ZERO-SURGERY CHARACTERIZES INFINITELY MANY KNOTS

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Abstract. We prove that 0 is a characterizing slope for infinitely many knots, namely the genus-1 knots whose knot Floer homology is 2-dimensional in the top Alexander grading, which we classified in recent work and which include all $(-3, 3, 2n+1)$ pretzel knots. This was previously only known for 5_2 and its mirror, as a corollary of that classification, and for the unknot, trefoils, and the figure eight by work of Gabai from 1987.

1. Introduction

A rational number $r \in \mathbb{Q}$ is said to be a *characterizing slope* for a knot $K \subset S^3$ if the orientation-preserving homeomorphism type of the manifold obtained via Dehn surgery on K of slope r uniquely determines K; that is,

$$\text{if } S^3_r(J) \cong S^3_r(K) \text{ then } J = K.$$

It seems very hard to prove for most knots that any given integral slope is characterizing. This is especially true for slope 0: in his celebrated 1987 work [Gab87], Gabai proved that $S^3_0(K)$ detects the genus of K and whether or not K is fibered, which immediately implies that 0-surgery characterizes the unknot (resolving the Property R Conjecture), trefoils, and figure eight. To our knowledge, the only other knots known to be characterized by their 0-surgeries are 5_2 and its mirror, which we proved in our recent work [BS22a]. The main result of this paper is that infinitely many knots are characterized by their 0-surgeries:

Theorem 1.1. *Let K be any of the knots*

$$15n_{43522},\ \mathrm{Wh}^-(T_{2,3}, 2),\ \mathrm{Wh}^+(T_{2,3}, 2),\ P(-3, 3, 2n+1)\ (n \in \mathbb{Z}),$$

or their mirrors. Then 0 is a characterizing slope for K.

Here, $\mathrm{Wh}^{\pm}(T_{2,3}, 2)$ is the 2-twisted Whitehead double of the right-handed trefoil, with a positive or a negative clasp, respectively, and the $P(-3, 3, 2n+1)$ are pretzel knots. See Figure 1.

By contrast, there are many knots that are not characterized by their 0-surgeries. Brakes [Bra80] gave the first pairs of examples, and later Osoinach [Oso06] used annulus twisting to construct infinite families of examples. In fact, there can be infinitely many knots K_n with pairwise diffeomorphic 0-traces $X_0(K_n)$, the result of attaching a 0-framed 2-handle to B^4 along K_n [AJOT13]. Knots

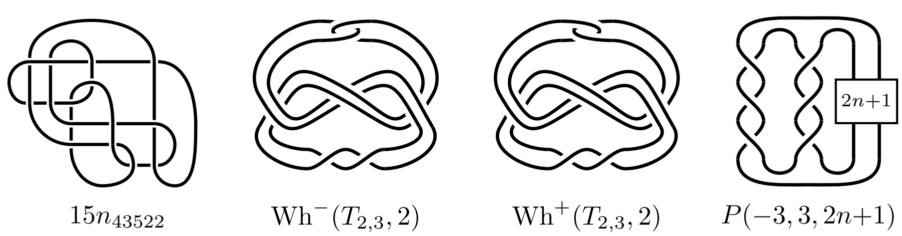

Figure 1. The knots that Theorem 1.1 says are characterized by their 0-surgeries.

which are not smoothly concordant, or which have different slice genera, can nonetheless have diffeomorphic 0-surgeries [Yas15] or even 0-traces [MP18, Pic19]. Indeed, Piccirillo [Pic20] famously proved that the Conway knot is not slice by exhibiting a non-slice knot with the same 0-trace. Recently, Manolescu and Piccirillo [MP21] have given a systematic construction of pairs of knots with the same 0-surgeries, and used it as a source of potentially exotic 4-spheres.

In general, a major difficulty in Floer-theoretic approaches to proving that some integral slope characterizes a knot K is that one must first identify all knots with the same knot Floer homology as K, and this was out of reach until recently for all but a handful of knots. However, Theorem 1.1 is made possible by our recent classification [BS22b] of all genus-1 *nearly fibered* knots:

Theorem 1.2 ([BS22b, Theorem 1.2])**.** *Let $K \subset S^3$ be a genus-1 knot with $\dim_{\mathbb{Q}} \widehat{HFK}(K,1) = 2$. Then up to mirroring K must be one of*

$$5_2,\ 15n_{43522},\ \mathrm{Wh}^-(T_{2,3},2) \tag{1.1}$$

or

$$\mathrm{Wh}^+(T_{2,3},2),\ P(-3,3,2n+1)\ (n \in \mathbb{Z}), \tag{1.2}$$

where the knots in (1.1) have Alexander polynomial $\Delta_K(t) = 2t - 3 + 2t^{-1}$ and determinant $|\Delta_K(-1)| = 7$, and those in (1.2) have Alexander polynomial $\Delta_K(t) = -2t + 5 - 2t^{-1}$ and determinant $|\Delta_K(-1)| = 9$.

For example, we were able to use this classification to prove in [BS22a] that all rational slopes besides the positive integers (i.e., not just 0) are characterizing for 5_2:

Theorem 1.3 ([BS22a, Theorem 1.1])**.** *Every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is a characterizing slope for 5_2.*

We do not expect anything as strong as Theorem 1.3 to hold for the knots in Theorem 1.1. Indeed, Baker and Motegi [BM18, Example 4.1] proved that $P(-3,3,5)$ is not characterized by any non-zero integer surgeries. On the other hand, Theorem 1.1 gives an affirmative answer to [BM18, Question 4.4], which asked whether 0 might be a characterizing slope for $P(-3,3,5)$.

In this paper we assume some background in Heegaard Floer homology, but the Floer-theoretic techniques we use were all present in [BS22a]; the casual reader may be relieved to know that unlike in [BS22a], we make no use of the "mapping cone" formula for the Heegaard Floer homology of surgeries on a knot. On the other hand, Floer theoretic invariants cannot distinguish the 0-surgeries on any of the pretzel knots $P(-3,3,2n+1)$, so we will eventually need to introduce some perturbative invariants defined by Ohtsuki [Oht10] which can tell them apart.

Organization. Theorem 1.1 is proved in several steps. In Section 2 we prove some general facts about 0-surgery on knots of genus one, and then we use these in Section 3 to prove Theorem 3.1, stating that 0-surgery characterizes $15n_{43522}$ and $\mathrm{Wh}^-(T_{2,3},2)$ as well as their mirrors. In Section 4, we use JSJ decompositions to deal with $\mathrm{Wh}^+(T_{2,3},2)$ and its mirror in Theorem 4.3. Then in Section 5 we use Ohtsuki's invariants to prove in Theorem 5.4 that 0 is a characterizing slope for each of the pretzel knots $P(-3,3,2n+1)$. We prove as a bonus in Proposition 5.5 that r-surgery distinguishes these pretzel knots for any $r \in \mathbb{Q}$.

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2. Zero-surgery on genus-one knots

We begin by introducing some general results that will let us reduce Theorem 1.1 to the case where J is one of the knots listed in Theorem 1.2.

Proposition 2.1. *Let $K \subset S^3$ be a knot with Seifert genus 1, and suppose for some other knot $J \subset S^3$ that there is an orientation-preserving homeomorphism*

$$S^3_0(K) \cong S^3_0(J).$$

Then J has genus 1 and the same Alexander polynomial as K, and moreover

$$\dim_{\mathbb{F}} \widehat{HFK}(K,1) = \dim_{\mathbb{F}} \widehat{HFK}(J,1)$$

over any field $\mathbb{F}$.

Proof. The manifold $S^3_0(J)$ determines the Alexander polynomial of J, because the infinite cyclic covers of both $S^3_0(J)$ and the knot exterior $S^3 \setminus N(J)$ have the same first homology as $\mathbb{Z}[t^{\pm 1}]$-modules, so $\Delta_K(t) = \Delta_J(t)$. Gabai [Gab87] proved that it also determines the Seifert genus $g(J)$, so $g(J) = g(K) = 1$.

We now study the Heegaard Floer homology of various surgeries on K, which for the remainder of this proof we will always take with coefficients in a fixed field $\mathbb{F}$. We recall that there is a smooth concordance invariant $V_0(K) \in \mathbb{Z}$, defined by Rasmussen [Ras03], which can be extracted from the knot Floer complex $CFK^\infty(K)$. Its precise definition does not matter here, except to note that it appears in computing the Heegaard Floer correction terms of surgeries on K, by a formula of Ni and Wu [NW15, Proposition 1.6] which implies

$$d(S^3_1(K)) = -2V_0(K) \tag{2.1}$$

as a special case.

The correction terms of the zero-surgery on K satisfy

$$\begin{aligned} d_{1/2}(S^3_0(K)) &= \ \ \tfrac{1}{2} - 2V_0(K) \\ d_{-1/2}(S^3_0(K)) &= -\tfrac{1}{2} + 2V_0(\overline{K}), \end{aligned}$$

by [OS03, Proposition 4.12] and (2.1). The same is true for J, and these correction terms for $S^3_0(K)$ and $S^3_0(J)$ must agree since $S^3_0(K) \cong S^3_0(J)$, so we have

$$V_0(K) = V_0(J). \tag{2.2}$$

Now since $g(K) = 1$ we can apply [BS22a, Lemma 2.8] to see that $HF^+_{\mathrm{red}}(S^3_1(K))$ is an $\mathbb{F}[U]$-module with trivial U-action, and that

$$\dim HF^+_{\mathrm{red}}(S^3_1(K)) = \dim \widehat{HFK}(K,1) - V_0(K).$$

This means that

$$HF^+(S^3_1(K)) \cong \frac{\mathbb{F}[U,U^{-1}]}{U \cdot \mathbb{F}[U]} \oplus \mathbb{F}^{\dim \widehat{HFK}(K,1) - V_0(K)}$$

as ungraded $\mathbb{F}[U]$-modules, so from the exact triangle

$$\cdots \to \widehat{HF}(S^3_1(K)) \to HF^+(S^3_1(K)) \xrightarrow{U} HF^+(S^3_1(K)) \to \cdots$$

we deduce that

$$\dim \widehat{HF}(S^3_1(K)) = 2\left(\dim \widehat{HFK}(K,1) - V_0(K)\right) + 1.$$

Now we apply the surgery exact triangle

$$\cdots \to \widehat{HF}(S^3) \to \widehat{HF}(S^3_0(K)) \to \widehat{HF}(S^3_1(K)) \to \cdots$$

to see that

$$\dim \widehat{HF}(S^3_0(K)) = 2\left(\dim \widehat{HFK}(K,1) - V_0(K)\right) + 1 \pm 1. \tag{2.3}$$

The same is true for J since $g(J) = 1$ as well, namely

$$\dim \widehat{HF}(S^3_0(J)) = 2\left(\dim \widehat{HFK}(J,1) - V_0(J)\right) + 1 \pm 1. \tag{2.4}$$

But $\widehat{HF}(S^3_0(K)) \cong \widehat{HF}(S^3_0(J))$ since the two manifolds are the same, so we combine (2.3) and (2.4) together with (2.2) to get

$$2\left(\dim \widehat{HFK}(K,1) - \dim \widehat{HFK}(J,1)\right) \in \{-2,0,2\}. \tag{2.5}$$

Now we recall that $\widehat{HFK}(K)$ carries a $\mathbb{Z}$-valued Maslov grading, and that each $\widehat{HFK}(K,i)$ has Euler characteristic equal to the t^i-coefficient of $\Delta_K(t)$. Since $\Delta_K(t) = \Delta_J(t)$, this means that

$$\chi(\widehat{HFK}(K,1)) = \chi(\widehat{HFK}(J,1)),$$

and in particular this implies that

$$\dim \widehat{HFK}(K,1) \equiv \dim \widehat{HFK}(J,1) \pmod 2.$$

But then the left side of (2.5) is a multiple of 4, so it must be zero, and thus $\dim \widehat{HFK}(K,1) = \dim \widehat{HFK}(J,1)$ as claimed. □

Remark 2.2. The analogue of the $\widehat{HFK}$ claim in Proposition 2.1 for $g \geq 2$ is that if $S^3_0(K) \cong S^3_0(J)$ then $\widehat{HFK}(K,g) \cong \widehat{HFK}(J,g)$. This has long been known because in that case [OS04, Corollary 4.5] identifies $\widehat{HFK}(K,g)$ with $HF^+(S^3_0(K), \mathfrak{s}_{g-1})$ for a certain Spin^c structure $\mathfrak{s}_{g-1}$.

3. The determinant-7 case

Proposition 2.1 allows us to take care of the knots in Theorem 1.2 with Alexander polynomial $2t - 3 + 2t^{-1}$, using only classical invariants from now on.

Theorem 3.1. *Let K be one of $15n_{43522}$, $\mathrm{Wh}^-(T_{2,3},2)$, or their mirrors. If $S^3_0(K) \cong S^3_0(J)$ for some knot J, then J is isotopic to K.*

Proof. In each case we have $\Delta_K(t) = 2t - 3 + 2t^{-1}$ and $\dim_{\mathbb{Q}} \widehat{HFK}(K,1) = 2$. Thus Proposition 2.1 says that the same is true of J, and then by Theorem 1.2 we know that J must be one of the knots listed in (1.1) up to mirroring. In fact, it cannot be 5_2 or its mirror, because we know from Theorem 1.3 that 0 is a characterizing slope for each of these.

Next, we claim that J cannot be isotopic to the mirror $\overline{K}$. Indeed, if this is the case then

$$S^3_0(K) \cong S^3_0(\overline{K}) \cong -S^3_0(K),$$

so if $\chi : H_1(S^3_0(K)) \cong \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the unique surjection then the Casson–Gordon invariant $\sigma_1(S^3_0(K), \chi)$ (see [CG78]) must be zero. This invariant is equal to minus the signature of K [CG78, Lemma 3.1], so it follows that $\sigma(K) = 0$. However, this is impossible because $\Delta_K(t)$ has a conjugate pair of simple roots on the unit circle, at

$$t = \tfrac{1}{4}(3 \pm i\sqrt{7}),$$

and these are its only roots. Thus the Tristram–Levine signature $\sigma_K(-1) = \sigma(K)$ must be ± 2, giving a contradiction.

It now remains to be shown that if K is $15n_{43522}$ or its mirror, then J cannot be $\mathrm{Wh}^-(T_{2,3},2)$ or its mirror, and vice versa. In other words, we need to show that

$$\pm S^3_0(15n_{43522}) \not\cong \pm S^3_0(\mathrm{Wh}^-(T_{2,3},2)),$$

and we do this by checking that they have different fundamental groups. This can be done in SnapPy [CDGW] by counting 6-fold covers of each:

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In[1]: M = Manifold('15n43522(0,1)')
In[2]: N = Manifold('16n696530(0,1)')
In[3]: len(M.covers(6))
Out[3]: 3
In[4]: len(N.covers(6))
Out[4]: 21
```

In particular, the fundamental groups of each have different numbers of index-6 subgroups, so they cannot be homeomorphic. □

Remark 3.2. Even with Proposition 2.1, we will need more than just classical invariants to address the knots in Theorem 1.2 with Alexander polynomial $-2t+5-2t^{-1}$. For example, if P is one of the pretzel knots $P(-3,3,2n+1)$, then P is slice and so $\sigma(P)=0$, meaning that the arguments used in Theorem 3.1 cannot even distinguish the 0-surgery on P from the 0-surgery on its mirror.

4. The determinant-9 case, part 1

We now turn to the knots in Theorem 1.2 with Alexander polynomial $-2t+5-2t^{-1}$. In order to do this, we will first discuss the JSJ decompositions of their 0-surgeries.

Lemma 4.1. *Let Y be the result of 0-surgery on $P(-3,3,2n+1)$ for some $n\in\mathbb{Z}$. Then Y is a graph manifold: it has a single, non-separating JSJ torus, whose complement is Seifert fibered over the annulus.*

Proof. We know that Y is toroidal, because if Σ is a genus-1 Seifert surface for $P=P(-3,3,2n+1)$ then it extends to a non-separating torus $\hat\Sigma$ after performing 0-surgery on P, and $\hat\Sigma$ is incompressible by [Gab87, Corollary 8.2]. Since P is a Montesinos knot other than a trefoil, Ichihara and Jong [IJ10] proved that $S^3_0(P)$ cannot be toroidal and Seifert fibered, so Y is not Seifert fibered. On the other hand, if we cut Y open along the torus $\hat\Sigma$ then Cantwell and Conlon [CC93, Theorem 1.5] proved that the resulting manifold is the complement of the $(2,4)$-torus link $T_{2,4}\subset S^3$, which is Seifert fibered over the annulus. □

Lemma 4.2. *Let Y be the result of 0-surgery on $\mathrm{Wh}^+(T_{2,3},2)$. Then Y is a graph manifold, and its JSJ decomposition consists of two pieces: one piece is the exterior of $T_{2,3}$, and the other is Seifert fibered over a pair of pants.*

Proof. Let $W=\mathrm{Wh}^+(T_{2,3},2)$. We observe that W is a satellite, with companion $C=T_{2,3}$; its pattern P has winding number 0, hence is not a 0- or 1-bridge braid in the solid torus $V=S^1\times D^2$. This means that 0-surgery on the pattern $P\subset V$ produces a manifold with incompressible torus boundary, by [Gab89, Theorem 1.1]. Thus the companion torus $T=\partial N(C)$ in the exterior of W remains incompressible in $Y=S^3_0(W)$. In particular T is one of the JSJ tori of $S^3_0(W)$, and moreover it separates $S^3_0(W)$ into the union of $S^3\setminus N(T_{2,3})$ (which is Seifert fibered) and $V_0(P)$.

We claim that $V_0(P)$ is not Seifert fibered. Indeed, if it were then all but at most one Dehn filling of its boundary would also be Seifert fibered. But for any n we can realize one of these Dehn fillings by doing $(0,\frac{1}{n})$-surgery on the Whitehead link, and these are homeomorphic to 0-surgeries on infinitely many different twist knots. The only twist knots with a toroidal, Seifert fibered surgery are the trefoils [IJ10], however, so $V_0(P)$ cannot be Seifert fibered after all.

On the other hand, that the pattern P has a genus-1 Seifert surface Σ which lies entirely inside V, and which extends to a non-separating, incompressible torus $\hat\Sigma$ in $V_0(P)\subset S^3_0(W)$. According to [BS22b, Theorem 7.1], if we cut $S^3_0(W)$ open along $\hat{\Sigma}$ then we are left with the complement of the $(2,4)$-cable of $T_{2,3}$, where the companion torus is the same torus T discussed above. It follows that cutting $V_0(P)$ along $\hat{\Sigma}$ produces the complement of a $(2,4)$-torus link in the solid torus, and this is Seifert fibered over a pair of pants. We conclude that T and $\hat{\Sigma}$ are the JSJ tori of $S^3_0(W)$, and that $S^3_0(W)$ has the claimed JSJ decomposition. □

Lemmas 4.1 and 4.2 make it easy to distinguish 0-surgery on $\mathrm{Wh}^+(T_{2,3},2)$ from the 0-surgeries on the $P(-3,3,2n+1)$ pretzel knots.

Theorem 4.3. *Let K be either $\mathrm{Wh}^+(T_{2,3},2)$ or its mirror. If $S^3_0(J) \cong S^3_0(K)$ for some knot $J \subset S^3$, then J is isotopic to K.*

Proof. By Proposition 2.1, we see that J has genus 1 and top knot Floer homology

$$\widehat{HFK}(J,1;\mathbb{Q}) \cong \widehat{HFK}(K,1;\mathbb{Q}) \cong \mathbb{Q}^2,$$

and its Alexander polynomial is $-2t+5-2t^{-1}$. According to Theorem 1.2, we therefore know that J is either K, its mirror $\overline{K}$, or some pretzel knot $P(-3,3,2n+1)$. (We note here that the mirror of $P(-3,3,2n+1)$ is $P(-3,3,-2n-1)$.)

In order to show that J cannot be $\overline{K}$, we consider the JSJ decompositions of

$$S^3_0(K) \quad \text{and} \quad S^3_0(\overline{K}) \cong -S^3_0(K).$$

One of these two manifolds is $S^3_0(\mathrm{Wh}^+(T_{2,3},2))$, and by Lemma 4.2 its JSJ decomposition consists of two pieces, one of which is the exterior of $T_{2,3}$ and the other of which is not a knot complement. But then the other manifold decomposes into the exterior of $T_{-2,3}$ and another piece, which is again not a knot complement. By the uniqueness of the JSJ decomposition, any orientation-preserving homeomorphism $S^3_0(K) \xrightarrow{\cong} -S^3_0(K)$ would have to restrict to an orientation-preserving homeomorphism

$$S^3 \setminus N(T_{2,3}) \cong S^3 \setminus N(T_{-2,3}),$$

and this is impossible.

Now if $J = P(-3,3,2n+1)$ then Lemma 4.1 says that the JSJ decomposition of $S^3_0(J)$ consists of a single Seifert fibered piece. This does not match the decomposition of $S^3_0(K)$, so again we must have $S^3_0(K) \not\cong S^3_0(J)$. We have now shown that J cannot be either $\bar{K}$ or any of the pretzel knots $P(-3,3,2n+1)$, so J must be isotopic to K after all. □

5. The determinant-9 case, part 2

In this section we prove that 0 is a characterizing slope for each pretzel knot $P(-3,3,2n+1)$. We begin with the following.

Lemma 5.1. *If $S^3_0(J) \cong S^3_0(P(-3,3,2n+1))$ for some $n \in \mathbb{Z}$, then J is isotopic to the pretzel knot $P(-3,3,2m+1)$ for some $m \in \mathbb{Z}$.*

Proof. Just as in the proof of Theorem 4.3, we apply Proposition 2.1 and Theorem 1.2 to see that if we write $W = \mathrm{Wh}^+(T_{2,3},2)$ then J must be one of

$$W,\ \overline{W},\ \text{or } P(-3,3,2m+1)\ (m \in \mathbb{Z}).$$

On the other hand, Theorem 4.3 tells us that

$$S^3_0(W) \not\cong S^3_0(P(-3,3,2n+1)) \quad \text{and} \quad S^3_0(\overline{W}) \not\cong S^3_0(P(-3,3,2n+1)),$$

so J cannot be W or $\overline{W}$, hence it must be some $P(-3,3,2m+1)$. □

In order to distinguish the 3-manifolds $S^3_0(P(-3,3,2n+1))$ for different values of n, we use Ohtsuki's perturbative invariants of 3-manifolds M with $b_1(M)=1$ [Oht10], which take the form of a power series

$$\tau(M;c)=\sum_{\ell=0}^{\infty}\lambda_\ell(M;c)(q-1)^\ell \in \mathbb{C}[[q-1]]$$

that can be evaluated at $c=0$ or at any root c of the Alexander polynomial $\Delta_M(t)$. Each $\lambda_\ell(M;c)$ is itself an invariant of M, and $\lambda_0(M;c)$ is determined by the Alexander polynomial of M [Oht10, Proposition 5.3], so we will compute $\lambda_1(S^3_0(P(-3,3,2n+1)),0)$.

According to the discussion in [Oht10, §1], we have

$$\lambda_\ell(S^3_0(K);c) = -\frac{1}{2}\cdot\frac{1+c}{1-c}\left(\operatorname*{Res}_{t=c}\frac{(1-t^{-1})^2P_\ell(t)}{\Delta_K(t)^{2\ell+1}}\right),$$

where the Laurent polynomials $P_\ell(t)$ are the coefficients of the loop expansion

$$J_n(K;q)=\sum_{\ell=0}^{\infty}\frac{P_\ell(q^n)}{\Delta_K(q^n)^{2\ell+1}}(q-1)^\ell$$

of the colored Jones polynomial. We have $P_0(t)=1$ regardless of K, and then Ohtsuki [Oht04, Proposition 6.1] computed that

$$P_1(t) = -(t^{1/2}-t^{-1/2})^2\cdot\hat{\Theta}_K(t), \tag{5.1}$$

where the last factor

$$\hat{\Theta}_K(t)=\frac{\Theta_K(t,1)}{(t^{1/2}-t^{-1/2})^2}\in\mathbb{Q}[t,t^{-1}]$$

is a specialization of a polynomial called the "2-loop polynomial" $\Theta_K(t_1,t_2)$ arising from the Kontsevich integral of K. (We note that the polynomial $J_n(K;q)$ in [Oht10] is the same as the one denoted $V_n(K;q)$ in [Oht04] – both are normalized to take the value 1 when K is the unknot – and also that (5.1) may differ from the value in [Oht10] by a sign, but this only changes the invariants $\lambda_1(S^3_0(K);c)$ that we will compute by an overall sign.)

The calculation of these polynomials was described in part by Ohtsuki [Oht07], including a computation of both $\Theta_K(t_1,t_2)$ and $\hat{\Theta}_K(t)$ when K is a 3-stranded pretzel knot:

Lemma 5.2 ([Oht07, Example 3.6])**.** *For the pretzel knot $K=P(p,q,r)$, if we let*

$$d=\frac{pq+qr+rp+1}{4}$$

then the reduced 2-loop polynomial of K is given by

$$\hat{\Theta}_K(t)=\tfrac{1}{16}\big((p+q+r)(4d+1)+pqr\big)\left(-2-\frac{2d+1}{3}(t-2+t^{-1})\right).$$

Applying Lemma 5.2 when $(p,q,r)=(-3,3,2n+1)$, we have $d=-2$ and then

$$\hat{\Theta}_{P(-3,3,2n+1)}(t) = -(2n+1)\left(t-4+t^{-1}\right), \tag{5.2}$$

whence for $K=P(-3,3,2n+1)$ we have $\Delta_K(t)=-2t+5-2t^{-1}$ and

$$\begin{aligned} P_1(t) &= -(t-2+t^{-1})\cdot\hat{\Theta}_K(t)\\ &= (2n+1)(t-2+t^{-1})(t-4+t^{-1})\\ &= (2n+1)(t^2-6t+10-6t^{-1}+t^{-2})\\ &= (2n+1)\left(\tfrac{1}{4}\Delta_K(t)^2+\tfrac{1}{2}\Delta_K(t)-\tfrac{3}{4}\right). \end{aligned} \tag{5.3}$$

The reason for writing it this way is that we can compute $\lambda_1(S^3_0(K),0)$ via the following lemma.

Lemma 5.3 ([Oht10, Proposition 1.7(2)])**.** *Suppose that the Alexander polynomial of K has degree 1, and write*

$$\Delta_K(t) = b_0 - b_1(t - 2 + t^{-1}),$$
$$P_1(t) = f(t)\Delta_K(t)^3 + a_2\Delta_K(t)^2 + a_1\Delta_K(t) + a_0$$

for some constants $b_0, b_1, a_0, a_1, a_2 \in \mathbb{Q}$ and Laurent polynomial $f(t)$. Then

$$\lambda_1(S^3_0(K); 0) = -\frac{d}{2} + \frac{a_2}{2b_1}$$

where d is the constant term of $(t - 2 + t^{-1})f(t)$.

Theorem 5.4. *Fix an integer $n \in \mathbb{Z}$. If $S^3_0(P(-3, 3, 2n+1)) \cong S^3_0(K)$ for some knot $K \in S^3$, then K is isotopic to $P(-3, 3, 2n + 1)$.*

Proof. Lemma 5.1 guarantees that K is $P(-3, 3, 2m + 1)$ for some $m \in \mathbb{Z}$. We use Lemma 5.3 for $P(-3, 3, 2n + 1)$: we have $(b_0, b_1) = (1, 2)$, and (5.3) tells us that

$$(f(t), a_2, a_1, a_0) = \left(0, \frac{2n+1}{4}, \frac{2n+1}{2}, -\frac{3(2n+1)}{4}\right).$$

The constant term of $(t - 2 + t^{-1})f(t) = 0$ is $d = 0$, so we end up with

$$\lambda_1(S^3_0(P(-3, 3, 2n + 1)); 0) = \frac{a_2}{2b_1} = \frac{2n+1}{16}.$$

But then an identical calculation says that

$$\lambda_1(S^3_0(P(-3, 3, 2m + 1)); 0) = \frac{2m+1}{16},$$

and since these two invariants agree, we must have $m = n$. □

In fact, we can distinguish surgeries of any slope on these pretzel knots.

Proposition 5.5. *If $r \in \mathbb{Q}$ is non-zero and m and n are distinct integers, then*

$$S^3_r(P(-3, 3, 2m + 1)) \not\cong S^3_r(P(-3, 3, 2n + 1)).$$

Proof. This uses an LMO invariant obstruction due to Ito [Ito20], just as in [BS22a, §7]: both knots have the same Conway polynomial $\nabla_K(z) = 1 - 2z^2$, with the same z^4-coefficient

$$a_4(P(-3, 3, 2m + 1)) = a_4(P(-3, 3, 2n + 1)) = 0.$$

Thus if their r-surgeries are homeomorphic, then by [Ito20, Corollary 1.3(iv)] these knots must have the same finite type invariants

$$v_3(P(-3, 3, 2m + 1)) = v_3(P(-3, 3, 2n + 1)).$$

But Ohtsuki [Oht07, Proposition 1.1] proved that $v_3(K) = \frac{1}{2}\hat{\Theta}_K(1)$, and so (5.2) says that

$$v_3(P(-3, 3, 2n + 1)) = 2n + 1,$$

hence these pretzel knots have different v_3 invariants unless $m = n$. (We note that Ohtsuki's normalization of v_3 differs from Ito's by a scalar, but this does not affect the argument.) □

We remark that Ito's obstruction cannot be used to prove Theorem 5.4, however, because it only applies to non-zero surgeries. Moreover, Proposition 5.5 does not prove that non-zero slopes are characterizing for these pretzel knots, because for example the Heegaard Floer homology of $S^3_r(K) \cong S^3_r(P(-3, 3, 2n + 1))$ may not suffice to determine $\widehat{HFK}(K)$ when $r \neq 0$.

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