Curve-Counting and Mirror Symmetry

Emily Clader

Curve-counting is a subject that dates back hundreds, even thousands, of years. Broadly speaking, its goal is to answer questions about the number of curves in some ambient space that satisfy prescribed conditions, such as the following:

- How many conics pass through five given points in the plane?
- How many lines pass through four given lines in three-space?
- How many lines lie on the quintic threefold
$$\{z_1^5 + z_2^5 + \cdots + z_5^5 = 0\}$$
in $\mathbb{CP}^4$?

The answer to the first of these questions was known to the ancient Greeks: given five (sufficiently general) points in $\mathbb{R}^2$, there is exactly one conic that passes through all five of them. The method by which the ancient Greeks would have arrived at this answer is by an explicit construction, given the coordinates of the five points, of the conic that passes through them.

The modern perspective on curve-counting is somewhat different. Rather than seeking explicit constructions of the curves being counted—which can be unnecessarily cumbersome if our ultimate goal is simply enumeration—one instead searches for answers that are deformation-invariant: for example, a count of conics through five given points that remains unchanged if the five points are slightly varied. This property not only allows us to answer entire families of questions simultaneously (not just "how many conics pass through *these* five points?" but "how many conics pass through *any* five general points?"), but it also introduces the possibility of answering a difficult question by deforming it to a simpler one.

For instance, suppose one wishes to answer the second question posed at the beginning of the article: how many lines pass through four given lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{R}^3$? If the answer to this question is deformation-invariant, then we can deform our lines until they meet in pairs, so that $\ell_1 \cap \ell_2 = \{P\}$ and $\ell_3 \cap \ell_4 = \{Q\}$. At this point, a bit of reflection is enough to see that there are exactly two lines passing through all four of our original lines: one that joins P and Q, and one where the plane spanned by ℓ_1 and ℓ_2 meets the plane spanned by ℓ_3 and ℓ_4.

The deformation-invariance of enumerations like this one was first proposed by Hermann Schubert in the 1870s under the name *Prinzip der Erhaltung der Anzahl*, or *principle of convseration of number* [Sch79]. But exactly when does this principle hold, and why? A rigorous explanation of Schubert's enumerations was missing from the mathematical literature for decades, and the hunt for such an explanation was deemed important enough to appear on Hilbert's famous list of 23 unsolved problems that shaped twentieth-century mathematics, in which the fifteenth problem is listed (in the English translation that appeared in the *Bulletin of the AMS* in 1902) as "rigorous foundation of Schubert's enumerative calculus."

The solution to Hilbert's fifteenth problem came in the second half of the twentieth century, with the twin developments of moduli spaces and intersection theory. A moduli space, roughly speaking, is a geometric space (often a variety or manifold) in which each point corresponds to an object of some type being studied. For example, someone wishing to study the number of conics passing through five points in the plane might form a moduli space $\mathcal{M}$ in which each point corresponds to a plane conic. From this perspective, the conics passing through a given point form a subvariety of $\mathcal{M}$, and the original enumerative question is reinterpreted as a count of the number of intersection points of the five corresponding subvarieties. The advantage of this reframing is that it allows curve-counting questions to be attacked via the tools of intersection theory, a deep mathematical subject studying the structure of intersections within an ambient variety that was developed (in large part with precisely the application to Hilbert's fifteenth problem in mind) over many decades in the early twentieth century.

Through the lens of intersection theory, one can see more clearly the sense in which curve counts are—or are not—deformation-invariant. First, an intersection theory

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problem generally only has a deformation-invariant answer when one works over an algebraically closed field (for instance, $\mathbb{C}$ rather than $\mathbb{R}$) and within a compact ambient variety. To illustrate the first of these limitations, consider the intersection of the parabola $y = x^2$ and the line $y = c$ in $\mathbb{R}^2$. These curves intersect in two points for $c > 0$, but for $c < 0$, the intersection points are only visible if we work not in $\mathbb{R}^2$ but in $\mathbb{C}^2$. In fact, so long as we count intersections "with multiplicity," the parabola $y = x^2$ and the line $y = c$ in $\mathbb{C}^2$ meet in exactly two points for any choice of c.

To see why compactness is necessary, consider the question "in how many points do two lines in $\mathbb{R}^2$ intersect?" The answer to this question can change when the lines are deformed, because the lines can become parallel, which effectively means that their point of intersection has "fallen off" the noncompact ambient space $\mathbb{R}^2$. To avoid this phenomenon, one should replace $\mathbb{R}^2$ by its compactification $\mathbb{RP}^2$, in which any two lines indeed meet in a single point—so long as they are not the same line.

This brings us to one final issue of deformation-invariance that intersection theory is equipped to solve: can an intersection still be said to be deformation-invariant if the subvarieties are deformed so far that they meet along an entire curve? For instance, is there a sensible way in which to interpret the "number of intersection points" of two identical lines in $\mathbb{RP}^2$ as 1, so that this number is truly insensitive to deformations of the lines? The answer to this question is "yes," and it is precisely what the subject of excess intersection theory addresses.

Applying these ideas to the context of enumerative geometry led mathematicians, in the late twentieth century, to express curve counts as certain intersection numbers on a moduli space that are now called *Gromov–Witten invariants*. This development allowed the deformation-invariance of curve counts to finally be expressed in a robust and rigorous way, but the work was far from over. In particular, the project of actually computing Gromov–Witten invariants is difficult and ongoing, and moreover, there are other methods of formalizing curve counts (such as Donaldson–Thomas theory) whose relationship to Gromov–Witten theory is not obvious. One breakthrough in the subject came in the 1990s from an unexpected interaction between curve-counting and the theoretical physics of string theory, and in the decades since then, this interdisciplinary connection has continued to yield fruit.

The Moduli Space of Stable Maps

We begin our journey toward defining Gromov–Witten invariants by fixing an ambient space X in which we will count curves. For the reasons mentioned above, we will always assume that X is compact and the ground field is $\mathbb{C}$; for instance, if our goal is to count conics passing through five given points in the plane, "the plane" refers to $X = \mathbb{CP}^2$. We also fix the degree β of the curves being counted and the number n of incidence conditions being imposed, so in the above example, $\beta = 2$ and $n = 5$. To put things a bit more precisely, X should be a smooth projective variety and β an element of $H_2(X;\mathbb{Z})$, so setting $\beta = 2$ in our example really means $\beta = 2L$, where L is the homology class of a line in $\mathbb{CP}^2$. What we will count is maps $f : C \to X$, where C is a curve, $f_*[C] = \beta$, and the image of f satisfies the requisite incidence conditions. In order for our count to be finite in general, we must fix one further piece of information: the genus g of the source curve C. In our example of conics through five points, this choice is forced upon us if we want our count to include the embedded irreducible conics in $\mathbb{CP}^2$, since the genus-degree formula implies that $g = 0$ for these.

Having fixed the data of X, g, β, and n, we now define a moduli space in which we will interpret our curve count as an intersection theory problem. As we have seen, we should look for a compact moduli space if we want any hope that our count will be deformation-invariant. Unfortunately, this means that we cannot restrict ourselves to including only maps $f : C \to X$ for which C is a smooth curve, nor for which f is an embedding, even if these are the types of maps we really care about; the issue is that these "nice" maps may degenerate to less nice ones.

To produce a compact moduli space, one must allow some degeneracies. This can be done while keeping the singularities of the curves mild; specifically, we will consider nodal curves, which can roughly be viewed as the result of gluing together a collection of smooth curves at finitely many pairs of points, as illustrated in Figure 1. The trade-off for the mildness of these singularities is that the map $f : C \to X$ may become quite degenerate, possibly collapsing entire components to a point. The result is the following key player in our story.

Definition 1. The *moduli space of stable maps* is the set $\overline{\mathcal{M}}_{g,n}(X,\beta)$ consisting of (isomorphism classes of) tuples $(C; x_1, \dots, x_n; f)$, where

- C is a nodal curve of arithmetic genus g;
- $x_1, \dots, x_n \in C$ are distinct and not nodes;
- $f : C \to X$ is a morphism with $f_*[C] = \beta$;
- the data $(C; x_1, \dots, x_n; f)$ has finitely many automorphisms.

We will abbreviate $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n}(X,\beta)$ for now. The last condition in the definition may appear technical—and relies on a definition of "isomorphism" that we have not specified—but, as we will see momentarily, it turns out to be crucial in ensuring that the moduli space is well-behaved from a geometric perspective.

But what is $\overline{\mathcal{M}}$, as a geometric object? Thus far, we have defined it as a set, but what makes it a "moduli space" is

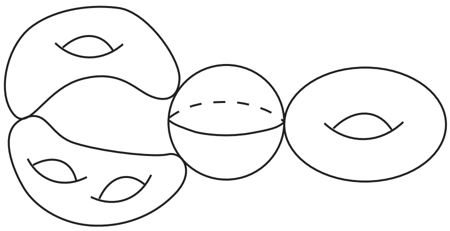

Figure 1. A nodal curve of arithmetic genus five.

that it can be given a geometry that encodes how tuples $(C; x_1, \dots, x_n; f)$ can continuously vary. For example, once $\overline{\mathcal{M}}$ is given a geometric structure, it makes sense to speak of a "path" in $\overline{\mathcal{M}}$, and the elements of $\overline{\mathcal{M}}$ that one passes through while walking along this path should form a one-parameter family of tuples $(C; x_1, \dots, x_n; f)$. Figure 2 is a cartoon illustration of this phenomenon.

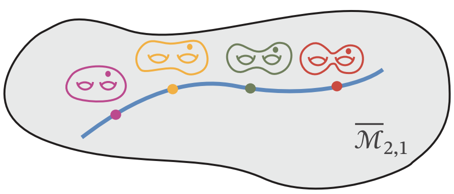

Figure 2. A one-parameter family of curves.

Thus, in order to give $\overline{\mathcal{M}}$ a geometry, one must first decide upon a definition of *family* of stable maps over any base B; a one-parameter family, for instance, is a family in which B is a line. From here, what it means to say that the geometry of $\overline{\mathcal{M}}$ encodes how stable maps vary is that, for any base B, there is a bijection

$$\{\text{families of stable maps over } B\}/\cong \quad \updownarrow \quad \{\text{morphisms } B \to \overline{\mathcal{M}}\}. \tag{1}$$

This, in particular, relies on giving $\overline{\mathcal{M}}$ a geometry in order to make the notion of "morphism" meaningful.

Up to this point, we have been purposefully vague about what we mean by "geometry." A topologist might hope that $\overline{\mathcal{M}}$ is a manifold, or an algebraic geometer might hope that it is a variety. Unfortunately, neither can be the case: a manifold or variety $\overline{\mathcal{M}}$ for which (1) is a bijection does not, in fact, exist. The root of the problem lies in the existence of automorphisms of stable maps, which allow one to construct families in which every stable map in the family is isomorphic to every other (so they should correspond to constant maps on the right-hand side of (1)), but which are nevertheless nontrivial as families. See Figure 3 for a cartoon illustration.

There is a fix to this problem, which is to give $\overline{\mathcal{M}}$ the structure of an *orbifold* (or, in more modern language, a

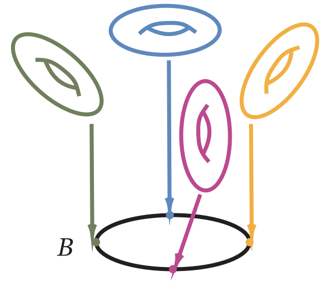

Figure 3. A nontrivial family of curves in which every fiber is isomorphic. (The tori over each point of B should be viewed as fitting together into something of a Möbius band, so that as one moves in a full circle around B, the initial torus is identified with the final torus via a nontrivial automorphism.)

Deligne–Mumford stack); very roughly, this is a space that looks locally like the quotient of a manifold by a finite group. In the setting of $\overline{\mathcal{M}}$, these finite groups are the automorphism groups of stable maps, which helps to explain why we insisted that a stable map have finitely many automorphisms. Equipped with the more general notion of morphism in the orbifold setting, a bijection as in (1) indeed exists.

Gromov–Witten Invariants

Now that we have a moduli space, our goal is to use it to count genus-g, degree-β curves in X that satisfy a collection of n incidence conditions—that is, that pass through a collection of n prescribed subvarieties $Y_1, \dots, Y_n \subseteq X$. In order to do so, we first define *evaluation maps*

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

for each $i \in \{1, \dots, n\}$, by

$$\mathrm{ev}_i(C; x_1, \dots, x_n; f) = f(x_i).$$

Then $\mathrm{ev}_i^{-1}(Y_i)$ is the set of stable maps $f : C \to X$ whose image passes through Y_i at $f(x_i)$, so one way in which to encode our desired curve count might be to count the number of points in the intersection

$$\mathrm{ev}_1^{-1}(Y_1) \cap \mathrm{ev}_2^{-1}(Y_2) \cap \cdots \cap \mathrm{ev}_n^{-1}(Y_n), \tag{2}$$

provided this intersection is finite. Inverse images generally preserve codimension, and intersections generally add codimension, so if $\mathrm{codim}_X(Y_i) = d_i$, we would expect (2) to be finite—that is, to have dimension zero—when

$$d_1 + \cdots + d_n = \dim(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

A more refined version of (2), which captures its insensitivity to deformations of the subvarieties Y_i, would be to consider instead the cohomology class $\gamma_i := [Y_i] \in H^{d_i}(X; \mathbb{Q})$. We then interpret (2) in cohomology by replacing inverse image with pullback and intersection with cup product, yielding the following preliminary definition of a Gromov–Witten invariant.

Preliminary Definition 1. Fix g, n, and β, and fix $\gamma_i \in H^{d_i}(X;\mathbb{Q})$ for $i = 1, \dots, n$. Assume that $d \coloneqq d_1 + \cdots + d_n = \dim(\overline{\mathcal{M}}_{g,n}(X,\beta))$. Then the associated *Gromov–Witten invariant* is given by evaluating the cohomology class

$$\mathrm{ev}_1^*(\gamma_1) \smile \cdots \smile \mathrm{ev}_n^*(\gamma_n) \in H^d(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$$

on the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to yield an element of $H_0(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q}) \cong \mathbb{Q}$. We denote this evaluation by

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} \mathrm{ev}_1^*(\gamma_1) \smile \cdots \smile \mathrm{ev}_n^*(\gamma_n) \in \mathbb{Q}.$$

We will see shortly that this definition has some serious deficiencies that will need to be repaired, but taking it as a working definition for the moment, one sees that using it requires first of all knowing the dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. What is this dimension? Let us walk through how it might be computed, momentarily taking $n = 0$ for simplicity.

If $\overline{\mathcal{M}}_{g,0}(X,\beta)$ were a smooth manifold, then its dimension would be the same as the dimension of its tangent space at any point. Thinking of a point in $\overline{\mathcal{M}}_{g,0}(X,\beta)$ as a stable map $f : C \to X$, and a tangent vector as an "infinitesimal family" of stable maps containing this one, we arrive at the perspective that the dimension of $\overline{\mathcal{M}}_{g,0}(X,\beta)$ should be the dimension of the space of "infinitesimal deformations" of any given stable map. In the special case where $f : C \to X$ is an embedding of a smooth curve, the space of such infinitesimal deformations can be identified with the space of sections of the normal bundle of $C \subseteq X$, the intuition being that a section of the normal bundle gives a direction in which each point in C can deform.

This reasoning leads to the guess that $\dim(\overline{\mathcal{M}}_{g,0}(X,\beta))$ is the dimension of the vector space $H^0(N_{C/X})$ of sections of the normal bundle for any stable map $f : C \to X$. This guess cannot be correct, however, because the dimension of this vector space depends on $f : C \to X$. What is independent of f is the difference

$$\dim(H^0(N_{C/X})) - \dim(H^1(N_{C/X})),$$

which equals

$$(\dim X - 3)(1-g) + \int_\beta c_1(T_X). \tag{3}$$

(The interested reader with some background in algebraic geometry is encouraged to verify this computation; the key ingredients are the short exact sequence

$$0 \to T_C \to f^*T_X \to N_{C/X} \to 0$$

and the Riemann–Roch theorem.)

We refer to the quantity (3) as the *virtual dimension* of $\overline{\mathcal{M}}_{g,0}(X,\beta)$. Similar reasoning applies when n is nonzero, yielding the following.

Definition 2. The *virtual dimension* of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, denoted $\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta))$, is the integer

$$(\dim X - 3)(1-g) + \int_\beta c_1(T_X) + n.$$

The idea, from a deformation-theoretic perspective, is that while $H^0(N_{C/X})$ parameterizes infinitesimal deformations of a stable map, $H^1(N_{C/X})$ parameterizes obstructions to extending these infinitesimal deformations to honest deformations over some base. We thus denote it by $\mathrm{Ob}(C,f)$ in the case where $n = 0$, and we denote the analogue more generally by $\mathrm{Ob}(C, x_1, \dots, x_n, f)$.

If it happens that $\mathrm{Ob}(C, x_1, \dots, x_n, f) = 0$ for all stable maps in the moduli space, then the virtual dimension is the dimension of the space of infinitesimal deformations, which, by the above reasoning, is equal to the dimension of the moduli space. This happens, albeit rarely; for instance, it happens when $X = \mathbb{CP}^m$ and $g = 0$, as well as when X is a single point.

If, however, $\mathrm{Ob}(C, x_1, \dots, x_n, f) \neq 0$ for some stable maps in the moduli space, then the dimension of the space of infinitesimal deformations of these stable maps is higher than the virtual dimension, and it may vary from one stable map to another. This is a reflection of the fact that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can have multiple irreducible components of different dimensions, all of which are bounded below by the virtual dimension.

A simplified perspective may give a flavor of these ideas: imagine that $\overline{\mathcal{M}}$ is given by the vanishing of r equations in a smooth d-dimensional variety. Then the dimension that one would expect $\overline{\mathcal{M}}$ to have is $d - r$, and this is certainly true when $r = 0$, in the same way that the dimension of $\overline{\mathcal{M}}$ is equal to the virtual dimension when the obstructions vanish. But when $r \neq 0$, dependencies among the defining equations may lead the actual dimension of $\overline{\mathcal{M}}$ to be larger than expected, or even to vary from one component of $\overline{\mathcal{M}}$ to another; for instance, the vanishing of the equations $xy = 0$ and $xz = 0$ in $\mathbb{R}^3$ consists of the line $y = z = 0$ (which has the expected dimension) together with the plane $x = 0$ (which has larger dimension).

With all of this in mind, we now see two problems with our preliminary definition of Gromov–Witten invariants. First, because $\overline{\mathcal{M}}_{g,n}(X,\beta)$ may have different components of different dimensions, it is unclear how to make sense of the condition that $d_1 + \cdots + d_n$ is equal to the dimension of the moduli space. And second, because it has multiple irreducible components, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is in general not smooth, so it need not have a fundamental class on which to evaluate.

The solution to these problems lies in constructing a *virtual fundamental class*, an element

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$$

that agrees with the fundamental class in the special situations where the moduli space is smooth of dimension equal to the virtual dimension. It is a hard theorem, first due to Behrend–Fantechi in the algebro-geometric setting [BF97] and to Li–Tian in the related setting of symplectic geometry [LT98], that a natural such class exists.

To give a very rough intuition in the simplified vision of $\overline{\mathcal{M}}$ as a vanishing locus inside a smooth d-dimensional variety Y, suppose that the r defining equations of $\overline{\mathcal{M}}$ correspond to a section s of a rank-r vector bundle E on Y, so that $\overline{\mathcal{M}} = \{s = 0\} \subseteq Y$. In this setting, what we seek is a $(d-r)$-dimensional homology class supported on $\overline{\mathcal{M}}$. If s meets the zero section of E transversally, this can be done by simply taking $[\{s = 0\}]$. On the opposite end of the spectrum, if s is identically zero, it can be done by taking $[Y] \cap c_r(E)$, where $c_r(E)$ denotes the top Chern class; roughly, this amounts to perturbing $s \equiv 0$ to a transverse section and then taking its zero locus. In practice, however, such a perturbation may not exist, and even if it does, it is unclear how to use the vanishing of the perturbation to produce a homology class supported on $\{s = 0\}$. The hard work of defining a virtual fundamental class lies in surmounting these difficulties, and doing so even when $\overline{\mathcal{M}}$ is not given as a vanishing locus in a smooth ambient variety.

Once the virtual fundamental class is constructed, we are at last ready to give the true definition of Gromov–Witten invariants.

Definition 3. Fix g, n, and β, and fix $\gamma_i \in H^{d_i}(X)$ for each $i = 1, \dots, n$. Assume that $d_1 + \cdots + d_n = \mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta))$. Then the associated *Gromov–Witten invariant* is

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \smile \cdots \smile \mathrm{ev}_n^*(\gamma_n).$$

At this point, we have gotten quite far afield of our initial goal of counting curves. Thus, at least two questions are in order. First, do Gromov–Witten invariants agree with a more naïve notion of curve counts, when the latter is possible? And second, how can Gromov–Witten invariants be computed?

The answer to the first question is sometimes—though admittedly rarely—yes. For instance, we have mentioned that the virtual fundamental class is the ordinary fundamental class on $\overline{\mathcal{M}}_{0,n}(\mathbb{CP}^m,\beta)$, and in this case, the locus of smooth embedded curves $f : C \to \mathbb{CP}^m$ is dense in the moduli space. Thus, the Gromov–Witten invariant associated to $\gamma_1, \dots, \gamma_n$ genuinely counts the number of genus-zero degree-β embedded curves in $\mathbb{CP}^m$ in which $x_1, \dots, x_n$ lie in generic subvarieties $Y_1, \dots, Y_n$ representing the cohomology classes $\gamma_1, \dots, \gamma_n$. As an example, one could calculate the number of conics through five generic points in $\mathbb{CP}^2$ as the Gromov–Witten invariant

$$\langle P \cdot P \cdot P \cdot P \cdot P \rangle_{0,5,2}, \tag{4}$$

where $P \in H^*(\mathbb{CP}^2)$ denotes the cohomology class of a point.

This situation is rare, though. In fact, the possibility that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has orbifold structure means that Gromov–Witten invariants are not even necessarily integers but rational numbers in general, so it is difficult to interpret them as counting anything at all. A particularly stark—yet fascinating—example occurs when X is a quintic threefold in $\mathbb{CP}^4$, for which $c_1(T_X) = 0$ and hence $\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = n$. Taking $n = 0$, one might hope to interpret the Gromov–Witten invariant

$$\langle\ \rangle_{0,0,\beta} = \int_{[\overline{\mathcal{M}}_{0,0}(X,\beta)]^{\mathrm{vir}}} 1 \tag{5}$$

as counting the number of degree-β embedded rational curves on X, but this cannot be the case. Indeed, for any divisor k of β, one can obtain a degree-β map $\mathbb{CP}^1 \to X$ by composing a k-fold cover $\mathbb{CP}^1 \to \mathbb{CP}^1$ with a degree-(β/k) map $\mathbb{CP}^1 \to X$, and there is a positive-dimensional family of such covers. This means that $\overline{\mathcal{M}}_{0,0}(X,\beta)$ has components of excess dimension, which contribute in a complicated way to (5). The BPS conjecture suggests a way to account for these contributions to extract integers (called Gopakumar–Vafa invariants) from Gromov–Witten invariants in this case, but this is a longer story that we will not delve into here.

Instead, we will simply content ourselves with attempting to compute Gromov–Witten invariants, out of the philosophy that this is a worthwhile goal even when we are not in the rare situations when the invariants are enumerative. In particular, Gromov–Witten invariants have beautiful structure that is worth studying in its own right; we will see one example of this at the end of the article when we discuss mirror symmetry. Various other connections to theoretical physics as well as to more classical algebro-geometric subjects like the moduli space of curves have motivated mathematicians to study Gromov–Witten theory. Thus, as often happens in mathematics, our initial goal (curve-counting) has led us to an object (Gromov–Witten invariants) that is interesting regardless of the extent to which it actually achieves the goal.

How, then, to calculate Gromov–Witten invariants? This is a difficult question, in many cases prohibitively difficult, but there are important situations in which computation is possible. The first such situation we consider is when $X = \mathbb{CP}^2$ and $g = 0$, which was famously addressed by Kontsevich.

Kontsevich's Formula

To explain Kontsevich's computation, we first note that in the case where the target $X = \bullet$ is a single point, the moduli space $\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(\bullet, 0)$ was studied well before the advent of Gromov–Witten theory, and remains a topic of active research. This study is advantageous to us because, for any target X, there is a map $\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$, so—at least in good situations where $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a smooth manifold—an understanding of the cohomology of $\overline{\mathcal{M}}_{g,n}$ can be pulled back to yield information about the Gromov–Witten invariants of X. A particularly good situation occurs when $g = 0$ and $n \geq 4$; then, there is a morphism

$$p : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,4}$$

whose codomain can be understood very concretely. First, when C is smooth, there is a unique isomorphism $C \cong \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ sending (x_1, x_2, x_3, x_4) to $(0, 1, \infty, q)$ for some q, so the locus of smooth curves in $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The compactification, then, must be $\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1$, and indeed, the three missing points correspond to the nodal curves shown in Figure 4, which we denote by $D_{12|34}$, $D_{13|24}$, and $D_{14|23}$, respectively.

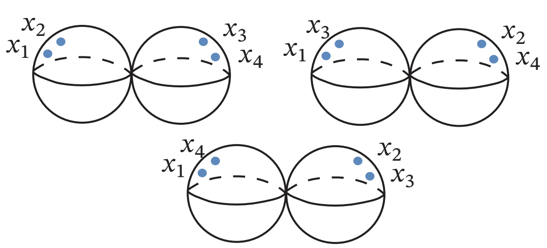

Figure 4. The three nodal curves in $\overline{\mathcal{M}}_{0,4}$.

Because any two points in $\mathbb{CP}^1$ are equivalent in cohomology, one obtains a relation

$$[D_{12|34}] = [D_{13|24}] = [D_{14|23}] \in H^2(\overline{\mathcal{M}}_{0,4}). \qquad (6)$$

Pulling this relation back under p yields a corresponding relation among three cohomology classes on $\overline{\mathcal{M}}_{0,n}(X,\beta)$. Namely, for any decomposition $\{1, \ldots, n\} = I \sqcup J$ and $\beta = \beta_1 + \beta_2$ of the marked points and degree, let $\widetilde{D}_{I,\beta_1|J,\beta_2} \subseteq \overline{\mathcal{M}}_{0,n}(X,\beta)$ denote the subvariety whose general element is a curve with two components, one containing the marked points indexed by I on which the degree of f is β_1, and the other containing the remaining marked points and degree. Then the pullback of (6) under p shows that the sum

$$\sum_{\substack{I \sqcup J = [n] \\ \beta_1 + \beta_2 = \beta \\ 1,2 \in I \text{ and } 3,4 \in J}} [\widetilde{D}_{I,\beta_1|J,\beta_2}] \in H^2(\overline{\mathcal{M}}_{0,n}(X,\beta))$$

is equal to the corresponding sum where we instead insist that $1, 3 \in I$ and $2, 4 \in J$, and also to the sum where we insist $1, 4 \in I$ and $2, 3 \in J$. In particular, multipyling $\mathrm{ev}_1^*(\gamma_1) \smile \cdots \smile \mathrm{ev}_n^*(\gamma_n)$ by any of these three cohomology classes before integrating should yield the same answer. Furthermore, $\widetilde{D}_{I,\beta_1|J,\beta_2}$ can be interpreted as a fiber product of the two moduli spaces $\overline{\mathcal{M}}_{0,|I|+1}(X,\beta_1)$ and $\overline{\mathcal{M}}_{0,|J|+1}(X,\beta_2)$, so one can deduce an equality between three sums, each of whose terms is a product of two Gromov–Witten invariants.

This yields a host of relations among genus-zero Gromov–Witten invariants, collectively known as the *WDVV relations*. When $X = \mathbb{CP}^2$, an example of one of the resulting relations (after a bit of simplification) is the following, in which P denotes the cohomology class of a point, L the cohomology class of a line, and 1 the fundamental class:

$$\begin{aligned} &\langle P \cdot P\rangle_{0,2,1}\langle P \cdot P\rangle_{0,2,1} \\ &+\langle L \cdot L \cdot 1\rangle_{0,3,0}\langle P \cdot P \cdot P \cdot P \cdot P\rangle_{0,5,2} \\ &=2\langle P \cdot P\rangle_{0,2,1}\langle P \cdot P\rangle_{0,2,1}. \end{aligned}$$

Interpreting the invariant $\langle P \cdot P\rangle_{0,2,1}$ as the number of lines through two points in $\mathbb{CP}^2$, it should be intuitively believable that it equals 1. Similarly, interpreting the invariant $\langle L \cdot L \cdot 1\rangle_{0,3,0}$ as the number of intersection points of two lines in $\mathbb{CP}^2$ suggests (correctly) that this invariant equals 1, as well. Thus, the above relation implies

$$\langle P \cdot P \cdot P \cdot P \cdot P\rangle_{0,5,2} = 1,$$

which recovers, via much more modern machinery, the ancient Greeks' assertion that there is a unique conic passing through five general points in the (complex, projective) plane. In a celebrated theorem from the early days of Gromov–Witten theory [Kon92], Kontsevich generalized the above computation to interpret the WDVV relations on $\mathbb{CP}^2$ as a recursion that effectively computes all of the numbers N_d of rational degree-d curves through $3d-1$ general points in $\mathbb{CP}^2$, requiring only the base case of $N_1 = 1$.

Although Kontsevich's proof was entirely mathematical, there is a different interpretation of his result that passes through the unexpected world of theoretical physics—more specifically, string theory. Curves arise in that setting as the "worldsheet" traced out by a string as it travels through spacetime, and Gromov–Witten invariants appear in the definition of a structure known as the "quantum product" on $H^*(X)$. The WDVV equations turn out to be equivalent to the condition that this product is associative, so Kontsevich's formula can be interpreted as a consequence of this associativity for $X = \mathbb{CP}^2$.

The proof of Kontsevich's theorem was a true triumph; prior to this work, only a handful of the numbers N_d could be computed, and then only by difficult ad hoc methods. But this early success of the interplay between enumerative geometry and string theory was only the beginning of a decades-long story that continues to reveal new structures and bring new computations within reach. A second key moment came with the discovery of mirror symmetry, to which we now turn. Although our discussion will be necessarily brief, the reader can find much more on the connection between physics and curve-counting in surveys such as [Kat06] and [Cla17].

Mirror Symmetry

In the language of theoretical physics, string theories provide an example of an object known as an $N = 2$ superconformal field theory, where the $N = 2$ refers to the presence of two "supersymmetries." One way in which to construct such a theory, called the nonlinear sigma model, takes as input a three-dimensional compact complex manifold X with trivial canonical bundle (that is, a *Calabi–Yau threefold*) together with a complexified Kähler class $\omega \in H^2(X;\mathbb{C})$.

In fact, the data of (X,ω) determines not just a superconformal field theory but an ordering of the two supersymmetries, with the two possible choices of ordering referred to as the "A-model" and "B-model" of the theory. Each of these models gives an induced choice of generators for a distinguished two-dimensional subalgebra of the theory's infinitesimal symmetries, and the eigenspaces of these generators can be mathematically identified with $H^q(X,\Lambda^p T_X)$ and $H^q(X,\Omega^p_X)$.

The statement of mirror symmetry, from a physical perspective, is that there should exist a "mirror" pair $(X^\vee,\omega^\vee)$ for which the associated nonlinear sigma model is the same superconformal field theory but with the opposite ordering of the supersymmetries. This would in particular imply isomorphisms of eigenspaces

$$H^q(X,\Lambda^p T_X) \cong H^q(X^\vee,\Omega^p_{X^\vee})$$
$$H^q(X,\Omega^p_X) \cong H^q(X^\vee,\Lambda^p T_{X^\vee}).$$

In the special case when $p = q = 1$, deformation theory interprets $H^1(X,T_X)$ as the space of infinitesimal deformations of the complex structure on X and $H^1(X,\Omega_X)$ as the space of infinitesimal deformations of the Kähler class. This leads to a more refined version of the mirror conjecture: there should be an isomorphism between a neighborhood of (X,ω) in the moduli space of complex structures on the underlying manifold of X, and a neighborhood of $(X^\vee,\omega^\vee)$ in the moduli space of Kähler structures on $X^\vee$.

The predictions of mirror symmetry extend still deeper, and this is where the enumerative geometry comes in. The superconformal field theory associated to (X,ω) admits two types of "correlation functions," which can be interpreted mathematically as connections on certain vector bundles: in the A-model, this is the *quantum connection* on a vector bundle over the moduli space of complex structures, and in the B-model, it is the *Gauss–Manin connection* on a vector bundle over the moduli space of Kähler structures. The quantum connection can be defined in terms of genus-zero Gromov–Witten invariants, whereas the Gauss–Manin connection is a more well-studied and explicit object that uses period integrals to measure how integral homology deforms relative to Kähler structure. In this context, mirror symmetry predicts an exchange of the A-model and B-model connections on (X,ω) and $(X^\vee,\omega^\vee)$, yielding an equality between a generating function of genus-zero Gromov–Witten invariants of X and certain B-model information from $X^\vee$ that can be exactly calculated.

As a particular key example, one can take X to be the quintic threefold—that is, the vanishing locus of the equation

$$Q(z_1,\dots,z_5) = z_1^5 + \cdots + z_5^5$$

in $\mathbb{CP}^4$. Then mirror symmetry—due in this setting to the 1991 work of physicists Candelas, de la Ossa, Green, and Parkes [CdlOGP91]—predicted that a certain generating function of the numbers n_d of degree-d rational curves in X could be recovered via explicit elementary transformations from a simple hypergeometric series. This conjecture shocked the mathematical community, since it not only would imply that the numbers n_d (which, after accounting for multiple cover contributions, can be related to the genus-zero Gromov–Witten invariants of X) can be effectively computed, but it would reveal that these numbers admit a deep and unexpected structure.

The first mathematical proof of the above mirror conjecture was provided by Givental [Giv96]. (A different viewpoint also appeared at approximately the same time in work of Lian–Liu–Yau [LLY99], but we will focus on Givental's formulation here.) To state the result, it is helpful to introduce a generalization of Gromov–Witten invariants known as descendent integrals; these are integrals

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\phi_1)\psi_1^{a_1}\cdots \mathrm{ev}_n^*(\phi_n)\psi_n^{a_n},$$

where $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta))$ is the first Chern class the line bundle whose fiber over $(C;x_1,\dots,x_n;f)$ is the cotangent line $T^*_{x_i}C$. (These "psi-classes" show up naturally in the theory, from non-transverse intersections of the classes $\widetilde{D}_{I,\beta_1|J_{\beta_2}}$ mentioned in the previous section.) The genus-zero descendent invariants of X can be packaged into a generating series

$$J(\mathbf{t}) = z + \mathbf{t} + \sum_{n,\beta,\mu} \frac{q^\beta}{n!}\left\langle \mathbf{t}(\psi),\dots,\mathbf{t}(\psi),\frac{\phi_\mu}{z-\psi}\right\rangle_{0,n+1,\beta}\phi^\mu,$$

which takes as input $\mathbf{t} \in H^*(X)[z]$ and produces as output a formal series in q, z, and z^{-1} with coefficients in $H^*(X)$.

Another such formal series, this time an entirely explicit one, is the power series expansion of

$$I = z \sum_{\beta \geq 0} \frac{\prod_{b=1}^{5\beta}(5H + bz)}{\prod_{b=1}^{\beta}(H + bz)^5},$$

where $H \in H^*(X)$ denotes the restriction to X of the hyperplane class on $\mathbb{CP}^4$. Givental's mirror theorem states that these two functions differ by a change of variables.

Theorem 1 (Givental's mirror theorem). *Let I_+ denote the part of I with non-negative powers of z. Then*

$$I = J\Big(- z + I_+\Big).$$

Though these generating functions are complex at first glance, the moral of the theorem is simple: by comparing coefficients of monomials on the two sides of the theorem, certain genus-zero Gromov–Witten invariants of X can be calculated. In fact, basic relationships determine all of the genus-zero Gromov–Witten invariants of X in terms of the ones that appear in this equality, so the mirror theorem provides a way to calculate the genus-zero Gromov–Witten theory of the quintic threefold in its entirety.

Givental's original proof of the mirror theorem was a tour de force involving, among other tools, an equivariant localization formula that can be used to express the Gromov–Witten invariants of X as a complicated sum over graphs. Although this localization method had been previously employed by Kontsevich [Kon95] to compute particular numbers n_d, the combinatorial complexity of the graph sum increases incredibly quickly; one aspect of Givental's contribution was an ingenious method of organizing the sum.

On the other hand, in the years since Givental's work, a new perspective on the mirror theorem has emerged that makes the proof somewhat more transparent: Gromov–Witten theory can be viewed as just one among a family of theories depending on a positive rational parameter ϵ, and the mirror theorem arises in this context as a relationship between the theories when $\epsilon \to \infty$ and $\epsilon \to 0$. This more general theory, known as *quasimap theory* and developed in the work of Ciocan-Fontanine, Kim, and Maulik [CFKM14, CFK14] (building on previous work by Marian–Oprea–Pandharipande [MOP11]), is the topic of the next and final section of the article.

Wall-Crossing and Beyond

To motivate the definition of quasimaps, note that a stable map $f : C \to X$ to the quintic $X \subseteq \mathbb{CP}^4$ can be described via its five coordinates. When $C \cong \mathbb{CP}^1$, these are five polynomials in the coordinates $[x : y] \in \mathbb{CP}^1$; more generally, they can be viewed as five sections $f_1, \dots, f_5 \in H^0(L)$ of the line bundle $L = f^*\mathcal{O}_{\mathbb{CP}^4}(1)$. In order to ensure that these give a well-defined map to $X \subseteq \mathbb{CP}^4$, these sections must satisfy

1. $f_1, \dots, f_5$ have no common zeroes (so that f gives a map to $\mathbb{CP}^4$), and
2. $Q(f_1, \dots, f_5) = 0$ (so that f lands in X).

Furthermore, the condition that f is stable can be unpacked in terms of the sections: it amounts to insisting that

3. $\deg(L) > 0$ on any genus-zero component of C with fewer than three special points.

(A "special point," here, is a point that is either a node or one of the marked points $x_1, \dots, x_n$.)

Now, let ϵ be a positive rational number. An *ϵ-stable quasimap* to X is a curve C equipped with a line bundle L and five sections $f_1, \dots, f_5 \in H^0(L)$ satisfying variants of the above conditions in which $f_1, \dots, f_5$ are allowed to have common zeroes to a limited extent controlled by ϵ. More precisely, they must satisfy

1′. $f_1, \dots, f_5$ may have common zeroes, but only finitely many and only at nonspecial points, and the order of any common zero must be $\leq 1/\epsilon$;

2. $Q(f_1, \dots, f_5) = 0$;

3′. $\deg(L) > 0$ on any genus-zero component with two special points, and $\deg(L) > 1/\epsilon$ on any genus-zero component with one special point.

Note, in particular, that taking $\epsilon \to \infty$ recovers the definition of ordinary stable maps. On the other hand, when $\epsilon \to 0$, common zeroes of arbitrarily high order are allowed, and the geometry of the curve is correspondingly simplified; in particular, condition 3′ implies that C has no rational tails (genus-zero components with a single special point).

The key observation of Ciocan-Fontanine and Kim is that there exists a generating function $J^\epsilon(\mathbf{t})$ of genus-zero ϵ-stable quasimaps for any ϵ, which agrees with Givental's J-function when $\epsilon \to \infty$ and with Givental's I-function when $\epsilon \to 0$ and $\mathbf{t} = 0$. The mirror theorem then becomes a special case of a more general result relating the functions J^ϵ for different values of ϵ to one another. The advantage of this increased generality is that, since the theory only changes at discrete values of ϵ (namely, when $1/\epsilon$ is an integer), which are sometimes referred to as the "walls" of the theory, the problem of relating $J^{\epsilon\to\infty}$ to $J^{\epsilon\to 0}$ can instead be tackled by understanding how quasimap theory changes when ϵ crosses each wall individually.

In addition to providing an enlightening new perspective on Givental's mirror theorem, quasimap theory also gives a hint as to a difficult further question: given that the mirror theorem, as stated above, applies only in genus zero, is there an analogous statement in higher genus?

One answer to this question is provided by Ciocan-Fontanine and Kim's work on higher-genus wall-crossing

[CFK17]. To state their result, denote by $\overline{\mathcal{M}}^{\epsilon}_{g,n}(X,\beta)$ the moduli space of ϵ-stable quasimaps to X. For any tuple $\vec{\beta} = (\beta_0, \dots, \beta_k)$ of nonnegative integers with $\beta_0 + \cdots + \beta_k = \beta$, there is a map

$$c_{\vec{\beta}} : \overline{\mathcal{M}}_{g,n+k}(X,\beta_0) \to \overline{\mathcal{M}}^{\epsilon}_{g,n}(X,\beta)$$

that, roughly speaking, replaces the last k marked points of an ordinary stable map with common zeroes of $f_1, \dots, f_5$ of orders $\beta_1, \dots, \beta_k$. The essence of the higher-genus wall-crossing theorem is that the map $c_{\vec{\beta}}$ relates the virtual fundamental classes of the ordinary and the ϵ-stable moduli spaces to one another, with a correction coming from genus-zero data

Theorem 2. *Let J^{ϵ}_{+} denote the part of $J^{\epsilon}(0)$ with non-negative powers of z, and let $\mu^{\epsilon}_{\beta}(z)$ denote the coefficient of q^{β} in $-z+J^{\epsilon}_{+}$. Then*

$$[\overline{\mathcal{M}}^{\epsilon}_{g,n}(X,\beta)]^{\mathrm{vir}} = \sum_{\beta_0+\ldots+\beta_k=\beta} \frac{1}{k!} c_{\vec{\beta}*}\left(\prod_{i=1}^{k} \mathrm{ev}^{*}_{n+i}(\mu^{\epsilon}_{\beta_i}(-\psi_{n+i})) \frown [\overline{\mathcal{M}}_{g,n+k}(X,\beta_0)]^{\mathrm{vir}}\right).$$

This theorem—first proved by Ciocan-Fontanine and Kim via the techniques of virtual pushforwards and later reproved by the author with Janda and Ruan via more formal machinery that can be applied in greater generality [CJR17, CJR21b]—is much more powerful than the original mirror theorem, not only because it applies in all genus, but because it relates not merely enumerative invariants but the virtual fundamental classes themselves. On the other hand, on a surface level, perhaps a statement like this should not come as a complete surprise: the geometric difference between quasimap theory and ordinary Gromov–Witten theory can be understood in terms of what sorts of rational tails are allowed, which is genus-zero information and thus can be encoded in the coefficients of the function J^{ϵ}.

Even with the new perspective provided by wall-crossing, however, it is not obvious how to use mirror symmetry to compute the Gromov–Witten invariants of the quintic threefold in higher genus, because the higher-genus quasimap theory for $\epsilon \to 0$ must still be computed. Physicists Bershadsky, Cecotti, Ooguri, and Vafa conjectured what the answer should be as early as 1993, proposing an explicit formula for the generating functions of genus-one and genus-two Gromov–Witten invariants that relied on structural properties inherent in the B-model [BCOV97]. Ten years after this prediction, a mathematical proof in genus one was given by Zinger [Zin09], and another ten years later, the genus-two prediction was proven by Chen–Guo–Janda–Ruan [GJR17, CJR21a]. The immense amount of work that it has taken to achieve these increases in genus is one indication of the depth of the mirror symmetry phenomenon, and the wealth of mysteries that it continues to hold.

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