

WEIGHT 2 COMPACTLY SUPPORTED COHOMOLOGY OF MODULI SPACES OF CURVES

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Abstract

We study the weight 2 graded piece of the compactly supported rational cohomology of the moduli spaces of curves $\mathcal{M}_{g,n}$ and show that this can be computed as the cohomology of a graph complex that is closely related to graph complexes arising in the study of embedding spaces. For $n = 0$, we express this cohomology in terms of $W_0 H_c^\bullet(\mathcal{M}_{g',n'})$ for $g' \leq g$ and $n' \leq 2$, and thereby produce several new infinite families of nonvanishing unstable cohomology groups on $\mathcal{M}_g$, including the first such families in odd degrees. In particular, we show that $\dim H^{4g-k}(\mathcal{M}_g)$ grows at least exponentially with g for $k \in \{8, 9, 11, 12, 14, 15, 16, 18, 19\}$.

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1. Introduction

The weight 0 compactly supported cohomology $W_0 H_c^\bullet(\mathcal{M}_{g,n})$ is naturally isomorphic to the cohomology of the genus g part of the standard commutative graph complex with n marked external vertices (see [8], [9]). Similar graph complexes arise

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in the study of rational homology and homotopy groups of higher-dimensional long links (see [32]). In this article, we use commutative graph complexes with additional decorations on the external vertices, similar to those appearing in the embedding calculus (see [16]) to study the next nontrivial weight-graded piece of the compactly supported cohomology of $\mathcal{M}_{g,n}$. Throughout, all cohomology groups are taken with rational coefficients.

The weight k graded piece of $H_c^\bullet(\mathcal{M}_{g,n})$ is

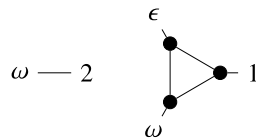
$$\mathrm{gr}_k H_c^\bullet(\mathcal{M}_{g,n}) := W_k H_c^\bullet(\mathcal{M}_{g,n}) / W_{k-1} H_c^\bullet(\mathcal{M}_{g,n}).$$

Note that $\mathrm{gr}_1 H_c^\bullet(\mathcal{M}_{g,n})$ vanishes for all g and n ; this follows from the vanishing of $H^1(\overline{\mathcal{M}}_{g,n})$ for all g and n via Deligne's weight spectral sequence. The next nontrivial weight-graded piece is $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$, which we study by combining graph complex techniques with the presentations and pullback formulas for $H^2(\overline{\mathcal{M}}_{g,n})$ from [2] and [22].

Our first main result expresses $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ as the cohomology of a graph complex $X_{g,n}$ generated by possibly disconnected simple graphs without loops or multiple edges, with features and decorations as follows. Say that the 1-valent vertices are *external* and all other vertices are *internal*.

- All internal vertices are at least trivalent.
- Each external vertex has a label from $\{1, \dots, n, \epsilon, \omega\}$.
- Each label $1, \dots, n$ appears exactly once, and the label ω appears exactly twice.
- The graph obtained by joining all external vertices marked ϵ or ω is connected and has genus g .
- There is no connected component consisting of a single edge connecting two external vertices with markings $\{\epsilon, \omega\}$ or $\{\omega, \omega\}$.

The edges that do not contain an external vertex with marking from $\{1, \dots, n\}$ are called *structural*. Each generator is equipped with a total ordering of its structural edges, and reordering the structural edges induces multiplication by the sign of the permutation. Note that some connected components may not have any internal vertices. The *degree* of a generator for $X_{g,n}$ is the number of structural edges plus 1. In figures, we replace each external vertex with its label and omit the edge ordering. For example, the following is a generator of $X_{2,2}$ in degree 6:



The differential δ on $X_{g,n}$ is a sum of two parts $\delta = \delta_{\text{split}} + \delta_{\text{join}}$. Here δ_{split} is a sum over all vertices and over all ways of splitting the vertex into two vertices joined by an edge so that the new vertices are at least trivalent:

$$\delta_{\text{split}} \Gamma = \sum_{v \text{ vertex}} \Gamma \text{ split } v, \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \mapsto \sum \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad (1)$$

The part δ_{join} glues together a subset S of the ϵ - and ω -decorated external vertices, where

- $|S| \geq 2$, and S contains either 0 or 1 of the ω -decorated vertices, and then attaches an edge to a new external vertex. The new edge is decorated by ω if S contains an ω -decorated vertex and by ϵ otherwise:

$$\delta_{\text{join}} \begin{array}{c} \textcircled{\Gamma} \\ \diagup \quad \diagdown \end{array} = \sum_S \begin{array}{c} \textcircled{\Gamma} \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} \quad \epsilon \text{ or } \omega$$

Each graph produced by the differential has one new structural edge, which is taken first in the ordering; the relative ordering of the remaining edges is preserved.

THEOREM 1.1

For all g and n such that $2g + n \geq 3$ and $(g, n) \neq (1, 1)$, there is an $\mathbb{S}_n$ -equivariant isomorphism

$$\text{gr}_2 H_c^\bullet(\mathcal{M}_{g,n}) \cong H^\bullet(X_{g,n}, \delta).$$

This isomorphism at the level of cohomology is achieved through a zigzag of quasi-isomorphisms of complexes, relating the row of Deligne's weight spectral sequence that computes $\text{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ to the graph complex $X_{g,n}$.

The cohomology of such commutative graph complexes is far from fully understood. Nevertheless, a partial understanding of this graph cohomology is enough to yield new results about the unstable cohomology of $\mathcal{M}_{g,n}$ in weight 2, just as in weight 0. For $n = 0$, we express the weight 2 compactly supported cohomology of $\mathcal{M}_g$ in terms of the weight 0 compactly supported cohomology of $\mathcal{M}_{g',1}$ and $\mathcal{M}_{g',2}$ for $g' \leq g$ as follows. Let

$$\mathbb{V}_{\bullet, \bullet} := \bigwedge^2 \left(\bigoplus_{g,k} W_0 H_c^k(\mathcal{M}_{g,1}) \right),$$

bigraded so that an element of $W_0 H_c^k(\mathcal{M}_{g,1}) \wedge W_0 H_c^{k'}(\mathcal{M}_{g',1})$ is in degree $(g + g', k + k')$. Let V_{as} denote the antisymmetric part of an $\mathbb{S}_2$ -representation V . By convention, we set $H_c^\bullet(\mathcal{M}_{0,2}) := 0$.

THEOREM 1.2

For $g \geq 2$, there is an isomorphism of $\mathbb{Q}$ -vector spaces

$$\text{gr}_2 H_c^k(\mathcal{M}_g) \cong \mathbb{V}_{g,k-3} \oplus W_0 H_c^{k-3}(\mathcal{M}_{g-1,2})_{\text{as}} \oplus \mathbb{V}_{g-1,k-4} \oplus W_0 H_c^{k-4}(\mathcal{M}_{g-2,2})_{\text{as}}.$$

Remark 1.3

Note that the Hodge structure on $\text{gr}_2 H_c^k(\mathcal{M}_g)$ is pure Tate; this follows from Deligne's weight spectral sequence associated to $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$, which abuts to $\text{gr}_\bullet H_c^k(\mathcal{M}_g)$, together with the fact that $H^2(\overline{\mathcal{M}}_{g',n'})$ is pure Tate for all g' and n' (see Section 2.3 below). The Hodge structure on $\text{gr}_2 H_c^k(\mathcal{M}_g)$ is therefore obtained from the $\mathbb{Q}$ -vector space description above by tensoring with $\mathbb{Q}(-1)$. For any variety or Deligne–Mumford stack X for which $\text{gr}_1 H_c^*(X) = 0$, such as $\mathcal{M}_g$, the subspace $W_2 H_c^*(X)$ is an extension of $\text{gr}_2 H_c^*(X)$ by $W_0 H_c^*(X)$, in the category of mixed Hodge structures. For $X = \mathcal{M}_g$, one might naturally expect that this extension is trivial. To prove this, it would be enough to show that $W_2 H_c^*(\mathcal{M}_g)$ is the mixed Hodge realization of a mixed Tate motive over $\mathbb{Z}$, by [13, (1.6.11)].

To understand the geometric meaning of the first two summands in this theorem, let $D_g^\circ \subset \overline{\mathcal{M}}_g$ be the locally closed substack parameterizing 1-nodal curves. We have a natural identification

$$W_0 H_c^k(D_g^\circ) \cong \mathbb{V}_{g,k} \oplus W_0 H_c^k(\mathcal{M}_{g-1,2})_{\text{as}}.$$

The cup product with the first Chern class of the normal bundle followed by the push-forward for the open inclusion ι of D_g° in the boundary $\partial \mathcal{M}_g := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ and the coboundary map δ for the excision sequence give

$$W_0 H_c^\bullet(D_g^\circ) \xrightarrow{c_1 \smile} \text{gr}_2 H_c^{\bullet+2}(D_g^\circ) \xrightarrow{\iota_*} \text{gr}_2 H_c^{\bullet+2} \partial \mathcal{M}_g \xrightarrow{\delta} \text{gr}_2 H_c^{\bullet+3}(\mathcal{M}_g).$$

We expect that the geometric content of Theorem 1.2 is that this composition is injective and the image accounts for the first two summands in Theorem 1.2. As a first step toward seeing this, note that c_1 is represented by a smooth differential form that extends to the boundary. This pulls back to the normalization of $\partial \mathcal{M}_g$ and hence acts on every page of the weight spectral sequence abutting to $\text{gr} H_c^\bullet(\mathcal{M}_g)$ (cf. the proof of Theorem 1.6 in Section 8.4). This action can be interpreted graphically by following our zigzag of quasi-isomorphisms. We do not have an analogous expected geometric interpretation for the other two summands. (For a graphical illustration of the simplest nontrivial cohomology classes produced by Theorem 1.2, see Section 3.4.)

For applications, we will use only subspaces of $\mathbb{V}_{\bullet,\bullet}$ that arise from the images of known classes under pullback to the universal curve $W_0H_c^\bullet(\mathcal{M}_g) \hookrightarrow W_0H_c^\bullet(\mathcal{M}_{g,1})$. Recall that $W_0H_c^{2g}(\mathcal{M}_g)$ is nonvanishing for $g = 3$, $g = 5$, and $g \geq 7$, and its dimension grows at least exponentially with g (see [8, Theorem 1.1]); the proof relies on connections with Grothendieck–Teichmüller theory (see [39]) and a deep result of Brown [5]. Here, we show furthermore that $W_0H_c^{2g+3}(\mathcal{M}_g)$ is nonvanishing for $g = 6$ and $g \geq 8$, and its dimension again grows at least exponentially with g (see Proposition 2.4). It follows that the dimensions of $\mathbb{V}_{g,2g+k}$ and $\mathbb{V}_{g,2g+k+3}$ grow at least exponentially with g whenever $W_0H_c^{2g'+k}(\mathcal{M}_{g'}) \neq 0$ for some g' . Combining Theorem 1.2 with what is currently known about $\bigoplus_g W_0H_c^\bullet(\mathcal{M}_g)$ and applying Poincaré duality, we have the following corollary.

COROLLARY 1.4

The following unstable rational cohomology groups are nonzero:

- (1) $H^{4g-8}(\mathcal{M}_g)$ for $g = 9$ and $g \geq 11$;
- (2) $H^{4g-9}(\mathcal{M}_g)$ for $g = 6$ and $g \geq 8$;
- (3) $H^{4g-11}(\mathcal{M}_g)$ for $g = 10$ and $g \geq 12$;
- (4) $H^{4g-12}(\mathcal{M}_g)$ for $g = 9$ and $g \geq 11$;
- (5) $H^{4g-14}(\mathcal{M}_g)$ for $g = 15$ and $g \geq 17$;
- (6) $H^{4g-15}(\mathcal{M}_g)$ for $g = 14$ and $g \geq 16$;
- (7) $H^{4g-16}(\mathcal{M}_g)$ for $g = 13$, $g = 15$, and $g \geq 17$;
- (8) $H^{4g-18}(\mathcal{M}_g)$ for $g = 17$ and $g \geq 19$;
- (9) $H^{4g-19}(\mathcal{M}_g)$ for $g = 16$ and $g \geq 18$.

In each case, the dimension of $\mathrm{gr}_2 H^{4g-k}(\mathcal{M}_g)$ grows at least exponentially with g .

In particular, this gives the first infinite collections of nonvanishing odd cohomology groups on $\mathcal{M}_g$. Only a few sporadic examples were known previously, such as $H^5(\mathcal{M}_4)$ (see [33]), and $H^{15}(\mathcal{M}_6)$, $H^{23}(\mathcal{M}_8)$, and $H^{27}(\mathcal{M}_{10})$ (see [8]). For the proof of Corollary 1.4, see Section 2.7.

Remark 1.5

For context, let us recall that $H^\bullet(\mathcal{M}_g)$ is fully understood in the stable range, consisting of degrees up to $2\lfloor g/3 \rfloor$, where it agrees with the polynomial ring on the κ -classes (see [28], [37]). The Euler characteristic computations of Harer and Zagier [20] show that the dimension of the unstable cohomology of $\mathcal{M}_g$ grows super-exponentially with g . This is far larger than the tautological subring generated by the κ -classes, which is nonzero precisely in the even degrees up to $2g - 4$ (see [15], [27]). The problem of understanding the rest of the cohomology of $\mathcal{M}_g$, starting with the degrees in which it is zero and nonzero, is highlighted in [29, Section 9].

Until recently, there were only two instances of cohomology groups that were known to be nonzero due to the presence of nontautological classes: $H^6(\mathcal{M}_3)$ (see [26]) and $H^5(\mathcal{M}_4)$, each of which is 1-dimensional. It was previously conjectured by Kontsevich [24] and independently by Church, Farb, and Putman [11] that $H^{4g-5-k}(\mathcal{M}_g)$ vanishes for $g \gg 0$, when k is fixed. This is true for $k < 0$ for virtual cohomological dimension reasons and for $k = 0$ by [10] and [30]. The main result of [8] disproved the conjecture for $k = 1$. Corollary 1.4 shows that the conjecture is false in nine more cases, namely for $k \in \{3, 4, 6, 7, 9, 10, 11, 13, 14\}$. See Table 1, at the end of the introduction, for a summary of what is now known about the vanishing and nonvanishing of $H^{4g-5-k}(\mathcal{M}_g)$.

Corollary 1.4 uses only the subspace of $\mathbb{V}_{\bullet,\bullet}$ that comes from $W_0 H_c^\bullet(\mathcal{M}_g) \hookrightarrow W_0 H_c^\bullet(\mathcal{M}_{g,1})$. Explicit computations show that this inclusion is an isomorphism for $g < 7$. For $g = 8$, the Euler characteristic formulas in [7] show that it is not surjective, but we do not know in which degrees the cokernel is supported. Similarly, $W_0 H_c^\bullet(\mathcal{M}_{g,2})_{\text{as}}$ vanishes for $g < 7$, and for $g = 8$, its Euler characteristic is nonzero but we do not know in which degrees it is supported. By Theorem 1.1, any future improvement in our understanding of $W_0 H_c^\bullet(\mathcal{M}_g)$, $W_0 H_c^\bullet(\mathcal{M}_{g,1})$, and $W_0 H_c^\bullet(\mathcal{M}_{g,2})_{\text{as}}$ will yield a corresponding improvement in our understanding of $\text{gr}_2 H_c^\bullet(\mathcal{M}_g)$. For an inspiring sense of what one might reasonably hope or expect to be true about $W_0 H_c^\bullet(\mathcal{M}_g)$, see [6, Conjecture 1, Table 2].

For $n \geq 1$ we do not have an analogue of Theorem 1.2 expressing $\text{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ in terms of $W_0 H_c^\bullet(\mathcal{M}_{g',n'})$. Nevertheless, we produce injections that prove nonvanishing and lower bounds on dimensions in many cases. We begin with $n = 1$. Recall that the pullback $\pi^*: \text{gr}_k H^\bullet(\mathcal{M}_g) \rightarrow \text{gr}_k H_c^\bullet(\mathcal{M}_{g,1})$ is injective, as is the composition

$$\text{gr}_k H_c^\bullet(\mathcal{M}_g) \xrightarrow{\pi^*} \text{gr}_k H_c^\bullet(\mathcal{M}_{g,1}) \xrightarrow{\psi \wedge} \text{gr}_{k+2} H_c^{\bullet+2}(\mathcal{M}_{g,1}),$$

because further composition with the Gysin pushforward is multiplication by $2g - 2$. For weight $k = 0$, we strengthen this by showing that $\psi \wedge$ is itself injective.

THEOREM 1.6

Let $g \geq 2$. Multiplication by the ψ -class at the marking yields an injection

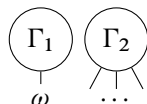
$$\psi \wedge: W_0 H_c^\bullet(\mathcal{M}_{g,1}) \rightarrow \text{gr}_2 H_c^{\bullet+2}(\mathcal{M}_{g,1}). \quad (2)$$

Furthermore, the image of the injection

$$\pi^*: \text{gr}_2 H_c^\bullet(\mathcal{M}_g) \rightarrow \text{gr}_2 H_c^\bullet(\mathcal{M}_{g,1}), \quad (3)$$

intersects trivially with that of (2).

For $n \geq 2$, we show that a quasi-isomorphic subcomplex of $X_{g,n}$ can be split into several direct summands, as detailed in Section 6. For some of these summands one may evaluate the cohomology explicitly in terms of known data or weight 0 cohomology. The summands whose cohomology can be computed correspond, in a sense to be made precise below, to graphs of the form



where Γ_1 is a connected component with one ω -decoration and no other external vertices. The genus g of the overall graph is the sum of the genera g_1 and g_2 of Γ_1 and Γ_2 , respectively. The cohomology of the summands with $g_2 = 0$ and $g_2 = 1$ are explicitly understood, and the complementary part Γ_1 contributes a tensor factor that is identified with $W_0 H_c^\bullet(\mathcal{M}_{g_1,1})$. In this way, one arrives at the following result.

THEOREM 1.7

For $g \geq 2$ and $n \geq 2$, one has an injection

$$H_c^{n-3}(\mathcal{M}_{0,n}) \otimes W_0 H_c^{\bullet-n}(\mathcal{M}_{g,1}) \oplus \mathbb{B}_n \otimes W_0 H_c^{\bullet-n-3}(\mathcal{M}_{g-1,1}) \rightarrow \mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n}),$$

where $\mathbb{B}_n$ is a vector space defined in (46) below, which has dimension at least $(n-2)!$ for $n \geq 3$.

2. Preliminaries

2.1. Notation and conventions

We work over the rational numbers $\mathbb{Q}$. All vector spaces are understood to be $\mathbb{Q}$ -vector spaces, and likewise all homology and cohomology groups are taken with $\mathbb{Q}$ -coefficients. If V is a finite-dimensional vector space with the action of a finite group G , then we identify the invariant subspace V^G with the coinvariant space V_G , by averaging over the group action.

For a graded vector space V and an integer k , let $V[k]$ be the graded vector space obtained by shifting all degrees by k ; that is, the degree j part of V is the degree $j - k$ part of $V[k]$.

When working with graded vector spaces, we use the standard Koszul sign convention. That is, for V and W graded vector spaces the isomorphism $V \otimes W \cong W \otimes V$ sends a tensor product $v \otimes w$ of homogeneous elements $v \in V$, $w \in W$ of degrees $|v|$, $|w|$ to the element $(-1)^{|v||w|} w \otimes v$. Furthermore, let $f \in \mathrm{Hom}(V, V')$

Table 1. A summary of vanishing and nonvanishing results for the rational cohomology groups $H^{4g-5-k}(\mathcal{M}_g)$. Black and dark gray boxes denote previously known and new nonvanishing groups, respectively. The white boxes denote groups that are known to vanish, and the light gray boxes denote those that are as yet unknown.

g, k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
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and $g \in \text{Hom}(W, W')$ be two linear maps between graded vector spaces, of homogeneous degree $|f|$ and $|g|$, respectively. Then we define the linear map $f \otimes g \in \text{Hom}(V \otimes V', W \otimes W')$ such that, for homogeneous $v \in V$ and $w \in W$,

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

The phrase “differential graded” is abbreviated dg. We follow cohomological grading conventions, so all differentials have degree $+1$ unless stated otherwise. The Koszul sign rule will be particularly important when we consider the tensor product of dg vector spaces (V, d_V) and (W, d_W) . The differential on $V \otimes W$ is then

$$d_{V \otimes W} := d_V \otimes \text{id}_W + \text{id}_V \otimes d_W.$$

When no confusion seems possible, we denote this by $d_V + d_W$, with the Koszul sign convention understood.

We will consider many instances of spaces that depend on a genus g and a number of marked points n , among them the moduli spaces $\overline{\mathcal{M}}_{g,n}$ and the dg vector spaces $X_{g,n}$. When $n = 0$, we follow the usual convention of omitting this from the notation; for example, we write $\overline{\mathcal{M}}_g := \overline{\mathcal{M}}_{g,0}$ and $X_g := X_{g,0}$.

2.2. Quasi-isomorphisms and acyclicity

A morphism of dg vector spaces is a quasi-isomorphism if it induces an isomorphism on cohomology, and a dg vector space (V, d) is acyclic if its cohomology vanishes. We will make repeated use of the following elementary sufficient criteria for morphisms of dg vector spaces to be quasi-isomorphisms, and for dg vector spaces to be acyclic.

LEMMA 2.1

Let (V, d) be a dg vector space with $V_1 \subset V$ a dg subspace. Suppose that there is a decomposition of graded vector spaces $V = V_0 \oplus V_1$. Let π_1 be the projection onto V_1 , and let $D = \pi_1 \circ d : V_0 \rightarrow V_1$. Then $\ker D \subset V$ and $V_0 \oplus \operatorname{im} D \subset V$ are dg subvector spaces. Moreover,

- (1) if D is surjective, then $\ker D \rightarrow V$ is a quasi-isomorphism, and
 - (2) if D is injective, then $V \rightarrow V/(V_0 \oplus \operatorname{im} D) = \operatorname{coker} D$ is a quasi-isomorphism.
- In particular, if D is an isomorphism, then (V, d) is acyclic.

Proof

The differential d on V is a sum of three pieces d_0 , D , and d_1 :

$$\begin{array}{ccc} d_0 & & d_1 \\ \downarrow & \xrightarrow{D} & \downarrow \\ V_0 & & V_1 \end{array} .$$

The fact that $d^2 = 0$ translates into the three equations:

$$d_0^2 = 0, \quad d_1^2 = 0, \quad d_1 D + D d_0 = 0.$$

It follows that $\ker D$ and $V_0 \oplus \operatorname{im} D$ are dg subspaces of V .

Next, we show that if D is an isomorphism, then (V, d) is acyclic. Let $x \in V$ be a cocycle, and write $x = x_0 + x_1$, with $x_i \in V_i$. Since x is a cocycle, we have $d_0 x_0 = 0$ and $D x_0 + d_1 x_1 = 0$. Thus

$$D x_0 = -d_1 x_1 = -d_1 D D^{-1} x_1 = D d_0 D^{-1} x_1.$$

Applying D^{-1} to both sides shows that $x_0 = d_0 D^{-1} x_1$, and hence $x = d D^{-1} x_1$ is a coboundary, as required.

Statements (1) and (2) follow by applying this criterion for acyclicity to $V/\ker D$ and $V_0 \oplus \operatorname{im} D$, respectively. $\square$

2.3. Deligne's weight spectral sequence

The natural starting point for understanding the weight-graded compactly supported cohomology of a smooth variety or Deligne–Mumford stack with a given normal

crossings compactification is the Poincaré dual of Deligne's weight spectral sequence (see [12, Section 3.2]; cf. [31, Example 3.5]). We briefly recall this construction in the special case of the Deligne–Mumford compactification of $\mathcal{M}_{g,n}$ by the space of stable n -marked curves $\overline{\mathcal{M}}_{g,n}$. It is naturally stratified by the topological type, which is encoded combinatorially in the dual graph of the curve. Each stratum thus corresponds to a stable n -marked graph Γ of genus g , that is, a graph with n legs labeled $1, \dots, n$ in which each vertex v is labeled by a nonnegative integer g_v such that $2g_v + n_v \geq 3$, where n_v denotes the valence of the vertex v , and such that $g = h^1(\Gamma) + \sum_v g_v$.

For each such stable n -marked graph Γ of genus g , the locus of curves with dual graph Γ is the image of the natural gluing map

$$\xi_\Gamma: \mathcal{M}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,n},$$

where $\mathcal{M}_\Gamma := \prod_{v \in V(\Gamma)} \mathcal{M}_{g_v, n_v}$. This gluing extends naturally to $\overline{\mathcal{M}}_\Gamma := \prod_v \overline{\mathcal{M}}_{g_v, n_v}$. Let $D_\Gamma := \xi_\Gamma(\prod_v \overline{\mathcal{M}}_{g_v, n_v})$ be the image of this extension; it is the closure of the locus of curves with dual graph Γ . The normalization of D_Γ is the smooth and proper Deligne–Mumford stack

$$\widetilde{D}_\Gamma = \overline{\mathcal{M}}_\Gamma / \text{Aut}(\Gamma).$$

Note that the codimension of D_Γ in $\overline{\mathcal{M}}_{g,n}$ is the number of edges in Γ . Let

$$\widetilde{D}^j := \bigsqcup_{|E(\Gamma)|=j} \widetilde{D}_\Gamma.$$

The image of a small neighborhood of a point in $\widetilde{D}^j$ is contained in precisely j analytic branches of the boundary divisor $\partial \mathcal{M}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. The monodromy action on these branches defines a local system of rank j on $\widetilde{D}^j$ whose determinant is denoted ϵ^j . The E_1 -page of the weight spectral sequence for $H_c^\bullet(\mathcal{M}_{g,n})$ is expressed naturally in terms of cohomology with coefficients in this local system:

$$E_1^{j,k} \cong H^k(\widetilde{D}^j, \epsilon^j).$$

Note that the branches of D that contain the image of a small neighborhood of a point in $\widetilde{D}_\Gamma$ are naturally identified with the edges of Γ , and ϵ^j is trivialized by pullback to $\overline{\mathcal{M}}_\Gamma$. In this way, the local system ϵ^j is identified with the determinant of the permutation action of $\text{Aut}(\Gamma)$ on $E(\Gamma)$. Thus, we have

$$E_1^{j,k} \cong \bigoplus_{|E(\Gamma)|=j} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\text{Aut}(\Gamma)}.$$

The spectral sequence degenerates at E_2 . Hence $\text{gr}_k H_c^{j+k}(\mathcal{M}_{g,n})$ is canonically identified with H^j of the complex

$$\begin{aligned}
\cdots \rightarrow \bigoplus_{|\Gamma|=j-1} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\text{Aut}(\Gamma)} &\xrightarrow{d_{j-1}} \bigoplus_{|\Gamma|=j} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\text{Aut}(\Gamma)} \\
&\xrightarrow{d_j} \bigoplus_{|E(\Gamma)|=j+1} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\text{Aut}(\Gamma)} \rightarrow \cdots.
\end{aligned} \tag{4}$$

All of the data in this spectral sequence is neatly encoded in the language of modular operads (see [18]; cf. [31, Example 3.9]). More precisely, the cohomology groups $H^\bullet(\overline{\mathcal{M}}_{g,n})$ naturally form a modular cooperad, whose Feynman transform evaluated at (g, n) is the direct sum, over all weights k , of the complex (4). Using the Künneth decomposition, one may encode generators as linear combinations of graphs Γ whose vertices v are labeled with elements of $H^{k_v}(\overline{\mathcal{M}}_{g_v, n_v})$ such that $\sum_v k_v = k$. The result is now known as the *Getzler–Kapranov graph complex* and denoted $\mathbf{GK}_{g,n}^k$ (see, e.g., [1, Section 6.1] and [21]).

2.4. Modular cooperads and Feynman transform

We briefly recall the notion of modular cooperads and their Feynman transforms, and refer the reader to [18] for details. A stable $\mathbb{S}$ -module is a collection of dg vector spaces $\mathcal{P} = \{\mathcal{P}(g, n)\}$, for $g, n \geq 0$ and $2g + n \geq 3$, with each $\mathcal{P}(g, n)$ equipped with an action of $\mathbb{S}_n$.

Let $\mathcal{P}$ be a stable $\mathbb{S}$ -module. For a stable graph Γ , we may define the tensor product

$$\otimes_\Gamma \mathcal{P} := \otimes_{v \in V(\Gamma)} \mathcal{P}(g_v, n_v).$$

Note that $\text{Aut}(\Gamma)$ acts naturally on $\otimes_\Gamma \mathcal{P}$. A modular cooperad is a stable $\mathbb{S}$ -module together with a morphism

$$\mathcal{P}(g, n) \rightarrow \otimes_\Gamma \mathcal{P}$$

for every stable graph of genus g with n legs, satisfying suitable compatibility relations.

To a modular cooperad $\mathcal{P}$, one naturally associates the Feynman transform $\mathcal{F}\mathcal{P}$, which is a modular cooperad whose underlying stable $\mathbb{S}$ -module is

$$\mathcal{F}\mathcal{P}(g, n) \cong \bigoplus_{[\Gamma]} \left(\bigotimes_{\Gamma} \mathcal{P} \otimes \mathbb{Q}[-1]^{\otimes |E(\Gamma)|} \right)_{\text{Aut}(\Gamma)}.$$

Here, the sum is over isomorphism classes of stable graphs of genus g with n legs, $|E(\Gamma)|$ is the set of edges of Γ , and the action of the automorphism group is diagonal on $\bigotimes_\Gamma$ and $\mathbb{Q}[-1]^{\otimes |E(\Gamma)|}$ by permutation of factors. Elements of $\mathcal{F}\mathcal{P}(g, n)$ can be understood as isomorphism classes of graphs whose vertices are decorated by suitable

elements of $\mathcal{P}$. In cases where $\mathcal{P}$ has no differential, the differential on $\mathcal{F}\mathcal{P}(g, n)$ has two pieces,

$$\delta = \delta_{\text{split}} + \delta_{\text{loop}}.$$

The piece δ_{split} splits vertices similar to (1), using the cooperadic composition of $\mathcal{P}$ on decorations. Similarly, δ_{loop} creates a loop at a vertex

$$\delta_{\text{loop}} : \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \text{---} \end{array} \mapsto \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \text{---} \quad \circ \end{array}$$

using the modular cooperadic map $\mathcal{P}(g, n) \rightarrow \mathcal{P}(g-1, n+2)$ on the decoration.

2.5. The Getzler–Kapranov graph complex

The cohomology groups $H^\bullet(\overline{\mathcal{M}}_{g,n})$ of the Deligne–Mumford compactified moduli spaces $\overline{\mathcal{M}}_{g,n}$ assemble into a modular cooperad that we shall denote $H(\overline{\mathcal{M}})$. We consider the Feynman transform $\mathcal{F}H(\overline{\mathcal{M}})$ of this modular cooperad.

An element of $\mathcal{F}H(\overline{\mathcal{M}})(g, n)$ is a linear combination of stable n -marked graphs of genus g whose vertices v are decorated by elements of $H(\overline{\mathcal{M}}_{g_v, n_v})$. As recalled in the preceding section, the differential has two parts:

$$\delta = \delta_{\text{split}} + \delta_{\text{loop}}.$$

The dg vector spaces $\mathcal{F}H(\overline{\mathcal{M}})(g, n)$ inherit the weight grading from $H(\overline{\mathcal{M}})$, where the weight of a graph is the sum of the degrees of its vertex decorations.

DEFINITION 2.2

The weight k Getzler–Kapranov graph complex is

$$\mathbf{GK}_{g,n}^k := \text{gr}_k \mathcal{F}H(\overline{\mathcal{M}})(g, n).$$

Note that $\mathbf{GK}_{g,n}^k$ is also naturally identified with (4), the weight k row on the E_1 -page of the weight spectral sequence for $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$, and hence there is a canonical isomorphism

$$H^\bullet \mathbf{GK}_{g,n}^k \cong \text{gr}_k H_c^\bullet(\mathcal{M}_{g,n}).$$

2.6. A nonvanishing result in weight 0

Here we state and prove Proposition 2.4, a nonvanishing and exponential growth result for $W_0 H_c^{2g+3}(\mathcal{M}_g)$. This proposition, combined with Theorem 1.2 and the results on $W_0 H_c^\bullet(\mathcal{M}_g)$ in [8], gives the nonvanishing statements in Corollary 1.4.

Let $\mathbf{fGC}_{2,\text{conn}}$ be the full commutative graph complex studied in [39]. The elements of $\mathbf{fGC}_{2,\text{conn}}$ are (possibly infinite) formal $\mathbb{Q}$ -linear combinations of isomorphism classes of connected graphs with vertices of any valence. It is a dg Lie algebra with a combinatorially defined bracket, and the differential δ is the bracket with a single edge. The subspace $\mathbf{GC}_2$ spanned by stable graphs is a direct summand, and the restriction of δ is exactly δ_{split} . The cohomology of $\mathbf{fGC}_{2,\text{conn}}$ is

$$H(\mathbf{fGC}_{2,\text{conn}}) = H(\mathbf{GC}_2) \oplus \bigoplus_{k \geq 1} \mathbb{Q}[L_{4k+1}], \quad (5)$$

where L_n denotes a loop of n edges, in which every vertex has valence 2 (see [39, Proposition 3.4]).

Recall that the cohomological degree of a graph in $\mathbf{fGC}_{2,\text{conn}}$ of genus g with e edges is $e - 2g$. There is an extra differential ∇ on $\mathbf{fGC}_{2,\text{conn}}$, given by bracket with a loop edge, studied in [23].

LEMMA 2.3

Every class in $H^0(\mathbf{GC}_2)$ is represented by a δ -cocycle $F \in \mathbf{fGC}_{2,\text{conn}}$ such that $\nabla F = 0$.

Proof

Let $F \in \mathbf{fGC}_{2,\text{conn}}$ be a δ -cocycle representing a nontrivial class in $H(\mathbf{GC}_2)$. We must show that there is some $F' \in \mathbf{fGC}_{2,\text{conn}}$ such that $F' - F$ is δ -exact and $\nabla F' = 0$. First, note that $\delta \nabla F = -\nabla \delta F = 0$. Since $H(\mathbf{fGC}_{2,\text{conn}})$ vanishes in negative degrees (see [39, Theorem 1.1]), we see that ∇F is δ -exact; that is, there is $F_1 \in \mathbf{fGC}_{2,\text{conn}}$ such that $\delta F_1 = \nabla F$. Moreover, $\delta \nabla F_1 = -\nabla \delta F_1 = -\nabla^2 F = 0$. Continuing in this way, the number of edges remains fixed and the genus increases by 1 at each step, so eventually we arrive at a sequence $(F = F_0, \dots, F_n)$

$$\delta F_n = \nabla F_{n-1} \quad \text{and} \quad \nabla(F_n) = 0. \quad (6)$$

Then, since $\mathbf{fGC}_{2,\text{conn}}$ is ∇ -acyclic in genus greater than 1 (see [23, Corollary 3]), there is some G_n such that $\nabla G_n = F_n$. If $n > 1$, then we can replace F_n with $F_{n-1} + \delta G_n$ to get a shorter sequence that satisfies (6). Repeating the process, we arrive at such a sequence with $n = 1$. Choose G such that $\nabla G = F_1$, and set $F' = F + \delta G$. $\square$

Recall from [8] that $W_0 H_c^\bullet(\mathcal{M}_g)$ is identified with the genus g part of the cohomology of the complex of commutative stable graphs $\mathbf{GC}_2$ and the dimension of $W_0 H_c^{2g}(\mathcal{M}_g)$ grows at least exponentially with g .

PROPOSITION 2.4

The weight 0 cohomology group $W_0 H_c^{2g+3}(\mathcal{M}_g)$ is nonzero for $g = 6$ and $g \geq 8$. Moreover, its dimension grows at least exponentially with g .

Proof

Let us denote the genus g part of GC_2 by $\mathrm{GC}_2^{(g)}$. The main result of [8] shows that $W_0 H_c^{2g+3}(\mathcal{M}_g) \cong H^k(\mathrm{GC}_2^{(g)})$. We recall that $H^0(\mathrm{GC}_2)$ is naturally identified with the Grothendieck–Teichmüller Lie algebra (see [39]). Using this identification, it follows from the main results of [5] that $H^0(\mathrm{GC}_2)$ contains a free Lie subalgebra generated by classes $\sigma_g \in H^0(\mathrm{GC}_2^{(g)})$ for odd $g \geq 3$ (see [6, (1.3)]).

We consider two linear maps induced by the Lie bracket:

$$[\sigma_3, \cdot]: H^0(\mathrm{GC}_2) \rightarrow H^0(\mathrm{GC}_2) \quad \text{and} \quad [L_5, \cdot]: H^0(\mathrm{GC}_2) \rightarrow H^3(\mathrm{GC}_2).$$

The natural target of $[L_5, \cdot]$ is $H^3(\mathfrak{fGC}_{2,\mathrm{conn}})$, but the image of any nontrivial class has genus greater than 1; by (5), it must be in the summand $H^3(\mathrm{GC}_2)$. We claim that the kernel of $[L_5, \cdot]$ is contained in the kernel of $[\sigma_3, \cdot]$. The proposition follows from the claim, since the free Lie algebra generated by $\sigma_5, \sigma_7, \dots$ has trivial intersection with the kernel of $[\sigma_3, \cdot]$ and hence also with the kernel of $[L_5, \cdot]$. It remains to prove the claim.

Let $F \in \mathfrak{fGC}_{2,\mathrm{conn}}$ be a cocycle of genus g representing a class in $H^0(\mathrm{GC}_2^{(g)})$. Suppose that $[L_5, F]$ is exact, and write it as $\delta(\lambda)$, with $\lambda \in \mathfrak{fGC}_{2,\mathrm{conn}}$ of genus $g+1$. We must show that $[\sigma_3, F]$ is exact. By Lemma 2.3, we may assume that $\nabla F = 0$.

Recall that there is a graph γ of genus 2 such that $(\delta + \nabla)(L_5 + \gamma) = \sigma_3$ (as in [23, Figure 2]). Let $\zeta := [\gamma, F] - \nabla\lambda$. Then $\zeta = [L_5 + \gamma, F] - (\delta + \nabla)\lambda$, and, using the assumption that $\nabla F = 0$, it follows that

$$(\delta + \nabla)\zeta = [\sigma_3, F]. \quad (7)$$

Note that ζ is of genus $g+2$ and $[\sigma_3, F]$ is of genus $g+3$, and δ leaves the genus invariant, while ∇ increases it by 1. Hence, comparing terms of like genus in (7), we find that separately $\delta\zeta = 0$ and $\nabla\zeta = [\sigma_3, F]$. Let D denote the vertex deletion operation introduced by Živković in [40, Section 3]. Then $\nabla = \delta D - D\delta$ (see [40, Proposition 3.3]) and hence

$$[\sigma_3, F] = \nabla\zeta = (\delta D - D\delta)\zeta = \delta(D\zeta). \quad (8)$$

This shows that $[\sigma_3, F]$ is exact, as required, and proves the proposition. We note that [40, Proposition 3.3] is stated in a version of the graph complex that also contains disconnected graphs. However, since the connected graphs span a direct summand with respect to δ , given that (8) holds in the larger complex, it must also hold for the connected part. $\square$

2.7. The support of $H^\bullet(X_{g,n})$

Here we deduce bounds on the support of $H^\bullet(X_{g,n})$ from Theorem 1.1, using mixed Hodge theory and known vanishing statements for $H_c^\bullet(\mathcal{M}_{g,n})$. We also explain how Corollary 1.4 follows from Theorem 1.2.

COROLLARY 2.5

The graph cohomology $H^\bullet(X_{g,n})$ is supported in degrees between $\max\{2g, 2g - 2 + n\}$ and $3g - 2 + n$.

Proof

By Theorem 1.1, we must show that $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ is supported in the indicated range of degrees. By Poincaré duality, we have

$$\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n}) \cong \mathrm{gr}_{6g-8+2n} H^{6g-6+2n-\bullet}(\mathcal{M}_{g,n})^\vee. \quad (9)$$

To see the upper bound, recall (see [12]) that $\mathrm{gr}_* H^k$ is supported in weights $* \leq 2k$. Hence the right-hand side of (9) vanishes when $\bullet > 3g - 2 + n$. For $n \geq 2$, the lower bound also follows from (9), since the virtual cohomological dimension (vcd) of $\mathcal{M}_{g,n}$ is $4g - 4 + n$. The lower bounds for $n = 0$ and 1 are similar; one uses that the vcds of $\mathcal{M}_{g,1}$ and $\mathcal{M}_g$ are $4g - 3$ and $4g - 5$, respectively, and that $H^{4g-3}(\mathcal{M}_{g,1})$ and $H^{4g-5}(\mathcal{M}_g)$ both vanish (see [10], [30]). $\square$

Finally, we explain how Corollary 1.4 follows from Theorem 1.2 and Proposition 2.4.

Proof of Corollary 1.4

We start by recalling that

- $W_0 H_c^{2g}(\mathcal{M}_g)$ is nonvanishing for $g = 3$, $g = 5$, and $g \geq 7$, by [8], and
- $W_0 H_c^{2g+3}(\mathcal{M}_g)$ is nonvanishing for $g = 6$ and $g \geq 8$, by Proposition 2.4.

In both cases the dimension grows exponentially with g . Furthermore, we know that $W_0 H_c^{27}(\mathcal{M}_{10}) \neq 0$; the corresponding graph cohomology computation is due to Bar-Natan and McKay [3]. For this proof only, let us set

$$\mathbb{W}_0 := \bigoplus_g W_0 H_c^{2g}(\mathcal{M}_g), \quad \mathbb{W}_3 := \bigoplus_g W_0 H_c^{2g+3}(\mathcal{M}_g),$$

$$\mathbb{W}_7 := W_0 H_c^{27} \mathcal{M}_{10}.$$

Then $\bigwedge^2 \mathbb{W}_0$ contributes to $\mathbb{V}_{g,2g}$ for $g = 8$ and $g \geq 10$, and the dimension of this subspace grows exponentially with g . Considering the third summand in Theorem 1.2, this shows that $\mathrm{gr}_2 H_c^{2g+2}(\mathcal{M}_g)$ is nonzero for $g = 9$ and $g \geq 11$, and grows exponentially with g . The corresponding statement for $H^{4g-8}(\mathcal{M}_g)$ follows

by Poincaré duality since $H^{4g-8}(\mathcal{M}_g) \cong H_c^{2g+2}(\mathcal{M}_g)^\vee$. This proves Corollary 1.4 for $H^{4g-8}(\mathcal{M}_g)$.

Corollary 1.4 for $H^{4g-9}(\mathcal{M}_g)$ is an immediate consequence of Proposition 2.4 since $H^{4g-9}(\mathcal{M}_g) \cong H_c^{2g+3}(\mathcal{M}_g)^\vee$.

The remaining cases of Corollary 1.4 are similar to the case of $H^{4g-8}(\mathcal{M}_g)$. The subspace $\mathbb{W}_\mu \wedge \mathbb{W}_3$ contributes to $\mathbb{V}_{g,2g+3}$ for $g = 9$ and $g \geq 11$, and the dimension of this contribution grows exponentially with g . Considering the third summand in Theorem 1.2 and applying Poincaré duality shows that $H^{4g-11}(\mathcal{M}_g)$ is nonzero for $g = 10$ and $g \geq 12$, and its dimension grows exponentially with g . The corresponding statements for $H^{4g-12}(\mathcal{M}_g)$ are proved similarly, using the first summand in Theorem 1.2.

The next three cases, for $H^{4g-14}(\mathcal{M}_g)$, $H^{4g-15}(\mathcal{M}_g)$, and $H^{4g-16}(\mathcal{M}_g)$, use the contributions of $\bigwedge^2 \mathbb{W}_3$ and $\mathbb{W}_0 \wedge \mathbb{W}_7$ to $\mathbb{V}_{g,2g+6}$ and $\mathbb{V}_{g,2g+7}$, respectively. The final two cases, for $H^{4g-18}(\mathcal{M}_g)$ and $H^{4g-19}(\mathcal{M}_g)$, are deduced similarly from the contribution of $\mathbb{W}_3 \wedge \mathbb{W}_7$ to $\mathbb{V}_{g,2g+10}$. $\square$

2.8. A vanishing result in weight 0

Our main motivation is to use what is known in weight 0 to prove new results about the weight 2 cohomology $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$. However, it is worth noting that the information flows meaningfully in both directions, via Theorem 1.2. For instance, applying basic results from mixed Hodge theory to the weight 2 cohomology, we deduce the following vanishing result in weight 0.

PROPOSITION 2.6

The cohomology $W_0 H_c^{3g-1}(\mathcal{M}_{g,2})_{\mathrm{as}}$ vanishes for all $g \geq 1$. Also,

$$\dim \bigoplus_g W_0 H_c^{3g-2}(\mathcal{M}_{g,1}) \leq 1.$$

In other words, the graph complex $G^{(g,2)}$ computing $W_0 H_c(\mathcal{M}_{g,2})$ (see Section 4.1) has no antisymmetric cohomology in the top degree, corresponding to trivalent graphs with two marked points. Furthermore the top degree cohomology of $\bigoplus_g G^{(g,1)}$, corresponding to trivalent graphs, is at most 1-dimensional.

Proof

The antisymmetric part $W_0 H_c^{3g-1}(\mathcal{M}_{g,2})_{\mathrm{as}}$ injects into $\mathrm{gr}_2 H_c^{3g+2}(\mathcal{M}_{g+1})$, via the second summand in Theorem 1.2, and the Poincaré dual of $\mathrm{gr}_2 H_c^{3g+2}(\mathcal{M}_{g+1})$ is $\mathrm{gr}_{6g-2} H^{3g-2}(\mathcal{M}_{g+1})$, which vanishes because $\mathrm{gr}_* H^k$ is supported in weights $* \leq 2k$ (see [12]). Similarly, the first summand in Theorem 1.2 gives an injection from

$\bigwedge^2 \bigoplus_g W_0 H_c^{3g-2}(\mathcal{M}_{g,1})$ into $\bigoplus_{g'} \text{gr}_2 H_c^{3g'-1}(\mathcal{M}_{g'})$, which vanishes for the same weight reasons. $\square$

It is not known whether $W_0 H^{3g-2}(\mathcal{M}_{g,1})$ vanishes, or equivalently, whether every trivalent graph with one marked point is a coboundary in $G^{(g,1)}$, for all g .

3. The weight 2 Getzler–Kapranov graph complex

3.1. Generators and relations

The decorated graphs that generate $\text{GK}_{g,n}^2$ have a particularly simple description. Each generator has an underlying stable graph of genus g and comes with a total ordering of the structural edges, that is, the edges that are not incident to the external vertices labeled $1, \dots, n$, subject to the relation that reordering the structural edges is multiplication by the sign of the permutation. In addition, each vertex v is decorated by an element of $H^{k_v}(\overline{\mathcal{M}}_{g_v, n_v})$, where g_v and n_v are the genus and valence of v , and $\sum_v k_v = 2$.

Since $H^0(\overline{\mathcal{M}}) = \mathbb{Q}$ and $H^1(\overline{\mathcal{M}}) = 0$ for each moduli space $\overline{\mathcal{M}}$ attached to a vertex, any generator for $\text{GK}_{g,n}^2$ has a unique vertex v with decoration in $H^2(\overline{\mathcal{M}}_{g_v, n_v})$, which we call the *special vertex*. After rescaling, we may assume that all other vertices are decorated with the unit $1 \in H^0(\overline{\mathcal{M}})$; for simplicity, we omit these trivial decorations. We can thus give a finite generating set for $\text{GK}_{g,n}^2$ by specifying a finite generating set for $H^2(\overline{\mathcal{M}}_{g', n'})$ for all g' and n' .

Recall that $H^2(\overline{\mathcal{M}}_{g,n})$ is generated by the *tautological classes*: κ , $\psi_1, \dots, \psi_n$, δ_{irr} , and $\delta_{a,A} = \delta_{g-a, A^c}$, for subsets $A \subset \{1, \dots, n\}$ and $0 \leq a \leq g$ such that $2a + |S| \geq 2$ and $2(g-a) + |A^c| \geq 2$. Thus $\text{GK}_{g,n}^2$ is generated by graphs in which the special vertex v is decorated by one of these tautological classes in $H^2(\overline{\mathcal{M}}_{g_v, n_v})$. Here, the set $\{1, \dots, n_v\}$ is implicitly identified with the set of half-edges incident to v .

For $g \geq 3$, the tautological classes form a basis $H^2(\overline{\mathcal{M}}_{g,n})$; there are no further relations. However, for $g \leq 2$, the tautological classes satisfy relations as follows (see [2, Theorem 2.2]).

For $g = 2$, there is one relation:

$$5\kappa = 5 \sum_{i=1}^n \psi_i + \delta_{\text{irr}} - 5 \sum_A \delta_{0,A} + 7 \sum_A \delta_{1,A}.$$

For $g = 1$, there are $n + 1$ relations:

$$\kappa = \sum_{i=1}^n \psi_i - \sum_{|A| \geq 2} \delta_{0,A} \quad \text{and} \quad 12\psi_i = \delta_{\text{irr}} + 12 \sum_{i \in A} \delta_{0,A}.$$

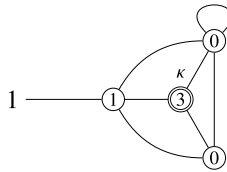
For $g = 0$, the relations are generated by

$$\kappa = \sum_{A \not\ni x, y} (|A| - 1) \delta_{0, A}, \quad \psi_i = \sum_{i \in A \not\ni x, y} \delta_{0, A}, \quad \text{and} \quad \delta_{\text{irr}} = 0.$$

Here i and $x \neq y$ run over elements of $\{1, \dots, n\}$. See also Proposition 3.5 for another presentation of $H^2(\overline{\mathcal{M}}_{0, n})$.

Example 3.1

The following is a generator in degree 9 for $\mathbf{GK}_{8,1}^2$. The inscribed numbers in vertices v represent the genera g_v :



The special vertex is drawn with a double circle, and decorated by $\kappa \in H^2(\overline{\mathcal{M}}_{3,3})$.

3.2. Graphical depiction of boundary and ψ -class decorations

We use suggestive graphical depictions of the decorations on the special vertex, indicating a decoration ψ_i with an arrow on the corresponding half-edge:

$$\begin{array}{c} \psi_i \\ \nearrow \\ \text{---} \textcircled{h} \text{---} i \\ \searrow \end{array} =: \begin{array}{c} \text{---} \textcircled{h} \text{---} \\ \nearrow \quad \searrow \end{array}$$

To indicate a decoration $\delta_{a,A}$, we replace the special vertex with two vertices connected by a marked edge:

$$\begin{array}{c} \delta_{a,A} \\ \nearrow \\ \text{---} \textcircled{h} \text{---} \\ \searrow \end{array} =: \begin{array}{c} \text{---} \textcircled{a} \text{---} \textcircled{b} \text{---} \\ \nearrow \quad \searrow \end{array} \quad (10)$$

Here $b = h - a$, and A is the subset of the half-edges at the special vertex that are connected to the vertex labeled a .

3.3. Graphical depiction of the differential

The differential $\delta = \delta_{\text{split}} + \delta_{\text{loop}}$ on $\mathbf{GK}_{g,n}^k$ acts on each generator by taking a sum over the vertices. Roughly speaking, it amounts to splitting vertices and attaching loops at vertices in all allowable ways and then pulling back cohomological decorations along the corresponding clutching and gluing maps between moduli spaces. When

the decoration on a given vertex is trivial, the differential acts just like the differential on the standard commutative graph complex.

Thus, we may depict the contribution “ Γ split v ” from a nonspecial vertex v with $g_v = h$ as

$$\text{graph with vertex } h \text{ and four half-edges} \mapsto \sum_{a+b=h} \sum \text{graph with vertices } a \text{ and } b \text{ and four half-edges}. \quad (11)$$

On the right-hand side, the second sum is taken over all ways of distributing the half-edges incident to the special vertex over the two new vertices of genus a and b such that the resulting graph is stable. Similarly, we may depict the contribution to δ_{loop} from a nonspecial vertex v as

$$\text{graph with vertex } h \text{ and four half-edges} \mapsto \text{graph with vertex } h-1 \text{ and four half-edges, plus a loop}, \quad (12)$$

with the right-hand side being understood as zero if $h = 0$. In both cases, the edge orderings on the right-hand side are fixed so that the newly created edge is first and the relative order of the other edges is unchanged.

Next, we turn to the special vertex. In what follows, it will be convenient to consider curves with marked points labeled by a finite set S that is not necessarily identified with $\{1, \dots, n\}$. We follow the usual notational convention (e.g., from [2]), writing $\mathcal{M}_{g,S}$ for the moduli space of smooth curves of genus g with $|S|$ marked points labeled by a bijection to S , and $\overline{\mathcal{M}}_{g,S}$ for its compactification by stable curves.

To describe the differential, one needs to understand the pullback of the classes in $H^2(\overline{\mathcal{M}}_{h,k})$ under the maps

$$\xi: \overline{\mathcal{M}}_{h-1, S \cup \{t, t'\}} \rightarrow \overline{\mathcal{M}}_{h,S} \quad \text{and} \quad \theta: \overline{\mathcal{M}}_{a, A \cup \{q\}} \rightarrow \overline{\mathcal{M}}_{h,S},$$

where the latter is obtained by attaching a fixed curve of genus $h - a$ with marked points labeled by $A^c \cup \{r\}$, where $A^c := S \setminus A$. (The map θ depends on the choice of this curve, but only up to homotopy, so the pullback map on cohomology is well defined.) The relevant formulas are well known (see [2, Lemmas 3.2 and 3.3]).

First, for the pullback θ^* , which is related to δ_{split} at the special vertex, we have

$$\theta^*(\kappa) = \kappa, \quad \theta^*(\delta_{\text{irr}}) = \delta_{\text{irr}}, \quad \text{and} \quad \theta^*(\psi_i) = \begin{cases} \psi_i & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting formulas for δ_{split} when applied to graphs in which the special vertex is decorated with κ , δ_{irr} , or ψ_i can then be depicted graphically in a way that naturally generalizes (11). For example, when the special vertex is decorated with ψ_i , we have

$$\delta_{split} : \text{diagram} \mapsto \sum_{a+b=h} \sum \text{diagram} \quad (13)$$

Here, the second sum is again taken over all ways of distributing the half-edges incident to the special vertex over the two new vertices of genus a and b such that the resulting graph is stable.

Next, we discuss the formulas for $\theta^*(\delta_{b,B})$. In the special case where $A = S$, we have

$$\theta^*(\delta_{b,B}) = \begin{cases} \delta_{2b-h, S \cup \{q\}} - \psi_q & \text{if } (b, B) = (a, S) \text{ or } (b, B) = (h-a, \emptyset), \\ \delta_{b,B} + \delta_{b+a-h, B \cup \{q\}} & \text{otherwise.} \end{cases}$$

If $A \neq S$, then

$$\theta^*(\delta_{b,B}) = \begin{cases} -\psi_q & \text{if } (b, B) = (a, A) \text{ or } (b, B) = (h-a, A^c), \\ \delta_{b,B} & \text{if } B \subset A \text{ and } (b, B) \neq (a, A), \\ \delta_{b+a-h, (B \setminus A^c) \cup \{q\}} & \text{if } B \supset A^c \text{ and } (b, B) \neq (h-a, A^c), \\ 0 & \text{otherwise.} \end{cases}$$

Again, the resulting action of δ_{split} on a special vertex with decoration $\delta_{b,B}$ has a convenient graphic depiction:

$$\delta_{split} : \text{diagram} \mapsto \sum_{b'+b''=b} \sum \text{diagram} + \sum_{a'+a''=a} \sum \text{diagram} - \text{diagram} - \text{diagram} \quad (14)$$

Next, consider pullback ξ^* , which is related to δ_{loop} at the special vertex. We have

$$\begin{aligned} \xi^* \kappa &= \kappa, & \xi^* \psi_j &= \psi_j, & \text{and} \\ \xi^* \delta_{a,A} &= \begin{cases} \delta_{a,A} & \text{if } h = 2a, A = S = \emptyset, \\ \delta_{a,A} + \delta_{a-1, A \cup \{t, t'\}} & \text{otherwise.} \end{cases} \end{aligned}$$

These algebraic rules correspond to the following pictorial definition of δ_{loop} :

$$\begin{aligned} \text{diagram} &\mapsto \text{diagram} & \text{diagram} &\mapsto \text{diagram} \\ \text{diagram} &\mapsto \text{diagram} + \text{diagram} \end{aligned}$$

3.5. The quotient $\overline{\mathbf{GK}}$

Let $I_{g,n} \subset \mathbf{GK}_{g,n}$ be the graded subspace spanned by graphs that have at least one vertex of positive genus. Then define the quotient

$$\overline{\mathbf{GK}}_{g,n} = \mathbf{GK}_{g,n} / (I_{g,n} + \delta I_{g,n}).$$

In other words, $\overline{\mathbf{GK}}_{g,n}$ is generated by graphs, all of whose vertices are of genus 0. Among those graphs we impose relations of the form $\delta_{\text{loop}} \Gamma$, for Γ running over graphs with exactly one vertex of genus 1. The quotient $\overline{\mathbf{GK}}_{g,n}$ inherits the weight grading from $\mathbf{GK}_{g,n}$, where the weight is the sum of the cohomological degrees of the vertex decorations. We denote the summand of weight k by

$$\overline{\mathbf{GK}}_{g,n}^k \subset \overline{\mathbf{GK}}_{g,n}.$$

In particular, the weight 0 piece $\overline{\mathbf{GK}}_{g,n}^0$ is generated by graphs without loop edges, all of whose vertices have genus 0. After rescaling, we may assume that all vertex decorations are trivial, that is, each decoration is the identity $1 \in H^0(\overline{\mathcal{M}}_{0,k})$. One main result of [8] and [9] is that the quotient $\mathbf{GK}_{g,n}^0 \rightarrow \overline{\mathbf{GK}}_{g,n}^0$ is a quasi-isomorphism for all g, n .

In this paper, we prove an analogous result in weight 2. The weight 2 piece $\overline{\mathbf{GK}}_{g,n}^2$ is generated by graphs with all vertices of genus 0, and one special vertex decorated by a tautological generator for $H^2(\overline{\mathcal{M}}_{0,k})$. We quotient by graphs with a loop edge at a nonspecial vertex. Furthermore, if there is a loop edge at the special vertex, then we impose the additional relations $\delta_{0,A} = 0$ for sets of half-edges A that contain both half-edges of the loop. (Since $\delta_{0,A} = \delta_{0,A^c}$ in $H^2(\overline{\mathcal{M}}_{0,k})$, this also kills graphs labeled with $\delta_{0,A}$, where A contains neither half-edge of the loop.)

THEOREM 3.3

The quotient map $\mathbf{GK}_{g,n}^2 \rightarrow \overline{\mathbf{GK}}_{g,n}^2$ is a quasi-isomorphism for all g, n except $g = n = 1$.

For $g = n = 1$, the right-hand complex is acyclic and the left-hand complex has 1-dimensional cohomology, spanned by the class of a single-vertex graph with decoration δ_{irr} :

$$\begin{array}{c} \delta_{\text{irr}} \\ \textcircled{1} \text{---} 1 \end{array}$$

3.6. Resolutions of $H^2(\overline{\mathcal{M}}_{0,n})$

In the proof of Theorem 3.3, we use the following presentation for $H^2(\overline{\mathcal{M}}_{0,n})$. We believe this is well known to experts. However, lacking a suitable reference, we include a short proof.

LEMMA 3.4

The cohomology group $H^2(\overline{\mathcal{M}}_{0,n})$ is generated by boundary classes $\delta_{0,A}$ for subsets $A \subset \{1, \dots, n\}$ with $2 \leq |A| \leq n-2$, and the ψ -classes $\psi_1, \dots, \psi_n$ modulo the relations

- (1) $\delta_{0,A} = \delta_{0,A^c}$,
- (2) $\psi_i + \psi_j = \sum_{i \in A, j \in A^c} \delta_{0,A}$.

Proof

These classes generate $H^2(\overline{\mathcal{M}}_{0,n})$ and $\dim H^2(\overline{\mathcal{M}}_{0,n}) = 2^{n-1} - \binom{n}{2} - 1$ (see [22]). The relations $\delta_{0,A} = \delta_{0,A^c}$ are standard, and the quotient by these relations has dimension $2^{n-1} - 1$. The remaining $\binom{n}{2}$ relations are independent, so it suffices to show that they hold in $H^2(\overline{\mathcal{M}}_{0,n})$. For $n = 3$, this is trivial. We proceed by induction on n . Assume that (2) holds in $H^2(\overline{\mathcal{M}}_{0,n})$. Let $\pi: \overline{\mathcal{M}}_{0,n+1} \rightarrow \overline{\mathcal{M}}_{0,n}$ be the universal curve. Then apply the pullback formulas $\pi^* \psi_i = \psi_i - \delta_{0,\{i,n+1\}}$ and $\pi^* \delta_{0,A} = \delta_{0,A} + \delta_{0,A \cup \{n+1\}}$ (see [2, Lemma 3.1]) to deduce that (2) holds in $H^2(\overline{\mathcal{M}}_{0,n+1})$. $\square$

The following proposition is an immediate consequence of Lemma 3.4.

PROPOSITION 3.5

There is an $\mathbb{S}_n$ -equivariant short exact sequence

$$0 \rightarrow \bigoplus_{1 \leq i < j \leq n} \mathbb{Q} E_{ij} \rightarrow \left(\bigoplus_{1 \leq i \leq n} \mathbb{Q} \psi_i \right) \oplus \bigoplus_{1 \in A} \mathbb{Q} \delta_{0,A} \rightarrow H^2(\overline{\mathcal{M}}_{0,n}) \rightarrow 0$$

in which the first arrow is given by $E_{ij} \mapsto \psi_i + \psi_j - \sum_{i \in A, j \in A^c} \delta_{0,A}$. Note that here we identify $\delta_{0,A} = \delta_{0,A^c}$.

The $\mathbb{S}_n$ -action is given by permuting all labels and setting $E_{ij} = E_{ji}$ for $i > j$ and $\delta_{0,A} = \delta_{0,A^c}$ when $1 \notin A$.

4. Filtration by the number of vertices

The main idea of the proof of Theorem 3.3 is to use the filtration on $\mathbf{GK}_{g,n}$ by the number of vertices and look at the associated spectral sequence. The associated graded of this filtration $\text{gr } \mathbf{GK}_{g,n}$ can be identified with $(\mathbf{GK}_{g,n}, \delta_{\text{loop}})$; that is, we kill the part of the differential splitting vertices. As discussed in Section 4.3, this complex splits into subcomplexes that are direct sums of tensor products of complexes associated to single vertices. We begin by studying these single vertex complexes

$$V_{g,n} = \bigoplus_h (H^\bullet(\overline{\mathcal{M}}_{g-h,n+2h}) \otimes \mathbb{Q}[-1]^{\otimes h})_{\mathbb{S}_2 \wr \mathbb{S}_h}, \quad (15)$$

where the wreath product $\mathbb{S}_2 \wr \mathbb{S}_h$ acts by permuting the last $2h$ marked points and the factors $\mathbb{Q}[-1]^{\otimes h}$, introducing signs via the Koszul sign rule (see Section 2.1). The weight grading on $V_{g,n}$ is given by the cohomological degree in $H^\bullet(\overline{\mathcal{M}}_{g-h,n+2h})$, and the complex decomposes into a direct sum of weight-graded pieces. We denote the piece of weight k by $V_{g,n}^k$; in this paper, we focus on the cases $k = 0$ and $k = 2$.

4.1. The single vertex complex in weight 0

In weight 0, we have

$$H(V_{g,n}^0) = \begin{cases} \mathbb{Q} & \text{for } g = 0, n \geq 3, \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

To see this, note that

$$(H^0(\overline{\mathcal{M}}_{g-h,n+2h}) \otimes \mathbb{Q}[-1]^{\otimes h})_{\mathbb{S}_2 \wr \mathbb{S}_h} = \begin{cases} \mathbb{Q} & \text{for } h \in \{0, 1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential on $V_{g,n}^0$ is nontrivial when g is positive.

The complex $\overline{\mathbf{GK}}_{g,n}^0$ is generated by connected simple graphs of genus g with n marked legs. Following [9], we denote this

$$G^{(g,n)} := \overline{\mathbf{GK}}_{g,n}^0.$$

One can deduce from (16) that $\mathbf{GK}_{g,n}^0 \rightarrow \overline{\mathbf{GK}}_{g,n}^0$ is a quasi-isomorphism. However, we do not give the argument now. Instead, we proceed directly to the weight 2 case and merely remark that the weight 0 case can be handled by a simplified version of the argument in Section 4.3, filtering by the number of vertices and using (16) in place of Proposition 4.2. Thus, the methods presented here give an alternative approach to the following result.

THEOREM 4.1 ([8], [9])

The graph cohomology $H(G^{(g,n)}) = H(\overline{\mathbf{GK}}_{g,n}^0)$ is identified with the weight 0 part $W_0 H_c^\bullet(\mathcal{M}_{g,n})$ of the compactly supported cohomology of the open moduli space of curves.

4.2. The single vertex complex in weight 2

Our proof of Theorem 3.3 will combine (16) with the following analogous statement in weight 2.

PROPOSITION 4.2

For $g \geq 2$ and all n , one has

$$H(V_{g,n}^2) = 0.$$

For $g = 1$, one has

$$H(V_{1,n}^2) \cong \mathbb{Q}\delta_{\text{irr}} \oplus (H^2(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{S}_2} / \xi^* H^2(\overline{\mathcal{M}}_{1,n})).$$

For $g = 0$, the differential on $V_{0,n}^2$ vanishes.

We write $V_{g,n}^2$ in the form

$$0 \rightarrow H^2(\overline{\mathcal{M}}_{g,n}) \rightarrow H^2(\overline{\mathcal{M}}_{g-1,n+2})_{\mathbb{S}_2} \rightarrow H^2(\overline{\mathcal{M}}_{g-2,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2} \rightarrow H^2(\overline{\mathcal{M}}_{g-3,n})_{\mathbb{S}_2 \wr \mathbb{S}_3} \rightarrow \cdots \quad (17)$$

Here, the action of $\mathbb{S}_2 \wr \mathbb{S}_h$ is understood to include a sign that accounts for the permutation of the tensor factors $\mathbb{Q}[-1]^{\otimes h}$ in (15). To make this concrete, recall that $H^2(\overline{\mathcal{M}}_{g,n})$ is generated by the tautological classes κ , ψ_i , δ_{irr} and $\delta_{a,A} = \delta_{g-a,A^c}$, with relations for $g \leq 2$ as discussed in Section 3. Then the action of $\mathbb{S}_2 \wr \mathbb{S}_h$ is induced by a signed permutation action on this set of tautological generators, as follows. One may think of each generator as a graph with a single internal vertex of genus $g - h$ incident to h loop edges and n marked legs, equipped with a decoration by a tautological generator for $H^2(\overline{\mathcal{M}}_{g-h,n+2h})$. Then each copy of $\mathbb{S}_2$ acts by exchanging the half-edges in a loop (without sign). The quotient map to $\mathbb{S}_h$ is given by the permutation of the loop edges, and the induced permutation on generators is twisted by the sign of this $\mathbb{S}_h$ -action.

LEMMA 4.3

As a vector space, $V_{g,n}^2 = \bigoplus_h H^2(\overline{\mathcal{M}}_{g-h,n+2h})_{\mathbb{S}_2 \wr \mathbb{S}_h}$ is generated by the following classes:

- (1) κ or δ_{irr} , for $h \in \{0, 1\}$,
- (2) ψ_i , with $i \in \{1, \dots, n\}$, for $h \in \{0, 1\}$,
- (3) ψ_j with $j \notin \{1, \dots, n\}$, for $h \in \{1, 2\}$
- (4) $\delta_{a,A}$, for $h \in \{0, 1, 2, 3\}$.

In particular, the complex (17) vanishes beyond $h = 3$.

Proof

Any other tautological generator for $H^2(\overline{\mathcal{M}}_{g-h,n+2h})$ is preserved by an element of $\mathbb{S}_2 \wr \mathbb{S}_h$ that acts by a transposition on the set of loop edges. $\square$

Proof of Proposition 4.2 for $g \geq 4$

For $g \geq 4$, there are no nontrivial relations among the above generators for $V_{g,n}^2$, that is, all relations are induced by isomorphisms of marked and decorated graphs. In this case, we can see that the complex is acyclic by constructing a null-homotopy

as follows. Depict boundary classes and ψ -classes by marked edges and half-edges, respectively, as in Section 3. Thus the generators of $V_{g,n}^2$ are depicted by graphs that have either one or two vertices. Say that a vertex v is *active* if either $g_v > 0$ or if there is a loop attached to v . Note that each vertex has at most one loop attached, since otherwise there is an odd automorphism interchanging two loops.

For each loop edge, there is a natural contraction, in which the loop edge is deleted and the genus of the attached vertex is increased by 1. The orientation on the resulting graph is determined as follows: after reordering, we may assume that the contracted loop was first in the loop order, and then the remaining edges keep their induced ordering. The null-homotopy takes the sum of all such contractions, divided by the number of active vertices. $\square$

When $g = 0$, there is nothing to prove. When $g = 1, 2$, or 3 , additional arguments are required. For simplicity, we give complete proofs for $n \geq 1$; the cases where $n = 0$ are similar. Throughout the remainder of the proof, we use S to denote an arbitrary subset of $\{1, \dots, n\}$, subject to restrictions as specifically indicated. For instance, “ $\delta_{0,S}, 1 \in S$ ” refers to the 2^{n-1} elements of the form $\delta_{0,S}$ where S is a subset of $\{1, \dots, n\}$ that contains 1.

Proof of Proposition 4.2 for $g = 3$

The complex $V_{3,n}^2$ has the form

$$H^2(\overline{\mathcal{M}}_{3,n}) \rightarrow H^2(\overline{\mathcal{M}}_{2,n+2})_{\mathbb{S}_2} \rightarrow H^2(\overline{\mathcal{M}}_{1,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2} \rightarrow H^2(\overline{\mathcal{M}}_{0,n+6})_{\mathbb{S}_2 \wr \mathbb{S}_3}. \quad (18)$$

By [2, Theorem 2.10], we know that $H^2(\overline{\mathcal{M}}_{3,n}) \rightarrow H^2(\overline{\mathcal{M}}_{2,n+2})_{\mathbb{S}_2}$ is injective. We now check exactness at the remaining places, working from right to left. Let $s, s', t, t',$ and u, u' be the pairs of markings added to $\{1, \dots, n\}$ at each step. We claim that $H^0(\overline{\mathcal{M}}_{0,n+6})_{\mathbb{S}_2 \wr \mathbb{S}_3}$ has a basis consisting of the classes

$$\delta_{0,S \cup \{s,s',t\}}, \quad 1 \in S.$$

This can be seen by taking coinvariants of the resolution in Proposition 3.5; each E_{ij} is stabilized by an element of $\mathbb{S}_2 \wr \mathbb{S}_3$ that acts by a simple transposition on the set of three loops, and hence the coinvariants of $\bigoplus_{i,j} \mathbb{Q} E_{ij}$ vanish.

Next, recall that $H^2(\overline{\mathcal{M}}_{1,n+4})$ is freely generated by boundary classes, which are permuted by $\mathbb{S}_2 \wr \mathbb{S}_2$. Taking coinvariants, we see that $H^2(\overline{\mathcal{M}}_{1,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}$ has a basis consisting of:

$$\delta_{0,S \cup \{s\}}, \quad |S| \geq 1; \quad \delta_{0,S \cup \{s,s'\}}; \quad \delta_{0,S \cup \{s,s',t\}}.$$

Working with respect to these bases, the same chain homotopy argument used for $g \geq 4$ shows that (18) is exact at $H^2(\overline{\mathcal{M}}_{0,n+6})_{\mathbb{S}_2 \wr \mathbb{S}_3}$ and at $H^2(\overline{\mathcal{M}}_{1,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}$. It

remains to check that it is exact at $H^2(\overline{\mathcal{M}}_{2,n+2})_{\mathbb{S}_2}$. We prove this when $n > 0$ by counting dimensions and computing the Euler characteristic; the argument for $n = 0$ is similar.

Assume that $n > 0$. Then $H^2(\overline{\mathcal{M}}_{2,n+2})_{\mathbb{S}_2}$ has a basis given by $\psi_1, \dots, \psi_n, \psi_s$, and δ_{irr} together with:

$$\delta_{0,S}, \quad |S| \geq 2; \quad \delta_{1,S}; \quad \delta_{2,S}; \quad \delta_{0,S \cup \{s\}}, \quad |S| \geq 1; \quad \delta_{1,S \cup \{s\}}, \quad 1 \in S.$$

The dimension of $H^2(\overline{\mathcal{M}}_{3,n})$ is $2 \cdot 2^n + 1$. Counting the bases above shows that

$$\chi(V_{3,n}^2) = (2 \cdot 2^n + 1) - (4 \cdot 2^n + 2^{n-1}) + (3 \cdot 2^n - 1) - 2^{n-1},$$

which is zero, as required. $\square$

Proof of Proposition 4.2 for $g = 2$

The complex $V_{2,n}^2$ has the form

$$H^2(\overline{\mathcal{M}}_{2,n}) \rightarrow H^2(\overline{\mathcal{M}}_{1,n+2})_{\mathbb{S}_2} \rightarrow H^2(\overline{\mathcal{M}}_{0,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}.$$

By [2, Theorem 2.10], we know that the first arrow is injective. We proceed to prove exactness at the remaining two places, working from right to left.

Say s, s' and t, t' are the two pairs of markings added at each step. First, we claim that $H^2(\overline{\mathcal{M}}_{0,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}$ has a basis consisting of:

$$\delta_{0,S \cup \{s,s'\}}, \quad 1 \in S; \quad \delta_{0,S \cup \{s\}}, \quad |S| \geq 2. \quad (19)$$

To see this, consider the $(\mathbb{S}_2 \wr \mathbb{S}_2)$ -coinvariants of the resolution of $H^2(\overline{\mathcal{M}}_{0,n+4})$ from Proposition 3.5. The coinvariant space of $\bigoplus_{i,j} \mathbb{Q}E_{ij}$ has a basis consisting of $E_{ss'}$ together with $E_{1s}, \dots, E_{ns}$. Similarly, the coinvariant space of $(\bigoplus_i \mathbb{Q}\psi_i) \oplus (\bigoplus_S \mathbb{Q}\delta_{0,S})$ has a basis consisting of:

$$\psi_s; \quad \delta_{0,S \cup \{s,s'\}}, \quad 1 \in S; \quad \delta_{0,S \cup \{s\}}, \quad |S| \geq 1.$$

By examining the differential, one checks that $E_{s,s'}$ can be used to eliminate ψ_s , while E_{is} can be used to eliminate $\delta_{0,\{i,s\}}$ for $i \in \{1, \dots, n\}$. Hence the images of the remaining basis elements listed in (19) form a basis for $H^2(\overline{\mathcal{M}}_{0,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}$, as claimed.

Next, recall that $H^2(\overline{\mathcal{M}}_{1,n+2})$ is freely generated by boundary classes, which are permuted by the $\mathbb{S}_2$ -action. One then sees that the coinvariant space $H^2(\overline{\mathcal{M}}_{1,n+2})_{\mathbb{S}_2}$ has a basis consisting of:

$$\delta_{\text{irr}}; \quad \delta_{0,S}, \quad |S| \geq 2; \quad \delta_{0,S \cup \{s\}}, \quad |S| \geq 1; \quad \delta_{1,S}. \quad (20)$$

Working with these bases, the same chain homotopy argument that works for $g \geq 4$ gives a splitting of the map to $H^2(\overline{\mathcal{M}}_{0,n+4})_{\mathbb{S}_2 \wr \mathbb{S}_2}$. It remains to check that $V_{2,n}^2$ is

exact at $H^2(\overline{\mathcal{M}}_{1,n+2})_{\mathbb{S}_2}$. Just as for $g = 3$, we do this by counting dimensions and computing the Euler characteristic. For simplicity, we assume that $n \geq 1$; the $n = 0$ case is similar.

The dimension of $H^2(\overline{\mathcal{M}}_{2,n})$ is $2^n + 2^{n-1}$. Counting the bases (19) and (20) then shows that

$$\chi(V_{2,n}^2) = (2^n + 2^{n-1}) - (3 \cdot 2^n - n - 1) + (2^n + 2^{n-1} - n - 1),$$

which is zero, as required. $\square$

We use the following lemma in the proof of Proposition 4.2 for $g = 1$, and also in the proof of Proposition 5.3.

LEMMA 4.4

The classes $\delta_{0,S}$ in $H^2(\overline{\mathcal{M}}_{1,n})$ for $S \subset \{1, \dots, n\}$, $|S| \geq 2$ have linearly independent images in $H^2(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{S}_2}$, where $\mathbb{S}_2$ acts by transposing the last two marked points.

Proof

These classes are $\mathbb{S}_2$ -invariant, so it suffices to show that they are linearly independent in $H^2(\overline{\mathcal{M}}_{0,n+2})$. We claim that there is a curve C in $\overline{\mathcal{M}}_{0,n+2}$ that intersects $\delta_{0,S}$ at finitely many points and is disjoint from $\delta_{0,S'}$ for all other subsets $S' \subset \{1, \dots, n\}$. Fix $n + 2$ general points in $\mathbb{P}^1$. Let $\mathbb{G}_m$ act so that the coordinates of the points in S are multiplied by z and the rest are multiplied by z^{-1} , and let C be the closure of this $\mathbb{G}_m$ -orbit. The image of $\mathbb{G}_m$ is contained in the open moduli space $\mathcal{M}_{0,n+2}$, and the remaining two points of C are general in $\delta_{0,S}$. This proves the claim, and the lemma follows. $\square$

Proof of Proposition 4.2 for $g = 1$

We now consider the two-term complex $V_{1,n}^2$,

$$H^2(\overline{\mathcal{M}}_{1,n}) \rightarrow H^2(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{S}_2}.$$

Recall that $H^2(\overline{\mathcal{M}}_{1,n})$ is freely generated by the boundary classes δ_{irr} and $\delta_{0,S}$, $|S| \geq 2$. Also, $H^2(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{S}_2}$ is generated by the classes

$$\delta_{0,S}, \quad |S| \geq 2; \quad \delta_{0,S \cup \{s\}}, \quad |S| \geq 1, |S^c| \geq 1.$$

The differential is then given by $\delta_{\text{irr}} \mapsto 0$ and $\delta_{0,S} \mapsto \delta_{0,S}$. The statement of Proposition 4.2 for $g = 1$ reduces to saying that the kernel of the differential is generated by δ_{irr} . This follows from Lemma 4.4, since the classes $\delta_{0,S}$, $|S| \geq 2$ have linearly independent images in $H^2(\overline{\mathcal{M}}_{0,n+2})_{\mathbb{S}_2}$. This completes the proof of Proposition 4.2. $\square$

4.3. The filtered map of filtered complexes

We now prove Theorem 3.3, showing that the projection

$$\mathrm{GK}_{g,n}^2 \rightarrow \overline{\mathrm{GK}}_{g,n}^2 \quad (21)$$

is a quasi-isomorphism for any $(g, n) \neq (1, 1)$. We do this by filtering both sides by the number of vertices and showing that the induced map between spectral sequences is an isomorphism at E_2 . From this it follows that (21) is a quasi-isomorphism by the spectral sequence comparison theorem (see [38, Theorem 5.2.12]).

Proof of Theorem 3.3

Let $\mathcal{F}^p \mathrm{GK}_{g,n}^2 \subset \mathrm{GK}_{g,n}^2$ and $\mathcal{F}^p \overline{\mathrm{GK}}_{g,n}^2 \subset \overline{\mathrm{GK}}_{g,n}^2$ be the dg subvector spaces of $\mathrm{GK}_{g,n}^2$ and $\overline{\mathrm{GK}}_{g,n}^2$, respectively, spanned by graphs with at least p vertices. So $\mathcal{F}^\bullet \mathrm{GK}_{g,n}^2$ and $\mathcal{F}^\bullet \overline{\mathrm{GK}}_{g,n}^2$ are decreasing filtrations of dg vector spaces, and (21) respects the filtrations. We now consider the spectral sequences associated to these filtrations and the induced map between them.

On E_0 , we have the associated graded dg vector spaces

$$\mathrm{gr} \mathrm{GK}_{g,n}^2 \cong (\mathrm{GK}_{g,n}^2, \delta_{\mathrm{loop}}) \quad \text{and} \quad \mathrm{gr} \overline{\mathrm{GK}}_{g,n}^2 \cong (\overline{\mathrm{GK}}_{g,n}^2, 0).$$

Say that the *loopless core* of a stable graph is the stable graph obtained by contracting all loop edges and increasing the genera of the adjacent vertices accordingly. Note that δ_{loop} does not change the loopless core, and hence $(\mathrm{GK}_{g,n}^2, \delta_{\mathrm{loop}})$ decomposes as a direct sum

$$(\mathrm{GK}_{g,n}^2, \delta_{\mathrm{loop}}) \cong \bigoplus_{\Gamma} (\mathrm{GK}_{\Gamma}, \delta_{\mathrm{loop}}),$$

where Γ ranges over isomorphism classes of n -marked stable graphs of genus g without loop edges and GK_{Γ} is the dg subspace spanned by graphs with loopless core Γ .

Since δ_{loop} acts independently on all vertices, we furthermore have the isomorphism of dg vector spaces

$$(\mathrm{GK}_{\Gamma}, \delta_{\mathrm{loop}}) \cong \left(\bigotimes_{v \in \mathrm{Vert}(\Gamma)} V_{g_v, n_v}^{k_v} \otimes \mathbb{Q}[-1]^{\otimes E(\Gamma)} \right)_{\mathrm{Aut}(\Gamma)},$$

where $E(\Gamma)$ is the set of structural edges of Γ . Since taking coinvariants with respect to finite group actions commutes with taking cohomology, we find that

$$H(\mathrm{GK}_{\Gamma}, \delta_{\mathrm{loop}}) \cong \left(\bigotimes_{v \in \mathrm{Vert}(\Gamma)} H(V_{g_v, n_v}^{k_v}) \otimes \mathbb{Q}[-1]^{\otimes E(\Gamma)} \right)_{\mathrm{Aut}(\Gamma)}.$$

Recall that $k_v = 0$ for all vertices except one special vertex w which has $k_w = 2$. By (16), the right-hand side vanishes if Γ has a nonspecial vertex of positive genus. Otherwise, the right-hand side is

$$(H(V_{g,w,n_w}^2) \otimes \mathbb{Q}[-1]^{\otimes E(\Gamma)})_{\text{Aut}(\Gamma)}.$$

In other words, $H(\text{GK}_{g,n}^2, \delta_{\text{loop}})$ can be identified with a graded vector space of stable graphs without loop edges where all vertices have genus 0 except possibly one special vertex w that is decorated by $H(V_{g,w,n_w}^2)$, oriented by a total ordering of the structural edges.

The differential on the E_1 -page of our spectral sequence is then induced by the part δ_{split} of the original differential. Our goal is to show that the projection

$$(H(\text{GK}_{g,n}^2, \delta_{\text{loop}}), \delta_{\text{split}}) \rightarrow (\overline{\text{GK}}_{g,n}^2, \delta_{\text{split}}) \quad (22)$$

is a quasi-isomorphism. The right-hand side is also a graded vector space of stable graphs. Now all vertices have genus 0, but loop edges are allowed, and one special vertex w is decorated by $H^2(\overline{\mathcal{M}}_{0,n_w})$, modulo the image of δ_{loop} . Applying Proposition 4.2, we see that (22) is surjective, with kernel $K_{g,n}$ generated by stable graphs in which all vertices have genus 0 except one special vertex of genus 1 that is decorated with δ_{irr} . Equivalently, $K_{g,n}$ is a commutative graph complex generated by graphs with one special vertex in which all vertices have valence at least 3 except the special vertex, which may also have valence 1 or 2. We prove the acyclicity of $K_{g,n}$ as follows.

Let $K_1 \subset K_{g,n}$ be the dg subspace spanned by graphs in which the special vertex has valence 1, and let $K_{\geq 2} \subset K_{g,n}$ be the complementary subspace spanned by graphs having a special vertex of valence at least 2. The differential δ_{split} decomposes accordingly into the following pieces

$$\begin{array}{ccc} \curvearrowright & & \curvearrowright \\ K_{\geq 2} & \xrightarrow{s} & K_1 \end{array}.$$

The part s splits off the special vertex as shown,

$$s: \begin{array}{c} \diagup \\ \diagdown \end{array} \circ \mapsto \begin{array}{c} \diagup \\ \diagdown \end{array} \circ \circ,$$

with the special vertex marked by double circles. It is clear that this map $s: K_{\geq 2} \rightarrow K_1$ is injective. It is also surjective if $(g, n) \neq (1, 1)$. Applying Lemma 2.1, it follows that $K_{g,n}$ is acyclic, as required. $\square$

5. A zigzag of quasi-isomorphisms of graph complexes

In this section, we discuss the graph complex $X_{g,n}$ in more detail. We then describe a zigzag of quasi-isomorphisms relating $\overline{\text{GK}}_{g,n}^2$ to $X_{g,n}$ and thereby prove Theorem 1.1.

Recall that we consider $X_{g,n}$ only for nonnegative integers g and n such that $2g + n \geq 3$.

5.1. Generators for $X_{g,n}$

The generators for $X_{g,n}$ are simple graphs without loops or multiple edges, in which no vertices have valence 2. The vertices of valence at least 3 are *internal* and those of valence 1 are *external*. Each external vertex is decorated with an element from the set $\{\epsilon, \omega, 1, \dots, n\}$ such that

- each label $1, \dots, n$ appears exactly once and the label ω appears exactly twice;
- the graph obtained by joining all external vertices labeled ϵ or ω is connected and has genus g .

Say that an edge with two external vertices labeled a and b is an (a, b) -edge. We further require that

- no connected component is an (ϵ, ω) - or (ω, ω) -edge.

An edge is *structural* if it does not contain an external vertex with label from $\{1, \dots, n\}$. Note that there are exactly n nonstructural edges. The degree of a graph is the number of structural edges plus 1. Each generator comes with a total ordering of the structural edges, and we impose the relation that permuting the structural edges is multiplication by the sign of the permutation.

The differential δ on $X_{g,n}$ is a sum of two parts $\delta = \delta_{\text{split}} + \delta_{\text{join}}$. Here δ_{split} is a sum over all vertices and over all ways of splitting the vertex into two vertices joined by an edge:

$$\delta_{\text{split}} \Gamma = \sum_{v \text{ vertex}} \Gamma \text{ split } v, \quad \text{---} \times \mapsto \sum \text{---} \bullet \text{---} \times$$

The part δ_{join} glues together a subset S of the ϵ - and ω -decorated external vertices, such that $|S| \geq 2$, and S contains either 0 or 1 of the ω -decorated vertices. It then attaches an edge to a new external vertex decorated by ϵ or ω , respectively:

$$\delta_{\text{join}} \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \Gamma = \sum_S \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \Gamma \begin{array}{c} \bullet \\ \text{---} \\ \epsilon \text{ or } \omega \end{array}.$$

In both parts, there is precisely one new structural edge, and the ordering is chosen so that the new edge comes first and the relative ordering of the old edges is preserved. It is straightforward to verify that applying δ_{split} and δ_{join} to generators for $X_{g,n}$ produces a sum of generators for $X_{g,n}$, and that δ squares to zero (as do δ_{split} and δ_{join}).

In pictures, we draw graphs in $X_{g,n}$ with filled black vertices, to distinguish them from graphs in $\mathbf{GK}_{g,n}$.

Remark 5.1

The graph complex $X_{g,n}$ is closely related to the graph complexes studied by Fresse,

Turchin, and the second author [16] in the context of the embedding calculus. For suitable submanifolds $M \subset \mathbb{R}^m$ they showed that the rational homotopy groups of the (long) embedding spaces modulo immersions $\overline{\text{Emb}}_{\partial}(M, \mathbb{R}^N)$ can be expressed as the cohomology of a graph complex $\text{HGC}_{\bar{A}, N}$, in which external vertices are labeled by elements of the augmentation ideal $\bar{A}$ of a dg commutative algebra model of the one-point compactification $M \cup \{\infty\}$. Furthermore, the cohomology groups of $\overline{\text{Emb}}_{\partial}(M, \mathbb{R}^N)$ are computed by the Chevalley complex of $\text{HGC}_{\bar{A}, N}$, and this latter complex can be identified with a complex of (possibly) disconnected graphs.

In particular, if M is a union of n copies of $\mathbb{R}^k$ and $S^{2\ell}$, then the corresponding model $\bar{A}$ has a basis consisting of ϵ in degree 0, ω in degree 2ℓ , and $\{\omega_1, \dots, \omega_n\}$ in degree k , with product given by $\epsilon^2 = \epsilon$, $\epsilon\omega = \omega$, and $\epsilon\omega_j = \omega^2 = \omega_i\omega = \omega_i\omega_j = 0$.

Then, up to unimportant degree shifts, $X_{g,n}$ may be identified with a subquotient of the Chevalley complex of $\text{HGC}_{\bar{A}, N}$, for N even, corresponding to genus g graphs without loop edges that contain each decoration ω_j exactly once, and that are connected after fusing the ϵ - and ω -decorated legs. We shall not need this connection to the embedding calculus elsewhere in the paper, and leave the more precise comparison of the cohomology of $X_{g,n}$ and the cohomology of embedding spaces to future work.

5.2. A resolution of $\overline{\text{GK}}_{g,n}^2$

Our goal is to relate $\overline{\text{GK}}_{g,n}^2$ to $X_{g,n}$ by a zigzag of quasi-isomorphisms. To this end, we now construct a quasi-isomorphism $\widetilde{\text{GK}}_{g,n}^2 \xrightarrow{\sim} \overline{\text{GK}}_{g,n}^2$ such that $\widetilde{\text{GK}}_{g,n}^2$ also maps naturally onto $X_{g,n}$. Roughly speaking $\widetilde{\text{GK}}_{g,n}^2$ is constructed from $\overline{\text{GK}}_{g,n}^2$ by resolving the space of decorations $H^2(\overline{\mathcal{M}}_{0,k})$, as in Proposition 3.5. One can also construct a quasi-isomorphism $X_{g,n} \rightarrow \overline{\text{GK}}_{g,n}^2$ (see Appendix A). The construction of the map is explicit, but the proof that it is a quasi-isomorphism uses the resolution $\widetilde{\text{GK}}_{g,n}^2 \xrightarrow{\sim} \overline{\text{GK}}_{g,n}^2$, and so we have chosen to focus on the zigzag.

Concretely, $\widetilde{\text{GK}}_{g,n}^2$ is a graph complex analogous to $\text{GK}_{g,n}^2$ generated by graphs with the following features. All vertices have genus 0 and there is exactly one special vertex with extra decoration. There are no loop edges at nonspecial vertices. The special vertex is decorated with one of the following:

- a symbol $\delta_{0,A}$, where A is a subset of the k half-edges at the special vertex such that $2 \leq |S| \leq k-2$;
- a symbol ψ_i , where i is a half-edge at the special vertex; or
- a symbol E_{ij} , where i, j are distinct half-edges at the special vertex.

We impose the relations $\delta_{0,A} = \delta_{0,A^c}$ and $E_{ij} = E_{ji}$. Furthermore, if there is a loop at the special vertex with half-edges t and t' , then we impose the *loop relations*:

$$\begin{aligned} \psi_i &= 0; & \psi_t &= \psi_{t'} = \frac{1}{2} \sum_{t \in A, t' \notin A} \delta_{0,A}; & E_{it} &= E_{it'} = 0; \\ \delta_{0,A} &= 0 & \text{if } t, t' \in S \text{ (or } t, t' \in S^c). \end{aligned} \quad (23)$$

As usual, each generator comes equipped with an ordering of the structural edges, and permuting this ordering induces multiplication by the sign of the permutation. Formally, these decorations should be interpreted as elements of a dg vector space resolving $H^2(\overline{\mathcal{M}}_{0,k})$, as in Proposition 3.5. In this resolution, $\delta_{0,A}$ and ψ_i have degree 2, while E_{ij} has degree 1. Thus, a generator with n structural edges has degree $n + 2$ unless its decoration is E_{ij} , in which case the degree is $n + 1$.

We continue to depict the decoration at the special vertex by adding combinatorial features to the graph. The depictions of ψ_i and $\delta_{0,S}$ are exactly as in Section 3.2. We depict the decoration E_{ij} , with i and j distinct half-edges incident to the special vertex, by marking the two half-edges with arrows as follows:

$$E_{ij} =: \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ \circ \\ \downarrow \end{array}$$

The loop relations (23) then take on the following graphical form:

$$\begin{array}{c} \text{loop with arrow} \\ \circ \end{array} = 0 \quad \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} = \frac{1}{2} \sum \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} \quad \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} = \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} = 0 \quad \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} = 0 \quad (24)$$

Note that any graph with two or more loops at the special vertex is set to zero by these relations.

The differential on $\widetilde{\text{GK}}_{g,n}^2$ is the sum of two parts

$$\delta = \delta_{\text{split}} + \delta_{\text{res}},$$

where δ_{res} encodes the resolution of $H^2(\overline{\mathcal{M}}_{0,n})$ from Proposition 3.4. Recall that the resolution is given by

$$E_{ij} \mapsto \psi_i + \psi_j - \sum_{i \in S, j \in S^c} \delta_{0,S}.$$

The differential δ_{res} applies this formula to the E_{ij} -decoration at the special vertex, with a conventional sign $(-1)^e$, where e is the number of structural edges. Graphically, this may be depicted as

$$\delta_{\text{res}}: \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} \mapsto (-1)^e \left(\begin{array}{c} \text{loop with arrow} \\ \circ \end{array} + \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} - \sum \begin{array}{c} \text{loop with arrow} \\ \circ \end{array} \right) \quad (25)$$

If the special vertex is decorated by $\delta_{0,A}$ or ψ_i , then δ_{res} vanishes. Note that δ_{res} is homogeneous of degree 1.

The part δ_{split} splits undecorated vertices as usual. When the special vertex is split, this is done as follows.

- If decorated by ψ_i , the special vertex is again split as in (13).
- If decorated by $\delta_{0,S}$, the special vertex is split as in (14).
- If decorated by E_{ij} , some subset S of half-edges is split off that does not contain both i and j (but might contain neither of them), as shown:

$$\delta_{\text{split}} : \text{diagram} \mapsto \sum \text{diagram}_1 + \sum \text{diagram}_2 + \sum \text{diagram}_3 \quad (26)$$

PROPOSITION 5.2

The differential on $\widetilde{\mathbf{GK}}_{g,n}^2$ is well defined and squares to zero.

Proof

First, we need to show that this differential is well defined, that is, that it respects the loop relations (24). That it respects the first and third relations is relatively straightforward. To see that it respects the second loop relation, note that the differential applied to the left-hand side is

$$\sum \text{diagram}_1 + \sum \text{diagram}_2 \quad (27)$$

The differential applied to the right-hand side produces

$$-\sum \text{diagram}_1 + \sum \text{diagram}_2 + \sum \text{diagram}_3 \quad (28)$$

The first term in (27) agrees with that in (28), with the sign due to interchanging the two edges. The second term agrees with the second term in (28) due to the relations. Each summand in the third term appears twice, with opposite sign, by applying δ_{split} to two different terms in the sum appearing in (24). The verification that the differential respects the fourth loop relation is similar.

Now we must check that the differential squares to zero. Suppose that Γ is a generator for $\widetilde{\mathbf{GK}}_{g,n}^2$. We must show that

$$(\delta_{\text{split}} + \delta_{\text{res}})^2 \Gamma = \delta_{\text{split}}^2 \Gamma + (\delta_{\text{split}} \delta_{\text{res}} + \delta_{\text{res}} \delta_{\text{split}}) \Gamma = 0.$$

The argument for cancellation of terms that arise from splitting nonspecial vertices is identical to the proof that the differential squares to zero on the ordinary commutative graph complex. The same argument adapts easily to cases where the special vertex is decorated with ψ_i or $\delta_{0,A}$, because in these cases $\delta_{\text{res}} = 0$.

Suppose that the special vertex is decorated by E_{ij} . The standard argument shows that $\delta_{\text{split}}^2 \Gamma = 0$. It remains to show that $(\delta_{\text{split}} \delta_{\text{res}} + \delta_{\text{res}} \delta_{\text{split}}) \Gamma = 0$. Applying δ_{res} to the right-hand side of (26) gives $(-1)^{e+1}$ times

$$\begin{aligned} & \sum \text{[diagram 1]} + \sum \text{[diagram 2]} - \sum \text{[diagram 3]} + \sum \text{[diagram 4]} + \sum \text{[diagram 5]} \\ & - \sum \text{[diagram 6]} + \sum \text{[diagram 7]} + \sum \text{[diagram 8]} - \sum \text{[diagram 9]} - \sum \text{[diagram 10]} . \end{aligned} \quad (29)$$

We need to compare this to δ_{split} applied to the right-hand side of (25). The fourth and seventh terms together match δ_{split} applied to the second term of (25), with opposite sign. Likewise, the second and the eighth term of (29) match δ_{split} applied to the first term of (25). The cancellation of the remaining terms is similar. $\square$

PROPOSITION 5.3

There is a quasi-isomorphism

$$\widetilde{\text{GK}}_{g,n}^2 \rightarrow \overline{\text{GK}}_{g,n}^2$$

that sets E_{ij} to zero and imposes the relation $\psi_i + \psi_j - \sum_{i \in S, j \in S^c} \delta_{0,S}$.

Proof

We first verify that this projection is a well-defined map of complexes. It clearly respects the differential on all generators where the special vertex is decorated by ψ or $\delta_{0,A}$. Consider a generator $\Gamma \in \widetilde{\text{GK}}_{g,n}^2$ whose special vertex is decorated by E_{ij} . Then $\delta_{\text{split}} \Gamma$ also has an E_{ij} -decorated special vertex and is hence sent to zero. Furthermore, $\delta_{\text{res}} \Gamma$ is sent to zero since its special vertex is just decorated by the relation between ψ - and δ -classes.

To see that this projection is a quasi-isomorphism, we filter both sides by the number of vertices and consider the associated spectral sequences. We claim that the map $\widetilde{\text{GK}}_{g,n}^2 \rightarrow \overline{\text{GK}}_{g,n}^2$ induces a quasi-isomorphism already on the associated graded complex. Proceeding as in Section 4.3, one can see that it therefore suffices to consider the complex associated to the special vertex. The differential on the associated graded of $\widetilde{\text{GK}}_{g,n}^2$ is zero, and the differential on the associated graded of $\overline{\text{GK}}_{g,n}^2$ does not add any edges. We can therefore consider separate cases according to whether or not there is a loop at the special vertex.

If there is no loop, then the loop relations listed above do not come into play, and the statement boils down to Proposition 3.5, which says that

$$\bigoplus_{1 \leq i < j \leq k} \mathbb{Q}E_{ij} \rightarrow \left(\bigoplus_{1 \leq i \leq k} \mathbb{Q}\psi_i \right) \oplus \bigoplus \mathbb{Q}\delta_{0,S}$$

is a resolution of $H^2(\overline{\mathcal{M}}_{0,k})$.

It remains to consider the case where there is a loop at the special vertex. Let t, t' denote the last two marked points on $\overline{\mathcal{M}}_{0,k+2}$. We must show that the complex

$$\bigoplus_{1 \leq i < j \leq k} \mathbb{Q}E_{ij} \rightarrow \bigoplus_{A \subset \{1, \dots, k\}} \mathbb{Q}\delta_{0,A \cup \{t\}} \rightarrow H^2(\overline{\mathcal{M}}_{0,k+2})_{\mathbb{S}_2} / \sim \quad (30)$$

is exact, where “ $\sim$ ” is the quotient by the loop relations $\delta_{0,A} = 0$ for $A \subset \{1, \dots, k\}$. To see this, we begin by taking the $\mathbb{S}_2$ -coinvariants of the resolution of $H^2(\overline{\mathcal{M}}_{0,k+2})$ from Proposition 3.5. Then the $\mathbb{S}_2$ -coinvariant space of the first term $\bigoplus \mathbb{Q}E_{ij}$ has a basis consisting of

$$E_{ij}, \quad 1 \leq i < j \leq k; \quad E_{it}, \quad 1 \leq i \leq k; \quad E_{tt'}.$$

Similarly, the $\mathbb{S}_2$ -coinvariant space of the second term $(\bigoplus \mathbb{Q}\psi_i) \oplus (\bigoplus \mathbb{Q}\delta_{0,A})$ has a basis consisting of

$$\begin{aligned} \psi_i, \quad 1 \leq i \leq k; \quad \psi_t; \quad \delta_{0,A}, \quad A \subset \{1, \dots, k\}, \quad |S| \geq 2; \\ \delta_{0,A \cup \{t\}}, \quad A \subset \{1, \dots, k\}, \quad |A| \geq 1, \quad |A^c| \geq 1. \end{aligned}$$

By Lemma 4.4, the subspace spanned by the basis elements $\delta_{0,A}$, for $A \subset \{1, \dots, k\}$, $|A| \geq 2$ maps isomorphically onto its image in $H^2(\overline{\mathcal{M}}_{0,k+2})_{\mathbb{S}_2}$, which is precisely the span of the loop relations. We can then use E_{it} to eliminate ψ_i and $E_{tt'}$ to eliminate ψ_t and see that (30) is exact, as required. $\square$

5.3. Completing the zigzag

We now complete the zigzag from $\overline{\mathbf{GK}}_{g,n}^2$ to $X_{g,n}$ by producing a quasi-isomorphism

$$\Phi : \widetilde{\mathbf{GK}}_{g,n} \xrightarrow{\sim} X_{g,n}.$$

Let Γ be a graph in $\widetilde{\mathbf{GK}}_{g,n}$ with e structural edges. Say that the half-edges incident to external vertices are *legs*.

- If the special vertex of Γ is decorated by $\delta_{0,A}$, then we define $\Phi(\Gamma) = 0$.
- If the special vertex of Γ is decorated by E_{ij} , then we let $\Phi(\Gamma)$ be the marked graph obtained by deleting the special vertex, making the half-edges incident to the special vertex into legs, and decorating the external vertices on i, j by ω and the other newly created external vertices by ϵ :

Diagram (31) illustrates the map Φ . On the left, a vertex is shown with several half-edges. Two half-edges are labeled i and j . An arrow labeled Φ points to the right. On the right, the image of the vertex is a star graph where the half-edges are paired into edges. The edges are labeled ω and ϵ . The half-edges i and j are now part of the ω edges.

(31)

Note that the structural edges in Γ and in $\Phi(\Gamma)$ are in 1-1 correspondence; we retain their ordering.

- If the special vertex of Γ is decorated by ψ_i , then we define $\Phi(\Gamma)$ by cutting the edge and pairing each of its half-edges with an ω -decorated leg, with an overall sign of $(-1)^e$:

Diagram (32) shows the map Φ for a decorated vertex. On the left, a vertex is decorated with a circle and a dot, and has half-edges labeled i and j . An arrow labeled Φ points to the right. On the right, the image is a star graph where the half-edges are paired into edges. The edges are labeled ω and ϵ . The half-edges i and j are now part of the ω edges. The overall sign is $(-1)^e$.

(32)

Note in particular that if the half-edge opposite i is a leg, say, incident to an external vertex with some marking j , then $\Phi(\Gamma)$ will have a connected component that is an (ω, j) -edge:

Diagram (33) shows the map Φ for a half-edge incident to an external vertex. On the left, a half-edge is labeled j and is incident to an external vertex. An arrow labeled Φ points to the right. On the right, the image is a star graph where the half-edges are paired into edges. The edges are labeled ω and ϵ . The half-edges i and j are now part of the ω edges. The overall sign is $(-1)^e$.

(33)

In both cases (32) and (33), $\Phi(\Gamma)$ has one more edge than Γ . We may assume, without loss of generality, that the decorated edge (of which i is one half-edge) is the first in the ordering of edges. Then we order the structural edges in $\Phi(\Gamma)$ such that the edge containing i is first, the edge containing the opposite half-edge of i is second, and the relative order of the remaining edges is unchanged.

LEMMA 5.4

The map Φ is a map of complexes.

Proof

It is clear that the map Φ respects the degrees and genera and is hence a well-defined map of graded vector spaces. We need to verify that it commutes with the differentials,

$$\delta\Phi(\Gamma) = \Phi(\delta\Gamma).$$

Recall that the differential on $\widetilde{\mathbf{GK}}_{g,n}$ is $\delta = \delta_{\text{split}} + \delta_{\text{res}}$, while the differential on $X_{g,n}$ is $\delta = \delta_{\text{split}} + \delta_{\text{join}}$. We furthermore decompose the operation δ_{split} on $\widetilde{\mathbf{GK}}_{g,n}$ into two terms,

$$\delta_{\text{split}} = \delta_{\text{split}}^s + \delta_{\text{split}}^o,$$

with δ_{split}^s splitting the special vertex and δ_{split}^o splitting the other vertices. Similarly, we write $\delta_{\text{split}}\Phi(\Gamma) = (\delta_{\text{split}}^s + \delta_{\text{split}}^o)\Phi(\Gamma)$ in $X_{g,n}$ with δ_{split}^s splitting the vertex that is

the image of the special vertex, when the decoration on Γ is ψ_i , and δ_{split}^o splitting the other vertices. (When the decoration on Γ is E_{ij} , we set $\delta_{\text{split}}^s \Phi(\Gamma) = 0$.) Since away from the special vertex the graph is not altered by Φ , we have

$$\Phi(\delta_{\text{split}}^o \Gamma) = \delta_{\text{split}}^o \Phi(\Gamma).$$

Keep in mind that if Γ has a ψ_j -decoration then $\Phi(\Gamma)$ has one more structural edge than Γ , producing an additional minus sign upon applying δ_{split}^o . However, due to the sign $(-1)^e$ in the definition of Φ the signs on both sides of the above equation still agree. Next, we then need to check that

$$(\delta_{\text{split}}^s + \delta_{\text{join}}) \Phi(\Gamma) = \Phi((\delta_{\text{split}}^s + \delta_{\text{res}}) \Gamma).$$

We consider cases according to the decoration at the special vertex of Γ .

First, suppose that the special vertex of Γ is decorated by $\delta_{0,A}$. Then we have $\Phi(\Gamma) = 0$ and $\delta_{\text{res}} \Gamma = 0$, so we need to check that

$$\Phi(\delta_{\text{split}}^s \Gamma) = 0.$$

The only terms of $\delta_{\text{split}}^s \Gamma$ that do not themselves carry a δ -decoration (and are hence sent to zero by Φ) are those appearing in the last line of (14). Both of these terms are mapped to the same graph via Φ as depicted in (32), with opposite orderings of the two ω -edges, and hence the matching terms cancel.

Next, suppose that the special vertex of Γ is decorated by ψ_i . We have $\delta_{\text{res}} \Gamma = 0$, and, since $\Phi(\Gamma)$ has no ϵ -legs, $\delta_{\text{join}} \Phi(\Gamma) = 0$. We need to check that

$$\delta_{\text{split}}^s \Phi(\Gamma) = \Phi(\delta_{\text{split}}^s \Gamma).$$

Applying the definitions shows that both sides have the same form

$$\sum \text{graph} \quad \omega \quad \omega - .$$

Finally, suppose that the special vertex of Γ is decorated by E_{ij} . The differential δ_{res} sends Γ to $(-1)^e$ times a graph whose special vertex is decorated by $\psi_i + \psi_j - \sum_{i \in S, j \in S^c} \delta_{0,S}$. The $\delta_{0,S}$ -terms can be dropped upon applying Φ . The two ψ -terms produce graphs

$$\begin{array}{c} i \quad \omega \quad j \\ \omega \quad \omega \end{array} + \begin{array}{c} i \quad \omega \quad j \\ \omega \quad \omega \end{array}, \quad (34)$$

where one has to remember that in each case the ω -edge adjacent to the depicted vertex is the first in the ordering. Furthermore, note that the two factors $(-1)^e$ from

the definitions of Φ and δ_{res} cancel. Next consider the terms $\delta_{\text{split}}^s \Gamma$, which are given by replacing the special vertex by two vertices in one of three ways as follows:

$$\sum \text{diagram 1} + \sum \text{diagram 2} + \sum \text{diagram 3}$$

Applying Φ , we obtain terms

$$\sum \text{diagram 4} + \sum \text{diagram 5} + \sum \text{diagram 6} . \quad (35)$$

Note that in all terms there is at least one ϵ -leg. Next we look at $\Phi(\Gamma)$. In this case $\delta_{\text{split}}^s \Phi(\Gamma) = 0$, since Φ removes the special vertex. Applying δ_{join} to the right-hand side of (31) produces several graphs, by fusing a subset of ϵ -legs together to produce a new ϵ -leg, or fusing a subset of the ϵ -legs to one ω -leg. The two cases in which all the ϵ -hairs are fused to one ω -leg precisely contribute (34). The cases where one or more ϵ -legs remain contribute the terms (35). To confirm the signs, note that in all cases considered the newly added edge is the first in the ordering.

Hence we conclude that $\delta_{\text{join}} \Phi(\Gamma) = \Phi(\delta_{\text{split}}^s \Gamma + \delta_{\text{res}} \Gamma)$, as required. $\square$

PROPOSITION 5.5

The map Φ is a surjective quasi-isomorphism with kernel generated by graphs in which the special vertex is decorated by some $\delta_{0,S}$, and by symmetric combinations of graphs with ψ -decorations on either half of some edge:

$$\text{diagram 7} + \text{diagram 8} .$$

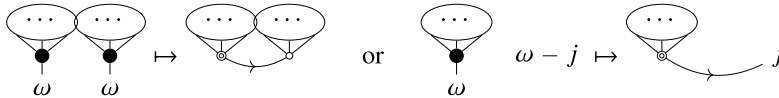
Proof

We begin by showing that Φ is surjective. Let Γ be a graph in $X_{g,n}$. We will construct a graph $\tilde{\Gamma} \in \widetilde{\mathbf{GK}}_{g,n}$ such that $\Phi(\tilde{\Gamma}) = \Gamma$. First, consider the case where Γ has at least one ϵ -decorated external vertex. Then we build $\tilde{\Gamma}$ by joining the ϵ - and ω -decorated external vertices to a new internal vertex, decorated by E_{ij} , with i, j corresponding to the legs at the two ω -decorations:

$$\text{diagram 9} \mapsto \text{diagram 10} .$$

This cannot produce a “forbidden graph” with decoration E_{it} or $E_{it'}$, where t and t' are the half-edges of a loop, because, in defining $X_{g,n}$, we excluded graphs with (ϵ, ω) - and (ω, ω) -edges. By construction, $\Phi(\tilde{\Gamma}) = \Gamma$.

It remains to consider the case where Γ has no ϵ -decorated external vertices. Then we build $\tilde{\Gamma}$ by deleting the two ω -decorated external vertices and their incident legs, joining the two unpaired half-edges into a new edge, and adding a ψ -decoration on one of the two, as shown:



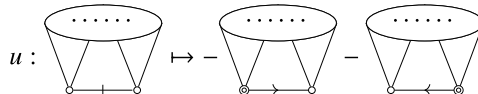
In either case, $\Phi(\tilde{\Gamma}) = \Gamma$. Here we also use that at least one of the ω -legs is adjacent to an internal vertex. Indeed, if this were not the case, then the graph would be either a single (ω, ω) -edge, or the union of an $(\omega, 1)$ -edge and an $(\omega, 2)$ -edge. In either case, we would have $2g + n < 3$, which we have excluded from consideration.

The map takes graphs with decoration $\delta_{0,S}$ to zero, and the orientation data ensures that symmetric combinations of ψ -decorations on paired half-edges also map to zero. Otherwise, distinct generators for $\widetilde{\text{GK}}_{g,n}$ map to distinct generators of $X_{g,n}$, so nothing else is in the kernel.

It remains to check that $J := \ker \Phi$ is acyclic. Decompose $J = J_\delta \oplus J_\psi$, where J_δ and J_ψ are linear combinations of graphs with δ - or ψ -decorations, respectively. The differential δ_{split} on J then splits into the following pieces:

$$\begin{array}{c} \curvearrowright \\ J_\delta \end{array} \xrightarrow{u} \begin{array}{c} \curvearrowright \\ J_\psi \end{array}.$$

Note that u appears in the last terms in (14), and takes the form



In these graphical depictions, u replaces a marked edge with a symmetric combination of ψ -decorations on its paired half-edges. This map gives a bijection between bases for J_δ and for J_ψ , and is hence an isomorphism. It then follows by Lemma 2.1 that J is acyclic, as required. $\square$

Combining Propositions 5.3 and 5.5 gives the desired zigzag of quasi-isomorphisms $\widetilde{\text{GK}}_{g,n} \xleftarrow{\Phi} \widetilde{\text{GK}}_{g,n} \xrightarrow{\Phi} X_{g,n}$.

6. A quasi-isomorphic subcomplex

Although we do not fully understand the cohomology of $X_{g,n}$, we can describe many nontrivial classes. We do so by identifying a quasi-isomorphic subcomplex $X_{g,n}^*$ and constructing an involution on $X_{g,n}^*$ that simplifies the differential. In this simplified subcomplex, we find direct summands whose cohomology we can either compute or bound from below.

6.1. A quasi-isomorphic subcomplex

Recall that every connected component of a generator Γ for $X_{g,n}$ contains an external vertex labeled by ϵ or ω .

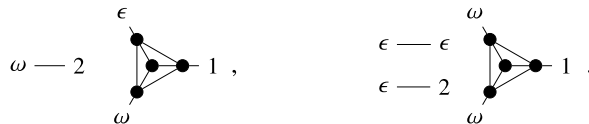
DEFINITION 6.1

A connected component that contains an ω -decoration is an ω -component. All other connected components are ϵ -components.

DEFINITION 6.2

Let $X_{g,n}^* \subset X_{g,n}$ be the subspace spanned by generators in which the union of all ϵ -components contains no internal vertices and at most one decoration from $\{1, \dots, n\}$.

In other words, if Γ is a generator for $X_{g,n}^*$, then the union of its ϵ -components is either empty or consists of an (ϵ, ϵ) -edge, an (ϵ, j) -edge, or one of each. Here are two examples:



LEMMA 6.3

The graded subspace $X_{g,n}^* \subset X_{g,n}$ is a subcomplex.

Proof

Let Γ be a generator for $X_{g,n}^*$. The differential is a sum of two parts $\delta_{\text{split}} + \delta_{\text{join}}$. The part δ_{split} acts separately on the internal vertices of each connected component. It cannot create internal vertices in a component that does not already have any, and it does change the decorations. So $\delta_{\text{split}}\Gamma$ is a linear combination of generators for $X_{g,n}^*$.

Now consider the terms that appear in $\delta_{\text{join}}\Gamma$, which are obtained by joining external vertices of one or several components. If at least one of the joined components is an ω -component, then the resulting graph is a generator for $X_{g,n}^*$. However,

if all components that are joined are ϵ -components, then the resulting graph has either a loop or two ϵ -labeled external vertices adjacent to the same internal vertex, and hence is zero in $X_{g,n}$. $\square$

PROPOSITION 6.4

The inclusion $X_{g,n}^ \subset X_{g,n}$ is a quasi-isomorphism.*

Proof

Filter the complexes $X_{g,n}^*$ and $X_{g,n}$ by the number of internal vertices in ω -components. We claim that the inclusion induces a quasi-isomorphism between the associated graded complexes with respect to this filtration.

The associated graded complexes have differential $\delta_{\text{join}}^{\epsilon,\epsilon} + \delta_{\text{split}}^{\epsilon}$, with $\delta_{\text{join}}^{\epsilon,\epsilon}$ the part of $\delta_{\text{join}}^{\epsilon}$ (as defined in the proof of Lemma 6.3) that joins ϵ -legs of ϵ -components only and $\delta_{\text{split}}^{\epsilon}$ the part of δ_{split} that splits vertices in ϵ -components only. Note that the differential on $X_{g,n}^*$ is zero. Furthermore, we have the direct sum decomposition

$$(X_{g,n}, \delta_{\text{join}}^{\epsilon,\epsilon} + \delta_{\text{split}}^{\epsilon}) = X_{g,n}^* \oplus (W, \delta_{\text{join}}^{\epsilon,\epsilon} + \delta_{\text{split}}^{\epsilon}),$$

with $W \subset X_{g,n}$ spanned by graphs that are not generators for $X_{g,n}^*$. Generators of W are graphs in which the ϵ -components contain an internal vertex or more than one leg labeled $1, \dots, n$. We further decompose

$$W = W' \oplus W'',$$

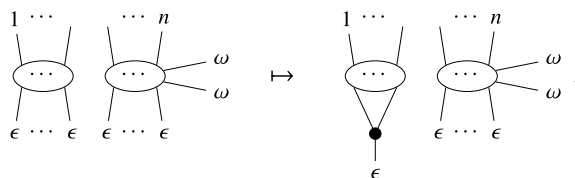
where W' is spanned by graphs that do not contain an (ϵ, ϵ) -edge, and W'' is spanned by those that do. Note that W'' is isomorphic (up to degree shift) to the W' summand that arises in $X_{g-1,n}$, so it suffices to show that W' is acyclic. To this end, consider the decomposition of graded vector spaces

$$W' = \begin{array}{ccc} \downarrow & \xleftarrow{\delta_{\text{join}}^{\epsilon,\epsilon}} & \downarrow \\ W'_1 & \oplus & W'_{\geq 2} \end{array},$$

with W'_1 (resp., $W'_{\geq 2}$) spanned by graphs that have 1 (resp., at least 2) ϵ -decorations in ϵ -components. Using again Lemma 2.1, it suffices to check that the map (part of the differential $\delta_{\text{join}}^{\epsilon,\epsilon}$)

$$W'_{\geq 2} \rightarrow W'_1$$

is an isomorphism. Combinatorially, this map joins all ϵ -decorated external vertices in ϵ -components, attaching a new internal vertex together with an ϵ -leg:



This map is injective; it has a one-sided inverse obtained by removing both new vertices and the edge between them, and adding an external vertex labeled ϵ to each of the dangling edges. This map is also surjective, since the unique ϵ -decoration in an ϵ -component of a graph in W'_1 must be adjacent to an internal vertex. (If it was adjacent to a j -decorated leg the graph would not be in W , and (ϵ, ω) -edges are forbidden in $X_{g,n}$.) $\square$

We note that in the above proof the presence of the part of the differential $\delta_{\text{join}}^\omega$ played no role. The same proof also shows the following auxiliary result, which we will use in the proof of Proposition 7.2.

LEMMA 6.5

The inclusion $(X_g^\star, \delta_{\text{join}}^\epsilon + \delta_{\text{split}}) \rightarrow (X_g, \delta_{\text{join}}^\epsilon + \delta_{\text{split}})$ is a quasi-isomorphism of dg vector spaces.

6.2. A simplifying involution

We now construct an involution of $X_{g,n}^\star$ that simplifies the differential. This involution is given by reattaching subsets of the ϵ -decorated external vertices in all possible ways and motivates the introduction of the subcomplex $X_{g,n}^\star$ (see Remark 6.6).

Let Γ be a generator for $X_{g,n}^\star$. We write $R_S(\Gamma)$ for the sum of all graphs obtained by reattaching a subset S of the ϵ -decorated external vertices of Γ to internal vertices, in all possible ways without forming loop edges:

$$\underbrace{\Gamma}_{\substack{\sim \\ S}} \mapsto R_S(\Gamma) = \sum \text{graphs with } \Gamma \text{ and } S \text{ reattached}.$$

The structural edges of each graph in $R_S(\Gamma)$ are in bijection with those of Γ , and we keep the given ordering.

Remark 6.6

The restriction on the union of all ϵ -components of graphs in $X_{g,n}^\star$ (see Definition 6.2) guarantees that the reattachment operation does not produce connected components

without any ϵ - or ω -decoration. For this reason, R_S is well defined on $X_{g,n}^*$. It is not well defined on $X_{g,n}$.

We now consider the graded endomorphism Ξ of $X_{g,n}^*$, given by

$$\Xi(\Gamma) = (-1)^{\#\epsilon} \sum_S R_S(\Gamma),$$

where the sum runs over all subsets of the set of ϵ -decorations, and $\#\epsilon$ is the number of ϵ -decorations in Γ .

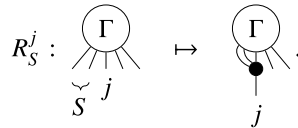
LEMMA 6.7

The map Ξ is an involution, that is, it is invertible and $\Xi^{-1} = \Xi$.

Proof

Any graph appearing in $\Xi \circ \Xi(\Gamma)$ is obtained by reattaching some subset S of the ϵ -decorated legs to internal vertices. Each such graph appears $2^{|S|}$ different ways, with signs that cancel unless $S = \emptyset$. $\square$

For $\Gamma \in X_{g,n}^*$ a graph, S a nonempty subset of its ϵ -decorated external vertices, and $j = 1, \dots, n$, let $R_S^j(\Gamma)$ be the graph obtained by reattaching the external vertices in S to the midpoint of the edge that contains the j -decoration:



This operation creates precisely one new structural edge, which we take to be first in the ordering, preserving the relative ordering of the remaining edges. Note that the operation also makes sense if applied not to a numbered leg, but an ω -labeled leg, and we shall denote the sum of such operations applied to the two ω -legs by R_S^ω .

We also define

$$\tilde{\delta}_{\text{join}}^j(\Gamma) = \sum_{|S| \geq 1} |S| \cdot R_S^j(\Gamma),$$

where the sum runs over all nonempty subsets of the set of ϵ -legs of Γ . We then have the following result.

PROPOSITION 6.8

Let

$$\tilde{\delta} := \delta_{\text{split}} + \delta_{\text{join}}^{\epsilon} - \sum_{j=1}^n \tilde{\delta}_{\text{join}}^j.$$

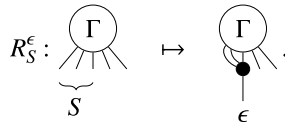
Then $\tilde{\delta} = \Xi \circ \delta \circ \Xi$. In particular, Ξ defines an isomorphism of dg vector spaces

$$\Xi : (X_{g,n}^{\star}, \tilde{\delta}) \rightarrow (X_{g,n}^{\star}, \delta).$$

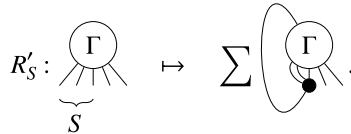
Proof

The proof is similar to that of [36, Lemma 3.6]; the argument there goes through essentially unchanged, even though the complexes $X_{g,n}^{\star}$ we consider here are different. We give only a condensed sketch of the proof.

We introduce the following notation. For S a subset of the ϵ -legs in some graph $\Gamma \in X_{g,n}^{\star}$, we denote by $R_S^{\epsilon}(\Gamma)$ the graph obtained by reattaching the legs in S to a new ϵ -decorated vertex:



Furthermore, we denote by $R'_S(\Gamma)$ the sum of graphs obtained from $R_S^{\epsilon}(\Gamma)$ by attaching the newly formed ϵ -leg to an internal vertex of Γ :



Using the fact that $\Xi^{-1} = \Xi$, one computes that

$$\begin{aligned} (\Xi \circ \delta_{\text{split}} \circ \Xi)(\Gamma) &= \delta_{\text{split}}(\Gamma) + \sum_{|S| \geq 2} |S| \cdot R_S^{\epsilon}(\Gamma) + \sum_{|S| \geq 2} (|S| - 1) \cdot R'_S(\Gamma) \\ &\quad + \sum_{|S| \geq 1} \sum_{j \in \{1, \dots, n, \omega\}} R_S^j(\Gamma). \end{aligned}$$

Here S runs over subsets of the set of ϵ -legs. Similarly, one may compute

$$(\Xi \circ \delta_{\text{join}}^{\epsilon} \circ \Xi)(\Gamma) = \sum_{|S| \geq 2} (1 - |S|) \cdot R_S^{\epsilon}(\Gamma) + \sum_{|S| \geq 2} (1 - |S|) \cdot R'_S(\Gamma).$$

Together, we find that

$$\begin{aligned} (\Xi \circ (\delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \circ \Xi)(\Gamma) &= (\delta_{\text{split}} + \delta_{\text{join}}^\epsilon + \delta_{\text{join}}^\omega)\Gamma + \sum_{|S| \geq 1} \sum_{j \in \{1, \dots, n\}} R_S^j(\Gamma) \\ &= \delta(\Gamma) + \sum_{|S| \geq 1} \sum_{j \in \{1, \dots, n\}} R_S^j(\Gamma). \end{aligned}$$

Then one may finally compute that

$$\left(\Xi \circ \sum_{|S| \geq 1} R_S^j \circ \Xi \right)(\Gamma) = \sum_{|S| \geq 1} |S| \cdot R_S^j(\Gamma) = \tilde{\delta}_{\text{join}}^j(\Gamma).$$

Putting the above computations together, we find that

$$\Xi \circ \tilde{\delta} \circ \Xi = \Xi \circ \left(\delta_{\text{split}} + \delta_{\text{join}}^\epsilon - \sum_j \tilde{\delta}_{\text{join}}^j \right) \circ \Xi = \delta + \sum_j \delta_{\text{join}}^j - \sum_j \delta_{\text{join}}^j = \delta,$$

which proves the proposition. $\square$

7. The weight 2 compactly supported cohomology of $\mathcal{M}_g$

We now focus on the special case where $n = 0$. Using the quasi-isomorphic subcomplex $X_g^\star \subset X_g$, with its simplified differential $\tilde{\delta}$, we shall see that the cohomology of X_g can be fully expressed through the cohomology of the ordinary commutative graph complexes $G^{(g', n')}$ of Section 4.1, with $n' = 1, 2$ and $g' = g, g - 1, g - 2$.

When $n = 0$, the differential $\tilde{\delta}$ in Proposition 6.8 further simplifies to $\delta_{\text{split}} + \delta_{\text{join}}^\epsilon$. We define the subcomplex

$$H_g \subset (X_g^\star, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \quad (36)$$

spanned by graphs with either no ϵ -decorations, or exactly two on an (ϵ, ϵ) -edge. Note that $\delta_{\text{join}}^\epsilon$ vanishes on H_g .

We also introduce the space

$$G^{(\bullet, n)} = \bigoplus_g G^{(g, n)},$$

which inherits an additional grading by the genus. We also define the exterior product

$$\mathbb{W}_\bullet = \bigwedge^2 G^{(\bullet, 1)}.$$

This comes with a grading inherited from the genus grading; let $\mathbb{W}_g$ be the homogeneous part of genus g .

LEMMA 7.1

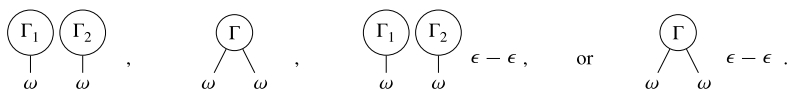
The subspace H_g is a subcomplex with respect to $\tilde{\delta} = \delta_{\text{split}} + \delta_{\text{join}}^\epsilon$, and $(H_g, \tilde{\delta})$ is isomorphic to

$$\mathbb{W}_g[-3] \oplus G_{\text{as}}^{(g-1,2)}[-3] \oplus \mathbb{W}_{g-1}[-4] \oplus G_{\text{as}}^{(g-2,2)}[-4].$$

Here the subscript $(-)_{\text{as}}$ refers to taking the antisymmetric part under the $\mathbb{S}_2$ -action permuting the labels 1, 2.

Proof

Graphs in H_g have exactly two ω -decorated legs, and every connected component has an ω - or ϵ -decorated leg. There are four possibilities, with and without an (ϵ, ϵ) -edge, and with one or two ω -components:



The graded vector space H_g decomposes into a direct sum of four subcomplexes accordingly; these subcomplexes are identified with $\mathbb{W}_g[-3]$, $G_{\text{as}}^{(g-1,2)}[-3]$, $\mathbb{W}_{g-1}[-4]$, and $G_{\text{as}}^{(g-2,2)}[-4]$, respectively, since $\delta_{\text{join}}^\epsilon$ vanishes on H_g . $\square$

PROPOSITION 7.2

The inclusion (36) is a quasi-isomorphism.

Proof

First note that one has a commutative diagram of morphisms of dg vector spaces:

$$\begin{array}{ccc} (H_g, \delta_{\text{split}}) & \xrightarrow{(36)} & (X_g^\star, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \\ & \searrow & \downarrow \sim \\ & & (X_g, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \end{array}$$

The vertical map is a quasi-isomorphism by Lemma 6.5. It hence suffices to show that the diagonal inclusion

$$(H_g, \delta_{\text{split}}) \rightarrow (X_g, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon)$$

is a quasi-isomorphism. Each complex splits, as in the proof of Proposition 6.4, into one piece generated by graphs that do not have any (ϵ, ϵ) -edge and a complementary piece spanned by those that do. So, we write H'_g and X'_g for the subcomplexes of H_g

and X_g , respectively, spanned by graphs that do not have any (ϵ, ϵ) -edge. Similarly, we write H_g'' and X_g'' for the subcomplexes spanned by graphs with an (ϵ, ϵ) -edge. We then have

$$H_g \cong H_g' \oplus H_g'', \quad X_g \cong X_g' \oplus X_g''.$$

Note that $H_g'' \cong H_{g-1}'[-1]$ and $X_g'' \cong X_{g-1}'[-1]$, so it suffices to show that $H_g' \subset X_g'$ is a quasi-isomorphism.

We furthermore have a direct sum decomposition of complexes

$$(X_g', \delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \cong (H_g', \delta_{\text{split}}) \oplus (U, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon),$$

where $U \subset X_g'$ is the subcomplex spanned by graphs with at least one ϵ -leg. It remains to show that U is acyclic.

We proceed as in Proposition 6.4 and decompose

$$U = \begin{array}{ccc} \curvearrowright & \xleftarrow{D} & \curvearrowright \\ U_1 & \oplus & U_{\geq 2} \end{array},$$

with the graded subspace $U_1 \subset U$ (resp., $U_{\geq 2} \subset U$) being spanned by graphs with exactly one (resp., at least two) ϵ -legs. The drawn arrows indicate various parts of the differential. In particular, the part

$$D : U_{\geq 2} \rightarrow U_1$$

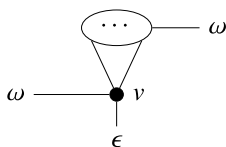
arising from $\delta_{\text{join}}^\epsilon$ joins all ϵ -legs into one:

$$D : \begin{array}{c} \omega \text{ --- } \text{---} \text{---} \text{---} \omega \\ \quad \diagdown \quad \quad \diagup \\ \quad \epsilon \quad \cdots \quad \epsilon \end{array} \mapsto \begin{array}{c} \omega \text{ --- } \text{---} \text{---} \text{---} \omega \\ \quad \diagdown \quad \quad \diagup \\ \quad \bullet \\ \quad \quad \epsilon \end{array}.$$

This part is injective, since the operation can be undone by removing the two newly added vertices, and adding back the ϵ -decorations on legs. By Lemma 2.1, we hence conclude that the projection

$$U \rightarrow \text{coker } D$$

is a quasi-isomorphism. The cokernel of D is spanned by graphs in U_1 such that the unique ϵ -leg is connected to a vertex v that is connected to an ω -leg. (Recall that we forbade (ϵ, ω) -edges in the definition of $X_{g,n}$.)



However, we can continue in the same fashion and filter the cokernel of D as

$$\text{coker}(D) = U'_1 \begin{array}{c} \curvearrowright \\ \leftarrow D' \\ \oplus \\ \curvearrowright \end{array} U'_2 ,$$

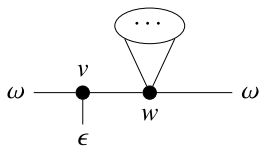
with U'_1 (resp., U'_2) being spanned by graphs for which the vertex v has valence 3 (resp., at least 4). The part of the differential

$$D' : U'_2 \rightarrow U'_1$$

is injective. Hence, applying Lemma 2.1 again, we find that the projection

$$\text{coker}(D) \rightarrow \text{coker}(D')$$

is a quasi-isomorphism. The cokernel of D' is spanned by graphs in U'_1 such that the unique vertex w neighboring v has an ω -hair attached:



Repeating the argument once more, we split

$$\text{coker}(D') = U''_1 \begin{array}{c} \curvearrowright \\ \leftarrow D'' \\ \oplus \\ \curvearrowright \end{array} U''_2 ,$$

with U''_1 (resp., U''_2) being spanned by graphs for which the vertex w has valence 3 (resp., at least 4). Now the piece of the differential

$$D'' : U''_2 \rightarrow U''_1$$

is a bijection, since there are only two ω -legs. Hence by Lemma 2.1 we have that $\text{coker}(D')$ is acyclic, and hence so are $\text{coker}(D)$ and U . This proves the proposition. $\square$

Proof of Theorem 1.2

By Propositions 6.4, 6.8, and 7.2, the composition

$$H_g \hookrightarrow (X_g^\star, \delta_{\text{split}} + \delta_{\text{join}}^\epsilon) \xrightarrow{\Phi} (X_g^\star, \delta_{\text{split}} + \delta_{\text{join}}) \hookrightarrow X_g$$

is a quasi-isomorphism. The cohomology of the right-hand side is $\text{gr}^2 H_c^\bullet(\mathcal{M}_g)$. The cohomology of H_g is expressed via Lemma 7.1 through the cohomology of the complexes $G^{(g', n')}$ that compute $\text{gr}^0 H_c^\bullet(\mathcal{M}_{g', n'})$ by Theorem 4.1, and Theorem 1.2 follows. $\square$

8. Direct summands with marked points

Let Γ be a generator for $X_{g,n}^\star$, and let us now assume that $n \geq 1$. Recall that each such generator has some number of ϵ -decorations and precisely two ω -decorations. The remaining external vertices are decorated by a bijection to $\{1, \dots, n\}$. Each component has at least one decoration from $\{\epsilon, \omega\}$.

DEFINITION 8.1

A connected component of Γ is isolated if it contains no decorations from $\{1, \dots, n, \epsilon\}$.

In other words, a component is isolated if all of its external vertices are decorated with ω . Components that do have a decoration from $\{1, \dots, n, \epsilon\}$ are *nonisolated*. We decompose $X_{g,n}^\star$ according to the number of ω -decorations that are contained in nonisolated components.

DEFINITION 8.2

Let $H_{g,n}$, $J_{g,n}$, and $K_{g,n}$ be the graded subspaces of $X_{g,n}^\star$ spanned by graphs with 0, 1, or 2 of their ω -decorations contained in nonisolated components, respectively.

No term of the differential $\tilde{\delta}$ can make an isolated connected component into a nonisolated one or vice versa, so we have a direct sum decomposition of dg vector spaces

$$(X_{g,n}^\star, \tilde{\delta}) = H_{g,n} \oplus J_{g,n} \oplus K_{g,n}. \quad (37)$$

We will study the cohomology of the summands separately. Unfortunately, we have nothing to say about $H(K_{g,n})$.

8.1. The cohomology of $H_{g,n}$

The summand $H_{g,n}$ contributes to the cohomology of $X_{g,n}^\star$ only when $n = 1$, and that contribution is well understood.

LEMMA 8.3

We have $H_{g,1} \cong H_g$ and $H_{g,n} = 0$ for $n \geq 2$.

Proof

By the definition of $X_{g,n}^*$, the nonisolated components that do not contain ω -decorations consist of either an (ϵ, ϵ) -edge, an (ϵ, j) -edge, or one of each. If $n \geq 2$, then none of the generators for $X_{g,n}^*$ are in $H_{g,n}$. Moreover, the generators for $H_{g,1}$ are precisely the graphs of the form

$$\bigcirc \Gamma \quad \epsilon - 1 ,$$

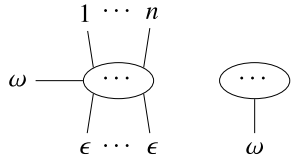
where Γ is a (possibly disconnected) generator for H_g , and the lemma follows. $\square$

We remark that the identification $H_{g,1} \cong H_g$ of the lemma is induced by the pullback under the forgetful map $\overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ (see Appendix A.1 below).

Combining Lemmas 7.1 and 8.3, we can express the cohomology of $H_{g,n}$ through the graph cohomology $H(G^{(g',n')})$, or equivalently through $W_0 H_c(\mathcal{M}_{g',n'})$.

8.2. The graphs in $J_{g,n}$

Each generator for $J_{g,n}$ has exactly one isolated component, and can be drawn as:



The differential acts by δ_{split} on the isolated component, so $J_{g,n}$ decomposes as a direct sum of subcomplexes determined by the genus of the isolated component. The subcomplex where the nonisolated part has genus h can be written as a tensor product $J_{h,n}^* \otimes G^{(h',1)}[-1]$, where $h + h' = g$ and $J_{h,n}^*$ is a graph complex perfectly analogous to $X_{h,n}^*$, except that each generator has exactly one ω -decoration instead of two, and this ω -decoration must be part of a nonisolated component.¹

We then have a decomposition

$$J_{g,n} = \bigoplus_{h+h'=g} J_{h,n}^* \otimes G^{(h',1)}[-1]. \quad (38)$$

The degree shift on the factor $G^{(h',1)}$ is due to the additional structural edge that contains the ω -decoration. The degree of a generator for $J_{h,n}^*$ is the number of structural

¹The latter condition is automatically satisfied if $n \geq 2$.

edges plus 1, just as for $X_{g',n}$. The cohomology of $J_{h,n}^\star$ is difficult to evaluate when h is large; we leave its study for $h \geq 2$ to future work. Theorems 1.6 and 1.7 use only the cohomology of $J_{0,n}^\star$ and $J_{1,n}^\star$, which we now describe.

8.3. The cohomology of $J_{0,n}^\star$

We now explain how the cohomology of $J_{0,n}^\star$ is closely related to $W_0H_c^\bullet(\mathcal{M}_{0,n})$ and $W_0H_c^\bullet(\mathcal{M}_{0,n+1})$. Let $V_1 \subset J_{0,n}^\star$ be the subcomplex generated by graphs with no ϵ -decorations. Each generator for V_1 is a rooted tree with n leaves labeled $\{1, \dots, n\}$ and a root labeled ω . Let V_0 be the complementary graded subspace generated by graphs with an ϵ -decoration. Since the genus is zero, there are no (ϵ, ϵ) -edges, and no connected component can have both an ϵ -decoration and also an ω -decoration. So each generator for V_0 has two connected components, an (ϵ, j) -edge and an ω -rooted tree with $(n-1)$ leaves labeled $\{1, \dots, \hat{j}, \dots, n\}$. As an immediate consequence, we see that

$$J_{0,0}^\star = 0 \quad \text{and} \quad J_{0,1}^\star \cong \mathbb{Q}[-1], \quad \text{spanned by the graph } \begin{array}{c} 1 \\ | \\ \omega \end{array}.$$

We now compute the cohomology of $J_{0,n}^\star$ for $n \geq 2$.

PROPOSITION 8.4

For $n \geq 2$, $H(J_{0,n}^\star, \delta)$ has dimension $(n-2)!$ and is concentrated in degree $n-1$. More precisely, there is an isomorphism of $\mathbb{S}_n$ -modules

$$H^{n-1}(J_{0,n}^\star, \delta) \cong H_c^{n-3}(\mathcal{M}_{0,n}) \cong \text{Lie}((n)) \otimes \text{sgn}_n$$

with the n -ary part of the cyclic Lie operad $\text{Lie}((n))$.

Proof

The differential on $J_{0,n}^\star$ splits into pieces as

$$\begin{array}{c} \curvearrowright \\ V_0 \end{array} \xrightarrow{\delta_{\text{join}}} \begin{array}{c} \curvearrowright \\ V_1 \end{array},$$

where the internal arrows on V_0 and V_1 are δ_{split} . Note that $V_1 \cong G^{(0,n+1)}[-2]$, and $(V_0, \delta_{\text{split}}) \cong \bigoplus_{j=1}^n G^{(0,n)}[-2]$. Therefore, $H(V_1) \cong W_0H_c(\mathcal{M}_{0,n+1})[-2]$ has dimension $(n-1)!$, supported in degree n , and $H(V_0, \delta_{\text{split}})$ has dimension $n(n-2)!$, supported in degree $n-1$. By Lemma 2.1(1), the first statement of Proposition 8.4 follows if we can show that

$$\delta_{\text{join}}: H(V_0, \delta_{\text{split}}) \rightarrow H(V_1)$$

is surjective. For $1 \leq j \leq n$, let $D_j \subset \mathcal{M}_{0,n+1}$ be the locally closed divisor parameterizing 1-nodal curves where j and ω collide. Then $D_j \cong \mathcal{M}_{0,n}$, the union $\mathcal{M}_{0,n+1} \sqcup$

$(\bigsqcup_j D_j)$ is open in $\overline{\mathcal{M}}_{0,n+1}$, and $\bigsqcup_j D_j$ is closed in this union. Then δ_{join} is naturally identified with the excision coboundary map

$$\delta: \bigoplus_j W_0 H_c^\bullet(D_j) \rightarrow W_0 H_c^{\bullet+1}(\mathcal{M}_{0,n+1}).$$

We must show that δ is surjective. We will give a short algebraic proof of this surjectivity.

To this end, it will be convenient to work with the $\mathbb{S}_n$ -modules

$$\tilde{J}_{0,n}^\star := J_{0,n}^\star \otimes \text{sgn}_n.$$

Tensoring with sgn_n can be nicely incorporated in the sign conventions on graph complexes; we take generators for $\tilde{J}_{0,n}^\star$ to be graphs with a total ordering of all edges (not just the structural edges), and impose the relation that permuting the edges induces multiplication by the sign of the permutation. Decompose $\tilde{J}_{0,n}^\star \cong \tilde{V}_0 \oplus \tilde{V}_1$, as above.

Then we can identify the cohomology groups with subspaces of free Lie algebras

$$H(\tilde{V}_0, \tilde{\delta}_{\text{split}}) = \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n)[-n], \quad H(\tilde{V}_1) = \text{Lie}(x_1, \dots, x_n)[-n-1]$$

with $\text{Lie}(x_1, \dots, x_n)$ the part of the free Lie algebra with generators $x_1, \dots, x_n$ for which each generator appears exactly once. Here the identification between (trivalent) trees and Lie words is such that the root of our tree is taken to be ω , every vertex is replaced by one Lie bracket, and the j -labeled leaf is replaced by x_j ; for example,

$$\begin{array}{c} \omega \\ | \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \quad \quad 1 \quad 3 \end{array} \rightarrow [x_2, [x_1, x_3]].$$

The differential is then given by

$$(F_1, \dots, F_n) \rightarrow \sum_{j=1}^n [x_j, F_j], \tag{39}$$

which is surjective, as required.

For the second statement of Proposition 8.4, we take for granted the well-known fact that $H_c^{n-3}(\mathcal{M}_{0,n}) \cong \text{Lie}((n)) \otimes \text{sgn}_n$. It then remains to check that the kernel of

(39) can be identified with $\text{Lie}((n))$ as an $\mathbb{S}_n$ -module. We may do this by providing an injective map

$$\iota : \text{Lie}((n)) \rightarrow \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n)$$

whose image is in the kernel of (39). To construct ι , we use the following notation for elements of $\text{Lie}((n))$. We have that $\text{Lie}((n)) \cong \text{Lie}(x_1, \dots, x_{n-1})$ and we write

$$f(x_1, \dots, x_n) := \langle f_n(x_1, \dots, x_{n-1}), x_n \rangle \in \text{Lie}((n))$$

for the element corresponding to a Lie expression $f_n(x_1, \dots, x_{n-1}) \in \text{Lie}(x_1, \dots, x_{n-1})$. The $\mathbb{S}_n$ -action on $\text{Lie}((n))$ is defined by permuting variables, considering $\langle -, - \rangle$ formally as an invariant inner product. In particular, for each j we have Lie expressions $f_j(x_1, \dots, \hat{x}_j, \dots, x_n) \in \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n)$ such that

$$f(x_1, \dots, x_n) = \langle f_j(x_1, \dots, \hat{x}_j, \dots, x_n), x_j \rangle.$$

We then define the map ι such that

$$\iota(f) := (f_1, f_2, \dots, f_n) \in \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n).$$

It is clear that ι is an injection since f may be recovered from any of the f_j .

We are hence left with checking that the composition of ι and (39) is zero. To this end, note that in $\text{Lie}((n+1))$ we have the equality

$$\begin{aligned} \langle [x_j, f_j(x_1, \dots, \hat{x}_j, \dots, x_n)], x_{n+1} \rangle &= -\langle f_j(x_1, \dots, \hat{x}_j, \dots, x_n), [x_j, x_{n+1}] \rangle \\ &= f(x_1, \dots, [x_{n+1}, x_j], \dots, x_n) \\ &= \langle f_n(x_1, \dots, [x_{n+1}, x_j], \dots, x_{n-1}), x_n \rangle, \end{aligned}$$

for $j = 1, \dots, n-1$, using invariance of our formal inner product $\langle -, - \rangle$ and the definition of the f_j . Thus in $\text{Lie}((n+1))$ we have that

$$\begin{aligned} &\left\langle \sum_{j=1}^n [x_j, f_j(x_1, \dots, \hat{x}_j, \dots, x_n)], x_{n+1} \right\rangle \\ &= \left\langle -[x_{n+1}, f_n(x_1, \dots, x_{n-1})] + \sum_{j=1}^{n-1} f_n(x_1, \dots, [x_{n+1}, x_j], \dots, x_{n-1}), x_n \right\rangle \\ &= 0 \end{aligned}$$

by the Jacobi identity. Hence $\sum_{j=1}^n [x_j, f_j(x_1, \dots, \hat{x}_j, \dots, x_n)] = 0$ as desired and the second statement of Proposition 8.4 follows. $\square$

The second statement of Proposition 8.4 was kindly pointed out to us by the anonymous referee, whom we thank for her or his contribution.

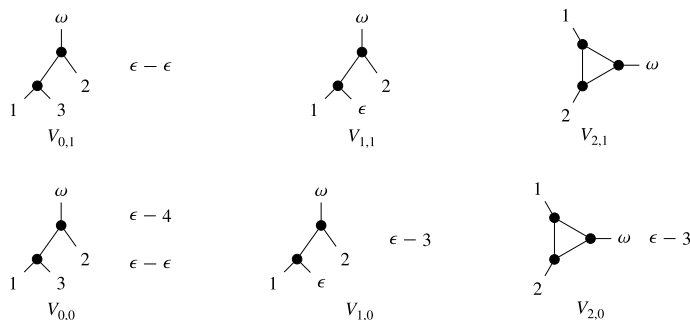
Example 8.5

In particular, Proposition 8.4 states that $H(J_{0,2}^\star)$ is 1-dimensional, concentrated in degree 1. Tracing the proof, one can see that a representative is given by the cocycle

$$\begin{array}{ccc} \omega & \text{---} & 1 \\ \epsilon & \text{---} & 2 \end{array} - \begin{array}{ccc} \omega & \text{---} & 2 \\ \epsilon & \text{---} & 1 \end{array}.$$

8.4. The cohomology of $J_{1,n}^\star$

Each generator for $J_{1,n}^\star$ has one of six combinatorial types: with or without an (ϵ, j) -edge, and with 0, 1, or 2 other ϵ -decorations. The following diagram depicts one generator of each type (for varying values of n). We denote the corresponding graded subspaces of $J_{1,n}^\star$ by $V_{i,k}$, as shown. The notation is chosen so that a generator for $V_{i,k}$ has $1-k$ edges of type (ϵ, j) and $2-i$ ϵ -decorations on other components:



Let us filter $J_{1,n}^\star$ by the number of ϵ -decorations and consider the associated spectral sequence. On the E_0 -page (the associated graded), the differential is δ_{split} , which preserves each of the six combinatorial types. So E_1 is the direct sum of the homologies of the associated graph complexes, with respect to δ_{split} . Each term then has an evident interpretation in terms of weight 0 cohomology of moduli spaces; for instance,

$$H(V_{1,0}) \cong \bigoplus_{j=1}^n W_0 H_c(\mathcal{M}_{0,n+1})[-3] \quad \text{and} \quad H(V_{2,1}) \cong W_0 H_c(\mathcal{M}_{1,n+1})[-2].$$

Note that the cohomology of $V_{i,k}$ is supported in degree $n + i + k$.

The differential on E_1 is the part of δ_{join} that reduces the number of ϵ -decorations by exactly 1. This further decomposes as two parts, one that eliminates an (ϵ, j) -edge, and one that decreases the number of other ϵ -decorations by 1. Therefore, the E_1 -page is the total complex of a diagram of the following form:

$$\begin{array}{ccccc}
 W_0 H_c(\mathcal{M}_{0,n+1})[-3] & \longrightarrow & W_0 H_c(\mathcal{M}_{0,n+2})[-3] & \longrightarrow & W_0 H_c(\mathcal{M}_{1,n+1})[-2] \\
 \uparrow & & \uparrow & & \uparrow \\
 \bigoplus_{j=1}^n W_0 H_c(\mathcal{M}_{0,n})[-3] & \longrightarrow & \bigoplus_{j=1}^n W_0 H_c(\mathcal{M}_{0,n+1})[-3] & \longrightarrow & \bigoplus_{j=1}^n W_0 H_c(\mathcal{M}_{1,n})[-2].
 \end{array} \quad (40)$$

The arrows are once again coboundary maps in excision sequences arising from stratifications of moduli spaces.

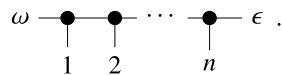
As in our computation of $H(J_{0,n}^\star)$, we find it helpful to tensor once again with sgn_n and then give an algebraic interpretation for the resulting diagram. Let $\tilde{J}_{1,n}^\star := J_{1,n}^\star \otimes \text{sgn}_n$, with differential $\tilde{\delta} = \tilde{\delta}_{\text{split}} + \tilde{\delta}_{\text{join}}$. Similarly, let $\tilde{V}_{i,k} := V_{i,k} \otimes \text{sgn}_n$. Recall that tensoring with sgn_n amounts to equipping each generator with a total ordering of all of its edges, not just the structural edges, and imposing the usual relation that reordering the edges is multiplication by the sign of the induced permutation. We then have

$$H(\tilde{V}_{0,1}, \tilde{\delta}_{\text{split}}) \cong \text{Lie}(x_1, \dots, x_n)[-n-1],$$

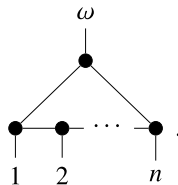
$$H(\tilde{V}_{1,1}, \tilde{\delta}_{\text{split}}) \cong \text{Assoc}(x_1, \dots, x_n)[-n-2], \quad (41)$$

$$H(\tilde{V}_{2,1}, \tilde{\delta}_{\text{split}}) \cong \text{Assoc}(x_1, \dots, x_n)_{\mathbb{S}_2}[-n-3]. \quad (42)$$

Here, $\text{Assoc}(x_1, \dots, x_n)$ is the subspace of the free associative algebra spanned by words in which each variable appears exactly once, and $\mathbb{S}_2$ acts by reversing each word (corresponding to the symmetry reversing the orientation of a based loop). Under the identification (41), the associative word $x_1 \cdots x_n$ corresponds to the graph



Similarly, under the identification (42) the associative word $x_1 \cdots x_n$ corresponds to the graph



The cohomology groups in the second row of (40) are similar, but one must take a direct sum over the decorations j that appear in an (ϵ, j) -edge, and omit the variable x_j , that is, $H(\tilde{V}_{0,0}, \delta_{\text{split}}) \cong \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n)[-n]$, and so on.

With these identifications, the E_1 -page of our spectral sequence (see diagram (40)) hence becomes isomorphic to the total complex of the following diagram, from which we omit degree shifts for notational brevity:²

$$\begin{array}{ccccc} \text{Lie}(x_1, \dots, x_n) & \longrightarrow & \text{Assoc}(x_1, \dots, x_n) & \longrightarrow & \text{Assoc}(x_1, \dots, x_n)_{\mathbb{S}_2} \\ \uparrow & & \uparrow & & \uparrow \\ \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n) & \longrightarrow & \bigoplus_{j=1}^n \text{Assoc}(x_1, \dots, \hat{x}_j, \dots, x_n) & \longrightarrow & \bigoplus_{j=1}^n \text{Assoc}(x_1, \dots, \hat{x}_j, \dots, x_n)_{\mathbb{S}_2}. \end{array} \quad (43)$$

Each vertical arrow in (43) is defined by the formula

$$(F_1, \dots, F_n) \mapsto \sum_{j=1}^n [x_j, F_j]. \quad (44)$$

The horizontal arrows are the canonical inclusions and projections, so that the cohomology of each row of (43) is concentrated in the middle term. It is hence easy to check (for example, using Lemma 2.1 again) that the total complex of (43) is quasi-isomorphic to the two-term complex built from its row-wise cohomology. It hence remains to study these middle cohomology groups of each row, and the induced map between them.

To this end, let $\text{Pois}(x_1, \dots, x_n)$ be the part of the free Poisson algebra in $x_1, \dots, x_n$ in which each x_j appears exactly once. Its elements are linear combinations of Poisson expressions of the form

$$F(x_1, \dots, x_n) = f_1(x_1, \dots, x_n) \wedge \cdots \wedge f_k(x_1, \dots, x_n),$$

where the f_j are Lie words, such that each variable x_j appears exactly once in F . Let $\text{Pois}^k(x_1, \dots, x_n)$ be the subspace spanned by expressions that are products of exactly k Lie words. There is a natural map

$$D_n^k : \bigoplus_{j=1}^n \text{Pois}^k(x_1, \dots, \hat{x}_j, \dots, x_n) \rightarrow \text{Pois}^k(x_1, \dots, x_n), \quad (45)$$

given again by the expression (44), that is, $(F_1, \dots, F_n) \mapsto \sum_{j=1}^n [x_j, F_j]$. Let us define

$$\mathbb{A}_n^k := \ker D_n^k, \quad \mathbb{B}_n^k := \text{coker } D_n^k$$

²The bottom left space is concentrated in degree n , and each arrow increases the degree by 1.

and

$$\mathbb{A}_n := \bigoplus_{j \geq 1} \mathbb{A}_n^{2j+1}, \quad \mathbb{B}_n := \bigoplus_{j \geq 1} \mathbb{B}_n^{2j+1}. \quad (46)$$

All objects here are naturally $\mathbb{S}_n$ -representations, by changing the variable indices and order of summands on the left-hand side of (45). For example, we have $\mathbb{A}_1 = \mathbb{B}_1 = \mathbb{A}_2 = \mathbb{B}_2 = \mathbb{A}_3 = 0$, $\mathbb{B}_3$ and $\mathbb{A}_4$ are 1-dimensional trivial representations, and $\mathbb{B}_4$ is a 3-dimensional irreducible representation of $\mathbb{S}_4$.

PROPOSITION 8.6

We have $J_{1,0}^\star = 0$. For $n \geq 1$, we have an isomorphism of graded $\mathbb{S}_n$ -modules

$$H(J_{1,n}^\star, \delta) \cong (\mathbb{A}_n[-n-1] \oplus \mathbb{B}_n[-n-2]) \otimes \text{sgn}_n.$$

Proof

Equivalently, we must show that $H(\tilde{J}_{1,n}^\star, \tilde{\delta}) \cong \mathbb{A}_n[-n-1] \oplus \mathbb{B}_n[-n-2]$. Poincaré–Birkhoff–Witt gives an $\mathbb{S}_n$ -equivariant isomorphism $\text{Pois}(x_1, \dots, x_n) \cong \text{Assoc}(x_1, \dots, x_n)$. Applying this to the center and right-hand columns of (43), we get the diagram

$$\begin{array}{ccccc} \text{Lie}(x_1, \dots, x_n) & \longrightarrow & \text{Pois}(x_1, \dots, x_n) & \longrightarrow & \text{Pois}(x_1, \dots, x_n)_{\mathbb{S}_2} \\ \uparrow & & \uparrow & & \uparrow \\ \bigoplus_{j=1}^n \text{Lie}(x_1, \dots, \hat{x}_j, \dots, x_n) & \longrightarrow & \bigoplus_{j=1}^n \text{Pois}(x_1, \dots, \hat{x}_j, \dots, x_n) & \longrightarrow & \bigoplus_{j=1}^n \text{Pois}(x_1, \dots, \hat{x}_j, \dots, x_n)_{\mathbb{S}_2}. \end{array} \quad (47)$$

The vertical arrows in (47) are still given by (44). Furthermore, note that taking $\mathbb{S}_2$ -coinvariants in the right-hand column is the same as reducing to Poisson words containing evenly many Lie words; that is,

$$\text{Pois}(x_1, \dots, x_n)_{\mathbb{S}_2} \cong \bigoplus_{j \geq 0} \text{Pois}^{2j}(x_1, \dots, x_n).$$

Then the total complex of (47) is quasi-isomorphic to the complex of its row-wise cohomology groups:

$$\begin{array}{ccc} \bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, x_n) & & (48) \\ \uparrow \Sigma_k D_n^k & & \\ \bigoplus_{j=1}^n \bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, \hat{x}_j, \dots, x_n) & & \end{array}$$

The cohomology of (48) is the kernel plus the cokernel of the differential $\sum_k D_n^k$, that is, the direct sum of $\mathbb{A}_n$ and $\mathbb{B}_n$, as defined above. The proposition follows, after suitably accounting for the required degree shifts. $\square$

The above description of the spaces $\mathbb{A}_n$ and $\mathbb{B}_n$ is not very explicit. However, we can at least provide the following lower bound on the dimensions.

LEMMA 8.7

For $n \geq 3$, we have

$$\dim \mathbb{B}_n \geq (n-2)!.$$

Proof

From the definition of $\mathbb{B}_n$, we have

$$\begin{aligned} \dim \mathbb{B}_n &\geq \dim \left(\bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, x_n) \right) \\ &\quad - \dim \left(\bigoplus_{j=1}^n \bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, \hat{x}_j, \dots, x_n) \right). \end{aligned}$$

From the top row of (43), we have

$$\begin{aligned} \dim \left(\bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, x_n) \right) &= \dim(\text{Assoc}(x_1, \dots, x_n)) \\ &\quad - \dim(\text{Assoc}(x_1, \dots, x_n)_{\mathbb{S}_2}) \\ &\quad - \dim(\text{Lie}(x_1, \dots, x_n)) \\ &= n! - \frac{n!}{2} - (n-1)!, \end{aligned}$$

which simplifies to $\frac{1}{2}(n-2)(n-1)!$. Here, we have used that the action of $\mathbb{S}_2$ on associative words of length at least 2 is faithful in computing $\dim(\text{Assoc}(x_1, \dots, x_n)_{\mathbb{S}_2}) = \frac{n!}{2}$. By the same computation (valid for $n \geq 3$) we also see that

$$\dim \left(\bigoplus_{j=1}^n \bigoplus_{k \geq 1} \text{Pois}^{2k+1}(x_1, \dots, \hat{x}_j, \dots, x_n) \right) = \frac{1}{2}n(n-3)(n-2)!.$$

Hence

$$\dim \mathbb{B}_n \geq \frac{1}{2}((n-2)(n-1) - n(n-3))(n-2)!,$$

which is $(n-2)!$, as required. $\square$

Proof of Theorems 1.6 and 1.7

By (37), we know that $H(H_{g,n}) \oplus H(J_{g,n})$ injects into the cohomology of $X_{g,n}^*$. The latter is identified with the cohomology of $X_{g,n}$, and hence with $\text{gr}_2 H_c(\mathcal{M}_{g,n})$. The

summand $H(H_{g,n})$ is evaluated by Lemma 8.3. It contributes only for $n = 0, 1$. For $n = 1$, it produces an injection of the form (3). This injection agrees with the pullback map π^* , as shown in Appendix A.1.

For the summand $H(J_{g,n})$, we can use that by the decomposition (38) we have an injection of

$$H(J_{0,n}^\star) \otimes H(G^{(g,1)})[-1] \oplus H(J_{1,n}^\star) \otimes H(G^{(g-1,1)})[-1] \quad (49)$$

into its cohomology. The second factors in the tensor products compute the weight 0 cohomology of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g-1,1}$, respectively. The first factors $H(J_{0,n}^\star)$ and $H(J_{1,n}^\star)$ are evaluated by Propositions 8.4 and 8.6, respectively.

For $n = 1$, one only has a contribution from $H(J_{0,1}^\star) \cong \mathbb{Q}[-1]$, while $H(J_{1,1}^\star) = 0$. This produces the injection (2). To see that this injection indeed corresponds to multiplication with the ψ -class at the marking as is claimed in Theorem 1.6, we have to trace the representatives. Say that $\Gamma \in G^{(g,1)}$ is a cocycle representing a cohomology class in $W_0 H_c^k(\mathcal{M}_{g,1})$. Then the corresponding class in $H(X_{g,1}^\star, \tilde{\delta})$ is represented by

$$\Gamma' = \begin{array}{c} \textcircled{\Gamma} \\ | \\ \omega \end{array} \quad \omega - 1,$$

The action of the morphism Ξ on this cocycle is trivial since there are no ϵ -legs. Hence Γ' is also a cocycle in $(X_{g,1}, \tilde{\delta})$ and in $X_{g,1}$. From (33), we see that the corresponding class in $\widetilde{\mathbf{GK}}_{g,1}^2$ is obtained, up to a sign, by decorating the unique leg in Γ with a ψ -class. We claim that this is the graphical representation of multiplying with a ψ -class at the marking in $H_c^\bullet(\mathcal{M}_{g,1})$. To see this, note that $H^\bullet$ acts on $H_c^\bullet$ via cup product, and recall that the weight spectral sequence abutting to $\mathrm{gr} H^\bullet(\mathcal{M}_{g,1})$ is induced by a natural filtration on the sheaf of smooth differential forms with logarithmic poles along the boundary of $\overline{\mathcal{M}}_{g,1}$ (see [25, Chapter 4, Section 3]). It is related via Poincaré duality to the spectral sequence that we study here, abutting to $\mathrm{gr} H_c^\bullet(\mathcal{M}_{g,1})$, which is likewise induced by a natural filtration on currents. The ψ -class is of pure weight 2 in $H^2(\overline{\mathcal{M}}_{g,1})$ and is represented by a smooth and closed differential form η without logarithmic poles. The cup product $\psi \cup: \mathrm{gr}_\bullet H_c^\bullet(\mathcal{M}_{g,1}) \rightarrow \mathrm{gr}_{\bullet+2} H_c^{\bullet+2}(\mathcal{M}_{g,1})$ is therefore induced by the action of η on currents, which also induces a compatible action on every term on every page in our weight spectral sequence. The ξ_Γ -pullback of ψ is the corresponding ψ -class on $\overline{\mathcal{M}}_\Gamma$, and it follows that $\psi \cup$ is induced by multiplication by ψ on every term of $\mathbf{GK}_{g,1}$. The claim follows by descending from $\mathbf{GK}_{g,1}^2$ to $\overline{\mathbf{GK}}_{g,1}^2$ and then lifting to $\widetilde{\mathbf{GK}}_{g,1}^2$. This finishes the proof of Theorem 1.6.

To show Theorem 1.7, we consider again the injection from (49) to $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$, for $n \geq 2$. Then Proposition 8.4 directly produces the first summand in the formula of Theorem 1.7, and Lemma 8.6 produces the second summand. The assertion on the dimension of $\mathbb{B}_n$ is shown in Lemma 8.7. $\square$

Remark 8.8

Proposition 8.6 yields an additional summand to $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ corresponding to the subspace

$$(\mathbb{A}_n[-n-1] \otimes \mathrm{sgn}_n) \subset H(J_{1,n}^\star).$$

Lacking lower bounds on the dimension of $\mathbb{A}_n$, we omitted this summand from the statement of Theorem 1.7.

8.4.1. Examples

Let us illustrate Theorem 1.7 with pictures of the simplest nontrivial cohomology classes that it provides. First, consider the first summand in Theorem 1.7 for $(g, n) = (3, 2)$. This summand has dimension 1, concentrated in degree 8. A generating cocycle in $(X_{3,2}^\star, \delta)$ is given by the linear combination

$$\begin{array}{ccc} \omega - 1 & & \omega - 2 \\ \epsilon - 2 & & \epsilon - 1 \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \omega \end{array} - \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \omega \end{array}$$

Here the component with internal vertices is the generator of $H^6(G^{(3,1)})$, which is isomorphic to $sW_0 H_c^6(\mathcal{M}_{3,1})$. The disjoint union of two isolated edges corresponds to the generator of $H(J_{0,2}^\star)$ of Example 8.5. (Note that $H(J_{0,2}^\star)$ has dimension 1, by Proposition 8.4.) The corresponding cocycle in $(X_{3,2}^\star, \delta)$ is obtained by applying the involution Ξ . Up to a conventional overall sign that we omit, it is:

$$\begin{array}{ccc} \omega - 1 & & \omega - 1 \\ \epsilon - 2 & & 2 \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \omega \end{array} + 3 \cdot \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \omega \end{array} + \begin{array}{ccc} \omega - 1 & & \omega - 1 \\ & & 2 \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \omega \end{array} - (1 \leftrightarrow 2).$$

Here “ $(1 \leftrightarrow 2)$ ” stands for the same terms, with the labels 1 and 2 interchanged. The corresponding cocycle in $\mathrm{GK}_{3,2}^2$ is then

$$3 \cdot \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ 1 \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \\ 1 \end{array} - (1 \leftrightarrow 2).$$

By [34], we know that $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{3,2}) \cong \mathbb{Q} \oplus \mathrm{sgn}_2$, supported in degree 8. Our computation shows that the summand sgn_2 is naturally identified with $H(J_{0,2}^\star) \otimes H(G^{(3,1)})[-1]$.

Next, we turn to the second summand in Theorem 1.7. The first nontrivial class is found in $\mathrm{gr}_2 H_c^{12}(\mathcal{M}_{4,3})$. It corresponds to the generator of $\mathbb{B}_3 \otimes H^6(G^{(3,n)})[-1]$, and is represented by the following degree 12 cocycle in $(X_{4,3}^\star, \tilde{\delta})$:

$$\sum_{\sigma \in \mathbb{S}_3} \mathrm{sgn}(\sigma) \begin{array}{c} \omega \\ \diagup \quad \diagdown \\ \sigma(1) \quad \bullet \\ \diagup \quad \diagdown \\ \sigma(2) \quad \bullet \\ \diagup \quad \diagdown \\ \sigma(3) \quad \epsilon \end{array} \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \omega \end{array}.$$

Finally, as explained in Remark 8.8, there is one more summand of $\mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ coming from $\mathbb{A}_n$ which is not mentioned in Theorem 1.7. The first case where this summand is nontrivial arises when $(g, n) = (4, 4)$. We then have $\mathbb{A}_4 \cong \mathbb{Q}$, and the graph cocycle in $(X_{4,4}^\star, \tilde{\delta})$ corresponding to $\mathbb{A}_4 \otimes H^6(G^{(3,1)})[-1]$ has the form

$$\sum_{\sigma \in \mathbb{S}_4} \mathrm{sgn}(\sigma) \begin{array}{c} \omega \\ \diagup \quad \diagdown \\ \sigma(1) \quad \bullet \\ \diagup \quad \diagdown \\ \sigma(2) \quad \bullet \\ \diagup \quad \diagdown \\ \sigma(3) \quad \epsilon \end{array} \quad \begin{array}{c} \sigma(4) - \epsilon \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \\ \omega \end{array}.$$

Appendix A. A direct map $X_{g,n} \rightarrow \overline{\mathrm{GK}}_{g,n}^2$

We have connected the complexes $X_{g,n}$ and $\overline{\mathrm{GK}}_{g,n}^2$ through a zigzag of quasi-isomorphisms. Here, we construct a direct map $F: X_{g,n} \rightarrow \overline{\mathrm{GK}}_{g,n}^2$ that fits into a homotopy commutative triangle with the maps in the zigzag.

The map F is useful for giving geometrically meaningful representatives in $\overline{\mathrm{GK}}_{g,n}^2$ of the cohomology classes we have constructed in $X_{g,n}$. By Theorem 3.3, for $(g, n) \neq (1, 1)$, the projection $\mathrm{GK}_{g,n}^2 \rightarrow \overline{\mathrm{GK}}_{g,n}^2$ is a quasi-isomorphism. Hence each cocycle in $\overline{\mathrm{GK}}_{g,n}^2$ lifts to a cocycle on $\mathrm{GK}_{g,n}^2$, uniquely up to exact terms. We now turn to the construction of the map F , which we define initially as a map of graded vector spaces.

Let Γ be a generator for $X_{g,n}$ with e structural edges. First, suppose that Γ has no ϵ -decorations. At least one of the ω -decorations must be adjacent to an internal vertex. If both ω -decorations are adjacent to internal vertices, then $F(\gamma)$ is obtained by joining the ω -legs to form a new edge, and decorating either half-edge by ψ_i , in the antisymmetric linear combination:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} \mapsto (-1)^e \frac{1}{2} \left(\begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} - \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} \right) \quad (50)$$

To fix the sign, in this case and in all following cases, draw Γ so that the left-hand ω -edge precedes the right-hand ω -edge in the ordering. Then the new edge comes first, the relative order of the remaining edges is preserved, and the sign is as indicated. Otherwise, if there are no ϵ -decorations and only one ω -decoration is adjacent to an internal vertex, then F is given simply by

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \omega \end{array} \begin{array}{c} j \\ | \\ \omega \end{array} \mapsto -(-1)^e \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} j \\ | \\ \omega \end{array}$$

It remains to consider the cases where Γ has a positive number of ϵ -decorations. First, suppose that both of the ω -decorations are adjacent to internal vertices. By the orientation relations, we may assume that these internal vertices are distinct. Call them v and v' . Then let Γ_v be the graph formed by making v the special vertex, attaching all ϵ - and ω -decorated legs to that vertex and decorating it by $\delta_{0,S}$ with S the union of the half-edges adjacent to the ϵ -decorations and the half-edge toward v' . Then $F(\Gamma) = \frac{1}{2}(\Gamma_v - \Gamma_{v'})$. Pictorially:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array} \mapsto (-1)^e \frac{1}{2} \left(\begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} - \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} \omega \\ \omega \end{array} \right). \quad (51)$$

Next, if exactly one ω -decoration is adjacent to an internal vertex v , then we define Γ_v as above and set $F(\gamma) = \frac{1}{2}\Gamma_v$:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \omega \end{array} \begin{array}{c} j \\ | \\ \omega \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array} \mapsto (-1)^e \frac{1}{2} \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \begin{array}{c} j \\ | \\ \omega \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array} \begin{array}{c} \epsilon \\ | \end{array}.$$

Finally, if neither ω -decoration is adjacent to an internal vertex, then we set $F(\gamma) = 0$.

PROPOSITION A.1

The map $F : X_{g,n} \rightarrow \overline{\mathbf{GK}}_{g,n}^2$ is a map of complexes and fits into a homotopy commutative triangle

$$\begin{array}{ccc}
 \widetilde{\mathbf{GK}}_{g,n} & \xrightarrow{\pi} & \overline{\mathbf{GK}}_{g,n}^2 \\
 \Phi \downarrow & \nearrow F & \\
 X_{g,n} & &
 \end{array}$$

Proof

We begin by constructing a degree -1 map of graded vector spaces

$$h: \widetilde{\mathbf{GK}}_{g,n}^2 \rightarrow \overline{\mathbf{GK}}_{g,n}^2.$$

and show that it satisfies the required equation (52) to give a chain homotopy from $F \circ \Phi$ to π . We will then use h to show that F is a map of complexes, and thereby prove the proposition.

Let Γ be a generator for $\widetilde{\mathbf{GK}}_{g,n}^2$. We define $h(\Gamma)$ by a local replacement at the special vertex as follows. Suppose that the special vertex of Γ is decorated by ψ_i , and i is not joined to a leg, so the half-edge is joined to a half-edge at an internal vertex v . Then we let $h(\Gamma)$ be the graph obtained by merging the special vertex with v , and decorating the new special vertex by $\frac{1}{2}\delta_{0,S}$, where S is the set of half-edges incident to v . Pictorially:

$$h: \begin{array}{c} \nearrow \circ \rightarrow \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array} \mapsto \frac{1}{2} \begin{array}{c} \nearrow \circ + \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array}$$

To fix the sign, we assume here that the ψ -decorated edge is first in the ordering. In all other cases we set $h(\Gamma) = 0$.

We claim that h satisfies the required equation to give a chain homotopy from $F \circ \Phi$ to π , that is:

$$F(\Phi(\Gamma)) - \pi(\Gamma) = \delta h(\Gamma) + h(\delta(\Gamma)). \quad (52)$$

We prove this case by case, according to the decoration on the special vertex of Γ .

First, suppose that the decoration on the special vertex of the graph $\Gamma \in \widetilde{\mathbf{GK}}_{g,n}$ is $\delta_{0,S}$. Then $\Phi(\Gamma) = 0$ and $h(\Gamma) = 0$. Furthermore, $\pi(\Gamma)$ is the class in $\overline{\mathbf{GK}}_{g,n}^2$ represented by the decorated graph Γ . Note that $\delta\Gamma$ consists of some terms with $\delta_{0,T}$ -decorations, plus terms with ψ -decorations, and only the latter contribute upon applying h . Furthermore, they contribute exactly Γ . Pictorially:

$$h(\delta \begin{array}{c} \nearrow \circ + \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array}) = h \left(- \begin{array}{c} \nearrow \circ \rightarrow \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array} - \begin{array}{c} \nearrow \circ \leftarrow \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array} + (\dots) \right) = -2 \frac{1}{2} \begin{array}{c} \nearrow \circ + \circ \nwarrow \\ \nearrow \quad \quad \nwarrow \end{array},$$

with $(\dots)$ representing terms killed by $h(-)$.

Next, suppose that the special vertex of Γ is decorated by ψ_i , with i a half-edge incident to the special vertex. Then $\pi(\Gamma)$ is once again the class in $\widetilde{\mathbf{GK}}_{g,n}^2$ represented by the decorated graph Γ . If the half-edge i connects to a leg, then $F(\Phi(\Gamma)) = \Gamma$ and $h(\Gamma) = h(\delta\Gamma) = 0$, so (52) holds.

If the half-edge i points toward an internal vertex, then $F(\Phi(\Gamma))$ is obtained by antisymmetrizing over the two ψ -decorations one can put on the i -edge,

$$F(\Phi(\Gamma)) = \frac{1}{2} \left(\begin{array}{c} \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} - \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \circ \text{---} \end{array} \right).$$

In this case, we have

$$\delta h(\Gamma) + h(\delta\Gamma) = -\frac{1}{2} \left(\begin{array}{c} \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \circ \text{---} \end{array} \right),$$

so (52) holds.

Finally, suppose that the special vertex of Γ is decorated by E_{ij} , with i, j half-edges incident to the special vertex. Then $\pi(\Gamma) = 0$ and $h(\Gamma) = 0$. We must show that $F(\Phi(\Gamma)) = h(\delta\Gamma)$, and we consider cases according to whether i and j are joined to legs. If both are joined to legs, then $F(\Phi(\Gamma)) = h(\delta\Gamma) = 0$. If only j is joined to a leg, then one computes, with e the number of structural edges of Γ ,

$$F(\Phi(\Gamma)) = (-1)^e \frac{1}{2} \begin{array}{c} \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}, \quad \text{and} \quad h(\delta\Gamma) = h(\delta_{res}\Gamma) = (-1)^e \frac{1}{2} \begin{array}{c} \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array},$$

as required. If neither i nor j connects to a leg, then they connect to distinct internal vertices and one computes

$$F(\Phi(\Gamma)) = (-1)^e \frac{1}{2} \left(\begin{array}{c} \text{---} \circ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} - \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \circ \text{---} \end{array} \right) = h(\delta\Gamma).$$

This completes the proof of (52).

Having proved (52), we have that for all $\Gamma \in \widetilde{\mathbf{GK}}_{g,n}$,

$$\delta F(\Phi(\Gamma)) - F(\delta\Phi(\Gamma)) = 0.$$

By Proposition 5.5, the map Φ is surjective, and hence $\delta \circ F = F \circ \delta$. We conclude that F is a map of chain complexes, and h is chain homotopy from $F \circ \Phi$ to π , as required. $\square$

A.1. The pullback map

Recall that the projection $\pi: \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ is proper and extends to a morphism on the stable curves compactifications, also denoted $\pi: \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$. Recall also that the boundary divisors $\partial\mathcal{M}_{g,1} := \overline{\mathcal{M}}_{g,1} \setminus \mathcal{M}_{g,1}$ and $\partial\mathcal{M}_g := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ have normal crossings. Applying the “fundamental simplicial constructions” as in [14, Section II.I] gives an induced morphism π^* from the simplicial resolution of the constant sheaf $\mathbb{Q}$ on $\partial\mathcal{M}_g$ to that of the constant sheaf $\mathbb{Q}$ on $\partial\mathcal{M}_{g,1}$, induced by the normal crossings structure. After realizing the compactly supported cohomology of $\mathcal{M}_{g,1}$ and of $\mathcal{M}_g$ as the reduced cohomology of the mapping cones for the inclusions $\partial\mathcal{M}_{g,1} \rightarrow \overline{\mathcal{M}}_{g,1}$ and $\partial\mathcal{M}_g \rightarrow \overline{\mathcal{M}}_g$, respectively, we get an induced pullback morphism between the weight spectral sequences that abuts to $\pi^*: \mathrm{gr} H_c(\mathcal{M}_g) \rightarrow \mathrm{gr} H_c(\mathcal{M}_{g,1})$.

We consider the graphical interpretation of the induced map on the E_1 -page of the weight spectral sequence, that is, the induced map between the Getzler–Kapranov graph complexes. We claim that $\pi^*: \mathbf{GK}_{g,0} \rightarrow \mathbf{GK}_{g,1}$ is given by the formula

$$\pi^* \Gamma = \sum_v p_v(\Gamma), \quad (53)$$

where the sum is over all vertices of the graph $\Gamma \in \mathbf{GK}_{g,0}$, and $p_v(\Gamma)$ is obtained from Γ by attaching a leg labeled by 1 to a vertex v . If the old decoration of v is $\alpha \in H^i(\overline{\mathcal{M}}_{h,k})$, then the new decoration on the vertex is given by $\pi^* \alpha \in H^i(\overline{\mathcal{M}}_{h,k+1})$, with $\pi: \overline{\mathcal{M}}_{h,k+1} \rightarrow \overline{\mathcal{M}}_{h,k}$ forgetting the new marking. To see this, note that if Γ has j edges, then the underlying graphs of $p_v(\Gamma)$, obtained by attaching a leg labeled 1 to a vertex v of Γ , correspond precisely to the strata of codimension j that map onto the codimension j stratum $\xi_\Gamma(\mathcal{M}_\Gamma) \subset \overline{\mathcal{M}}_\Gamma$, and the operation on decorations that we have described corresponds, via the Künneth decomposition, to the induced pullback on the cohomology of strata in the simplicial resolution.

The same formula (53) also defines a pullback operation

$$\pi^*: \overline{\mathbf{GK}}_{g,0} \rightarrow \overline{\mathbf{GK}}_{g,1},$$

and the projection map $\mathbf{GK}_{g,n} \rightarrow \overline{\mathbf{GK}}_{g,n}$ intertwines the two operations π^* . We furthermore define another operation

$$\varpi^*: (H_g, \tilde{\delta}) \rightarrow (X_{g,1}^*, \tilde{\delta})$$

by taking minus the disjoint union with an $(\epsilon, 1)$ -edge,

$$\varpi^*: \bigcirc \Gamma \mapsto - \bigcirc \Gamma \quad \epsilon = 1.$$

LEMMA A.2

The following diagram commutes:

$$\begin{array}{ccccccc}
 (X_{g,1}^*, \tilde{\delta}) & \xrightarrow{\Xi} & (X_{g,1}^*, \delta) & \hookrightarrow & X_{g,1} & \xrightarrow{F} & \overline{\text{GK}}_{g,1} \\
 \varpi^* \uparrow & & & & & & \uparrow \pi^* \\
 (H_g, \tilde{\delta}) & \xrightarrow{\Xi} & (X_g^*, \delta) & \hookrightarrow & X_g & \xrightarrow{F} & \overline{\text{GK}}_g
 \end{array} \quad (54)$$

Proof sketch

A graph $\Gamma \in H_g$ contains either no ϵ -leg, or exactly two, on an (ϵ, ϵ) -edge. We consider both cases in turn. Suppose first that $\Gamma \in H_g$ contains no ϵ -leg. Then its image under the first two horizontal maps in (54) is just the same unaltered graph Γ . Applying F produces the linear combination (50). Applying π^* afterwards produces a linear combination of graphs of the form

$$-(-1)^e \frac{1}{2} \left(\sum \text{graph}_1 - \sum \text{graph}_2 - \text{graph}_3 + \text{graph}_4 \right) \quad (55)$$

The diagrams in (55) show four graph configurations. Each graph has a dashed vertical line labeled '1' at the bottom, representing a 1-labeled leg. The first two graphs are connected by a minus sign, and the last two by a plus sign. The first graph has a vertex with a loop and a 1-labeled leg. The second graph has a vertex with a loop and a 1-labeled leg. The third graph has a vertex with a loop and a 1-labeled leg. The fourth graph has a vertex with a loop and a 1-labeled leg.

In the first two sums one sums over all ways of attaching the 1-labeled leg to a vertex, including the special vertex. Here we are using [2, Proposition 3.1(ii)] to compute the pullback of the ψ -class, and the additional $\delta_{0,S}$ -terms there produce the last two summands. To compare, let us now follow our graph Γ along the lower rim of (54). The vertical arrow ϖ^* adds one $(\epsilon, 1)$ -edge to Γ . Call the resulting graph $-\Gamma' \in X_{g,1}^*$. Applying Ξ then produces Γ' plus the sum of all graphs obtained by attaching a 1-labeled leg to Γ . Applying F produces from this latter sum the two sums in (55). Applying F to Γ' produces the last two summands in (55) via (51). This shows commutativity of (54) for $\Gamma \in H_g$ without an (ϵ, ϵ) -edge. The proof in the case where $\Gamma \in H_g$ has an (ϵ, ϵ) -edge is essentially similar, although the map Ξ produces more terms in the presence of an (ϵ, ϵ) -edge. $\square$

Appendix B. Numerical results

We record the cohomology groups of the graph complexes $G^{(g,n)}$, $X_{g,n}$, and $J_{g,n}^*$, with the characters of their $\mathbb{S}_n$ -representations for small g and n , obtained by calculations in Sage. For example, the entry $t^{11}s_{2,1} + t^{12}s_{1,1,1}$ in Figure 1 for $(g,n) = (4,3)$ indicates that $W_0 H_c^{11}(\mathcal{M}_{4,3})$ is the 2-dimensional irreducible representation of $\mathbb{S}_3$ with character $s_{2,1}$, $W_0 H_c^{12}(\mathcal{M}_{4,3})$ is the sign representation, and $W_0 H_c^\bullet(\mathcal{M}_{4,3})$ vanishes in all other degrees. The full cohomology of $\mathcal{M}_{g,n}$ is well understood for $g = 0$, since $\mathcal{M}_{0,n}$ is a hyperplane arrangement complement, and also for $g = 1$ (see [17], [19]). For $g = 2$, we understand that Tommasi [35] has computed the cohomol-

g,n	0	1	2	3	4	5	6
0	-	-	-	s_3	$t^1 s_{2,2}$	$t^2 s_{3,1,1}$	$t^3 (s_{3,3} + s_{4,1,1} + s_{2,2,1,1})$
1	-	0	0	$t^3 s_{1,1,1}$	$t^4 s_{3,1}$	$t^5 (s_5 + s_{3,2} + s_{2,2,1} + s_{1,1,1,1,1})$	
2	0	0	$t^5 s_2$	0	$t^6 s_4 + t^7 s_{2,1,1}$		
3	t^6	$t^6 s_1$	0	$t^9 s_3$	$t^9 s_{2,1,1} + t^{10} s_{2,2}$		
4	0	0	0	$t^{11} s_{2,1} + t^{12} s_{1,1,1}$			
5	t^{10}	$t^{10} s_1$	$t^{14} s_2$	$t^{13} s_3 + t^{14} s_{2,1}$			
6	t^{15}	$t^{15} s_1$					

Figure 1. The $\mathbb{S}_n$ -equivariant Poincaré polynomials of $H^\bullet(G(g,n)) \cong W_0 H_c^\bullet(\mathcal{M}_{g,n})$ for small g and n .

g,n	0	1	2	3	4	5	6
0	-	-	-	0	$t^2 s_4$	$t^3 s_{3,2}$	$t^4 (s_{3,2,1} + s_{4,1,1})$
1	-	0	0	0	$t^5 s_{2,1,1}$	$t^6 (s_{3,1,1} + s_{3,2} + s_{4,1})$	
2	0	0	0	$t^7 (s_3 + s_{2,1})$	$t^8 s_4$		
3	0	$t^8 s_1$	$t^8 (s_2 + s_{1,1})$	$2t^9 s_3$			
4	0	0	0	$2t^{12} s_{1,1,1}$			
5	0	$t^{12} s_1$	$t^{12} (s_2 + s_{1,1}) + 2t^{14} s_2$				
6	0	$2t^{15} s_1 + t^{17} s_1$					

Figure 2. The $\mathbb{S}_n$ -equivariant Poincaré polynomials of $H^\bullet(X_{g,n}) \cong \mathrm{gr}_2 H_c^\bullet(\mathcal{M}_{g,n})$ for small g and n .

g,n	1	2	3	4	5	6
0	$t s_1$	$t s_{1,1}$	$t^2 s_3$	$t^3 s_{2,2}$	$t^4 s_{3,1,1}$	$t^5 (s_{4,1,1} + s_{3,3} + s_{2,2,1,1})$
1	0	0	$t^5 s_{1,1,1}$	$t^5 s_{1,1,1,1} + t^6 s_{3,1}$	$t^6 s_{3,1,1} + t^7 (s_5 + s_{3,2} + s_{2,2,1} + s_{1,1,1,1,1})$	
2	0	$t^7 s_2$	$t^7 s_{2,1}$	$t^8 (2s_4 + s_{3,1}) + t^9 s_{2,1,1}$		
3	$t^8 s_1$	$t^8 s_{1,1}$	$t^9 s_3 + t^{11} s_3$			
4	0	0	$t^{12} s_{1,1,1} + t^{13} (s_3 + s_{2,1}) + t^{14} s_{1,1,1}$			
5	$t^{12} s_1$	$t^{12} s_{1,1} + t^{14} (s_2 + s_{1,1}) + t^{16} s_2$				

Figure 3. The $\mathbb{S}_n$ -equivariant Poincaré polynomials of $H^\bullet(J_{g,n}^*)$ for small g and n .

ogy of $\mathcal{M}_{2,n}$, for $n \leq 5$ (cf. [4] for weight 0 and $n \leq 4$). The additional computations in Figure 1 were previously known as $\mathbb{Q}$ -vector spaces (see [9, Appendix A]). We include the details here to facilitate comparison with the computations in Figures 2 and 3. To the best of our understanding, the computations in Figure 2 are new for $g \geq 3$, as are all of the computations in Figure 3.

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