Complex Analysis and Operator Theory (2023) 17:93
https://doi.org/10.1007/s11785-023-01398-w

Complex Analysis and Operator Theory

Cohn–Elkies Functions from Gabor Frames

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Received: 30 December 2022 / Accepted: 21 July 2023

Abstract

We investigate the relation between two different mathematical problems: the construction of bounds on sphere packing density using Cohn–Elkies functions and the construction of Gabor frames for signal analysis. In particular, we present a general construction of Cohn–Elkies functions in arbitrary dimension, obtained from an approximate Wexel–Raz dual for Gabor frames with Gaussian window.

1 Introduction

In this paper we compare two seemingly different mathematical problems, showing that they share a deep connection: the construction of bounds on the density of sphere packings in Euclidean spaces, and the construction of Gabor frames for signal analysis.

The best currently available construction of bounds on the density of sphere packings is provided by the method introduced in [7], based on the construction of (radial) functions that vanish at the points of the lattice (or periodic set) with specific decay conditions and sign conditions on the function and its Fourier transform. We refer to such functions as *Cohn–Elkies functions*. This method was especially successful in Viazovska's explicit construction, using modular forms, of one such Cohn–Elkies function proving the optimality of the E_8 lattice for the sphere packing problem in dimension 8, see [19]. This construction was then adapted in [9] to prove the optimality of the Leech lattice in dimension 24. Despite these remarkable achievements, in general explicit geometric constructions of Cohn–Elkies functions remain elusive,

Communicated by Uwe Kaehler.

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Published online: 03 August 2023

through a numerical approximation algorithm using linear programming is described in [7].

On the other hand, Gabor frames provide systems of filters for signal analysis that have good encoding and decoding properties, though they do not consist of orthogonal bases [11]. A Gabor system is typically constructed by applying translation and modulation operators parameterized by the points of a lattice (or more general sets including periodic sets) to a window function with nice properties (for instance a Gaussian). The main question then is whether a Gabor system constructed in this way satisfies the frame condition (hence has good properties for signal analysis). This property depends crucially on the lattice. A good way of analyzing Gabor frames and properties equivalent to the frame condition is through Wexel–Raz duality [12]. This leads to a characterization of the frame condition for a Gabor system in $\mathbb{R}^n$ in terms of an entire function in $\mathbb{C}^n$ that vanishes at points of the lattice and is related to the Bargmann transform of the Wexel–Raz dual window function.

These two problems share the underlying question of the construction of a function vanishing at points of a lattice, with assigned properties in terms of the closely related Fourier and Bargmann (or short-time Fourier) transform. In the case of Cohn–Elkies functions one usually assumes that the function is radial, hence vanishing on spheres containing lattice points, while in the Gabor frame problem one typically deals with functions vanishing on hyperplanes containing lattice points, in the sense of the general construction of [16]. In fact, as remarked in [7], the radial hypothesis in the Cohn–Elkies case is not necessary, and we will consider more general such functions.

There is another important direct relation between these two questions. A special class of Gabor frames, called Grassmannian frames [18], have the property that they minimize (over lattices) the maximal correlation between the functions in the Gabor system. These are frames that most closely resemble the properties of orthogonal frames. It turns out that the optimization problem for the construction of Grassmannian frames is the same as the optimization problem for lattices achieving maximal sphere packing density.

Given these relations between the two questions, it is natural to ask whether one can use techniques from Gabor frame analysis to provide a different geometric approach to the construction of Cohn–Elkies functions. In this paper we show that this is indeed the case and that Wexel–Raz duality for Gabor frames provides a new approach to the construction of Cohn–Elkies functions.

It is important to notice here the role of lattices. In the context of the sphere packing density problem, it is expected that lattice solutions will be only a low-dimensional feature, with the maximal density achievable by lattices diverging from the maximal sphere packing density in higher-dimensions. The known cases of dimensions 1, 2, 3, 8, 24 are the only dimensions where an explicit lattice solution is known, and may be the only ones. Thus, focusing on the possibility of lattice solutions is clearly very restrictive. A conjecture of Zassenhaus expects the maximal density in any dimension to be attainable by periodic packings, that is, sphere packings with sphere centers on periodic sets (unions of translates of lattices). It is known that periodic packings can approximate arbitrarily well the greatest packing density. After discussing the case of lattices, we show in the last section of this paper how to adapt the construction to the case of periodic sets.

The construction of Cohn–Elkies functions that we discuss in this paper uses a lattice $L \subset \mathbb{R}^n$ (whose dual $L^\vee$ is the lattice whose density one wants to probe), together with a choice of an auxiliary lattice $K \subset \mathbb{R}^n$ chosen so that $\Lambda = L \times K$ gives a Gabor frame for a Gaussian window. It is in general difficult to obtain explicit constructions of Wexel–Raz dual windows for Gabor frames. Indeed, even for the case of a Gaussian window, we need to use an approximate dual. It is interesting to notice that in both the problem of constructing Cohn–Elkies functions and the problem of constructing Wexel–Raz dual windows, cases where direct explicit constructions are known involve the use of modular forms: in dimension 8 and 24 for the Cohn–Elkies problem [9, 19], and for the Wexel–Raz duality in dimension one (that is, for lattices in $\mathbb{R}^2$), where the canonical dual window is expressible explicitly in terms of lattice theta functions [15].

In the rest of this introductory section we present these two problems in more detail, and we recall the background material that we need for our main construction, which we present in the following section.

1.1 Cohn–Elkies Functions

In [7], Cohn and Elkies obtained a bound on the density of sphere packings in terms of radial functions with assigned decay and sign properties of the function and its Fourier transform. The same concept was independently introduced in [10]. Viazovska's explicit modular forms construction [19] of such a function famously solved the sphere packing problem in dimension 8, and a generalization of the same method also gave a solution in dimension 24 [9].

Definition 1.1 A Cohn–Elkies function of dimension $n \in \mathbb{N}$ and of size $\ell \in \mathbb{R}^*_+$ is a real-valued Schwartz function $f(x)$ with real valued Fourier transform $(\mathfrak{F}f)(\xi)$, such that

(1) $f(x) \leq 0$ for all $\|x\| \geq \ell$;
(2) $(\mathfrak{F}f)(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$;
(3) $(\mathfrak{F}f)(0) > 0$.

Note that condition $(\mathfrak{F}f)(\xi) \geq 0$, with $(\mathfrak{F}f)$ not identically zero, implies $f(0) > 0$.

Remark 1.2 In [7] the Cohn–Elkies functions are assumed to be real-valued radial functions, $f(x) = f_0(\|x\|)$, for all $x \in \mathbb{R}^n$, with $f_0 \in L^2([0,\infty), r^{n-1}dr)$ satisfying a rapid decay condition. In this case the Fourier transform is automatically real-valued and radial, by the description of Fourier transform of radial functions as Hankel transform. Also in [7] a more general decay condition is assumed for the Cohn–Elkies functions, weaker than the Schwartz condition we consider here, which suffices for the use of the Poisson summation formula. In fact, the condition was further generalized in [8]. Here we consider the more restrictive class of Schwartz functions, as in [19], but one can replace this hypothesis with decay conditions as in [7] or [8].

A sphere packing $\mathcal{P}_L$ based on a lattice $L \subset \mathbb{R}^n$ is a packing of spheres S^{n-1} centered at the lattice points, with sphere diameters equal to the length ℓ_L of the shortest lattice vector. The *density* $\Delta_{\mathcal{P}}$ of a sphere packing $\mathcal{P}$ is the fraction of volume occupied by spheres, hence in the case of a lattice packing it is given by the ratio

$$\Delta_{\mathcal{P}_L} = \frac{\mathrm{Vol}(B_1^n(0))}{|L|} \left(\frac{\ell_L^n}{2}\right)^n , \tag{1.1}$$

where $|L| = \mathrm{Vol}(\mathbb{R}^n/L)$ is the covolume of the lattice and

$$\mathrm{Vol}(B_1^n(0)) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

is the volume of the unit ball in $\mathbb{R}^n$. In the case of a periodic lattice, based on a periodic set consisting of N translations of a lattice L, the density is similarly described, with $|L|$ replaced by $|L|/N$ in (1.1). The *center-density* is defined as $\delta_{\mathcal{P}} = \Delta_{\mathcal{P}}/\mathrm{Vol}(B_1^n(0))$. Thus, for a sphere packing $\mathcal{P}_L$ based on a lattice $L \subset \mathbb{R}^n$, the center-density is given by

$$\delta_L = \left(\frac{\ell_L}{2}\right)^n \frac{1}{|L|} , \tag{1.2}$$

with ℓ_L the shortest length of L.

Theorem 3.2 of [7] shows that the existence of a Cohn–Elkies function of dimension $n \in \mathbb{N}$ and size $\ell \in \mathbb{R}_+^*$ gives a bound on the center-density $\delta_{\mathcal{P}}$

$$\delta_{\mathcal{P}} \leq \left(\frac{\ell}{2}\right)^n \frac{f(0)}{(\mathfrak{F}f)(0)} , \tag{1.3}$$

for any arbitrary sphere packing $\mathcal{P}$ in $\mathbb{R}^n$.

Remark 1.3 Note that in the sphere packing problem, the lattice covolume $|L|$ is fixed and can be taken $|L| = 1$. Here we leave $|L|$ written explicitly to highlight the dependence of the construction on $|L|$. The reader should assume that it has a fixed value.

Definition 1.4 Let $L \subset \mathbb{R}^n$ be a lattice with shortest length ℓ_L. A Cohn–Elkies function of dimension $n \in \mathbb{N}$ and size ℓ_L is special if in addition to the properties of Definition 1.1 it also satisfies

$$\frac{1}{|L|} = \frac{f(0)}{(\mathfrak{F}f)(0)} . \tag{1.4}$$

Lemma 1.5 *Given a lattice $L \subset \mathbb{R}^n$, suppose there is an associated special Cohn–Elkies function of dimension $n \in \mathbb{N}$ and size ℓ_L, with ℓ_L the shortest length of L. Then the lattice L realizes the maximal density for sphere packings in $\mathbb{R}^n$.*

Proof As in [7], from the Poisson summation formula

$$\sum_{\lambda \in L} f(x+\lambda) = \frac{1}{|L|} \sum_{\lambda' \in L^\vee} e^{-2\pi i \langle x, \lambda' \rangle} (\mathfrak{F}f)(\lambda') ,$$

with $L^\vee$ the dual lattice, one obtains that

$$\sum_{\lambda \in L} f(\lambda) \le f(0)$$

since each term with $\lambda \neq 0$ in the sum is non-positive, as ℓ_L is the shortest length of L. On the other hand

$$\frac{1}{|L|} \sum_{\lambda' \in L^\vee} (\mathfrak{F} f)(\lambda') \ge \frac{1}{|L|} (\mathfrak{F} f)(0)\,,$$

as all the other terms are non-negative. Thus, we have

$$f(0) - \frac{1}{|L|} (\mathfrak{F} f)(0) \ge 0$$

which gives the estimate

$$\frac{1}{|L|} \le \frac{f(0)}{(\mathfrak{F} f)(0)}\,.$$

The lattice packing is optimal if it achieves the Cohn–Elkies bound

$$\delta_L = \left(\frac{\ell_L}{2}\right)^n \frac{f(0)}{(\mathfrak{F} f)(0)}$$

determined by the Cohn–Elkies function, hence if the above inequality is optimized. □

We also recall the following observation from [7].

Corollary 1.6 *Given a lattice $L \subset \mathbb{R}^n$ with shortest length ℓ_L and covolume $|L|$, a special Cohn–Elkies function of dimension $n \in \mathbb{N}$ and size ℓ_L vanishes on all the nonzero vectors of L and its Fourier transform vanishes on all the nonzero vectors of the dual lattice $L^\vee$.*

Proof Since (1.4) holds, the Poisson summation formula gives

$$\sum_{\lambda \in L \smallsetminus \{0\}} f(\lambda) = \frac{1}{|L|} \sum_{\lambda' \in L^\vee \smallsetminus \{0\}} (\mathfrak{F} f)(\lambda')\,,$$

but on the left-hand-side all the terms are non-positive while on the right-hand-side all the terms are non-negative, hence all terms vanish. □

We have formulated here Lemma 1.5 and Corollary 1.6 in the lattice case. For the analogous formulation in the case of periodic sets see [7].

1.2 Gabor Frames

The construction of good frames is a fundamental question in signal analysis. Unlike orthogonal bases, frames are overdetermined and have some amount of redundancy, but they also have important properties, such as optimization of the uncertainty principle (localization in both position and frequency variables). The frame condition ensures good encoding (via the frame operator) and decoding properties. In particular, we focus here on Gabor frames, obtained by acting on a window function via translation and modulation operators. The crucial question of when a Gabor system obtained by translation and modulation of a window function satisfies the frame condition is completely understood in the case of Gabor frames in $L^2(\mathbb{R})$ with lattices $\Lambda \subset \mathbb{R}^2$, while a full characterization in higher dimensions remains a more complicated problem.

Definition 1.7 (1) Given a window function ϕ in $L^2(\mathbb{R}^n)$ and a lattice $\Lambda \subset \mathbb{R}^{2n}$, the Gabor system $\mathcal{G}(\phi, \Lambda) = \{\pi_\lambda \phi\}_{\lambda\in\Lambda}$ consists of the collection of functions

$$\pi_\lambda \phi(x) = e^{2\pi i \langle \eta, x\rangle} \phi(x - \xi)\,, \tag{1.5}$$

for $\lambda = (\xi, \eta) \in \Lambda$.

(2) The Gabor system $\mathcal{G}(\phi, \Lambda)$ is a frame (satisfies the frame condition) if there are constants $C, C' > 0$ such that, for all $f \in L^2(\mathbb{R}^d)$

$$C\, \|f\|_{L^2(\mathbb{R}^d)} \le \sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda \phi\rangle|^2 \le C'\, \|f\|_{L^2(\mathbb{R}^d)}\,. \tag{1.6}$$

(3) The Gabor system $\mathcal{G}(\phi, \Lambda)$ is a Bessel sequence if the upper inequality of (1.6) holds,

$$\sum_{\lambda\in\Lambda} |\langle f, \pi_\lambda \phi\rangle|^2 \le C' \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

The frame operator $\mathcal{S} = \mathcal{S}_{\phi,\Lambda}$ associated to the Gabor system $\mathcal{G}(\phi, \Lambda)$ is given by

$$\mathcal{S} f = \sum_{\lambda\in\Lambda} \langle f, \pi_\lambda \phi\rangle\, \pi_\lambda \phi\,. \tag{1.7}$$

The Gabor system $\mathcal{G}(\phi, \Lambda)$ is a frame iff $\mathcal{S}_{\phi,\Lambda}$ is both bounded and invertible on $L^2(\mathbb{R}^n)$ and a Bessel sequence if it is bounded.

1.3 Adjoint Lattice

The adjoint lattice plays a crucial role in the Wexel–Raz duality for Gabor frames and in the equivalent characterization of the frame condition in terms of sampling and interpolation of entire functions.

Definition 1.8 Given a lattice $\Lambda \subset \mathbb{R}^{2n}$, the adjoint lattice Λ^o is given by

$$\Lambda^o = \{\lambda' \in \mathbb{R}^{2n} \mid \pi_\lambda \circ \pi_{\lambda'} = \pi_{\lambda'} \circ \pi_\lambda\,,\ \forall \lambda \in \Lambda\}\,. \tag{1.8}$$

with the translation-modulation operators π_λ as in (1.5).

We have the following equivalent description of the adjoint lattice (see Lemma 4.3.3 of [12]).

Lemma 1.9 *For $\Lambda = A\mathbb{Z}^{2n}$ with $A \in \mathrm{GL}_{2n}(\mathbb{R})$, the adjoint lattice is given by*

$$\Lambda^o = J^{-1}\,(A^t)^{-1}\,\mathbb{Z}^{2n} \tag{1.9}$$

with

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\,. \tag{1.10}$$

Proof This simply follows from the fact that, for $\lambda = (\lambda_1, \lambda_2)$ and $\lambda' = (\lambda_1', \lambda_2')$,

$$\pi_{\lambda'} \circ \pi_\lambda = e^{2\pi i (\langle \lambda_1, \lambda_2' \rangle - \langle \lambda_2, \lambda_1' \rangle)} \pi_\lambda \circ \pi_{\lambda'}$$

where the condition $1 = e^{2\pi i \langle Ak, J\lambda' \rangle}$ with $k \in \mathbb{Z}^{2n}$ holds iff $\langle Ak, J\lambda' \rangle = \langle k, A^t J\lambda' \rangle \in \mathbb{Z}$ for all $k \in \mathbb{Z}^{2n}$, which gives $\lambda' \in J^{-1}\,(A^t)^{-1}\,\mathbb{Z}^{2n}$.

Note that the covolume satisfies $|\Lambda| = Vol(\mathbb{R}^{2n}/\Lambda) = |\det A|$, for a lattice of the form $\Lambda = A\mathbb{Z}^{2n}$ for $A \in \mathrm{GL}_{2n}(\mathbb{R})$, and for the adjoint lattice $|\Lambda^o| = |\Lambda|^{-1}$.

Remark 1.10 In the case of a split lattice, namely a lattice $\Lambda \subset \mathbb{R}^{2n}$ of the form $\Lambda = L_1 \times L_2$, with L_1, L_2 lattices in $\mathbb{R}^n$, the adjoint lattice is of the form

$$\Lambda^o = L_2^\vee \times L_1^\vee, \tag{1.11}$$

where $L_i^\vee$ are the dual lattices of the L_i in $\mathbb{R}^n$.

1.4 Wexel–Raz Duality for Gabor Frames

The frame condition for a Gabor system $\mathcal{G}(\phi, \Lambda)$ can be characterized in terms of a duality relation, namely the existence of a dual window function γ with the property that the Gabor systems $\mathcal{G}(\phi, \Lambda)$ and $\mathcal{G}(\gamma, \Lambda)$ are mutually orthogonal (Wexel–Raz biorthogonality relation).

Definition 1.11 For a Gabor system $\mathcal{G}(\phi, \Lambda)$ in $L^2(\mathbb{R}^n)$ that is a Bessel sequence, a Wexel–Raz dual window γ is a window function that satisfies the reconstruction identity

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi_\lambda \phi \rangle \pi_\lambda \gamma\,. \tag{1.12}$$

Dual windows are not unique. In particular, the *canonical dual window* is the one obtained from the frame operator (1.7) by $\gamma_{\phi,\Lambda} = \mathcal{S}^{-1}_{\phi,\Lambda}\phi$. In this case, while the frame operator (1.7) provides the encoding

$$\mathcal{S}_{\phi,\Lambda} : f \mapsto \sum_{\lambda\in\Lambda} \langle f, \pi_\lambda \phi\rangle\, \pi_\lambda \phi\,, \tag{1.13}$$

the canonical Wexel–Raz dual provides the corresponding decoding operator

$$\mathcal{S}^{-1}_{\phi,\Lambda} : f \mapsto \sum_{\lambda\in\Lambda} \langle f, \pi_\lambda \gamma_{\phi,\Lambda}\rangle\, \pi_\lambda \gamma_{\phi,\Lambda}\,.$$

Dual windows can be characterized in terms of a vanishing property of their short-time Fourier transform on the adjoint lattice.

Definition 1.12 For a window function $\phi \in L^2(\mathbb{R}^n)$ the short-time Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ is given by

$$(V_\phi f)(w) := \int_{\mathbb{R}^n} f(t)\bar{\phi}(t-u)e^{-2\pi i\langle v,t\rangle}dt = \langle f, \pi_w \phi\rangle, \tag{1.14}$$

for $w = (u, v) \in \mathbb{R}^{2n}$.

The short-time Fourier transform satisfies

$$V_{\pi_w\phi}(\pi_w f)(z) = e^{2\pi i\langle z, J\cdot w\rangle} V_\phi f(z)\,,$$

with J as in (1.10), and

$$\langle V_\phi f, V_\gamma h\rangle_{L^2(\mathbb{R}^{2n}} = \langle f, h\rangle_{L^2(\mathbb{R}^n)} \overline{\langle \phi, \gamma\rangle}_{L^2(\mathbb{R}^n)}\,.$$

The phase factor $e^{2\pi i\langle\lambda, J\cdot z\rangle}$ satisfies,

$$e^{2\pi i\langle\lambda, J\cdot z\rangle} = 1\, \forall\lambda\in\Lambda \;\Leftrightarrow\; z \in \Lambda^o\,. \tag{1.15}$$

We then have the following characterization of Wexel–Raz dual windows, see Theorem 4.4.1 of [12].

Lemma 1.13 *For a Gabor system $\mathcal{G}(\phi,\Lambda)$ in $L^2(\mathbb{R}^n)$ that is a Bessel sequence, a Wexel–Raz dual window γ is a window function that satisfies*

$$\frac{1}{|\Lambda|}\langle\gamma, \pi_{\lambda'}\phi\rangle = \delta_{\lambda',0}\,, \quad \forall\lambda' \in \Lambda^o\,. \tag{1.16}$$

Proof We recall briefly the proof that (1.12) implies (1.16), and we refer the reader to [12] for a more detailed account. One first shows that if for two window functions ϕ and γ in $L^2(\mathbb{R}^n)$ both Gabor systems $\mathcal{G}(\phi, \Lambda)$ and $\mathcal{G}(\gamma, \Lambda)$ are Bessel sequences and

$$\sum_{\lambda' \in \Lambda^o} |V_\phi \gamma(\lambda')| < \infty \,,$$

then the Poisson summation formula gives

$$\sum_{\lambda \in \Lambda} V_\phi f(z+\lambda) \overline{V_\gamma h(z+\lambda)} = \frac{1}{|\Lambda|} \sum_{\lambda' \in \Lambda^o} V_\phi \gamma(\lambda') \overline{V_f h(\lambda')} e^{2\pi i \langle \lambda', Jz \rangle} \,,$$

for all $z \in \mathbb{R}^{2n}$ and for any $f, h \in L^2(\mathbb{R}^n)$, see Theorem 4.3.2 of [12]. For a dual window γ one then writes

$$\langle f, h \rangle = \sum_{\lambda \in \Lambda} \langle \pi_z^* f, \pi_\lambda \phi \rangle \langle \pi_\lambda \gamma, \pi_z^* h \rangle = \sum_{\lambda \in \Lambda} V_\phi f(z+\lambda) \overline{V_\gamma h(z+\lambda)}$$

where the latter must be a constant function of $z \in \mathbb{R}^{2n}$ hence with Fourier coefficients

$$\frac{1}{|\Lambda|} V_\phi \gamma(\lambda') \overline{V_f h(\lambda')} = \langle f, h \rangle \delta_{\lambda', 0} \,,$$

see Theorem 4.4.1 of [12]. Thus, dual windows that satisfy (1.12) also satisfy the relation (1.16). □

The Gabor frame condition can then be equivalently formulated in terms of Wexel–Raz duality as follows (see Theorem 4.4.1 of [12]).

Proposition 1.14 *For a Gabor system $\mathcal{G}(\phi, \Lambda)$ in $L^2(\mathbb{R}^n)$ the following properties are equivalent:*

(1) *$\mathcal{G}(\phi, \Lambda)$ is a frame;*
(2) *$\mathcal{G}(\phi, \Lambda^o)$ is a Bessel sequence and there is a Wexel–Raz dual window $\gamma_{\phi,\Lambda} \in L^2(\mathbb{R}^n)$ (satisfying (1.16)) such that $\mathcal{G}(\gamma_{\phi,\Lambda}, \Lambda)$ is also a Bessel sequence.*

Thus, the problem of verifying the frame condition for Gabor systems is equivalently rephrased as the problem of constructing Wexel–Raz dual windows satisfying the interpolating condition (1.16) on the adjoint lattice.

1.5 Grassmannian Gabor Frames and Sphere Packings

In [18] a special class of frames is introduced that have the property of minimizing correlation. Namely, frames $\{\psi_\alpha\}_{\alpha \in \mathcal{I}}$ such that the maximal correlation $|\langle \psi_\alpha, \psi_\beta \rangle|$ over all $\alpha \neq \beta \in \mathcal{I}$ is as small as possible for a fixed redundancy. Such frames are called *Grassmannian frames* (see Definition 1.18 below).

This question can be seen as follows; an orthonormal frame has no redundancy and the basis elements are completely uncorrelated. Frames in general have redundancy and for a fixed amount of redundancy this minimization problem is addressing the question of how closely such a frame can resemble an orthonormal frame, in the sense of having as little correlation as possible among the basis elements.

Remark 1.15 In a finite dimensional Hilbert space of dimension n, a generating set $\{f_k\}_{k=1}^N$ has redundancy $\rho = N/n$. In this finite dimensional case, the problem of constructing Grassmannian frames is shown in [18] to be equivalent to the problem of finding an arrangement of N lines with largest possible angles between them. This is in turn equivalent to constructing a spherical code with fixed number N of points and with largest possible minimal angle φ.

The notion of redundancy can be extended to the infinite dimensional case in such a way that it agrees with the simple expression $\rho = N/n$ in finite dimensions, as in Remark 1.15.

Definition 1.16 Let $\{z_k\}_{k\in\mathbb{N}}$ be a fixed choice of points in $\mathbb{R}^{2n}$ and let $B_k(z_k) = \{x \in \mathbb{R}^n \mid \|z - z_k\| \leq k\}$. Given a lattice $\Lambda \subset \mathbb{R}^{2n}$, let $\Lambda_k = \Lambda \cap B_k(z_k)$. The redundancy of $\mathcal{G}(\phi, \Lambda)$ is defined as

$$\rho(\phi, \Lambda) := \left(\lim_{k\to\infty} \frac{1}{\#\Lambda_k} \sum_{\lambda\in\Lambda_k} \langle \pi_\lambda \phi, \mathcal{S}_{\phi,\Lambda}^{-1} \pi_\lambda \phi \rangle \right)^{-1} . \tag{1.17}$$

One defines $\rho^{\pm}(\phi, \Lambda)$ as the limsup/liminf when the limit (1.17) does not exist.

It is shown in [1] that the redundancy $\rho(\phi, \Lambda)$ of a Gabor frame is equal to its “density of label sets" $D(\phi, \Lambda)$, which is defined as

$$D(\phi, \Lambda) := \lim_{k\to\infty} \frac{\#\Lambda_k}{(2k)^{2n}} . \tag{1.18}$$

The definition of redundancy recalled above applies to sets $\Lambda \subset \mathbb{R}^{2n}$ that are not necessarily lattices. In the case of lattices the notion simplifies.

Remark 1.17 With all the $z_n = 0$ we have the lattice covolume

$$\lim_{k\to\infty} \frac{\#(\Lambda \cap B_k(0))}{Vol(B_k(0))} = \frac{1}{|\Lambda|} ,$$

so that the redundancy is simply given by

$$D(\phi, \Lambda) = \frac{Vol(B_1(0))}{2^{2n}\, |\Lambda|} ,$$

where for the unit ball $B_1(0) \subset \mathbb{R}^{2n}$ we have $Vol(B_1(0)) = \frac{\pi^n}{n!}$. Thus, considering Gabor frames $\mathcal{G}(\phi, \Lambda)$ with fixed redundancy $\rho(\phi, \Lambda) = \rho$ corresponds to considering lattices Λ with fixed covolume. The Gabor frame condition implies that the density $D(\phi, \Lambda) \geq 1$ so we can assume a fixed covolume $|\Lambda| \leq 1$.

Definition 1.18 For a fixed window function $\phi \in L^2(\mathbb{R}^n)$, a Gabor frame $\mathcal{G}(\phi, \Lambda)$ for a lattice $\Lambda \subset \mathbb{R}^{2n}$ is a Grassmannian frame if it minimizes the maximal correlation

$$Corr(\phi, \Lambda) := \max_{\lambda \in \Lambda \setminus \{0\}} |\langle \phi, \pi_\lambda \phi \rangle| \,, \tag{1.19}$$

with the minimization taken over lattices Λ with fixed redundancy (fixed covolume).

The relation between the problem of Grassmannian Gabor frames and the problem of lattice optimizers for the sphere packing problem can then be formulated in the following way.

Lemma 1.19 *When the window $\phi_\alpha(x) = e^{-\alpha\|x\|^2}$ is a Gaussian, for a lattice $\Lambda \subset \mathbb{R}^{2n}$ let Λ_α denote the lattice*

$$\Lambda_\alpha := \left\{ \left(\lambda_1, \frac{\pi}{\alpha}\lambda_2\right) \in \mathbb{R}^{2n} \mid \lambda = (\lambda_1, \lambda_2) \in \Lambda \right\} . \tag{1.20}$$

For $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, we write λ_α for the corresponding point $\lambda_\alpha = (\lambda_1, \frac{\pi}{\alpha}\lambda_2) \in \Lambda_\alpha$. Searching for a lattice optimizer

$$\Lambda_{opt} = \arg\min_{\Lambda} \max_{\lambda \in \Lambda \setminus \{0\}} |\langle \phi_\alpha, \pi_\lambda \phi_\alpha \rangle| = \arg\min_{\Lambda} Corr(\phi, \Lambda)$$

is equivalent to searching for a lattice optimizer

$$\Lambda_{opt} = \arg\max_{\Lambda} \min_{\lambda \in \Lambda \setminus \{0\}} \|\lambda_\alpha\| = \arg\max_{\Lambda} \ell_{\Lambda_\alpha} \,,$$

maximizing the shortest length ℓ_{Λ_α} for fixed covolume of Λ, hence for an optimizer of the sphere packing density.

Proof We can write explicitly the correlation as

$$\begin{aligned}
\langle \phi_\alpha, \pi_z \phi_\alpha \rangle &= \int_{\mathbb{R}^n} e^{-\alpha\|x\|^2} e^{-\alpha\|x-u\|^2} e^{2\pi i x \cdot v} dx \\
&= e^{-\frac{\alpha}{2}\|u\|^2} \int_{\mathbb{R}^n} e^{-2\alpha\|x-u/2\|^2} e^{2\pi i x\cdot v} dx = e^{-\frac{\alpha}{2}\|u\|^2} e^{\pi i u \cdot v} \int_{\mathbb{R}^n} e^{-2\alpha\|x\|^2} e^{2\pi i x \cdot v} dx \\
&= e^{\pi i u\cdot v}\, e^{-\frac{\alpha}{2}\|u\|^2} \left(\frac{\pi}{2\alpha}\right)^{n/2} e^{-\frac{\pi^2}{2\alpha}\|v\|^2}
\end{aligned} \tag{1.21}$$

for $z = u + iv \in \mathbb{C}^n$, so that we have

$$|\langle \phi_\alpha, \pi_z \phi_\alpha \rangle| = \left(\frac{\pi}{2\alpha}\right)^{n/2} e^{-\frac{\alpha}{2}(\|u\|^2 + \frac{\pi^2}{\alpha^2}\|v\|^2)} . \tag{1.22}$$

We then have

$$|\langle \phi_\alpha, \pi_\lambda \phi_\alpha \rangle| = \left(\frac{\pi}{2\alpha}\right)^{n/2} e^{-\frac{\alpha}{2}\|\lambda_\alpha\|^2} .$$

Thus, the correlation $|\langle \phi_\alpha, \pi_\lambda \phi_\alpha \rangle|$ monotonically decreases as $\|\lambda_\alpha\|$ increases. Thus, the maximum is achieved on the set of shortest vectors in Λ_α. Then optimizing the lattice Λ by making the shortest length in Λ_α as large as possible corresponds to optimizing Λ by making the largest correlation $|\langle \phi_\alpha, \pi_\lambda \phi_\alpha \rangle|$ over the shortest length vectors as small as possible. □

2 Cohn–Elkies Functions from Wexel–Raz Duality

Using an approximate construction of a Wexel–Raz dual window for a Gabor system with split lattice and Gaussian window, we obtain a general construction of Cohn–Elkies functions associated to critical lattices in $\mathbb{R}^n$.

2.1 Approximation of Wexel–Raz Dual

We show that, given a Gaussian window function $\phi_\alpha(x) = e^{-\alpha\|x\|^2}$, such that $\mathcal{G}(\phi_\alpha, \Lambda)$ is a Gabor frame, there is a (non-canonical) dual window γ that is well approximated by a superposition of shifted copies of ϕ_α.

We first recall briefly an argument given in Theorems 1 and 2 of [6], which our statement in Proposition 2.1 below generalizes.

For a matrix $C \in \mathrm{GL}_n(\mathbb{R})$ and a function $f \in L^2(\mathbb{R}^n)$ the dilation of f by C is defined as

$$(D_C f)(x) := |\det(C)|^{1/2} f(Cx) . \tag{2.1}$$

Let ϕ be a compactly supported real-valued function in $\mathbb{R}^n$, with $\mathrm{supp}(\phi) \subseteq [0, N]^n$ for some $N \in \mathbb{N}$, that satisfies the partition of unity condition

$$\sum_{k\in\mathbb{Z}^n} \phi(x-k) = 1 , \quad \forall x \in \mathbb{R}^n . \tag{2.2}$$

By Theorems 1 and 2 of [6], if the matrices $C, B \in \mathrm{GL}_n(\mathbb{R})$ satisfy

$$\|C^t B\| \le \frac{1}{\sqrt{n}(2N-1)} , \tag{2.3}$$

then there is a finite subset $\mathcal{F} \subset \mathbb{Z}^n$ and a function

$$\gamma(x) = |\det(C^t B)| \left(\phi(x) + 2\sum_{k\in\mathcal{F}} \phi(x+k)\right) , \tag{2.4}$$

such that the dilated functions $D_{C^{-1}}\phi$ and $D_{C^{-1}}\gamma$ generate dual Gabor frames

$$\mathcal{G}(D_{C^{-1}}\phi, \Lambda) \quad \text{and} \quad \mathcal{G}(D_{C^{-1}}\gamma, \Lambda).$$

This result shows that γ is a dual window for $\mathcal{G}(\phi, \Lambda)$ by showing that the biorthogonality relation

$$\langle \gamma, \pi_\lambda \phi \rangle = \frac{1}{|\Lambda|}\delta_{\lambda,0}\,, \quad \forall \lambda \in \Lambda^o$$

can be equivalently stated as the property that

$$\sum_{k\in\mathbb{Z}^n} \phi(x - (B^t)^{-1}n - Ck)\gamma(x - Ck) = |\det(B)|\,\delta_{n,0}\,. \tag{2.5}$$

Then the key properties needed to show that this relation holds are the partition of unity relation (2.2) and the identity

$$\begin{aligned} 1 = \left(\sum_{n\in\Gamma} \phi(x+n)\right)^2 &= \frac{1}{|\det(C^t B)|}\sum_{n\in\Gamma}\phi(x+n)\gamma(x+n) \\ &= \sum_{n\in\Gamma}\phi(x+n)(\phi(x+n) + 2\sum_{\ell\in\mathcal{F}}\phi(x+n+\ell))\,. \end{aligned} \tag{2.6}$$

where $\Gamma = [0, N-1]^n \cap \mathbb{Z}^n$. For the vanishing cases of (2.5), one uses the fact that for $\|B\| \le (\sqrt{n}(2N-1))^{-1}$ the vanishing of (2.5) for $n \neq 0$ is guaranteed by non-overlapping supports.

The following statement adapts and generalizes this argument.

Proposition 2.1 *Let $\Lambda \subset \mathbb{R}^{2n}$ be a lattice of the form $\Lambda = L \times K$ with lattices $L, K \subset \mathbb{R}^n$, where $L = C\mathbb{Z}^n$ and $K = B\mathbb{Z}^n$ for some $C, B \in \mathrm{GL}_n(\mathbb{R})$. Let $\Gamma_\Omega := \mathbb{Z}^n \cap [-\Omega, \Omega]^n$. Let $\phi_\alpha(x) = e^{-\alpha\|x\|^2}$ be a Gaussian window function, such that $\mathcal{G}(\phi_\alpha, \Lambda)$ is a Gabor frame. For $\Omega > 0$ let χ_Ω be the characteristic function of the set $[-\Omega, \Omega]^n$ and let $\phi_{\alpha,\Omega}(x) := \chi_\Omega(x)\,\phi_\alpha(x)$. There exists a function*

$$\mu : \Gamma_\Omega \to \mathbb{N} \quad \textit{with} \quad \mu_{-k} = \mu_k \tag{2.7}$$

with the property that, if the matrices B, C satisfy

$$\|C^t B\| \le \frac{1}{\sqrt{n}(2\Omega - 1)}\,, \tag{2.8}$$

then the function

$$\gamma_\Omega(x) := |\det(C^t B)| \left(\phi_{\alpha,\Omega}(x) + 2\sum_{\ell\in\Gamma_\Omega} \mu_\ell\,\phi_{\alpha,\Omega}(x+\ell)\right) \tag{2.9}$$

is a dual frame for $\mathcal{G}(\phi_{\alpha,\Omega},\Lambda)$. Moreover, for any $\epsilon > 0$ and an $\Omega > 0$ such that $\sup|\phi_\alpha - \phi_{\alpha,\Omega}| < \epsilon$ *and*

$$2ne^{-\frac{\pi^2}{\alpha}n(2\Omega-1)^2} < \epsilon\,,$$

if (2.8) *holds, then the function*

$$\gamma(x) := |\det(C^t B)| \left(\phi_\alpha(x) + 2\sum_{\ell\in\Gamma_\Omega} \mu_\ell\, \phi_\alpha(x+\ell)\right) \tag{2.10}$$

satisfies the Wexel–Raz duality for $\mathcal{G}(\phi_\alpha,\Lambda)$ up to an overall error of size the maximum between ϵ and

$$|\det(C^t B)|(1+2\sum \mu_\ell)\epsilon\,.$$

Proof We need to adapt the argument of [6] recalled above in two ways: first to window functions supported in a set $[-\Omega,\Omega]^n$ and then further extend it from a truncated Gaussian that is compactly supported to an actual Gaussian.

We consider a window function that is compactly supported inside $[-\Omega,\Omega]^n$. We want to show that, in this case, the domain $\mathcal{F}$ that satisfies (2.6) can be taken to be symmetric $\mathcal{F} = \check{\mathcal{F}}$. To this purpose, it suffices to generalize the case of Theorem 1 of [6], with $C = 1$, since the general case is then obtained as in Theorem 2 of [6]. We assume that ϕ is a compactly supported window function with $\mathrm{supp}(\phi) \subset [-\Omega,\Omega]^n$, which satisfies the partition of unity condition (2.2). As described above, we want to construct a dual window γ that satisfies the Wexel–Raz duality expressed in the form (2.5) (with $C = 1$). Let $\Gamma_\Omega := [-\Omega,\Omega]^n \cap \mathbb{Z}^n$, and let $N_\Omega := \#\Gamma_\Omega$. Let $\Gamma_\Omega \simeq \{n_1,\ldots,n_{N_\Omega}\}$ be a choice of an enumeration (ordering) of the set Γ_Ω. As in (2.6), we write

$$\begin{aligned}
1 = \left(\sum_{n\in\Gamma_\Omega}\phi(x+n)\right)^2 &= (\phi(x+n_1)+\cdots \\
&\quad +\phi(x+n_{N_\Omega})\cdot(\phi(x+n_1)+\cdots+\phi(x+n_{N_\Omega}) \\
&= \phi(x+n_1)(\phi(x+n_1)+2\phi(x+n_2)+\cdots+2\phi(x+n_{N_\Omega}) \\
&\quad +\phi(x+n_2)(\phi(x+n_2)+2\phi(x+n_3)+\cdots 2\phi(x+n_{N_\Omega})+\cdots \\
&\quad +\phi(x+n_{N_\Omega})\phi(x+n_{N_\Omega})\,.
\end{aligned}$$

To obtain (2.6), we want to rewrite this as

$$1 = \frac{1}{|\det(B)|}\sum_{j=1}^{N_\Omega}\phi(x+n_j)\cdot\gamma_{n_j}(x)\,,$$

where, as in [6],

$$\gamma_m(x) = |\det(B)| \left(\phi(x+m) + 2 \sum_{i=1}^{n} \sum_{k \in E_i^m} \phi(x+k) \right) ,$$

where the sets E_i^m have the property that

$$\cup_i E_i^m = \{m' \in \Gamma_\Omega \,|\, m' > m\}$$

in the chosen ordering $\Gamma_\Omega \simeq \{n_1, \dots, n_{N_\Omega}\}$.

In [6] the lexicographic ordering is used on the positive quadrant $[0, \Omega]^n$, with

$$\{m' > m\} = \cup_i \{m' > m\}_i := \cup_i \{m' \,|\, m'_i > m_i \text{ and } m'_j = m_j \text{ for } i+1 \leq j \leq n\} ,$$

so that one has $E_i^m = \{m' > m\}_i$ with

$$E_i^m = \left\{ k \in \mathbb{Z}^n \,\middle|\, \begin{array}{ll} 0 \leq k_j \leq \Omega & j = 1, \dots, i-1 \\ m_i < k_i \leq \Omega & j = i \\ k_j = m_j & j = i+1, \dots, n \end{array} \right\}$$

Moreover, one then writes

$$\sum_{k \in E_i^m} \phi(x+k) = \sum_{k \in \mathcal{F}_i} \phi(x+k+m) ,$$

where

$$\mathcal{F}_i = \left\{ k \in \mathbb{Z}^n \,\middle|\, \begin{array}{ll} |k_j| \leq \Omega & j = 1, \dots, i-1 \\ 1 \leq k_i \leq \Omega & j = i \\ k_j = 0 & j = i+1, \dots, n \end{array} \right\} .$$

Let $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm\}^n$ be a sequence of n signs. In $[-\Omega, \Omega]^n$ let Q_ϵ denote the quadrant where each coordinate k_i has sign ϵ_i. We write $Q_+ = [0, \Omega]^n$ for the positive quadrant where all the $\epsilon_i = +$. We denote by $Y_\epsilon : Q_+ \to Q_\epsilon$ the bijection $Y_\epsilon(k) = \epsilon k := (\epsilon_i k_i)_{i=1}^n$. We have $\Gamma_\Omega = \cup_\epsilon \Gamma_\Omega^\epsilon$, where $\Gamma_\Omega^\epsilon = \Gamma_\Omega \cap Q_\epsilon$. We identify points of Γ_Ω with pairs (m, ϵ) with $m \in \Gamma_\Omega^+ = \Gamma_\Omega \cap Q_+$. We order the set $\{\epsilon\} = \{\pm\}^n$ lexicographically, with $- < +$, and we order $\Gamma_\Omega \cap Q_+$ lexicographically as in [6], so that we have in Γ_Ω

$$\{(m', \epsilon') > (m, \epsilon)\} = \cup_{\epsilon', i} \{(m', \epsilon') \,|\, \epsilon' > \epsilon \text{ or } \epsilon' = \epsilon \text{ and } m' > m\} = \cup_i \tilde{E}_i^{m,\epsilon} ,$$

where we have

$$\tilde{E}_i^{m,\epsilon} := \cup_{\epsilon' > \epsilon} \Gamma_\Omega^{\epsilon'} \cup E_i^{m,\epsilon} \quad \text{with} \quad E_i^{m,\epsilon} = Y_\epsilon(E_i^m) .$$

We can then again identify the sums

$$\sum_{k\in \tilde{E}_i^{m,\epsilon}} \phi(x+k) = \sum_{k\in \tilde{\mathcal{F}}_i^{\epsilon}} \phi(x+k+m)\,,$$

where

$$\tilde{\mathcal{F}}_i^{\epsilon} := \cup_{\epsilon'>\epsilon}\Gamma_\Omega^{\epsilon'} \cup Y_\epsilon(\mathcal{F}_i)\,.$$

Since there are overlaps between the sets $\tilde{\mathcal{F}}_i^{\epsilon}$, the points have finite non-negative integer multiplicities if we view the $\tilde{\mathcal{F}}_i^{\epsilon}$ as subsets of Γ_Ω, or we count all points with multiplicity one, and consider the disjoint unions

$$\mathcal{F}_i^{\epsilon} := \sqcup_{\epsilon'>\epsilon}\Gamma_\Omega^{\epsilon'} \sqcup Y_\epsilon(\mathcal{F}_i)\,.$$

Note that the set

$$\mathcal{F} := \sqcup_{\epsilon,i}\mathcal{F}_i^{\epsilon}$$

is invariant under the symmetry $\epsilon \leftrightarrow -\epsilon$. Thus, the corresponding set

$$\cup_{\epsilon,i}\tilde{\mathcal{F}}_i^{\epsilon}$$

consists of Γ_Ω with the appropriate multiplicites assigned to each of the points, and these multiplicites are invariant with respect to the symmetry $k \mapsto -k$. We can then write points of this set as (k, μ_k) with $k \in \Gamma_\Omega$ and $\mu_k \in \mathbb{N}$ the resulting multiplicity, namely the cardinality

$$\mu_k = \#\Pi_\Omega^{-1}(k) \tag{2.11}$$

of the fiber under the projection

$$\Pi_\Omega : \mathcal{F} = \sqcup_{\epsilon,i}\mathcal{F}_i^{\epsilon} \twoheadrightarrow \cup_{\epsilon,i}\tilde{\mathcal{F}}_i^{\epsilon} = \Gamma_\Omega\,,$$

satisfying $\mu_{-k} = \mu_k$. The assignment (2.11) determines a function $\mu : \Gamma_\Omega \to \mathbb{N}$ as in (2.7) with the desired properties. Indeed, we obtain the identity

$$1 = \left(\sum_{n\in\Gamma_\Omega} \phi(x+n)\right)^2 = \frac{1}{|\det(B)|}\sum_{k\in\mathbb{Z}^n} \phi(x+k+n)\gamma(x+n)\,,$$

where

$$\gamma(x) = |\det(B)|\left(\phi(x) + 2\sum_{m\in\Gamma_\Omega} \mu_m\, \phi(x+m)\right)\,.$$

This shows that we can obtain in this way a Wexel–Raz dual window for $\mathcal{G}(\phi_{\alpha,\Omega}, \Lambda)$, with the truncated Gaussian window $\phi_{\alpha,\Omega}$, given by (2.9).

We then need to further extend this result to the case of the Gaussian window ϕ_α. While Gaussians are not compactly supported and do not satisfy the partition of unity property, they can be well approximated by functions that satisfy both, with arbitrarily small error. In particular, for a one-dimensional Gaussian of the form

$$u(t) = \Delta \cdot \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha t^2} ,$$

with $\Delta > 0$, the partition of unity relation (2.2) holds up to an error term (see [2])

$$\sum_{k\in\mathbb{Z}} u(t - k\Delta) = 1 + 2\cos\left(\frac{2\pi t}{\Delta}\right) e^{-\frac{\pi^2}{\alpha\Delta^2}} . \tag{2.12}$$

The error terms add in the case of a multidimensional Gaussian. The approximation (2.10) is then obtained by applying (2.9) to a truncation $D_C\chi_\Omega \cdot \phi_\alpha$, where χ_Ω is the characteristic function of a set $[-\Omega, \Omega]^n$. Thus, for the Gaussian ϕ_α we have an error term on the partition of unity

$$|1 - \sum_{k\in\mathbb{Z}^n} \phi_\alpha(x + C^t Bk)| \leq 2ne^{-\frac{\pi^2}{\alpha\|C^t B\|^2}} .$$

Under the assumption that

$$\|C^t B\| \leq \frac{1}{\sqrt{n}(2\Omega - 1)} ,$$

this error in the partition of unity relation is bounded by

$$2ne^{-\frac{\pi^2}{\alpha}n(2\Omega-1)^2} .$$

Thus, for a given $\epsilon > 0$ we can choose an $\Omega > 0$ such that both

$$\sup |\phi_\alpha - \phi_{\alpha,\Omega}| < \epsilon ,$$

and the error in the partition of unity relation is $2ne^{-\frac{\pi^2}{\alpha}n(2\Omega-1)^2} < \epsilon$, so that the window function

$$\gamma(x) := |\det(C^t B)| \left(\phi_\alpha(x) + 2\sum_{m\in\Gamma_\Omega} \mu_m \phi_\alpha(x + m)\right)$$

satisfies the Wexel–Raz duality up to an error term of size at most $\max\{\epsilon, \det(C^t B)(1 + 2\sum \mu_\ell)\epsilon\}$. $\square$

Remark 2.2 Given a lattice $\Lambda = L \times K$ and Gabor frames $\mathcal{G}(\phi_\alpha, \Lambda)$, if γ is an approximate Wexel–Raz dual window constructed as above, for a chosen error size ϵ and a corresponding cutoff of size Ω, we refer to the pair (ϵ, Ω) as the size of the approximation.

2.2 Hermite Constant

As in (1.20), for a lattice $\Lambda \subset \mathbb{R}^{2n}$, we denote by $\Lambda_{2\sigma}$ the lattice

$$\Lambda_{2\sigma} := \left\{ \left(\lambda_1, \frac{\pi}{2\sigma}\lambda_2\right) \in \mathbb{R}^{2n} \mid \lambda = (\lambda_1, \lambda_2) \in \Lambda \right\}. \tag{2.13}$$

We also write $\Lambda^o_{2\sigma} := (\Lambda^o)_{2\sigma}$ where Λ^o is the adjoint lattice of Λ. (Note that this is not the same as $(\Lambda_{2\sigma})^o$, the adjoint lattice of $\Lambda_{2\sigma}$.) We set

$$C_{\Lambda,\sigma} := \left(\frac{4\sigma}{\pi}\right)^{n/2} Corr(\phi_{2\sigma}, \Lambda^o), \tag{2.14}$$

with $Corr(\phi_{2\sigma}, \Lambda^o)$ defined as in (1.19), with a Gaussian window $\phi_{2\sigma}(x) = e^{-2\sigma\|x\|^2}$.

Remark 2.3 The shortest length ℓ_Λ of a lattice in $\mathbb{R}^{2n}$ is bounded by

$$\ell_\Lambda^2 \leq \gamma_{2n} \cdot |\Lambda|^{1/n},$$

where γ_{2n} is the Hermite constant in $\mathbb{R}^{2n}$. A lattice Λ is *critical* if $\ell_\Lambda^2 = \gamma_{2n}|\Lambda|^{1/n}$. These realize the maximum lattice-packing density.

Lemma 2.4 *For a lattice $\Lambda \subset \mathbb{R}^{2n}$ such that $\Lambda^o_{2\sigma}$ is a critical lattice we have*

$$C_{\Lambda,\sigma} \leq e^{-n\frac{\sigma\,|\Lambda^o_{2\sigma}|^{1/n}}{\pi e}}. \tag{2.15}$$

Proof As in (1.21), we have

$$\langle \phi_{2\sigma}, \pi_z \phi_{2\sigma} \rangle = e^{\pi i u \cdot v}\, e^{-\sigma\|u\|^2} \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\frac{\pi^2}{4\sigma}\|v\|^2}, \tag{2.16}$$

so that we have

$$|\langle \phi_{2\sigma}, \pi_z \phi_{2\sigma} \rangle| = \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\sigma(\|u\|^2 + \frac{\pi^2}{4\sigma^2}\|v\|^2)}. \tag{2.17}$$

Thus, from (2.14) we obtain

$$C_{\Lambda,\sigma} = e^{-\sigma \ell^2_{\Lambda^o_{2\sigma}}}.$$

For a critical lattice we then have

$$C_{\Lambda,\sigma} = e^{-\sigma\gamma_{2n}|\Lambda^o_{2\sigma}|^{1/n}}.$$

Thus, an upper bound on $C_{\Lambda,\sigma}$ is obtained from a lower bound on the Hermite constant γ_{2n}. The Minkowski–Hlawka theorem [14] gives a lower bound for the Hermite constant of the form

$$\gamma_{2n} \geq \left(\frac{2\zeta(2n)}{Vol(B_1^{2n}(0))}\right)^{1/n},$$

where

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} \to 1$$

for $n \to \infty$, with the Bernoulli numbers satisfying $|B_{2n}| \sim \frac{(2n)!2}{(2\pi)^{2n}}$. This results in a linear estimate

$$\gamma_{2n} \geq \frac{n}{\pi e}.$$

□

We consider lattices $\Lambda \subset \mathbb{R}^{2n}$ of the form $\Lambda = L \times K$ with lattices $L, K \subset \mathbb{R}^n$, where $L = C\mathbb{Z}^n$ and $K = B\mathbb{Z}^n$ for some $C, B \in \mathrm{GL}_n(\mathbb{R})$, with $\Lambda_{2\sigma} = L \times \frac{\pi}{2\sigma}K$. In this case we have $\Lambda^o_{2\sigma} = (\Lambda^o)_{2\sigma} = K^\vee \times \frac{\pi}{2\sigma}L^\vee$, while $(\Lambda_{2\sigma})^o = \frac{2\sigma}{\pi}K^\vee \times L^\vee$. In this case we set

$$Corr(\phi_{2\sigma}, L^\vee) := \max_{\ell\in L^\vee} |\langle\phi_{2\sigma}, \pi_{i\ell}\phi_{2\sigma}\rangle|, \tag{2.18}$$

$$C_{L,\sigma} := \left(\frac{4\sigma}{\pi}\right)^{n/2} Corr(\phi_{2\sigma}, L^\vee_{2\sigma}), \tag{2.19}$$

with $L^\vee_{2\sigma} := (L^\vee)_{2\sigma} = \frac{\pi}{2\sigma}L^\vee$.

Corollary 2.5 *For a lattice $L \subset \mathbb{R}^n$ such that $L^\vee_{2\sigma}$ is a critical lattice in $\mathbb{R}^n$, we have*

$$C_{L,\sigma} \leq e^{-n\frac{\sigma|L^\vee_{2\sigma}|^{1/2n}}{2\pi e}}. \tag{2.20}$$

If $L^\vee$ is a critical lattice in $\mathbb{R}^n$, we correspondingly have

$$C_{L,\sigma} \leq e^{-n\frac{\frac{\pi^2}{4\sigma}|L^\vee|^{1/2n}}{2\pi e}}. \tag{2.21}$$

Proof As in (2.17) we have

$$|\langle \phi_{2\sigma}, \pi_{ix}\phi_{2\sigma}\rangle| = \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\frac{\pi^2}{4\sigma}\|x\|^2}, \tag{2.22}$$

with

$$\max_{\ell\in L^\vee} |\langle \phi_{2\sigma}, \pi_{i\ell}\phi_{2\sigma}\rangle| = \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\sigma\, \ell^2_{L^\vee_{2\sigma}}}.$$

If we assume that $L^\vee_{2\sigma}$ is a critical lattice in $\mathbb{R}^n$, we have $\ell^2_{L^\vee_{2\sigma}} = \gamma_n \cdot |L^\vee_{2\sigma}|^{1/2n}$, while if we assume that $L^\vee$ is a critical lattice, we have $\ell^2_{L^\vee} = \gamma_n \cdot |L^\vee|^{1/2n}$. Moreover, $\ell_{L^\vee_{2\sigma}} = \frac{\pi}{2\sigma}\ell_{L^\vee}$ and $|L^\vee_{2\sigma}| = \left(\frac{\pi}{2\sigma}\right)^n |L^\vee|$ so in this case we have

$$\ell^2_{L^\vee_{2\sigma}} = \left(\frac{\pi}{2\sigma}\right)^2 \ell^2_{L^\vee} = \gamma_n \cdot \left(\frac{\pi}{2\sigma}\right)^2 |L^\vee|^{1/2n} = \left(\frac{\pi}{2\sigma}\right)^{3/2} \gamma_n \cdot |L^\vee_{2\sigma}|^{1/2n}.$$

□

2.3 Preliminary Estimates

We discuss here some preliminaries for the main construction of Sect. 2.4.

For a function $f \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$ we denote by $T_u f$ the translate

$$(T_u f)(x) = f(x-u). \tag{2.23}$$

Let $\phi_\alpha(x) = e^{-\alpha\|x\|^2}$, with

$$\psi_\alpha(x) := \left(\frac{\pi}{\alpha}\right)^{n/2} e^{-\frac{\pi^2}{\alpha}\|x\|^2} = \mathfrak{F}(\phi_\alpha). \tag{2.24}$$

Lemma 2.6 *Consider a translate $g_\alpha = T_\ell \phi_\alpha$, for some $\ell \in \mathbb{R}^n$, and let $\phi_\beta(z) = e^{-\beta\|x\|^2}$ with $\psi_\beta := \mathfrak{F}\phi_\beta$. Let*

$$\kappa_{n,\Xi,\beta,\sigma} := \exp\left(-n\,\frac{\sigma\beta\,\Xi^{1/2n}}{2\pi e}\right). \tag{2.25}$$

For $\beta = \frac{\pi^2}{4\sigma^2}$ and α and σ in the range

$$\alpha \le q\cdot\pi \quad \textit{with} \quad q := \begin{cases} \dfrac{\sigma}{e}\,\Xi & \Xi \le 1 \\ \dfrac{\sigma}{e} & \Xi > 1, \end{cases} \tag{2.26}$$

the following estimate holds:

$$\int_{\mathbb{R}^n} g_\alpha(y)\psi_\beta(x-y)\,dy \geq \kappa_{n,\Xi,\beta,\sigma}\, g_\alpha(x)\,. \tag{2.27}$$

Proof We have

$$\psi_\beta(x) = \left(\frac{\pi}{\beta}\right)^{n/2} e^{-\frac{\pi^2}{\beta}\|x\|^2}$$

and for $g_\alpha(x) = e^{-\alpha\|x-\ell\|^2}$

$$\begin{aligned}
\left(\frac{\pi}{\beta}\right)^{n/2} \int_{\mathbb{R}^n} g_\alpha(y) e^{-\frac{\pi^2}{\beta}\|x-y\|^2} dy &= \left(\frac{\pi}{\beta}\right)^{n/2} \int_{\mathbb{R}^n} e^{-(\alpha\|y\|^2 + \frac{\pi^2}{\beta}\|y-(x-\ell)\|^2)} dy \\
&= \left(\frac{\pi}{\beta}\right)^{n/2} e^{-\frac{\alpha\frac{\pi^2}{\beta}}{\alpha+\frac{\pi^2}{\beta}}\|x-\ell\|^2} \int_{\mathbb{R}^n} e^{-(\alpha+\frac{\pi^2}{\beta})\|y-\frac{\alpha}{\alpha+\frac{\pi^2}{\beta}}(x-\ell)\|^2} dy \\
&= \left(\frac{\pi}{\beta}\right)^{n/2} \left(\frac{\pi}{\alpha+\frac{\pi^2}{\beta}}\right)^{n/2} e^{-\frac{\alpha\frac{\pi^2}{\beta}}{\alpha+\frac{\pi^2}{\beta}}\|x-\ell\|^2} \\
&= \left(\frac{\pi^2}{\beta\alpha+\pi^2}\right)^{n/2} \exp\left(-\frac{\alpha\pi^2}{\beta\alpha+\pi^2}\|x-\ell\|^2\right).
\end{aligned}$$

In order to verify the condition

$$\begin{aligned}
&\left(\frac{\pi^2}{\beta\alpha+\pi^2}\right)^{n/2} \cdot \exp\left(-\frac{\alpha\pi^2}{\beta\alpha+\pi^2}\|x-\ell\|^2\right) \\
&\quad \geq \exp\left(-\frac{n}{2}\,\frac{\sigma\beta\;\Xi^{1/2n}}{\pi e}\right) \cdot \exp\left(-\alpha\|x-\ell\|^2\right)
\end{aligned}$$

note that

$$\frac{\alpha\pi^2}{\beta\alpha+\pi^2} < \alpha$$

is always verified since $\alpha, \beta > 0$ hence $\alpha\pi^2 < \alpha\pi^2 + \beta\alpha^2$. Thus, it suffices to check when

$$\log\left(1+\frac{\beta\alpha}{\pi^2}\right) \leq \frac{\sigma\beta\;\Xi^{1/2n}}{\pi e}\,. \tag{2.28}$$

We are assuming that $\beta = \frac{\pi^2}{4\sigma^2}$, so the above gives

$$\log\left(1 + \frac{\alpha}{4\sigma^2}\right) \leq \frac{\pi\ \Xi^{1/2n}}{4\sigma e}\,.$$

When α and σ are in the range (2.26) this is satisfied. $\square$

Remark 2.7 We will be interested in the case where $\Xi = |L^\vee|$ for a lattice $L \subset \mathbb{R}^n$ with

$$\kappa_{n,\Xi,\beta,\sigma} := \exp\left(-n\,\frac{\sigma\beta|L^\vee|^{1/2n}}{2\pi e}\right)\,.$$

Checking (2.28) is then equivalent to checking when

$$\left(\frac{\pi^2}{\beta\alpha + \pi^2}\right)^{n/2} \geq \exp\left(-\frac{n}{2}\,\frac{\sigma\beta\,|L^\vee_{2\sigma}|^{1/2n}}{\pi e}\right)$$

and this is satisfied with $q = q_L$ in the range

$$\alpha \leq q_L \cdot \pi \quad \text{with} \quad q_L := \begin{cases} \dfrac{\sigma}{e}|L^\vee|\ |L^\vee| \leq 1 \\ \dfrac{\sigma}{e} \qquad |L^\vee| > 1\,. \end{cases} \tag{2.29}$$

2.4 Construction of Cohn–Elkies Functions

As above, let $\phi_\alpha(x) = e^{-\alpha\|x\|^2}$ with $\psi_\alpha = \mathfrak{F}(\phi_\alpha)$ as in (2.24). Let $\Lambda \subset \mathbb{R}^{2n}$ be a lattice, with $\Lambda_{2\sigma}$ the lattice (2.13) and $\Lambda^o_{2\sigma} = (\Lambda^o)_{2\sigma}$. We assume that the lattice Λ is such that the Gabor system $\mathcal{G}(\phi_\alpha, \Lambda_{2\sigma})$ satisfies the frame condition.

We consider in particular lattices $\Lambda = L \times K$, as above, with $\Lambda_{2\sigma} = L \times \frac{\pi}{2\sigma}K$ and $\Lambda^o_{2\sigma} = K^\vee \times \frac{\pi}{2\sigma}L^\vee$.

For $\Lambda = L \times K$, let $\mathcal{F} \subset L$ be a finite subset and let $\mu : \mathcal{F} \to \mathbb{N}$ be a function that assigns multiplicities to the points of $\mathcal{F}$. Let $\mathcal{D}_{\mathcal{F},\mu}$ be the function

$$\mathcal{D}_{\mathcal{F},\mu}(x) = 1 + 2\sum_{\ell\in\mathcal{F}} \mu_\ell\, e^{2\pi i\langle \ell, x\rangle}\,. \tag{2.30}$$

For example, for $n = 1$ and $\mathcal{F} = ([-N, N] \smallsetminus \{0\}) \cap \mathbb{Z}$, with all multiplicities equal to one, this is related to the usual Dirichlet kernel by

$$\mathcal{D}_{([-N,N]\smallsetminus\{0\})\cap\mathbb{Z},1}(t) = 2\sum_{k=-N}^{N} e^{2\pi i k x} - 1 = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} - 1\,.$$

We write

$$e_\ell(x) := e^{2\pi i \langle \ell, x\rangle}\,, \tag{2.31}$$

so that $\mathcal{D}_{\mathcal{F},\mu} = 1 + 2\sum_{\ell\in\mathcal{F}} \mu_\ell\, e_\ell$.

For a function $f \in L^2(\mathbb{R}^n)$ and $u \in \mathbb{R}^n$ we denote by $T_u f$ the translate as in (2.23) and we write

$$\mathbb{T}_{\mathcal{F},\mu} f := f + 2\sum_{\ell\in\mathcal{F}} \mu_\ell\, T_\ell f\,. \tag{2.32}$$

Consider then functions of the form

$$h_{\Lambda,\sigma}(x) := \left(\frac{4\sigma}{\pi}\right)^{n/2} |\langle \phi_{2\sigma}, \pi_{ix}\phi_{2\sigma}\rangle| - C_{L,\sigma} \tag{2.33}$$

with $C_{L,\sigma}$ as in (2.19), and $\phi_\sigma(x) := e^{-\sigma\|x\|^2}$, and

$$f_{\mathcal{F},\mu}(x) := \langle \mathbb{T}_{\mathcal{F},\mu}\gamma, \pi_{ix}\phi_\alpha\rangle \cdot h_\Lambda(x)\,. \tag{2.34}$$

with $\gamma = \gamma_{\phi_\alpha,\Lambda_{2\sigma}}$ a Wexel–Raz dual window for $\mathcal{G}(\phi_\alpha, \Lambda_{2\sigma})$.

Theorem 2.8 *Let $\Lambda = L \times K$ be a lattice in $\mathbb{R}^{2n}$ such that $L^\vee$ is a critical lattice in $\mathbb{R}^n$ and the lattice $K \subset \mathbb{R}^n$ is chosen so that the Gabor system $\mathcal{G}(\phi_\alpha, \Lambda_{2\sigma})$ satisfies the frame condition and (2.8) is satisfied. Let $\gamma = \gamma_{\phi_\alpha,\Lambda_{2\sigma}}$ be the approximation to a Wexel–Raz dual window with approximation size (ϵ, Ω) (see Remark 2.2) constructed as in Proposition 2.1. Consider a datum $(\mathcal{F}, \mu)$ given by the pair (Γ_Ω, μ) of Proposition 2.1. Then, for α and σ in the range (2.29), the function (2.34),*

$$f_{\Gamma_\Omega,\mu}(x) := \langle \mathbb{T}_{\Gamma_\Omega,\mu}\gamma, \pi_{ix}\phi_\alpha\rangle \cdot h_\Lambda(x)$$

is a Cohn–Elkies function of dimension n and size $\ell_{L^\vee}$, the shortest length of $L^\vee$.

Proof We first show that $f_{\Gamma_\Omega,\mu}$ is a real valued Schwartz function that satisfies $f_{\Gamma_\Omega,\mu} \leq 0$ for $\|x\| \geq \ell_{L^\vee}$. We have as in (2.22)

$$|\langle \phi_{2\sigma}, \pi_{ix}\phi_{2\sigma}\rangle| = \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\frac{\pi^2}{4\sigma}\|x\|^2}\,,$$

so that we have

$$h_\Lambda(x) = \phi_{\frac{\pi^2}{4\sigma^2}}(x) - C_{L,\sigma} = \phi_\sigma\left(\frac{\pi}{2\sigma}x\right) - C_{L,\sigma}\,.$$

Since $C_{L,\sigma} = e^{-\sigma \ell^2_{L^\vee_{2\sigma}}}$, we have

$$h_\Lambda(x) \leq 0 \quad \text{for} \quad \|x\| \geq \ell_{L^\vee_{2\sigma}} \frac{2\sigma}{\pi} = \ell_{L^\vee}\,.$$

For $z = u + iv \in \mathbb{C}^n$, we have

$$\langle \gamma, \pi_z \phi_\alpha \rangle = \int_{\mathbb{R}^n} \gamma(x) e^{2\pi i x \cdot v} \phi_\alpha(x-u)\, dx = \mathfrak{F}(\gamma \cdot T_u \phi_\alpha) = \mathfrak{F}(\gamma) \star (e_u \cdot \mathfrak{F}(\phi_\alpha)) ,$$

with e_u as in (2.31). Thus, we have

$$\langle \gamma, \pi_z \phi_\alpha \rangle|_{i\mathbb{R}^n} = \langle \gamma, \pi_{iv} \phi_\alpha \rangle = \mathfrak{F}(\gamma \cdot \phi_\alpha)(v) .$$

and we obtain

$$\langle \mathbb{T}_{\Gamma_\Omega,\mu} \gamma, \pi_{ix} \phi_\alpha \rangle = \langle \gamma, \pi_{ix} \phi_\alpha \rangle + 2 \sum_{\ell \in \mathcal{F}} \langle T_\ell \gamma, \pi_{ix} \phi_\alpha \rangle = \mathfrak{F}(\gamma) \star \mathfrak{F}(\phi_\alpha)$$
$$+2 \sum_{\ell \in \mathcal{F}} \mu_\ell \mathfrak{F}(T_\ell \gamma) \star \mathfrak{F}(\phi_\alpha) .$$

We also have

$$\mathfrak{F}(\mathbb{T}_{\Gamma_\Omega,\mu} \gamma) = \mathfrak{F}(\gamma) + 2 \sum_{\ell \in \mathcal{F}} \mu_\ell \mathfrak{F}(T_\ell \gamma) = \mathcal{D}_{\mathcal{F},\mu} \cdot \mathfrak{F}(\gamma) .$$

We consider here the case where the pair $(\mathcal{F}, \mu)$ is given by (Γ_Ω, μ) as in (2.7) in Proposition 2.1. Since both the set Γ_Ω and the multiplicity function $\mu : \Gamma_\Omega \to \mathbb{N}$ are invariant under the symmetry $x \mapsto -x$, the function $\mathcal{D}_{\Gamma_\Omega,\mu}$ also satisfies the symmetry

$$\mathcal{D}_{\Gamma_\Omega,\mu}(-x) = \mathcal{D}_{\Gamma_\Omega,\mu}(x) .$$

Since $\bar{e}_\ell(x) = e_\ell(-x)$, we also have $\bar{\mathcal{D}}_{\Gamma_\Omega,\mu}(x) = \mathcal{D}_{\Gamma_\Omega,\mu}(-x)$, hence $\mathcal{D}_{\Gamma_\Omega,\mu}$ is a real-valued even function.

We use as $\gamma(x)$ the Wexel–Raz dual window approximation of Proposition 2.1, given by (2.10). We then have

$$\mathfrak{F}(\gamma) = |\det(C^t B)| \, (\mathfrak{F}(\phi_\alpha) + 2 \sum_{\ell \in \Gamma_\Omega} \mu_\ell \, \mathfrak{F}(T_\ell \phi_\alpha)) = |\det(C^t B)| \, \mathcal{D}_{\Gamma_\Omega,\mu} \cdot \mathfrak{F}(\phi_\alpha) .$$

Thus, we obtain

$$f_{\Gamma_\Omega,\mu} = |\det(C^t B)| \, \mathcal{D}^2_{\Gamma_\Omega,\mu} \cdot \psi_\alpha \cdot (\phi_{\frac{\pi^2}{4\sigma^2}} - C_{L,\sigma}) ,$$

with $\psi_\alpha = \mathfrak{F}(\phi_\alpha)$. It is clear from this expression that $f_{\Gamma_\Omega,\mu}$ is a real valued Schwartz function. Since $|\det(C^t B)| \, \mathcal{D}^2_{\Gamma_\Omega,\mu}(x) \cdot \psi_\alpha(x) \geq 0$ for all $x \in \mathbb{R}^n$, while $\phi_{\frac{\pi^2}{4\sigma^2}}(x) \geq C_{L,\sigma}$ iff $\|x\| \leq \ell_{L^\vee}$, we have

$$f_{\Gamma_\Omega,\mu}(x) \leq 0 \quad \text{for} \quad \|x\| \geq \ell_{L^\vee}$$

and $f_{\Gamma_\Omega,\mu}(x) \geq 0$ otherwise.

When computing Fourier transforms, we interpret the Fourier transform $\mathfrak{F}(h_\Lambda)$ in the distributional sense, so that we have

$$\mathfrak{F}(h_\Lambda) = \psi_{\frac{\pi^2}{4\sigma^2}} - C_{L,\sigma}\,\delta_0\,,$$

with δ_0 the Dirac delta distribution centered at 0. The convolution product of the Dirac delta distribution δ_0 with a test function φ leaves the test function unchanged,

$$(\varphi\star\delta_0)(x) = \int_{\mathbb{R}^n} \varphi(x-u)\delta_0(u)\,du = \varphi(x)\,.$$

Thus, we obtain

$$\begin{aligned}\mathfrak{F}(f_{\Gamma_\Omega,\mu}) &= |\det(C^t B)|\,\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})\star\phi_\alpha\star(\psi_{\frac{\pi^2}{4\sigma^2}} - C_{L,\sigma}\delta_0)\\ &= |\det(C^t B)|\,\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})\star\phi_\alpha\star\psi_{\frac{\pi^2}{4\sigma^2}} - C_{L,\sigma}\,|\det(C^t B)|\,\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})\star\phi_\alpha\,.\end{aligned}$$

We have

$$\mathcal{D}^2_{\Gamma_\Omega,\mu} = \left(1 + 2\sum_{\ell\in\Gamma_\Omega}\mu_\ell e_\ell\right)^2 = 1 + 4\sum_{\ell\in\Gamma_\Omega}\mu_\ell e_\ell + 4\sum_{\ell,\ell'\in\Gamma_\Omega}\mu_\ell\mu_{\ell'}e_{\ell+\ell'}$$

so that the Fourier transform $\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})$, also interpreted in the distributional sense, gives

$$\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu}) = 1 + 4\sum_{\ell\in\Gamma_\Omega}\mu_\ell\delta_\ell + 4\sum_{\ell,\ell'\in\Gamma_\Omega}\mu_\ell\mu_{\ell'}\delta_{\ell+\ell'}\,,$$

with δ_{x_0} the Dirac delta centered at x_0.

The convolution product $\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})\star\phi_\alpha$ is then given by

$$\mathfrak{F}(\mathcal{D}^2_{\Gamma_\Omega,\mu})\star\phi_\alpha = \phi_\alpha + 4\sum_{\ell\in\Gamma_\Omega}\mu_\ell T_\ell\phi_\alpha + 4\sum_{\ell,\ell'\in\Gamma_\Omega}\mu_\ell\mu_{\ell'}T_{\ell+\ell'}\phi_\alpha\,.$$

Thus, we obtain a non-negative Fourier transform $\mathfrak{F}(f_{\Gamma_\Omega,\mu})(x) \geq 0$ for all $x \in \mathbb{R}^n$ if the following inequality holds, for all $\ell \in \Gamma_\Omega$ and all $x \in \mathbb{R}^n$:

$$(T_\ell\phi_\alpha\star\psi_{\frac{\pi^2}{4\sigma^2}})(x) \geq C_{L,\sigma}\;T_\ell\phi_\alpha(x)\,. \tag{2.35}$$

Since we are assuming that $L^\vee$ is a critical lattice in $\mathbb{R}^n$, we have as in (2.21)

$$C_{L,\sigma} \leq e^{-\frac{n}{2}\,\frac{\pi}{4\sigma e}|L^\vee|^{1/2n}}\,.$$

Then for α and σ in the range (2.29), we obtain from Lemma 2.6 that (2.35) is verified. $\square$

2.5 Lattice Solutions

The Cohn–Elkies functions constructed above are associated to a lattice $L \subset \mathbb{R}^n$ with the property that its dual $L^\vee$ is a critical lattice, namely one whose sphere packing density is maximal among lattices. The Voronoi algorithm provides a way to enumerate all these locally optimal solutions of the lattice-packing problem, by describing the space of lattices up to isometry in terms of positive definite quadratic forms and the identifying the local maxima of the density function with the vertices of the Ryshkov polyhedron [17]. In general these local maxima will not be actual solutions of the sphere packing problem, as the actual solution may not be achievable by a lattice. In terms of Cohn–Elkies functions, the property that the critical lattice $L^\vee$ is also an actual solution of the sphere packing problem is reflected in whether the additional property (1.4) can also be satisfied, as this provides a sufficient condition for the optimality of the lattice for the sphere packing problem (see Lemma 1.5).

In the case of the Cohn–Elkies functions of Theorem 2.8, this optimality condition can be described more explicitly as follows.

Proposition 2.9 *Setting*

$$\Upsilon_{\alpha,\mathcal{F},\mu} := 1 + 4\sum_{\ell\in\mathcal{F}} \mu_\ell e^{-\alpha\|\ell\|^2} + 4\sum_{\ell,\ell'\in\mathcal{F}} \mu_\ell\mu_{\ell'} e^{-\alpha\|\ell+\ell'\|^2} \tag{2.36}$$

we obtain that the condition

$$f_{\Gamma_\Omega,\mu}(0) = \frac{1}{|L^\vee|}(\mathfrak{F} f_{\Gamma_\Omega,\mu})(0) \tag{2.37}$$

is given by

$$\begin{aligned}\Upsilon_{\frac{\alpha}{2},\Gamma_\Omega,\mu} \cdot \left((1 - e^{-\sigma\ell^2_{L^\vee_{2\sigma}}}) \left(\frac{\pi}{2\alpha}\right)^{n/2} + \frac{1}{|L^\vee|} e^{-\sigma\ell^2_{L^\vee_{2\sigma}}} \right) \\ = \Upsilon_{\frac{4\sigma^2\alpha}{\alpha+4\sigma^2},\Gamma_\Omega,\mu} \cdot \frac{1}{|L^\vee|} \cdot \left(\frac{4\sigma^2}{\alpha+4\sigma^2}\right)^{n/2}.\end{aligned} \tag{2.38}$$

Proof We have

$$\begin{aligned} f_{\Gamma_\Omega,\mu}(0) &= \langle \mathbb{T}_{\Gamma_\Omega,\mu}\gamma, \phi_\alpha\rangle \cdot h_\Lambda(0) = \langle \mathbb{T}_{\Gamma_\Omega,\mu}\gamma, \phi_\alpha\rangle\, (1 - e^{-\sigma\ell^2_{L^\vee_{2\sigma}}}) \\ &= (1 - e^{-\sigma\ell^2_{L^\vee_{2\sigma}}})|\det(C^t B)| \left(\langle\gamma, \phi_\alpha\rangle + 2\sum_{\ell\in\mathcal{F}} \mu_\ell \langle T_\ell\gamma, \phi_\alpha\rangle \right) \end{aligned}$$

$$= (1 - e^{-\sigma \ell^2_{L^\vee_{2\sigma}}}) |\det(C^t B)| \left(\langle \phi_\alpha, \phi_\alpha \rangle + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell \langle T_\ell \phi_\alpha, \phi_\alpha \rangle \right.$$
$$\left. +4 \sum_{\ell, \ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} \langle T_{\ell+\ell'} \phi_\alpha, \phi_\alpha \rangle \right) .$$

Writing $e^{-(\alpha\|x\|^2 + \alpha\|x-\ell\|^2)} = e^{-\alpha\|\ell\|^2/2}\, e^{-2\alpha\|x - \frac{\ell}{2}\|^2}$ we get

$$f_{\Gamma_\Omega,\mu}(0) = (1 - e^{-\sigma \ell^2_{L^\vee_{2\sigma}}}) |\det(C^t B)| \left(\frac{\pi}{2\alpha}\right)^{n/2} \left(1 + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell e^{-\frac{\alpha}{2}\|\ell\|^2} \right.$$
$$\left. +4 \sum_{\ell, \ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} e^{-\frac{\alpha}{2}\|\ell+\ell'\|^2} \right) .$$

On the other hand we have

$$(\mathfrak{F} f_{\Gamma_\Omega,\mu})(0) = |\det(C^t B)| \left(\langle \phi_\alpha, \psi_{\frac{\pi^2}{4\sigma^2}} \rangle + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell \langle T_\ell \phi_\alpha, \psi_{\frac{\pi^2}{4\sigma^2}} \rangle \right.$$
$$\left. +4 \sum_{\ell, \ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} \langle T_{\ell+\ell'} \phi_\alpha, \psi_{\frac{\pi^2}{4\sigma^2}} \rangle \right)$$
$$- |\det(C^t B)|\, e^{-\sigma \ell^2_{L^\vee_{2\sigma}}} \left(1 + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell e^{-\frac{\alpha}{2}\|\ell\|^2} \right.$$
$$\left. +4 \sum_{\ell, \ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} e^{-\frac{\alpha}{2}\|\ell+\ell'\|^2} \right) .$$

We have

$$\left(\frac{4\sigma^2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-\alpha\|x-\ell\|^2} e^{-4\sigma^2\|x\|^2} dx = \left(\frac{4\sigma^2}{\alpha + 4\sigma^2}\right)^{n/2} e^{-\frac{4\sigma^2\alpha}{\alpha+4\sigma^2}\|\ell\|^2} ,$$

hence we obtain

$$(\mathfrak{F} f_{\Gamma_\Omega,\mu})(0) = |\det(C^t B)| \left(\frac{4\sigma^2}{\alpha + 4\sigma^2}\right)^{n/2} \left(1 + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell e^{-\frac{4\sigma^2\alpha}{\alpha+4\sigma^2}\|\ell\|^2} \right.$$
$$\left. +4 \sum_{\ell, \ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} e^{-\frac{4\sigma^2\alpha}{\alpha+4\sigma^2}\|\ell+\ell'\|^2} \right)$$

$$-|\det(C^t B)|\, e^{-\sigma \frac{\ell^2_{L^\vee}}{2\sigma}} \left(1 + 4 \sum_{\ell \in \Gamma_\Omega} \mu_\ell e^{-\frac{\alpha}{2}\|\ell\|^2} \right.$$
$$\left. +4 \sum_{\ell,\ell' \in \Gamma_\Omega} \mu_\ell \mu_{\ell'} e^{-\frac{\alpha}{2}\|\ell+\ell'\|^2} \right) .$$

Thus, using (2.36) we can write (2.37) in the form (2.38), where

$$|L^\vee|^{-1} = |L| = |\det(C)|.$$

□

Note that, while we always have the vanishing at lattice points of the function $\langle \gamma, \pi_z \phi \rangle$ by the Wexel–Raz duality, in general we do not have the vanishing of the $\langle \pi_\ell \gamma, \pi_z \phi \rangle$, hence of $f_{\Gamma_\Omega,\mu}$. This vanishing occurs in the case when this is a special Cohn–Elkies function (by Corollary 1.6) that is, when the identity of Proposition 2.9 holds.

3 From Lattices to Periodic Sets

The construction presented above uses essentially the fact that the sphere packing considered is a lattice sphere packing. Since one expects that only in very few dimensions the optimal sphere packing will be realized by a lattice, one would like to extend this method of construction of Cohn–Elkies functions adapted to lattices to the case of periodic sets, which are known to approximate the maximal density in any dimension. The Zassenhaus conjecture predicts that in every dimension the maximal density sphere packing can be realized by a periodic packing.

Definition 3.1 A periodic set Σ in $\mathbb{R}^n$ is a set for which there exist a finite collection $\{a_1, \ldots, a_N\}$ of vectors $a_i \in \mathbb{R}^n$ and a lattice $L \subset \mathbb{R}^n$ such that

$$\Sigma = \bigcup_{i=1}^{N} a_i + L \,. \tag{3.1}$$

The lattice L can be taken to be the maximal period lattice for Σ. The size of a periodic set is the minimal number N of translations such that the set can be represented in the form (3.1). A periodic set $\Sigma \subset \mathbb{R}^n$ of size N is critical if it is a maximizer of the sphere packing density in $\mathbb{R}^n$ among all periodic packings of size at most N.

The center density of a sphere packing with sphere centers places at the points of a periodic set Σ is given by

$$\delta_\Sigma = \frac{N\, \ell^n_\Sigma}{2^n |L|} \,, \tag{3.2}$$

with the minimal length ℓ_Σ, which is given by

$$\ell_\Sigma = \min\{\|\ell + a_i - a_j\| \mid \ell \in L,\ i, j = 1, \dots, N\}.$$

Definition 3.2 A periodic set $\Sigma \subset \mathbb{R}^n$ of size N is critical if it is a maximizer of the sphere packing density δ_Σ of (3.2) in $\mathbb{R}^n$, among all the periodic packings of size at most N, with fixed ratio $N/|L|$, that is, if it maximizes ℓ_Σ among all periodic sets of size at most N.

Gabor frames $\mathcal{G}(\phi, \Lambda)$ where Λ is not a lattice have been considered in signal analysis, though a lot less is known about them than in the lattice case. For Gabor frames with irregular and semi-regular $\Lambda \subset \mathbb{R}^2$ see for instance [3, 5, 13].

Here we need to consider a special type of irregular Gabor frames, namely semiregular frames with $\Lambda = \Sigma \times K \subset \mathbb{R}^{2n}$ where $\Sigma \subset \mathbb{R}^n$ is a periodic set and $K \subset \mathbb{R}^n$ is a lattice, and with a Gaussian window function.

Definition 3.3 A Gabor multisystem $\mathcal{G}(\phi_1, \dots, \phi_N, \Lambda_1, \dots, \Lambda_N)$ is defined as the union of the Gabor systems

$$\mathcal{G}(\phi_1, \dots, \phi_N, \Lambda_1, \dots, \Lambda_N) := \bigcup_{i=1}^{N} \mathcal{G}(\phi_i, \Lambda_i).$$

A multi-window Gabor system is a multisystem of the form $\mathcal{G}(\phi_1, \dots, \phi_N, \Lambda_1, \dots, \Lambda_N)$ where $\Lambda_i = \Lambda$ for all $i = 1, \dots, N$. We write $\mathcal{G}(\underline{\phi}, \Lambda)$ in this case.

Remark 3.4 Note that the Gabor system $\mathcal{G}(\phi, \Lambda)$ with $\Lambda = \Sigma \times K$ for $\Sigma = \cup_{i=1}^{N} a_i + L$ is the same as the multi-window system given by the union of the $\mathcal{G}(\phi_i, \Lambda_i)$ where $\phi_i = \pi_{a_i}\phi$ and with $\Lambda_i = L \times K$ for all $i = 1, \dots, N$.

Remark 3.5 Theorem 5.1 of [4] shows that a necessary condition for completeness of the Gabor multi-window system $\mathcal{G}(\underline{\phi}, L \times K)$, with $\phi_i = \pi_{a_i}\phi$ and ϕ a Gaussian, is the condition that $\det(A) \times \det(\overline{B}) < N$, where $L = A\mathbb{Z}^n$ and $K = B\mathbb{Z}^n$ for $A, B \in \mathrm{GL}_n(\mathbb{R})$ and with N the number of translations of the periodic set Σ.

In the case of a periodic set (3.1) we define

$$\Sigma^\vee := \bigcup_{i=1}^{N} a_i + L^\vee, \tag{3.3}$$

where $L^\vee$ is the dual lattice of L and the $\{a_i\}_{i=1}^N$ are the same translations as in Σ. We then set

$$Corr(\phi_{2\sigma}, \Sigma^\vee) := \max_{\ell, \ell' \in \Sigma^\vee, \ell \neq \ell'} |\langle \pi_{i\ell}\phi_{2\sigma}, \pi_{i\ell'}\phi_{2\sigma}\rangle|, \tag{3.4}$$

which directly generalizes (2.18) for lattices. We also define as before

$$C_{\Sigma,\sigma} := \left(\frac{4\sigma}{\pi}\right)^{n/2} \, Corr(\phi_{2\sigma}, \Sigma^\vee_{2\sigma}), \tag{3.5}$$

with $\Sigma^\vee_{2\sigma} := (\Sigma^\vee)_{2\sigma} = \frac{\pi}{2\sigma}\Sigma^\vee$.

Proposition 3.6 *Let $\Sigma \subset \mathbb{R}^n$ be a periodic set of size N with $\Sigma^\vee$ critical, and let $K \subset \mathbb{R}^n$ be a lattice such that the $\mathcal{G}(\phi_{j,\alpha}, L \times K)$ for $j = 1, \ldots, N$ are Gabor frames. Then a Cohn–Elkies functions in dimension n with size $\ell_{\Sigma^\vee}$ is given by $f_{\Gamma_\Omega,\mu} = \sum_{j=1}^N f_{j,\Gamma_\Omega,\mu}$, with*

$$f_{j,\Gamma_\Omega,\mu}(x) := \langle \mathbb{T}_{\Gamma_\Omega,\mu}\gamma_j, \pi_{ix}\phi_{j,\alpha}\rangle \cdot h_\Sigma(x),$$
$$h_\Sigma(x) = \phi_{\frac{\pi^2}{4\sigma^2}}(x) - C_{\Lambda,\sigma},$$

with α, σ in the range (2.29).

Proof In general a multi-window Gabor system $\mathcal{G}(\underline{\phi}, L \times K)$, with $\Lambda \subset \mathbb{R}^{2n}$ a lattice, satisfies the frame condition if there are constants $C, C' > 0$ such that for all $f \in L^2(\mathbb{R}^n)$

$$C\|f\|^2_{L^2(\mathbb{R}^n)} \leq \sum_{j=1}^N \sum_{\lambda\in L\times K} |\langle f, \pi_\lambda \pi_{a_j}\phi\rangle|^2 \leq C'\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Thus, if the individual Gabor systems $\mathcal{G}(\phi_j, L \times K)$ satisfy the Gabor frame condition then the multi-window system also does.

Since only the translations part $\Sigma \subset \mathbb{R}^n$ is a periodic set, while the modulation part $K \subset \mathbb{R}^n$ is an actual lattice, we have that the functions of the multi-window $\phi_j = \pi_{a_j}\phi = T_{a_j}\phi$ are translates of the Gaussian ϕ along the translations a_j of the periodic set $\Sigma = \cup_j a_j + L$.

Given the periodic set $\Sigma \subset \mathbb{R}^n$, suppose that the lattice $K \subset \mathbb{R}^n$ is chosen so that the condition of Remark 3.5 holds and the Gabor systems $\mathcal{G}(\phi_{j,\alpha}, L \times K)$, with $\phi_{j,\alpha} = T_{a_j}\phi_\alpha$ and ϕ_α a Gaussian are Gabor frames for all $i = 1, \ldots, N$.

Then proceeding as in Proposition 2.1, with $\Omega >> \|a_j\|$ for all $i = 1, \ldots, N$, we obtain approximate Wexel–Raz duals γ_j for each $\mathcal{G}(\phi_{j,\alpha}, L\times K)$. Note that the sets Γ_Ω and the multiplicity function $\mu : \Gamma_\Omega \to \mathbb{N}$ are unchanged, and we have approximate Wexel–Raz duals for the Gabor systems $\mathcal{G}(\phi_{j,\alpha}, L \times K)$ of the form

$$\gamma_j = |\det(A^t B)|\,(\phi_{j,\alpha} + 2\sum_{\ell\in\Gamma_\Omega} \mu_\ell T_\ell \phi_{j,\alpha}) = |\det(A^t B)|\,\mathbb{T}_{\Gamma_\Omega,\mu}\phi_{j,\alpha}.$$

As in (2.22) we have

$$|\langle \pi_{i\ell}\phi_{2\sigma}, \pi_{i\ell'}\phi_{2\sigma}\rangle| = \left(\frac{\pi}{4\sigma}\right)^{n/2} e^{-\frac{\pi^2}{4\sigma}\|\ell-\ell'\|^2}, \tag{3.6}$$

hence we obtain

$$C_{\Sigma,\sigma} = \max_{\ell,\ell'\in\Sigma^\vee,\ell\neq\ell'} e^{-\frac{\pi^2}{4\sigma}\|\ell-\ell'\|^2} = e^{-\frac{\pi^2}{4\sigma}\ell^2_{\Sigma^\vee}} . \tag{3.7}$$

We can assume without loss of generality that the lattices $L \times K$ we consider all have $L \subset \mathbb{R}^n$ with $|L| = 1$, with conditions such as Remark 3.5 for the Gabor frames property formulated as conditions on the choice of the auxiliary lattice $K \subset \mathbb{R}^n$.

Assuming that the periodic set $\Sigma^\vee$ is critical, we have

$$\ell^2_{\Sigma^\vee} \geq \gamma_n \geq \frac{n}{2\pi e} ,$$

since $\Sigma^\vee$ maximizes density among all packings by periodic sets of size at most N, hence its density is not worse than the optimal density among lattices. We use the same lower bound for the Hermite constant γ_n as in Corollary 2.5. Thus, we obtain

$$C_{\Sigma,\sigma} \leq e^{-n\frac{\pi}{8\sigma e}} , \tag{3.8}$$

as in (2.21). We can then use the same estimates of Lemma 2.6 and Remark 2.7 (with $|L| = |L^\vee| = 1$).

Lemma 2.6 holds for each $g_{j,\alpha} = T_\ell\phi_{j,\alpha}$, hence the same argument used in Theorem 2.8 shows that the functions

$$f_{j,\Gamma_\Omega,\mu}(x) := \langle \mathbb{T}_{\Gamma_\Omega,\mu}\gamma_j, \pi_{ix}\phi_{j,\alpha}\rangle \cdot h_\Sigma(x) ,$$

with

$$h_\Sigma(x) = \phi_{\frac{\pi^2}{4\sigma^2}}(x) - C_{\Lambda,\sigma} ,$$

with the parameters α, σ in the same range as in Theorem 2.8, are Cohen-Elkies functions in dimension n with size $\ell_{\Sigma^\vee}$, as in Definition 1.1, hence so is their sum $f_{\Gamma_\Omega,\mu} = \sum_{j=1}^N f_{j,\Gamma_\Omega,\mu}$. □

We showed in this section that our construction of Cohn–Elkies functions in arbitrary dimension extends from the case of Gabor frames based on lattices to the case of periodic sets. In all cases we use a window function that is a multidimensional Gaussian. It is possible that a similar construction may extend to other classes of window functions: for example, a natural generalization of the Gaussian case would be windows obtained from multidimensional Hermite functions. Our argument depends on the Gaussian shape of the window function in two ways: first in the relation between correlation and length, used in (1.22) in Lemma 1.19 and in (2.17) in Lemma 2.4, and then in the fundamental estimate of Lemma 2.6, where the Fourier transform properties of Gaussians play a crucial role. Thus, the present argument does not directly extend to non-Gaussian windows, though this does not exclude that a modification of the argument will lead to a more general construction for other classes of Gabor windows.

Acknowledgements The second author is supported by NSF grant DMS-2104330.

Author Contributions Both authors contributed equally to the research and the preparation of the paper.

Declarations

Conflict of interest The authors declare no competing interests.

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