{i}{2}}||_{L^2(\Gamma)} \leqslant C, \qquad \text{for } 0 \leqslant n + \frac{i}{2} \leqslant k+1 \qquad \text{and } i = 0, 1, \tag{73}$$

for a constant C depending only on the initial energy E_0.

Step 1. We first find a condition on T such that Assumption 1B (65) is satisfied. Suppose that the linear interpolation $\overline{\omega}_N$ is defined up to time $(k+1)\Delta t$. Then, by (72) and (73), we have

$$||\overline{\omega}_N||_{W^{1,\infty}(0,(k+1)\Delta t; L^2(\Gamma))} \leqslant C, \qquad ||\overline{\omega}_N||_{L^\infty(0,(k+1)\Delta t; H_0^2(\Gamma))} \leqslant C, \tag{74}$$

where C depends only on E_0 and is independent of N. Thus, following the method in [47], we obtain by an interpolation inequality that for all $t, t+\tau \in [0, (k+1)\Delta t]$ with $\tau > 0$,

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^1(\Gamma)} \leqslant C||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||^{1/2}_{L^2(\Gamma)} ||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||^{1/2}_{H^2(\Gamma)}. \tag{75}$$

Here, we used a Sobolev interpolation inequality, see for example Theorem 4.17 (pg. 79) of [1]. By the Lipschitz continuity of $\overline{\omega}_N$ taking values in $L^2(\Gamma)$ and by the boundedness of $\overline{\omega}_N$ in $H_0^2(\Gamma)$,

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^1(\Gamma)} \leqslant C \cdot \tau^{1/2} \tag{76}$$

for a constant C depending only on E_0 (and in particular, not depending on k or N). Therefore, setting $t = 0$ and $\tau = (k+1)\Delta t$ and using the continuous embedding of $H^1(\Gamma)$ into $C(\Gamma)$,

$$||\omega_N^{k+1} - \omega_0||_{C(\Gamma)} \leqslant C \cdot [(k+1)t]^{1/2} \leqslant C \cdot T^{1/2}, \tag{77}$$

where C depends only E_0. Because $|\omega_0| < R$, we can choose $T > 0$ sufficiently small so that

$$C \cdot T^{1/2} < R - ||\omega_0||_{C(\Gamma)}. \tag{78}$$

This will give the first part of Assumption 1B, which is (65).

Step 2. Next, we find a condition on T so that the remaining assumptions (66), (67), and (68) are satisfied. We do this by controlling the behavior of the structure displacement $\boldsymbol{\eta}$. First note that

$$||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\Omega_b)} \leqslant (\Delta t) \sum_{n=1}^{k+1} ||\dot{\boldsymbol{\eta}}_N^n||_{L^2(\Omega_b)} \leqslant C(k+1)(\Delta t) \leqslant CT,$$

for C depending only on E_0. By the odd extension defined in Definition 8.2,

$$||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\tilde{\Omega}_b)} \leqslant C \left(||\boldsymbol{\eta}_N^{k+1} - \boldsymbol{\eta}_0||_{L^2(\Omega_b)} + ||\omega_N^{k+1} - \omega_0||_{L^2(\Gamma)} \right) \leqslant CT,$$

for a constant C depending only on E_0, where the estimate $||\omega_N^{k+1} - \omega_0||_{L^2(\Gamma)} \leqslant CT$ follows from the bound (74). By regularization, we then have that for a constant depending only on δ and E_0,

$$||(\boldsymbol{\eta}_N^{k+1})^\delta - (\boldsymbol{\eta}_0)^\delta||_{H^3(\Omega_b)} \leqslant C(\delta, E_0) \cdot T.$$

By using the trace theorem and the continuous embedding of $H^2(\Gamma)$ into $C(\Gamma)$, we thus conclude that

$$||(\boldsymbol{\eta}_N^{k+1})^\delta|_\Gamma - (\boldsymbol{\eta}_0)^\delta|_\Gamma||_{C(\Gamma)} \leqslant C(\delta, E_0) \cdot T. \tag{79}$$

Since $H^2(\Omega_b)$ embeds continuously into $C(\Omega_b)$, we also have that

$$||\nabla(\boldsymbol{\eta}_N^{k+1})^\delta - \nabla(\boldsymbol{\eta}_0)^\delta||_{C(\Omega_b)} \leqslant C(\delta, E_0) \cdot T. \tag{80}$$

Note that $\det(\boldsymbol{I} + \boldsymbol{A})$ is a continuous function of the entries of $\boldsymbol{A}$. Also note that the matrix norms $|\boldsymbol{I}+\boldsymbol{A}|$ and $|(\boldsymbol{I}+\boldsymbol{A})^{-1}|$ are continuous functions of the matrix $\boldsymbol{A}$. Furthermore, we emphasize that the constant $C(\delta, E_0)$ depends only on δ and E_0 and hence is independent of k and N. This dependence on δ is allowable, since for this existence proof, δ is an arbitrary but fixed regularization parameter.

Thus, there exists T sufficiently small so that by (79) and (80), the remaining assumptions (66), (67), and (68) are satisfied, since these assumptions are all satisfied for the initial displacement $\boldsymbol{\eta}_0$. Furthermore, we can choose the constants c_0, c_1, c_2, and R_{max} (defined in the statement of those assumptions) independently of N and $n = 0, 1, ..., N$, because of the fact that the constant $C(\delta, E_0)$ in our estimates does not depend on k (satisfying $(k+1)\Delta t \leqslant T$) or N. $\square$

8 Compactness arguments

We next want to pass to the limit in the semidiscrete formulation for the approximate solutions, stated in (60) and (61). Because this is a nonlinear problem with geometric nonlinearities, we must obtain stronger convergence than just weak and weak* convergence in Proposition 7.1, in order to pass to the limit. To do this, we will use compactness arguments of two types: the classical Aubin-Lions compactness theorem for functions defined on fixed domains, and *generalized Aubin-Lions compactness arguments* introduced in [52] for functions defined on moving domains, see also [47]. We will first deal with compactness arguments for the plate displacement and the Biot domain displacement. Then, we will deal with compactness arguments for the fluid velocity defined on moving domains.

8.1 Compactness for Biot poroelastic medium displacement

We show strong convergence of the Biot structure displacements $\overline{\boldsymbol{\eta}}_N$ by using a standard Aubin-Lions compactness argument. In particular, we have the following strong convergence result for the Biot medium displacement:

Lemma 8.1. The following compact embedding holds true $W^{1,\infty}(0,T;L^2(\Omega_b)) \cap L^\infty(0,T;H^1(\Omega_b)) \subset\subset C(0,T;L^2(\Omega_b))$, which implies the existence of a subsequence such that

$$\overline{\boldsymbol{\eta}}_N \to \boldsymbol{\eta} \text{ strongly in } C(0,T;L^2(\Omega_b)).$$

Proof. The compact embedding above is a direct consequence of the standard Aubin-Lions compactness lemma in the case of $p = \infty$, which gives a stronger compact embedding into $C(0,T;L^2(\Omega_b))$ rather than just $L^\infty(0,T;L^2(\Omega_b))$. The fact that we can find a strongly convergent subsequence follows from this compact embedding, once we recall that $\{\overline{\boldsymbol{\eta}}_N\}_{N=1}^\infty$ are uniformly bounded in the Banach space $W^{1,\infty}(0,T;L^2(\Omega_b)) \cap L^\infty(0,T;H^1(\Omega_b))$ by the uniform energy estimates. $\square$

8.2 Compactness for the plate displacement

The uniform boundedness of the linear interpolation of the plate displacement $\overline{\omega}_N$ in $W^{1,\infty}(0,T;L^2(\Gamma))$ and $L^\infty(0,T;H_0^2(\Gamma))$ implies the following strong convergence result:

Proposition 8.1. Given arbitrary $0 < s < 2$, there exists a subsequence such that the following strong convergences hold:

$$\overline{\omega}_N \to \omega, \qquad \text{in } C(0,T;H^s(\Gamma)),$$
$$\omega_N \to \omega, \qquad \text{in } L^\infty(0,T;H^s(\Gamma)).$$

Proof. Using the same argument as in Step 1 of the proof of Lemma 7.2, one can show the following uniform estimate for the linear interpolations $\overline{\omega}_N$ and $\tau > 0$, $t, t+\tau \in [0,T]$:

$$||\overline{\omega}_N(t+\tau) - \overline{\omega}_N(t)||_{H^{2\alpha}(\Gamma)} \leqslant C\tau^{1-\alpha}, \qquad \text{for } 0 < \alpha < 1,$$

where the constant C is independent of N, but can depend on the choice of α. Because C is independent of N, the estimate implies that for a given arbitrary $\alpha \in (0,1)$, the functions $\overline{\omega}_N$ are uniformly bounded as functions in $C^{0,1-\alpha}(0,T;H^{2\alpha}(\Gamma))$. Hence, the strong convergence of $\overline{\omega}_N$ follows directly from the Arzela-Ascoli theorem and the fact that $H^{2\alpha}$ embeds compactly into any $H^{2\alpha-\epsilon}$ for $\epsilon > 0$, once we choose $\alpha \in (0,1)$ and $\epsilon > 0$ appropriately so that $2\alpha - \epsilon = s$ for a given arbitrary $0 < s < 2$. Hence, we obtain the desired strong convergence, as the equicontinuity condition for the Arzela-Ascoli theorem follows from the above estimate.

To show a similar strong convergence result for ω_N, we must show that

$$||\omega_N(t) - \overline{\omega}_N(t)||_{L^\infty(0,T;H^s(\Gamma))} \to 0,$$

for arbitrary $0 < s < 2$. Once we observe that $\overline{\omega}_N(n\Delta t) = \omega_N(t)$ for $n\Delta t \leqslant t < (n+1)\Delta t$, this follows immediately from the above Hölder continuity estimate, as

$$||\omega_N(t) - \overline{\omega}_N(t)||_{L^\infty(0,T;H^s(\Gamma))} \leqslant C(\Delta t)^{1-\frac{s}{2}} \to 0, \qquad \text{as } N \to \infty.$$

Thus, ω_N and $\overline{\omega}_N$ have the same limit in $L^\infty(0,T;H^s(\Gamma))$ for $0 < s < 2$.

□

Next, we will obtain compactness for the Biot velocity, plate velocity, pore pressure, and fluid velocity. Because the test space (52) has the pore pressure and fluid velocity decoupled from the Biot/plate velocity, we can handle the compactness argument for each of these quantities separately. In particular, we recall the definition of the discrete test space from (52) and note that we can decouple this test space into three smaller test spaces, one for the Biot/plate displacement/velocity, one for the pore pressure, and one for the fluid velocity. In the next section we show compactness results for the Biot velocity and plate velocity, which must be treated together since they are coupled by a kinematic coupling condition at the plate interface Γ.

8.3 Compactness for the Biot velocity and plate velocity

Theorem 8.1. For $-1/2 < s < 0$, there exists a subsequence such that

$$(\boldsymbol{\xi}_N, \zeta_N) \to (\boldsymbol{\xi}, \zeta) \;\text{ strongly in } L^2(0,T;H^{-s}(\Omega_b) \times H^{-s}(\Gamma)).$$

Proof. We will establish this result by using a compactness criterion for piecewise constant functions due to Dreher and Jüngel [28]. To simplify arguments, we define a slightly more regular Biot/plate velocity test space:

$$\mathcal{Q}_v = \{(\boldsymbol{\psi}, \varphi) \in (V_d \cap H^2(\Omega_b)) \times H_0^2(\Gamma) : \boldsymbol{\psi} = \varphi \boldsymbol{e}_y \text{ on } \Gamma\}. \tag{81}$$

We will use the following chain of embeddings

$$L^2(\Omega_b) \times L^2(\Gamma) \subset\subset H^{-s}(\Omega_b) \times H^{-s}(\Gamma) \subset \mathcal{Q}_v',$$

where the first embedding is compact, as required for the Dreher-Jüngel compactness criterion [28].

Let $\tau_{\Delta t}$ denote the time shift $\tau_{\Delta t} f(t,\cdot) = f(t-\Delta t,\cdot)$ for a function f defined on $[0,T]$. As required by the Dreher-Jüngel compactness criterion [28], to obtain compactness we must verify that the following inequality is satisfied for a uniform constant C and for all $\Delta t = T/N$:

$$\left|\left|\frac{\tau_{\Delta t}(\boldsymbol{\xi}_N,\zeta_N)-(\boldsymbol{\xi}_N,\zeta_N)}{\Delta t}\right|\right|_{L^1(\tau,T;\mathcal{Q}_v')} + ||(\boldsymbol{\xi}_N,\zeta_N)||_{L^\infty(0,T;L^2(\Omega_b)\times L^2(\Gamma))} \leqslant C. \tag{82}$$

The second term in this inequality is uniformly bounded by Lemma 7.1, which gives exactly the uniform boundednenss of $(\boldsymbol{\xi}_N,\zeta_N)$ in $L^\infty(0,T;L^2(\Omega_b)\times L^2(\Gamma))$.

To deal with the first term in (82) we use the coupled semidiscrete formulation (60), (61) and set the test functions $\boldsymbol{v}$ and r for the fluid velocity and Biot pore pressure to be zero because we are considering only the Biot and plate velocities. We obtain that for all test functions $(\boldsymbol{\psi},\varphi)\in\mathcal{Q}_v$, where $\mathcal{Q}_v$ is defined in (81), the following holds:

$$\begin{aligned}
&\rho_b\int_{\Omega_b}\left(\frac{\boldsymbol{\xi}_N^{n+1}-\boldsymbol{\xi}_N^n}{\Delta t}\right)\cdot\boldsymbol{\psi} + \rho_p\int_\Gamma\left(\frac{\zeta_N^{n+1}-\zeta_N^n}{\Delta t}\right)\cdot\varphi\\
&= -\int_\Gamma\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{u}_N^n - p_N^{n+1}\right)(\boldsymbol{\psi}\cdot\boldsymbol{n}^{\omega_N^n}) - \int_\Gamma\frac{\beta}{\mathcal{J}_\Gamma^{\omega_N^n}}(\zeta_N^{n+1}\boldsymbol{e}_y-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}^{\omega_N^n}(\boldsymbol{\psi}\cdot\boldsymbol{\tau}^{\omega_N^n})\\
&- 2\mu_e\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\eta}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi}) - \lambda_e\int_{\Omega_b}(\nabla\cdot\boldsymbol{\eta}_N^{n+1})(\nabla\cdot\boldsymbol{\psi}) - 2\mu_v\int_{\Omega_b}\boldsymbol{D}(\boldsymbol{\xi}_N^{n+1}):\boldsymbol{D}(\boldsymbol{\psi})\\
&- \lambda_v\int_{\Omega_b}(\nabla\cdot\boldsymbol{\xi}_N^{n+1})(\nabla\cdot\boldsymbol{\psi}) + \alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi} - \int_\Gamma\Delta\omega_N^{n+\frac{1}{2}}\cdot\Delta\varphi.
\end{aligned}$$

The estimate for the first term in (82) will follow if we can estimate the right-hand side in terms of the $\mathcal{Q}_v'$ norm. For this purpose consider an arbitrary $||(\boldsymbol{\psi},\varphi)||_{\mathcal{Q}_v}\leqslant 1$, so that $||\boldsymbol{\psi}||_{H^2(\Omega_b)}\leqslant 1$ and $||\varphi||_{H_0^2(\Gamma)}\leqslant 1$. By the uniform estimates in Lemma 7.1 and the regularity of the test functions in (81), it is clear that the terms on the right hand side are all uniformly bounded by a constant C, independent of $||(\boldsymbol{\psi},\varphi)||_{\mathcal{Q}_v}\leqslant 1$, except possibly the term

$$\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}.$$

To estimate this term we recall the definitions

$$\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta), \qquad \nabla^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi} = \mathrm{tr}\left[\nabla\boldsymbol{\psi}\cdot(I+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}\right].$$

By assumption 2C (68) and the fact that $||\boldsymbol{\psi}||_{H^1(\Omega_b)}\leqslant 1$, we have that $||\nabla^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}||_{L^2(\Omega_b)}$ is uniformly bounded, while by the boundedness of $\boldsymbol{\eta}_N^n$ in $H^1(\Omega_b)$, we have that $|\mathcal{J}_b^{(\eta_N^n)^\delta}|\leqslant C$. Therefore, using the fact that p_N is uniformly bounded in $L^\infty(0,T;L^2(\Omega_b))$, we obtain the desired estimate

$$\left|\alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}p_N^{n+1}\nabla_b^{(\eta_N^n)^\delta}\cdot\boldsymbol{\psi}\right|\leqslant C.$$

Finally, we conclude that

$$\left|\left|\frac{(\boldsymbol{\xi}_N^{n+1},\zeta_N^{n+1})-(\boldsymbol{\xi}_N^n,\zeta_N^n)}{\Delta t}\right|\right|_{\mathcal{Q}_v'}\leqslant C, \quad \text{for a constant } C \text{ that is independent of } n \text{ and } N,$$

and since
$$\sum_{n=1}^{N-1}(\Delta t)\left|\left|\frac{(\boldsymbol{\xi}_N^{n+1},\zeta_N^{n+1})-(\boldsymbol{\xi}_N^n,\zeta_N^n)}{\Delta t}\right|\right|_{\mathcal{Q}_v'} \leqslant (\Delta t)\sum_{n=1}^{N-1} C \leqslant CT,$$
we conclude that (82) holds for a uniform constant C. This establishes the desired result.

□

8.4 Compactness for the pore pressure

Theorem 8.2. There exists a subsequence such that
$$p_N \to p \text{ strongly in } L^2(0,T;L^2(\Omega_b)).$$

Proof. The proof is based on a similar application of the Dreher-Jüngel compactness criterion for piecewise constant functions [28] as in the previous compactness result. We first observe that we have the following chain of embeddings $H^1(\Omega_b) \subset\subset L^2(\Omega_b) \subset (V_p \cap H^2(\Omega_b))'$, and so by the Dreher-Jüngel compactness criterion [28] it suffices to show that the following inequality holds for a constant C independent of N:
$$\left|\left|\frac{\tau_{\Delta t}p_N - p_N}{\Delta t}\right|\right|_{L^1(\Delta t,T;(V_p\cap H^2(\Omega_b))')} + ||p_N||_{L^2(0,T;H^1(\Omega_b))} \leqslant C. \tag{83}$$

To obtain this estimate, we observe that the approximate solutions for the pore pressure satisfy the following weak formulation for all test functions $r \in V_p$, where V_p is defined by (33):
$$c_0\int_{\Omega_b}\left(\frac{p_N^{n+1}-p_N^n}{\Delta t}\right)\cdot r - \alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta} r - \alpha\int_\Gamma(\dot{\boldsymbol{\eta}}_N^{n+1}\cdot\boldsymbol{n}^{(\omega_N^n)^\delta})r$$
$$+\kappa\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta} r - \int_\Gamma[(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}^{\omega_N^n}]r = 0.$$

We use more regularity for the test space $V_p \cap H^2(\Omega_b)$ to make the following estimates simpler. We compute that for any $r \in V_p \cap H^2(\Omega_b)$ we have
$$c_0\int_{\Omega_b}\left(\frac{p_N^{n+1}-p_N^n}{\Delta t}\right)\cdot r = \alpha\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\boldsymbol{\xi}_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta} r + \alpha\int_\Gamma(\zeta_N^{n+1}\boldsymbol{e}_y\cdot\boldsymbol{n}^{(\omega_N^n)^\delta})r$$
$$-\kappa\int_{\Omega_b}\mathcal{J}_b^{(\eta_N^n)^\delta}\nabla_b^{(\eta_N^n)^\delta}p_N^{n+1}\cdot\nabla_b^{(\eta_N^n)^\delta} r + \int_\Gamma[(\boldsymbol{u}_N^{n+1}-\zeta_N^{n+1}\boldsymbol{e}_y)\cdot\boldsymbol{n}^{\omega_N^n}]r.$$

We estimate the right hand side for $||r||_{V_p\cap H^2(\Omega_b)} \leqslant 1$. Recall that $\mathcal{J}_b^{(\eta_N^n)^\delta} = \det(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)$,
$$\nabla_b^{(\eta_N^n)^\delta} r = \left(\frac{\partial r}{\partial\tilde{x}},\frac{\partial r}{\partial\tilde{y}}\right)\cdot(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}, \quad \text{and} \quad \nabla_b^{(\eta_N^n)^\delta} p_N^{n+1} = \left(\frac{\partial p_N^{n+1}}{\partial\tilde{x}},\frac{\partial p_N^{n+1}}{\partial\tilde{y}}\right)\cdot(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}.$$

We have by Assumption 2C (68) that $|(\boldsymbol{I}+\nabla(\boldsymbol{\eta}_N^n)^\delta)^{-1}|$ is uniformly bounded, and furthermore, $\mathcal{J}_b^{(\eta_N^n)^\delta}$ is positive and bounded above. By combining these facts with standard estimates we obtain that
$$\left|\left|\frac{p_N^{n+1}-p_N^n}{\Delta t}\right|\right|_{(V_p\cap H^2(\Omega_b))'} \leqslant C \quad \text{for a constant } C \text{ that is independent of } n \text{ and } N.$$

Combining this with the fact that p_N is uniformly bounded in $L^2(0,T;H^1(\Omega_b))$ gives the desired estimate (83). □

8.5 Compactness for the fluid velocity

We will obtain convergence of the fluid velocity along a subsequence by using a *generalized Aubin-Lions compactness theorem* for functions defined on moving domains [52]. The reason we must use a generalized Aubin-Lions compactness theorem is that the approximate fluid velocities are defined on different time-dependent fluid domains. To prepare for an application of the generalized Aubin-Lions compactness argument we will *map our approximate fluid problem back onto the physical domain*

$$\Omega^n_{f,N} = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega^n_N(x)\},$$

where we redefine the fluid velocity solution and test spaces as follows:

$$V^{n+1}_N = \{\boldsymbol{u} \in H^1(\Omega^n_{f,N}) : \nabla \cdot \boldsymbol{u} = 0 \text{ on } \Omega^n_{f,N}, \boldsymbol{u} = 0 \text{ on } \partial\Omega^n_{f,N} \backslash \Gamma^n_N\}, \quad Q^n_N = V^{n+1}_N \cap H^3(\Omega^n_{f,N}). \tag{84}$$

The approximate fluid velocity $\boldsymbol{u}^{n+1}_N \in V^{n+1}_N$ on the physical domain satisfies the following semidiscrete formulation:

$$\begin{aligned}
&\int_{\Omega^n_{f,N}} \frac{\boldsymbol{u}^{n+1}_N - \tilde{\boldsymbol{u}}^n_N}{\Delta t} \cdot \boldsymbol{v} + 2\nu \int_{\Omega^n_{f,N}} \boldsymbol{D}(\boldsymbol{u}^{n+1}_N) : \boldsymbol{D}(\boldsymbol{v}) \\
&+ \frac{1}{2} \int_{\Omega^n_{f,N}} \left[\left(\left(\tilde{\boldsymbol{u}}^n_N - \zeta^{n+\frac{1}{2}}_N \frac{R+y}{R+\omega^n_N} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{u}^{n+1}_N \right) \cdot \boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}^n_N - \zeta^{n+\frac{1}{2}}_N \frac{R+y}{R+\omega^n_N} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{v} \right) \cdot \boldsymbol{u}^{n+1}_N \right] \\
&+ \frac{1}{2R} \int_{\Omega^n_{f,N}} \frac{R}{R+\omega^n_N} \zeta^{n+\frac{1}{2}}_N \boldsymbol{u}^{n+1}_N \cdot \boldsymbol{v} + \frac{1}{2} \int_{\Gamma^n_N} (\boldsymbol{u}^{n+1}_N - \dot{\boldsymbol{\eta}}^{n+1}_N) \cdot \boldsymbol{n} (\tilde{\boldsymbol{u}}^n_N \cdot \boldsymbol{v}) \\
&- \int_{\Gamma^n_N} \left(\frac{1}{2} \boldsymbol{u}^{n+1}_N \cdot \tilde{\boldsymbol{u}}^n_N - p^{n+1}_N \right) (\boldsymbol{v} \cdot \boldsymbol{n}) - \beta \int_{\Gamma^n_N} (\dot{\boldsymbol{\eta}}^{n+1}_N - \boldsymbol{u}^{n+1}_N) \cdot \boldsymbol{\tau} (\boldsymbol{v} \cdot \boldsymbol{\tau}) = 0, \ \forall \boldsymbol{v} \in Q^n_N
\end{aligned} \tag{85}$$

where

$$\tilde{\boldsymbol{u}}^n_N = \boldsymbol{u}^n_N \circ \boldsymbol{\Phi}^{\omega^{n-1}_N}_f \circ (\boldsymbol{\Phi}^{\omega^n_N}_f)^{-1},$$

$\boldsymbol{u}^n_N$ is originally defined on $\Omega^{n-1}_{f,N}$, and the ALE map $\boldsymbol{\Phi}^{\omega^n_N}_f : \Omega_f \to \Omega^n_{f,N}$ is defined by (16).

To be able to compare functions on different physical domains we introduce a maximal domain Ω^M_f which contains all the physical domains. The existence of such a domain, and the extensions of the velocity functions onto the maximal domain are discussed next.

8.5.1 Extension to maximal domain

We consider the following *maximal fluid domain* which contains all the physical fluid domains:

$$\Omega^M_f = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant M(x)\},$$

where the function $M(x)$ is obtained from the following proposition, established in Lemma 2.5 in [62] and Lemma 4.5 in [52] in the context of fluid-structure interaction between an incompressible viscous Newtonian fluid and an elastic Koiter shell:

Proposition 8.2. There exists smooth functions $m(x)$ and $M(x)$ defined on $\Gamma = [0,L]$, satisfying $m(0) = m(L) = M(0) = M(L) = 0$, such that

$$m(x) \leqslant \omega^n_N(x) \leqslant M(x), \qquad \text{for all } x \in [0,L], N, \text{ and } n = 0,1,...,N.$$

Furthermore, there exist smooth functions $m^{n,l}_N(x)$ and $M^{n,l}_N(x)$ defined for positive integers N, $n = 0,1,...,N-1$ and $l = 0,1,...,N-n$, such that

1. $m_N^{n,l}(x) \leqslant \omega_N^{n+i}(x) \leqslant M_N^{n,l}(x), \qquad$ for all $x \in [0,L]$ and $i = 0,1,...,l$.
2. $M_N^{n,l}(x) - m_N^{n,l}(x) \leqslant C\sqrt{l\Delta t}, \qquad$ for all $x \in [0,L]$.
3. $||M_N^{n,l}(x) - m_N^{n,l}(x)||_{L^2(\Gamma)} \leqslant C(l\Delta t)$,

where C is independent of n, l, and N. Finally, the functions $M_N^{n,l}(x)$ and $m_N^{n,l}(x)$ for all n, l, and N, are Lipschitz continuous with a Lipschitz constant that is uniformly bounded above by some constant $L > 0$ independent of n, l, and N.

Once the maximal fluid domain is defined, we can extend the fluid velocities $\boldsymbol{u}_N^n$ from $\Omega_{f,N}^n$ to this common maximal domain Ω_f^M, using extensions by zero in $\Omega_f^M \cap (\Omega_{f,N}^n)^c$. Notice that since $\omega_N^n(x)$ are all uniformly Lipschitz, the extensions by zero of the H^1 functions $\boldsymbol{u}_N^n$ defined on Lipschitz domains to Ω_f^M are uniformly bounded in $H^s(\Omega_f^M)$ for all s such that $0 < s < 1/2$. Indeed, we have the following lemma, which follows from Theorem 2.7 in [46].

Lemma 8.2. The approximate fluid velocities $\{\boldsymbol{u}_N\}_{N=1}^\infty$ defined on the maximal fluid domain Ω_f^M by extension by zero are uniformly bounded in $L^2(0,T;H^s(\Omega_f^M))$ for $s \in (0,1/2)$.

8.5.2 Velocity convergence via a generalized Aubin-Lions compactness argument

We now show strong convergence as $N \to \infty$ along a subsequence of the approximate fluid velocities $\boldsymbol{u}_N$, which are now functions in time defined on the fixed maximal domain Ω_f^M.

Proposition 8.3. The sequence $\boldsymbol{u}_N$ is relatively compact in $L^2(0,T;L^2(\Omega_f^M))$.

Proof. The proof is based on using the generalized Aubin-Lions compactness theorem, Theorem 3.1 in [52], for problems on moving domains. For this purpose we define the Hilbert spaces V and H from the statement of the theorem to be

$$H = L^2(\Omega_f^M), \qquad V = H^s(\Omega_f^M), \qquad \text{for } 0 < s < 1/2,$$

where we note that indeed $V \subset\subset H$ as required by Theorem 3.1 in [52]. Additionally, the spaces $(V_{\Delta t}^n, Q_{\Delta t}^n)$ from the statement of the theorem correspond to our spaces (V_N^n, Q_N^n) defined by (84). Notice that $V_N^n \times Q_N^n$ embeds continuously into $V \times V$ as required by the statement of Theorem 3.1 in [52], where the embedding can be achieved by the extension by zero operator to the maximal domain Ω_f^M, uniformly in n and N.

To obtain compactness of the sequence $\boldsymbol{u}_N$ in $L^2(0,T,H)$, by Theorem 3.1 in [52], seven properties need to be satisfied by the sequence $\boldsymbol{u}_N$ and the spaces V_N^n and Q_N^n. They are called Properties A1-3, B, and C1-3.

The proof that approximate solutions $\boldsymbol{u}_N$ satisfy Properties A1-3 and C1-3 is analogous to the corresponding proof in [52] (Section 4.2). The main difficulty is to verify Property B, which is a condition on equicontinuity of $\boldsymbol{u}_N$, stated as follows:

Property B, [52]. There exists a constant $C > 0$ independent of N such that

$$\left\| P_N^n \frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^n}{\Delta t} \right\|_{(Q_N^n)'} \leqslant C\left(1 + ||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right), \qquad \text{for all } n = 0,1,...,N-1. \tag{86}$$

The sequence $\boldsymbol{u}_N$ constructed in this manuscript, however, *does not satisfy this property*. Never the less, $\boldsymbol{u}_N$ satisfy the following *generalized Property B* which implies the desired equicontinuity under which the generalized Aubin-Lions theorem from [52] still holds:

Generalized Property B. There exist a constant C independent of n and N, an exponent p, $1 \leqslant p < 2$, and a sequence of nonnegative numbers $\{a_N^n\}_{n=0}^{N-1}$ for each N, satisfying $(\Delta t)\sum_{n=0}^{N-1}|a_N^n|^2 \leqslant C$ uniformly in N, such that

$$\left|\left|P_N^n \frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\right|\right|_{(Q_N^n)'} \leqslant C\left(a_N^n + ||\boldsymbol{u}_N^n||_{V_N^n} + ||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right)^p, \qquad \text{for all } n = 0, 1, ..., N-1, \tag{87}$$

where P_N^n denotes the orthogonal projection onto the closed subspace $\overline{Q_N^n}^H$ of the Hilbert space H.

Indeed, with this Generalized Property B the compactness theorem, Theorem 3.1 in [52] still holds, as we still obtain the essential equicontinuity estimate needed in the proof. In particular, for the original form of Property B in (86), one has from Lemma 3.1 in [52] the following equicontinuity estimate for a constant $C > 0$ that is independent of N:

$$||P_{\Delta t}^{n,l}(\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n)||_{(Q_N^{n,l})'} \leqslant C\sqrt{l\Delta t}.$$

With the generalized form of Property B that we use above in (87), the same arguments as in the proof of Lemma 3.1 in [52] will still give rise to the following equicontinuity estimate for a constant $C > 0$ that is independent of N:

$$||P_{\Delta t}^{n,l}(\boldsymbol{u}_N^{n+l} - \boldsymbol{u}_N^n)||_{(Q_N^{n,l})'} \leqslant C(l\Delta t)^{1-\frac{p}{2}},$$

where the generalized Aubin-Lions compactness theorem on moving domains still holds with this new equicontinuity estimate. This is because $1 \leqslant p < 2$ and hence, $C(l\Delta t)^{1-\frac{p}{2}}$ still converges to zero as $\Delta t \to 0$.

We can now complete the proof of Proposition 8.3 by verifying that our sequence $\boldsymbol{u}_N^n$ indeed satisfies the Generalized Property B.

Verification that $\boldsymbol{u}_N^n$ satisfies the Generalized Property B. First, recall that by definition,

$$\left|\left|P_N^n \frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\right|\right|_{(Q_N^n)'} = \max_{||\boldsymbol{v}||_{Q_N^n}\leqslant 1}\left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}_N^n}{\Delta t}\cdot \boldsymbol{v}d\boldsymbol{x}\right|. \tag{88}$$

To estimate the right hand-side, we use

$$\left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\boldsymbol{u}^n}{\Delta t}\cdot \boldsymbol{v}d\boldsymbol{x}\right| \leqslant \left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\tilde{\boldsymbol{u}}_N^n}{\Delta t}\cdot \boldsymbol{v}d\boldsymbol{x}\right| + \left|\int_{\Omega_{f,N}^n}\frac{\tilde{\boldsymbol{u}}_N^{n}-\boldsymbol{u}_N^n}{\Delta t}\cdot \boldsymbol{v}d\boldsymbol{x}\right|. \tag{89}$$

To estimate the first term on the right hand-side we use the semidiscrete formulation for the fluid velocity on the physical domain given by (85) to obtain

$$\begin{aligned}
&\left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\tilde{\boldsymbol{u}}_N^n}{\Delta t}\cdot \boldsymbol{v}d\boldsymbol{x}\right| \leqslant 2\nu\left|\int_{\Omega_{f,N}^n}\boldsymbol{D}(\boldsymbol{u}_N^{n+1}):\boldsymbol{D}(\boldsymbol{v})\right| \\
&+\frac{1}{2}\left|\int_{\Omega_{f,N}^n}\left[\left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla\boldsymbol{u}_N^{n+1}\right)\cdot\boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}_N^n - \zeta_N^{n+\frac{1}{2}}\frac{R+y}{R+\omega_N^n}\boldsymbol{e}_y\right)\cdot\nabla\boldsymbol{v}\right)\cdot\boldsymbol{u}_N^{n+1}\right]\right| \\
&+\frac{1}{2R}\left|\int_{\Omega_{f,N}^n}\frac{R}{R+\omega_N^n}\zeta_N^{n+\frac{1}{2}}\boldsymbol{u}_N^{n+1}\cdot\boldsymbol{v}\right| + \frac{1}{2}\left|\int_{\Gamma_N^n}(\boldsymbol{u}_N^{n+1}-\dot{\boldsymbol{\eta}}_N^{n+1})\cdot\boldsymbol{n}(\tilde{\boldsymbol{u}}_N^n\cdot\boldsymbol{v})\right| \\
&+\left|\int_{\Gamma_N^n}\left(\frac{1}{2}\boldsymbol{u}_N^{n+1}\cdot\tilde{\boldsymbol{u}}_N^n - p_N^{n+1}\right)(\boldsymbol{v}\cdot\boldsymbol{n})\right| + \beta\left|\int_{\Gamma_N^n}(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}(\boldsymbol{v}\cdot\boldsymbol{\tau})\right|.
\end{aligned} \tag{90}$$

We can bound the terms on the right hand-side uniformly in n, N, and $||\boldsymbol{v}||_{Q^n_N} \leqslant 1$ as follows. By the boundedness of $\boldsymbol{u}^{n+1}_N$ in the uniform energy estimates we immediately have

$$2\nu \left| \int_{\Omega^n_{f,N}} \boldsymbol{D}(\boldsymbol{u}^{n+1}_N) : \boldsymbol{D}(\boldsymbol{v}) \right| \leqslant C||\boldsymbol{u}^{n+1}_N||_{H^1(\Omega^n_{f,N})}.$$

The second term on the right hand-side of the above inequality is bounded as follows. First notice that because $||\boldsymbol{v}||_{Q^n_N} \leqslant 1$, and by the definition of Q^n_N in (84), we have that $\boldsymbol{v}$ is bounded in $H^3(\Omega^n_{f,N})$, and hence, $\boldsymbol{v}$ and $\nabla\boldsymbol{v}$ are bounded pointwise. Furthermore, by the boundedness of the fluid velocity $\boldsymbol{u}^n_N$ on the reference domain due to the uniform energy estimate, and by the uniform boundedness of the Jacobian of the ALE map $\Phi_f^{\omega^n_N}$, we obtain the following bound:

$$\begin{aligned}
&\frac{1}{2}\left| \int_{\Omega^n_{f,N}} \left[\left(\left(\tilde{\boldsymbol{u}}^n_N - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R+\omega^n_N} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{u}^{n+1}_N \right) \cdot \boldsymbol{v} - \left(\left(\tilde{\boldsymbol{u}}^n_N - \zeta_N^{n+\frac{1}{2}} \frac{R+y}{R+\omega^n_N} \boldsymbol{e}_y \right) \cdot \nabla \boldsymbol{v} \right) \cdot \boldsymbol{u}^{n+1}_N \right] \right| \\
&\quad \leqslant C \left(||\tilde{\boldsymbol{u}}^n_N||_{L^2(\Omega^n_{f,N})} + ||\zeta^{n+\frac{1}{2}}||_{L^2(\Gamma)} \right) ||\boldsymbol{u}^{n+1}_N||_{H^1(\Omega^n_{f,N})} \cdot ||\boldsymbol{v}||_{H^3(\Omega^n_{f,N})} \leqslant C||\boldsymbol{u}^{n+1}_N||_{H^1(\Omega^n_{f,N})}.
\end{aligned}$$

Similarly, the next term in (90) is bounded as follows:

$$\frac{1}{2R}\left| \int_{\Omega^n_{f,N}} \frac{R}{R+\omega^n_N} \zeta_N^{n+\frac{1}{2}} \boldsymbol{u}^{n+1}_N \cdot \boldsymbol{v} \right| \leqslant C||\zeta_N^{n+\frac{1}{2}}||_{L^2(\Gamma)} ||\boldsymbol{u}^{n+1}_N||_{L^2(\Omega^n_{f,N})} \cdot ||\boldsymbol{v}||_{H^3(\Omega^n_{f,N})} \leqslant C||\boldsymbol{u}^{n+1}_N||_{L^2(\Omega^n_{f,N})}.$$

To bound the next term we observe that $||\dot{\boldsymbol{\eta}}^{n+1}_N||_{L^2(\Gamma)}$ is bounded uniformly and furthermore, the arc length element on Γ^n_N is uniformly bounded pointwise since η^n_N is uniformly bounded in $H^2_0(\Gamma)$. Therefore, by using the trace inequality on Ω_f we have the following estimate:

$$\begin{aligned}
&\frac{1}{2}\left| \int_{\Gamma^n_N} (\boldsymbol{u}^{n+1}_N - \dot{\boldsymbol{\eta}}^{n+1}_N) \cdot \boldsymbol{n}(\tilde{\boldsymbol{u}}^n_N \cdot \boldsymbol{v}) \right| \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{L^4(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{L^4(\Gamma)} \cdot ||\boldsymbol{v}||_{L^2(\Gamma)} + ||\dot{\boldsymbol{\eta}}^{n+1}_N||_{L^2(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{L^4(\Gamma)} \cdot ||\boldsymbol{v}||_{L^4(\Gamma)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{H^{1/4}(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{H^{1/4}(\Gamma)} \cdot ||\boldsymbol{v}||_{H^1(\Omega_f)} + ||\dot{\boldsymbol{\eta}}^{n+1}_N||_{L^2(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{H^{1/4}(\Gamma)} \cdot ||\boldsymbol{v}||_{H^{1/4}(\Gamma)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{H^{3/4}(\Omega_f)} \cdot ||\boldsymbol{u}^n_N||_{H^{3/4}(\Omega_f)} + ||\dot{\boldsymbol{\eta}}^{n+1}_N||_{L^2(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{H^{3/4}(\Omega_f)} \right) \cdot ||\boldsymbol{v}||_{H^3(\Omega_f)} \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||^{1/4}_{L^2(\Omega_f)} ||\boldsymbol{u}^{n+1}_N||^{3/4}_{H^1(\Omega_f)} ||\boldsymbol{u}^n_N||^{1/4}_{L^2(\Omega_f)} ||\boldsymbol{u}^n_N||^{3/4}_{H^1(\Omega_f)} + ||\dot{\boldsymbol{\eta}}^{n+1}_N||_{L^2(\Gamma)} ||\boldsymbol{u}^n_N||^{1/4}_{L^2(\Omega_f)} ||\boldsymbol{u}^n_N||^{3/4}_{H^1(\Omega_f)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||^{3/4}_{H^1(\Omega_f)} \cdot ||\boldsymbol{u}^n_N||^{3/4}_{H^1(\Omega_f)} + ||\boldsymbol{u}^n_N||^{3/4}_{H^1(\Omega_f)} \right) \leqslant C \left[1 + \left(||\boldsymbol{u}^n_N||_{V^n_N} + ||\boldsymbol{u}^{n+1}_N||_{V^n_N} \right)^{3/2} \right].
\end{aligned}$$

The second to last term in (90) is estimated as follows:

$$\begin{aligned}
&\left| \int_{\Gamma^n_N} \left(\frac{1}{2} \boldsymbol{u}^{n+1}_N \cdot \tilde{\boldsymbol{u}}^n_N - p^{n+1}_N \right) (\boldsymbol{v} \cdot \boldsymbol{n}) \right| \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{L^4(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{L^4(\Gamma)} \cdot ||\boldsymbol{v}||_{L^2(\Gamma)} + ||p^{n+1}_N||_{L^2(\Gamma)} \cdot ||\boldsymbol{v}||_{L^2(\Gamma)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{H^{1/4}(\Gamma)} \cdot ||\boldsymbol{u}^n_N||_{H^{1/4}(\Gamma)} \cdot ||\boldsymbol{v}||_{H^1(\Omega_f)} + ||p^{n+1}_N||_{H^1(\Omega_b)} \cdot ||\boldsymbol{v}||_{H^1(\Omega_f)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||_{H^{3/4}(\Omega_f)} \cdot ||\boldsymbol{u}^n_N||_{H^{3/4}(\Omega_f)} + ||p^{n+1}_N||_{H^1(\Omega_b)} \right) \\
&\quad \leqslant C \left(||\boldsymbol{u}^{n+1}_N||^{1/4}_{L^2(\Omega_f)} ||\boldsymbol{u}^{n+1}_N||^{3/4}_{H^1(\Omega_f)} \cdot ||\boldsymbol{u}^n_N||^{1/4}_{L^2(\Omega_f)} ||\boldsymbol{u}^n_N||^{3/4}_{H^1(\Omega_f)} + ||p^{n+1}_N||_{H^1(\Omega_b)} \right) \\
&\quad \leqslant C \left[1 + \left(||p^{n+1}_N||_{H^1(\Omega_b)} + ||\boldsymbol{u}^n_N||_{V^n_N} + ||\boldsymbol{u}^{n+1}_N||_{V^{n+1}_N} \right)^{3/2} \right].
\end{aligned}$$

Finally, we estimate the last term

$$\beta\left|\int_{\Gamma_N^n}(\dot{\boldsymbol{\eta}}_N^{n+1}-\boldsymbol{u}_N^{n+1})\cdot\boldsymbol{\tau}(\boldsymbol{v}\cdot\boldsymbol{\tau})\right|\leqslant C\left(||\dot{\boldsymbol{\eta}}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}+||\boldsymbol{u}_N^{n+1}||_{L^2(\Gamma)}\cdot||\boldsymbol{v}||_{L^2(\Gamma)}\right)$$
$$\leqslant C\left(1+||\boldsymbol{u}_N^{n+1}||_{H^1(\Omega_f)}\right).$$

Therefore, we obtain the final estimate of the first term in (89) which implies the existence of a constant C independent of n and N, such that

$$\max_{||\boldsymbol{v}||_{Q_N^n}\leqslant 1}\left|\int_{\Omega_{f,N}^n}\frac{\boldsymbol{u}_N^{n+1}-\tilde{\boldsymbol{u}}_N^n}{\Delta t}\cdot\boldsymbol{v}d\boldsymbol{x}\right|\leqslant C\left(a_N^n+||\boldsymbol{u}_N^n||_{V_N^n}+||\boldsymbol{u}_N^{n+1}||_{V_N^{n+1}}\right)^{3/2},$$
$$\text{for } a_N^n:=1+||p_N^{n+1}||_{H^1(\Omega_b)},\ \text{where } (\Delta t)\sum_{n=0}^{N-1}|a_N^n|^2\leqslant 2\left[(\Delta t)N+||p_N||^2_{L^2(0,T;H^1(\Omega_b))}\right]\leqslant C. \quad (91)$$

To complete the estimate (89), it remains to show that the second term $\left|\int_{\Omega_{f,N}^n}\frac{\tilde{\boldsymbol{u}}_N^n-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{v}d\boldsymbol{x}\right|$ is uniformly bounded. This follows from the same estimates as those presented in [52] which show that there exists a constant C independent of n and N, such that

$$\max_{||\boldsymbol{v}||_{Q_N^n}\leqslant 1}\left|\int_{\Omega_{f,N}^n}\frac{\tilde{\boldsymbol{u}}_N^n-\boldsymbol{u}_N^n}{\Delta t}\cdot\boldsymbol{v}d\boldsymbol{x}\right|\leqslant C. \quad (92)$$

Combining (91) and (92) with (88) and (89) establishes Generalized Property B and completes the proof of Proposition 8.3. □

9 Passing to the limit in the regularized weak formulation

We have so far established the following strong convergence results:

$$\begin{aligned}
&\boldsymbol{\eta}_N\to\boldsymbol{\eta}, \quad \text{in } C(0,T;L^2(\Omega_b)),\\
&\omega_N\to\omega, \quad \text{in } L^\infty(0,T;H^s(\Gamma)) \text{ for } 0<s<2,\\
&\zeta_N^*\to\zeta, \quad \text{in } L^2(0,T;H^{-s}(\Gamma)),\ \text{for } -1/2<s<0,\\
&\zeta_N\to\zeta, \quad \text{in } L^2(0,T;H^{-s}(\Gamma)),\ \text{for } -1/2<s<0,\\
&\boldsymbol{\xi}_N\to\boldsymbol{\xi}, \quad \text{in } L^2(0,T;H^{-s}(\Omega_b)),\ \text{for } -1/2<s<0,\\
&\boldsymbol{u}_N\to\boldsymbol{u}, \quad \text{in } L^2(0,T;L^2(\Omega_f^M)), \qquad p_N\to p, \quad \text{in } L^2(0,T;L^2(\Omega_b)),
\end{aligned}$$

where ζ_N^* and ζ_N converge to the same limit in $L^2(0,T;H^{-s}(\Gamma))$ for $-1/2<s<0$ due to the numerical dissipation estimates $\sum_{n=1}^N||\zeta_N^n-\zeta_N^{n-\frac{1}{2}}||^2_{L^2(\Gamma)}\leqslant C$, which imply that $||\zeta_N-\zeta_N^*||_{L^2(0,T;L^2(\Gamma))}\to 0$.

These strong convergence results will be used to pass to the limit in the semidiscrete formulation of the coupled problem (60) and show that the limit satisfies the weak formulation of the regularized problem. Before we can do this, there are two more convergence results that need to be established. One is a strong convergence result for the traces for the fluid velocity on the boundary of the fluid domain, and the other is a convergence result for the test functions, which are defined on approximate moving domains.

We start with the convergence result for the trace of the fluid velocity $\hat{\boldsymbol{u}}_N|_\Gamma$ along Γ.

9.1 Strong convergence of the fluid velocity traces on Γ

Proposition 9.1. The traces $\hat{\boldsymbol{u}}_N|_\Gamma$ of the approximate fluid velocities on Γ converge to the trace of the limiting fluid velocity on Γ as $N \to \infty$:

$$\hat{\boldsymbol{u}}_N|_\Gamma \to \hat{\boldsymbol{u}}|_\Gamma, \qquad \text{in } L^2(0,T;H^{s-\frac{1}{2}}(\Gamma)), \qquad \text{for } s \in (0,1),$$

where $\hat{\boldsymbol{u}}_N = \boldsymbol{u}_N \circ \Phi_f^{\tau_{\Delta t}\omega_N}$ and $\hat{\boldsymbol{u}} = \boldsymbol{u} \circ \Phi_f^{\omega}$.

To prove Proposition 9.1, we will use the following elementary lemma.

Lemma 9.1. Suppose that the functions $\{f_n\}_{n=1}^{\infty}$ and f are all uniformly bounded in $L^2(0,T;H^1(\Omega_f))$ and $f_n \to f$ in $L^2(0,T;L^2(\Omega_f))$. Then, $f_n \to f$ in $L^2(0,T;H^s(\Omega_f))$ for $s \in (0,1)$ and hence $f_n|_\Gamma \to f|_\Gamma$ in $L^2(0,T;H^{s-\frac{1}{2}}(\Gamma))$ for $s \in (1/2,1)$.

Proof of Lemma 9.1. For $s \in (0,1)$, we compute using Sobolev interpolation that

$$||f_n - f||^2_{L^2(0,T;H^s(\Omega_f))} = \int_0^T ||(f_n - f)(t)||^2_{H^s(\Omega_f)} dt$$
$$\leqslant \int_0^T ||(f_n - f)(t)||^{2(1-s)}_{L^2(\Omega_f)} \cdot ||(f_n - f)(t)||^{2s}_{H^1(\Omega_f)} dt \leqslant ||f_n - f||^{2(1-s)}_{L^2(0,T;L^2(\Omega_f))} \cdot ||f_n - f||^{2s}_{L^2(0,T;H^1(\Omega_f))}.$$

The result then follows from the fact that $||f_n - f||_{L^2(0,T;H^1(\Omega_f))} \leqslant C$ for a constant C that does not depend on N, the assumption that $||f_n - f||_{L^2(0,T;L^2(\Omega_f))} \to 0$ as $N \to \infty$, and the trace embedding which gives that $||f_n|_\Gamma - f|_\Gamma||^2_{L^2(0,T;H^{s-\frac{1}{2}}(\Gamma))} \leqslant ||f_n - f||^2_{L^2(0,T;H^s(\Omega_f))}$ for $s \in (1/2,1)$. $\square$

We can use the elementary lemma above to show the desired strong convergence of the fluid velocity traces.

Proof of Proposition 9.1. We would like to combine the fact that $\boldsymbol{u}_N \to \boldsymbol{u}$ in $L^2(0,T;L^2(\Omega_f^M))$ with the fact that $\boldsymbol{u}_N$ and $\boldsymbol{u}$ are all uniformly bounded in $L^2(0,T;H^1(\Omega_f(t)))$ for all N, to deduce strong convergence of the traces of the fluid velocities using the previous elementary lemma. We do this in the following steps.

Step 1. We show that $\hat{\boldsymbol{u}}_N \to \hat{\boldsymbol{u}}$ on $L^2(0,T;L^2(\Omega_f))$, for $\hat{\boldsymbol{u}}_N$ and $\hat{\boldsymbol{u}}$ defined on the reference fluid domain.

To prove this, we compute $||\hat{\boldsymbol{u}}_N - \hat{\boldsymbol{u}}||^2_{L^2(0,T;L^2(\Omega_f))}$ using the functions $\boldsymbol{u}_N$ and $\boldsymbol{u}$ which are defined on the maximal domain Ω_f^M:

$$||\hat{\boldsymbol{u}}_N - \hat{\boldsymbol{u}}||^2_{L^2(0,T;L^2(\Omega_f))} = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}_N\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\omega\right)\right|^2$$
$$\leqslant 2(I_1 + I_2),$$

where

$$I_1 = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}_N\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right)\right|^2,$$
$$I_2 = \int_0^T \int_{\Omega_f} \left|\boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t,x,y+\left(1+\frac{y}{R}\right)\omega\right)\right|^2.$$

We show that $I_1 \to 0$ as $N \to \infty$ by using the fact that $1 + \frac{\omega_N^n}{R}$ is uniformly bounded from above by a positive constant, and the fact that Ω_f^M contains all of the domains $\Omega_{f,N}^n$, so that we can estimate:

$$\begin{aligned} I_1 &= \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_{f,N}^n} \left(1 + \frac{\omega_N^n}{R}\right) |\boldsymbol{u}_N^{n+1} - \boldsymbol{u}|^2 \leqslant C \sum_{n=0}^{N-1} \int_{n\Delta t}^{(n+1)\Delta t} \int_{\Omega_{f,N}^n} |\boldsymbol{u}_N^{n+1} - \boldsymbol{u}|^2 \\ &\leqslant C||\boldsymbol{u}_N - \boldsymbol{u}||^2_{L^2(0,T;L^2(\Omega_f^M))} \to 0. \end{aligned}$$

For I_2, we break up the integral into two parts:

$$I_2 = I_{2,1} + I_{2,2},$$

where

$$\begin{aligned} I_{2,1} &= \int_0^T \int_0^L \int_{-R}^{\min(0,y^*(t,x))} \left|\boldsymbol{u}\left(t, x, y + \left(1 + \frac{y}{R}\right)\tau_{\Delta t}\omega_N\right) - \boldsymbol{u}\left(t, x, y + \left(1 + \frac{y}{R}\right)\omega\right)\right|^2, \\ I_{2,2} &= \int_0^T \int_0^L \int_{\min(0,y^*(t,x))}^0 \left|\boldsymbol{u}\left(t, x, y + \left(1 + \frac{y}{R}\right)\tau_{\Delta t}\omega_N\right)\right|^2, \end{aligned}$$

for $y^*(t,x) = \frac{\omega - \tau_{\Delta t}\omega_N}{R + \tau_{\Delta t}\omega_N}$. We can interpret $y^*(t,x)$ as the y value for which $y + \left(1 + \frac{y}{R}\right)\tau_{\Delta t}\omega_N = \omega$. Now, note that

$$\begin{aligned} I_{2,1} &\leqslant \int_0^T \int_0^L \int_{-R}^{\min(0,y^*(t,x))} \left(\int_{y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N}^{y+\left(1+\frac{y}{R}\right)\omega} |\partial_y \boldsymbol{u}(t,x,y')| dy'\right)^2 \\ &\leqslant \int_0^T \int_0^L \int_{-R}^{\min(0,y^*(t,x))} \left(\int_{y+\left(1+\frac{y}{R}\right)\tau_{\Delta t}\omega_N}^{y+\left(1+\frac{y}{R}\right)\omega} |\partial_r \boldsymbol{u}(t,x,y')|^2 dy'\right) \cdot \left(1 + \frac{y}{R}\right) \cdot |\omega - \tau_{\Delta t}\omega_N|. \end{aligned}$$

We note that $\tau_{\Delta t}\omega_N \to \omega$ pointwise uniformly on $[0,T] \times \Gamma$ as $N \to \infty$ by Proposition 8.1, which implies $\overline{\omega}_N \to \omega$ in $C(0,T;H^s(\Gamma))$ for $0 < s < 2$, and by the estimate (76). Combining this with the fact that $||\nabla \boldsymbol{u}||_{L^2(0,T;L^2(\Omega_f^\omega(t)))}$ is bounded, we have that $I_{2,1} \to 0$ as $N \to \infty$.

Next, by Poincare's inequality,

$$\begin{aligned} I_{2,2} &\leqslant \int_0^T \int_0^L |\min(0, y^*(t,x))| \cdot \max_{w \in [-R, \omega(t,x)]} |\boldsymbol{u}(t,x,w)|^2 \\ &\leqslant \int_0^T \int_0^L |\min(0, y^*(t,x))| \cdot \int_{-R}^{\omega(t,x)} |\partial_r \boldsymbol{u}(t,x,y')|^2 dy', \end{aligned}$$

so we conclude that $I_{2,2} \to 0$ as $N \to \infty$ by the fact that $|\min(0, y^*(t,x))| \to 0$ uniformly on $[0,T] \times \Gamma$, and by the boundedness of $||\nabla \boldsymbol{u}||_{L^2(0,T;L^2(\Omega_f^\omega(t))}$. Thus, we have that $||\hat{\boldsymbol{u}}_N - \hat{\boldsymbol{u}}||_{L^2(0,T;L^2(\Omega_f)))} \to 0$.

Step 2. We claim that the functions $\hat{\boldsymbol{u}}_N$ for positive integers N and $\hat{\boldsymbol{u}}$ are all uniformly bounded in $L^2(0,T;H^1(\Omega_f))$. Recall from Lemma 7.1 that the approximate solutions $\hat{\boldsymbol{u}}_N$ are uniformly bounded in $L^2(0,T;H^1(\Omega_f))$. Since $\hat{\boldsymbol{u}}$ is the strong limit of $\hat{\boldsymbol{u}}_N$ in $L^2(0,T;L^2(\Omega_f))$ and $\hat{\boldsymbol{u}}_N$ converge weakly in $L^2(0,T;H^1(\Omega_f))$ along a subsequence to a weak limit which hence must also be $\hat{\boldsymbol{u}}$, we conclude that $\hat{\boldsymbol{u}}$ is also in $L^2(0,T;H^1(\Omega_f))$, which establishes the desired result of this step.

Step 3. From Step 1, we have that $\hat{\boldsymbol{u}}_N \to \hat{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega_f))$ and from Step 2, the functions $\hat{\boldsymbol{u}}_N$ and $\hat{\boldsymbol{u}}$ are bounded in $L^2(0,T;H^1(\Omega_f))$ independently of N, so we can conclude the proof of Proposition 9.1 by using Lemma 9.1. $\square$

9.2 Convergence of the test functions on approximate fluid domains

The main difficulty in passing to the limit will be the test functions for the fluid velocity. In particular, on the fixed reference domain Ω_f for the fluid, we note that the test functions for the fluid velocity in (38) satisfy $\nabla_f^\omega \cdot \boldsymbol{v} = 0$ on Ω_f, where ω is the solution for the plate displacement. However, the test functions for the fluid velocity in the semidiscrete formulation in the semidiscrete test space $\mathcal{Q}_N^{n+1}$, defined by (52), satisfy $\nabla^{\omega_N^n} \cdot \boldsymbol{v} = 0$ on Ω_f. Hence, we need a way of comparing test functions in $\mathcal{Q}_N^{n+1}$ to test functions in the actual test space $\mathcal{V}_{\text{test}}^\omega$.

To do this, recall that we have defined the maximal domain Ω_f^M that contains all of the numerical fluid domains $\Omega_{f,N}^n$. We then propose to work with the test functions that are defined on Ω_f^M, and are constructed in such a way that the restrictions of those test functions to the domain defined by the plate displacement ω, and composed with the ALE mapping Φ_f^ω defined in (16), gives a space of test functions $\mathcal{X}_f^\omega$ that is dense in the fluid velocity test space $\mathcal{V}_f^\omega$. The space of all such test functions defined on Ω_f^M is denoted by $\mathcal{X}$ and it is defined as follows.

The test space $\mathcal{X}$: The test space $\mathcal{X}$ consists of functions $\boldsymbol{v} \in C_c^1([0,T); H^1(\Omega_f^M))$ satisfying the following properties for each $t \in [0,T)$:

1. For each $t \in [0,T)$, $\boldsymbol{v}(t)$ is a smooth vector-valued function on Ω_f^M.
2. $\nabla \cdot \boldsymbol{v}(t) = 0$ on Ω_f^M for all $t \in [0,T)$.
3. $\boldsymbol{v}(t) = 0$ on $\partial\Omega_f^M \backslash \Gamma_M$ for all $t \in [0,T)$, where $\Gamma_M = \{(x, M(x)) : 0 \leqslant x \leqslant L\}$ is the top boundary of the maximal fluid domain Ω_f^M.

Given $\boldsymbol{v} \in \mathcal{X}$, define

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}|_{\Omega_f^\omega} \circ \boldsymbol{\Phi}_f^\omega \quad \text{and} \quad \tilde{\boldsymbol{v}}_N = \boldsymbol{v}|_{\Omega_f^{\omega_N}} \circ \boldsymbol{\Phi}_f^{\omega_N}. \tag{93}$$

The test functions $\tilde{\boldsymbol{v}}$ are dense in the fluid velocity test space $\mathcal{V}_f^\omega$ associated with the fixed domain formulation, and the test functions $\tilde{\boldsymbol{v}}_N$ restricted to $[n\Delta t, (n+1)\Delta t)$ are dense in $V_f^{\omega_N^n}$, where $V_f^{\omega_N^n}$ is the velocity test space for the semidiscretized problem(s) given in (52). Therefore, for each fixed N, we can consider the semidiscrete formulation with the test function $\tilde{\boldsymbol{v}}_N$, which we emphasize is discontinuous in time, due to the jumps in ω_N at each $n\Delta t$. To pass to the limit as $N \to \infty$ we can use the same approach as in Lemma 7.1 in [47] and Lemma 2.8 in [62], to obtain the following strong convergence results of the velocity test functions $\tilde{\boldsymbol{v}}_N$ and their gradients, which will allow us to pass to the limit in the semidiscrete weak formulations:

Proposition 9.2. Consider $\boldsymbol{v} \in \mathcal{X}$, and $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}_N$ defined in (93). Then

$$\tilde{\boldsymbol{v}}_N \to \boldsymbol{v}, \qquad \nabla\tilde{\boldsymbol{v}}_N \to \nabla\boldsymbol{v},$$

pointwise, uniformly on $[0,T] \times \Omega_f$, as $N \to \infty$.

Remark 9.1. We emphasize that we were able to construct such a test space $\mathcal{X}$ because in the definition of the full test space $\mathcal{V}_{\text{test}}^\omega$ in (38), the only component of the test space whose definition depends on the plate displacement is the fluid velocity, and fortunately, this fluid velocity component of the test space is *decoupled from the other components.* This is a feature of fluid-poroelastic structure interaction problems. In the purely elastic case of FSI, the fluid velocity test space is coupled to that of the structure, and the construction of the test functions that converge on the approximate fluid domains in more involving, see e.g., [20, 47, 49].

9.3 Passing to the limit

We are now in a position to pass to the limit in the semidiscrete formulation. From (60) we obtain that for all $(\tilde{\boldsymbol{v}}_N, \varphi, \boldsymbol{\psi}, r)$ in the test space with $\boldsymbol{v} \in \mathcal{X}$, the following holds:

$$
\begin{aligned}
&\int_0^T \int_{\Omega_f} \left(1 + \frac{\tau_{\Delta t}\omega_N}{R}\right) \partial_t \overline{\boldsymbol{u}}_N \cdot \tilde{\boldsymbol{v}}_N + \frac{1}{2} \int_0^T \int_{\Omega_f} \left(1 + \frac{\tau_{\Delta t}\omega_N}{R}\right) \Bigg[\left(\left(\tau_{\Delta t}\boldsymbol{u}_N - \zeta_N \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\tau_{\Delta t}\omega_N} \boldsymbol{u}_N \right) \cdot \tilde{\boldsymbol{v}}_N \\
&- \left(\left(\tau_{\Delta t}\boldsymbol{u}_N - \zeta_N \frac{R+y}{R} \boldsymbol{e}_y \right) \cdot \nabla_f^{\tau_{\Delta t}\omega_N} \tilde{\boldsymbol{v}}_N \right) \cdot \boldsymbol{u}_N \Bigg] + \frac{1}{2R} \int_0^T \int_{\Omega_f} \zeta_N \boldsymbol{u}_N \cdot \tilde{\boldsymbol{v}}_N \\
&+ \frac{1}{2} \int_0^T \int_\Gamma (\boldsymbol{u}_N - \zeta_N^* \boldsymbol{e}_y) \cdot \boldsymbol{n}^{\tau_{\Delta t}\omega_N} (\tau_{\Delta t}\boldsymbol{u}_N \cdot \tilde{\boldsymbol{v}}_N) + 2\nu \int_0^T \int_{\Omega_f} \left(1 + \frac{\tau_{\Delta t}\omega_N}{R}\right) \boldsymbol{D}_f^{\tau_{\Delta t}\omega_N}(\boldsymbol{u}_N) : \boldsymbol{D}_f^{\tau_{\Delta t}\omega_N}(\tilde{\boldsymbol{v}}_N) \\
&+ \int_0^T \int_\Gamma \left(\frac{1}{2} \boldsymbol{u}_N \cdot \tau_{\Delta t}\boldsymbol{u}_N - p_N \right) (\boldsymbol{\psi} - \tilde{\boldsymbol{v}}_N) \cdot \boldsymbol{n}^{\tau_{\Delta t}\omega_N} + \frac{\beta}{\mathcal{J}_\Gamma^{\tau_{\Delta t}\omega_N}} \int_0^T \int_\Gamma (\zeta_N^* \boldsymbol{e}_y - \boldsymbol{u}_N) \cdot \boldsymbol{\tau}^{\tau_{\Delta t}\omega_N} (\boldsymbol{\psi} - \tilde{\boldsymbol{v}}_N) \cdot \boldsymbol{\tau}^{\tau_{\Delta t}\omega_N} \\
&+ \rho_b \int_0^T \int_{\Omega_b} \left(\frac{\boldsymbol{\xi}_N - \tau_{\Delta t}\boldsymbol{\xi}_N}{\Delta t} \right) \cdot \boldsymbol{\psi} + \rho_p \int_0^T \int_\Gamma \partial_t \overline{\zeta}_N \cdot \varphi + 2\mu_e \int_0^T \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_N) : \boldsymbol{D}(\boldsymbol{\psi}) \\
&+ \lambda_e \int_0^T \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_N)(\nabla \cdot \boldsymbol{\psi}) + 2\mu_v \int_0^T \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}_N) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_0^T \int_{\Omega_b} (\nabla \cdot \boldsymbol{\xi}_N)(\nabla \cdot \boldsymbol{\psi}) \\
&- \alpha \int_0^T \int_{\Omega_b} \mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta} p_N \nabla_b^{(\tau_{\Delta t}\eta_N)^\delta} \cdot \boldsymbol{\psi} + c_0 \int_0^T \int_{\Omega_b} \partial_t \overline{p}_N \cdot r - \alpha \int_0^T \int_{\Omega_b} \mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta} \boldsymbol{\xi}_N \cdot \nabla_b^{(\tau_{\Delta t}\eta_N)^\delta} r \\
&- \alpha \int_0^T \int_\Gamma (\zeta_N^* \boldsymbol{e}_y \cdot \boldsymbol{n}^{(\tau_{\Delta t}\omega_N)^\delta}) r + \kappa \int_0^T \int_{\Omega_b} \mathcal{J}_b^{(\tau_{\Delta t}\eta_N)^\delta} \nabla_b^{(\tau_{\Delta t}\eta_N)^\delta} p_N \cdot \nabla_b^{(\tau_{\Delta t}\eta_N)^\delta} r \\
&- \int_0^T \int_\Gamma [(\boldsymbol{u}_N - \zeta_N^* \boldsymbol{e}_y) \cdot \boldsymbol{n}^{\tau_{\Delta t}\omega_N}] r + \int_0^T \int_\Gamma \Delta\omega_N \cdot \Delta\varphi = 0.
\end{aligned}
$$

Using the strong convergence results established above, combined with the previously established weak convergence results in Proposition 7.1, we can pass to the limit in all of the terms in the semidiscrete weak formulation except those involving time derivatives. However, we can handle these by a discrete integration by parts. For example, for the first integral, we can use a discrete integration by parts to obtain:

$$
\begin{aligned}
&\int_0^T \int_{\Omega_f} \left(1 + \frac{\tau_{\Delta t}\omega_N}{R}\right) \partial_t \overline{\boldsymbol{u}}_N \cdot \tilde{\boldsymbol{v}}_N \\
&\to - \int_0^T \int_{\Omega_f} \left(1 + \frac{\omega}{R}\right) \boldsymbol{u} \cdot \partial_t \tilde{\boldsymbol{v}} - \frac{1}{R} \int_0^T \int_{\Omega_f} (\partial_t \omega) \boldsymbol{u} \cdot \tilde{\boldsymbol{v}} - \int_{\Omega_f} \left(1 + \frac{\omega_0}{R}\right) \boldsymbol{u}(0) \cdot \tilde{\boldsymbol{v}}(0),
\end{aligned}
$$

where $\tilde{\boldsymbol{v}}_N = \boldsymbol{v} \circ \Phi_f^{\tau_{\Delta t}\omega_N}$ and $\tilde{\boldsymbol{v}} = \boldsymbol{v} \circ \Phi_f^{\omega}$ for $\boldsymbol{v} \in \mathcal{X}$. See for example pg. 79-81 in [62].

The limiting weak formulation holds for all velocity test functions in the smooth test space, which can be extended to the general test space $\mathcal{V}_{\text{test}}^{\omega}$ defined in (38) by using a density argument. Therefore, we have shown that the approximate weak solutions converge, up to a subsequence, to a weak solution to the regularized problem, as stated in Theorem 5.1.

This completes the main result of this manuscript, stated in Theorem 5.1 providing existence of a weak solution to the nonlinearly coupled, regularized fluid-poroviscoelastic structure interaction problem, given in Definition 5.5.

We conclude this section by making the important observation that the weak solution that we have constructed to the regularized FPSI problem satisfies the desired energy estimate. This will be important for showing weak-classical consistency in the next section, and can be shown easily by using the discrete energy estimate for the approximate solutions.

Proposition 9.3. (Energy estimate for the limiting solution to the regularized problem.) The weak solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ constructed from the splitting scheme as the limit of approximate solutions satisfies the following energy estimate for almost every $t \in [0, T]$:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_f(t)} |\boldsymbol{u}|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta})|^2 \\
&+ \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}|^2 + \frac{1}{2}\rho_p \int_{\Gamma} |\zeta|^2 + \frac{1}{2}\int_{\Gamma} |\Delta\omega|^2 + 2\nu \int_0^t \int_{\Omega_f(s)} |\boldsymbol{D}(\boldsymbol{u})|^2 \\
&+ 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi})|^2 + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}|^2 + \kappa \int_0^t \int_{\Omega_b^\delta(s)} |\nabla p|^2 + \beta \int_0^t \int_{\Gamma(s)} |(\zeta \boldsymbol{e}_y - \boldsymbol{u}) \cdot \boldsymbol{\tau})|^2 \leqslant E_0,
\end{aligned} \tag{94}$$

where E_0 is the initial energy of the problem.

Proof. The approximate solutions $(\boldsymbol{u}_N, \boldsymbol{\eta}_N, p_N, \omega_N)$ satisfy the following energy inequality:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_{f,N}(t)} |\boldsymbol{u}_N|^2 + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}_N|^2 + \frac{1}{2}c_0 \int_{\Omega_b} |p_N|^2 + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_N)|^2 \\
&+ \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_N|^2 + \frac{1}{2}\rho_p \int_{\Gamma} |\zeta_N|^2 + \frac{1}{2}\int_{\Gamma} |\Delta\omega_N|^2 + 2\nu \int_0^t \int_{\Omega_{f,N}(s)} |\boldsymbol{D}(\boldsymbol{u}_N)|^2 \\
&+ 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_N)|^2 + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}_N|^2 + \kappa \int_0^t \int_{\Omega_{b,N}^\delta(s)} |\nabla p_N|^2 + \beta \int_0^t \int_{\Gamma(s)} |(\zeta_N^* \boldsymbol{e}_y - \boldsymbol{u}_N) \cdot \boldsymbol{\tau}|^2 \leqslant E_0.
\end{aligned}$$

By using the weak and weak-star convergences of the approximate solutions, stated in Proposition 7.1 and lower semicontinuity, we can pass to the limit in the energy inequality, one recovers the energy inequality (94). ☐

10 Weak-classical consistency

We have now shown the existence of weak solutions to the regularized FPSI problem (40). However, it is not clear that the solutions to this regularized problem are physically relevant, since the regularized weak formulation is not equivalent to the original weak formulation without the regularization. However, we will demonstrate the following weak-classical consistency result: given a spatially and temporally smooth solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ to the FPSI problem, then the weak solutions to the regularized problem with regularization parameter δ, which we will denote by $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$, converge to the smooth solution as $\delta \to 0$.

10.1 Notation

Since we will have to use spatial convolution of the solution to the regularized problem $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$, and spatial convolution of the smooth solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$, we introduce the following notation to avoid additional superscripts involving δ.

1. Recall that $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$ denotes the weak solutions to the regularized problem (40);

2. We will use $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ to denote a spatially and temporally smooth solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ to the original FPSI problem (2), (3), (6), (7), (11)-(15);

3. We will use the superscript δ notation

$$\boldsymbol{\eta}_\delta^\delta = (\boldsymbol{\eta}_\delta)^\delta := \delta^{-2}\boldsymbol{\eta}_\delta * \sigma(x/\delta)$$

   to denote the spatial convolution defined by (23) of the weak solution to the regularized problem with the smooth convolution δ kernel;

4. Similarly, we will use

$$\boldsymbol{\eta}^\delta = (\boldsymbol{\eta})^\delta = \delta^{-2}\boldsymbol{\eta} * \sigma(x/\delta) \tag{95}$$

   to denote the spatial convolution of the classical solution $\boldsymbol{\eta}$ with the convolution kernel;

5. We will use superscript δ to denote the physical Biot domain under the regularized displacement:

$$\Omega_{b,\delta}^\delta(t) = (\boldsymbol{I} + \boldsymbol{\eta}_\delta^\delta(t))(\Omega_b). \tag{96}$$

Weak formulations reformulated. We note that even though the weak formulation (22) and the regularized weak formulation (40) are stated up until a fixed final time T, we can reformulate the weak formulation for almost every time $t \in [0, T]$ by using a cutoff function (see for example the proof of Lemma 12.2 in the appendix where this is done explicitly).

Thus, the **classical (temporally and spatially smooth) solution** $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ satisfies the following *non-regularized weak formulation* for almost all $t \in [0, T]$, for all test functions $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{V}_{\text{test}}$ with the (moving domain) test space $\mathcal{V}_{\text{test}}$ defined in (37):

$$\begin{aligned}
&- \int_0^t \int_{\Omega_f(s)} \boldsymbol{u} \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^t \int_{\Omega_f(s)} [((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}] + \frac{1}{2}\int_0^t \int_{\Gamma_1(s)} (\boldsymbol{u} \cdot \boldsymbol{n} - 2\boldsymbol{\xi}_1 \cdot \boldsymbol{n})\boldsymbol{u} \cdot \boldsymbol{v} \\
&+ 2\nu \int_0^t \int_{\Omega_f(s)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_0^t \int_{\Gamma_1(s)} \left(\frac{1}{2}|\boldsymbol{u}|^2 - p\right)(\psi_n - v_n) + \beta \int_0^t \int_{\Gamma_1(s)} (\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{t}(\psi_t - v_t) \\
&- \rho_p \int_0^t \int_\Gamma \partial_t \omega \cdot \partial_t \varphi + \int_0^t \int_\Gamma \Delta\omega \cdot \Delta\varphi - \rho_b \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta}_1 \cdot \partial_t \boldsymbol{\psi} + 2\mu_e \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_1) : \boldsymbol{D}(\boldsymbol{\psi}) \\
&+ \lambda_e \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_1)(\nabla \cdot \boldsymbol{\psi}) + 2\mu_v \int_0^t \int_{\Omega_b} \boldsymbol{D}(\partial_t \boldsymbol{\eta}) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_0^t \int_{\Omega_b} (\nabla \cdot \partial_t \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\psi}) \\
&- \alpha \int_0^t \int_{\Omega_b(s)} p \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^t \int_{\Omega_b} p \partial_t r - \alpha \int_0^t \int_{\Omega_b(s)} \frac{D}{Dt}\boldsymbol{\eta}_1 \cdot \nabla r - \alpha \int_0^t \int_{\Gamma(s)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{n}) r \\
&+ \kappa \int_0^t \int_{\Omega_b(s)} \nabla p \cdot \nabla r - \int_0^t \int_{\Gamma_1(s)} ((\boldsymbol{u} - \boldsymbol{\xi}_1) \cdot \boldsymbol{n}) r \\
&= - \int_{\Omega_f(t)} \boldsymbol{u}(t) \cdot \boldsymbol{v}(t) - \rho_p \int_\Gamma \zeta(t) \cdot \boldsymbol{\psi}(t) - \rho_b \int_{\Omega_b} \boldsymbol{\xi}(t) \cdot \boldsymbol{\psi}(t) - c_0 \int_{\Omega_b} p(t) \cdot r(t) \\
&+ \int_{\Omega_f(0)} \boldsymbol{u}_0 \cdot \boldsymbol{v}(0) + \rho_p \int_\Gamma \beta_0 \cdot \boldsymbol{\psi}(0) + \rho_b \int_{\Omega_b} \boldsymbol{\xi}_0 \cdot \boldsymbol{\psi}(0) + c_0 \int_{\Omega_b} p_0 \cdot r(0).
\end{aligned} \tag{97}$$

Similarly, the **solution to the regularized FPSI problem** $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$ satisfies the following *regularized weak formulation* for every test function $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) \in \mathcal{V}_{\text{test}}$, and for almost every

$t \in [0, T_\delta]$ where the final time T_δ potentially depends on δ:

$$
\begin{aligned}
&-\int_0^t \int_{\Omega_{f,\delta}(s)} \boldsymbol{u}_\delta \cdot \partial_t \boldsymbol{v} + \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(s)} [((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{u}_\delta) \cdot \boldsymbol{v} - ((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{v}) \cdot \boldsymbol{u}_\delta] \\
&+ \frac{1}{2}\int_0^t \int_{\Gamma_\delta(s)} (\boldsymbol{u}_\delta \cdot \boldsymbol{n} - 2\boldsymbol{\xi}_\delta \cdot \boldsymbol{n})\boldsymbol{u}_\delta \cdot \boldsymbol{v} + 2\nu \int_0^t \int_{\Omega_{f,\delta}(s)} \boldsymbol{D}(\boldsymbol{u}_\delta) : \boldsymbol{D}(\boldsymbol{v}) \\
&+ \int_0^t \int_{\Gamma_\delta(s)} \left(\frac{1}{2}|\boldsymbol{u}_\delta|^2 - p_\delta\right)(\psi_n - v_n) + \beta \int_0^t \int_{\Gamma_\delta(s)} (\boldsymbol{\xi}_\delta - \boldsymbol{u}_\delta) \cdot \boldsymbol{t}(\psi_t - v_t) \\
&- \rho_p \int_0^t \int_\Gamma \partial_t \omega_\delta \cdot \partial_t \varphi + \int_0^t \int_\Gamma \Delta\omega_\delta \cdot \Delta\varphi - \rho_b \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta}_\delta \cdot \partial_t \boldsymbol{\psi} + 2\mu_e \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_\delta) : \boldsymbol{D}(\boldsymbol{\psi}) \\
&+ \lambda_e \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_\delta)(\nabla \cdot \boldsymbol{\psi}) + 2\mu_v \int_0^t \int_{\Omega_b} \boldsymbol{D}(\partial_t \boldsymbol{\eta}_\delta) : \boldsymbol{D}(\boldsymbol{\psi}) + \lambda_v \int_0^t \int_{\Omega_b} (\nabla \cdot \partial_t \boldsymbol{\eta}_\delta)(\nabla \cdot \boldsymbol{\psi}) \\
&- \alpha \int_0^t \int_{\Omega_{b,\delta}^\delta(s)} p_\delta \nabla \cdot \boldsymbol{\psi} - c_0 \int_0^t \int_{\Omega_b} p_\delta \partial_t r - \alpha \int_0^t \int_{\Omega_{b,\delta}^\delta(s)} \frac{D^\delta}{Dt}\boldsymbol{\eta}_\delta \cdot \nabla r - \alpha \int_0^t \int_{\Gamma_\delta(s)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}) r \\
&+ \kappa \int_0^t \int_{\Omega_{b,\delta}^\delta(s)} \nabla p_\delta \cdot \nabla r - \int_0^t \int_{\Gamma_\delta(s)} ((\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n}) r \\
&= -\int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \boldsymbol{v}(t) - \rho_p \int_\Gamma \zeta_\delta(t) \cdot \varphi(t) - \rho_b \int_{\Omega_b} \boldsymbol{\xi}_\delta(t) \cdot \boldsymbol{\psi}(t) - c_0 \int_{\Omega_b} p_\delta(t) \cdot r(t) \\
&+ \int_{\Omega_f(0)} \boldsymbol{u}_0 \cdot \boldsymbol{v}(0) + \rho_p \int_\Gamma \beta_0 \cdot \varphi(0) + \rho_b \int_{\Omega_b} \boldsymbol{\xi}_0 \cdot \boldsymbol{\psi}(0) + c_0 \int_{\Omega_b} p_0 \cdot r(0), \qquad (98)
\end{aligned}
$$

where $\frac{D^\delta}{Dt}$ is the material derivative with respect to the regularized Biot displacement. We remark that while our existence proof in the previous sections holds for both a purely elastic and viscoelastic Biot medium, our weak-classical consistency result will hold in the specific case of a Biot *poroviscoelastic* medium so that the viscoelasticity parameters μ_v and λ_v are strictly positive, and hence, the plate velocity $\zeta_\delta \boldsymbol{e}_y$ in the weak formulation is equivalently the trace of the Biot medium velocity $\boldsymbol{\xi}_\delta \in L^2(0,T; H^1(\Omega_b))$ along Γ.

10.2 Statement of the result

In the remainder of the manuscript, we will prove the weak-classical consistency result. Before stating the result, we need to introduce some additional notation. Namely, to prove the weak-classical consistency, we will subtract the weak formulations for the two solutions $\boldsymbol{u}$ and $\boldsymbol{u}_\delta$ and test formally with the difference of the two solutions $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\delta$. However, the functions $\boldsymbol{u}$ and $\boldsymbol{u}_\delta$ are defined on different domains, and hence, the difference $\boldsymbol{u} - \boldsymbol{u}_\delta$ is not well-defined. Therefore, we will have to use a transformation to bring a divergence-free function defined on one fluid domain to a divergence-free function on another fluid domain.

For this purpose consider the two fluid domains

$$\Omega_f(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega(t,x)\},$$

$$\Omega_{f,\delta}(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega_\delta(t,x)\},$$

that are associated to the plate displacements ω and ω_δ.

We define a map between $\Omega_f(t)$ and $\Omega_{f,\delta}(t)$, and a transformation that sends functions on one domain to functions on the other domain as follows. Let $\psi_\delta(t) : \Omega_{f,\delta}(t) \to \Omega_f(t)$ be the mapping defined by

$$\psi_\delta(t,x,y) = (t,x,\gamma_\delta(t,x)(R+y)-R), \text{ where } \gamma_\delta(t,x) = \frac{R+\omega(t,x)}{R+\omega_\delta(t,x)}. \tag{99}$$

This mapping, unfortunately, does not preserve the divergence free condition. However, if we calculate the gradient of the composite mapped function we get

$$\nabla(\boldsymbol{u}\circ\psi_\delta) = [(\nabla\boldsymbol{u})\circ\psi_\delta]J_\delta \tag{100}$$

where

$$J_\delta(t,x,y) = \begin{pmatrix} 1 & 0 \\ (R+y)\partial_x\gamma_\delta(t,x) & \gamma_\delta(t,x)\end{pmatrix}. \tag{101}$$

Similarly, for the regularized problem we define

$$\tilde{J}_\delta = J_\delta\circ\psi_\delta^{-1} = \begin{pmatrix} 1 & 0 \\ (R+y)\gamma_\delta^{-1}\partial_x\gamma_\delta(t,x) & \gamma_\delta(t,x)\end{pmatrix}. \tag{102}$$

These Jacobian matrices will now be used to define the transformations that map divergence free functions to divergence free functions.

Definition 10.1. Part I: Given a divergence-free function $\boldsymbol{u}$ on $\Omega_f(t)$, the following transformation $\hat{}\, : \boldsymbol{u}\mapsto\hat{\boldsymbol{u}}$ maps $\boldsymbol{u}$ to a divergence free function $\hat{\boldsymbol{u}}$ on $\Omega_{f,\delta}(t)$:

$$\hat{\boldsymbol{u}} = \gamma_\delta J_\delta^{-1}\cdot(\boldsymbol{u}\circ\psi_\delta). \tag{103}$$

Part II: Given a divergence-free function $\boldsymbol{u}_\delta$ on $\Omega_{f,\delta}(t)$ the following transformation $\check{}\, : \boldsymbol{u}_\delta\mapsto\check{\boldsymbol{u}}_\delta$ maps $\boldsymbol{u}_\delta$ to a divergence free function $\check{\boldsymbol{u}}_\delta$ on $\Omega_f(t)$:

$$\check{\boldsymbol{u}}_\delta = \gamma_\delta^{-1}\tilde{J}_\delta\cdot(\boldsymbol{u}_\delta\circ\psi_\delta^{-1}). \tag{104}$$

Remark 10.1. Both transformations preserve the trace of functions along Γ.

Note that even though the definition of $\hat{\boldsymbol{u}}$ depends on δ, we will not explicitly notate this dependence, as δ will be clear from the context. We now state the weak-classical consistency result.

Theorem 10.1. (Weak-classical consistency) Let $(\boldsymbol{\eta}_0,\boldsymbol{\xi}_0,\omega_0,\zeta_0,p_0,\boldsymbol{u}_0)$ be smooth initial data for the nonlinearly coupled FPSI problem (2), (3), (6), (7), (11)-(15). Suppose $(\boldsymbol{\eta},\omega,p,\boldsymbol{u})$ is a classical (temporally and spatially smooth) solution to this FPSI problem on the time interval $[0,T]$. Let $(\boldsymbol{\eta}_\delta,\omega_\delta,p_\delta,\boldsymbol{u}_\delta)$ denote the weak solution to the regularized FPSI problem (40) with regularity parameter δ.

Then the following holds true:

1. $(\boldsymbol{\eta}_\delta,\omega_\delta,p_\delta,\boldsymbol{u}_\delta)$ is uniformly defined on the time interval $[0,T]$ for all $\delta>0$;
2. The energy norm of the difference between the two solutions $E_\delta(t)$ converges to zero as $\delta\to0$, for all $t\in[0,T]$, where

$$\begin{aligned} E_\delta(t) := {} & ||(\hat{\boldsymbol{u}}_1-\boldsymbol{u}_\delta)(t)||^2_{L^2(\Omega_{f,\delta}(t))} + \int_0^t ||\boldsymbol{D}(\hat{\boldsymbol{u}}_1-\boldsymbol{u}_\delta)(s)||^2_{L^2(\Omega_{f,\delta}(s))}ds \\ & + ||(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)(t)||^2_{L^2(\Gamma)} + ||(\omega-\omega_\delta)(t)||^2_{H^2(\Gamma)} + ||(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)(t)||^2_{L^2(\Omega_b)} \\ & + ||\boldsymbol{D}(\boldsymbol{\eta}-\boldsymbol{\eta}_\delta)(t)||^2_{L^2(\Omega_b)} + ||(\nabla\cdot(\boldsymbol{\eta}-\boldsymbol{\eta}_\delta))(t)||_{L^2(\Omega_b)} + \int_0^t ||\boldsymbol{D}(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)(s)||^2_{L^2(\Omega_b)}ds \\ & + \int_0^t ||\nabla\cdot(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)(s)||_{L^2(\Omega_b)} + ||(p-p_\delta)(t)||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla(p-p_\delta)(s)||^2_{L^2(\Omega^\delta_{b,\delta}(s))}. \end{aligned} \tag{105}$$

Preview of the main steps of the proof of weak-classical consistency. The proof is based on Gronwall's inequality for $E_\delta(t)$. However, there are several obstacles to applying Gronwall's inequality due to the fact that we are working on a moving domain problem. We summarize those main obstacles, and the main ideas behind their resolution here.

The main idea is to estimate the energy difference between $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ and $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$, defined in (105) and obtain an estimate for $E_\delta(t)$ in terms of $E_\delta(0)$, the integral of $E_\delta(s)$ for times $s \in [0, t]$, and other terms that have sufficiently strong convergence in δ as $\delta \to 0$:

$$E_\delta(t) \leqslant C\left(\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)} + \int_0^t E_\delta(t)ds\right) \tag{106}$$

and then apply Gronwall's inequality to obtain

$$E_\delta(t) \leqslant C\delta^3 e^{Ct},$$

where C is independent of δ, and conclude that $E_\delta(t) \to 0$ as $\delta \to 0$. We remark that the factor of δ^3 appearing in the Gronwall estimate comes from an estimate of the convergence rate of the spatial convolution $\boldsymbol{\eta}^\delta$ to $\boldsymbol{\eta}$ in $H^1(\Omega_b)$, which we establish in the upcoming Lemma 10.2.

To do this, we will test the weak formulations for $\boldsymbol{u}$ and $\boldsymbol{u}_\delta$ with appropriate test functions and use the energy inequality (9.3). More precisely, the main steps in the proof are:

1. Test the non-regularized weak formulation (97) for the classical solution $(\boldsymbol{u}, \boldsymbol{\eta}, p, \omega)$ with the "difference" of $(\boldsymbol{u}, \partial_t\boldsymbol{\eta}, p, \partial_t\omega)$ and $(\boldsymbol{u}_\delta, \partial_t\boldsymbol{\eta}_\delta, p_\delta, \partial_t\omega_\delta)$, where the notion of the difference between these two solutions will be made precise in Section 10.3;

2. Test the regularized weak formulation (98) for $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$ with $(\boldsymbol{u}, \partial_t\boldsymbol{\eta}, p, \partial_t\omega)$;

3. Rewrite the energy inequality (94) for $(\boldsymbol{u}_\delta, \boldsymbol{\eta}_\delta, p_\delta, \omega_\delta)$ so that it parallels the terms in the weak formulation (98);

4. Combine the equations from Step 1, Step 2, and Step 3. This will give us an expression that we can analyze term by term in order to obtain estimate (107) for the energy difference $E_\delta(t)$. Details will be presented in Section 10.3;

5. Construct a bootstrap argument. Namely, at a first pass, Gronwall's inequality is proven locally in time, namely, on the interval $[0, T_\delta]$ along which the assumptions on the determinant of the Jacobian of the transformation from the moving domain to the fixed domain is bounded for the solution of the regularized problem. However, we need the Gronwall's inequality to hold along the entire time interval $[0, T]$, along which the classical solution is defined. This will be done by a construction of a bootstrap argument, see Section 10.4.

6. Apply Gronwall's inequality to (107) holding on $[0, T]$ to obtain the following bound for $E_\delta(t)$:

   $$E_\delta(t) \leqslant C\delta^3 e^{Ct},$$

   where C is independent of δ, and conclude that $E_\delta(t) \to 0$ as $\delta \to 0$.

Before we start with the proof of weak-classical consistency, we emphasize that there are two main **mathematical difficulties** that need to be addressed in the proof:

1. In step 1 above, we want to test (97) with the difference of $(\boldsymbol{u}, \partial_t\boldsymbol{\eta}, p, \partial_t\omega)$ and $(\boldsymbol{u}_\delta, \partial_t\boldsymbol{\eta}_\delta, p_\delta, \partial_t\omega_\delta)$. This is formal because the test functions in $\mathcal{V}_{\text{test}}$, defined in (37), must be continuously differentiable in time, and furthermore, for the fluid velocities, the difference between $\boldsymbol{u}$ and $\boldsymbol{u}_\delta$ does not make sense, since these functions are defined on different fluid domains. Thus, we must carefully define which test functions we will use. This is addressed at the beginning of Section 10.3 below.

2. As mentioned in step 5 above, the regularized weak formulation involves integrals on the physical time-dependent Biot domain $\Omega^\delta_{b,\delta}(t)$, which give an extra factor of $\det(\boldsymbol{I} + \nabla\boldsymbol{\eta}^\delta_\delta)$ in the integrand from the Jacobian, when the integrals are transferred to the fixed reference Biot domain Ω_b. This factor cannot be estimated in the finite energy space where $\boldsymbol{\eta}_\delta$ is only bounded uniformly in δ in the function space $L^\infty(0,T;H^1(\Omega_b))$. To obtain pointwise estimates of this term that hold on the time interval $[0,T]$, where T is independent of δ, we need to use a *bootstrap argument* to get from the local pointwise estimates on $[0,T_\delta]$, where T_δ depends on δ, to the global, uniform estimates on $[0,T]$. This is addressed in Section 10.4 below.

10.3 Gronwall's Inequality

We show that the following Gronwall's inequality holds for almost all $t \in [0,T_\delta]$, where T_δ depends on δ. Later on we will use a bootstrap argument to show that the weak-classical consistency holds uniformly, on the entire interval $[0,T]$ on which the classical solution exists.

Lemma 10.1. Gronwall's estimate. Let (131), (132), and (133) hold for almost all $t \in [0,T_\delta]$. Furthermore, let $\boldsymbol{\eta}$ and $\boldsymbol{\eta}^\delta$ denote the smooth solution and its regularization, defined on $[0,T]$, and E_δ be the energy norm difference (105). Then the following inequality hold:

$$E_\delta(t) \leqslant C\left(\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)}ds + \int_0^t E_\delta(t)ds\right), \tag{107}$$

where $E_\delta(t)$ is defined by (105). Furthermore,

$$E_\delta(t) \leqslant C\delta^3 e^{Ct}.$$

To prove Gronwall's inequality, we want to test the non-regularized weak formulation formally with the difference between $(\boldsymbol{u}, \partial_t\boldsymbol{\eta}, p, \partial_t\omega)$ and $(\boldsymbol{u}_\delta, \partial_t\boldsymbol{\eta}_\delta, p_\delta, \partial_t\omega_\delta)$. However, there are two reasons why this is not rigorously justified. First, $\partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}_\delta$ is not a continuously differentiable function in time as is required for the test functions, and hence, we will use a *convolution in time* and pass to the limit as the convolution parameter goes to zero. Second, the fluid velocities give an additional difficulty, as the fluid velocities are defined on *time-dependent moving domains.* Thus, we must transfer the fluid velocities between different time-dependent domains in order to make sense of the "difference" between $\boldsymbol{u}$ and $\boldsymbol{u}_\delta$ as a test function. Furthermore, the way in which we do this transformation and the way in which we perform the convolution in time must both respect the divergence-free nature of the fluid velocity on the time-dependent domain. We will address both of these difficulties as follows.

Construction of appropriate test functions $(\boldsymbol{u}, \partial_t\boldsymbol{\eta}, p, \partial_t\omega) - (\boldsymbol{u}_\delta, \partial_t\boldsymbol{\eta}_\delta, p_\delta, \partial_t\omega_\delta)$:

Difficulty 1: Lack of regularity in time. We address the first difficulty by defining a convolution in time. This will allow us to regularize $\partial_t(\boldsymbol{\eta} - \boldsymbol{\eta}_\delta) = \boldsymbol{\xi} - \boldsymbol{\xi}_\delta$, $p - p_\delta$, and $\partial_t(\omega - \omega_\delta) = \zeta - \zeta_\delta$ so that these functions are continuously differentiable in time. Since the classical solution is already continuously differentiable in time, we only need to regularize the weak solutions to the regularized problem. Because these differences are all defined on fixed domains, we can use a standard convolution in time.

Convolution in time. Let $j(\cdot) : \mathbb{R} \to \mathbb{R}$ be a compactly supported even function with $\text{supp}(j) \subset [-1, 1]$ and $\displaystyle\int_{\mathbb{R}} j = 1$, and we define $j_\nu(t) = \nu^{-1} j(\nu^{-1} t)$, where $\nu > 0$ is the convolution parameter in time.

Consider $\nu > 0$. Extend $\boldsymbol{\xi}_\delta$, p_δ, and ζ_δ to the larger interval $[-\nu, T + \nu]$ by reflecting across $t = 0$ and $t = T$. For example, define:

$$\boldsymbol{\xi}_\delta(t) = \boldsymbol{\xi}_\delta(-t), \quad \text{for } t \in [-\nu, 0],$$
$$\boldsymbol{\xi}_\delta(t) = \boldsymbol{\xi}_\delta(2T - t), \quad \text{for } t \in [T, T + \nu].$$

Convolution in time is then defined by:

$$(\boldsymbol{\xi}_\delta)_\nu(t) = \boldsymbol{\xi}_\delta(t, \cdot) * j_\nu = \int_{\mathbb{R}} \boldsymbol{\xi}_\delta(s) j_\nu(t - s) ds, \text{ for } t \in [0, T].$$

The convolutions $(p_\delta)_\nu$ and $(\zeta_\delta)_\nu$ are defined similarly. With these definitions we can now test with $\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu$, $p - (p_\delta)_\nu$, and $\zeta - (\zeta_\delta)_\nu$.

Difficulty 2: Velocities are defined on moving domains. Because the fluid velocities are defined on moving time-dependent domains, we cannot directly apply a convolution in time. We must first be able to transform fluid velocities from one domain to another, while preserving the divergence-free condition, and then convolve in time. The transformation of fluid velocities from one domain to another, while preserving the divergence-free condition, will be performed using the following matrix:

$$K(s, t, x, y) = \begin{pmatrix} \frac{R + \omega(s,x)}{R + \omega(t,x)} & 0 \\ -(R + y)\partial_x \left(\frac{R + \omega(s,x)}{R + \omega(t,x)} \right) & 1 \end{pmatrix}. \tag{108}$$

This matrix has the following essential property: if $\boldsymbol{u}(x, y)$ is a divergence-free function on the domain $\Omega_f(s)$ defined by the structure displacement $\omega(s, x)$, then the function

$$K(s, t, x, y)\boldsymbol{u} \left(x, \frac{R + \omega(s, x)}{R + \omega(t, x)} (R + y) - R \right)$$

is a divergence-free vector field on the domain $\Omega_f(t)$ defined by the structure displacement $\omega(t, x)$.

Combined transformation of fluid velocities and convolution in time: We can now use this transformation to convolve in time, as follows. We extend $\boldsymbol{u}_\delta$ to $[-\nu, T + \nu]$ by reflection, as above, and define, for $t \in [0, T]$,

$$(\boldsymbol{u}_\delta)_\nu(t) = \int_{\mathbb{R}} K_{2,\delta}(s, t, x, y)\boldsymbol{u}_\delta \left(s, x, \frac{R + \omega_\delta(s, x)}{R + \omega_\delta(t, x)} (R + y) - R \right) j_\nu(t - s) ds. \tag{109}$$

For a divergence-free function $\boldsymbol{v}$, extended as above in time to $[-\nu, T + \nu]$, we can define $\boldsymbol{v}_\nu$ on $\Omega_f(t)$ analogously by

$$\boldsymbol{v}_\nu(t) = \int_{\mathbb{R}} K_1(s, t, x, y)\boldsymbol{v} \left(s, x, \frac{R + \omega(s, x)}{R + \omega(t, x)} (R + y) - R \right) j_\nu(t - s) ds.$$

Here, $K_1(s, t, x, y)$ and $K_{2,\delta}(s, t, x, y)$ are defined as $K(s, t, x, y)$ with the choices of $\omega = \omega$ and $\omega = \omega_\delta$ respectively. An example of such a function $\boldsymbol{v}$ which will be convenient to consider on $\Omega_f(t)$

is the function $\breve{\boldsymbol{u}}_\delta$ defined on $\Omega_f(t)$, which is the function $\boldsymbol{u}_\delta$ defined on $\Omega_{f,\delta}(t)$ transferred in a divergence-free manner, as described above, onto the domain $\Omega_f(t)$. Specifically,

$$\breve{\boldsymbol{u}}_\delta(t,x,y) = \begin{pmatrix} \frac{R+\omega_\delta(t,x)}{R+\omega(t,x)} & 0 \\ -(R+y)\partial_x\left(\frac{R+\omega_\delta(t,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix} \cdot \boldsymbol{u}\left(x, \frac{R+\omega_\delta(t,x)}{R+\omega(t,x)}(R+y) - R\right).$$

We present the main properties of $(\boldsymbol{u}_\delta)_\nu$ in the proposition below, which are a specific case of Lemma 2.6 in [57].

Proposition 10.1. Fix an arbitrary $\delta > 0$. Given $\boldsymbol{u}_\delta \in L^2(0,T;H^1(\Omega_b(t))$ and $\omega, \omega_\delta \in H_0^2(\Gamma)$, the following properties hold:

- Divergence-free condition: $\operatorname{div}[(\boldsymbol{u}_\delta)_\nu] = 0$ and $\operatorname{div}[(\breve{\boldsymbol{u}}_\delta)_\nu] = 0$, $\forall \nu > 0$ and $\forall t \in [0,T]$;

- Convergence properties:

$$\begin{aligned}
&(\boldsymbol{u}_\delta)_\nu \to \boldsymbol{u}_\delta && \text{strongly in } L^p(0,T;L^q(\Omega_{f,\delta}(t))), && \text{for all } p\in[1,\infty), q\in[1,2),\\
&(\breve{\boldsymbol{u}}_\delta)_\nu \to \breve{\boldsymbol{u}}_\delta && \text{strongly in } L^p(0,T;L^q(\Omega_{f,1}(t))), && \text{for all } p\in[1,\infty), q\in[1,2),\\
&(\boldsymbol{u}_\delta)_\nu \rightharpoonup \boldsymbol{u}_\delta && \text{weakly in } L^2(0,T;W^{1,p}(\Omega_{f,\delta}(t))), && \text{for all } p\in[1,2),\\
&(\breve{\boldsymbol{u}}_\delta)_\nu \rightharpoonup \breve{\boldsymbol{u}}_\delta && \text{weakly in } L^2(0,T;W^{1,p}(\Omega_{f,1}(t))), && \text{for all } p\in[1,2).
\end{aligned}$$

Proof. **(Proof of Gronwall's estimate.)**

We begin by testing the weak formulation (97) for the classical solution $(\boldsymbol{u},\boldsymbol{\eta},p,\omega)$ to the original non-regularized problem with

$$\boldsymbol{v} = \boldsymbol{u} - (\breve{\boldsymbol{u}}_\delta)_\nu, \quad \varphi = \zeta - (\zeta_\delta)_\nu, \quad \boldsymbol{\psi} = \boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu, \quad r = p - (p_\delta)_\nu, \tag{110}$$

and then test the regularized weak formulations (98) for the weak solutions $(\boldsymbol{u}_\delta,\boldsymbol{\eta}_\delta,p_\delta,\omega_\delta)$ with

$$\boldsymbol{v} = \widehat{\boldsymbol{u}}, \quad \varphi = \zeta, \quad \boldsymbol{\psi} = \boldsymbol{\xi}, \quad r = p. \tag{111}$$

Next, we rewrite the energy estimate in Proposition 9.3, which holds for the function $\boldsymbol{u}_\delta$, in a more convenient form by adding extra terms that will cancel out, in order to have the energy inequality parallel the weak formulation term by term. In particular, we have that for almost every $t \in [0,T_\delta]$,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_{f,\delta}(t)} |\boldsymbol{u}_\delta|^2 + \frac{1}{2}\int_0^t \int_{\Omega_f(s)} [((\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta)\cdot\boldsymbol{u}_\delta - ((\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta] + \frac{1}{2}\int_0^t\int_{\Gamma_\delta(s)} (\boldsymbol{u}_\delta\cdot\boldsymbol{n} - 2\boldsymbol{\xi}_\delta\cdot\boldsymbol{n})\boldsymbol{u}_\delta\cdot\boldsymbol{u}_\delta\\
&+ 2\nu\int_0^t\int_{\Omega_{f,\delta}(s)} |\boldsymbol{D}(\boldsymbol{u}_\delta)|^2 + \int_0^t\int_{\Gamma_\delta(s)} \left(\frac{1}{2}|\boldsymbol{u}_\delta|^2 - p_\delta\right)(\boldsymbol{\xi}_\delta - \boldsymbol{u}_\delta)\cdot\boldsymbol{n} + \beta\int_0^t\int_{\Gamma_\delta(s)} |(\boldsymbol{\xi}_\delta - \boldsymbol{u}_\delta)\cdot\boldsymbol{t})|^2\\
&\qquad + \frac{1}{2}\rho_p\int_\Gamma |\boldsymbol{\xi}_\delta|^2 + \frac{1}{2}\int_\Gamma |\Delta\omega_\delta|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\boldsymbol{\xi}_\delta|^2 + \mu_e\int_0^t\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_\delta)(s)|^2\\
&\qquad + \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_\delta(s)|^2 + 2\mu_v\int_0^t\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\xi}_\delta)|^2 + \lambda_v\int_0^t\int_{\Omega_b}|\nabla\cdot\boldsymbol{\xi}_\delta|^2\\
&- \alpha\int_0^t\int_{\Omega^\delta_{b,\delta}(s)} p_\delta\nabla\cdot\boldsymbol{\xi}_\delta + \frac{1}{2}c_0\int_0^t\int_{\Omega_b}|p_\delta(s)|^2 - \alpha\int_0^t\int_{\Omega^\delta_{b,\delta}(s)} \frac{D^\delta}{Dt}\boldsymbol{\eta}_\delta\cdot\nabla p_\delta - \alpha\int_0^t\int_{\Gamma^\delta_\delta(s)}(\boldsymbol{\xi}_\delta\cdot\boldsymbol{n})p_\delta\\
&\quad + \kappa\int_0^t\int_{\Omega^\delta_{b,\delta}(s)}|\nabla p_\delta|^2 - \int_0^t\int_{\Gamma_\delta(s)}((\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta)\cdot\boldsymbol{n})p_\delta \leqslant \frac{1}{2}\int_{\Omega_f(0)}|\boldsymbol{u}_0|^2 + \frac{1}{2}\rho_p\int_\Gamma|\boldsymbol{\xi}_0|^2\\
&\quad + \frac{1}{2}\int_\Gamma|\Delta\omega_0|^2 + \frac{1}{2}\rho_b\int_{\Omega_b}|\boldsymbol{\xi}_0|^2 + \mu_e\int_{\Omega_b}|\boldsymbol{D}(\boldsymbol{\eta}_0)|^2 + \frac{1}{2}\lambda_e\int_{\Omega_b}|\nabla\cdot\boldsymbol{\eta}_0|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_0|^2.
\end{aligned} \tag{112}$$

Finally, we combine the weak formulation for $\boldsymbol{u}$ tested with (110), subtract the regularized weak formulation for $\boldsymbol{u}_\delta$ tested with (111), and add the energy estimate (112) for $\boldsymbol{u}_\delta$ to obtain an expression of the form

$$\sum_{i=1}^{18} T_i \leqslant 0, \tag{113}$$

where the terms T_i are given below. We have to estimate each term, and the combined estimate will give the Gronwall's inequality (107). To make this section more concise, we summarize the final estimates here, and present details of the derivation of these terms and the estimates in Appendix 12.2.

Term T1. Term T_1 is defined as follows:

$$T_1 = -\int_0^t \int_{\Omega_f(t)} \boldsymbol{u} \cdot \partial_t [\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu] - \frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n}) \boldsymbol{u} \cdot [\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu] + \int_{\Omega_f(t)} \boldsymbol{u}(s) \cdot [\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu](s) \tag{114}$$

$$- \int_{\Omega_f(0)} \boldsymbol{u}(0) \cdot [\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu](0) - \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t \widehat{\boldsymbol{u}} - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}_\delta) \boldsymbol{u}_\delta \cdot \widehat{\boldsymbol{u}}$$
$$+ \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(s) \cdot \widehat{\boldsymbol{u}}(s) - \int_{\Omega_f(0)} \boldsymbol{u}_\delta(0) \cdot \widehat{\boldsymbol{u}}(0) + \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |\boldsymbol{u}_\delta(s)|^2 - \frac{1}{2}\int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_0|^2. \tag{115}$$

This term is estimated so that after taking the limit as $\nu \to 0$, the contribution of this term becomes

$$T_1 = \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)(t)|^2 + R_1,$$

where

$$|R_1| \leqslant \epsilon \int_0^T ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))}$$
$$+ C(\epsilon)\left(\int_0^T ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^T ||\partial_t \omega - \partial_t \omega_\delta||^2_{L^2(\Gamma)} + \int_0^T ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).$$

Term T2. Term T_2 is defined as follows:

$$T_2 = \frac{1}{2}\int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) \cdot [\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu] - \frac{1}{2}\int_0^t \int_{\Omega_f(t)} (\boldsymbol{u} \cdot \nabla)[\boldsymbol{u} - (\check{\boldsymbol{u}}_\delta)_\nu] \cdot \boldsymbol{u}$$
$$- \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} ((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{u}_\delta) \cdot (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta) + \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} ((\boldsymbol{u}_\delta \cdot \nabla)(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)) \cdot \boldsymbol{u}_\delta. \tag{116}$$

After taking the limit $\nu \to 0$, term T_2 can be estimated as follows:

$$|T_2| \leqslant \epsilon \int_0^t ||\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).$$

Term T3. Term T_3 is defined as follows:

$$
\begin{aligned}
T_3 =& \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{u}\cdot\boldsymbol{n}-\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu] - \frac{1}{2}\int_0^t\int_{\Gamma_\delta(t)} (\boldsymbol{u}_\delta\cdot\boldsymbol{n}_\delta-\boldsymbol{\xi}_\delta\cdot\boldsymbol{n}_\delta)\boldsymbol{u}_\delta\cdot\widehat{\boldsymbol{u}} \\
&+ \frac{1}{2}\int_0^t\int_{\Gamma(t)} |\boldsymbol{u}|^2(\boldsymbol{\xi}\cdot\boldsymbol{n}-\boldsymbol{u}\cdot\boldsymbol{n}) - \frac{1}{2}\int_0^t\int_{\Gamma(t)} |\boldsymbol{u}|^2[(\boldsymbol{\xi}_\delta)_\nu\cdot\boldsymbol{n}-(\breve{\boldsymbol{u}}_\delta)_\nu\cdot\boldsymbol{n}] - \frac{1}{2}\int_0^t\int_{\Gamma_\delta(t)} |\boldsymbol{u}_\delta|^2(\boldsymbol{\xi}\cdot\boldsymbol{n}_\delta-\widehat{\boldsymbol{u}}\cdot\boldsymbol{n}_\delta) \\
&+ \frac{1}{2}\int_0^t\int_{\Gamma_\delta(t)} (\boldsymbol{u}_\delta\cdot\boldsymbol{n}-\boldsymbol{\xi}_\delta\cdot\boldsymbol{n})|\boldsymbol{u}_\delta|^2 + \frac{1}{2}\int_0^t\int_{\Gamma_\delta(t)} |\boldsymbol{u}_\delta|^2(\boldsymbol{\xi}_\delta\cdot\boldsymbol{n}-\boldsymbol{u}_\delta\cdot\boldsymbol{n}). \qquad (117)
\end{aligned}
$$

After taking the limit $\nu \to 0$, term $T3$ can be estimated as follows:

$$
\begin{aligned}
|T_3| \leqslant& \epsilon\int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} \\
&+ C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi}-\boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).
\end{aligned}
$$

Term T4. Term T_4 is defined as follows:

$$
T_4 = 2\nu\int_0^t\int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu) - 2\nu\int_0^t\int_{\Omega_{f,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_\delta):\boldsymbol{D}(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta). \qquad (118)
$$

After taking the limit $\nu \to 0$, term $T4$ can be estimated as follows:

$$
T_4 = 2\nu\int_0^t\int_{\Omega_{f,\delta}(t)} |\boldsymbol{D}(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)|^2 + R_4,
$$

where

$$
|R_4| \leqslant \epsilon\int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).
$$

Term T5. Term T_5 is defined as follows:

$$
T_5 = \beta\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}-\boldsymbol{u})_t[(\boldsymbol{\xi}-(\boldsymbol{\xi}_\delta)_\nu)_t-(\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu)_t] - \beta\int_0^t\int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta-\boldsymbol{u}_\delta)_t[(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)_t-(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)_t]
$$

After taking the limit $\nu \to 0$, term T_5 can be estimated as follows:

$$
T_5 = \beta\int_0^t\int_{\Gamma_\delta(t)} |(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)_t-(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)_t|^2 + R_5,
$$

where

$$
|R_5| \leqslant \epsilon\int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi}-\boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)}\right).
$$

Terms T6-T8. Terms T_6-T_8 are defined as follows:

$$T_6 = -\rho_p \int_0^t \int_\Gamma \zeta \cdot \partial_t \left[\zeta - (\zeta_\delta)_\nu\right] + \rho_p \int_\Gamma \zeta(s) \cdot \left[\zeta(s) - (\zeta_\delta)_\nu(s)\right] - \rho_p \int_\Gamma \zeta(0) \cdot \left[\zeta(0) - (\zeta_\delta)_\nu(0)\right]$$
$$+ \rho_p \int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t \zeta - \rho_p \int_\Gamma \zeta_\delta(s) \cdot \zeta(s) + \rho_p \int_\Gamma \zeta_\delta(0) \cdot \zeta(0) + \frac{1}{2}\rho_p \int_\Gamma |\zeta_\delta(s)|^2 - \frac{1}{2}\rho_p \int_\Gamma |\zeta_0|^2. \tag{119}$$

$$T_7 = \int_0^t \int_\Gamma \Delta\omega \cdot \Delta\left[\zeta - (\zeta_\delta)_\nu\right] - \int_0^t \int_\Gamma \Delta\omega_\delta \cdot \Delta\zeta + \frac{1}{2}\int_\Gamma |\Delta\omega_\delta(s)|^2 - \frac{1}{2}\int_\Gamma |\Delta\omega_0|^2. \tag{120}$$

$$T_8 = -\rho_b \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta} \cdot \partial_t \left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right] + \rho_b \int_{\Omega_b} \boldsymbol{\xi}(s) \cdot \left[\boldsymbol{\xi}(s) - (\boldsymbol{\xi}_\delta)_\nu(s)\right] - \rho_b \int_{\Omega_b} \boldsymbol{\xi}(0) \cdot \left[\boldsymbol{\xi}(0) - (\boldsymbol{\xi}_\delta)_\nu(0)\right]$$
$$+ \rho_b \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta}_\delta \cdot \partial_t \boldsymbol{\xi} - \rho_b \int_{\Omega_b} \boldsymbol{\xi}_\delta(s) \cdot \boldsymbol{\xi}(s) + \rho_b \int_{\Omega_b} \boldsymbol{\xi}_\delta(0) \cdot \boldsymbol{\xi}(0) + \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}_\delta(s)|^2 - \frac{1}{2}\rho_b \int_{\Omega_b} |\boldsymbol{\xi}_0|^2. \tag{121}$$

After taking the limit $\nu \to 0$, the terms T_6-T_8 become:

$$T_6 = \frac{1}{2}\rho_p \int_\Gamma |(\zeta - \zeta_2)(t)|^2, \quad T_7 = \frac{1}{2}\int_\Gamma |\Delta(\omega - \omega_2)(t)|^2, \quad T_8 = \frac{1}{2}\rho_b \int_{\Omega_b} |(\boldsymbol{\xi} - \boldsymbol{\xi}_2)(t)|^2.$$

Terms T9-T12. Terms T_9-T_{12} are defined as follows:

$$T_9 = 2\mu_e \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}) : \boldsymbol{D}\left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right] - 2\mu_e \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\eta}_\delta) : \boldsymbol{D}(\boldsymbol{\xi}) + \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_\delta)(s)|^2 - \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta}_0)|^2.$$
$$T_{10} = \lambda_e \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta})\left(\nabla \cdot \left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right]\right) - \lambda_e \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\eta}_\delta)(\nabla \cdot \boldsymbol{\xi}) + \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_\delta(s)|^2 - \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot \boldsymbol{\eta}_0|^2.$$
$$T_{11} = 2\mu_v \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}) : \boldsymbol{D}\left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right] - 2\mu_v \int_0^t \int_{\Omega_b} \boldsymbol{D}(\boldsymbol{\xi}_\delta) : \boldsymbol{D}(\boldsymbol{\xi}) + 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi}_\delta)|^2.$$
$$T_{12} = \lambda_v \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\xi})\left(\nabla \cdot \left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right]\right) - \lambda_v \int_0^t \int_{\Omega_b} (\nabla \cdot \boldsymbol{\xi}_\delta)(\nabla \cdot \boldsymbol{\xi}) + \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot \boldsymbol{\xi}_\delta|^2. \tag{122}$$

Because $\boldsymbol{\xi}_\delta \in L^2(0,T;H^1(\Omega_b))$ where Ω_b is a fixed domain, we have that $(\boldsymbol{\xi}_\delta)_\nu \to \boldsymbol{\xi}_\delta$ strongly in $L^2(0,T;H^1(\Omega_b))$. Hence, as $\nu \to 0$, we have that Terms 9-12 converge to the following:

$$T_9 = \mu_e \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\eta} - \boldsymbol{\eta}_\delta)(t)|^2, \qquad T_{10} = \frac{1}{2}\lambda_e \int_{\Omega_b} |\nabla \cdot (\boldsymbol{\eta} - \boldsymbol{\eta}_\delta)(t)|^2,$$

$$T_{11} = 2\mu_v \int_0^t \int_{\Omega_b} |\boldsymbol{D}(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)|^2, \qquad T_{12} = \lambda_v \int_0^t \int_{\Omega_b} |\nabla \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)|^2.$$

Term T13. Term T_{13} is defined as follows:

$$T_{13} = -\alpha \int_0^t \int_{\Omega_b(t)} p\left(\nabla \cdot \left[\boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu\right]\right) + \alpha \int_0^t \int_{\Omega^\delta_{b,\delta}(t)} p_\delta \left(\nabla \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)\right).$$

After taking the limit $\nu \to 0$, term T_{13} can be estimated as follows:

$$
\begin{aligned}
|T_{13}| \leqslant & C(\epsilon)\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \epsilon\int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||^2_{L^2(\Omega_b)} \\
& + C(\epsilon)\left(\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||p - p_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}
$$

Term T14. Term T_{14} is defined as follows:

$$
\begin{aligned}
T_{14} = & - c_0\int_0^t\int_{\Omega_b} p \cdot \partial_t\left[p - (p_\delta)_\nu\right] + c_0\int_{\Omega_b} p(s)\cdot[p(s) - (p_\delta)_\nu(s)] - c_0\int_{\Omega_b} p_0\cdot[p(0) - (p_\delta)_\nu(0)] \\
& + c_0\int_0^t\int_{\Omega_b} p_\delta\cdot\partial_t p - c_0\int_{\Omega_b} p_\delta(s)\cdot p(s) + c_0\int_{\Omega_b}|p_0|^2 + \frac{1}{2}c_0\int_{\Omega_b}|p_\delta(s)|^2 - \frac{1}{2}c_0\int_{\Omega_b}|p_0|^2. \quad (123)
\end{aligned}
$$

This term can be handled in the same way as Terms 6-8. In the limit as $\nu \to 0$, the contribution from this term is

$$
T_{14} = \frac{1}{2}c_0\int_{\Omega_b}|(p - p_\delta)(\tau)|^2.
$$

Term T15. Term T_{15} is defined as follows:

$$
T_{15} = - \alpha\int_0^t\int_{\Omega_b(t)} \boldsymbol{\xi}\cdot\nabla\left[p - (p_\delta)_\nu\right] + \alpha\int_0^t\int_{\Omega^\delta_{b,\delta}(t)} \boldsymbol{\xi}_\delta\cdot\nabla(p - p_\delta). \quad (124)
$$

After taking the limit $\nu \to 0$, term T_{15} can be estimated as follows:

$$
\begin{aligned}
|T_{15}| \leqslant & \epsilon\int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))} + C(\epsilon)\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} \\
& + C(\epsilon)\left(\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}
$$

Term T16. Term T_{16} is defined as follows:

$$
T_{16} = - \alpha\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\left[p - (p_\delta)_\nu\right] + \alpha\int_0^t\int_{\Gamma^\delta_\delta(t)} (\boldsymbol{\xi}_\delta\cdot\boldsymbol{n})(p - p_\delta). \quad (125)
$$

After passing to the limit as $\nu \to 0$, this term can be estimated as follows:

$$
\begin{aligned}
|T_{16}| \leqslant & \ \epsilon\left(\int_0^t ||\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\xi}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla p - \nabla p_\delta||^2_{L^2(\tilde{\Omega}_{b,2,\delta}(t))}\right) + C(\epsilon)\left(\int_0^t ||p - p_\delta||^2_{L^2(\Omega_b)}\right. \\
& \left. + \int_0^t ||\nabla\boldsymbol{\eta} - \nabla\tilde{\boldsymbol{\eta}}_1||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}
$$

Term T17. Term T_{17} is defined as follows:

$$
T_{17} = \kappa\int_0^t\int_{\Omega_b(t)} \nabla p\cdot\nabla\left[p - (p_\delta)_\nu\right] - \kappa\int_0^t\int_{\Omega^\delta_{b,\delta}(t)} \nabla p_\delta\cdot\nabla(p - p_\delta). \quad (126)
$$

This term can be estimated as follows:

$$T_{17} \leqslant \kappa \int_0^t \int_{\Omega^\delta_{b,\delta}(t)} |\nabla(p - p_\delta)|^2 + R_{17},$$

where the remainder is bounded by

$$\begin{aligned}|R_{17}| \leqslant & \epsilon \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} \\ & + C(\epsilon) \left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} \right).\end{aligned}$$

Term T18. Term T_{18} is defined as follows:

$$\begin{aligned}T_{18} = & \int_0^t \int_{\Gamma(t)} p(\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma(t)} p[(\boldsymbol{u}_\delta)_\nu - (\boldsymbol{\xi}_\delta)_\nu] \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma_\delta(t)} p_\delta (\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n} \\ & + \int_0^t \int_{\Gamma_\delta(t)} p_\delta(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma(t)} ((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n})[p - (p_\delta)_\nu] + \int_0^t \int_{\Gamma_\delta(t)} ((\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n})(p - p_\delta).\end{aligned}$$

This term can be estimated as follows:

$$\begin{aligned}|T_{18}| \leqslant & \epsilon \left(\int_0^t ||\boldsymbol{D}(\hat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||_{L^2(\Omega_b)} + \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} \right) \\ & + C(\epsilon) \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}.\end{aligned}$$

The combined estimates for the terms T_1-T_{18} give the estimate (107). This finishes the proof of the Gronwall's estimate presented in Lemma 10.1. $\square$

All that is left to show to complete the proof of weak-classical consistency stated in Theorem 10.1, is to argue that Gronwall's inequality (107) holds for all $t \in [0, T]$ where T is independent of δ. This will also imply the first statement in the theorem, which states that $(\boldsymbol{\eta}_\delta, \omega_\delta, p_\delta, \boldsymbol{u}_\delta)$ is uniformly defined on the time interval $[0, T]$ for all $\delta > 0$. In order to do this we use a bootstrap argument presented in the next subsection.

10.4 Bootstrap argument

To obtain the desired Gronwall estimate, we need the following uniform bounds on the factor $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}^\delta_\delta)$, which appears in the regularized weak formulation (40) defined on the fixed reference domain Ω_b:

$$\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}^\delta_\delta) \geqslant c, \tag{127}$$

$$0 < c \leqslant |\boldsymbol{I} + \nabla \boldsymbol{\eta}^\delta_\delta| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}, \tag{128}$$

$$|\nabla \boldsymbol{\eta}^\delta_\delta| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}, \tag{129}$$

which need to hold *for all* $t \in [0, T]$ *where* $T > 0$ *is independent of* δ. Notice that we only have uniform boundedness of $\boldsymbol{\eta}_\delta$ with respect to δ in $L^\infty(0, T; H^1(\Omega_b))$, which implies that $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}^\delta_\delta)$ is uniformly bounded with respect to δ only in $L^\infty(0, T; L^1(\Omega_b))$, which is insufficient for estimating any integrands with this factor.

To get around this difficulty we use the following strategy. Recall that by the way the weak solution to the regularized problem was constructed using the splitting scheme, we have that there exists a sufficiently small constant c (uniform in δ) such that

$$\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta) \geqslant c > 0, \tag{130}$$

for all $t \in [0, T_\delta]$ where $T_\delta > 0$ *may depend on* δ. This estimate holds at least locally, although not locally uniformly, for each $\delta > 0$. In fact, similarly, the following three estimates hold locally, for $t \in [0, T_\delta]$, where T_δ *may depend on* δ, with positive constants c and C that are independent of δ:

$$\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta) \geqslant c, \tag{131}$$

$$0 < c \leqslant |\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}, \tag{132}$$

$$|\nabla \boldsymbol{\eta}_\delta^\delta| \leqslant C, \qquad \text{pointwise in } \overline{\Omega_b}. \tag{133}$$

These estimates imply that for sufficiently small $c > 0$, the following inequality also holds locally, for all $t \in [0, T_\delta]$:

$$0 < C^{-1} \leqslant |(\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^{-1}| \leqslant c^{-1}. \tag{134}$$

Let $[0, T]$ denote the time interval on which the classical solution $\boldsymbol{\eta}$ exists. Then, we can choose $c > 0$ and $C > 0$ so that the inequalities (130)-(134) also hold for the classical solution *for all* $t \in [0, T]$.

We will now show how to use a bootstrap argument to deduce that the time interval on which estimates (130)-(134) hold for the regularized weak solution of the regularized problem $\boldsymbol{\eta}_\delta^\delta$ can, in fact, be extended to the entire interval $[0, T]$, namely, that estimates (130)-(134) hold globally, uniformly in δ, where T is independent of δ.

This will follow if we can show that $\nabla \boldsymbol{\eta}$ and $\nabla \boldsymbol{\eta}_\delta^\delta$ are pointwise uniformly "close", i.e.,

$$|(\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta^\delta)(t, x)| \to 0 \text{ pointwise uniformly in } [0, T] \times \Omega_b \text{ as } \delta \to 0. \tag{135}$$

To obtain this estimate we start with the main proof of Gronwall's inequality *under the assumptions that* (131), (132), *and* (133) *are locally valid for* $t \in [0, T_\delta]$:

$$E_\delta(t) \leqslant C_1 \int_0^t ||(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(s)||^2_{H^1(\Omega_b)} ds + C_2 \int_0^t E_\delta(s) ds, \tag{136}$$

where the constants C_1 and C_2 are independent of δ. Then, by Lemma 10.2 below, we obtain that the first term on the right hand-side above can be estimated as follows:

$$||\boldsymbol{\eta}^\delta - \boldsymbol{\eta}||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}, \qquad \text{for all } t \in [0, T],$$

since the classical solution $\boldsymbol{\eta}$ is spatially smooth, and $\boldsymbol{\eta}^\delta$ is the convolution of $\boldsymbol{\eta}$ with the smooth δ kernel, defined in (95). With this essential observation, the Gronwall estimate based on (136) gives

$$E_\delta(t) \leqslant C_1 \left(\int_0^t ||(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(s)||^2_{H^1(\Omega_b)} ds \right) e^{C_2 t} \leqslant C_1 \left(\int_0^T ||(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(s)||^2_{H^1(\Omega_b)} ds \right) e^{C_2 t} \leqslant C\delta^3 e^{C_2 t}.$$

By the definition of $E_\delta(t)$ and an application of Poincare's and Korn's inequalities on Ω_b, see Proposition 6.1, this implies that the following terms in the definition of $E_\delta(t)$

$$||(\boldsymbol{\eta} - \boldsymbol{\eta}_\delta)(t)||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}, \text{ and } ||(\omega - \omega_\delta)(t)||_{H^2(\Gamma)} \leqslant C\delta^{3/2} \to 0 \text{ as } \delta \to 0$$

converge to zero as $\delta \to 0$ at a rate of $\delta^{3/2}$, as long as the assumptions (131), (132), and (133) hold. Therefore, by Hölder's inequality, for sufficiently small $\delta > 0$, we can prove that the following estimate holds:

$$|(\nabla \boldsymbol{\eta}^{\delta} - \nabla \boldsymbol{\eta}_{\delta}^{\delta})(t,x)| = \left| \int_{\tilde{\Omega}_b} (\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_{\delta})(t,y) \sigma_{\delta}(x-y) dy \right| \leqslant C\delta^{3/2} \cdot \delta^{-1} \to 0, \tag{137}$$
$$\text{pointwise uniformly in } [0,T] \times \Omega_b \text{ as } \delta \to 0,$$

where C is independent of δ. More precisely, notice that the convolution integral in (137) is defined on the domain $\tilde{\Omega}_b$, which is triple the size of the domain Ω_b. Furthermore, we recall that the convolution is defined using odd extensions as in Definition 8.2. Thus, by the definition of the odd extensions of $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_{\delta}$ to the larger domain $\tilde{\Omega}_b$, we get

$$||\boldsymbol{\eta} - \boldsymbol{\eta}_{\delta}||_{H^1(\tilde{\Omega}_b)} \leqslant C \left(||\boldsymbol{\eta} - \boldsymbol{\eta}_{\delta}||_{H^1(\Omega_b)} + ||\omega - \omega_{\delta}||_{H^1(\Gamma)} \right).$$

In addition, since we have extended the functions $\boldsymbol{\eta}$ and $\boldsymbol{\eta}_{\delta}$ to the larger domain $\tilde{\Omega}_b$, the estimate (137) holds for all δ such that $\{(x,y) \in \mathbb{R}^2 : \text{dist}((x,y), \Omega_b) \leqslant \delta\} \subset \tilde{\Omega}_b$. Thus, $|(\nabla \boldsymbol{\eta}^{\delta} - \nabla \boldsymbol{\eta}_{\delta}^{\delta})(t,x)| \to 0$ pointwise uniformly in $[0,T]$ as $\delta \to 0$.

To obtain (135) it suffices to show that $|(\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^{\delta})(t,x)| \to 0$ pointwise uniformly in $[0,T]$ as $\delta \to 0$. This follows from Lemma 10.2. Namely, Lemma 10.2 implies

$$|(\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^{\delta})(t,x)| \leqslant C\delta \to 0 \text{ pointwise uniformly in } [0,T] \times \Omega_b \text{ as } \delta \to 0. \tag{138}$$

So combining (138) with (137), we get (135).

Now we use a bootstrap argument on $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}_{\delta}^{\delta})$ by continuity, since we have that $\det(\boldsymbol{I} + \nabla \boldsymbol{\eta}) \geqslant c > 0$ up to a final time $T > 0$. Similarly, for all sufficiently small δ, the assumptions (132) and (133) will also hold up to the final time $T > 0$, as we can also bootstrap these two conditions (132) and (133) similarly. This closes the bootstrap argument, and so we obtain that the estimate (131), and similarly the estimates (132) and (133), hold uniformly up to the final time $T > 0$ uniformly in δ.

We end this section by proving the following lemma, which establishes convergence of the spatial convolution of the classical solution $\boldsymbol{\eta}$ in $H^1(\Omega_b)$, which is needed for the bootstrap argument described above.

Lemma 10.2. Let $\boldsymbol{\eta} \in L^{\infty}(0,T;V_d)$ be an arbitrary but fixed smooth function in time and space on $[0,T] \times \overline{\Omega_b}$, where V_d is defined in (31). Then, there exists a constant C independent of $\delta > 0$, depending only on $\boldsymbol{\eta}$, such that

$$\max_{t \in [0,T]} ||\boldsymbol{\eta}^{\delta} - \boldsymbol{\eta}||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}, \text{ and } |\nabla \boldsymbol{\eta}^{\delta} - \nabla \boldsymbol{\eta}| \leqslant C\delta \quad \forall x \in \overline{\Omega}_b \text{ and } \forall t \in [0,T].$$

Remark 10.2. More generally, if f is a smooth function on $\mathbb{R}^2$ with sufficient decay at infinity, such as a Schwartz function, then the argument below shows that the function $\tilde{f}$ defined by

$$\tilde{f} = f * \sigma_{\delta} \qquad \text{on } \mathbb{R}^2$$

would satisfy $||\tilde{f} - f||_{H^1(\Omega_b)} \leqslant C\delta^2$ for a constant C. However, because we are working on a bounded domain Ω_b, we must use an odd extension to define the spatial convolution of $\boldsymbol{\eta}$. Since the odd extension of $\boldsymbol{\eta}$ to the larger domain $\tilde{\Omega}_b$ is not necessarily smooth on $\tilde{\Omega}_b$ even if $\boldsymbol{\eta}$ is a smooth function on $\overline{\Omega_b}$, we incur a loss in our estimate due to potential irregularities of the odd extension due to the behavior of the initial function $\boldsymbol{\eta}$ near the boundary $\partial\Omega_b$, which gives rise to the convergence rate $\delta^{3/2}$ instead of the optimal rate of convergence δ^2.

Proof. Separate the domain $\Omega_b = (0, L) \times (0, R)$ into two parts:

$$\Omega_{b,1} = (\delta, L - \delta) \times (\delta, R - \delta), \qquad \Omega_{b,2} = \Omega_b \backslash \Omega_{b,1}.$$

For $\boldsymbol{x} \in \Omega_{b,1}$, we note that because the convolution kernel σ_δ is radially symmetric,

$$(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(\boldsymbol{x}) = \int_{\Omega_b} \left(\frac{1}{2}\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \boldsymbol{\eta}(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\eta}(\boldsymbol{x} - \boldsymbol{x}') \right) \sigma_\delta(\boldsymbol{x}') d\boldsymbol{x}',$$

$$(\nabla\boldsymbol{\eta}^\delta - \nabla\boldsymbol{\eta})(\boldsymbol{x}) = \int_{\Omega_b} \left(\frac{1}{2}\nabla\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \nabla\boldsymbol{\eta}(\boldsymbol{x}) + \frac{1}{2}\nabla\boldsymbol{\eta}(\boldsymbol{x} - \boldsymbol{x}') \right) \sigma_\delta(\boldsymbol{x}') d\boldsymbol{x}'.$$

For $\boldsymbol{x} \in \Omega_{b,1}$, these points are at least δ away from the boundary. Therefore, we have the following estimate for the discretized second derivative:

$$\left| \frac{1}{2}\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \boldsymbol{\eta}(x) + \frac{1}{2}\boldsymbol{\eta}(\boldsymbol{x} - \boldsymbol{x}') \right| \leqslant C\delta^2 \quad \text{ for } |\boldsymbol{x}'| \leqslant \delta,$$

and similarly for $\nabla\boldsymbol{\eta}$, by using the fact that $\boldsymbol{\eta}$ is spatially smooth in $\overline{\Omega_b}$. Therefore,

$$|(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(\boldsymbol{x})| \leqslant C\delta^2, \quad |(\nabla\boldsymbol{\eta}^\delta - \nabla\boldsymbol{\eta})(\boldsymbol{x})| \leqslant C\delta^2, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,1}, \tag{139}$$

for a constant C depending only on $\boldsymbol{\eta}$.

For $\boldsymbol{x} \in \Omega_{b,2}$ we cannot use the same estimate, since after extending $\boldsymbol{\eta}$ to the larger domain $\tilde{\Omega}_b$, the extended function on $\tilde{\Omega}_b$ does not necessarily have a continuous second derivative, as a result of the properties of odd extension, and in fact, there may be discontinuities of the second derivative along the boundary $\partial\Omega_b$. However, $\nabla\boldsymbol{\eta}$ on the larger domain $\tilde{\Omega}_b$ is still *Lipschitz continuous.* Thus, we instead use the equations:

$$(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(\boldsymbol{x}) = \int_{\Omega_b} (\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \boldsymbol{\eta}(\boldsymbol{x}))\sigma_\delta(\boldsymbol{x}') d\boldsymbol{x}',$$

$$(\nabla\boldsymbol{\eta}^\delta - \nabla\boldsymbol{\eta})(\boldsymbol{x}) = \int_{\Omega_b} (\nabla\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \nabla\boldsymbol{\eta}(\boldsymbol{x}))\sigma_\delta(\boldsymbol{x}') d\boldsymbol{x}'.$$

Since $\boldsymbol{x} \in \Omega_{b,2}$, even if $|\boldsymbol{x}'| \leqslant \delta$, we may have that $\boldsymbol{x} + \boldsymbol{x}'$ is outside of Ω_b. However, due to the Lipschitz continuity of $\nabla\boldsymbol{\eta}$ on the larger domain $\tilde{\Omega}_b$, we still have the estimates

$$|\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \boldsymbol{\eta}(\boldsymbol{x})| \leqslant C\delta, \quad |\nabla\boldsymbol{\eta}(\boldsymbol{x} + \boldsymbol{x}') - \nabla\boldsymbol{\eta}(\boldsymbol{x})| \leqslant C\delta, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,2}, |\boldsymbol{x}'| \leqslant \delta,$$

which give

$$|(\boldsymbol{\eta}^\delta - \boldsymbol{\eta})(\boldsymbol{x})| \leqslant C\delta, \quad |(\nabla\boldsymbol{\eta}^\delta - \nabla\boldsymbol{\eta})(\boldsymbol{x})| \leqslant C\delta, \qquad \text{for } \boldsymbol{x} \in \Omega_{b,2}. \tag{140}$$

The area of $\Omega_{b,2}$ is bounded by $(2R + 2L)\delta$, so by (139) and (140), we have $||\boldsymbol{\eta}^\delta - \boldsymbol{\eta}||_{H^1(\Omega_b)} \leqslant C\delta^{3/2}$ for a spatially smooth function $\boldsymbol{\eta}$ on $\overline{\Omega_b}$, where C depends only on the norms of up to the second spatial derivative of $\boldsymbol{\eta}$ on $\overline{\Omega_b}$. The generalization of this result to a function $\boldsymbol{\eta}$ that also depends on time and is spatially smooth in both space and time follows analogously.

□

This completes the proof of the weak-classical consistency results. This proof effectively shows that the weak solutions that we have constructed to the regularized FPSI problem converge (in the energy norm on a uniform time interval) as the regularization parameter goes to zero to a classical solution of the original (non-regularized) FPSI problem when such a classical solution to the original FPSI problem exists.

11 Conclusions

In this manuscript we proved the existence of a weak solution to a fluid-structure interaction problem between the flow of an incompressible, viscous fluid and a multi-layered poroelastic/poroviscoelastic structure consisting of the Biot equations of poro(visco)elasticity and a thin, reticular interface with mass and elastic energy, which is transparent to fluid flow. The fluid and multilayered structure are nonlinearly coupled, giving rise to significant difficulties in the existence proof, associated with the geometric nonlinearity of the coupled problem. The existence proof is constructive, and it consists of two major steps. In the fist step we proved the existence of a weak solution to a regularized problem in the class of finite energy solutions. In the second step we showed that the solution of this regularized problem converges to a classical solution to the original, nonregularized probroblem as the regularization parameter tends to zero, as long as the original problem possesses a classical solution. While the proof of the existence of a weak solution to the regularized problem only requires that the Biot structure is poroelastic, additional regularity of the Biot poroelatic medium is required to prove the weak-classical consistency-the Biot structure is assumed to be poroviscoelastic. This weak-classical consistency result also shows that the solution we constructed is unique in the sense of weak-classical uniqueness.

An interesting extension of this work is to consider the singular limit as the thin interface thickness converges to zero, and investigate the existence of a weak solution to the FSI problem between the Navier-Stokes equations for an incompressible, viscous fluid and the Biot equations of poroviscoelasticity, nonlinearly coupled over the moving interface. Preliminary results indicate that this will be possible under certain assumptions.

12 Appendix

12.1 Weak continuity of solutions to the regularized PFSI problem

In this appendix, we show a result related to weak continuity of solutions to the regularized FPSI problem, namely, we will show that as $\nu \to 0$:

$$\int_{\Omega_{f,\delta}(0)} \hat{\boldsymbol{u}}(0) \cdot (\boldsymbol{u}_\delta)_\nu(0) \to \int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_0|^2, \quad \text{and} \quad \int_{\Omega_{f,\delta}(t)} \hat{\boldsymbol{u}}(t) \cdot (\boldsymbol{u}_\delta)_\nu(t) \to \int_{\Omega_{f,\delta}(t)} \hat{\boldsymbol{u}}(t) \cdot \boldsymbol{u}_\delta(t),$$

for almost all points $0 < t \leqslant T$.

This result will be used in Section 12.2 to estimate the first term T_1 in (113) in the Gronwall's estimate. We will show weak continuity through the following series of lemmas.

Lemma 12.1. Let $\omega \in L^\infty(0,T;H_0^2(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))$ with

$$\min_{t\in[0,T],x\in[0,L]} R + \omega(t,x) > 0,$$

define the moving fluid domain $\Omega_f^\omega(t)$. Then, given $\boldsymbol{u} \in L^2(0,T;H^1(\Omega_f^\omega(t))) \cap L^\infty(0,T;L^2(\Omega_f^\omega(t)))$ where $\Omega_f^\omega(t) = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant L, -R \leqslant y \leqslant \omega(t,x)\}$, we have that

$$||\boldsymbol{u}_\nu(t,x,y) - \boldsymbol{u}(t,x,y)||_{L^2(\Omega_f^\omega(t))} \to 0 \qquad \text{as } \nu \to 0,$$

for almost all $t \in [0,T]$.

Proof. Recall that in the case of real-valued functions, one shows convergence of the convolution to the function itself almost everywhere by using the Lebesgue differentiation theorem [29]. To apply the theorem in this context, we need to apply it to a function taking values in a *fixed* Banach space rather than a time-dependent Banach space.

As a result, we consider the following function,

$$\boldsymbol{v}(t,x,y) = K(t,0,x,y)\boldsymbol{u}\left(t, x, \frac{R+\omega(t,x)}{R+\omega(0,x)}(R+y) - R\right),$$

where we have pulled the fluid velocity back to the fixed initial domain $\Omega_f^\omega(0)$. We recall the definition of $K(s,t,z,r)$ from (108) and its inverse:

$$K(s,t,x,y) = \begin{pmatrix} \frac{R+\omega(s,x)}{R+\omega(t,x)} & 0 \\ -(R+y)\partial_x\left(\frac{R+\omega(s,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix}, \quad K^{-1}(s,t,x,y) = \begin{pmatrix} \frac{R+\omega(t,x)}{R+\omega(s,x)} & 0 \\ (R+y)\frac{R+\omega(t,x)}{R+\omega(s,x)}\partial_x\left(\frac{R+\omega(s,x)}{R+\omega(t,x)}\right) & 1 \end{pmatrix}.$$

By the uniform boundedness of $R+\omega(t,x)$ and $|\partial_x\omega(t,x)|$, and $\min_{t\in[0,T],x\in[0,L]} R+\omega(t,x) > 0$, it is immediate to see that $\boldsymbol{v}(t,z,r)$ is in $L^\infty(0,T;L^2(\Omega_f^\omega(0)))$, where we emphasize that $L^2(\Omega_f^\omega(0))$ is a fixed function space that no longer depends on time.

By Lebesgue's differentiation theorem, almost every $t \in [0,T]$ is a Lebesgue point satisfying

$$\lim_{\nu\to 0}\frac{1}{2\nu}\int_{t-\nu}^{t+\nu} ||\boldsymbol{v}(t,\cdot) - \boldsymbol{v}(s,\cdot)||_{L^2(\Omega_f^\omega(0))} ds \to 0. \tag{141}$$

Recall that by definition (109),

$$\boldsymbol{u}_\nu(t,x,y) = \int_{\mathbb{R}} K(s,t,x,y)\boldsymbol{u}\left(s, x, \frac{R+\omega(s,x)}{R+\omega(t,x)}(R+y) - R\right) j_\nu(t-s)ds.$$

Thus, we compute

$$\begin{aligned}\boldsymbol{u}_\nu(t,x,y) - \boldsymbol{u}(t,x,y) &= \int_{\mathbb{R}} \left(K(s,t,x,y)\boldsymbol{u}\left(s, x, \frac{R+\omega(s,x)}{R+\omega(t,x)}(R+y) - R\right) - \boldsymbol{u}(t,x,y)\right)\cdot j_\nu(t-s)ds \\ &:= I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned} I_1 = \int_{\mathbb{R}} K^{-1}&\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\cdot \\ &\left(\boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) - \boldsymbol{v}\left(t, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right) j_\nu(t-s)ds,\end{aligned}$$

$$\begin{aligned} I_2 = \int_{\mathbb{R}} &\left(K(s,t,x,y)K^{-1}\left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right)\cdot \\ &\boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) j_\nu(t-s)ds.\end{aligned}$$

We estimate each of these terms as follows. For I_1, we compute that

$$K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) = \begin{pmatrix} \frac{R+\omega(0,x)}{R+\omega(t,x)} & 0 \\ (R+y)\left(\frac{R+\omega(0,x)}{R+\omega(t,x)}\right)^2\partial_x\left(\frac{R+\omega(t,x)}{R+\omega(0,x)}\right) & 1 \end{pmatrix},$$

which we note is uniformly bounded on $[0,T]$. Hence, using the fact that $|j_\nu(t-s)| \leqslant \frac{1}{\nu}$, we get

$$
\begin{aligned}
||I_1||_{L^2(\Omega_f^\omega(t))} &\leqslant C \cdot \frac{1}{\nu} \int_{t-\nu}^{t+\nu} \left|\left| \boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) - \boldsymbol{v}\left(t, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right|\right|_{L^2(\Omega_f^\omega(t))} ds \\
&\leqslant C \cdot \frac{1}{\nu} \int_{t-\nu}^{t+\nu} \left(\frac{R+\omega(t,x)}{R+\omega(0,x)}\right)^{1/2} ||\boldsymbol{v}(s,x,y) - \boldsymbol{v}(t,x,y)||_{L^2(\Omega_f^\omega(0))} ds \to 0,
\end{aligned}
$$

as $\nu \to 0$ if t is a Lebesgue point, by (141) and the uniform boundedness of $\frac{R+\omega(t,x)}{R+\omega(0,x)}$ on $[0,T]$.

To estimate I_2, we can use the continuity in time of ω and $\partial_x \omega$ to calculate that

$$
\left| K(s,t,x,y) K^{-1}\left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right) - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right| \to 0,
$$

uniformly in (x,y) as $s \to t$. Now, we estimate

$$
\begin{aligned}
||I_2||_{L^2(\Omega_f^\omega(t))} \leqslant& \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1}\left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right. \\
&\left. - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right| \\
&\cdot \left|\left| \boldsymbol{v}\left(s, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right|\right|_{L^2(\Omega_f^\omega(t))} \cdot j_\nu(t-s) ds \\
\leqslant& \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1}\left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right. \\
&\left. - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right| \\
&\cdot \left(\frac{R+\omega(t,x)}{R+\omega(0,x)}\right)^{1/2} \cdot ||\boldsymbol{v}\,(s,x,y)||_{L^2(\Omega_f^\omega(0))} \cdot j_\nu(t-s) ds \\
\leqslant& C \int_{\mathbb{R}} \max_{x,y \in \Omega_f^\omega(t)} \left| K(s,t,x,y) K^{-1}\left(s, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right. \\
&\left. - K^{-1}\left(t, 0, x, \frac{R+\omega(0,x)}{R+\omega(t,x)}(R+y) - R\right)\right| \cdot j_\nu(t-s) ds,
\end{aligned}
$$

where we used the fact that $\boldsymbol{v} \in L^\infty(0,T; L^2(\Omega_f^\omega(0)))$. Thus, we conclude that $||I_2||_{L^2(\Omega_f^\omega(t))} \to 0$ as $\nu \to 0$. This completes the proof. $\square$

We also have a weak continuity lemma, which states that the value of $\boldsymbol{u}_\delta$ tested against any function in the fluid function space has a continuity property as $t \to 0$.

Lemma 12.2. Consider an arbitrary $\boldsymbol{q} \in C^1(0,T; V_{f,\delta}(t))$ and the weak solution $\boldsymbol{u}_\delta$ to the regularized problem for arbitrary δ, where $V_{f,\delta}(t)$ is defined by the displacement ω_δ and (26). There exists a measure zero subset S of $[0,T]$ (depending on δ) such that

$$
\lim_{t \to 0, t \in [0,T] \cap S^c} \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \boldsymbol{q}(t) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0).
$$

Proof. Consider the following function for each $\tau \in [0,T]$ and $\alpha > 0$, given by

$$
J_{\tau,\nu}(t) = 1 - \int_0^t j_\nu(s-\tau) ds, \tag{142}
$$

and note that $J'_{\tau,\nu}(t) = -j_\nu(t-\tau)$. We want to test the regularized weak formulation for $\boldsymbol{u}_\delta$ with the test function $J_{\tau,\nu}(t)\boldsymbol{q}$ for certain admissible choices of τ. To see which τ we want to choose, we define the function

$$\boldsymbol{w}(t,x,y) = \frac{R+\omega_\delta(t)}{R+\omega_\delta(0)} \cdot \boldsymbol{u}_\delta\left(t, x, \frac{R+\omega_\delta(t)}{R+\omega_\delta(0)}(R+y) - R\right) \cdot \boldsymbol{q}\left(t, x, \frac{R+\omega_\delta(t)}{R+\omega_\delta(0)}(R+y) - R\right).$$

We claim that $\boldsymbol{w} \in L^\infty(0,T;L^1(\Omega_{f,\delta}(0)))$. To see this, we compute by a change of variables that

$$||\boldsymbol{w}(t,x,y)||_{L^1(\Omega_{f,\delta}(0))} = \int_{\Omega_{f,\delta}(t)} |\boldsymbol{u}_\delta(t,x,y) \cdot \boldsymbol{q}(t,x,y)|,$$

and we then use the fact that $\boldsymbol{u}_\delta, \boldsymbol{q} \in L^\infty(0,T;L^2(\Omega_{f,\delta}(t)))$.

Hence, by the Lebesgue differentiation theorem, there exists a measurable subset $S \subset [0,T]$ of measure zero such that every point in $[0,T] \cap S^c$ is a Lebesgue point of $\boldsymbol{w}$, in the sense that

$$\lim_{\nu\to 0} \frac{1}{2\nu} \int_{\tau-\nu}^{\tau+\nu} ||\boldsymbol{w}(\tau,\cdot) - \boldsymbol{w}(s,\cdot)||_{L^1(\Omega_{f,\delta}(0))} ds \to 0. \tag{143}$$

for every $\tau \in [0,T] \cap S^c$. These are the τ for which we will consider the test function $J_{\tau,\nu}(t)\boldsymbol{q}$. For the test functions for the Biot medium and the plate, we will take these test functions to be zero. Hence, in the regularized weak formulation (98), we will test with $(\boldsymbol{v}, \varphi, \boldsymbol{\psi}, r) = (J_{\tau,\nu}(t)\boldsymbol{q}, 0, 0, 0)$.

Hence, we obtain the following equality:

$$\begin{aligned}
&-\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t(J_{\tau,\nu}(t)\boldsymbol{q}) + \frac{1}{2}\int_0^T \int_{\Omega_{f,\delta}(t)} [((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{u}_\delta) \cdot (J_{\tau,\nu}(t)\boldsymbol{q}) - ((\boldsymbol{u}_\delta \cdot \nabla)(J_{\tau,\nu}(t)\boldsymbol{q})) \cdot \boldsymbol{u}_\delta] \\
&+ \frac{1}{2}\int_0^T \int_{\Gamma_\delta(t)} (\boldsymbol{u}_\delta \cdot \boldsymbol{n} - 2\boldsymbol{\xi}_\delta \cdot \boldsymbol{n})\boldsymbol{u}_\delta \cdot (J_{\tau,\nu}(t)\boldsymbol{q}) + 2\nu \int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_\delta) : \boldsymbol{D}(J_{\tau,\nu}(t)\boldsymbol{q}) \\
&- \int_0^T \int_{\Gamma_\delta(t)} \left(\frac{1}{2}|\boldsymbol{u}_\delta|^2 - p_\delta\right) J_{\tau,\nu}(t) q_n - \beta \int_0^T \int_{\Gamma_\delta(t)} [(\xi_\delta)_\tau - (u_\delta)_\tau] \cdot J_{\tau,\nu}(t) q_\tau = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot J_{\tau,\nu}(0)\boldsymbol{q}(0).
\end{aligned}$$

Consider $\tau \in (0,T) \cap S^c$. We want to pass to the limit as $\nu \to 0$, and then pass to the limit as $\tau \to 0$, in order to obtain the desired result.

First, we pass to the limit as $\nu \to 0$. We handle the convergences as follows.

First term: We will show that because τ is a Lebesgue point of $\boldsymbol{w}$,

$$-\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t(J_{\tau,\nu}(t)\boldsymbol{q}) \to \int_{\Omega_{f,\delta}(\tau)} \boldsymbol{u}_\delta(\tau)\boldsymbol{q}(\tau) - \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t \boldsymbol{q}, \qquad \text{as } \nu \to 0.$$

We compute that

$$-\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t(J_{\tau,\nu}(t)\boldsymbol{q}) = \int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot j_\nu(t-\tau)\boldsymbol{q} - \int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot J_{\tau,\nu}(t)\partial_t \boldsymbol{q}.$$

It is easy to see that

$$\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta J_{\tau,\nu}(t)\partial_t \boldsymbol{q} \to \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \partial_t \boldsymbol{q}.$$

So it remains to show that

$$\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot j_\nu(t-\tau)\boldsymbol{q} \to \int_{\Omega_{f,\delta}(\tau)} \boldsymbol{u}_\delta(\tau)\boldsymbol{q}(\tau), \qquad \text{as } \nu \to 0.$$

By a change of variables, we compute that

$$\int_0^T \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot j_\nu(t-\tau)\boldsymbol{q}$$
$$= \int_0^T \int_{\Omega_{f,\delta}(\tau)} \frac{R+\omega_\delta(t)}{R+\omega_\delta(\tau)} \cdot \boldsymbol{u}_\delta \left(t, x, \frac{R+\omega_\delta(t)}{R+\omega_\delta(\tau)}(R+y) - R\right) \cdot j_\nu(t-\tau)\boldsymbol{q}\left(t, x, \frac{R+\omega_\delta(t)}{R+\omega_\delta(\tau)}(R+y) - R\right)$$
$$= \int_0^T \int_{\Omega_{f,\delta}(\tau)} \frac{R+\omega_\delta(0)}{R+\omega_\delta(\tau)} \boldsymbol{w}\left(t, x, \frac{R+\omega_\delta(0)}{R+\omega_\delta(\tau)}(R+y) - R\right) \cdot j_\nu(t-\tau) = \int_0^T \int_{\Omega_{f,\delta}(0)} \boldsymbol{w}(t,x,y) \cdot j_\nu(t-\tau).$$

By (143), we have that

$$\int_0^T \int_{\Omega_{f,\delta}(0)} \boldsymbol{w}(t,x,y) \cdot j_\nu(t-\tau) \to \int_{\Omega_{f,\delta}(0)} \boldsymbol{w}(\tau,x,y) = \int_{\Omega_{f,\delta}(\tau)} \boldsymbol{u}_\delta(\tau) \cdot \boldsymbol{q}(\tau),$$

which establishes the desired convergence.
Final term: It is immediate to see that for all sufficiently small $\nu > 0$,

$$\int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot J_{\tau,\nu}(0)\boldsymbol{q}(0) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0).$$

We can now easily take $\nu \to 0$ in the remaining terms to obtain that for any $\tau \in (0,T) \cap S^c$,

$$\int_{\Omega_{f,\delta}(\tau)} \boldsymbol{u}_\delta(\tau) \cdot \boldsymbol{q}(\tau) - \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t \boldsymbol{q} + \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} [((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{u}_\delta) \cdot \boldsymbol{q} - ((\boldsymbol{u}_\delta \cdot \nabla)\boldsymbol{q}) \cdot \boldsymbol{u}_\delta]$$
$$+ \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{u}_\delta \cdot \boldsymbol{n} - 2\boldsymbol{\xi}_\delta \cdot \boldsymbol{n})\boldsymbol{u}_\delta \cdot \boldsymbol{q} + 2\nu \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_\delta) : \boldsymbol{D}(\boldsymbol{q})$$
$$- \int_0^t \int_{\Gamma_\delta(t)} \left(\frac{1}{2}|\boldsymbol{u}_\delta|^2 - p_\delta\right) q_n - \beta \int_0^t \int_{\Gamma_\delta(t)} [(\xi_\delta)_\tau - (u_\delta)_\tau] \cdot q_\tau = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \boldsymbol{q}(0).$$

Passing to the limit as $\tau \to 0$ with $\tau \in (0,T) \cap S^c$ gives the desired result.

$\square$

Lemma 12.3. Let $\boldsymbol{u}_0$ be divergence free and smooth on $\overline{\Omega_f(0)}$. Define

$$\tilde{\boldsymbol{q}}(t,x,y) = K_\delta(0,t,x,y)\boldsymbol{u}_0\left(x, \frac{R+\omega_\delta(0,x)}{R+\omega_\delta(t,x)}(R+y) - R\right), \tag{144}$$

where K_δ is given by (108). Then, there exists a sequence of functions $\tilde{\boldsymbol{q}}_m \in C_c^1(0,T; V_{f,\delta}(t))$, with $V_{f,\delta}(t)$ determined by the plate displacement ω_δ via the definition (26), such that

$$\max_{0 \leqslant t \leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,\delta}(t))} \to 0, \qquad \text{as } m \to \infty.$$

Proof. There exists a rectangular two-dimensional maximal domain Ω_M of the form $[0,L] \times [-R, R_{max}]$ for some positive constant R_{max} that contains all of the domains $\Omega_{f,\delta}(t)$ for $t \in [0,T]$. We will extend $\tilde{\boldsymbol{q}}$ to the maximal spacetime domain $[0,T] \times \Omega_M$ by extending vertically in the radial direction by the trace of $\tilde{\boldsymbol{q}}$ along $\Gamma_\delta(t)$. In particular, we define

$$\tilde{\boldsymbol{q}}(t,x,y) = K(0,t,x,\omega_\delta(t,x))\boldsymbol{u}_0(x, \omega_\delta(0,x)), \qquad \text{for } (t,x,y) \in ([0,T] \times \Omega_M) - ([0,T] \times \Omega_{f,\delta}(t)). \tag{145}$$

Note that this extension preserves the *divergence free* property.

We have the following two claims about the extended function, considered as a function on the *fixed* maximal domain Ω_M. First, we claim that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$. Second, we claim that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$. To see that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$, we note that ω_δ and $\partial_x\omega_\delta$ are bounded uniformly pointwise, and furthermore $\boldsymbol{u}_0$ and its first spatial derivatives are bounded by assumption. In addition, $\partial_x^2\omega_\delta \in L^\infty(0,T;L^2(\Gamma))$, which allows us to conclude that $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M))$.

Next, we want to verify that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$. Consider any $t \in [0,T]$ and consider any $s \in [0,T]$ with $s \neq t$. We define the following regions:

$$A(s,t) = \Omega_f^M \cap (\Omega_{f,\delta}(s) \cup \Omega_{f,\delta}(t))^c,\ B(s,t) = [\Omega_{f,\delta}(s) \cap (\Omega_{f,\delta}(t))^c] \cup [(\Omega_{f,\delta}(s))^c \cap \Omega_{f,\delta}(t)],$$
$$C(s,t) = \Omega_{f,\delta}(s) \cap \Omega_{f,\delta}(t).$$

Consider $\epsilon > 0$. We want to find $h > 0$ such that

$$||\tilde{\boldsymbol{q}}(t,\cdot) - \tilde{\boldsymbol{q}}(s,\cdot)||^2_{L^2(\Omega_M)} \leqslant \epsilon, \qquad \text{for all } s \in (t-h,t+h) \cap [0,T]. \tag{146}$$

We compute that

$$||\tilde{\boldsymbol{q}}(t,\cdot) - \tilde{\boldsymbol{q}}(s,\cdot)||^2_{L^2(\Omega_M)} = \int_{A(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 + \int_{B(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2$$
$$+ \int_{C(s,t)} |\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 = I_A + I_B + I_C. \tag{147}$$

We estimate each of the terms I_A, I_B, and I_C separately.

For I_A, we recall that we are extending by the trace as in (145) on $A(s,t)$, so we have that

$$I_A = \int_{A(s,t)} |K_\delta(0,t,x,\omega_\delta(t,x)) - K_\delta(0,s,x,\omega_\delta(s,x))|^2 \cdot |\boldsymbol{u}_0(x,\omega_\delta(0,x))|^2.$$

We have that $|\boldsymbol{u}_0(x,\omega_\delta(0,x))| \leqslant M_1$ for some constant M_1 by the fact that $\boldsymbol{u}_0$ is continuous on $\overline{\Omega_f(0)}$. By continuity, we can choose $h > 0$ sufficiently small so that

$$|K_\delta(0,t,x,\omega_\delta(t,x)) - K_\delta(0,s,x,\omega_\delta(s,x))|^2 < \frac{\epsilon}{3M_1^2(R+R_{max})L}, \qquad \text{for all } s \in (t-h,t+h) \cap [0,T].$$

Thus, for all $s \in (t-h,t+h) \cap [0,T]$,

$$I_A \leqslant |A(s,t)| \cdot \frac{\epsilon}{3(R+R_{max})L} \leqslant \frac{\epsilon}{3}.$$

For I_B, we will use the fact that ω_δ does not change much in time over small time intervals, by continuity. We note that there exists a uniform constant M_2 such that $|\tilde{\boldsymbol{q}}| \leqslant M_2$ on $[0,T] \times \Omega_M$. Hence,

$$I_B = \int_{B(s,t)} |\tilde{\boldsymbol{q}}(t,z,r) - \tilde{\boldsymbol{q}}(s,z,r)|^2 \leqslant |B(s,t)| \cdot 4M_2^2 = 4M_2^2 \int_0^L |\omega_\delta(t,x) - \omega_\delta(s,x)| dx.$$

Because $\omega_\delta \in L^\infty(0,T;H_0^2(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))$, there exists $h > 0$ sufficiently small such that

$$|\omega_\delta(t,x) - \omega_\delta(s,x)| \leqslant \frac{\epsilon}{12M_2^2 L}, \qquad \text{for all } x \in [0,L] \text{ and } s \in (t-h,t+h) \cap [0,T].$$

This allows us to conclude that $I_B \leqslant \frac{\epsilon}{3}$, for all $s \in (t-h, t+h) \cap [0,T]$.

For I_C, we refer to the definition of $\tilde{\boldsymbol{q}}$ in (144) and note that $K_\delta(0,t,x,y)$ is continuous in time uniformly in $(x,y) \in [0,L] \times [-R, R_{max}]$, $\boldsymbol{u}_0$ is uniformly continuous as a function on $\overline{\Omega_f(0)}$, and $\omega_\delta(t,x)$ is continuous in time uniformly in $x \in [0,L]$. Hence, there exists $h > 0$ sufficiently small such that

$$|\tilde{\boldsymbol{q}}(t,x,y) - \tilde{\boldsymbol{q}}(s,x,y)|^2 \leqslant \frac{\epsilon}{3(R+R_{max})L}, \qquad \text{for all } (x,y) \in C(s,t) \text{ and } s \in (t-h,t+h) \cap [0,T],$$

which gives the desired result that $I_C \leqslant \frac{\epsilon}{3}$ for all $s \in (t-h,t+h) \cap [0,T]$. Thus, by using (147), we have established (146).

Since $\tilde{\boldsymbol{q}} \in L^\infty(0,T;H^1(\Omega_M)) \cap C(0,T;L^2(\Omega_M))$, we can extend $\tilde{\boldsymbol{q}}$ to a continuous function on all of $\mathbb{R}$ as follows. We can find an increasing sequence T_m with $T_m \to T$ as $m \to \infty$, such that $\tilde{\boldsymbol{q}}(T_m) \in H^1(\Omega_M)$ for all m. Define an extension $\hat{\boldsymbol{q}}_m$ for each m to all of $\mathbb{R}$ by $\hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}$ if $t \in [0,T_m]$,

$$\hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}(0), \qquad \text{if } t < 0, \qquad \hat{\boldsymbol{q}}_m = \tilde{\boldsymbol{q}}(T_m), \qquad \text{if } t > T_m.$$

Define

$$\tilde{\boldsymbol{q}}_m = \hat{\boldsymbol{q}}_m * j_{1/m},$$

where the convolution is a convolution in time with j_ν for $\alpha = 1/m$. Because $\hat{\boldsymbol{q}}_m \in L^\infty(0,T;H^1(\Omega_M)) \cap C(0,T;L^2(\Omega_M))$ with $\hat{\boldsymbol{q}}_m$ being divergence free for every $t \in [0,T]$, we have that $\tilde{\boldsymbol{q}}_m$ restricted to $\bigcup_{t\in[0,T]} \{t\} \times \Omega_{f,\delta}(t)$ gives a function in $C^1([0,T);V_{f,\delta}(t))$, where $V_{f,\delta}(t)$ is the space defined in (26) with the plate displacement ω_δ. The fact that

$$\max_{0\leqslant t\leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,\delta}(t))} \to 0, \qquad \text{as } m \to \infty,$$

follows from the uniform continuity of $\tilde{\boldsymbol{q}}$ on $[0,T]$ as a function taking values in $L^2(\Omega_M)$, convergence properties of convolutions, and the fact that $\tilde{\boldsymbol{q}} \in C(0,T;L^2(\Omega_M))$ which gives the convergence

$$\max_{t\in[T_m,T]} ||\tilde{\boldsymbol{q}}(T) - \tilde{\boldsymbol{q}}(t)||_{L^2(\Omega_M)} \to 0, \qquad \text{as } m \to \infty.$$

□

Lemma 12.4. For the function $\tilde{\boldsymbol{q}}$ defined in (144), there exists a measure zero subset S of $[0,T]$ such that

$$\lim_{t\to 0, t\in[0,T]\cap S^c} \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}(t) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0).$$

Proof. Note that because $\partial_t \tilde{\boldsymbol{q}}$ is not necessarily in $H^1(\Omega_{f,\delta}(t))$, $\tilde{\boldsymbol{q}}$ is not a valid test function. Thus, we use the sequence $\tilde{\boldsymbol{q}}_m \in C^1(0,T;\mathcal{V}_{f,\delta}(t))$ from Lemma 12.3, which satisfies

$$\max_{0\leqslant t\leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_m||_{L^2(\Omega_{f,\delta}(t))} \to 0, \qquad \text{as } m \to \infty.$$

We can then apply Lemma 12.2 to each of the test functions $\tilde{\boldsymbol{q}}_m$, to deduce that there exists a measure zero subset S_m of $[0,T]$ such that

$$\lim_{t\to 0, t\in[0,T]\cap S_m^c} \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}_m(t) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_m(0).$$

In addition, by uniform boundedness, $\boldsymbol{u}_\delta \in L^\infty(0,T; L^2(\Omega_{f,\delta}(t)))$, and hence, there exists a measure zero subset S_0 of $[0,T]$, and a positive constant C such that $||\boldsymbol{u}_0||_{L^2(\Omega_{f,\delta}(0))} \leqslant C$, and

$$||\boldsymbol{u}_\delta(t)||_{L^2(\Omega_{f,\delta}(t))} \leqslant C, \qquad \text{for all } t \in S_0^c. \tag{148}$$

Define $S = S_0 \cup \bigcup_{m \geqslant 1} S_m$, which is also a measure zero subset of $[0,T]$. Then, for each m,

$$\lim_{t \to 0, t \in [0,T] \cap S^c} \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}_m(t) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_m(0). \tag{149}$$

By passing to the limit in m, we claim that in addition,

$$\lim_{t \to 0, t \in [0,T] \cap S^c} \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}(t) = \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0).$$

To see this, consider $\epsilon > 0$. We claim that there exists $h > 0$ sufficiently small such that for all $t \in (0,h) \cap S^c$,

$$\left| \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) \right| < \epsilon.$$

We can choose M sufficiently large such that $\max_{0 \leqslant t \leqslant T} ||\tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}_M||_{L^2(\Omega_{f,\delta}(t))} < \dfrac{\epsilon}{3C}$, where C is defined by (148). Therefore, for all $t \in [0,T] \cap S^c$,

$$\left| \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}_M(t) \right| < \frac{\epsilon}{3}.$$

In addition,

$$\left| \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) - \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_M(0) \right| < \frac{\epsilon}{3}.$$

By applying (149) with $m = M$, we can choose $h > 0$ sufficiently small such that for all $t \in (0,h) \cap S^c$,

$$\left| \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}_M(t) - \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}_M(0) \right| < \frac{\epsilon}{3}.$$

Thus, by applying the triangle inequality, we have that for all $t \in (0,h) \cap S^c$,

$$\left| \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \tilde{\boldsymbol{q}}(t) - \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) \right| < \epsilon,$$

which establishes the desired result.

□

We can now prove the final result of this appendix. We recall the definition of $\widehat{\boldsymbol{u}}$ from (103).

Lemma 12.5. In the limit as $\nu \to 0$ we have the following convergence results:

$$\int_{\Omega_{f,\delta}(0)} \widehat{\boldsymbol{u}}(0) \cdot (\boldsymbol{u}_\delta)_\nu(0) \to \int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_0|^2, \quad \text{and} \quad \int_{\Omega_{f,\delta}(t)} \widehat{\boldsymbol{u}}(t) \cdot (\boldsymbol{u}_\delta)_\nu(t) \to \int_{\Omega_{f,\delta}(t)} \widehat{\boldsymbol{u}}(t) \cdot \boldsymbol{u}_\delta(t),$$

for almost all points $t \in (0,T]$.

Proof. The second convergence for almost all points $t \in (0, T]$ follows directly from Lemma 12.1 and the fact that $\hat{\boldsymbol{u}} \in L^\infty(0, T; L^2(\Omega_{f,\delta}(t)))$.

So we just need to verify the convergence at $t = 0$. To do this, we note that $\hat{\boldsymbol{u}}(0) = \boldsymbol{u}_0$. Hence,

$$\begin{aligned}
&\int_{\Omega_{f,\delta}(0)} \hat{\boldsymbol{u}}(0) \cdot (\boldsymbol{u}_\delta)_\nu(0) \\
&\quad = \int_{\Omega^{\omega_0}} \left(\int_{\mathbb{R}} K_\delta(s, 0, x, y) \boldsymbol{u}_\delta \left(s, x, \frac{R + \omega_\delta(s, x)}{R + \omega_\delta(0, x)} (R + y) - R \right) j_\delta(t - s) ds \right) \boldsymbol{u}_0(x, y) dx dy \\
&\quad = \int_{\mathbb{R}} \left(\int_{\Omega^{\omega_0}} K_\delta(s, 0, x, y) \boldsymbol{u}_\delta \left(s, x, \frac{R + \omega_\delta(s, x)}{R + \omega_\delta(0, x)} (R + y) - R \right) \cdot \boldsymbol{u}_0(x, y) dx dy \right) j_\nu(t - s) ds \\
&\quad = \int_{\mathbb{R}} \left(\int_{\Omega_{f,\delta}(s)} \boldsymbol{u}_\delta(s, x, y) \cdot \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} K_\delta^t \left(s, 0, x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) \right. \\
&\qquad \left. \cdot \boldsymbol{u}_0 \left(x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) dx dy \right) j_\nu(t - s) ds.
\end{aligned}$$

We compute

$$\begin{aligned}
&\frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \cdot K_\delta^t \left(s, 0, x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) = \begin{pmatrix} 1 & (R + y) \nabla \left(\frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \right) \\ 0 & \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} & 0 \\ -(R + y) \nabla \left(\frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \right) & 1 \end{pmatrix} + \begin{pmatrix} 1 - \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} & (R + y) \nabla \left(\frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \right) \\ (R + y) \nabla \left(\frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} \right) & \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} - 1 \end{pmatrix} \\
&:= K_\delta(0, s, x, y) + R_\delta(0, s, x, y).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\Omega_{f,\delta}(0)} \hat{\boldsymbol{u}}(0) \cdot (\boldsymbol{u}_\delta)_\nu(0) \\
&= \int_{\mathbb{R}} \left(\int_{\Omega_{f,\delta}(s)} \boldsymbol{u}_\delta(s, x, y) \cdot K_\delta(0, s, x, y) \boldsymbol{u}_0 \left(x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) dx dy \right) j_\nu(t - s) ds \\
&+ \int_{\mathbb{R}} \left(\int_{\Omega_{f,\delta}(s)} \boldsymbol{u}_\delta(s, x, y) \cdot R_\delta(0, s, x, y) \boldsymbol{u}_0 \left(x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) dx dy \right) j_\nu(t - s) ds = I_{K,\delta} + I_{R,\delta}.
\end{aligned}$$

Note that

$$I_{K,\delta} = \int_{\mathbb{R}} \left(\int_{\Omega_{f,\delta}(s)} \boldsymbol{u}_\delta(s, x, y) \cdot \tilde{\boldsymbol{q}}(s, x, y) dx dy \right) j_\nu(t - s) ds$$

where $\tilde{\boldsymbol{q}}$ is defined by (144). Since $\boldsymbol{u}_\delta(s) = \boldsymbol{u}_\delta(-s)$ so that $\omega_\delta(s) = \omega_\delta(-s)$ for $s \leqslant 0$ (see the extension procedure), we conclude by Lemma 12.4 that

$$I_{K,\delta} \to \int_{\Omega_{f,\delta}(0)} \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{q}}(0) = \int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_0|^2, \qquad \text{as } \nu \to 0.$$

So it suffices to show that $I_{R,\delta} \to 0$ as $\nu \to 0$. This follows from the fact that $|R_\delta| \to 0$ uniformly as $s \to 0$. In particular,

$$\int_{\Omega_{f,\delta}(s)} \left| \boldsymbol{u}_\delta(s, x, y) \cdot \boldsymbol{u}_0 \left(x, \frac{R + \omega_\delta(0, x)}{R + \omega_\delta(s, x)} (R + y) - R \right) \right| dx dy \leqslant C, \qquad \text{for almost all } s \in [0, T],$$

by the boundedness of $\boldsymbol{u}_\delta \in L^\infty(0,T;L^2(\Omega_{f,\delta}(t))$ and the fact that $\boldsymbol{u}_0$ is uniformly bounded. In addition, by the continuity properties of ω_δ in time, we have that

$$\max_{\overline{(x,y)\in\Omega_{f,\delta}(s)}} |R_\delta(0,s,x,y)| \to 0, \qquad \text{as } s \to 0,$$

which implies that $I_{R,\delta} \to 0$ as $\nu \to 0$. This completes the proof.

We will use this result in the next section to estimate the first term T_1, see (113) in the Gronwall's estimate.

12.2 Gronwall's terms estimates

In this appendix we provide details of the derivation of the terms appearing in (113) and the calculations providing the desired estimates of the terms in (113) used to prove Gronwall's estimate in Section 10.3.

Term T1. To derive term T_1, defined in (115), we first multiply the weak formulation (97) for $\boldsymbol{u}$ with the test function $\boldsymbol{v} = \boldsymbol{u} - (\breve{\boldsymbol{u}}_\delta)_\nu$ to obtain the terms:

$$\begin{aligned} T_{1,1} = &-\int_0^t \int_{\Omega_f(t)} \boldsymbol{u}\cdot\partial_t\,[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu] - \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu] \\ &+ \int_{\Omega_f(t)} \boldsymbol{u}(t)\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu](t) - \int_{\Omega_f(0)} \boldsymbol{u}(0)\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu](0), \end{aligned}$$

where $\Omega_f(0)$ is the fluid domain corresponding to the initial structure displacement ω_0. We note that $\boldsymbol{u}$ is smooth in time and $(\breve{\boldsymbol{u}}_\delta)_\nu$ is differentiable in time as a result of the time convolution. Thus, by the Reynold's transport theorem,

$$T_{1,1} = \int_0^t \int_{\Omega_f(t)} \partial_t\boldsymbol{u}\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu] + \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot[\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu].$$

Because $\boldsymbol{u}$ is smooth and by the weak convergence properties of $(\breve{\boldsymbol{u}}_\delta)_\nu$ in Proposition 10.1,

$$T_{1,1} = \int_0^t \int_{\Omega_f(t)} \partial_t\boldsymbol{u}\cdot[\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta] + \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot[\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta] + K_{1,1,\nu},$$

where $K_{1,1,\nu} \to 0$ as $\nu \to 0$. Using estimates as found in [57], we can transfer the first integral from $\Omega_1(t)$ to $\Omega_{f,\delta}(t)$ at the cost of an additional term, so that

$$T_{1,1} = \int_0^t \int_{\Omega_{f,\delta}(t)} \partial_t\hat{\boldsymbol{u}}\cdot(\hat{\boldsymbol{u}}-\boldsymbol{u}_\delta) + \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta) + \tilde{R}_1 + K_{1,1,\nu},$$

where

$$\begin{aligned} |\tilde{R}_1| \leqslant \epsilon &\int_0^t ||\hat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} \\ &+ C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\partial_t\omega-\partial_t\omega_\delta||^2_{L^2(\Gamma)} + \int_0^t ||\hat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right). \end{aligned}$$

Thus, by using Proposition 10.1 again,

$$T_{1,1} = \int_0^t \int_{\Omega_{f,\delta}(t)} \partial_t\hat{\boldsymbol{u}}\cdot(\hat{\boldsymbol{u}}-(\boldsymbol{u}_\delta)_\nu) + \frac{1}{2}\int_0^t\int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot(\boldsymbol{u}-(\breve{\boldsymbol{u}}_\delta)_\nu) + \tilde{R}_1 + K_{1,1,\nu}, \qquad (150)$$

where $K_{1,1,\nu} \to 0$ as $\nu \to 0$.

Next, we test the regularized weak formulation for $\boldsymbol{u}_\delta$ with $\widehat{\boldsymbol{u}}$ and obtain the following terms:

$$T_{1,2} = -\int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta \cdot \partial_t \widehat{\boldsymbol{u}} - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}_\delta)\boldsymbol{u}_\delta \cdot \widehat{\boldsymbol{u}} + \int_{\Omega_{f,\delta}(t)} \boldsymbol{u}_\delta(t) \cdot \widehat{\boldsymbol{u}}(t) - \int_{\Omega_f(0)} \boldsymbol{u}_\delta(0) \cdot \widehat{\boldsymbol{u}}(0).$$

We want to integrate by parts in time, but $\boldsymbol{u}_\delta$ is not necessarily smooth in time. Thus, we replace $\boldsymbol{u}_\delta$ by its time regularization $(\boldsymbol{u}_\delta)_\nu$ at the cost of a term $K_{1,2,\nu}$ which goes to zero as $\nu \to 0$ by Proposition 10.1. Combining this with the Reynold's transport theorem, we get:

$$T_{1,2} = \int_0^t \int_{\Omega_{f,\delta}(t)} \partial_t \left[(\boldsymbol{u}_\delta)_\nu\right] \cdot \widehat{\boldsymbol{u}} + \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}_\delta)(\boldsymbol{u}_\delta)_\nu \cdot \widehat{\boldsymbol{u}} + K_{1,2,\nu}, \tag{151}$$

where $K_{1,2,\nu} \to 0$ as $\nu \to 0$.

Now, from the energy inequality, we obtain the terms

$$T_{1,3} = \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |\boldsymbol{u}_\delta(t)|^2 - \frac{1}{2}\int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_\delta(0)|^2. \tag{152}$$

Using the Reynold's transport theorem, the total contribution $T_1 = T_{1,1} - T_{1,2} + T_{1,3}$ is

$$\begin{aligned} T_1 = {} & \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |\widehat{\boldsymbol{u}}(t)|^2 - \frac{1}{2}\int_{\Omega_{f,\delta}(0)} |\widehat{\boldsymbol{u}}(0)|^2 - \int_{\Omega_{f,\delta}(t)} (\widehat{\boldsymbol{u}} \cdot (\boldsymbol{u}_\delta)_\nu)(t) + \int_{\Omega_{f,\delta}(0)} (\widehat{\boldsymbol{u}} \cdot (\boldsymbol{u}_\delta)_\nu)(0) \\ & + \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |\boldsymbol{u}_\delta(t)|^2 - \frac{1}{2}\int_{\Omega_{f,\delta}(0)} |\boldsymbol{u}_\delta(0)|^2 - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}_\delta)\widehat{\boldsymbol{u}} \cdot (\widehat{\boldsymbol{u}} - (\boldsymbol{u}_\delta)_\nu) \\ & + \frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot (\boldsymbol{u} - (\breve{\boldsymbol{u}}_\delta)_\nu) + \tilde{R}_1 + K_{1,1,\nu} + K_{1,2,\nu}. \end{aligned}$$

By Proposition 10.1, $(\boldsymbol{u}_\delta)_\nu$ and $(\breve{\boldsymbol{u}}_\delta)_\nu$ converge weakly to $\boldsymbol{u}_\delta$ and $\breve{\boldsymbol{u}}_\delta$ respectively, weakly in $L^2(0,T,W^{1,p}(\Omega_{f,\delta}(t)))$ and $L^2(0,T,W^{1,p}(\Omega_{f,1}(t)))$ for all $p \in [1,2)$. Furthermore, by Lemma 12.5 proved in the appendix above, we have that

$$\int_{\Omega_{f,\delta}(0)} (\widehat{\boldsymbol{u}} \cdot (\boldsymbol{u}_\delta)_\nu)(0) \to \int_{\Omega_{f,\delta}(0)} (\widehat{\boldsymbol{u}} \cdot \boldsymbol{u}_\delta)(0), \quad \int_{\Omega_{f,\delta}(t)} (\widehat{\boldsymbol{u}} \cdot (\boldsymbol{u}_\delta)_\nu)(t) \to \int_{\Omega_{f,\delta}(t)} (\widehat{\boldsymbol{u}} \cdot \boldsymbol{u}_\delta)(t). \tag{153}$$

Thus, taking the limit as $\nu \to 0$, the contribution of this term is now

$$\begin{aligned} T_1 = {} & \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)(t)|^2 - \frac{1}{2}\int_{\Omega_{f,\delta}(0)} |(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)(0)|^2 \\ & - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta \cdot \boldsymbol{n}_\delta)\widehat{\boldsymbol{u}} \cdot (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta) + \frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi} \cdot \boldsymbol{n})\boldsymbol{u} \cdot (\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) + \tilde{R}_1. \end{aligned}$$

Since $\widehat{\boldsymbol{u}}(0) = \boldsymbol{u}_\delta(0) = \boldsymbol{u}_0$, we obtain after some standard estimates that

$$T_1 = \frac{1}{2}\int_{\Omega_{f,\delta}(t)} |(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)(t)|^2 + R_1,$$

where

$$\begin{aligned} |R_1| \leqslant {} & \epsilon \int_0^T ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_{2.\delta}||^2_{H^1(\Omega_{f,\delta}(t))} \\ & + C(\epsilon)\left(\int_0^T ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^T ||\partial_t \omega - \partial_t \omega_\delta||^2_{L^2(\Gamma)} + \int_0^T ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right). \end{aligned}$$

This completes the calculations associated with term T_1.

Term T2. To estimate term T_2, defined in (116) above, we notice that since $(\breve{\boldsymbol{u}}_\delta)_\nu$ converges weakly to $\breve{\boldsymbol{u}}_\delta$ in $L^2(0,T;W^{1,p}(\Omega_{f,\delta}(t)))$ for $p \in [1,2)$ by Proposition 10.1, and because $\boldsymbol{u}$ is smooth, as $\nu \to 0$, we have that T_2 converges to

$$T_2 := \frac{1}{2}\int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta) - \frac{1}{2}\int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta))\cdot\boldsymbol{u}$$
$$- \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} ((\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta)\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) + \frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} ((\boldsymbol{u}_\delta\cdot\nabla)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta))\cdot\boldsymbol{u}_\delta.$$

We note that the quantity $\frac{1}{2}\int_0^t \int_{\Omega_{f,\delta}(t)} ((\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta)\cdot\boldsymbol{u}_\delta$, is well-defined because $\boldsymbol{u}_\delta \in L^\infty(0,T;L^2(\Omega_{f,\delta}(t)))\cap L^2(0,T;H^1(\Omega_{f,\delta}(t)))$, which by interpolation is in $L^4(0,T;H^{1/2}(\Omega_{f,\delta}(t)))$, and hence by Sobolev inequalities embeds into $L^4(0,T;L^4(\Omega_{f,\delta}(t)))$.

We want to transfer the integrals

$$\int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta), \qquad \int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta))\cdot\boldsymbol{u}, \tag{154}$$

to integrals on $\Omega_{f,\delta}(t)$ by using the map $\psi_\delta : \Omega_{f,\delta}(t) \to \Omega_f(t)$ defined by (99). We use

$$\widehat{\boldsymbol{u}} = \gamma_\delta J_\delta^{-1}\cdot(\boldsymbol{u}\circ\psi_\delta), \qquad \widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta = \gamma_\delta J_\delta^{-1}\cdot((\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta)\circ\psi_\delta),$$

where we recall the definitions of the appropriate terms from (99), (101), (103), and (104).

Following arguments found in [57], we obtain the following estimates. We have, using (100), that

$$\begin{aligned}
\int_0^t \int_{\Omega_f(t)} &((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta) = \int_0^t \int_{\Omega_{f,\delta}(t)} \gamma_\delta[(\nabla(\boldsymbol{u}\circ\psi_\delta))J_\delta^{-1}(\boldsymbol{u}\circ\psi_\delta)]\cdot(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta)\circ\psi_\delta \\
&= \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}}]\cdot[\gamma_\delta^{-1}J_\delta(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)] \\
&= \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) - \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}}]\cdot[(I-\gamma_\delta^{-1}J_\delta)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)] \\
&= \int_0^t \int_{\Omega_{f,\delta}(t)} ((\nabla\widehat{\boldsymbol{u}})\widehat{\boldsymbol{u}})\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) + \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla((I-\gamma_\delta J_\delta^{-1})(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}})\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) \\
&- \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}}]\cdot[(I-\gamma_\delta^{-1}J_\delta)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)] \\
&= \int_0^t \int_{\Omega_{f,\delta}(t)} [(\widehat{\boldsymbol{u}}\cdot\nabla)\widehat{\boldsymbol{u}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) + R_{2,1},
\end{aligned} \tag{155}$$

where

$$R_{2,1} = \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla((I-\gamma_\delta J_\delta^{-1})(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}})\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) - \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u}\circ\psi_\delta))\widehat{\boldsymbol{u}}]\cdot[(I-\gamma_\delta^{-1}J_\delta)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)].$$

In the following estimates, we will repeatedly use the following inequalities, which hold for a constant C that is independent of δ:

$$|\gamma_\delta^{-1}J_\delta - I| \leqslant C(|\gamma_\delta^{-1}-1| + |\nabla\gamma_\delta|) \leqslant C||\omega-\omega_\delta||_{H^2(\Gamma)},$$

$$|\gamma_\delta J_\delta^{-1} - I| \leqslant C(|\gamma_\delta - 1| + |\nabla\gamma_\delta|) \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)},$$
$$|\nabla(\gamma_\delta J_\delta^{-1})| \leqslant C(|\partial_x\gamma_\delta| + |\partial_{xx}\gamma_\delta|) \leqslant C(||\omega - \omega_\delta||_{H^2(\Gamma)} + |\partial_{xx}(\omega - \omega_\delta)|), \tag{156}$$

so that

$$||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,\delta}(t))} \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)}. \tag{157}$$

To obtain (156), we estimate $|\partial_{xx}\gamma_\delta|$ by using the fact that ω is smooth so that $|\partial_{xx}\omega| \leqslant C$ and a direct computation of $\partial_{xx}\gamma_\delta$. Using these estimates, the Leibniz rule, and the smoothness of $\boldsymbol{u}$, we get

$$\left|\int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla((I - \gamma_\delta J_\delta^{-1})(\boldsymbol{u} \circ \psi_\delta))\widehat{\boldsymbol{u}}) \cdot (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)\right|$$
$$\leqslant C \int_0^t ||\omega - \omega_\delta||_{H^2(\Gamma)} ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||_{L^2(\Omega_{f,\delta}(t))} \leqslant C \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).$$

By using (100), and the fact that $|J_\delta| \leqslant C$ is uniformly bounded, due to the fact that $|J_\delta| \leqslant C(1 + ||\omega - \omega_\delta||_{H^2(\Gamma)}) \leqslant C$ is uniformly bounded, we obtain a similar estimate:

$$\int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\boldsymbol{u} \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot [(I - \gamma_\delta^{-1} J_\delta)(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)] \leqslant C \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).$$

Thus, we obtain

$$|R_1| \leqslant C \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right). \tag{158}$$

We now focus on the second integral in (154). By using (100) we obtain

$$\int_0^t \int_{\Omega_f(t)} ((\boldsymbol{u} \cdot \nabla)(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta)) \cdot \boldsymbol{u} = \int_0^t \int_{\Omega_{f,\delta}(t)} \gamma_\delta [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta)) J_\delta^{-1} (\boldsymbol{u} \circ \psi_\delta)] \cdot (\boldsymbol{u} \circ \psi_\delta)$$
$$= \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot (\gamma_\delta^{-1} J_\delta \widehat{\boldsymbol{u}})$$
$$= \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot \widehat{\boldsymbol{u}} - \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot [(I - \gamma_\delta^{-1} J_\delta)\widehat{\boldsymbol{u}}]$$
$$= \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)\widehat{\boldsymbol{u}}) \cdot \widehat{\boldsymbol{u}} + \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla[(I - \gamma_\delta J_\delta^{-1})((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta)]\widehat{\boldsymbol{u}}) \cdot \widehat{\boldsymbol{u}}$$
$$- \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot [(I - \gamma_\delta^{-1} J_\delta)\widehat{\boldsymbol{u}}]$$
$$= \int_0^t \int_{\Omega_{f,\delta}(t)} ((\widehat{\boldsymbol{u}} \cdot \nabla)(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)) \cdot \widehat{\boldsymbol{u}} + R_{2,2}, \tag{159}$$

where

$$R_{2,2} := \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla[(I - \gamma_\delta J_\delta^{-1})((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta)]\widehat{\boldsymbol{u}}) \cdot \widehat{\boldsymbol{u}} - \int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta))\widehat{\boldsymbol{u}}] \cdot [(I - \gamma_\delta^{-1} J_\delta)\widehat{\boldsymbol{u}}].$$

To estimate $R_{2,2}$, we will use the following inequalities:

$$|(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi| = |\gamma_\delta^{-1} J_\delta \cdot (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)| \leqslant C|\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta|,$$
$$|\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta)| = |\nabla(\gamma_\delta^{-1} J_\delta \cdot (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta))| \leqslant |\nabla(\gamma_\delta^{-1} J_\delta)| \cdot |\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta| + |\gamma_\delta^{-1} J_\delta| \cdot |\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)|$$
$$\leqslant C(|\nabla(\gamma_\delta^{-1} J_\delta)| \cdot |\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta| + |\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)|).$$

From the fact that $\max\left(|I-\gamma_\delta^{-1}J_\delta|, |I-\gamma_\delta J_\delta^{-1}|\right) \leqslant C\min\left(1, ||\omega-\omega_\delta||_{H^2(\Gamma)}\right)$, we obtain:

$$
\begin{aligned}
|R_{2,2}| &\leqslant C\left(\int_0^t\int_{\Omega_{f,\delta}(t)} |\nabla(\gamma_\delta J_\delta^{-1})|\cdot|(\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta)\circ\psi_\delta| + \int_0^t\int_{\Omega_{f,\delta}(t)} |I-\gamma_\delta J_\delta^{-1}|\cdot|\nabla((\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta)\circ\psi_\delta)|\right.\\
&\qquad\left. + \int_0^t\int_{\Omega_{f,\delta}(t)} |I-\gamma_\delta^{-1}J_\delta|\cdot|\nabla((\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta)\circ\psi_\delta)|\right)\\
&\leqslant C\left(\int_0^t\int_{\Omega_{f,\delta}(t)} \left(|\nabla(\gamma_\delta J_\delta^{-1})| + |\nabla(\gamma_\delta^{-1}J_\delta)|\right)\cdot|\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta| + \int_0^t\int_{\Omega_{f,\delta}(t)} ||\omega-\omega_\delta||_{H^2(\Gamma)}\cdot|\nabla(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)|\right)\\
&\leqslant \epsilon\int_0^t ||\nabla(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right). \qquad (160)
\end{aligned}
$$

In the last line, we use the following estimates, derived similarly as for (157),

$$
\begin{gathered}
|\nabla(\gamma_\delta^{-1}J_\delta)| \leqslant C(|\partial_x(\gamma_\delta^{-1})| + |\partial_x\gamma_\delta| + |\partial_{xx}\gamma_\delta|) \leqslant C(||\omega-\omega_\delta||_{H^2(\Gamma)} + |\partial_{xx}(\omega-\omega_\delta)|),\\
||\nabla(\gamma_\delta^{-1}J_\delta)||_{L^2(\Omega_{f,\delta}(t))} \leqslant C||\omega-\omega_\delta||_{H^2(\Gamma)}.
\end{gathered}
$$

Therefore, for the expression in (116), after transferring the integrals (155) and (159) and estimating $R_{2,1}$ (158) and $R_{2,2}$ (160), the remaining terms are:

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_{\Omega_{f,\delta}(t)} [(\widehat{\boldsymbol{u}}\cdot\nabla)\widehat{\boldsymbol{u}}]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) - [(\widehat{\boldsymbol{u}}\cdot\nabla)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)]\cdot\widehat{\boldsymbol{u}}\\
&-\frac{1}{2}\int_0^t\int_{\Omega_{f,\delta}(t))} [(\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta]\cdot(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta) - [(\boldsymbol{u}_\delta\cdot\nabla)(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)]\cdot\boldsymbol{u}_\delta\\
&=\frac{1}{2}\int_0^t\int_{\Omega_{f,\delta}(t)} [((\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)\cdot\nabla)\boldsymbol{u}_\delta]\cdot\widehat{\boldsymbol{u}} - \frac{1}{2}\int_0^t\int_{\Omega_{f,\delta}(t)} [((\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)\cdot\nabla)\widehat{\boldsymbol{u}}]\cdot\boldsymbol{u}_\delta.
\end{aligned}
$$

In absolute values, the right hand-side can be bounded as follows:

$$
\leqslant \epsilon\int_0^t ||\nabla(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}.
$$

Combining this estimate with (158) and (160) we obtain

$$
|T_2| \leqslant \epsilon\int_0^t ||\nabla(\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).
$$

Term T3. To estimate term T_3 defined in (117), we start by noting that because $\boldsymbol{u}$ and $\boldsymbol{\xi}$ are smooth, we can pass to the limit as $\nu \to 0$ using Proposition 10.1 and the fact that $(\boldsymbol{\xi}_\delta)_\nu \to \boldsymbol{\xi}_\delta$ strongly in $L^2(0,T;H^1(\Omega_b))$, so that we can ultimately just test with $\boldsymbol{v} = \boldsymbol{u}-\breve{\boldsymbol{u}}_\delta$ and $\boldsymbol{\psi} = \boldsymbol{\xi}-\boldsymbol{\xi}_\delta$. In the regularized weak formulation for $\boldsymbol{u}_\delta$, we test with $\boldsymbol{u}$ and $\boldsymbol{\xi}$. Note that both test functions $\boldsymbol{u}-\breve{\boldsymbol{u}}_\delta$ and $\widehat{\boldsymbol{u}}-\boldsymbol{u}_\delta$ have the same trace along $\Gamma(t)$ and $\Gamma_\delta(t)$ respectively, which we will formally denote by $\boldsymbol{u}-\boldsymbol{u}_\delta$ along the reference configuration of the interface Γ. Combining the resulting expressions, we have the following contribution of T_3 in the limit as $\nu \to 0$:

$$
\begin{aligned}
T_3 = &\frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{u}\cdot\boldsymbol{n} - \boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{u}\cdot(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{u}_\delta\cdot\boldsymbol{n}_\delta - \boldsymbol{\xi}_\delta\cdot\boldsymbol{n}_\delta)\boldsymbol{u}_\delta\cdot\widehat{\boldsymbol{u}} \\
&+ \frac{1}{2}\int_0^t \int_{\Gamma(t)} |\boldsymbol{u}|^2(\boldsymbol{\xi}\cdot\boldsymbol{n} - \boldsymbol{u}\cdot\boldsymbol{n}) - \frac{1}{2}\int_0^t \int_{\Gamma(t)} |\boldsymbol{u}|^2(\boldsymbol{\xi}_\delta\cdot\boldsymbol{n} - \breve{\boldsymbol{u}}_\delta\cdot\boldsymbol{n}) \\
&- \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} |\boldsymbol{u}_\delta|^2(\boldsymbol{\xi}\cdot\boldsymbol{n}_\delta - \widehat{\boldsymbol{u}}\cdot\boldsymbol{n}_\delta) = \frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n} - \boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\breve{\boldsymbol{u}}_\delta \\
&- \frac{1}{2}\int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi}_\delta\cdot\boldsymbol{n} - \breve{\boldsymbol{u}}_\delta\cdot\boldsymbol{n})|\boldsymbol{u}|^2 - \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n}_\delta - \widehat{\boldsymbol{u}}\cdot\boldsymbol{n}_\delta)|\boldsymbol{u}_\delta|^2 \\
&+ \frac{1}{2}\int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta\cdot\boldsymbol{n}_\delta - \boldsymbol{u}_\delta\cdot\boldsymbol{n}_\delta)\boldsymbol{u}_\delta\cdot\widehat{\boldsymbol{u}} = R_{3,1} + R_{3,2},
\end{aligned}
$$

where

$$
R_{3,1} = \frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{u})_y \boldsymbol{u}_\delta\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta) - \frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{\xi}_\delta - \boldsymbol{u}_\delta)_y \boldsymbol{u}\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta),
$$
$$
R_{3,2} = \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega(\boldsymbol{u})_x \boldsymbol{u}\cdot\boldsymbol{u}_\delta - \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega(\boldsymbol{u}_\delta)_x |\boldsymbol{u}|^2 - \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega_\delta(\boldsymbol{u})_x |\boldsymbol{u}_\delta|^2 + \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega_\delta(\boldsymbol{u}_\delta)_x \boldsymbol{u}\cdot\boldsymbol{u}_\delta.
$$

We estimate $R_{3,1}$ as follows: decompose $R_{3,1}$ as $R_{3,1} = R_{3,1,1} + R_{3,1,2}$, where

$$
R_{3,1,1} = -\frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{\xi})_y(\boldsymbol{u} - \boldsymbol{u}_\delta)\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta) + \frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_y \boldsymbol{u}\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta),
$$
$$
R_{3,1,2} = \frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{u})_y(\boldsymbol{u} - \boldsymbol{u}_\delta)\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta) - \frac{1}{2}\int_0^t \int_\Gamma (\boldsymbol{u} - \boldsymbol{u}_\delta)_y \boldsymbol{u}\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta).
$$

By interpolation,

$$
\begin{aligned}
|R_{3,1,1}| &\leqslant C\left(\int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^{1/2}_{L^2(\Omega_{f,\delta}(t))} ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^{3/2}_{H^1(\Omega_{f,\delta}(t))} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||_{L^2(\Gamma)} ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||_{H^1(\Omega_{f,\delta}(t))}\right) \\
&\leqslant \epsilon \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)}\right).
\end{aligned}
$$

By using the same interpolation inequality, we obtain the following estimate for $R_{3,1,2}$.

$$
|R_{3,1,2}| \leqslant \epsilon \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} + C(\epsilon)\int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}.
$$

We estimate $R_{3,2}$ by first rewriting $R_{3,2}$ as follows:

$$
\begin{aligned}
R_{3,2} = &-\frac{1}{2}\int_0^t \int_\Gamma (\partial_x\omega - \partial_x\omega_\delta)(\boldsymbol{u})_x \boldsymbol{u}\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta) - \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega_\delta(\boldsymbol{u})_x(\boldsymbol{u} - \boldsymbol{u}_\delta)\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta) \\
&+ \frac{1}{2}\int_0^t \int_\Gamma (\partial_x\omega - \partial_x\omega_\delta)(\boldsymbol{u} - \boldsymbol{u}_\delta)_x |\boldsymbol{u}|^2 + \frac{1}{2}\int_0^t \int_\Gamma \partial_x\omega_\delta(\boldsymbol{u} - \boldsymbol{u}_\delta)_x \boldsymbol{u}\cdot(\boldsymbol{u} - \boldsymbol{u}_\delta).
\end{aligned}
$$

By interpolation, by the boundedness of $|\partial_x\omega|$ and $|\partial_x\omega_\delta|$, and by the smoothness of $\boldsymbol{u}$, we get:

$$
|R_{3,2}| \leqslant \epsilon \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} + C(\epsilon)\left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))}\right).
$$

Hence, by combining the two estimates we get the final estimate for T_3:

$$|T_3| \leqslant \epsilon \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{H^1(\Omega_{f,\delta}(t))} + C(\epsilon) \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))} \right).$$

Term T4. To estimate term T_4, defined in (118), we again use Proposition 10.1 to pass to the limit as $\nu \to 0$ so that the contribution from T_4 is

$$T_4 := 2\nu \int_0^t \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) - 2\nu \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{D}(\boldsymbol{u}_\delta) : \boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta). \tag{161}$$

We want to transfer the integral on $\Omega_1(t)$ to $\Omega_{f,\delta}(t)$. Recalling (100), we have that

$$\int_0^t \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) = \int_0^t \int_{\Omega_{f,\delta}(t)} \gamma_\delta [\nabla(\boldsymbol{u} \circ \psi_\delta) J_\delta^{-1}]^s : [\nabla((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta) J_\delta^{-1}]^s,$$

where the superscript 's' notation denotes a symmetrization. Following the procedure in [57], we break up the integral as

$$\int_0^t \int_{\Omega_f(t)} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) = \int_0^t \int_{\Omega_{f,\delta}(t)} \boldsymbol{D}(\widehat{\boldsymbol{u}}) : \boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta) + R_{4,1} + R_{4,2} + R_{4,3} + R_{4,4}, \tag{162}$$

where

$$\begin{aligned}
R_{4,1} &= \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla(\boldsymbol{u} \circ \psi_\delta) J_\delta^{-1})^s : [\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)(J_\delta^{-1} - I) + (J_\delta - I)\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta) J_\delta^{-1}]^s, \\
R_{4,2} &= \int_0^t \int_{\Omega_{f,\delta}(t)} [(I - \gamma_\delta J_\delta^{-1})\nabla(\boldsymbol{u} \circ \psi_\delta) + \nabla(\boldsymbol{u} \circ \psi_\delta)(J_\delta^{-1} - I)]^s : \boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta), \\
R_{4,3} &= \int_0^t \int_{\Omega_{f,\delta}(t)} (\nabla(\boldsymbol{u} \circ \psi_\delta) J_\delta^{-1})^s : (\gamma_\delta \nabla(\gamma_\delta^{-1} J_\delta)(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta) J_\delta^{-1})^s, \\
R_{4,4} &= -\int_0^t \int_{\Omega_{f,\delta}(t)} [(\nabla(\gamma_\delta J_\delta^{-1}))\boldsymbol{u} \circ \psi_\delta]^s : \boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta).
\end{aligned}$$

To verify this equality, one can use the Leibniz rule, the definition $\widehat{\boldsymbol{u}} = \gamma_\delta J_\delta^{-1} \cdot (\boldsymbol{u} \circ \psi_\delta)$, and the identity $\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta = \gamma_\delta J_\delta^{-1} \cdot ((\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta) \circ \psi_\delta)$.

We now estimate the terms $R_{4,1}$-$R_{4,4}$. For this purpose we will use the following inequalities:

$$\begin{gathered}
|J_\delta^{-1}| \leqslant C(1 + |\partial_x \gamma_\delta|), \quad |J_\delta^{-1} - I| \leqslant C(|\gamma_\delta^{-1} - 1| + |\partial_x \gamma_\delta|), \\
|J_\delta - I| \leqslant C(|\gamma_\delta - 1| + |\partial_x \gamma_\delta|), \quad |\gamma_\delta J_\delta^{-1} - I| \leqslant C(|\gamma_\delta - 1| + |\partial_x \gamma_\delta|).
\end{gathered}$$

and, recalling the definition of γ_δ in (99), we have the following inequalities:

$$|\gamma_\delta - 1| \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)}, \qquad |\gamma_\delta^{-1} - 1| \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)},$$

$$|\partial_x \gamma_\delta| \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)}, \qquad |\partial_x(\gamma_\delta^{-1})| \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)}.$$

Because $|J_\delta^{-1}| \leqslant C(1 + |\partial_x \gamma_\delta|) \leqslant C$ since $|\partial_x \gamma_\delta|$ is bounded, and because $\boldsymbol{u}$ is smooth,

$$\begin{aligned}
|R_{4,1}| &\leqslant C \int_0^t ||\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} (||\gamma_\delta^{-1} - 1||_{L^2(\Omega_{f,\delta}(t))} + ||\gamma_\delta - 1||_{L^2(\Omega_{f,\delta}(t))} + ||\partial_x \gamma_\delta||_{L^2(\Omega_{f,\delta}(t))}) \\
&\leqslant \epsilon \int_0^t ||\nabla(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}.
\end{aligned}$$

We also have that

$$|R_{4,2}| \leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}.$$

For $R_{4,3}$ and $R_{4,4}$, we compute that

$$\nabla(\gamma_\delta^{-1} J_\delta) = \nabla \begin{pmatrix} \gamma_\delta^{-1} & 0 \\ (R+y)\gamma_\delta^{-1}\partial_x \gamma_\delta & 1 \end{pmatrix}, \qquad \nabla(\gamma_\delta J_\delta^{-1}) = \nabla \begin{pmatrix} \gamma_\delta & 0 \\ -(R+y)\partial_x \gamma_\delta & 1 \end{pmatrix}.$$

Therefore,

$$|\nabla(\gamma_\delta^{-1} J_\delta)| \leqslant C(|\partial_x(\gamma_\delta^{-1})| + |\partial_x \gamma_\delta| + |\partial_{xx}\gamma_\delta|), \qquad |\nabla(\gamma_\delta J_\delta^{-1})| \leqslant C(|\partial_x \gamma_\delta| + |\partial_{xx}\gamma_\delta|),$$

where we can estimate

$$|\partial_{xx}\gamma_\delta| \leqslant C(||\omega - \omega_\delta||_{H^2(\Gamma)} |\partial_{xx}\omega| + |\partial_{xx}(\omega - \omega_\delta)| + ||\omega - \omega_\delta||_{H^2(\Gamma)}).$$

So since $||\partial_{xx}\omega||_{L^2(\Omega_{f,\delta}(t))} \leqslant C$ since ω is uniformly bounded in $H^2(\Gamma)$, we have that

$$\begin{aligned} |R_{4,3}| &\leqslant C \int_0^t ||\nabla(\gamma_\delta^{-1} J_\delta)||_{L^2(\Omega_{f,\delta}(t))} ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||_{L^2(\Omega_{f,\delta}(t))} \leqslant C \int_0^t ||\omega - \omega_\delta||_{H^2(\Gamma)} ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||_{L^2(\Omega_{f,\delta}(t))} \\ &\leqslant C \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))} \right). \end{aligned}$$

Similarly, using $||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,\delta}(t))} \leqslant C||\omega - \omega_\delta||_{H^2(\Gamma)}$, we have the following estimate for R_4:

$$\begin{aligned} |R_{4,4}| &\leqslant C \int_0^t ||\nabla(\gamma_\delta J_\delta^{-1})||_{L^2(\Omega_{f,\delta}(t))} ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} \\ &\leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}. \end{aligned}$$

We now have the final estimate of T_4, obtained after using (161) and (162) as follows:

$$T_4 = 2\nu \int_0^t \int_{\Omega_{f,\delta}(t)} |\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)|^2 + R_4,$$

where

$$|R_4| \leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta||^2_{L^2(\Omega_{f,\delta}(t))} \right).$$

Term T5. Similarly as before, after passing to the limit as $\nu \to 0$ in term T_5, defined by (163), the contribution of this term is

$$T_5 = \beta \int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi} - \boldsymbol{u})_\tau [(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta)_\tau] - \beta \int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi}_\delta - \boldsymbol{u}_\delta)_\tau [(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)_\tau]. \quad (163)$$

We note that when we test the weak formulation for $\boldsymbol{u}$ with $\boldsymbol{v} = \boldsymbol{u} - (\breve{\boldsymbol{u}}_\delta)_\nu$ and $\boldsymbol{\psi} = \boldsymbol{\xi} - (\boldsymbol{\xi}_\delta)_\nu$, we can pass to the limit as $\nu \to 0$ to obtain the first term in T_5 above, by using similar arguments involving Proposition 10.1, as for the previously considered terms. This term can now be rewritten as follows:

$$T_5 = \beta \int_0^t \int_{\Gamma_\delta(t)} |(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)_\tau|^2 + R_5,$$

where

$$R_5 = \beta \int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi} - \boldsymbol{u})_\tau [(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta)_\tau] - \beta \int_0^t \int_{\Gamma_\delta(t)} (\boldsymbol{\xi} - \widehat{\boldsymbol{u}})_\tau [(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)_\tau].$$

Denote the arc length elements of $\Gamma_t(t)$ and $\Gamma_\delta(t)$ respectively by $\mathcal{J}_\Gamma^\omega = \sqrt{1 + |\partial_x \omega|^2}$ and $\mathcal{J}_\Gamma^{\omega_\delta} = \sqrt{1 + |\partial_x \omega_\delta|^2}$, and we denote the tangent vectors to $\Gamma(t)$ and $\Gamma_\delta(t)$ respectively by $\boldsymbol{\tau}_1 = \frac{1}{\mathcal{J}_\Gamma^\omega}(1, \partial_x \omega)$ and $\boldsymbol{\tau}_\delta = \frac{1}{\mathcal{J}_\Gamma^{\omega_\delta}}(1, \partial_x \omega_\delta)$. We can now rewrite R_5 by writing everything in terms of the x and y components. For this purpose, recall that $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_\delta$ along the interface displace in only the y direction. We formally express the common trace of $\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta$ and $\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta$ along the reference configuration of the interface Γ by $\boldsymbol{u} - \boldsymbol{u}_\delta$. Thus,

$$\begin{aligned} R_5 = \beta &\int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{u}) \cdot (1, \partial_x \omega)[(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) - (\boldsymbol{u} - \boldsymbol{u}_\delta)] \cdot \boldsymbol{\tau}_1 \\ - \beta &\int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{u}) \cdot (1, \partial_x \omega_\delta)[(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) - (\boldsymbol{u} - \boldsymbol{u}_\delta)] \cdot \boldsymbol{\tau}_\delta. \end{aligned}$$

In the previous step, we used the fact that when transferred back to the reference configuration Ω_f, $\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta$ and $\boldsymbol{u} - \breve{\boldsymbol{u}}_\delta$ have the same trace along Γ. Thus, $R_5 = R_{5,1} + R_{5,2}$, where

$$\begin{aligned} R_{5,1} &= \beta \int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{u})_y (\partial_x \omega - \partial_x \omega_\delta)[(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) - (\boldsymbol{u} - \boldsymbol{u}_\delta)] \cdot \boldsymbol{\tau}_1, \\ R_{5,2} &= \beta \int_0^t \int_\Gamma (\boldsymbol{\xi} - \boldsymbol{u}) \cdot (1, \partial_x \omega_\delta)[(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) - (\boldsymbol{u} - \boldsymbol{u}_\delta)] \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_\delta). \end{aligned}$$

We can use the fact that $|\partial_x \omega|$ and $|\partial_x \omega_\delta|$ are uniformly bounded to obtain the following estimates:

$$|R_{5,1}| \leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)} \right),$$

where we used the trace inequality, Poincare's inequality, and Korn's inequality for the fluid. For the second term $R_{6,2}$, we use the estimate $|\boldsymbol{\tau}^\omega - \boldsymbol{\tau}^{\omega_\delta}| \leqslant C|\partial_x \omega - \partial_x \omega_\delta|$ to obtain

$$|R_{5,2}| \leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)} \right).$$

Hence,

$$T_5 = \beta \int_0^t \int_{\Gamma_\delta(t)} |(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)_\tau - (\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)_\tau|^2 + R_6,$$

where

$$|R_5| \leqslant \epsilon \int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||_{L^2(\Omega_{f,\delta}(t))} + C(\epsilon) \left(\int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\boldsymbol{\xi} - \boldsymbol{\xi}_\delta||^2_{L^2(\Gamma)} \right).$$

Terms T6-T8. We will present estimates only for term T_6, defined in (119), as the same procedure will hold for T_7 and T_8. Since ζ and ζ_δ are weakly continuous in $L^2(\Gamma)$, by the weak formulation, we get:

$$\int_0^t \int_\Gamma \zeta \cdot \partial_t [(\zeta_\delta)_\nu] + \int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t \zeta = \int_0^t \int_\Gamma \zeta \cdot \partial_t [(\zeta_\delta)_\nu] + \int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t [(\zeta)_\nu] - \int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t [(\zeta)_\nu - \zeta]$$
$$\to \int_\Gamma \zeta(t) \cdot \zeta_\delta(t) - \int_\Gamma |\zeta_0|^2.$$

This follows from Lemma 2.5 in [57], which implies:

$$\int_0^t \int_\Gamma \zeta \cdot \partial_t [(\zeta_\delta)_\nu] + \int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t [(\zeta)_\nu] \to \int_\Gamma \zeta(t) \cdot \zeta_\delta(t) - \int_\Gamma |\zeta_0|^2, \qquad \text{as } \nu \to 0,$$

and from the fact that ζ is smooth in space and time, which implies

$$\int_0^t \int_\Gamma \zeta_\delta \cdot \partial_t [(\zeta)_\nu - \zeta] \to 0, \qquad \text{as } \nu \to 0.$$

Furthermore, because $\zeta(0) = \zeta_\delta(0) = \zeta_0$ weak continuity of ζ_δ at $t = 0$ implies that $\int_\Gamma \zeta(0) \cdot [\zeta(0) - (\zeta_\delta)_\nu(0)] \to 0$ as $\nu \to 0$. Similarly, $\int_\Gamma \zeta(\tau) \cdot [\zeta(\tau) - (\zeta_\delta)_\nu(\tau)] \to 0$ as $\nu \to 0$ for almost every $t \in [0, T]$. Hence, as $\nu \to 0$, the contribution from T_6 is

$$T_6 = \frac{1}{2}\rho_p \int_\Gamma |(\zeta - \zeta_2)(\tau)|^2.$$

Similarly, the contributions from T_7 and T_8 as $\nu \to 0$ are

$$T_7 = \frac{1}{2} \int_\Gamma |\Delta(\omega - \omega_2)(\tau)|^2, \qquad T_8 = \frac{1}{2}\rho_b \int_{\Omega_b} |(\boldsymbol{\xi} - \boldsymbol{\xi}_2)(\tau)|^2.$$

Terms T9-T12. Since these calculations are straight forward, a discussion about the limiting expressions as $\nu \to 0$ for terms T_9-T_{12} was presented earlier, just under (122).

Term T13. Similarly as before, by taking the limit as $\nu \to 0$, we have that

$$T_{13} = -\alpha \int_0^t \int_{\Omega_{b,1}(t)} p \nabla \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) + \alpha \int_0^t \int_{\Omega^\delta_{b,\delta}(t)} p_\delta \nabla \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta).$$

To estimate this term we use (21) and the matrix identity $\boldsymbol{B}^{-1} = \frac{1}{\det(\boldsymbol{B})} \boldsymbol{B}^C$ to obtain

$$|T_{13}| = \alpha \left| \int_0^t \int_{\Omega_b} \mathcal{J}_b^\eta p \nabla_b^\eta \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) - \int_0^t \int_{\Omega_b} \mathcal{J}_b^{\eta_\delta^\delta} p_\delta \nabla_b^{\eta_\delta^\delta} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) \right|$$
$$= \alpha \left| \int_0^t \int_{\Omega_b} p \cdot \operatorname{tr}\left(\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta})^C\right) - \int_0^t \int_{\Omega_b} p_\delta \cdot \operatorname{tr}\left(\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^C\right) \right| \leqslant R_{13,1} + R_{13,2},$$

where the superscript "C" denotes the cofactor matrix. The integrals $R_{13,1}$ and $R_{13,2}$ are defined as follows:

$$R_{13,1} = \alpha \left| \int_0^t \int_{\Omega_b} p \cdot \operatorname{tr}\left(\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) \cdot (\nabla(\boldsymbol{\eta} - \boldsymbol{\eta}_\delta^\delta))^C\right) \right|,$$
$$R_{13,2} = \alpha \left| \int_0^t \int_{\Omega_b} (p - p_\delta) \cdot \operatorname{tr}\left(\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^C\right) \right|.$$

In the previous calculations, we observe that the cofactor matrix operation is linear when the matrices are two by two. Using the fact that p is smooth, the assumption (133), and the fact that

$$||\nabla \boldsymbol{\eta}^\delta - \nabla \boldsymbol{\eta}_\delta^\delta||_{L^2(\Omega_b)} \leqslant C||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_2||_{L^2(\tilde{\Omega}_b)} \leqslant C\left(||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||_{L^2(\Omega_b)} + ||\omega - \omega_\delta||_{H^2(\Gamma)}\right) \qquad (164)$$

for a constant C independent of δ, by Young's convolution inequality and the definition of odd extension to the larger domain $\tilde{\Omega}_b$ in Definition 8.2, we obtain the estimates on $R_{13,1}$ and $R_{13,2}$:

$$\begin{aligned}
R_{13,1} &\leqslant \epsilon \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||^2_{L^2(\Omega_b)} + C(\epsilon)\left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta^\delta||^2_{L^2(\Omega_b)}\right) \\
&\leqslant C(\epsilon)\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)} + \epsilon \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||^2_{L^2(\Omega_b)} \\
&\quad + C(\epsilon)\left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}\right), \\
R_{13,2} &\leqslant \epsilon \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||^2_{L^2(\Omega_b)} + C(\epsilon)\left(\int_0^t ||p - p_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}$$

Therefore, the final estimate for T_{13} is as follows:

$$\begin{aligned}
|T_{13}| &\leqslant C(\epsilon)\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)} + \epsilon \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||^2_{L^2(\Omega_b)} \\
&\quad + C(\epsilon)\left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||p - p_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}$$

Term T14. This term can be handled in the same way as terms T_6-T_8.

Term T15. We pass to the limit as $\nu \to 0$ in (124) to obtain:

$$T_{15} = -\alpha \int_0^t \int_{\Omega_{b,1}(t)} \frac{D}{Dt}\boldsymbol{\eta} \cdot \nabla(p - p_\delta) + \alpha \int_0^t \int_{\Omega^\delta_{b,\delta}(t)} \frac{D^\delta}{Dt}\boldsymbol{\eta}_\delta \cdot \nabla(p - p_\delta).$$

To estimate this term we pull back to the reference domain and use (21) and the cofactor formula for the matrix inverse to obtain:

$$\begin{aligned}
|T_{15}| &= \alpha \left| \int_0^t \int_{\Omega_b} \mathcal{J}_b^\eta \partial_t \boldsymbol{\eta} \cdot \nabla_b^\eta (p - p_\delta) - \int_0^t \int_{\Omega_b} \mathcal{J}_b^{\eta_\delta^\delta} \partial_t \boldsymbol{\eta}_\delta \cdot \nabla_b^{\eta_\delta^\delta}(p - p_\delta) \right| \\
&= \alpha \left| \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta} \cdot \left[\nabla(p - p_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta})^C\right] - \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta}_\delta \cdot \left[\nabla(p - p_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^C\right] \right| \leqslant R_{15,1} + R_{15,2},
\end{aligned}$$

where

$$\begin{aligned}
R_{15,1} &= \alpha \left| \int_0^t \int_{\Omega_b} \partial_t \boldsymbol{\eta} \cdot \left[\nabla(p - p_\delta) \cdot (\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta^\delta)^C\right] \right|, \\
R_{15,2} &= \alpha \left| \int_0^t \int_{\Omega_b} (\partial_t \boldsymbol{\eta} - \partial_t \boldsymbol{\eta}_\delta) \cdot \left[\nabla(p - p_\delta) \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^C\right] \right|.
\end{aligned}$$

To estimate $R_{15,1}$, we use (131), (132), and the convolution inequality (164) to obtain:

$$\begin{aligned}
R_{15,1} &\leqslant \epsilon \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} + C(\epsilon)\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} \\
&\quad + C(\epsilon)\left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}\right).
\end{aligned}$$

Here, we also used the following estimate on the norm of the gradient of the pressure on the reference domain and on the moving domain, which is obtained by using (131), (132), and (21):

$$
\begin{aligned}
||\nabla(p-p_\delta)(t)||^2_{L^2(\Omega_b)} &= \int_{\Omega_b} |\nabla(p-p_\delta)|^2 = \int_{\Omega_b} \mathcal{J}_b^{\eta_\delta^\delta} |\nabla_b^{\eta_\delta^\delta}(p-p_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)|^2 \cdot (\mathcal{J}_b^{\eta_\delta^\delta})^{-1} \\
&\leqslant C \int_{\Omega_b} \mathcal{J}_b^{\eta_\delta^\delta} |\nabla_b^{\eta_\delta^\delta}(p-p_\delta)|^2 = C||\nabla(p-p_\delta)(t)||^2_{L^2(\Omega_{b,\delta}^\delta(t))},
\end{aligned}
\tag{165}
$$

where constant C is independent of δ and $t \in [0, T_\delta]$.

The estimate of $R_{15,2}$ is straight forward:

$$
R_{15,2} \leqslant \epsilon \int_0^t ||\nabla(p-p_\delta)||^2_{L^2(\Omega_{b,\delta}^\delta(t))} + C(\epsilon)\int_0^t ||\partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)}.
$$

From here we get the final estimate of T_{15}:

$$
\begin{aligned}
|T_{15}| \leqslant\ & \epsilon \int_0^t ||\nabla(p-p_\delta)||^2_{L^2(\Omega_{b,\delta}^\delta(t))} + C(\epsilon)\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} \\
&+ C(\epsilon)\left(\int_0^t ||\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega-\omega_\delta||^2_{H^2(\Gamma)} + \int_0^t ||\partial_t\boldsymbol{\eta} - \partial_t\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}
$$

Term T16. To estimate T_{16} defined in (125) we start by passing to the limit as $\nu \to 0$ to obtain

$$
T_{16} = -\alpha \int_0^t \int_{\Gamma(t)} (\boldsymbol{\xi}\cdot\boldsymbol{n})(p-p_\delta) + \alpha \int_0^t \int_{\Gamma_\delta^\delta(t)} (\boldsymbol{\xi}_\delta\cdot\boldsymbol{n}_\delta^\delta)(p-p_\delta),
$$

where $\boldsymbol{n}_\delta^\delta$ is the upward pointing normal vector to $\Gamma_\delta^\delta(t)$. We integrate by parts to obtain that $|T_{16}| \leqslant R_{16,1} + R_{16,2}$, where

$$
R_{16,1} := \alpha \left| \int_0^t \int_{\Omega_{b,1}(t)} (\nabla\cdot\boldsymbol{\xi})(p-p_\delta) - \int_0^t \int_{\Omega_{b,\delta}^\delta(t)} (\nabla\cdot\boldsymbol{\xi}_\delta)(p-p_\delta) \right|,
$$

$$
R_{16,2} := \alpha \left| \int_0^t \int_{\Omega_{b,1}(t)} \boldsymbol{\xi}\cdot\nabla(p-p_\delta) - \int_0^t \int_{\Omega_{b,\delta}^\delta(t)} \boldsymbol{\xi}_\delta\cdot\nabla(p-p_\delta) \right|.
$$

By using (21) and the bootstrap assumption (133), we have that

$$
\begin{aligned}
R_{16,1} &= \alpha \left| \int_0^t \int_{\Omega_b} \mathcal{J}_b^\eta (\mathrm{tr}(\nabla_b^\eta \boldsymbol{\xi}))(p-p_\delta) - \int_0^t \int_{\Omega_b} \mathcal{J}_b^{\eta_\delta^\delta} (\mathrm{tr}(\nabla_b^{\eta_\delta^\delta}\boldsymbol{\xi}_\delta))(p-p_\delta) \right| \\
&= \alpha \left| \int_0^t \int_{\Omega_b} \mathrm{tr}(\nabla\boldsymbol{\xi}\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta})^C)(p-p_\delta) - \int_0^t \int_{\Omega_b} \mathrm{tr}(\nabla\boldsymbol{\xi}_\delta\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C)(p-p_\delta) \right| \\
&\leqslant \alpha \left| \int_0^t \int_{\Omega_b} \mathrm{tr}(\nabla\boldsymbol{\xi}\cdot(\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta)^C)(p-p_\delta) \right| + \alpha \left| \int_0^t \int_{\Omega_b} \mathrm{tr}(\nabla(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C)(p-p_\delta) \right| \\
&\leqslant C\int_0^t ||\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta||_{L^2(\Omega_b)} \cdot ||p-p_\delta||_{L^2(\Omega_b)} + C\int_0^t ||\nabla\boldsymbol{\xi}-\nabla\boldsymbol{\xi}_\delta||_{L^2(\Omega_b)} \cdot ||p-p_\delta||_{L^2(\Omega_b)}.
\end{aligned}
$$

For $R_{16,2}$, we compute

$$
\begin{aligned}
R_{16,2} &= \alpha\left|\int_0^t\int_{\Omega_b}\boldsymbol{\xi}\cdot\left[\nabla(p-p_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta})^C\right]-\int_0^t\int_{\Omega_b}\boldsymbol{\xi}_\delta\cdot\left[\nabla(p-p_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C\right]\right|\\
&\leqslant \alpha\left|\int_0^t\int_{\Omega_b}\boldsymbol{\xi}\cdot\left[\nabla(p-p_\delta)\cdot(\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta)^C\right]\right|+\alpha\left|\int_0^t\int_{\Omega_b}(\boldsymbol{\xi}-\boldsymbol{\xi}_\delta)\cdot\left[\nabla(p-p_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C\right]\right|\\
&\leqslant C\int_0^t||\nabla p-\nabla p_\delta||_{L^2(\Omega_b)}\cdot||\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta||_{L^2(\Omega_b)}+C\int_0^t||\boldsymbol{\xi}-\boldsymbol{\xi}_\delta||_{L^2(\Omega_b)}\cdot||\nabla p-\nabla p_\delta||_{L^2(\Omega_b)}.
\end{aligned}
$$

By the convolution inequality (164) and the previous estimate on the gradient of the pressure (165), we conclude that

$$
\begin{aligned}
|T_{16}| \leqslant \epsilon&\left(\int_0^t||\nabla\boldsymbol{\xi}-\nabla\boldsymbol{\xi}_\delta||^2_{L^2(\Omega_b)}+\int_0^t||\nabla p-\nabla p_\delta||^2_{L^2(\Omega^\delta_{b,\delta}(t))}\right)+C(\epsilon)\left(\int_0^t||p-p_\delta||^2_{L^2(\Omega_b)}\right.\\
&\left.+\int_0^t||\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)}+\int_0^t||\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)}+\int_0^t||\omega-\omega_\delta||^2_{H^2(\Gamma)}+\int_0^t||\boldsymbol{\xi}-\boldsymbol{\xi}_\delta||^2_{L^2(\Omega_b)}\right).
\end{aligned}
$$

Term T17. To estimate term T_{17} defined in (126) we use (21) to compute

$$
\begin{aligned}
T_{17} &= \kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^\eta\nabla_b^\eta p\cdot\nabla_b^\eta(p-p_\delta)-\kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^{\eta_\delta^\delta}\nabla_b^{\eta_\delta^\delta}p_\delta\cdot\nabla_b^{\eta_\delta^\delta}(p-p_\delta)\\
&= \kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^{\eta_\delta^\delta}\nabla_b^{\eta_\delta^\delta}(p-p_\delta)\cdot\nabla_b^{\eta_\delta^\delta}(p-p_\delta)+I_1+I_2=\kappa\int_0^t\int_{\Omega^\delta_{b,\delta}(t)}|\nabla(p-p_\delta)|^2+R_{17,1}+R_{17,2},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{17,1} &= \kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^\eta\nabla_b^\eta p\cdot\nabla_b^\eta(p-p_\delta)-\kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^{\eta_\delta^\delta}\nabla_b^\eta p\cdot\nabla_b^{\eta_\delta^\delta}(p-p_\delta),\\
R_{17,2} &= \kappa\int_0^t\int_{\Omega_b}\mathcal{J}_b^{\eta_\delta^\delta}(\nabla_b^\eta p-\nabla_b^{\eta_\delta^\delta}p)\cdot\nabla_b^{\eta_\delta^\delta}(p-p_\delta).
\end{aligned}
$$

To estimate $R_{17,1}$, we use (21) to obtain

$$
R_{17,1}=\kappa\int_0^t\int_{\Omega_b}\nabla_b^\eta p\cdot\left(\nabla(p-p_\delta)\cdot\left[(\boldsymbol{I}+\nabla\boldsymbol{\eta})^C-(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C\right]\right).
$$

Because $\boldsymbol{\eta}$ is smooth, $|\nabla_b^\eta p|\leqslant C$ uniformly in space and time. Therefore,

$$
|R_{17,1}|\leqslant C\int_0^t||\nabla(p-p_\delta)||_{L^2(\Omega_b)}\cdot||(\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta)^C||_{L^2(\Omega_b)}.
$$

Using the estimate in (165), we obtain the desired estimate that

$$
|R_{17,1}|\leqslant\epsilon\int_0^t||\nabla(p-p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))}+C(\epsilon)\int_0^t||\nabla\boldsymbol{\eta}-\nabla\boldsymbol{\eta}_\delta^\delta||^2_{L^2(\Omega_b)}.
$$

To estimate $R_{17,2}$, we use the bootstrap assumption (133) that there exists a constant C (independent of δ) such that $|\nabla\boldsymbol{\eta}_\delta^\delta|\leqslant C$ pointwise for $t\in[0,T_\delta]$. Therefore, $|(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta)^C|$ is pointwise uniformly bounded in space and time on the time interval $[0,T_\delta]$. Thus, by (21),

$$
R_{17,2}=\kappa\int_0^t\int_{\Omega_b}(\nabla_b^\eta p-\nabla_b^{\eta_\delta^\delta}p)\cdot\left[\nabla(p-p_\delta)\cdot(\boldsymbol{I}+\nabla\boldsymbol{\eta}_\delta^\delta)^C\right]
$$

and hence

$$|R_{17,2}| \leqslant C \int_0^t ||\nabla_b^{\eta} p - \nabla_b^{\eta_\delta^\delta} p||_{L^2(\Omega_b)} \cdot ||\nabla(p - p_\delta)||_{L^2(\Omega_b)}.$$

We estimate the first pressure term by using (21) to obtain

$$\begin{aligned}
||\nabla_b^{\eta} p - \nabla_b^{\eta_\delta^\delta} p||^2_{L^2(\Omega_b)} &= \int_{\Omega_b} \left|\nabla p \cdot \left[(\boldsymbol{I} + \nabla \boldsymbol{\eta})^{-1} - (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^{-1}\right]\right|^2 \\
&= \int_{\Omega_b} \left|\nabla p \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^{-1} [(\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)(\boldsymbol{I} + \nabla \boldsymbol{\eta})^{-1} - \boldsymbol{I}]\right|^2 \\
&= \int_{\Omega_b} \left|\nabla p \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^{-1} [(\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta) - (\boldsymbol{I} + \nabla \boldsymbol{\eta})](\boldsymbol{I} + \nabla \boldsymbol{\eta})^{-1}\right|^2 \\
&= \int_{\Omega_b} \left|\nabla p \cdot (\boldsymbol{I} + \nabla \boldsymbol{\eta}_\delta^\delta)^{-1} (\nabla \boldsymbol{\eta}_\delta^\delta - \nabla \boldsymbol{\eta})(\boldsymbol{I} + \nabla \boldsymbol{\eta})^{-1}\right|^2.
\end{aligned}$$

Using the fact that p is smooth and the bootstrap assumption (132), we have that

$$||\nabla_b^{\eta} p - \nabla_b^{\eta_\delta^\delta} p||^2_{L^2(\Omega_b)} \leqslant C||\nabla \boldsymbol{\eta}_\delta^\delta - \nabla \boldsymbol{\eta}||^2_{L^2(\Omega_b)}.$$

Therefore, combining this with (165) we obtain

$$R_{17,2} \leqslant \epsilon \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} + C(\epsilon) \int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta^\delta||^2_{L^2(\Omega_b)}.$$

The final estimate of T_{17} now follows after the application of the convolution inequality (164):

$$T_{17} \leqslant \kappa \int_0^t \int_{\Omega^\delta_{b,\delta}(t)} |\nabla(p - p_\delta)|^2 + R_{17},$$

where the remainder is bounded by

$$\begin{aligned}
|R_{17}| &\leqslant \epsilon \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} \\
&\quad + C(\epsilon) \left(\int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}^\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\nabla \boldsymbol{\eta} - \nabla \boldsymbol{\eta}_\delta||^2_{L^2(\Omega_b)} + \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)} \right).
\end{aligned}$$

Term 18. Here want to estimate

$$\begin{aligned}
T_{18} = &\int_0^t \int_{\Gamma(t)} p(\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma(t)} p(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma_\delta(t)} p_\delta(\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n}_\delta + \int_0^t \int_{\Gamma_\delta(t)} p_\delta(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n}_\delta \\
&- \int_0^t \int_{\Gamma(t)} ((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n})(p - p_\delta) + \int_0^t \int_{\Gamma_\delta(t)} ((\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n}_\delta)(p - p_\delta) \\
&= - \int_0^t \int_{\Gamma(t)} p(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n} - \int_0^t \int_{\Gamma_\delta(t)} p_\delta(\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n}_\delta + \int_0^t \int_{\Gamma(t)} ((\boldsymbol{u} - \boldsymbol{\xi}) \cdot \boldsymbol{n}) p_\delta + \int_0^t \int_{\Gamma_\delta(t)} ((\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot \boldsymbol{n}_\delta) p.
\end{aligned}$$

By mapping all of the integrals back to the reference domain Γ, we obtain

$$\begin{aligned}
T_{18} &= -\int_0^t \int_\Gamma p(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot (-\partial_x \omega, 1) - \int_0^t \int_\Gamma p_\delta(\boldsymbol{u} - \boldsymbol{\xi}) \cdot (-\partial_x \omega_\delta, 1) \\
&\quad + \int_0^t \int_\Gamma p_\delta(\boldsymbol{u} - \boldsymbol{\xi}) \cdot (-\partial_x \omega, 1) + \int_0^t \int_\Gamma p(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta) \cdot (-\partial_x \omega_\delta, 1) \\
&= \int_0^t \int_\Gamma p(\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta)_x \cdot (\partial_x \omega - \partial_x \omega_\delta) - \int_0^t \int_\Gamma p_\delta(\boldsymbol{u} - \boldsymbol{\xi})_x \cdot (\partial_x \omega - \partial_x \omega_\delta) \\
&= -\int_0^t \int_\Gamma p[(\boldsymbol{u} - \boldsymbol{\xi})_x - (\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta)_x] \cdot (\partial_x \omega - \partial_x \omega_\delta) + \int_0^t \int_\Gamma (p - p_\delta)(\boldsymbol{u} - \boldsymbol{\xi})_x \cdot (\partial_x \omega - \partial_x \omega_\delta).
\end{aligned}$$

The absolute value is bounded as follows:

$$\begin{aligned}
&\left| \int_0^t \int_\Gamma p[(\boldsymbol{u} - \boldsymbol{\xi})_x - (\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta)_x] \cdot (\partial_x \omega - \partial_x \omega_\delta) \right| + \left| \int_0^t \int_\Gamma (p - p_\delta)(\boldsymbol{u} - \boldsymbol{\xi})_x \cdot (\partial_x \omega - \partial_x \omega_\delta) \right| \\
&\leqslant C \left(\int_0^t ||(\boldsymbol{u} - \boldsymbol{\xi})_x - (\boldsymbol{u}_\delta - \boldsymbol{\xi}_\delta)_x||_{L^2(\Gamma)} ||\partial_x \omega - \partial_x \omega_\delta||_{L^2(\Gamma)} + \int_0^t ||p - p_\delta||_{L^2(\Gamma)} ||\partial_x \omega - \partial_x \omega_\delta||_{L^2(\Gamma)} \right).
\end{aligned}$$

After the application of the trace theorem, Poincare's inequality, and Korn's inequality we obtain the final estimate:

$$\begin{aligned}
|T_{18}| &\leqslant \epsilon \left(\int_0^t ||\boldsymbol{D}(\widehat{\boldsymbol{u}} - \boldsymbol{u}_\delta)||^2_{L^2(\Omega_{f,\delta}(t))} + \int_0^t ||\nabla(\boldsymbol{\xi} - \boldsymbol{\xi}_\delta)||_{L^2(\Omega_b)} + \int_0^t ||\nabla(p - p_\delta)||^2_{L^2(\Omega^\delta_{b,\delta}(t))} \right) \\
&\quad + C(\epsilon) \int_0^t ||\omega - \omega_\delta||^2_{H^2(\Gamma)}.
\end{aligned}$$

□

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