Emergent modified gravity: Polarized Gowdy model on a torus

Martin Bojowald [*] and Erick I. Duque [†]

Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, Pennsylvania 16802, USA

(Received 13 August 2024; accepted 29 October 2024; published 2 December 2024)

New covariant theories of emergent modified gravity exist not only in spherically symmetric models, as previously found, but also in polarized Gowdy systems that have a local propagating degree of freedom. Several explicit versions are derived here, depending on various modification functions. These models do not have instabilities from higher time derivatives, and a large subset is compatible with gravitational waves and minimally coupled massless matter fields traveling at the same speed. Interpreted as models of loop quantum gravity, covariant Hamiltonian constraints derived from the covariance conditions found in polarized Gowdy systems are more restricted than those in spherical symmetry, requiring new forms of holonomy modifications with an anisotropy dependence that has not been considered before. Assuming homogeneous space, the models provide access to the full anisotropy parameters of modified Bianchi I dynamics, in which case different fates of the classical singularity are realized depending on the specific class of modifications.

DOI: 10.1103/PhysRevD.110.124001

I. INTRODUCTION

The canonical formulation of spherically symmetric general relativity has recently been shown [1,2] to allow a larger class of modifications than is suggested by the more common setting of covariant action principles. In this framework of emergent modified gravity, it is possible to couple perfect fluids [3], electromagnetism [4,5], and scalar matter [6,7] to the new spacetime geometries, including local degrees of freedom in the latter case. Here, we show that it is also possible to extend spherical symmetry to a polarized Gowdy symmetry that includes local gravitational degrees of freedom. This extension makes it possible to study properties of gravitational waves in this new set of covariant spacetime theories.

Building on previous canonical developments, starting with the classic [8] and using more recent contributions [9], emergent modified gravity constructs consistent gravitational dynamics and corresponding spacetime geometries by modifying the Hamiltonian constraint of general relativity and implementing all covariance conditions. A candidate for the spatial metric of a spacetime geometry is provided by the structure function in the Poisson bracket of two Hamiltonian constraints, which is required to be proportional to the diffeomorphism constraint as one of the consistency conditions. Canonical gauge transformations of the candidate spatial metric must then agree with coordinate transformations in a compatible spacetime geometry, forming the second consistency condition that had been formulated for the general case and analyzed for the first time in [2]. These constructions allow for the possibility that the spatial metric (or a triad) is not one of the fundamental fields of a phase-space formulation. It is derived from Hamilton's equations generated by the constraints and not presupposed, giving it the status of an emergent geometrical object. This feature is the main difference with standard action principles in metric or other formulations and makes this approach to modified gravity more general than previous constructions. Examples of new physical implications include the possibility of nonsingular black-hole solutions [10,11], covariant modified newtonian dynamics (MOND)-like effects [12], and new types of signature change [13].

There is a large variety of potential physical effects that depend on choices of modification functions. The physical origin of an implication such as nonsingular behavior or MOND-like effects is then related to the underlying motivation for such a choice, for instance in properties of canonical quantum gravity that could impose curvature bounds and therefore imply nonsingular behavior, or renormalization of quantum gravity in canonical form, which could imply logarithmic terms in quantum modifications relevant on intermediate distance scales where MOND would be relevant. Scalar quasinormal modes on a background spacetime of spherically symmetric emergent modified gravity have been computed in [14], showing new characteristic features that could be used in the future to subject this framework to observational tests. However, all these promising results were obtained for spherically symmetric models

[*]Contact author: bojowald@psu.edu
[†]Contact author: eqd5272@psu.edu

of gravity. For a testable and phenomenologically viable framework, it is important to relax this symmetry condition. Here we present the first step toward a more general framework.

The constructions of the present paper lead to the first model of emergent modified gravity that does not obey spherical symmetry. Nevertheless, spherical symmetry can be realized as a special case, allowing us to draw conclusions about how generic specific features seen in the more symmetric context are within a broader setting. An example of interest is the form of holonomy-type modifications that are often used in order to model potential effects from loop quantum gravity. The specific form of these modifications within a consistent and covariant set of equations is restricted compared with what had been assumed previously in loop constructions. The extension to polarized Gowdy models performed here shows that compatible modifications require significant deviations from what might be suggested by loop quantum gravity. In particular, the holonomy length for strictly periodic modifications of the extrinsic-curvature dependence does not directly depend on the volume or area of a symmetry orbit (all of space in a homogeneous cosmological model or a sphere at constant radius in black-hole models), but rather on its anisotropy parameters. General covariance therefore rules out the possibility that the holonomy length decreases as space or a spherical orbit expands, which would be a prerequisite to a nearly constant discreteness scale that does not increase to macroscopic sizes as the universe expands. Nevertheless, additional modification functions can be used in order to implement a dynamical suppression of holonomy modifications on classical scales, as discussed in detail in [15] for spherically symmetric models. The traditional picture of models of loop quantum gravity therefore has to be corrected in order to be compatible with a consistent spacetime geometry. Emergent modified gravity guides the way to a new understanding by a systematic classification of possible spacetime modifications in canonical form.

In addition, the new gravitational models found here are important in their own right because they have covariant equations with modifications that do not require higher-derivative terms and corresponding instabilities [16]. They are therefore potential alternatives to general relativity that could be used in comparisons with observations, provided the symmetry assumptions can be relaxed further. Polarized Gowdy symmetries constitute a first step in this direction, giving access to some properties of gravitational waves. In particular, we show that there is a class of modifications that implies the same propagation speed for gravitational waves and massless scalar matter traveling on the same background.

Unlike spherically symmetric models, which have a spatially homogeneous subset of Kantowski-Sachs models with a single anisotropy parameter, polarized Gowdy models give full access to the Bianchi I model with two anisotropy parameters. It is therefore possible to perform a more complete analysis of the big-bang singularity, which may be avoided depending on the type of modifications used. As a characteristic property, the classical Kasner exponents are preserved at large volume, and a nonsingular transition from collapse to expansion happens at the same time for all three spatial directions. We will present a detailed analysis of these questions in Sec. VI, after a brief review of canonical and emergent modified gravity in Sec. II and their application to polarized Gowdy models in Secs. III and IV with a summary of different classes of modifications in Sec. V. Implications for covariant holonomy modifications in models of loop quantum gravity can be found in Secs. III B 2–III B 4.

II. CLASSICAL THEORY

The classical polarized Gowdy system [17] is defined by spacetime line elements of the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{\theta\theta}(\mathrm{d}\theta + N^\theta \mathrm{d}t)^2 + q_{xx}\mathrm{d}x^2 + q_{yy}\mathrm{d}y^2 \quad (1)$$

with functions N, N^θ, and q_{ab} depending only on t and θ. All three spatial coordinates x, y, and θ take values in the range $[0, 2\pi)$ for the torus model with spatial slices $\Sigma \cong T^3 = S^1 \times S^1 \times S^1$. Solutions with periodic boundary conditions in θ can be interpreted as standing planar gravitational waves with transversal area element $\sqrt{q_{xx}q_{yy}}$, moving in the θ-direction in which the length measure is given by $q_{\theta\theta}$. Alternatively, solutions may be used as cosmological models with one direction of spatial inhomogeneity. (The periodicity condition in θ may be dropped, but it is part of the traditional Gowdy model.)

Equivalently, the spatial metric components q_{ab} can be parametrized by

$$q_{\theta\theta} = \frac{E^x E^y}{\varepsilon}, \qquad q_{xx} = \frac{E^y}{E^x}\varepsilon, \qquad q_{yy} = \frac{E^x}{E^y}\varepsilon \quad (2)$$

using the components E^x, E^y, and ε of a densitized triad

$$E^a_i \sigma_i \frac{\partial}{\partial x^a} = \varepsilon\sigma_3 \frac{\partial}{\partial\theta} + E^x\sigma_1\frac{\partial}{\partial x} + E^y\sigma_2\frac{\partial}{\partial y} \quad (3)$$

with Pauli matrices σ_i. In these variables, the transversal area element is given by $\sqrt{q_{xx}q_{yy}} = \varepsilon$.

For some purposes, it is conventional to write the metric in the diagonal case ($N^\theta = 0$) in the form

$$\mathrm{d}s^2 = e^{2a}(-\mathrm{d}T^2 + \mathrm{d}\theta^2) + T(e^{2W}\mathrm{d}x^2 + e^{-2W}\mathrm{d}y^2) \quad (4)$$

with a new time coordinate T. This conventional metric is associated with the canonical metric in a gauge defined by $\varepsilon = T$ and $N = \sqrt{q_{\theta\theta}}$, identifying $N^2 = q_{\theta\theta} =: e^{2a}$ and $W = \ln\sqrt{E^y/E^x}$.

If $E^x = E^y$ or $W = 0$, the geometry has an additional rotational symmetry in the transversal planes. This condition eliminates the local propagating degree of freedom present in the original model. If we replace $\mathrm{d}x^2+\mathrm{d}y^2$ with $\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2$ in the line element, we obtain the general form of spherically symmetric models. In this case, it is more common to choose a gauge fixing in which $q_{\vartheta\vartheta}$, analogous to ε in the polarized Gowdy model with $E^x=E^y$, is related to the radial coordinate instead of time. Moreover, from standard solutions in this case it follows that the static gauge choice in spherical symmetry, given by vanishing extrinsic curvature, implies an inverse relationship between the lapse function and the radial metric component.

Specific solutions of spherically symmetric models and the polarized Gowdy model, even for $W=0$, therefore appear quite different, but formally we will see that the models are closely related in their canonical properties. The main difference implied by using spherical symmetry orbits instead of planes is the presence of additional intrinsic-curvature terms in the Hamiltonian constraint that depend on spatial derivatives of $q_{\vartheta\vartheta}$ in a spherically symmetric model. It is less obvious that even the general polarized Gowdy model with $W\neq 0$ can be related to a spherically symmetric model, provided the gravitational variables in the latter case are coupled to a spherically symmetric scalar field. Seeing this relationship will require a suitable canonical transformation of the Gowdy variables.

A. Canonical formulation

The densitized-triad components are canonically conjugate to components of extrinsic curvature, implying canonical pairs (K_x,E^x), (K_y,E^y), and $(\mathcal{A},\varepsilon)$ and the symplectic structure

$$\Omega=\frac{1}{\tilde{\kappa}}\int\mathrm{d}\theta(\mathrm{d}K_x\wedge\mathrm{d}E^x+\mathrm{d}K_y\wedge\mathrm{d}E^y+\mathrm{d}\varepsilon\wedge\mathrm{d}\mathcal{A})\qquad(5)$$

with $\tilde{\kappa}=\kappa/(4\pi^2)=2G/\pi$ in terms of Newton's constant G. We will work in units such that $\tilde{\kappa}=1$.

The classical Hamiltonian and diffeomorphism constraints with a cosmological constant are [18]

$$\begin{aligned}H=&-\frac{1}{\sqrt{E^xE^y\varepsilon}}\Bigg[-\varepsilon E^xE^y\Lambda+K_xE^xK_yE^y+(K_xE^x+K_yE^y)\varepsilon\mathcal{A}+\frac{1}{2}\frac{\varepsilon^2}{E^xE^y}(E^x)'(E^y)'+\frac{1}{2}\frac{\varepsilon}{E^x}(E^x)'\varepsilon'+\frac{1}{2}\frac{\varepsilon}{E^y}(E^y)'\varepsilon'\\&-\frac{1}{4}\frac{\varepsilon^2}{(E^y)^2}((E^y)')^2-\frac{1}{4}\frac{\varepsilon^2}{(E^x)^2}((E^x)')^2-\frac{1}{4}(\varepsilon')^2-\varepsilon\varepsilon''\Bigg]\\=&-\frac{1}{\sqrt{E^xE^y\varepsilon}}(-\varepsilon E^xE^y\Lambda+E^xK_xE^yK_y+(E^xK_x+E^yK_y)\varepsilon\mathcal{A})-\frac{1}{4}\frac{1}{\sqrt{E^xE^y\varepsilon}}\left((\varepsilon')^2-4(\varepsilon(\ln\sqrt{E^y/E^x})')^2\right)+\left(\frac{\sqrt{\varepsilon}\varepsilon'}{\sqrt{E^xE^y}}\right)'\end{aligned}\qquad(6)$$

and

$$H_\theta=E^xK_x'+E^yK_y'-\mathcal{A}\varepsilon',\qquad(7)$$

where the primes are θ derivatives. The smeared constraints have Poisson brackets

$$\{H_\theta[N^\theta],H_\theta[M^\theta]\}=-H_\theta[M^\theta(N^\theta)'-N^\theta(M^\theta)'],\qquad(8)$$

$$\{H[N],H_\theta[M^\theta]\}=-H[M^\theta N'],\qquad(9)$$

$$\{H[N],H[M]\}=-H_\theta[q^{\theta\theta}(MN'-NM')]\qquad(10)$$

of hypersurface-deformation form, with structure function $q^{\theta\theta}=\varepsilon/(E^xE^y)$ directly given by the inverse metric component in the inhomogeneous direction. From general properties of canonical gauge systems [19,20] it then follows that the gauge transformations for the lapse function N and shift vector N^ϑ are given by

$$\delta_\epsilon N=\dot{\epsilon}^0+\epsilon^\theta N'-N^\theta(\epsilon^0)'\qquad(11)$$

and

$$\delta_\epsilon N^\theta=\dot{\epsilon}^\theta+\epsilon^\theta(N^\theta)'-N^\theta(\epsilon^\theta)'+q^{\theta\theta}(\epsilon^0N'-N(\epsilon^0)').\qquad(12)$$

In the classical theory it is clear that the inverse of the structure function, $q_{\theta\theta}=1/q^{\theta\theta}$, obeys a covariance condition as a component of the spacetime metric. More generally [2], covariance conditions can be directly formulated for phase-space functions such as a structure function in a modified theory. They implement the general condition that gauge transformations of any candidate spacetime metric component, generated by the canonical constraints, must be of the form of a Lie derivative by a spacetime vector field. Using explicit expressions for gauge transformations generated by the constraints on a given phase space, this general condition can be written as a set of partial differential equations that the constraints have to obey.

These covariance conditions, derived in [2] for spherical symmetry, are more complicated for polarized Gowdy models because the phase space is larger and the line element is a more complicated function of the phase-space variables. For the homogeneous component q_{xx} we obtain the conditions

$$\frac{1}{E^y}\left(\frac{\partial H}{\partial K'_y}-2\left(\frac{\partial H}{\partial K''_y}\right)'\right)-\frac{1}{E^x}\left(\frac{\partial H}{\partial K'_x}-2\left(\frac{\partial H}{\partial K''_x}\right)'\right)+\frac{1}{\varepsilon}\left(\frac{\partial H}{\partial \mathcal{A}'}-2\left(\frac{\partial H}{\partial \mathcal{A}''}\right)'\right)\bigg|_{\text{O.S.}}=0 \tag{13}$$

and

$$\frac{1}{E^y}\frac{\partial H}{\partial K''_y}-\frac{1}{E^x}\frac{\partial H}{\partial K''_x}+\frac{1}{\varepsilon}\frac{\partial H}{\partial \mathcal{A}''}\bigg|_{\text{O.S.}}=0, \tag{14}$$

where "O.S." indicates that the equations are required to hold on-shell, when constraints and equations of motion are satisfied. For our modified constraints, we assume spatial derivatives up to second order; otherwise, there would be additional terms in these equations. The $x \leftrightarrow y$ exchange symmetry of the constraint allows us to simplify these on-shell conditions to

$$\begin{aligned}\frac{\partial H}{\partial \mathcal{A}'}=\frac{\partial H}{\partial \mathcal{A}''}&=\frac{1}{E^y}\frac{\partial H}{\partial K'_y}-\frac{1}{E^x}\frac{\partial H}{\partial K'_x}\\&=\frac{1}{E^y}\frac{\partial H}{\partial K''_y}-\frac{1}{E^x}\frac{\partial H}{\partial K''_x}=0,\end{aligned} \tag{15}$$

which is clearly satisfied by the classical constraint even off-shell. The same condition is obtained from the other homogeneous component, q_{yy}.

For the inhomogeneous component, the covariance condition reads

$$\frac{\partial(\{q^{\theta\theta},H[\epsilon^0]\})}{\partial(\epsilon^0)'}\bigg|_{\text{O.S.}}=\frac{\partial(\{q^{\theta\theta},H[\epsilon^0]\})}{\partial(\epsilon^0)''}\bigg|_{\text{O.S.}}=\cdots=0, \tag{16}$$

which is also satisfied by the classical constraint because it does not contain any derivatives of K_x, K_y, or $\mathcal{A}$ that would introduce a dependence of the Poisson brackets on spatial derivatives of ϵ^0 upon integrating by parts. For this result, it is important to use the classical property that the structure function $q^{\theta\theta}$ is independent of the canonical variables conjugate to the triad components. This property is no longer required in emergent modified gravity.

The gauge transformations of lapse and shift, Eqs. (11) and (12), and the realization of the covariance conditions, Eqs. (15) and (16), ensure that the spacetime line element (1) is invariant, or the spacetime metric $g_{\mu\nu}$ is covariant in the sense that the canonical gauge transformations of the metric reproduce spacetime diffeomorphisms on-shell: We have

$$\delta_\epsilon g_{\mu\nu}|_{\text{O.S.}}=\mathcal{L}_\xi g_{\mu\nu}|_{\text{O.S.}}, \tag{17}$$

where the gauge functions, $(\epsilon^0,\epsilon^\theta)$, on the left-hand side are related to the 2-component vector generator, $\xi^\mu=(\xi^t,\xi^\theta)$, of the diffeomorphism on the right-hand side by

$$\xi^\mu=\epsilon^0 n^\mu+\epsilon^\theta s^\mu=\xi^t t^\mu+\xi^\theta s^\mu \tag{18}$$

with components

$$\xi^t=\frac{\epsilon^0}{N},\qquad \xi^\theta=\epsilon^\theta-\frac{\epsilon^0}{N}N^\theta. \tag{19}$$

B. New variables

It is convenient to perform the canonical transformation

$$\begin{aligned}P_{\bar W}&=K_xE^x-K_yE^y,\qquad \bar W=\ln\sqrt{\frac{E^y}{E^x}},\\ \bar a&=\sqrt{E^xE^y},\qquad K=\frac{K_xE^x+K_yE^y}{\sqrt{E^xE^y}},\end{aligned} \tag{20}$$

with $\bar W$ and K as the configuration variables, and $P_{\bar W}$ and $\bar a$ their respective conjugate momenta. The canonical pair $(\mathcal{A},\varepsilon)$ is left unchanged by this transformation.

The diffeomorphism constraint in these variables is form invariant,

$$\begin{aligned}H_\theta&=E^xK'_x+E^yK'_y-\mathcal{A}\varepsilon'\\&=\frac{1}{2}E^x\left(\frac{P_{\bar W}+K\sqrt{E^xE^y}}{E^x}\right)'+\frac{1}{2}E^y\left(\frac{-P_{\bar W}+K\sqrt{E^xE^y}}{E^y}\right)'-\mathcal{A}\varepsilon'\\&=\bar aK'+P_{\bar W}\bar W'-\mathcal{A}\varepsilon',\end{aligned} \tag{21}$$

while the Hamiltonian constraint (6) reads

$$
\begin{aligned}
H &= -\frac{1}{\sqrt{E^xE^y\varepsilon}}(-\varepsilon E^xE^y\Lambda + E^xK_xE^yK_y + (E^xK_x + E^yK_y)\varepsilon\mathcal{A}) \\
&\quad - \frac{1}{4}\frac{1}{\sqrt{E^xE^y\varepsilon}}\left((\varepsilon')^2 - 4(\varepsilon(\ln\sqrt{E^y/E^x})')^2\right) + \left(\frac{\sqrt{\varepsilon}\varepsilon'}{\sqrt{E^xE^y}}\right)' \\
&= -\sqrt{\varepsilon}\left[\bar{a}\left(-\Lambda + \frac{K^2}{4\varepsilon} - \frac{1}{4\varepsilon}\frac{P_{\bar{W}}^2}{\bar{a}^2} + K\frac{\mathcal{A}}{\bar{a}}\right) - \varepsilon\frac{(\bar{W}')^2}{\bar{a}} - \frac{1}{4\varepsilon}\frac{(\varepsilon')^2}{\bar{a}} + \frac{\bar{a}'\varepsilon'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}}\right] \\
&= -\sqrt{\varepsilon}\left[\bar{a}\left(-\Lambda + \frac{K^2}{4\varepsilon} + K\frac{\mathcal{A}}{\bar{a}}\right) - \frac{1}{4\varepsilon}\frac{(\varepsilon')^2}{\bar{a}} + \frac{\bar{a}'\varepsilon'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}}\right] + \frac{\sqrt{q^{\theta\theta}}}{2}\left(\frac{P_{\bar{W}}^2}{2\varepsilon} + 2\varepsilon(\bar{W}')^2\right)
\end{aligned} \tag{22}
$$

in these new variables. The first parenthesis resembles the Hamiltonian constraint of a spherically symmetric model, while the last term expresses $\bar{W}$ in the form of a scalar field. This relationship will be discussed in more detail in Sec. II D.

The spacetime metric

$$
\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{\theta\theta}(\mathrm{d}\theta + N^\theta\mathrm{d}t)^2 + q_{xx}\mathrm{d}x^2 + q_{yy}\mathrm{d}y^2 \tag{23}
$$

now has the spatial components

$$
q_{\theta\theta} = \frac{\bar{a}^2}{\varepsilon}, \qquad q_{xx} = e^{2\bar{W}}\varepsilon, \qquad q_{yy} = e^{-2\bar{W}}\varepsilon. \tag{24}
$$

The new variables therefore closely resemble the conventional choice used in (4).

C. Symmetries and observables

Given a potentially large class of modifications, it is useful to impose guiding principles such as the preservation of important symmetries of the classical system. For the models considered here, there are discrete as well as continuous symmetries.

1. Discrete symmetry

The constraints (6) are symmetric under the exchange $E^x \leftrightarrow E^y, K_x \leftrightarrow K_y$, while the full line element (1) has the same symmetry provided the coordinates are exchanged too, $x \leftrightarrow y$. The complete discrete transformation is then given by

$$
E^x \leftrightarrow E^y, \qquad K_x \leftrightarrow K_y, \qquad x \leftrightarrow y. \tag{25}
$$

This important symmetry implies the existence of an x-y plane of wave fronts, in which the two independent directions are interchangeable (while we do not have isotropy in this plane unless $E^x = E^y$). The modified theory should therefore retain this symmetry as an important characterization of the polarized Gowdy system. In the new variables, the discrete transformation takes the form

$$
P_{\bar{W}} \to -P_{\bar{W}}, \qquad \bar{W} \to -\bar{W}, \qquad x \leftrightarrow y, \tag{26}
$$

which is a symmetry of the system (21)–(24).

2. Continuous symmetries and related observables

Field observable. Another advantage of the new variables is that the constraint (22) is manifestly invariant under the transformation $\bar{W} \to \bar{W} + \omega$ where ω is a constant. Therefore, the phase-space functional

$$
G[\omega] = \int \mathrm{d}\theta\, \omega P_{\bar{W}} \tag{27}
$$

is a symmetry generator:

$$
\{G[\omega], H[N]\} = \{G[\omega], H_\theta[N^\theta]\} = 0, \tag{28}
$$

where we neglect boundary terms.

This property in turn implies that $G[\omega]$ is a conserved global charge because $\dot{G}[\omega] = \{G[\omega], H[N] + H_\theta[N^\theta]\} = 0$. Furthermore, as discussed in [6], the boundary terms that survive under the transformation of the local charge take the form $\dot{G} = -\partial_a J^a$, which takes the form $\partial_\mu J^\mu = \nabla_\mu J^\mu = 0$ of a covariant conservation law for a spacetime densitized 4-current with components

$$
J^t = G = P_{\bar{W}}, \qquad J^a = -\left(N\frac{\partial H}{\partial \bar{W}'}\right)' = -2\varepsilon^{3/2}\frac{\bar{W}'}{\bar{a}}. \tag{29}
$$

Mass observable: In the limit of $P_{\bar{W}} = \bar{W} = 0$, the expression

$$
\mathcal{M} = \frac{\sqrt{\varepsilon}}{2}\left(K^2 - \left(\frac{\varepsilon'}{2\bar{a}}\right)^2 + \frac{\Lambda\varepsilon}{3}\right) \tag{30}
$$

is a Dirac observable.

D. Analogy with spherical symmetry

In the new variables, the constraint (22) is close to the spherically symmetric constraint coupled to a scalar field.

In this subsection we will point out in detail how the two models are related.

In a spherically symmetric model, the spacetime line element can always be written as

$$ds^2 = -N^2 dt^2 + q_{xx}^{\rm sph}(dx + N^x dt)^2 + q_{\vartheta\vartheta}^{\rm sph} d\Omega^2, \tag{31}$$

where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ in spherical coordinates. (We use the conventional notation of calling the radial coordinate in a spherically symmetric model x when a coordinate choice has not been made yet. The notation as a radius r would then be reserved for the area radius in which case $q_{\vartheta\vartheta} = r^2$. We use the same letter x for one of the planar coordinates of polarized Gowdy models, but the context of a given model will make it clear which coordinate x refers to.) As initially developed for models of loop quantum gravity [21–23], it is convenient to parametrize the metric components $q_{xx}^{\rm sph}$ and $q_{\vartheta\vartheta}^{\rm sph}$ as

$$q_{xx}^{\rm sph} = \frac{(E^\varphi)^2}{E^x}, \qquad q_{\vartheta\vartheta}^{\rm sph} = E^x, \tag{32}$$

where E^x and E^φ are the radial and angular densitized-triad components, respectively. We assume $E^x > 0$, fixing the orientation of space.

The canonical pairs for spherically symmetric classical gravity are given by (K_φ, E^φ) and (K_x, E^x) where $2K_x$ and K_φ are components of extrinsic curvature. We have a further canonical pair (ϕ, P_ϕ) if scalar matter is coupled to the gravitational system. The basic Poisson brackets are given by

$$\{K_x(x), E^x(y)\} = \{K_\varphi(x), E^\varphi(y)\} = \{\phi(x), P_\phi(y)\} = \delta(x-y). \tag{33}$$

(Compared with other conventions, our scalar phase-space variables are divided by $\sqrt{4\pi}$, absorbing the remnant of a spherical integration. We use units in which Newton's constant, G, equals one. This convention is formally different from what we are using in Gowdy models, where $2G/\pi$ equals one. The discrepancy is necessary in order to take into account the difference in coordinate areas for the symmetry orbits, given by $4\pi^2$ in the toroidal Gowdy model and 4π in spherical symmetry, as well as the varying multiplicity of independent degrees of freedom in the homogeneous directions.)

The Hamiltonian constraint is given by

$$\begin{aligned} H^{\rm sph} = &-\frac{\sqrt{E^x}}{2}\left[E^\varphi\left(\frac{1}{E^x} + \frac{K_\varphi^2}{E^x} + 4K_\varphi\frac{K_x}{E^\varphi}\right) - \frac{1}{4E^x}\frac{((E^x)')^2}{E^\varphi} + \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi}\right] \\ &+ \frac{1}{2}\left(\frac{\sqrt{q^{xx}_{\rm sph}}}{E^x}{P_\phi}^2 + E^x\sqrt{q^{xx}_{\rm sph}}(\phi')^2 + \sqrt{q_{xx}^{\rm sph}}E^x V(\phi)\right), \end{aligned} \tag{34}$$

with a scalar potential $V(\phi)$ [or $\frac{1}{2}V(\phi)$, depending on conventions], and

$$H_x^{\rm sph} = E^\varphi K_\varphi' - K_x(E^x)' + P_\phi\phi' \tag{35}$$

is the diffeomorphism constraint. The primes denote derivatives with respect to the radial coordinate x, which is unrelated to the coordinates of the Gowdy model. These constraints are first class and have Poisson brackets of hypersurface-deformation form,

$$\{H_x^{\rm sph}[N^x], H_x^{\rm sph}[M^x]\} = H_x^{\rm sph}[N^x M^{x\prime} - N^{x\prime}M^x], \tag{36a}$$

$$\{H^{\rm sph}[N], H_x^{\rm sph}[M^x]\} = -H^{\rm sph}[M^x N'], \tag{36b}$$

$$\{H^{\rm sph}[N], H^{\rm sph}[M]\} = H_x^{\rm sph}[q^{xx}_{\rm sph}(NM' - N'M)] \tag{36c}$$

with the structure function $q^{xx}_{\rm sph} = E^x/(E^\varphi)^2$ equal to the inverse radial component of the spacetime metric.

The off-shell gauge transformations for lapse and shift

$$\begin{aligned} \delta_\epsilon N &= \dot\epsilon^0 + \epsilon^x N' - N^x(\epsilon^0)', \\ \delta_\epsilon N^x &= \dot\epsilon^x + \epsilon^x(N^x)' - N^x(\epsilon^x)' + q^{xx}_{\rm sph}(\epsilon^0 N' - N(\epsilon^0)') \end{aligned} \tag{37}$$

together with the realization of covariance conditions for spacetime,

$$\left.\frac{\partial H^{\rm sph}}{\partial K_x'}\right|_{\rm O.S.} = \left.\frac{\partial H^{\rm sph}}{\partial K_x''}\right|_{\rm O.S.} = \cdots = 0 \tag{38}$$

and

$$\left.\frac{\partial(\{q^{xx}_{\rm sph}, H^{\rm sph}[\epsilon^0]\})}{\partial(\epsilon^0)'}\right|_{\rm O.S.} = \left.\frac{\partial(\{q^{xx}_{\rm sph}, H^{\rm sph}[\epsilon^0]\})}{\partial(\epsilon^0)''}\right|_{\rm O.S.} = \cdots = 0, \tag{39}$$

which have been derived in [2] and are clearly satisfied, ensures that the line element (31) is invariant. Its coefficients then form a covariant metric tensor in the sense that its canonical gauge transformations reproduce spacetime diffeomorphisms on-shell:

$$\delta_\epsilon g_{\mu\nu}|_{\text{O.S.}} = \mathcal{L}_\xi g_{\mu\nu}. \tag{40}$$

The gauge functions (ϵ^0, ϵ^r) on the left-hand side are related to the 2-component vector generator $\xi^\mu = (\xi^t, \xi^r)$ of the diffeomorphism on the right-hand side by

$$\xi^\mu = \epsilon^0 n^\mu + \epsilon^x s^\mu = \xi^t t^\mu + \xi^x s^\mu \tag{41}$$

with

$$\xi^t = \frac{\epsilon^0}{N}, \qquad \xi^x = \epsilon^x - \frac{\epsilon^0}{N} N^x. \tag{42}$$

In addition, the realization of the covariance conditions for matter [6],

$$\frac{\partial H^{\text{sph}}}{\partial P'_\phi} = \frac{\partial H^{\text{sph}}}{\partial P''_\phi} = \cdots = 0, \tag{43}$$

ensures that the matter field transforms as a spacetime scalar in the sense that its canonical gauge transformations reproduce spacetime diffeomorphisms on-shell:

$$\delta_\epsilon \phi|_{\text{O.S.}} = \mathcal{L}_\xi \phi. \tag{44}$$

Finally, we note that the spherically symmetric system in the absence of a scalar potential permits the global symmetry generator

$$G^{\text{sph}}[\alpha] = \int \mathrm{d}x\, \alpha P_\phi, \tag{45}$$

with constant α. The gravitational mass observable is

$$\mathcal{M}^{\text{sph}} = \frac{\sqrt{E^x}}{2}\left(1 + K_\varphi^2 - \left(\frac{(E^x)'}{2E^\varphi}\right)^2 - \frac{\Lambda}{3}E^x\right), \tag{46}$$

which is a Dirac observable in the vacuum limit, $\phi = P_\phi = 0$.

We are now ready to identify the analog relationship between the Gowdy and the spherically symmetric models. By inspection, we find that relabeling the canonical pairs according to

$$(\mathcal{A}, \varepsilon) \to (K_x, E^x), \qquad (K, \bar{a}) \to (K_\varphi, E^\varphi),$$
$$(\bar{W}, P_{\bar{W}}) \to (\phi, P_\phi) \tag{47}$$

turns the Gowdy constraints (21) and (22) into

$$H = -\sqrt{E^x}\left[E^\varphi\left(\frac{K_\varphi^2}{4E^x} + K_\varphi\frac{K_x}{E^\varphi}\right) - \frac{1}{4E^x}\frac{((E^x)')^2}{E^\varphi} + \frac{(E^\varphi)'(E^x)'}{(E^\varphi)^2} - \frac{(E^x)''}{E^\varphi}\right] + \frac{\sqrt{q^{\theta\theta}}}{4E^x}{P_\phi}^2 + E^x\sqrt{q^{\theta\theta}}(\phi')^2 \tag{48}$$

and

$$H_\theta = E^\varphi K'_\varphi - K_x (E^x)' + P_\phi \phi', \tag{49}$$

respectively, and the Gowdy metric components (24) become

$$q_{\theta\theta} = \frac{(E^\varphi)^2}{E^x}, \qquad q_{xx} = e^{2\phi}E^x, \qquad q_{yy} = e^{-2\phi}E^x. \tag{50}$$

Up to a few numerical factors, all the terms in the Gowdy constraint (48) match those of the spherically symmetric constraint (34) except for the first and last terms of the latter: The inverse triad $1/E^x$ and the scalar potential V do not appear in the former. In the general modified constraints of the spherically symmetric system [6] these two terms are just the classical limits of modification functions that are in principle allowed to be different from what the classical dynamics requires. (The scalar potential may always be set equal to zero in order to define a specific model, while the $1/E^x$-term is a special case of the dilaton potential that would be a free function of E^x if the spherically symmetric model were generalized to two-dimensional dilaton gravity.) We thus conclude that the modified Gowdy constraint is equivalent to the spherically symmetric one up to the choice of modification functions. In arriving at this conclusion, we have implicitly assumed that all of the conditions imposed in [6] to obtain the general constraints apply to the Gowdy system as well. We now show that this is indeed the case.

The conditions for the modified theory considered in [6] are the following ones. (1) Anomaly freedom, (2) covariance conditions, (3) existence of a conserved matter current, and (4) existence of a vacuum mass observable. Anomaly freedom of the Gowdy model takes exactly the same form as in spherical symmetry because the structure function of the former, Eq. (50), is equivalent to that of the latter, Eq. (32). The covariance conditions of the Gowdy system, Eqs. (15) and (16), are also equivalent to the spherically symmetric ones, Eqs. (39) and (43), upon using the analog identification (47). Finally, the Gowdy symmetry generator (27) is identical to the spherically symmetric one (45) under the same identification, while the Dirac observables (30) and (46) are identical up to one term that in the modified theory is given by the classical limit of a modification function. Therefore, all the classes of general modified constraints obtained in [6] are also the results of applying these conditions to the Gowdy system, if we only invert the correspondence.

In fact, there is one additional condition of the Gowdy system that the spherically symmetric one does not have: The discrete symmetry discussed in Sec. II C 1. In this sense the Gowdy system is more restricted than the spherically symmetric one. Therefore, we can simply take the final results of [6] and impose the discrete symmetry on them.

III. LINEAR COMBINATION OF THE CONSTRAINTS

Before discussing general modifications, an interesting restricted case is given by linear combinations of the classical constraints with suitable phase-space dependent coefficients. By construction, this class of theories preserves the classical constraint surface but modifies gauge transformations and the dynamics, implying a nonclassical emergent spacetime metric if the covariance conditions are fulfilled.

A. Anomaly-free linear combination

We define a new candidate for the Hamiltonian constraint as

$$H^{(\mathrm{new})} = BH^{(\mathrm{old})} + AH_\theta \tag{51}$$

with suitable phase-space functions A and B, using the original constraints $H^{(\mathrm{old})}$ and H_θ of the classical theory and keeping the latter unchanged. We consider the phase-space dependence $B = B(K, \varepsilon, \bar{W})$. (For more details about the individual steps, see [6].)

The Leibniz rule allows us to reduce the new bracket $\{H^{(\mathrm{new})}[\epsilon_1], H^{(\mathrm{new})}[\epsilon_2]\}$ to Poisson brackets of the old constraints with the functions A and B. Using the derivative terms of the classical constraints, Poisson brackets relevant for the anomaly freedom and covariance conditions can be expanded by finitely many terms with different orders of θ-derivatives of the gauge functions. For instance, we can write

$$\{B, H^{(\mathrm{old})}[\bar{\epsilon}^0]\}|_{\mathrm{O.S.}} = \mathcal{B}\bar{\epsilon}^0 + \mathcal{B}^\theta \partial_\theta \bar{\epsilon}^0|_{\mathrm{O.S.}}. \tag{52}$$

with

$$\mathcal{B}^\theta = \sqrt{\varepsilon}\frac{\varepsilon'}{\bar{a}^2}\frac{\partial B}{\partial K}. \tag{53}$$

Anomaly freedom of the new constraints, using hypersurface-deformation brackets for the old constraints, then requires

$$A = -\mathcal{B}^\theta = -\sqrt{\varepsilon}\frac{\varepsilon'}{\bar{a}^2}\frac{\partial B}{\partial K} \tag{54}$$

because any term in $\{H^{(\mathrm{new})}[\epsilon_1], H^{(\mathrm{new})}[\epsilon_2]\}$ that is not proportional to the diffeomorphism constraint must cancel out.

Similarly, we can write

$$\{A, H^{(\mathrm{old})}[\bar{\epsilon}^0]\} = \mathcal{A}^0\bar{\epsilon}^0 + \mathcal{A}^\theta\partial_\theta\bar{\epsilon}^0 \tag{55}$$

in which anomaly freedom together with (54) implies

$$\mathcal{A}^\theta = -\frac{\varepsilon}{\bar{a}^2}\left(K\frac{\partial B}{\partial K} + \frac{(\varepsilon')^2}{\bar{a}^2}\frac{\partial^2 B}{\partial K^2}\right). \tag{56}$$

Using this new function, the bracket

$$\{\mathcal{A}^\theta, H^{(\mathrm{old})}[\bar{\epsilon}^0]\} = \Lambda^0\bar{\epsilon}^0 + \Lambda^\theta\partial_\theta\bar{\epsilon}^0 \tag{57}$$

requires

$$\Lambda^\theta = -\frac{\varepsilon^{3/2}\varepsilon'}{\bar{a}^4}\left(\frac{\partial B}{\partial K} + 3K\frac{\partial^2 B}{\partial K^2} + \frac{(\varepsilon')^2}{\bar{a}^2}\frac{\partial^3 B}{\partial K^3}\right). \tag{58}$$

The new structure function,

$$q^{\theta\theta}_{(\mathrm{new})} = B^2 q^{\theta\theta} + B\mathcal{A}^\theta, \tag{59}$$

follows from collecting all terms in the Poisson bracket of two Hamiltonian constraints that can contribute to the diffeomorphism constraint.

B. Covariant modified theory

Using the new structure function as an inverse spatial metric, the covariance condition is given by

$$\begin{aligned}\mathcal{C} &\equiv \Lambda^\theta - B^{-1}\mathcal{B}^\theta\mathcal{A}^\theta|_{\mathrm{O.S.}}\\ &= -\frac{\varepsilon^{3/2}\varepsilon'}{\bar{a}^4}\left(\frac{\partial B}{\partial K} + 3K\frac{\partial^2 B}{\partial K^2} + \frac{(\varepsilon')^2}{\bar{a}^2}\frac{\partial^3 B}{\partial K^3}\right)\\ &\quad + \frac{\varepsilon^{3/2}\varepsilon'}{\bar{a}^4}B^{-1}\frac{\partial B}{\partial K}\left(K\frac{\partial B}{\partial K} + \frac{(\varepsilon')^2}{\bar{a}^2}\frac{\partial^2 B}{\partial K^2}\right) = 0.\end{aligned} \tag{60}$$

We separate this condition into derivative terms,

$$\mathcal{C} = \mathcal{C}_\varepsilon\varepsilon' + \mathcal{C}_{\varepsilon\varepsilon\varepsilon}(\varepsilon')^3, \tag{61}$$

which must vanish individually. The equation $\mathcal{C}_\varepsilon = 0$ implies

$$K\left(\frac{\partial B}{\partial K}\right)^2 + B\left(K\frac{\partial^2 B}{\partial K^2} - \frac{\partial B}{\partial K}\right) = 0 \tag{62}$$

and is solved by

$$B = c_1\sqrt{c_2 \pm K^2}. \tag{63}$$

The equation $\mathcal{C}_{\varepsilon\varepsilon\varepsilon} = 0$ implies

$$B\frac{\partial^3 B}{\partial K^3} + 3\frac{\partial B}{\partial K}\frac{\partial^2 B}{\partial K^2} = 0 \tag{64}$$

and is solved by

$$B = \tilde{c}_1\sqrt{\tilde{c}_2 \pm K^2 + \tilde{c}_3 K}. \tag{65}$$

In these solutions, c_i and $\tilde{c}_i$ are free functions of $\bar{W}$ and ε. Their mutual consistency requires

$$B_s(K, \bar{W}, \varepsilon) = \lambda_0\sqrt{1 - s\lambda^2 K^2}, \tag{66}$$

which then implies

$$A_s = \lambda_0\sqrt{\varepsilon}\frac{\varepsilon'}{\bar{a}^2}\frac{s\lambda^2 K}{\sqrt{1 - s\lambda^2 K^2}} \tag{67}$$

such that

$$q^{\theta\theta}_{(\mathrm{new})} = \lambda_0^2\left(1 + \frac{s\lambda^2}{1 - s\lambda^2 K^2}\frac{(\varepsilon')^2}{\bar{a}^2}\right)\frac{\varepsilon}{\bar{a}^2}. \tag{68}$$

There are two remaining free functions, $\lambda_0 = \lambda_0(\bar{W}, \varepsilon)$ and $\lambda = \lambda(\bar{W}, \varepsilon)$, and we have separated the sign $s = \pm 1$ from the original solution, Eq. (63). Reality requires that $1 - s\lambda^2 K^2 \geq 0$, which may place an upper bound on K depending on s and λ. Finally, the discrete symmetry requires that both modification functions are even in $\bar{W}$: $\lambda_0(\bar{W}, \varepsilon) = \lambda_0(-\bar{W}, \varepsilon)$ and $\lambda(\bar{W}, \varepsilon) = \lambda(-\bar{W}, \varepsilon)$.

Since we now have complete solutions for A and B, we can derive the modified Hamiltonian constraint from (51):

$$H^{(\mathrm{new})} = -\lambda_0\sqrt{\varepsilon}\sqrt{1 - s\lambda^2 K^2}\left[\bar{a}\left(\frac{K^2}{4\varepsilon} - \frac{1}{4\varepsilon}\frac{P_{\bar{W}}^2}{\bar{a}^2} + \frac{\mathcal{A}}{\bar{a}}K\right) - \varepsilon\frac{(\bar{W}')^2}{\bar{a}} - \frac{1}{4\varepsilon}\frac{(\varepsilon')^2}{\bar{a}} + \frac{\bar{a}'\varepsilon'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}} \right.$$
$$\left. - \frac{\varepsilon'}{\bar{a}^2}\frac{s\lambda^2 K}{1 - s\lambda^2 K^2}(\bar{a}K' + P_{\bar{W}}\bar{W}' - \mathcal{A}\varepsilon')\right]. \tag{69}$$

The case $s = 1$, together with a reality condition for the constraint, implies a curvature bound $K < 1/\lambda$. The case $s = -1$ implies a possibility of signature change where $q^{xx}_{(\mathrm{new})}$ changes sign. [The inverse spatial metric is then determined by the absolute value of (51).]

1. Canonical transformations

For the case $s = 1$, a natural canonical transformation is

$$K \to \frac{\sin(\lambda K)}{\lambda}, \qquad \bar{a} \to \frac{\bar{a}}{\cos(\lambda K)},$$
$$\bar{W} \to \bar{W}, \qquad P_{\bar{W}} \to P_{\bar{W}} - \frac{\bar{a}}{\cos(\lambda K)}\frac{\partial}{\partial\bar{W}}\left(\frac{\sin(\lambda K)}{\lambda}\right),$$
$$\varepsilon \to \varepsilon, \qquad \mathcal{A} \to \mathcal{A} + \frac{\bar{a}}{\cos(\lambda K)}\frac{\partial}{\partial\varepsilon}\left(\frac{\sin(\lambda K)}{\lambda}\right), \tag{70}$$

under which the modified Hamiltonian constraint becomes

$$H^{(\mathrm{c})} = -\lambda_0\sqrt{\varepsilon}\left[\bar{a}\left(\frac{1}{4\varepsilon}\frac{\sin^2(\lambda K)}{\lambda^2} - \frac{1}{4\varepsilon}\cos^2(\lambda K)\left(\frac{P_{\bar{W}}}{\bar{a}} - \frac{\partial\ln\lambda}{\partial\bar{W}}K + \frac{\tan(\lambda K)}{\lambda}\frac{\partial\ln\lambda}{\partial\bar{W}}\right)^2 \right.\right.$$
$$\left. + \frac{\sin(2\lambda K)}{2\lambda}\left(\frac{\mathcal{A}}{\bar{a}} + \frac{\partial\ln\lambda}{\partial\varepsilon}K - \frac{\tan(\lambda K)}{\lambda}\frac{\partial\ln\lambda}{\partial\varepsilon}\right)\right)$$
$$- \varepsilon\frac{(\bar{W}')^2}{\bar{a}}\cos^2(\lambda K) + \frac{\bar{a}'\varepsilon'}{\bar{a}^2}\cos^2(\lambda K) - \lambda^2\frac{\sin(2\lambda K)}{2\lambda}\left(\frac{P_{\bar{W}}}{\bar{a}} - \frac{\partial\ln\lambda}{\partial\bar{W}}K\right)\frac{\bar{W}'\varepsilon'}{\bar{a}}$$
$$\left. - \left(\frac{\cos^2(\lambda K)}{4\varepsilon} - \lambda^2\frac{\sin(2\lambda K)}{2\lambda}\left(\frac{\mathcal{A}}{\bar{a}} + \frac{\partial\ln\lambda}{\partial\bar{\varepsilon}}K\right)\right)\frac{(\varepsilon')^2}{\bar{a}} - \frac{\varepsilon''}{\bar{a}}\cos^2(\lambda K)\right]. \tag{71}$$

A second canonical transformation,

$$\begin{aligned} K &\to \frac{\bar{\lambda}}{\lambda}K, \qquad \bar{a} \to \frac{\lambda}{\bar{\lambda}}\bar{a}, \\ \bar{W} &\to \bar{W}, \qquad P_{\bar{W}} \to P_{\bar{W}} + \frac{\bar{\lambda}}{\lambda}\bar{a}\frac{\partial \ln\lambda}{\partial \bar{W}}K, \\ \varepsilon &\to \varepsilon, \qquad \mathcal{A} \to \mathcal{A} - \frac{\bar{\lambda}}{\lambda}\bar{a}\frac{\partial\ln\lambda}{\partial\varepsilon}K, \end{aligned} \tag{72}$$

with constant $\bar{\lambda}$, renders the modified Hamiltonian constraint periodic in K:

$$\begin{aligned} H^{(\mathrm{cc})} = -\lambda_0\frac{\bar{\lambda}}{\lambda}\sqrt{\varepsilon}\Bigg[&\frac{\bar{a}}{4\varepsilon}\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} - \frac{\bar{a}}{4\varepsilon}\cos^2(\lambda K)\left(\frac{P_{\bar{W}}}{\bar{a}} + \frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\bar{W}}\right)^2 + \frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\left(\mathcal{A} - \bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\varepsilon}\right) \\ &- \varepsilon\cos^2(\bar{\lambda}K)\frac{(\bar{W}')^2}{\bar{a}} + \left(\frac{\cos^2(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\lambda}{\partial\bar{W}} - \bar{\lambda}^2\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\frac{P_{\bar{W}}}{\bar{a}}\right)\frac{\bar{W}'\varepsilon'}{\bar{a}} - \left(\frac{\cos^2(\bar{\lambda}K)}{4\varepsilon}\left(1 - 4\varepsilon\frac{\partial\ln\lambda}{\partial\varepsilon}\right) - \bar{\lambda}^2\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\frac{\mathcal{A}}{\bar{a}}\right)\frac{(\varepsilon')^2}{\bar{a}} \\ &+ \cos^2(\bar{\lambda}K)\left(\frac{\bar{a}'\varepsilon'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}}\right)\Bigg]. \end{aligned} \tag{73}$$

[The term $(\partial\ln\lambda/\partial\epsilon)K$ in (71) then disappears.] Unlike the phase-space coordinates in (69), the holonomylike coordinates of (71) imply a finite constraint at the curvature bound, implying a dynamics that can cross such a hypersurface of maximum curvature.

The expression

$$\sin(\bar{\lambda}K) = \sin\left(\bar{\lambda}(E^xK_x + E^yK_y)/\sqrt{E^xE^y}\right) \tag{74}$$

in the periodic version of the constraint always requires a nontrivial dependence on the densitized triads, in contrast to what appears in the corresponding K-dependent terms of spherically symmetric models, or of a restricted Gowdy system in which $E^x = E^y$ and $K_x = K_y$ and the argument of periodic functions is a single K-component. In the full polarized Gowdy model, some densitized-triad dependence always remains even if the initial function λ, which may depend on ε as well as $\bar{W}$, has been replaced by a constant $\bar{\lambda}$ using a canonical transformation. The specific phase-space function in (74) can be related to the (x,y)-contribution to the trace of the momentum tensor K_a^i canonically conjugate to E^a_i, given by $K^i_aE^a_i/\sqrt{|\det E|} = e^a_iK^i_a$. In general, however, the expression in emergent modified Gowdy models is not equal to the trace of extrinsic curvature in the resulting emergent spacetime for two reasons. First, the phase-space expressions E^a_i and K^i_a have modified geometrical meanings compared with the classical densitized triad and extrinsic curvature of spatial slices because the geometry is determined by the emergent metric. Second, the momentum tensor K^i_a with components that appear in (74) has been altered by several canonical transformations applied in our derivations.

2. Interpretation as holonomy modifications

Periodic K-dependent functions in modified constraints are often interpreted as potential effective descriptions of loop quantum gravity. This canonical approach to quantum gravity defines a Hilbert space of wave functions that depend on a gravitational connection A^i_a (the Ashtekar-Barbero connection) through matrix elements of SU(2) holonomies, $\mathcal{P}\exp(i\int A^i_a\sigma_i\mathrm{d}x^a)$ with SU(2)-generators σ_i. Since SU(2) is compact, it is possible to use a well-defined measure on the state space and represent holonomies as bounded operators. The Hamiltonian constraint should then also be expressed in terms of holonomy functions in order to act on the state space. This step requires modifications of the polynomial dependence of the classical constraint on A^i_a, for instance by instead using periodic functions as obtained from certain matrix elements of SU(2) holonomies. As a motivation, it is argued that the fundamental Hamiltonian constraint, for instance for the dynamics of some discrete quantum geometry, is expected to equal the classical constraint only at low curvature, while its general properties should be determined by consistency conditions within a framework of quantum gravity, such as representability on a given state space and anomaly freedom.

In models of loop quantum gravity, it is common to forego using operators and instead analyze the dynamics implied by modified constraints on a classical phase space. Periodic modifications of the connection or extrinsic-curvature dependence are still referred to as holonomy modifications in this context. However, relating the K-terms in (73) or (71) to traditional holonomy modifications in models of loop quantum gravity therefore requires some care because the additional dependence on E-components differs from what has usually been assumed. Compared with the traditional approach, there are two crucial new ingredients in our treatment: A strict imposition of covariance, and a detailed discussion of how canonical transformations can be used to relate different types of holonomy modifications (or in some cases to relate a supposed holonomy modification to the unmodified classical theory).

Any appearance of triad components in holonomylike terms in models of loop quantum gravity is usually motivated as a volume or area dependence of the coordinate length of a holonomy used to construct the Hamiltonian constraint. This property is not derived from fundamental operators but rather imposed phenomenologically, mainly in order to achieve certain desired properties in cosmological models such as classical behavior at large volume. In particular, dynamical solutions lead to large symmetry orbits, such as all of space in homogeneous models of an expanding universe or spherical orbits in nonrotating black-hole models. As a consequence, extrinsic-curvature components, given by linear combinations of time derivatives of the metric or triad components, can be large even in classical regimes. Their appearance in holonomies is then in danger of violating the classical limit on large length scales. This problem can be solved in an *ad hoc* manner by using a length parameter for holonomies that decreases with the size of increasing symmetry orbits, such that holonomy modifications are negligible even when some extrinsic-curvature or connection components become large. Heuristically, such a dependence can be motivated by lattice refinement [24], relating the holonomy length to a lattice structure in space that is being subdivided as the symmetry orbit expands, maintaining sufficiently short geometrical lengths of its edges.

Comparing with this motivation, the specific version of holonomylike terms of the form (74) found here, required for covariance, is crucially different: The coefficient functions of K_x and K_y can both be expressed in terms of $E^x/E^y = \sqrt{q_{yy}/q_{xx}} = e^{-2\bar{W}}$, which describes the geometrical anisotropy in the x-y plane but is independent of its area $\sqrt{q_{xx}q_{yy}} = \varepsilon$. Analyzing the general form of potential physical implications of this difference requires us to perform a detailed analysis of canonical transformations used here to arrive at the expression (74).

3. Phenomenology of holonomy modifications

In this context, it is useful to consider possible forms and interpretations of holonomy modifications for models of loop quantum gravity in the strictly isotropic context [25,26], in which spatial homogeneity eliminates the non-trivial covariance conditions. (See also [15] for a related discussion in spherical symmetry.) Extrinsic curvature (or a connection with its associated holonomies) reduced to isotropy has a single independent component, k, canonically conjugate to the independent densitized-triad component p. (We assume $p > 0$, fixing the orientation of space.) Classically, using the scale factor a, we have $k \propto \dot{a}$ and $p \propto a^2$. Holonomies for U(1), or suitable components of holonomies for SU(2), are then of the form $\exp(i\ell k)$ with the coordinate length ℓ of a spatial curve, derived from the general $\mathcal{P}\exp(i\int A^i_a \sigma_i \mathrm{d}x^a)$ for an isotropic $A^i_a \propto \delta^i_a$, with generators σ_i of the gauge group. The geometrical length of this curve in an expanding universe increases like ℓa and may therefore reach macroscopic values after a suitable amount of time. Similarly, $k \propto \dot{a} = aH$ with the Hubble parameter H is an approximately linear function of a in a universe dominated by dark energy or during inflation. The exponent ℓk is then large in a macroscopic universe, such that modifications would be noticeable on low curvature scales and contradict cosmological observations.

This problem can be solved phenomenologically by using a coordinate length or holonomy parameter $\ell \propto a^{-1} \propto p^{-1/2}$, such that the geometrical length is constant in an expanding universe. The relevant phase-space function $\exp(i\bar{\ell}k/\sqrt{p})$, with a constant $\bar{\ell}$, then depends on extrinsic-curvature and densitized-triad components. It is easier to quantize this expression if one first applies a canonical transformation that turns $k/\sqrt{p}$ into a basic canonical variable. Classically, this ratio is proportional to the Hubble parameter H, and the map from (k, p) to H can be completed to a canonical transformation by using the volume $V \propto a^3$ of some region in space, whose precise form does not matter thanks to homogeneity and isotropy. It is then possible to quantize $\exp(i\bar{\ell}H)$ to a simple translation operator in V.

Different versions of holonomy modifications are obtained by introducing periodic functions depending on different variables, such as k or H. The classical contribution to the isotropic Hamiltonian constraint can be written as $\sqrt{p}k^2 = p^{3/2}(k/\sqrt{p})^2 \propto VH^2$. Holonomy modifications may then be introduced for k or H (or any function of the form $p^q k$ with some exponent q), leading to dynamically inequivalent modifications of the form $\mathcal{H}_1 = \sqrt{p}\sin^2(\bar{\ell}k)/\bar{\ell}^2$ and $\mathcal{H}_2 = V\sin^2(\bar{\ell}H)/\bar{\ell}^2$, respectively. The latter can be transformed back to k-variables, implying a term proportional to $p^{3/2}\sin^2(\bar{\ell}k/\sqrt{p})/\bar{\ell}^2$ in which the decreasing length scale $\ell = \bar{\ell}/\sqrt{p}$ appears. Independently of canonical transformations, the different types of holonomy modifications can also be identified by analyzing equations of motion for small $\bar{\ell}$. From $\mathcal{H}_1$, we obtain $\dot{p} \propto \sqrt{p}k$ or $k \propto \dot{p}/\sqrt{p} \propto \dot{a}$, while $\mathcal{H}_2$ implies $\dot{V} \propto VH$ or $H \propto \dot{V}/V \propto \dot{a}/a$. Therefore, we do not have to know which canonical transformations may have been applied in order to determine how a given classical or modified constraint implies small or large values of holonomy modifications in classical regimes.

In isotropic models, the appearance of a scale-dependent holonomy length can be seen in two alternative ways: A dependence on the scale factor may directly appear in periodic functions, as in $\sin(\bar{\ell}k/\sqrt{p})$, or it may be implied by equations of motion that tell us whether an expression such as H in $\exp(i\bar{\ell}H)$ equals the classical basic phase-space variable k in the limit of small $\bar{\ell}$, or a different function such as H in which the potential growth of k as a function of a in some dynamical solutions is reduced.

More generally, the different status of a modification with scale-factor dependent ℓ can be seen in coefficients of the Hamiltonian constraint. In isotropic models, a holonomy modification can be implemented by directly replacing the classical k in the Hamiltonian constraint with $\ell^{-1}\sin(\ell k)$. For constant $\ell = \bar{\ell}$, the k-independent coefficient of this term retains its classical dependence on p. If ℓ depends on p, or if H is used instead of k, the p-dependence of the coefficient is modified along with the k-dependence. From the point of view of canonical structures, there is no difference between (k, p) and (H, V) if the relationship between k or H and classical extrinsic curvature is ignored (or unknown if one considers a generic modified theory). A modification of the form

$$\frac{\sin(\ell k_1)}{\ell} = \frac{\sin(\bar{\ell}k_2)}{\ell} = \frac{\bar{\ell}}{\ell}\frac{\sin(\bar{\ell}k_2)}{\bar{\ell}} \tag{75}$$

with triad-dependent $\ell/\bar{\ell}$, such that the map from k_1 to k_2 is part of a canonical transformation with $\ell k_1 = \bar{\ell}k_2$, can therefore be interpreted in two different ways, depending on whether k_1 or k_2 is closely related to classically reduced extrinsic curvature. If k_1 is extrinsic curvature, we have a triad-dependent holonomy length ℓ, and the small-k_1 limit reproduces the classical dependence of the coefficients because $\ell^{-1}\sin(\ell k_1) = k_1(1 + O(\ell^2 k_1^2))$. If k_2 is extrinsic curvature, we have constant holonomy length $\bar{\ell}$, and the classical triad-dependent coefficients of k_2 are modified because

$$\frac{\sin(\bar{\ell}k_2)}{\ell} = \frac{\bar{\ell}}{\ell}k_2(1 + O(\bar{\ell}^2 k_2^2)) = k_1(1 + O(\bar{\ell}^2 k_2^2)). \tag{76}$$

Instead of reducing the growing holonomy length in an expanding universe, the model is made compatible with the classical limit, producing the same k_1 to leading order, by modifying the triad-dependent coefficients of k_2-holonomy terms in the Hamiltonian constraint by factors of $\bar{\ell}/\ell$.

However, this classical limit, assuming small $\bar{\ell}k_2$, is in general only formal because it may not be guaranteed that this product is indeed small in expected classical regimes, such as a large isotropic universe. The limit is suitable as a classical one if $k_2 = H$, but not if $k_2 = k$. In isotropic models, the H-variable is therefore preferred. Therefore, k_1 rather than k_2 can be identified with extrinsic curvature in the classical limit, necessitating the application of a nonconstant holonomy function λ. Whether a canonical variable behaves like k or like H (or possibly a different version) follows from equations of motion generated by the modified Hamiltonian constraint.

4. Holonomy modifications in polarized Gowdy models

The possibility of applying canonical transformations in isotropic models is comparable to some of the steps in our derivation of covariant modifications of polarized Gowdy models. We have constant holonomy parameters $\bar{\lambda}$ or $\bar{\ell}$ in one form, and triad-dependent functions λ or ℓ in another one. In each case, both versions, if they are related by (70) or the canonical transformation that includes the mapping between H and k, are dynamically equivalent. But the two versions are not equivalent if the holonomy function, constant or nonconstant, multiplies the same phase-space function without applying a canonical transformation. Since the general form of a modified theory is defined only by its Hamiltonian constraint and does not contain an independent specification of what canonical transformations may have been used compared with the standard classical phase space, we should consider equations of motion in order to determine whether holonomy modifications depending on K in a polarized Gowdy model can include an area-dependent holonomy length.

For nonconstant $\lambda/\bar{\lambda}$, the ε-dependence of the coefficients in (73) differs from the classical one in the limit of $\bar{\lambda} \to 0$. As in (76), these terms signal deviations of K from the original component of extrinsic curvature: The equation of motion for ε (which is canonically conjugate to $\mathcal{A}$) is given by

$$\dot{\varepsilon} = \{\varepsilon, H^{(cc)}\} = -\lambda_0\frac{\bar{\lambda}}{\lambda}\sqrt{\varepsilon}\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}(1 + \bar{\lambda}^2(\varepsilon')^2/\bar{a}^2) \tag{77}$$

and implies

$$K \sim -\frac{\lambda}{\bar{\lambda}\lambda_0}\frac{\dot{\varepsilon}}{\sqrt{\varepsilon}} \tag{78}$$

for small $\bar{\lambda}$ if we assume $\lambda = \bar{\lambda}h(\varepsilon)$ with a $\bar{\lambda}$-independent holonomy function $h(\varepsilon)$. (In some classes of modified theories, h may also depend on the anisotropy parameter $\bar{W}$.) The formal classical limit requires small $\bar{\lambda}K$, but a specific regime in which classical behavior is expected may well imply large $\bar{\lambda}K$ if the area ε of symmetry orbits is large. If one chooses a λ that decreases with ε sufficiently quickly, Eq. (78) implies that the corresponding K of the modified theory increases less strongly than in a model with constant $\lambda = \bar{\lambda}$. It is easier to study the classical limit if the inverse of the canonical transformation (70) is applied, such that all holonomy terms now depend on λK with an explicit decreasing coefficient λ as a function of ε. As shown in the transition from a Hamiltonian constraint of the form (73) to an expression (71), ε-dependent modifications $\lambda/\bar{\lambda}$ of coefficients in the Hamiltonian constraint are then replaced with holonomylike terms with an ε-dependent function λ.

In isotropic models, these two versions, given by triad-dependent coefficients and triad-dependent holonomy length, respectively, are equivalent. In polarized Gowdy models, in which modifications are strongly restricted by covariance conditions, only the first viewpoint is available if a strict definition of holonomy modifications as periodic functions is used: Only (73), in which the coefficient functions are modified in their ε-dependence, is periodic in K, while (71), in which the function $\lambda(\varepsilon)$ appears in some of the holonomy terms, also contains nonperiodic contributions linear in K such as $K\partial \ln\lambda/\partial\varepsilon$. Effects of an ε-dependent holonomy length can therefore be inferred only indirectly when equations of motion are used, turning it into an on-shell property. Assigning an ε-dependent holonomy length directly to off-shell properties of the Hamiltonian constraint, as in isotropic models, is not possible unless one weakens the strict periodicity condition on holonomy modifications.

Another difference between isotropic and polarized Gowdy models appears in the specific form (74) interpreted in terms of components K_x and K_y of the momentum that appear in the phase-space function K. The only triad dependence allowed in this combination refers to anisotropy in the (x, y)-plane rather than its area. This specific dependence, just as properties of how an ε-dependent λ may appear in holonomy modifications, is implied by general covariance. The anisotropy dependence of holonomy modifications is therefore unavoidable, and unlike $\lambda(\varepsilon)$ it cannot be moved to coefficient functions. Moreover, holonomy modifications can only be implemented for the specific combination of K_x and K_y given by (74), but not separately for the two components K_x and K_y because all modified constraints allowed by covariance depend polynomially on the second phase-space variable, $P_{\bar{W}}$, that together with K represents K_x and K_y after our first canonical transformation.

Therefore, unlike in spherically symmetric models, the Hamiltonian constraint is not built out of basic holonomy operators that depend only on momentum components canonically conjugate to the densitized triad. There is always a necessary triad dependence given by the specific form of K that may appear in periodic terms as a linear combination of K_x and K_y with triad-dependent coefficients, derived from the covariance conditions. In loop quantum gravity, curvature (or connection) components and the triad are instead separated into basic holonomy and flux operators, which were used as building blocks of the first proposed operators for the Hamiltonian constraint [27]. More recent versions [28–30] use triad-dependent shift vectors in order to construct detailed properties of hypersurface deformations from operators, which is somewhat reminiscent of but conceptually unrelated to the triad dependence of holonomy-type expressions found here.

IV. GENERAL MODIFIED THEORY

Linear combinations of the classical constraints with phase-space dependent coefficients have revealed interesting properties of possible modifications of polarized Gowdy models. More generally, one may expect that individual terms in the Hamiltonian constraint can receive independent modifications. We now analyze this possibility within a setting of effective field theory in which we expand a generic Hamiltonian constraint in derivatives up to second order. The resulting expressions then determine gravitational theories of polarized Gowdy models compatible with the symmetry of general covariance, taking into account the possibility that the spacetime metric is not fundamental but rather emergent. New modifications are then possible even at the classical order of derivatives.

A. Constraint ansatz and the emergent spacetime metric

We consider modifications to the Gowdy system with phase-space variables $(\bar{W}, P_{\bar{W}})$, $(K, \bar{a})$, and $(\mathcal{A}, \varepsilon)$. If we modify the Hamiltonian constraint, then the constraint brackets (8)–(10) determine the inhomogeneous component of the spatial metric via $\tilde{q}_{\theta\theta} = 1/\tilde{q}^{\theta\theta}$, while the homogeneous components of the metric cannot be obtained in this way because they do not appear in the structure functions. The emergent spacetime line element is then given by

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \tilde{q}_{\theta\theta}(\mathrm{d}\theta + N^\theta \mathrm{d}t)^2 + \alpha_\varepsilon(\varepsilon)e^{2f_W(\varepsilon,\bar{W})}\mathrm{d}x^2 + \alpha_\varepsilon(\varepsilon)e^{-2f_W(\varepsilon,\bar{W})}\mathrm{d}y^2, \tag{79}$$

with $\tilde{q}_{\theta\theta}$ to be determined by anomaly freedom of the hypersurface deformation brackets, while we have partially chosen the form of the homogeneous components $\tilde{q}_{xx}$ and $\tilde{q}_{yy}$ based on their classical forms. We will discuss the free functions α_ε and f_W in due course. (If the structure function is negative in some regions, the inhomogeneous metric component is determined by the inverse of its absolute value, while $-N^2\mathrm{d}t^2$ is replaced by $-\sigma N^2\mathrm{d}t^2$ where σ is the sign of the structure function relative to the classical function, making the four-dimensional line element Euclidean in regions where $\sigma = -1$. For more details, see [2].)

We consider the following ansatz for the Hamiltonian constraint:

$$\tilde{H} = a_0 + e_{\bar{W}\bar{W}}(\bar{W}')^2 + e_{\bar{a}\bar{a}}(\bar{a}')^2 + e_{\varepsilon\varepsilon}(\varepsilon')^2 + e_{\bar{W}\bar{a}}\bar{W}'\bar{a}' + e_{\bar{W}\varepsilon}\bar{W}'\varepsilon' + e_{\bar{a}\varepsilon}\bar{a}'\varepsilon' + p_{\bar{a}\varepsilon}K'\varepsilon' + r_{\bar{a}\bar{a}}\bar{a}'K' + e_{2\bar{a}}\bar{a}'' + p_{2\bar{a}}K'' + e_{2\varepsilon}\varepsilon'', \tag{80}$$

where a_0, e_{ij}, e_{2i}, p_{ij}, and p_{2i} are all functions of the phase-space variables, but not of their derivatives. For the sake of tractability, we have omitted some terms that would be possible at second order in derivatives, such as terms containing $\bar{W}''$. Spatial derivatives of $\mathcal{A}$ and $P_{\bar{W}}$ have been omitted in anticipation of covariance condition (88), to be discussed shortly, which requires the constraint to be independent of them. Because of the discrete symmetry $\tilde{H}(\bar{W}, P_{\bar{W}}) = \tilde{H}(-\bar{W}, -P_{\bar{W}})$, we see that all the functions obey this even symmetry, except for $e_{\bar{W}\varepsilon}$ which should be odd.

Starting from this constraint ansatz we will obtain the conditions for it to satisfy the hypersurface-deformation brackets, Eqs. (8)–(10), with a possibly modified structure function, $\tilde{q}^{\theta\theta}$. We will then apply the covariance conditions in order to make sure that the new structure function can play the role of an inverse metric component in spacetime.

1. Canonical transformations I

To obtain distinct classes of Hamiltonian constraints for possible modified theories, it is crucial that we factor out canonical transformations that preserve the diffeomorphism constraint. If we do not take this extra care, we risk obtaining equivalent versions of the same theory that differ only in a choice of the phase-space coordinates. Two constraints differing only by a canonical transformation will look different and even the spacetime metric will do so too kinematically, but they in fact describe the same physical system.

We will therefore consider the following set of canonical transformations that preserves the diffeomorphism constraint:

$$\bar{W} = f_c^{\bar{W}}(\varepsilon, \tilde{\bar{W}}), \quad P_{\bar{W}} = \tilde{P}_{\bar{W}} \left(\frac{\partial f_c^{\bar{W}}}{\partial \tilde{\bar{W}}}\right)^{-1} - \tilde{\bar{a}} \frac{\partial f_c^K}{\partial \tilde{\bar{W}}} \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1}, \tag{81a}$$

$$K = f_c^K(\varepsilon, \tilde{\bar{W}}, \tilde{K}), \qquad E^\varphi = \tilde{E}^\varphi \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1}, \tag{81b}$$

$$\mathcal{A} = \frac{\partial(\alpha_c^2 \varepsilon)}{\partial \varepsilon} \tilde{\mathcal{A}} + \tilde{E}^\varphi \frac{\partial f_c^K}{\partial \varepsilon} \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1} + \tilde{P}_{\bar{W}} \frac{\partial f_c^{\bar{W}}}{\partial \varepsilon} \left(\frac{\partial f_c^{\bar{W}}}{\partial \tilde{\bar{W}}}\right)^{-1},$$
$$\tilde{\varepsilon} = \alpha_c^2(\varepsilon)\varepsilon, \tag{81c}$$

where the new phase-space variables are written with a tilde. A transformation with $f_c^K = \tilde{K}$, $f_c^{\bar{W}}(\varepsilon, \tilde{\bar{W}})$, and $\alpha_c(\varepsilon)$ can always be used to transform the homogeneous components of the metric in (79) from potentially modified expressions to their classical ones $\tilde{q}_{xx} = \varepsilon e^{2\bar{W}}$ and $\tilde{q}_{yy} = \varepsilon e^{-2\bar{W}}$. If we fix the classical form for these components, the residual canonical transformations are given by

$$\bar{W} = \tilde{\bar{W}}, \qquad P_{\bar{W}} = \tilde{P}_{\bar{W}} - \tilde{\bar{a}} \frac{\partial f_c^K}{\partial \tilde{\bar{W}}} \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1}, \tag{82a}$$

$$K = f_c^K(\varepsilon, \tilde{\bar{W}}, \tilde{K}), \qquad E^\varphi = \tilde{E}^\varphi \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1}, \tag{82b}$$

$$\mathcal{A} = \tilde{\mathcal{A}} + \tilde{E}^\varphi \frac{\partial f_c^K}{\partial \varepsilon} \left(\frac{\partial f_c^K}{\partial \tilde{K}}\right)^{-1}, \qquad \tilde{\varepsilon} = \varepsilon, \tag{82c}$$

where the new phase-space variables are again written with a tilde.

Under the previous canonical transformation, the emergent spacetime metric simplifies to

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + \tilde{q}_{\theta\theta}(\mathrm{d}\theta + N^\theta \mathrm{d}t)^2 + \varepsilon e^{2\bar{W}} \mathrm{d}x^2 + \varepsilon e^{-2\bar{W}} \mathrm{d}y^2, \tag{83}$$

while the constraint ansatz (80) does not acquire any new derivative terms.

2. Anomaly freedom and covariance conditions

Starting with a Hamiltonian constraint of the form (80) we impose anomaly freedom by requiring that, together with the unmodified diffeomorphism constraint, it reproduces the hypersurface-deformation brackets (8)–(10) up to a potentially modified structure function:

$$\{H_\theta[N^\theta], H_\theta[M^\theta]\} = -H_\theta[M^\theta(N^\theta)' - N^\theta(M^\theta)'], \tag{84}$$

$$\{\tilde{H}[N], H_\theta[M^\theta]\} = -\tilde{H}[M^\theta N'], \tag{85}$$

$$\{\tilde{H}[N], \tilde{H}[M]\} = -H_\theta[\tilde{q}^{\theta\theta}(MN' - NM')]. \tag{86}$$

Doing so restricts the functions in (80) by a set of partial differential equations for the modification functions. The same procedure reveals the dependence of the structure function $\tilde{q}^{\theta\theta}$ on the phase-space variables. Furthermore, the modified brackets (84)–(86) imply gauge transformation of the shift vector (12) according to

$$\delta_\epsilon N^\theta = \dot{\epsilon}^\theta + \epsilon^\theta (N^\theta)' - N^\theta (\epsilon^\theta)' + \tilde{q}^{\theta\theta}(\epsilon^0 N' - N(\epsilon^0)'), \tag{87}$$

which now involves the modified structure function as it should if it were to play the role of a component of the emergent spacetime metric.

Once the structure function has been obtained from the imposition of anomaly freedom, we have the full expression of a candidate spacetime metric as in (79). Because the homogeneous metric components retain their classical forms, the covariance conditions (15) remain unchanged. In the new phase-space variables they are given by

$$\frac{\partial \tilde{H}}{\partial \mathcal{A}'} = \frac{\partial \tilde{H}}{\partial \mathcal{A}''} = \frac{\partial \tilde{H}}{\partial P'_{\bar{W}}} = \frac{\partial \tilde{H}}{\partial P''_{\bar{W}}} = 0. \quad (88)$$

The inhomogeneous component (16) turns into highly nontrivial conditions,

$$\left.\frac{\partial(\{\tilde{q}^{\theta\theta}, \tilde{H}[\epsilon^0]\})}{\partial(\epsilon^0)'}\right|_{\text{O.S.}} = \left.\frac{\partial(\{\tilde{q}^{\theta\theta}, \tilde{H}[\epsilon^0]\})}{\partial(\epsilon^0)''}\right|_{\text{O.S.}} = \cdots = 0. \quad (89)$$

In addition to general covariance, we will require that the modified system retains the types of conserved quantities of the classical theory. We therefore impose the preservation of the symmetry generator $G[\omega]$ in (27), such that it commutes with the modified constraint,

$$\{G[\omega], \tilde{H}[N]\} = 0. \quad (90)$$

We will also demand that for $P_{\bar{W}}, \bar{W} \to 0$ a gravitational Dirac observable exists with (30) as its classical limit.

3. Canonical transformations II and additional guiding conditions

The imposition of anomaly freedom, covariance, and the gravitational symmetries all restrict the generic Hamiltonian constraint (80) by providing a large set of partial differential equations. These equations can be considerably simplified by a choice of phase-space coordinates, fixing the residual canonical transformations (82).

So far, all of the conditions we impose in the new variables are identical to those we chose when coupling scalar matter in spherical symmetry [6]. While these conditions are quite restrictive, they still do not allow a complete exact solution of the partial differential equations for modification functions. We therefore refer to additional conditions, most of which have also been used in the scalar case.

Classical $\bar{W}$ limit. One such condition applied in [6] was the compatibility of the constraint with a limit in which the corresponding class of modifications contains models with the classical equations of motion for the scalar matter, corresponding to the Klein-Gordon equation on a curved emergent spacetime. We can apply the same condition to the Gowdy model, thanks to its correspondence with the spherically symmetric system, by imposing compatibility of the constraint with a limit in which the class of modifications contains models with the classical equations of motion for $\bar{W}$ on an emergent background $\tilde{q}_{\theta\theta}$.

Classical constraint surface as a limit. Another additional condition considered in [6] was the compatibility of the constraint with a nontrivial limit in which the constraint surface took its classical form. This is the case if there is a limit in which the constraint contains the modifications from linear combinations of the classical constraints, as derived in the previous section.

Classes of constraints. With these conditions, we are in the position to obtain explicit expressions for the modified Hamiltonian constraint. The conditions of anomaly freedom, covariance, implementation of symmetries, and the factoring out of canonical transformations imply a set of differential equations that can be solved exactly if the additional conditions just described are considered.

However, if we implement the essential conditions we are left with some ambiguities. If one is not interested in the classical $\bar{W}$ limit, several of these ambiguities are removed, but one modification function remains unresolved. The vanishing of this function then leads to the compatibility of the classical constraint surface as a limit, thus describing our first class of constraints.

On the other hand, using a nontrivial choice for this modification function leads us to another class of constraints that is no longer compatible with the classical constraint surface as a limit, nor with the classical $\bar{W}$ limit. Lacking a positive characterization of these models, we simply call this set of modified theories the class of the second kind.

Finally, a third class of constraints can be obtained by imposing compatibility with the classical $\bar{W}$ limit.

These are precisely the three classes of constraints obtained in [6] for the spherically symmetric system coupled to scalar matter. We can then simply import the results and reinterpret the Hamiltonian constraint by its correspondence with the Gowdy system. The modified theory allows for modification functions that can be redefined and adapted to the Gowdy system.

Discrete symmetry. The Gowdy system has one further symmetry that is not obvious in the spherically symmetric system coupled to scalar matter. This is the discrete symmetry $P_{\bar{W}} \to -P_{\bar{W}}, \bar{W} \to -\bar{W}$. We will implement this symmetry in the classes of constraints imported from [6] and, therefore, obtain slightly simpler expressions.

B. Emergent modified gravity as a basis for quantization

Our generic Hamiltonian constraint contains only up to second-order spatial derivatives and uses the classical phase space, which implies that the equations of motion do not have higher time derivatives. Theories of this form, even though they are modified compared with general relativity, may therefore be considered classical gravitational systems that can be used as a basis for canonical quantization. Several additional conditions are then useful for different procedures of finding suitable constraint operators.

1. Partial Abelianization

Gravitational theories with a description as spacetime geometry require constraints that generate hypersurface deformations. The presence of structure functions is then a well-known obstacle toward quantization because an operator-valued structure function implies severe ordering problems in commutators of the constraints. This problem can be simplified if the original constraints can be replaced by linear combinations that replace the structure function by a constant, and perhaps setting it equal to zero in a partial Abelianization. In spherical symmetry, such a procedure has been proposed in [31], and then generalized in [2] by making it fully local.

For a systematic derivation of partial Abelianizations we make use of the procedure described in Sec. III, introducing a new phase-space function as a linear combination of the (now already modified) Hamiltonian constraint and the classical diffeomorphism constraint that replaces the Hamiltonian constraint of hypersurface-deformation brackets. The new constraints therefore have brackets that differ from the classical gravitational ones, and their gauge transformations do not correspond to hypersurface deformations. However, they have the same constraint surface as the original system, which can therefore be turned into a quantum description by this procedure.

We will make use of definition (51) for the constraint function

$$\tilde{H}^{(\mathrm{A})} = B\tilde{H} + AH_\theta. \tag{91}$$

Poisson brackets of B and A with the old Hamiltonian constraint are given by (52) and (55), and the latter is related to the former by

$$A = -\mathcal{B}^\theta(B) \tag{92}$$

as in (54). The only difference with the procedure in Sec. III is that we are not seeking a new covariant modified theory, but rather a partial Abelianization of the brackets of $H^{(\mathrm{A})}$ and H_θ. Therefore, we impose the condition that the new structure function (59) vanishes:

$$\tilde{q}^{\theta\theta}_{(\mathrm{A})} = B^2\tilde{q}^{\theta\theta} + B\mathcal{A}^\theta = 0. \tag{93}$$

We will apply this condition to the three classes of constraints derived below.

2. Point holonomies

In [6], it was possible to include point holonomies of the scalar field ϕ, given by periodic modification functions depending on this variable. Like partial Abelianizations, this property may be useful for quantizations because some of the basic fields can be represented by bounded operators, akin to a loop quantization.

Invoking the correspondence between spherical symmetry and the polarized Gowdy model, this result can be translated to point holonomies of $\bar{W}$ in the latter case. Recall that the relation between this variable and the original phase-space degrees of freedom is given by $\bar{W} = \ln\sqrt{E^y/E^x}$. This function depends on densitized-triad components rather than their momenta, classically related to extrinsic curvature or a connection, and the dependence is logarithmic.

Given the logarithmic dependence on triad components instead of linear combinations of extrinsic curvature, periodic modification functions of $\bar{W}$, or polymerizations of this variable, are rather different from what is usually assumed in models of loop quantum gravity, even compared with a polymerization of K in (74) which already showed several deviating features. But while a polymerization of $\bar{W}$ may not be directly motivated by traditional loop quantum gravity, we include this possibility here for completeness of the correspondence with spherical symmetry. New canonical quantizations could still be constructed in this way by exploiting the boundedness of operators quantizing a periodic function of $\bar{W}$.

V. CLASSES OF CONSTRAINTS

As derived in detail in [6], we consider different classes of constraints and the modified structure functions they imply, depending on which conditions are chosen in order to make the consistency equations explicitly solvable.

A. Constraints compatible with the classical constraint surface

Modified constraints that are compatible with the classical constraint surface in a suitable limit are direct generalizations of the models constructed in Sec. III from linear combinations of the classical constraints.

1. General constraint

As always, the expression for Hamiltonian constraints compatible with certain symmetry conditions may depend on modification functions that distinguish different cases of consistent constraints, but also on free functions that represent the freedom to apply canonical transformations. Here, we fix the latter choice by working with partially periodic modification functions in the phase-space variable K. In this class, the general expression of the Hamiltonian constraint is given by

$$\begin{aligned}\tilde{H}=&-\frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\bar{W})\sqrt{\varepsilon}\bigg[\bar{a}\bigg(-\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0+\bigg(\frac{\alpha_2}{4\varepsilon}c_f+\frac{1}{2}\frac{\partial c_f}{\partial\varepsilon}\bigg)\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}+\bigg(\frac{\mathcal{A}}{\bar{a}}-\frac{P_{\bar{W}}}{\bar{a}}\frac{\tan(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}-\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\varepsilon}\bigg)c_f\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\\&-\bigg(\frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})}+\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\bigg(\frac{\bar{\nu}}{\nu}c_{h3}+\frac{\partial\ln\lambda}{\partial\bar{W}}\bigg)\bigg)^2\frac{\alpha_3\nu^2}{4\varepsilon\bar{\nu}^2}c_f\cos^2(\bar{\lambda}K)\bigg)+\frac{(\varepsilon')^2}{\bar{a}}\bigg(\bar{\lambda}^2\frac{\mathcal{A}}{\bar{a}}\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\\&+\cos^2(\bar{\lambda}K)\bigg(\frac{\partial\ln\lambda}{\partial\varepsilon}-\frac{\alpha_2}{4\varepsilon}-\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\bigg(\frac{\bar{\nu}}{\nu}c_{h3}+\frac{\partial\ln\lambda}{\partial\bar{W}}+\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\frac{\varepsilon}{\alpha_3}\bigg)\bigg)\bigg)+\bigg(\frac{\varepsilon'\bar{a}'}{\bar{a}^2}-\frac{\varepsilon''}{\bar{a}}\bigg)\cos^2(\bar{\lambda}K)\\&+\cos^2(\bar{\lambda}K)\bigg(-\frac{1}{\bar{a}}\bigg(\bigg(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\bigg)'\bigg)^2\frac{\varepsilon}{\alpha_3}+\frac{\varepsilon'}{\bar{a}}\bigg(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\bigg)'\bigg(\frac{2\varepsilon}{\alpha_3}\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}+\frac{\bar{\nu}}{\nu}c_{h3}+\frac{\partial\ln\lambda}{\partial\bar{W}}-\frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})}\bar{\lambda}^2\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\bigg)\bigg)\bigg]\end{aligned} \tag{94}$$

with the structure function

$$\tilde{q}^{\theta\theta}=\bigg(c_f+\bigg(\frac{\bar{\lambda}\varepsilon'}{\bar{a}}\bigg)^2\bigg)\cos^2(\bar{\lambda}K)\frac{\bar{\lambda}^2}{\lambda^2}\lambda_0^2\cos^4(\bar{\nu}\bar{W})\frac{\varepsilon}{\bar{a}^2}. \tag{95}$$

All the nonclassical parameters are undetermined functions of ε only, except for $\bar{\lambda}$ and $\bar{\nu}$, which are constants, and λ_0 and λ, which can depend on both ε and $\bar{W}$. (This is the only class of constraints that allows λ to depend on $\bar{W}$.) The constraint (94) and its structure function (95) are symmetric under the discrete transformation $\bar{W}\to-\bar{W}$, $P_{\bar{W}}\to-P_{\bar{W}}$ only if the λ_0 and λ dependence on $\bar{W}$ is restricted by the discrete symmetry (26) to be of the form $\lambda_0(\varepsilon,\bar{W})=\lambda_0(\varepsilon,-\bar{W})$, and only if $c_{h3}=0$ because it is independent of $\bar{W}$. (Alternatively, the discrete transformation could be redefined as $\bar{W}\to-\bar{W}$, $P_{\bar{W}}\to-P_{\bar{W}}$, and $c_{h3}\to-c_{h3}$, in which case the constraint and structure function are symmetric even for nonzero c_{h3}.) The classical limit can be taken in different ways, the simplest one given by $\lambda\to\bar{\lambda}$ and $\nu\to\bar{\nu}$ followed by $\lambda_0,c_f,\alpha_2,\alpha_3\to1$, $\bar{\lambda},\bar{\nu}\to0$, and $\Lambda_0\to\Lambda$.

The inhomogeneous-field observable in this class is given by

$$G[\omega]=\int\mathrm{d}\theta\,\omega\frac{\nu}{\bar{\nu}}\bigg(\frac{P_{\bar{W}}}{\cos(\bar{\nu}\bar{W})}+\bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\bar{W}}\bigg), \tag{96}$$

where ω is a constant. The associated conserved current J^μ has the components

$$J^t=\frac{\nu}{\bar{\nu}}\bigg(\frac{P_{\bar{W}}}{\cos(\bar{\nu}\bar{W})}+\bar{a}\frac{\tan(\bar{\lambda}K_\varphi)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\bar{W}}\bigg), \tag{97}$$

$$\begin{aligned}J^\theta=&-\frac{\nu}{\bar{\nu}}\frac{\bar{\lambda}}{\lambda}\lambda_0\sqrt{\varepsilon}\cos^2(\bar{\nu}\bar{W})\cos^2(\bar{\lambda}K)\bigg(-\frac{2}{\bar{a}}\bigg(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\bigg)'\frac{\varepsilon}{\alpha_3}\\&+\frac{\varepsilon'}{\bar{a}}\bigg(\frac{2\varepsilon}{\alpha_3}\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}+\frac{\bar{\nu}}{\nu}c_{h3}+\frac{\partial\ln\lambda}{\partial\bar{W}}-\frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})}\bar{\lambda}^2\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\bigg)\bigg).\end{aligned} \tag{98}$$

When $P_{\bar{W}},\bar{W}\to0$, the homogeneous mass observable associated with (94) is given by

$$\begin{aligned}\mathcal{M}=&\,d_0+\frac{d_2}{8}\bigg(\exp\int\mathrm{d}\varepsilon\bigg(\frac{\alpha_2}{2\varepsilon}-\frac{\partial\ln\lambda^2}{\partial\varepsilon}\bigg)\bigg)\bigg(c_f\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}-\cos^2(\bar{\lambda}K)\bigg(\frac{\varepsilon'}{\bar{a}}\bigg)^2\bigg)\\&+\frac{d_2}{4}\int\mathrm{d}\varepsilon\bigg(\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0\exp\int\mathrm{d}\varepsilon\bigg(\frac{\alpha_2}{2\varepsilon}-\frac{\partial\ln\lambda^2}{\partial\varepsilon}\bigg)\bigg),\end{aligned} \tag{99}$$

where d_0 and d_2 are constants with classical limits given by $d_0\to0$ and $d_2\to1$.

2. Partial Abelianization

Following Sec. IV B 1 and using the constraint (94) with the structure function (95) we obtain

$$\tilde{q}^{(A)} = Q_0 + Q_\varepsilon(\varepsilon')^2 = 0 \tag{100}$$

for the Abelianization condition (93), where Q_0 and Q_ε are functions of B, as it appears in $\tilde{H}^{(A)} = B\tilde{H} + AH_\theta$, and of the phase-space variables, but not of their derivatives. Therefore, these two coefficients must vanish independently. The condition $Q_0 = 0$ implies the equation

$$B - \frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\frac{\partial B}{\partial K} = 0, \tag{101}$$

with the general solution

$$B = B_0\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}} \tag{102}$$

for an undetermined function $B_0(\varepsilon, \bar{W})$. The condition $Q_\varepsilon = 0$ implies the equation

$$2\bar{\lambda}^2 B + \bar{\lambda}\sin(2\bar{\lambda}K)\frac{\partial B}{\partial K} - 2\cos^2(\bar{\lambda}K)\frac{\partial^2 B}{\partial K^2} = 0. \tag{103}$$

By direct substitution we find that (102) solves this equation too. The Abelianized constraint is then given by

$$\begin{aligned}
\frac{\tilde{H}^{(A)}}{B_0} = &-\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\bar{W})\sqrt{\varepsilon}\Bigg[\bar{a}\left(-\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0 + \left(\frac{\alpha_2}{4\varepsilon}c_f + \frac{1}{2}\frac{\partial c_f}{\partial\varepsilon}\right)\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}\right.\\
&+\left(\frac{\mathcal{A}}{\bar{a}} - \frac{P_{\bar{W}}}{\bar{a}}\frac{\tan(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon} - \frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial\varepsilon}\right)c_f\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\\
&-\left(\frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})} + \frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\left(\frac{\bar{\nu}}{\nu}c_{h3} + \frac{\partial\ln\lambda}{\partial\bar{W}}\right)\right)^2\frac{\alpha_3}{4\varepsilon}\frac{\nu^2}{\bar{\nu}^2}c_f\cos^2(\bar{\lambda}K)\Bigg) + \frac{(\varepsilon')^2}{\bar{a}}\left(\bar{\lambda}^2\frac{\mathcal{A}}{\bar{a}}\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right.\\
&+\cos^2(\bar{\lambda}K)\left(\frac{\partial\ln\lambda}{\partial\varepsilon} - \frac{\alpha_2}{4\varepsilon} - \frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\left(\frac{\bar{\nu}}{\nu}c_{h3} + \frac{\partial\ln\lambda}{\partial\bar{W}} + \frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\frac{\varepsilon}{\alpha_3}\right)\right)\Bigg) + \left(\frac{\varepsilon'\bar{a}'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}}\right)\cos^2(\bar{\lambda}K)\\
&+\cos^2(\bar{\lambda}K)\left(-\frac{1}{\bar{a}}\left(\left(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\right)'\right)^2\frac{\varepsilon}{\alpha_3} + \frac{\varepsilon'}{\bar{a}}\left(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\right)'\left(\frac{2\varepsilon}{\alpha_3}\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right.\right.\\
&\left.\left.+\frac{\bar{\nu}}{\nu}c_{h3} + \frac{\partial\ln\lambda}{\partial\bar{W}} - \frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})}\bar{\lambda}^2\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\right)\right)\Bigg] - \frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\bar{W})\sqrt{\varepsilon}\varepsilon'(\bar{a}K' + P_{\bar{W}}\bar{W}' - \mathcal{A}\varepsilon').
\end{aligned} \tag{104}$$

The first line in (104) has a kinematical divergence at $K = \pi/(2\bar{\lambda})$ due to the overall tangent factor. This divergence can be removed if

$$\frac{\alpha_2}{4\varepsilon}c_f + \frac{1}{2}\frac{\partial c_f}{\partial\varepsilon} - \lambda^2\Lambda_0 = 0 \tag{105}$$

is satisfied because the relevant terms then combine to produce a $\cos^2$-factor and hence cancel the divergence of the tangent. If we interpret this condition as an equation for c_f, we must restrict λ to be a function of ε only. However, the solution to this equation is not compatible with the classical limit $c_f \to 1$. (For instance, for $\Lambda_0 = 0$ we have $c_f \propto \varepsilon^{-\alpha_2/2}$.) Therefore, we have to weaken the condition by neglecting the first term, and hence leave it as a divergent term of the Abelian constraint. The resulting equation for the partial resolution of the divergence,

$$\frac{1}{2}\frac{\partial c_f}{\partial\varepsilon} - \lambda^2\Lambda_0 = 0, \tag{106}$$

can now be directly integrated, yielding the modification function

$$c_f = 2\int\lambda^2\Lambda_0 \mathrm{d}\varepsilon. \tag{107}$$

If we choose the classical value of the cosmological constant $\Lambda_0 = \Lambda$, and $\lambda^2 = \Delta/\varepsilon$ with a constant Δ (sometimes used in models of loop quantum gravity), we obtain

$$c_f = 1 + 2\Lambda\Delta\ln\left(\frac{\varepsilon}{c_0}\right), \tag{108}$$

where c_0 is an integration constant. The correct classical limit is obtained for $\Delta \to 0$. The logarithmic dependence on ϵ is relevant on intermediate scales far from black-hole or cosmological horizons. It may then be related to MOND-like effects as shown in [12].

B. Constraints of the second kind

A second class of explicit modified constraints is obtained from a specific choice for one of the modification functions, so far without a detailed physical motivation. Nevertheless, this case is interesting because it can be used to show the variety of possible covariant theories.

1. General constraint

Again referring to more detailed derivations for a scalar field in spherical symmetry [6], we now have the modified Hamiltonian constraint

$$\begin{aligned}\tilde{H} = -\frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\bar{W})\sqrt{\varepsilon}\Bigg[&\bar{a}\left(-\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0+\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}\left(\left(\frac{\alpha_2}{4\varepsilon}-\frac{\partial\ln\lambda}{\partial\varepsilon}\right)c_f+\frac{1}{2}\frac{\partial c_f}{\partial\varepsilon}\right)\right)+\bar{a}\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\left(\left(\frac{\alpha_2}{2\varepsilon}-\frac{\partial\ln\lambda}{\partial\varepsilon}\right)\frac{\lambda}{\bar{\lambda}}q+\frac{\lambda}{\bar{\lambda}}\frac{\partial q}{\partial\varepsilon}\right)\\
&+\left(\mathcal{A}+\frac{P_{\bar{W}}}{\cos(\bar{\nu}\bar{W})}\left(\frac{\nu}{\bar{\nu}}c_{h3}-\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\right)\left(c_f\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}+\frac{\lambda}{\bar{\lambda}}q\cos(2\bar{\lambda}K)\right)\\
&-\frac{\nu^2}{\bar{\nu}^2}\frac{P_{\bar{W}}{}^2}{\bar{a}\cos^2(\bar{\nu}\bar{W})}\frac{\alpha_3}{4\varepsilon}\left(c_f\cos^2(\bar{\lambda}K)-2\frac{\lambda}{\bar{\lambda}}q\bar{\lambda}^2\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right)+\left(\frac{\varepsilon'\bar{a}'}{\bar{a}^2}-\frac{\varepsilon''}{\bar{a}}\right)\cos^2(\bar{\lambda}K)\\
&-\frac{(\varepsilon')^2}{\bar{a}}\left(\left(\frac{\alpha_2}{4\varepsilon}-\frac{\partial\ln\lambda}{\partial\varepsilon}\right)\cos^2(\bar{\lambda}K)-\left(\frac{\mathcal{A}}{\bar{a}}+\frac{P_{\bar{W}}}{\bar{a}\cos(\bar{\nu}\bar{W})}\left(\frac{\nu}{\bar{\nu}}c_{h3}-\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\right)\bar{\lambda}^2\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right.\\
&\left.+\frac{\nu^2}{\bar{\nu}^2}\frac{P_{\bar{W}}{}^2}{\bar{a}^2\cos^2(\bar{\nu}\bar{W})}\bar{\lambda}^2\frac{\alpha_3}{4\varepsilon}\cos^2(\bar{\lambda}K)\right)-\frac{1}{\bar{a}}\frac{\bar{\nu}^2}{\nu^2}\left(\left(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\right)'+\varepsilon'\left(\frac{\nu}{\bar{\nu}}c_{h3}-\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\right)^2\frac{\varepsilon}{\alpha_3}\Bigg],\end{aligned}\tag{109}$$

with structure function

$$\tilde{q}^{\theta\theta}=\frac{\bar{\lambda}^2}{\lambda^2}\lambda_0^2\left(\left(c_f+\left(\frac{\bar{\lambda}\varepsilon'}{\bar{a}}\right)^2\right)\cos^2(\bar{\lambda}K)-2\bar{\lambda}^2\frac{\lambda}{\bar{\lambda}}q\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right)\cos^4(\bar{\nu}\bar{W})\frac{\varepsilon}{\bar{a}^2}.\tag{110}$$

All the nonclassical parameters are undetermined functions of ε only, except for the parameters $\bar{\lambda}$ and $\bar{\nu}$ which are constants, and λ_0 which can depend on both ε and $\bar{W}$. The constraint (109) and its structure function (110) are symmetric under the discrete transformation $\bar{W}\to-\bar{W}$, $P_{\bar{W}}\to-P_{\bar{W}}$ only if the λ_0 dependence on $\bar{W}$ is restricted by the discrete symmetry (26) to the form $\lambda_0(\varepsilon,\bar{W})=\lambda_0(\varepsilon,-\bar{W})$, and only if $c_{h3}=0$ because it is independent of $\bar{W}$. (Alternatively, the discrete transformation can be redefined as $\bar{W}\to-\bar{W}$, $P_{\bar{W}}\to-P_{\bar{W}}$, and $c_{h3}\to-c_{h3}$, in which case the constraint and structure function are symmetric even for nonzero c_{h3}.) The classical limit can be taken in different ways, the simplest one given by $\lambda\to\bar{\lambda}$ and $\nu\to\bar{\nu}$, followed by $\lambda_0, c_f, \alpha_2, \alpha_3\to 1$, $q, \bar{\lambda}, \bar{\nu}\to 0$, and $\Lambda_0\to\Lambda$.

The inhomogeneous-field observable is

$$G[\omega]=\int \mathrm{d}\theta\omega\frac{\nu}{\bar{\nu}}\frac{P_{\bar{W}}}{\cos(\bar{\nu}\bar{W})},\tag{111}$$

where ω is a constant. The associated conserved current J^μ has the components

$$J^t=\frac{\nu}{\bar{\nu}}\frac{P_{\bar{W}}}{\cos(\bar{\nu}\bar{W})},\tag{112}$$

$$\begin{aligned}J^\theta=\frac{\bar{\nu}}{\nu}\frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\bar{W})\frac{2\varepsilon^{3/2}}{\alpha_3\bar{a}}&\left(\left(\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\right)'\right.\\
&\left.+\varepsilon'\left(\frac{\nu}{\bar{\nu}}c_{h3}-\frac{\sin(\bar{\nu}\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\right).\end{aligned}\tag{113}$$

When $P_{\bar{W}}, \bar{W}\to 0$, the homogeneous mass observable associated with (115) is given by

$$\begin{aligned}\mathcal{M}=d_0+\frac{d_2}{8}&\left(\exp\int\mathrm{d}\varepsilon\left(\frac{\alpha_2}{2\varepsilon}-\frac{\partial\ln\lambda^2}{\partial\varepsilon}\right)\right)\\
&\times\left(c_f\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}+2\frac{\lambda}{\bar{\lambda}}q\frac{\sin(2\bar{\lambda}K)}{\bar{\lambda}}-\cos^2(\bar{\lambda}K)\left(\frac{\varepsilon'}{\bar{a}}\right)^2\right)\\
&+\frac{d_2}{4}\int\mathrm{d}\varepsilon\left(\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0\exp\int\mathrm{d}\varepsilon\left(\frac{\alpha_2}{2\varepsilon}-\frac{\partial\ln\lambda^2}{\partial\varepsilon}\right)\right),\end{aligned}\tag{114}$$

where d_0 and d_2 are constants with classical limits given by $d_0\to 0$ and $d_2\to 1$. Most of these properties are similar to those in the first class, but explicit solutions in solvable cases, such as spatially homogeneous ones, can reveal crucial differences, as we will see in Sec. VI.

A key difference between the first and second classes can be seen easily in the structure functions (95) and (110), respectively. The first one is even in the curvature component K, while the second one contains an odd term, multiplied by the modification function q. The same behavior was possible in spherical symmetry [2] where it may have far-reaching implications for various particle effects [32]. In the Hamiltonian constraint, the q-terms show that there may be modifications linear in K (if a Taylor expansion is used for the trigonometric functions). Such terms can be more relevant than the classical quadratic terms as the curvature scale is increased. The second class of modified constraints for polarized Gowdy models shows that these interesting features are not restricted to spherical symmetry.

2. Partial Abelianization

The partial Abelianization of this constraint follows the same procedure as the last one. It requires exactly the same B-factor (102) and the associated A. However, a partial Abelianization is subject to the additional condition that the modification function q vanishes.

C. Constraints compatible with the classical-$\bar{W}$ limit

The third class has modified constraints that have a limit in which the field $\bar{W}$ behaves like a classical scalar field on the emergent (and nonclassical) spacetime. This case is useful because it allows us to make comparisons between the propagation speed of $\bar{W}$ as a polarized gravitational wave and the speed of a massless scalar field that may be coupled minimally.

1. General constraint

The modified Hamiltonian constraint is given by

$$\tilde{H} = -\frac{\bar{\lambda}}{\lambda}\lambda_0\cos^2(\bar{\nu}\,\bar{W})\sqrt{\varepsilon}\Bigg[\bar{a}\left(\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0 + \left(c_f\left(\frac{\alpha_2}{4\varepsilon} - \frac{\partial\ln\lambda}{\partial\varepsilon}\right) + \frac{1}{2}\frac{\partial c_f}{\partial\varepsilon}\right)\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2}\right) + \bar{a}\left(\frac{q}{2}\left(\frac{\alpha_2}{\varepsilon} - 2\frac{\partial\ln\lambda}{\partial\varepsilon}\right) + \frac{\lambda}{\bar{\lambda}}\frac{\partial q}{\partial\varepsilon}\right)\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}$$
$$+\left(\mathcal{A} - P_{\bar{W}}\frac{\tan(\bar{\nu}\,\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\left(c_f\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}} + \frac{\lambda}{\bar{\lambda}}q\cos(2\bar{\lambda}K)\right) + \frac{(\varepsilon')^2}{\bar{a}}\left(\left(\frac{\partial\ln\lambda}{\partial\varepsilon} - \frac{\alpha_2}{4\varepsilon}\right)\cos^2(\bar{\lambda}K)\right.$$
$$\left.+\bar{\lambda}^2\left(\frac{\mathcal{A}}{\bar{a}} - \frac{P_{\bar{W}}}{\bar{a}}\frac{\tan(\bar{\nu}\,\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right)\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right) + \left(\frac{\varepsilon'\bar{a}'}{\bar{a}^2} - \frac{\varepsilon''}{\bar{a}}\right)\cos^2(\bar{\lambda}K)\Bigg]$$
$$+\frac{\bar{\nu}^2}{\nu^2}\frac{\sqrt{\tilde{q}^{\theta\theta}}}{2}\left[\frac{{P_{\bar{W}}}^2}{\cos^2(\bar{\nu}\,\bar{W})}\frac{\alpha_3}{2\varepsilon} + \frac{2\varepsilon}{\alpha_3}\left(\left(\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\right)' - \frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\varepsilon'\right)^2\right], \tag{115}$$

with the structure function

$$\tilde{q}^{\theta\theta} = \left(\left(c_f + \left(\frac{\bar{\lambda}\varepsilon'}{\bar{a}}\right)^2\right)\cos^2(\bar{\lambda}K) - 2q\frac{\lambda}{\bar{\lambda}}\bar{\lambda}^2\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\right)\frac{\bar{\lambda}^2}{\lambda^2}\lambda_0^2\cos^4(\bar{\nu}\,\bar{W})\frac{\varepsilon}{\bar{a}^2} \tag{116}$$

appearing explicitly in the last line. All the nonclassical parameters are undetermined functions of ε only, except for the parameters $\bar{\lambda}$ and $\bar{\nu}$ which are constants, and λ_0 which can depend on both ε and $\bar{W}$. The constraint (115) and its structure function (116) are symmetric under the discrete transformation $\bar{W}\to-\bar{W}$, $P_{\bar{W}}\to-P_{\bar{W}}$ only if the λ_0 dependence on $\bar{W}$ is restricted by the discrete symmetry (26) to the form $\lambda_0(\varepsilon,\bar{W}) = \lambda_0(\varepsilon,-\bar{W})$. The classical limit can be taken in different ways, the simplest one given by $\lambda\to\bar{\lambda}$ and $\nu\to\bar{\nu}$, followed by $\lambda_0, c_f, \alpha_2, \alpha_3\to 1$, $q, \bar{\lambda}, \bar{\nu}\to 0$, and $\Lambda_0\to\Lambda$. The classical-$\bar{W}$ limit is obtained for $\nu=\bar{\nu}\to 0$ and $\alpha_3\to 1$. The last parenthesis in the Hamiltonian constraint then approaches the form of a classical scalar field propagating on the emergent spacetime with inhomogeneous spatial component $\tilde{q}_{\theta\theta}$.

The $\bar{W}$-field observable is given by

$$G[\omega] = \int \mathrm{d}\theta\omega\frac{\nu}{\bar{\nu}}\frac{P_{\bar{W}}}{\cos(\bar{\nu}\,\bar{W})}, \tag{117}$$

where ω is a constant. The associated conserved current J^μ has the components

$$J^t = \frac{\nu}{\bar{\nu}}\frac{P_{\bar{W}}}{\cos(\bar{\nu}\,\bar{W})}, \tag{118}$$

$$J^\theta == \frac{\bar{\nu}}{\nu}\sqrt{q^{\theta\theta}}\frac{2\varepsilon}{\alpha_3}\left(\left(\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\right)' - \varepsilon'\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\frac{\partial\ln\nu}{\partial\varepsilon}\right). \tag{119}$$

When $P_{\bar{W}}, \bar{W} \to 0$, the homogeneous mass observable associated with (115) is given by

$$\mathcal{M} = d_0 + \frac{d_2}{8}\left(\exp\int d\varepsilon\left(\frac{\alpha_2}{2\varepsilon} - \frac{\partial \ln\lambda^2}{\partial\varepsilon}\right)\right) \times \left(c_f \frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} + 2\frac{\lambda}{\bar{\lambda}} q \frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}} - \cos^2(\bar{\lambda}K)\left(\frac{\varepsilon'}{\bar{a}}\right)^2\right) + \frac{d_2}{4}\int d\varepsilon\left(\frac{\lambda^2}{\bar{\lambda}^2}\Lambda_0 \exp\int d\varepsilon\left(\frac{\alpha_2}{2\varepsilon} - \frac{\partial\ln\lambda^2}{\partial\varepsilon}\right)\right), \quad (120)$$

where d_0 and d_2 are constants with classical limits given by $d_0 \to 0$ and $d_2 \to 1$.

2. Partial Abelianization

The partial Abelianization of this constraint follows the same procedure as outlined in the first class of modified constraints. It implies the B-factor (102) and the associated A-factor, but in addition requires that the modification function q vanishes, as in the second class.

VI. DYNAMICAL SOLUTIONS WITH HOMOGENEOUS SPATIAL SLICES

First indications of possible physical effects of our modifications can be obtained by looking at properties of spatially homogeneous solutions. In this case, partial differential equations are replaced by ordinary ones that can often be solved more easily.

A. Classical constraint

For the sake of comparison, we first present useful gauge conditions and solutions in the classical case with a vanishing cosmological constant. Strict homogeneity then implies Kasner solutions, while an inhomogeneous solution for $\bar{W}$ can also be allowed.

1. Conformal gauge

In generally covariant theories, the form of solutions for the spacetime geometry depends on the coordinate choice used to express them. In canonical formulations, a coordinate system is presented as a gauge choice that prescribes the dependence of a suitable subset of metric components on the coordinates. Gowdy models as well as spherically symmetric systems have two constraints, given by the diffeomorphism constraint and the Hamiltonian constraint. In general, one should therefore choose two conditions in order to determine the gauge. These two conditions may not always be mutually consistent, in which case there is no coordinate system in which they can both be met. However, if all the constraints and equations of motion generated by them can be solved consistently, a valid solution is obtained. Solutions for metric components in terms of the coordinates introduced by the gauge choice then determine the line element in the corresponding coordinate system.

The coordinates of the conventional Gowdy metric (4) are associated with the gauge choice

$$N^\theta = 0, \qquad \varepsilon = T \quad (121)$$

with the time coordinate T. We impose this gauge and for now work with the classical constraint (22). The remaining metric components can be expressed in terms of the lapse function and the two fields

$$W = \ln\sqrt{E^y/E^x} = \bar{W}, \quad a = \ln\sqrt{E^x E^y/\varepsilon} = \ln\bar{a} - \frac{1}{2}\ln\varepsilon. \quad (122)$$

The on-shell conditions $H_\theta = 0$ and $H = 0$ in this gauge become

$$H_\theta = P_{\bar{W}}\bar{W}' + \bar{a}K' = 0, \quad (123)$$

$$H = \frac{P_{\bar{W}}{}^2 - 4T\bar{a}K\mathcal{A} - \bar{a}^2K^2 + 4T^2(\bar{W}')^2}{4\sqrt{T}\bar{a}} = 0. \quad (124)$$

Using the latter expression, we obtain an equation of motion

$$\partial_T(\bar{a}K) = \{\bar{a}K, H[N]\} = -NH, \quad (125)$$

which vanishes on-shell, such that $\bar{a}K = \mu$ where μ is a constant. The consistency equation $\dot{\varepsilon} = \partial\varepsilon/\partial T = 1$ can be solved for the lapse function

$$N = \frac{1}{K\sqrt{T}} = \frac{\mu^{-1}\bar{a}}{\sqrt{\varepsilon}} = \mu^{-1}\sqrt{q_{\theta\theta}}. \quad (126)$$

The equations of motion for $\bar{a}$ and $\bar{W}$, respectively, imply

$$\mathcal{A} = \mu\left(\frac{\dot{\bar{a}}}{\bar{a}} - \frac{1}{2T}\right), \quad P_{\bar{W}} = 2\,\mu T\dot{\bar{W}}. \quad (127)$$

Using these results, the on-shell conditions $H_\theta = 0$ and $H = 0$ can be rewritten as

$$a' = 2T\dot{W}W', \quad (128a)$$

$$\dot{a} = -\frac{1}{4T} + T\left(\dot{W}^2 + \frac{(W')^2}{\mu^2}\right), \quad (128b)$$

where we have used the identification (122).

The equation of motion $\ddot{W} = \{\{W, H[N]\}, H[N]\}$ requires some care because the lapse function (126) is phase-space dependent. In a first-order equation of motion, using a single Poisson bracket with $H[N]$, any term resulting from a nonzero Poisson bracket with N would be multiplied by H and therefore vanish on-shell. However, this argument does not apply to iterated Poisson brackets of some phase-space function with $H[N]$, where nonzero on-shell terms may contribute. Duly taking into account the phase-space dependence of N, such that the second $\{\cdot, H[N]\}$ acts on this function contained in the first $H[N]$, we obtain the second-order equation of motion

$$0 = \ddot{W} + \frac{\dot{W}}{T} - \frac{W''}{\mu^2}. \tag{129}$$

It can be checked that this equation is equivalent to what would be obtained in standard general relativity. [If N were treated as phase-space independent, we would instead obtain the bracket $\{\{W,H[N]\},H[N]\} = -\dot{W}/T + W''/\mu^2 + \dot{W}(1/(4T) - T\dot{W} - TW''/\mu^2)$, using the lapse function (126) only after computing the brackets. This expression has extra terms compared with the correct equation (129).]

These are the equations of motion for the polarized Gowdy system in conventional variables, which have the general solution [33]

$$W = \alpha + \beta \ln T + \sum_{n=1}^{\infty} [a_n J_0(nT) \sin(n\mu^{-2}\theta + \gamma_n) + b_n N_0(nT) \sin(n\mu^{-2}\theta + \delta_n)], \tag{130}$$

where α, β, a_n, b_n, γ_n, and δ_n are real constants, and J_0 and N_0 are Bessel and Neumann functions of the zeroth order, respectively. Given the solution for W, direct integration of (128) gives the expression for a. The spacetime line element in this gauge becomes

$$\mathrm{d}s^2 = \mu^{-2} e^{2a} (-\mathrm{d}T^2 + \mu^2 \mathrm{d}\theta^2) + T(e^{-2W} \mathrm{d}x^2 + e^{2W} \mathrm{d}y^2). \tag{131}$$

The square of the lapse function, $N^2 = \bar{a}^2/(\mu^2 T)$, exactly equals $q_{\theta\theta} = \bar{a}^2/T$ only if $\mu = \pm 1$. Thus, setting $\mu = 1$, imposing positivity of the lapse function in the region $T > 0$, we obtain a conformally flat metric for the $T - \theta$ components. With this choice, the equation of motion (129) implies that W-excitations travel at the speed of light.

Finally, note that using (127) and (130) and inserting this solution in the symmetry generator (27) with $\omega = 1$ we find that it corresponds to

$$G[1] = 4\pi\mu\beta, \tag{132}$$

where we used the periodicity conditions in θ. This is clearly a conserved quantity.

2. Homogeneous solution

For a spatially homogenous background, we consider the special case $W' = 0$. From (130), this implies that $a_n = b_n = 0$. The equations of motion (128) now have the solution

$$a = \frac{4\beta^2 - 1}{4} \ln\left(\frac{T}{T_0}\right), \tag{133}$$

$$\bar{a} = T_0^{1/4-\beta^2} T^{\beta^2+1/4}, \tag{134}$$

with a constant T_0, and hence

$$K = \mu T_0^{\beta^2-1/4} T^{-\beta^2-1/4}. \tag{135}$$

The spacetime metric (131) becomes

$$\mathrm{d}s^2 = \mu^{-2} \left(\frac{T}{T_0}\right)^{2\beta^2-1/2} (-\mathrm{d}T^2 + \mu^2 \mathrm{d}\theta^2) + e^{2\alpha} T^{1+2\beta} \mathrm{d}x^2 + e^{-2\alpha} T^{1-2\beta} \mathrm{d}y^2. \tag{136}$$

The constants μ, T_0, and α can be absorbed in the definition of coordinates. In proper time, defined by $\tau(T) \propto T^{\beta^2+3/4}$, we then have the line element

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^{2p_1} \mathrm{d}\theta^2 + \tau^{2p_2} \mathrm{d}x^2 + \tau^{2p_3} \mathrm{d}y^2 \tag{137}$$

with exponents

$$p_1 = \frac{\beta^2 - 1/4}{\beta^2 + 3/4}, \qquad p_2 = \frac{\beta + 1/2}{\beta^2 + 3/4}, \qquad p_3 = \frac{\beta - 1/2}{\beta^2 + 3/4} \tag{138}$$

that satisfy the Kasner relations $p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2$. The background Kasner behavior is therefore determined by the observable $G[1]$. If the periodic terms from a nonhomogeneous W in (130) are included, they describe a polarized gravitational wave traveling on a Kasner background.

3. The flat Kasner solution

As a special case, a flat solution within the Kasner class is defined by further taking $\alpha = 0$, $\beta = 1/2$, and $\mu = 1$. We obtain

$$a = 0, \qquad \bar{a} = \sqrt{T}, \qquad K = \mu/\sqrt{T}, \tag{139}$$

$$\bar{W} = \frac{1}{2} \ln T, \qquad P_{\bar{W}} = 1, \qquad \mathcal{A} = 0, \tag{140}$$

and the spacetime metric (131) becomes

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}\theta^2 + T^2\mathrm{d}x^2 + \mathrm{d}y^2. \tag{141}$$

The Ricci scalar vanishes, which means that this expression may be considered a vacuum solution. (The $T-x$ part is a two-dimensional Milne model.) Upon applying the coordinate transformation

$$t_{\mathrm{M}} = T\cosh x, \qquad x_{\mathrm{M}} = T\sinh x \tag{142}$$

with inverse

$$T^2 = t_{\mathrm{M}}^2 - x_{\mathrm{M}}^2, \qquad x = \operatorname{arctanh}\frac{x_{\mathrm{M}}}{t} \tag{143}$$

such that

$$-\mathrm{d}t_{\mathrm{M}}^2 + \mathrm{d}x_{\mathrm{M}}^2 = -\mathrm{d}T^2 + T^2\mathrm{d}x^2, \tag{144}$$

the Kasner-like line element (141) becomes Minkowskian,

$$\mathrm{d}s^2 = -\mathrm{d}t_{\mathrm{M}}^2 + \mathrm{d}\theta^2 + \mathrm{d}x_{\mathrm{M}}^2 + \mathrm{d}y^2. \tag{145}$$

While this result shows that the specific Kasner solution (141) is a locally flat spacetime, hypersurfaces of constant time T define a three-dimensional space with nonvanishing extrinsic curvature

$$K_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{1}{2}\{q_{ab}, H[1]\}\mathrm{d}x^a\mathrm{d}x^b = T\mathrm{d}x^2. \tag{146}$$

4. Homogeneous solution: Internal-time gauge

To compare the results of the different classes of constraints, we will evaluate them in the homogeneous case, $P'_{\bar{W}} = \bar{W}' = \bar{a}' = \varepsilon' = K' = \mathcal{A}' = N' = 0$, with vanishing cosmological constant, $\Lambda = 0$. It will be convenient to work with coordinates adapted to the full range of the curvature variable K, used as an internal time coordinate. We define this internal-time gauge by

$$N^\theta = 0, \qquad K = T_K, \tag{147}$$

with a new time coordinate T_K.

In the homogeneous case, as before, we obtain

$$\partial_{T_K}(\bar{a}K) = \{\bar{a}K, H[N]\} = -HN, \tag{148}$$

which vanishes on-shell and implies

$$\bar{a} = \frac{\mu}{T_K} = \frac{\mu}{K}, \tag{149}$$

for some constant μ. Because of homogeneity, the local version of the observable (27) is conserved, $\dot{G} = 2\pi\dot{P}_W = 0$. Therefore, any P_W in the constraints and equations of motion is time-independent and can be set equal to $P_W = 2\mu\beta$, defining the constant β in the internal-time gauge. Using this expression and the chain rule, we obtain the equations

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}K} = \frac{\dot{\varepsilon}}{\dot{K}} = -\frac{4\varepsilon}{(1+4\beta^2)K}, \tag{150}$$

$$\frac{\mathrm{d}\bar{W}}{\mathrm{d}K} = \frac{\dot{\bar{W}}}{\dot{K}} = -\frac{4\beta}{(1+4\beta^2)K}, \tag{151}$$

with solutions

$$\varepsilon = c_\varepsilon T_K^{-4/(1+4\beta^2)}, \tag{152}$$

$$\begin{aligned}\bar{W} &= c_w - \frac{4\beta}{1+4\beta^2}\ln T_K \\ &= \ln\left(e^{c_w}T_K^{-4\beta/(1+4\beta^2)}\right).\end{aligned} \tag{153}$$

The integration constants may be redefined as

$$\begin{aligned}c_\varepsilon &= \left(\mu T_0^{(4\beta^2-1)/4}\right)^{4/(4\beta^2+1)}, \\ e^{2c_w} &= e^{2\alpha}\left(\mu T_0^{(4\beta^2-1)/4}\right)^{8\beta/(4\beta^2+1)}.\end{aligned} \tag{154}$$

The on-shell condition $H_\theta = 0$ is trivial in the homogeneous case, while $H = 0$ greatly simplifies and can be solved for

$$\mathcal{A} = \mu\frac{4\beta^2-1}{4\varepsilon} = \frac{4\beta^2-1}{4c_\varepsilon}\mu T_K^{4\mu^2/(\mu^2+4\beta^2)}. \tag{155}$$

Finally, the lapse function is obtained by solving the consistency equation $\dot{K} = \partial K/\partial T_K = 1$,

$$N = -\frac{4}{1+4\beta^2}\sqrt{c_\varepsilon}T_K^{-2/(1+4\beta^2)-2}. \tag{156}$$

The negative value of the lapse function means that evolution runs from higher to lower values of T_K, similar to what happens in Schwarzschild coordinates in a black hole's interior. The spacetime metric (24) is then given by

$$\begin{aligned}\mathrm{d}s^2 = {}& c_\varepsilon^{-1}T_K^{2(1-4\beta^2)/(4\beta^2+1)}\left(-\left(\frac{4}{1+4\beta^2}\right)^2 c_\varepsilon^2 T_K^{-2(5+4\beta^2)/(4\beta^2+1)}\mathrm{d}T_K^2 + \mu^2\mathrm{d}\theta^2\right) \\ &+ c_\varepsilon\left(e^{2c_w}T_K^{-4(2\beta+1)/(4\beta^2+1)}\mathrm{d}x^2 + e^{-2c_w}T_K^{4(2\beta-1)/(4\beta^2+1)}\mathrm{d}y^2\right).\end{aligned} \tag{157}$$

The conventional time coordinate T and our curvature time T_K are related by

$$T_K = \mu T_0^{\beta^2-1/4} T^{-\beta^2-1/4}. \tag{158}$$

This coordinate transformation turns the metric (157) into (136). The flat Kasner solution defined by $\alpha = 0$, $\beta = 1/2$, and $\mu = 1$, in the present gauge implies

$$\varepsilon = T_K^{-2}, \tag{159}$$

$$\bar{W} = -\ln T_K. \tag{160}$$

The coordinate transformation relating the two gauges is then simplified to

$$T_K = 1/\sqrt{T} \tag{161}$$

and the spacetime metric is given by

$$\mathrm{d}s^2 = -4T_K^{-6}\mathrm{d}T_K^2 + \mathrm{d}\theta^2 + T_K^{-4}\mathrm{d}x^2 + \mathrm{d}y^2 \tag{162}$$

with extrinsic curvature

$$K_{ab}\mathrm{d}x^a\mathrm{d}x^b = T_K^{-2}\mathrm{d}x^2 \tag{163}$$

of constant-T_K slices. The coordinate singularity of (141) at $T \to 0_+$ is here given by $T_K \to \infty$, while the coordinate singularity of (162) at $T_K \to 0_+$ corresponds to $T \to \infty$.

B. Singularity-free solutions

We now consider the first two classes of modified theories, given by constraints compatible with the limit of reaching the classical constraint surface and constraints of the second kind. In both cases, the classical singularity of homogeneous solutions is removed.

1. New variables

We first use the modified constraint (69) with constant $\lambda = \bar{\lambda}$ and λ_0, choosing the gauge

$$N^\theta = 0, \qquad \varepsilon = T. \tag{164}$$

The on-shell conditions do not change under a linear combination of the constraints, and thus we have the classical constraint surface given by (124).

The consistency equation $\dot{\varepsilon} = 1$ can be solved for the lapse function

$$N = \frac{\lambda_0^{-1}}{\sqrt{\varepsilon}} \frac{1}{K\sqrt{1-\bar{\lambda}^2K^2}}. \tag{165}$$

Using this, we obtain

$$\partial_T(\bar{a}K) = \{\bar{a}K, \tilde{H}[N]\} \propto \tilde{H}, \tag{166}$$

which vanishes on-shell. Thus, $\bar{a}K = \mu$ with a constant μ.

The $T-\theta$ part of the line element is no longer conformally flat because we have the emergent metric component

$$\tilde{q}_{\theta\theta} = \lambda_0^{-2}\frac{\bar{a}^2}{T} = \lambda_0^{-2}e^{2a} \tag{167}$$

while

$$N^2 = \frac{(\mu\lambda_0)^{-2}}{1-\bar{\lambda}^2K^2}\frac{\bar{a}^2}{\varepsilon} = \frac{\mu^{-2}}{1-\bar{\lambda}^2K^2}\tilde{q}_{\theta\theta} = \mu^{-2}B^{-2}q_{\theta\theta}^{(\mathrm{cl})}, \tag{168}$$

where $q_{\theta\theta}^{(\mathrm{cl})} = \bar{a}^2/\varepsilon$ is the classical expression and $B = \lambda_0\sqrt{1-\bar{\lambda}^2K^2}$ is the factor in the linear combination (66). Because $\varepsilon' = 0$ in this gauge, the coefficient A in the linear combination vanishes. Thus, in this gauge the full Hamiltonian generator is identical to the classical one:

$$\tilde{H}[N] + H_\theta[N^\theta] = \tilde{H}\left[B^{-1}\sqrt{\mu^{-1}q_{\theta\theta}^{(\mathrm{cl})}}\right] = H\left[\sqrt{\mu^{-1}q_{\theta\theta}^{(\mathrm{cl})}}\right] = H[N^{(\mathrm{cl})}]. \tag{169}$$

This means that all of the equations of motion are identical to the classical ones, Eqs. (128) and (129), obtaining the same classical solutions (130).

However, even though all the phase-space solutions retain their classical forms, the resulting spacetime geometry is nonclassical because the structure function differs by a constant factor of λ_0^2 from the classical one and the lapse function differs from its classical expression more significantly. The resulting emergent spacetime line element is given by

$$\mathrm{d}s^2 = \frac{e^{2a}}{\mu^2\lambda_0^2}\left(-\frac{\mathrm{d}T^2}{1-\bar{\lambda}^2\mu^2e^{-2a}/T} + \mu^2\mathrm{d}\theta^2\right) + T(e^{2W}\mathrm{d}x^2 + e^{-2W}\mathrm{d}y^2), \tag{170}$$

where a and W are related to the phase-space variables by (122). In the limit $\bar{\lambda} \to 0$, $\lambda_0 \to 1$ we recover the classical solution whose curvature invariant $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ diverges as $T \to 0_+$. Seen from positive T, this singularity lies in the past where the x-y plane had collapsed to zero area $\varepsilon = T$.

In the modified case, the $T-\theta$ part of the metric is no longer conformally flat. There is a new singularity at $T = \bar{\lambda}^2e^{-2a} =: T_{\bar{\lambda}}$, a time later than the classical singularity at $T = 0_+$. This new singularity is therefore the relevant one in the positive-T branch, but it is not a physical singularity. To see this in detail, we will use a new gauge in the next subsection.

For now, the special case of a (modified) flat Kasner solution can be analyzed more easily. It is given by $\mu = 1$, $\beta = 1/2$, $\alpha = a_n = b_n = 0$, which implies $W = \frac{1}{2}\ln T$ and $a = 0$. The line element is then equal to

$$ds^2 = -\frac{dT^2}{1 - \bar{\lambda}^2/T} + d\theta^2 + T^2 dx^2 + dy^2, \quad (171)$$

where we have chosen $\lambda_0 = 1$ so as to recover the classical Kasner metric for large $T \gg \bar{\lambda}^2$. In the classical case $\bar{\lambda} \to 0$ this is the flat Kasner solution whose curvature invariants are finite. The solutions can be analytically extended across $T = 0$, but they are causally ill-behaved as they form closed timelike curves. However, for $\bar{\lambda} \neq 0$, the singularity $T = 0_+$ is hidden inside a region bounded by the new singularity at $T = \bar{\lambda}^2 > 0$. Curvature invariants can be used to support the expectation that this is only a coordinate singularity.

We first note that the Kasner solution (171) of the modified theory is not flat: It has the Ricci scalar

$$R = \frac{\bar{\lambda}^2}{T^3} \quad (172)$$

and the Kretschmann invariant

$$\mathcal{K} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = -\frac{1}{2}\frac{\bar{\lambda}^4}{T^{10}}\left(1 + T^4\left(1 - \frac{\bar{\lambda}^2}{T}\right)^2\right). \quad (173)$$

At $T = \bar{\lambda}^2$, both are finite. Moving across this value, a new gauge must be chosen, in which, as we will see in what follows, the physical singularity at $T = 0$ no longer appears. (This construction is similar to the nonsingular black-hole models in [10,11].) Any hypersurface of constant time T of the modified Kasner spacetime (171) defines a three-dimensional space with nonvanishing extrinsic curvature

$$K_{ab}dx^a dx^b = T\sqrt{1 - \frac{\bar{\lambda}^2}{T}}dx^2 \quad (174)$$

unless $T = \bar{\lambda}$. This specific value implies a hypersurface of time-reflection symmetry, which can be used to glue a time reverse of our solution at $T = \bar{\lambda}$. The classical singularity at $T = 0$ is then replaced by a transition from collapse to expansion.

If the interpretation of the classical flat solution as a vacuum spacetime is extended to the modified theory, it could suggest a vacuum different from the usual Minkowski one, being approximately flat only for $T \gg \bar{\lambda}^2$. However, in Sec. VI B 5 we will show that drawing such a conclusion based on only homogeneous models in a fixed gauge of internal time would not be justified. For now, we continue our analysis of homogeneous dynamics.

2. Periodic variables

We shall now use the modified constraint (71) with constant λ_0 and $\lambda = \bar{\lambda}$, reproducing the above results because this version will serve as a guide to obtaining the dynamical solutions of the other two constraints.

We again choose the gauge

$$N^\theta = 0, \qquad \varepsilon = T. \quad (175)$$

The on-shell conditions are

$$0 = \frac{1}{4\varepsilon}\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} - \frac{\cos^2(\bar{\lambda}K)}{4\varepsilon}\frac{P_{\bar{W}}{}^2}{\bar{a}^2} + \frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}}\frac{\mathcal{A}}{\bar{a}} - \varepsilon\frac{(\bar{W}')^2}{\bar{a}^2}\cos^2(\bar{\lambda}K), \quad (176)$$

$$0 = \bar{a}K' + P_{\bar{W}}\bar{W}', \quad (177)$$

and the consistency equation $\dot{\varepsilon} = 1$ can be solved for the lapse function

$$N = \frac{\lambda_0^{-1}}{\sqrt{\varepsilon}}\frac{2\bar{\lambda}}{\sin(2\bar{\lambda}K)}. \quad (178)$$

Using this result, we obtain

$$\partial_T\left(\bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\right) = \left\{\bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}, \tilde{H}[N]\right\} \propto \tilde{H}, \quad (179)$$

which vanishes on-shell. Thus, $\bar{a}\tan(\bar{\lambda}K)/\bar{\lambda} = \mu$ where μ is a constant. Because of the canonical transformation involved, the identification (122) changes to

$$\bar{W} = W,$$
$$a = \ln\sqrt{E^x E^y/\varepsilon} = \ln\left(\frac{\bar{a}}{\cos(\bar{\lambda}K)}\right) - \ln\sqrt{\varepsilon}. \quad (180)$$

Therefore,

$$\sin(\bar{\lambda}K) = \frac{\bar{\lambda}\mu}{\bar{a}/\cos(\bar{\lambda}K)} = \frac{\bar{\lambda}\mu}{\sqrt{\varepsilon}e^a}. \quad (181)$$

The equations of motion for $\bar{a}$ and $\bar{W}$, respectively, give

$$\mathcal{A} = \mu\partial_T\left(\ln\left(\frac{\bar{a}}{\cos(\bar{\lambda}K)}\right) - \ln\sqrt{T}\right) = \mu\dot{a}, \quad (182)$$

$$P_{\bar{W}} = 2\,\mu T\dot{\bar{W}}. \quad (183)$$

Using these results, the constraints $H_\theta = 0$ and $\tilde{H} = 0$ can be rewritten as

$$a' = 2T\dot{W}W', \qquad \dot{a} = -\frac{1}{4T} + T\left(\dot{W}^2 + \frac{(W')^2}{\mu^2}\right), \quad (184)$$

where we have used the identification (180). The equation of motion $\ddot{\bar{W}} = \{\{\bar{W}, \tilde{H}[N]\}, \tilde{H}[N]\}$, such that the brackets *do act* on the lapse as discussed before, can be rewritten as

$$0 = \ddot{W} + \frac{\dot{W}}{T} - \frac{W''}{\mu^2}. \quad (185)$$

These equations are identical to the classical ones. The emergent line element is then given by

$$\begin{aligned} ds^2 &= \lambda_0^{-2} e^{2a}(-\sec^2(\bar{\lambda}K)dT^2 + d\theta^2) \\ &\quad + T(e^{2W}dx^2 + e^{-2W}dy^2) \\ &= \lambda_0^{-2} e^{2a}\left(-\frac{dT^2}{1-\bar{\lambda}^2\mu^2/(Te^{2a})} + d\theta^2\right) \\ &\quad + T(e^{2W}dx^2 + e^{-2W}dy^2). \end{aligned} \quad (186)$$

Modified Kasner models are obtained for $W = \beta \ln T$, which implies $e^{2a} \propto T^{2\beta^2 - 1/2}$ as in the classical case. The line element then equals

$$\begin{aligned} ds^2 &= \lambda_0^{-2} T^{2\beta^2-1/2}\left(-\frac{dT^2}{1-\bar{\lambda}^2 T^{-2\beta^2-1/2}} + d\theta^2\right) \\ &\quad + T^{1+2\beta}dx^2 + T^{1-2\beta}dy^2 \end{aligned} \quad (187)$$

if we set μ^2 equal to the proportionality factor in e^{2a}. For constant but nonzero λ, proper time is now given by a hypergeometric function of T, which complicates any further analysis of general Kasner models. It is nevertheless possible to understand the general behavior.

To do so, we first introduce a new time coordinate $t = T^{\beta^2+3/4}$, such that

$$\begin{aligned} d\tau &\propto \frac{T^{\beta^2-1/4}}{\sqrt{1-\bar{\lambda}^2 T^{-2\beta^2-1/2}}} dT \\ &= \frac{1}{\beta^2+3/4} \frac{dt}{\sqrt{1-\bar{\lambda}^2 t^{-2(\beta^2+1/4)/(\beta^2+3/4)}}} \end{aligned} \quad (188)$$

according to the time component of (187). For large T, t and τ therefore proceed at almost the same rate, up to a constant rescaling. With respect to proper time, we now have the line element

$$ds^2 = -d\tau^2 + \lambda_0^{-2} t(\tau)^{2p_1} d\theta^2 + t(\tau)^{2p_2} dx^2 + t(\tau)^{2p_3} dy^2 \quad (189)$$

with Kasner exponents p_i as in (138), obeying the classical relations, and the inverse of a hypergeometric function (times t) for $t(\tau)$. [For $\bar{\lambda} = 1$, $t(\tau)$ is the inverse of $t\,{}_2F_1(1/2, 1/a; 1 + 1/a; t^a)$ if $a = -2(\beta^2 + 1/4)/(\beta^2 + 3/4) \neq -1$. If $a = -1$, $t(\tau)$ is the inverse of $\sqrt{(t-1)t} - \sinh^{-1}(t)$.]

For large $t(\tau)$ such that $\bar{\lambda} \ll T^{\beta^2+1/4} = t^{(\beta^2+1/4)/(\beta^2+3/4)}$, the behavior is close to the classical Kasner dynamics with the same relationship between the Kasner exponents and the conserved quantity β. For smaller t, however, there is a new effect because the relationship between t and τ is not one-to-one, in contrast to the classical solutions. We have

$$\frac{dt}{d\tau} = \lambda_0(\beta^2 + 3/4)\sqrt{1 - \bar{\lambda}^2 t^{-2(\beta^2+1/4)/(\beta^2+3/4)}} = 0 \quad (190)$$

at

$$t = t_{\bar{\lambda}} = \bar{\lambda}^{(\beta^2+3/4)/(\beta^2+1/4)}. \quad (191)$$

At the same value of t,

$$\begin{aligned} \frac{d^2t}{d\tau^2} &= \lambda_0 \bar{\lambda}^2(\beta^2 + 1/4)\frac{t^{-(3\beta^2+5/4)/(\beta^2+3/4)}}{\sqrt{1-\bar{\lambda}^2 t^{-2(\beta^2+1/4)/(\beta^2+3/4)}}}\frac{dt}{d\tau} \\ &= \lambda_0^2\bar{\lambda}^2(\beta^2+1/4)(\beta^2+3/4)t^{-(3\beta^2+5/4)/(\beta^2+3/4)} \\ &= \lambda_0^2(\beta^2+1/4)(\beta^2+3/4)t_{\bar{\lambda}}^{-1} > 0 \end{aligned} \quad (192)$$

such that $t(\tau)$ has a local minimum at the value $t(\tau) = t_{\bar{\lambda}}$. The full dynamics therefore describes nonsingular evolution of a collapsing Kasner model connected to an expanding Kasner model with the same exponents. All three spatial directions transition from collapse to expansion at the same time $\tau(t_{\bar{\lambda}})$. The behavior of $t(\tau)$ is illustrated in Figs. 1 and 2.

The special case of the modified flat Kasner model is given by $\mu = 1$, $\beta = 1/2$, $\alpha = a_n = b_n = 0$, which implies $W = \frac{1}{2}\ln T$ and $a = 0$. We have $\mathcal{A} = 0$ and

$$\frac{\sin(\bar{\lambda}K)}{\bar{\lambda}} = \frac{1}{\sqrt{T}}, \quad (193)$$

and the line element

$$ds^2 = \lambda_0^{-2}(-\sec^2(\bar{\lambda}K)dT^2 + d\theta^2) + T^2dx^2 + dy^2. \quad (194)$$

Here, proper time $\tau(T)$ can be integrated more easily but its inversion to $T(\tau)$ remains complicated.

3. Homogeneous solution: Internal-time gauge

Let us now use the inhomogeneous curvature component as an internal time, $T_K = K$. The two time coordinates are related to each other by

$$\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}} = \frac{1}{\sqrt{T}} \quad (195)$$

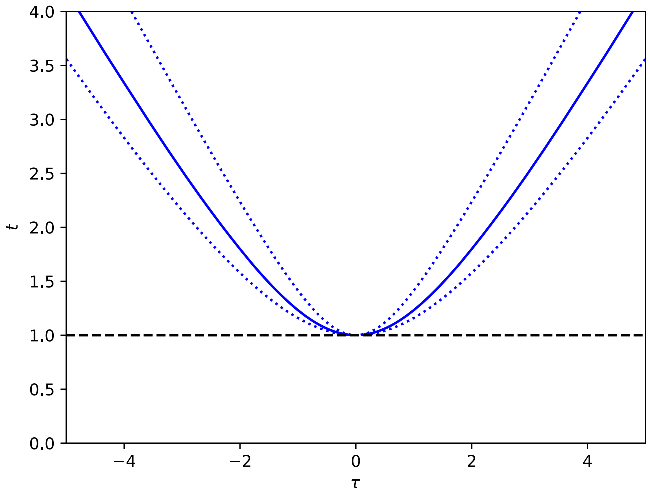

FIG. 1. The function of $t(\tau)$ with $\bar{\lambda} = 1$, obtained from the inverse of a hyperbolic function, shows the transition from collapse to expansion in Kasner models of emergent modified gravity. Its dependence on different values of β is shown here by the range of possible curves, with the upper bound given by large β such that $-2(\beta^2 + 1/4)/(\beta^2 + 3/4) \approx -2$ and the lower bound given by $\beta = 0$ such that $-2(\beta^2 + 1/4)/(\beta^2 + 3/4) = -2/3$ (dotted curves). The value $\beta = 1/2$ (solid line), such that $-2(\beta^2 + 1/4)/(\beta^2 + 3/4) = -1$, is close to the midpoint of this range.

such that

$$-2\frac{\bar{\lambda}^3}{\sin^3(\bar{\lambda}T_K)}\cos(\bar{\lambda}T_K)\mathrm{d}T_K = \mathrm{d}T. \tag{196}$$

Substituting in the line element for the modified flat Kasner model, we obtain

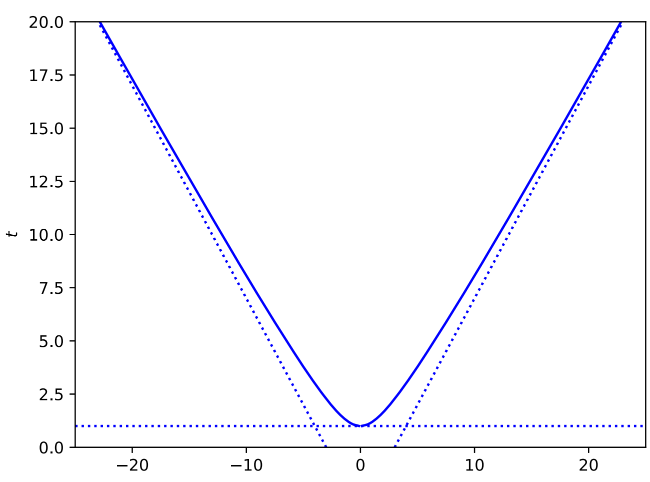

FIG. 2. The asymptotic behavior of the function of $t(\tau)$ for large τ is close to $t(\tau) = \pm\tau + t_\pm$ if $\bar{\lambda} = 1$. The value $\beta = 1/4$ has been used for the solid curve.

$$\mathrm{d}s^2 = \lambda_0^{-2}\left(-4\frac{\bar{\lambda}^6}{\sin^6(\bar{\lambda}T_K)}\mathrm{d}T_K^2 + \mathrm{d}\theta^2\right) + \frac{\bar{\lambda}^4}{\sin^4(\bar{\lambda}T_K)}\mathrm{d}x^2 + \mathrm{d}y^2, \tag{197}$$

which is indeed regular at maximum curvature, $T_K = \pi/2\bar{\lambda}$, defining a surface of reflection symmetry.

We now derive this result by directly solving the equations of motion in the internal-time gauge (147), rather than performing a coordinate transformation. For the homogeneous model, we set $P'_{\bar{W}} = \bar{W}' = \bar{a}' = \varepsilon' = K' = \mathcal{A}' = N' = 0$ and assume a vanishing cosmological constant, $\Lambda = 0$. We note that the modified constraints (94) and (109) are identical in the homogeneous case if the classical values for the functions $c_f, \alpha_2, \alpha_3 \to 1$, $q \to 0$, and $\Lambda_0 \to \Lambda \to 0$ are taken, with constant $\lambda = \bar{\lambda}$, $\nu = \bar{\nu}$, and λ_0. The results of the present and the following subsections then apply to both cases.

We first see that because of homogeneity the local version of the observable (27) is conserved, $\dot{G} = 2\pi\dot{P}_W = 0$, and we will write the momentum as $P_W = 2\mu\beta$ with constants μ and β. The on-shell condition $H_\theta = 0$ is trivially satisfied in this case, while $\tilde{H} = 0$ is solved by

$$\mathcal{A} = -\frac{\bar{a}}{4\varepsilon}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}} + \frac{\bar{\lambda}\cot(\bar{\lambda}K)}{4\varepsilon}\frac{4\mu^2\beta^2}{\bar{a}}. \tag{198}$$

We then obtain

$$\partial_T\left(\bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}\right) = \left\{\bar{a}\frac{\tan(\bar{\lambda}K)}{\bar{\lambda}}, \tilde{H}[N]\right\} \propto \tilde{H}, \tag{199}$$

which vanishes on-shell, such that $\bar{a}\tan(\bar{\lambda}K)/\bar{\lambda} = \mu$ where μ is a constant. Hence,

$$\mathcal{A} = (4\beta^2 - 1)\frac{\mu}{4\varepsilon}. \tag{200}$$

In combination with the chain rule we obtain the equations

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}K} = \frac{\dot{\varepsilon}}{\dot{K}} = -\frac{4\bar{\lambda}}{(1 + 4\beta^2)\tan(\bar{\lambda}K)}\varepsilon, \tag{201}$$

$$\frac{\mathrm{d}}{\mathrm{d}K}\left(\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\right) = -\frac{4\beta\bar{\lambda}}{(1 + 4\beta^2)\tan(\bar{\lambda}K)}, \tag{202}$$

solved by

$$\varepsilon = c_\varepsilon\left(\frac{\sin(\bar{\lambda}K)}{\bar{\lambda}}\right)^{-4/(1+4\beta^2)} \tag{203}$$

and

$$\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}} = \ln\left(e^{c_w}\frac{\sin(\bar{\lambda}K)}{\bar{\lambda}}\right)^{-4\beta/(1+4\beta^2)}. \tag{204}$$

For convenience, the integration constants may be redefined as

$$c_\varepsilon = (\mu T_0^{(4\beta^2-1)/4})^{4/(4\beta^2+1)} \tag{205}$$

and

$$e^{2c_w} = e^{2\alpha}(\mu T_0^{(4\beta^2-1)/4})^{8\beta/(4\beta^2+1)}. \tag{206}$$

Finally, the lapse function is obtained by solving the consistency equation $\dot{K} = 1$,

$$N = -\frac{4}{1+4\beta^2}\frac{\sqrt{c_\varepsilon}}{\lambda_0}\sec^2(\bar{\nu}\,\bar{W})\left(\frac{\sin(\bar{\lambda}K)}{\bar{\lambda}}\right)^{-2/(1+4\beta^2)-2}. \tag{207}$$

The spacetime line element is then given by

$$\begin{aligned} \mathrm{d}s^2 = {}& \sec^4(\bar{\nu}\,\bar{W})\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{4/(1+4\beta^2)-2}\left(-\left(\frac{4}{1+4\beta^2}\right)^2\frac{c_\varepsilon}{\lambda_0^2}\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{-8/(1+4\beta^2)-2}\mathrm{d}T_K^2 + \frac{\mu^2}{c_\varepsilon}\mathrm{d}\theta^2\right) \\ & + c_\varepsilon e^{2c_w}\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{-4(2\beta+1)/(4\beta^2+1)}\mathrm{d}x^2 + c_\varepsilon e^{-2c_w}\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{4(2\beta-1)/(4\beta^2+1)}\mathrm{d}y^2, \end{aligned} \tag{208}$$

where $\bar{W}$ is implicitly given by (204).

4. Modified flat Kasner solution

The Kasner solution $\mu = 1$, $\beta = 1/2$ is given by the simpler metric

$$\begin{aligned} \mathrm{d}s^2 = {}& \sec^4(\bar{\nu}\,\bar{W})\left(-\frac{4}{\lambda_0^2}\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{-6}\mathrm{d}T_K^2 + \mathrm{d}\theta^2\right) \\ & + \left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{-4}\mathrm{d}x^2 + \mathrm{d}y^2, \end{aligned} \tag{209}$$

where $\bar{W}$ can be obtained from inverting its relation with K,

$$\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}} = \exp\left(-\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}}\right). \tag{210}$$

Taking $\bar{\nu} \to 0$, the Kasner solution in this gauge (209) has the Ricci scalar

$$R = \bar{\lambda}^2\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^6, \tag{211}$$

and the Kretschmann invariant

$$\mathcal{K} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = -\frac{\bar{\lambda}^4}{8}\left(\frac{\sin(\bar{\lambda}T_K)}{\bar{\lambda}}\right)^{22}. \tag{212}$$

Both expressions are finite at $T_K = \pi/(2\bar{\lambda})$. The model is approximately flat only for $T_K \ll \bar{\lambda}$, and both curvature invariants vanish in the classical limit $\bar{\lambda} \to 0$.

A hypersurface of constant time T_K of the modified Kasner spacetime (209) defines a three-dimensional space with extrinsic curvature

$$K_{ab}\mathrm{d}x^a\mathrm{d}x^b = \lambda_0\frac{\bar{\lambda}^2\cos(\bar{\lambda}T_K)}{\sin^2(\bar{\lambda}T_K)}\mathrm{d}\theta^2, \tag{213}$$

which vanishes only at the maximum-curvature hypersurface.

5. Flat solution: Nonunique vacuum

From the above example one might conclude that the vacuum solution of this theory is different from Minkowski spacetime, as suggested for a similar case for instance in [34]. It is easy to see that such a statement is incorrect because flat spacetime, described by

$$\begin{aligned} & N = 1, \quad N^\theta = 0, \\ & \varepsilon = 1, \quad \bar{a} = 1, \quad \bar{W} = 0, \\ & \mathcal{A} = 0, \quad K = 0, \quad P_{\bar{W}} = 0, \end{aligned} \tag{214}$$

is a solution to the same theory if, to be specific, we take the classical values for all the modification functions except for λ and ν which we leave as arbitrary functions. This solution is excluded from the case of Kasner-like line elements by the assumption that ε can be used as a time variable, such that $\varepsilon = 1$ can be obtained only on one spacelike hypersurface but not across an entire spacetime region. Nevertheless, flat Minkowski spacetime is a solution of the same modified theory in which we obtained our Kasner spacetimes.

Minkowski spacetime is relevant because its local behavior describes the background spacetime of vacuum states in quantum field theory. According to the general meaning of "vacuum" in particle physics or general relativity, all Kasner models are vacuum solutions because they do not include matter. The case of $\beta = 1/2$ is special only because, classically, it happens to be locally equivalent to Minkowski spacetime and just appears written in non-Cartesian coordinates. The result that this correspondence is not realized in a modified theory only means that there is no longer a Kasner model related to flat Minkowski spacetime. It does not mean that Minkowski spacetime itself is modified or no longer appears as a solution, as demonstrated by the explicit counterexample of (214).

A distinguishing feature of (214) compared with any Kasner solution is that it is not only devoid of matter but also has a vanishing local gravitational degrees of freedom described by $(\bar{W}, P_{\bar{W}})$. We can formalize this property by making use of the definition of an effective stress energy, obtained from the Einstein tensor of the emergent spacetime metric. We find that the flat solution (214) has a vanishing net stress-energy tensor, while the modified flat Kasner solution has a nontrivial one, as shown by the nonvanishing Ricci scalar (211). Therefore, the solutions are distinguished from one another by their effective gravitational energy content. From this perspective, the standard Minkowski solution remains the preferred vacuum spacetime also in a modified theory. For $\bar{\lambda} \to 0$, the effective stress-energy tensor vanishes, and the modified flat Kasner solution approaches the strictly flat Minkowski solution.

The correct identification of a candidate vacuum solution therefore requires an extension of strict minisuperspace models to some inhomogeneity, which tells us that the nonzero $\bar{W}$ and $P_{\bar{W}}$ in Kasner models are homogeneous remnants of a propagating gravitational degree of freedom, and the correct identification of a covariant spacetime structure that defines curvature and effective stress energy. It is also important to have a gauge-invariant treatment that is not built on a fixed gauge choice such as an internal time, as such a choice might restrict the accessible solution space. None of these ingredients had been available in previous models of quantum cosmology. With some choices of modification functions, it might happen that strict Minkowski spacetime is no longer a solution or that the zero-mass limit of a black-hole solution differs from Minkowski spacetime as seen explicitly in an example in [15]. But such a conclusion cannot be drawn in a reliable manner in theories based on restricted gauge choices or on incomplete demonstrations of covariance properties.

C. Constraints compatible with the classical-$\bar{W}$ limit

We now use the constraint (115) in the internal time gauge (147) for the homogeneous case where $P'_{\bar{W}} = \bar{W}' = \bar{a}' = \varepsilon' = K' = \mathcal{A}' = N' = 0$. For simplicity we take the classical values for the following modification functions and assume a vanishing cosmological constant, $c_f, \alpha_2, \alpha_3 \to 1$, $q \to 0$, and $\Lambda_0 \to \Lambda \to 0$. We also set $\lambda_0, \lambda = \bar{\lambda}$, and $\nu = \bar{\nu}$ constant. With these values, the inhomogeneous component of the emergent spatial metric is given by

$$\tilde{q}_{\theta\theta} = \lambda_0^{-2} \cos^{-4}(\bar{\nu}\,\bar{W}) \frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)} \frac{1}{\varepsilon}. \tag{215}$$

As before, homogeneity implies that the local version of the observable (27) is conserved, $\dot{G} = 0$, and we shall write it as $G = 4\pi\mu\beta$ such that $P_{\bar{W}} = 2\mu\beta$, anticipating an integration constant (μ) that will be introduced in the process of solving equations of motion. The value of β then parametrizes the momentum.

The relevant equations of motion for recovering the emergent spacetime geometry are given by

$$\frac{\mathrm{d}\ln(\bar{a}^2/\cos^2(\bar{\lambda}K))}{\mathrm{d}(\sin(\bar{\lambda}K)/\bar{\lambda})} = -2\frac{\bar{\lambda}}{\sin(\bar{\lambda}K)} \left(\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} \cos^2(\bar{\lambda}K) \frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)} + 4\mu^2\beta^2 \frac{\cos(2\bar{\lambda}K)}{|\cos(\bar{\lambda}K)|} \right) \times \left(\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} \cos^2(\bar{\lambda}K) \frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)} + 4\mu^2\beta^2 |\cos(\bar{\lambda}K)| \right)^{-1} \tag{216}$$

as well as

$$\frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}K} = -4\frac{\sin(2\bar{\lambda}K)}{2\bar{\lambda}} \left(\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} |\cos(\bar{\lambda}K)| + 4\mu^2\beta^2 \frac{\cos^2(\bar{\lambda}K)}{\bar{a}^2} \right)^{-1} \tag{217}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}K} \left(\frac{\sin(\bar{\nu}\,\bar{W})}{\bar{\nu}} \right) = -4\mu\beta \frac{\cos^2(\bar{\lambda}K)}{\bar{a}} \left(\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} |\cos(\bar{\lambda}K)| + 4\mu^2\beta^2 \frac{\cos^2(\bar{\lambda}K)}{\bar{a}^2} \right)^{-1}, \tag{218}$$

where we have chosen K as an evolution parameter. Equation (216) can be solved exactly,

$$\frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)} = \frac{\mu^2}{4} \frac{\bar{\lambda}^2}{\sin^2(\bar{\lambda}K)} \left(1 - 4\beta^2 + \sqrt{(1-4\beta^2)^2 + \frac{16\beta^2}{|\cos(\bar{\lambda}K)|}} \right)^2. \tag{219}$$

The integration constant μ^2 and the sign of the square root have been chosen for the solution to match the classical one, Eq. (149), in the limit $\bar{\lambda} \to 0$.

The ratio (219) appears directly as a factor in the emergent metric component $q_{\theta\theta}$, given by (215). Near the maximum-curvature hypersurface, defined by $K \to \pi/(2\bar{\lambda})$, this expression diverges as $\sec(\bar{\lambda}K)$, and its internal-time derivative (216) diverges as $\sec^2(\bar{\lambda}K)$. Using this, we find that the right-hand side of (217) remains finite at the maximum-curvature hypersurface,

$$\frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}K} \approx -4\frac{\bar{\lambda}}{\sin(\bar{\lambda}K)}, \tag{220}$$

while that of (218) vanishes. We conclude that both ε and $\sin(\bar{\nu}\bar{W})$ remain finite, and hence the homogeneous components q_{xx} and q_{yy} are finite too.

We complete the gauge fixing by enforcing the consistency equation $\dot{K} = 1$ and solve it for the lapse function,

$$N = -\frac{\sec^2(\bar{\nu}\bar{W})}{\lambda_0} 4\sqrt{\varepsilon}\left(\frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)}|\cos(\bar{\lambda}K)|\frac{\sin^2(\bar{\lambda}K)}{\bar{\lambda}^2} + 4\mu^2\beta^2\right)^{-1}\frac{\bar{a}^2}{\cos^2(\bar{\lambda}K)}|\cos(\bar{\lambda}K)|, \tag{221}$$

which is finite at the maximum-curvature hypersurface provided $\bar{W} \neq -\pi/(2\bar{\nu})$.

Because of the divergence of (219) and its derivative (216), the emergent line element has a singular θ-component at the maximum-curvature hypersurface, and its time derivatives are singular there too. Thus, neglecting the time derivatives of the q_{xx} and q_{yy} components, a homogeneous line element of the form

$$\mathrm{d}s^2 = -N^2\mathrm{d}T_K^2 + \tilde{q}_{\theta\theta}\mathrm{d}\theta^2 + q_{xx}\mathrm{d}x^2 + q_{yy}\mathrm{d}y^2 \tag{222}$$

has the Ricci scalar

$$R \approx -\frac{\dot{\tilde{q}}_{\theta\theta}}{\tilde{q}_{\theta\theta}}\frac{\dot{N}}{N^3} - \frac{1}{2N^2}\left(\left(\frac{\dot{\tilde{q}}_{\theta\theta}}{\tilde{q}_{\theta\theta}}\right)^2 - 2\frac{\ddot{q}_{\theta\theta}}{q_{\theta\theta}}\right), \tag{223}$$

which diverges as $R \sim \sec^2(\bar{\lambda}T_K)$ near the maximum-curvature hypersurface, while the Kretschmann scalar takes the form

$$\mathcal{K} \approx -\frac{R^2}{2\tilde{q}_{\theta\theta}N^2}, \tag{224}$$

which diverges as $\mathcal{K} \sim \sec^3(\bar{\lambda}T_K)$ near the maximum-curvature hypersurface This constraint, unlike the other two versions considered in this paper, therefore implies a singular geometry at the maximum-curvature hypersurface.

VII. DISCUSSION

We have extended emergent modified gravity from spherically symmetric models to polarized Gowdy systems, preserving most of the qualitative features observed in previous publications. In particular, modification functions of the same number and type remain in the classes of modified constraints derived explicitly here, building on a relationship with models of a scalar field coupled to spherically symmetric gravity. Emergent modified gravity therefore is not restricted to spherical symmetry, and it is compatible with different kinds of local degrees of freedom from matter or gravity.

One class of models, compatible with the classical limit of the local gravitational degree of freedom, has a set of modification functions such that polarized gravitational waves travel on an emergent spacetime geometry just like a minimally coupled scalar field. The existence of these models shows that a nontrivial class of theories in emergent modified gravity has gravitational waves and matter (a minimally coupled massless scalar field propagating on the same geometry) traveling at the same speed. Emergent modified gravity is therefore compatible with strong observational restrictions on the difference of the two speeds [35–38]. Moreover, emergent modified gravity does not require higher time derivatives for nontrivial modifications, and is therefore free of related instabilities [16].

Compared with spherically symmetric models, polarized Gowdy systems have a large class of homogeneous solutions that correspond to the full Kasner dynamics of the Bianchi I model. We have derived consistent modifications of this dynamics with the correct classical limit at large volume but different behaviors at small volume. Some types of modifications lead to nonsingular evolution connecting collapsing and expanding Kasner dynamics, while models compatible with the classical limit for the local gravitational degree of freedom retain the classical big-bang singularity. In the nonsingular case, all three spatial directions transition from collapse to expansion at the same time. We demonstrated that the modified Kasner family may no longer include Minkowski spacetime, but that a different gauge choice not based on an internal time nevertheless shows that this geometry remains a solution of the modified theory. Discussions of possible vacuum states in a modified theory therefore require access to different gauge choices and cannot be made reliably in a deparametrized setting, as often used in quantum cosmology.

The restrictions on inhomogeneous terms in the covariant constraints, imposing the covariance requirement on an emergent spacetime metric distinct from the basic phase-space variables, demonstrates the nontrivial nature of modifications or quantizations of the polarized Gowdy model. In particular, a separate modification or quantization of a homogeneous Bianchi model coupled to linearized classical-type inhomogeneity, as proposed for instance in hybrid loop quantum cosmology [39–41], does not lead to covariant spacetime solutions because it is not contained in the general class of consistent models derived here. Modifications of the background dynamics, one of the key ingredients in cosmological models of loop quantum gravity, instead have to be reflected in coefficients of the inhomogeneous terms and in the corresponding emergent line element, as determined by strong covariance conditions. Midisuperspace quantizations of polarized Gowdy and related models, as in [42–49], would have to take into account the new holonomy behavior found in Eq. (74) in order to be compatible with a covariant semiclassical limit. The dependence of the holonomies on anisotropies (rather than areas or volumes as previously assumed in models of loop quantum gravity) then implies new phenomenological behaviors. These applications indicate that emergent modified gravity has important implications for classical as well as quantum models of gravity.

Our successful extension of emergent modified gravity from spherical symmetry to polarized Gowdy models is nontrivial, as previous attempts to generalize anomaly-free modifications of spherically symmetric models to Gowdy symmetries had failed [46]. The constructions shown here not only imply anomaly-free modified constraints, they also implement full covariance conditions. Together with the previous extension of vacuum spherically symmetric models to scalar matter [6] and perfect fluids [3], also within emergent modified gravity, these are the first nontrivial canonical modifications of gravitational models with local or matter degrees of freedom. Based on these examples, it seems that each new degree of freedom allows one additional modification function, without restricting the modification freedom of the remaining degrees of freedom. This success is encouraging, but the question remains open as to whether nontrivial versions of emergent modified gravity exist without any symmetry assumptions. As shown recently, some of the crucial equations that implement covariance for modified canonical theories hold in general because they follow from intrinsic properties of hypersurface deformations [50]. However, the solution space of these equations so far remains unexplored.

ACKNOWLEDGMENTS

The authors thank Manuel Díaz and Aidan Kelly for discussions. This work was supported in part by NSF Grant No. PHY-2206591.

[1] M. Bojowald and E. I. Duque, Classical Quantum Gravity **41**, 095008 (2024).

[2] M. Bojowald and E. I. Duque, Phys. Rev. D **108**, 084066 (2023).

[3] E. I. Duque, Phys. Rev. D **109**, 044014 (2024).

[4] E. I. Duque, arXiv:2407.14954.

[5] A. Alonso-Bardají and D. Brizuela, Phys. Rev. D **107**, 064067 (2023).

[6] M. Bojowald and E. I. Duque, Phys. Rev. D **109**, 084006 (2024).

[7] A. Alonso-Bardají and D. Brizuela, Phys. Rev. D **109**, 044065 (2024).

[8] S. A. Hojman, K. Kuchař, and C. Teitelboim, Ann. Phys. (N.Y.) **96**, 88 (1976).

[9] M. Bojowald, S. Brahma, and D.-H. Yeom, Phys. Rev. D **98**, 046015 (2018).

[10] A. Alonso-Bardají, D. Brizuela, and R. Vera, Phys. Lett. B **829**, 137075 (2022).

[11] A. Alonso-Bardají, D. Brizuela, and R. Vera, Phys. Rev. D **106**, 024035 (2022).

[12] M. Bojowald and E. I. Duque, Phys. Lett. B **847**, 138279 (2023).

[13] M. Bojowald, E. I. Duque, and D. Hartmann, Phys. Rev. D **109**, 084001 (2024).

[14] M. Bojowald, E. I. Duque, and S. Shankaranarayanan, arXiv:2410.17501.

[15] I. H. Belfaqih, M. Bojowald, S. Brahma, and E. I. Duque, arXiv:2407.12087.

[16] R. P. Woodard, Lect. Notes Phys. **720**, 403 (2007).

[17] R. H. Gowdy, Ann. Phys. (N.Y.) **83**, 203 (1974).

[18] K. Banerjee and G. Date, Classical Quantum Gravity **25**, 105014 (2008).

[19] J. M. Pons, D. C. Salisbury, and L. C. Shepley, Phys. Rev. D **55**, 658 (1997).

[20] M. Bojowald, *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity* (Cambridge University Press, Cambridge, England, 2010).

[21] M. Bojowald and H. A. Kastrup, Classical Quantum Gravity **17**, 3009 (2000).

[22] M. Bojowald, Classical Quantum Gravity **21**, 3733 (2004).

[23] M. Bojowald and R. Swiderski, Classical Quantum Gravity **23**, 2129 (2006).

[24] M. Bojowald, Gen. Relativ. Gravit. **38**, 1771 (2006).

[25] M. Bojowald, Classical Quantum Gravity **19**, 2717 (2002).

[26] M. Bojowald, Living Rev. Relativity **11**, 4 (2008).

[27] T. Thiemann, Phys. Lett. B **380**, 257 (1996).

[28] C. Tomlin and M. Varadarajan, Phys. Rev. D **87**, 044039 (2013).

[29] M. Varadarajan, Phys. Rev. D **87**, 044040 (2013).
[30] M. Varadarajan, Phys. Rev. D **97**, 106007 (2018).
[31] R. Gambini and J. Pullin, Phys. Rev. Lett. **110**, 211301 (2013).
[32] I. H. Belfaqih, M. Bojowald, S. Brahma, and E. I. Duque (to be published).
[33] B. K. Berger, Phys. Rev. D **11**, 2770 (1975).
[34] P. Singh, Int. J. Mod. Phys. D **25**, 1642001 (2016).
[35] L. Scientific *et al.*, Astrophys. J. **848**, L12 (2017).
[36] B. P. Abbott *et al.* (LIGO Scientific, Virgo, Fermi Gamma-Ray Burst Monitor, and INTEGRAL Collaborations), Astrophys. J. **848**, L13 (2017).
[37] D. A. Coulter *et al.*, Science **358**, 1556 (2017).
[38] A. Murguia-Berthier *et al.*, Astrophys. J. **848**, L34 (2017).
[39] M. Martín-Benito, L. J. Garay, and G. A. Mena Marugán, Phys. Rev. D **78**, 083516 (2008).
[40] L. J. Garay, M. Martín-Benito, and G. A. Mena Marugán, Phys. Rev. D **82**, 044048 (2010).
[41] M. Martín-Benito, D. Martín-de Blas, and G. A. Mena Marugán, Phys. Rev. D **83**, 084050 (2011).
[42] D. E. Neville, Phys. Rev. D **73**, 124005 (2006).
[43] K. Banerjee and G. Date, Classical Quantum Gravity **25**, 145004 (2008).
[44] F. Hinterleitner and S. Major, Classical Quantum Gravity **29**, 065019 (2012).
[45] F. Hinterleitner, arXiv:1703.03757.
[46] M. Bojowald and S. Brahma, Phys. Rev. D **92**, 065002 (2015).
[47] D. Martín-de Blas, J. Olmedo, and T. Pawlowski, Phys. Rev. D **96**, 106016 (2017).
[48] J. Ben Achour and S. Brahma, Phys. Rev. D **97**, 126003 (2018).
[49] F. Hinterleitner, arXiv:2403.11864.
[50] M. Bojowald, E. I. Duque, and A. Shah, arXiv:2410.18807.