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PERSPECTIVE

Tests of general relativity with future detectors

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Abstract

This "vision document" is about what the future has in store for tests of general relativity with gravitational wave detectors. I will make an honest attempt to answer this question by addressing the role of inspiral-based and ringdown-based tests; recent progress on quasinormal modes in modified theories of gravity; the complementarity between light ring tests and ringdown tests; and the interesting possibility of observing some of the nonlinear effects predicted by general relativity. I may well prove to be wrong. To quote Yogi Berra: "It's hard to make predictions, especially about the future".

Keywords General relativity · Gravitational waves · Black holes

Contents

Why should we test general relativity?

First of all, it is important to ask why we are doing this. Is there value in testing general relativity with higher and higher precision? I obviously believe that the answer is "yes" [1, 2], but your mileage may vary. While Clifford Will was a postdoc in Chicago, Subrahmanyan Chandrasekhar once asked him: "Why do you bother testing general

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relativity? We know that the theory is right." I recently heard over dinner with Bob Wald and Sean Carroll that Chandra must have made that remark more than once. When someone pointed out that Einstein was elated when the 1919 eclipse expedition confirmed the validity of his theory (in fact, he famously said that he had "palpitations" when he first computed the perihelion advance of Mercury), Chandra quipped: "That's because Einstein did not understand his own theory very well."

There is, of course, great value in testing the limits of validity of any of our physical theories. In his 1894 dedication of Ryerson Physical Laboratory in Chicago (quoted also in Weinberg's "Dreams of a final theory"), Albert Michelson stated: "While it is never safe to affirm that the future of physical science has no marvels in store even more astonishing than those of the past, it seems probable that most of the grand underlying principles have been firmly established and that further advances are to be sought chiefly in the rigorous application of these principles to all the phenomena which come under our notice. It is here that the science of measurement shows its importance - where quantitative work is more to be desired than qualitative work. An eminent physicist [probably Lord Kelvin?] remarked that *the future truths of physical science are to be looked for in the sixth place of decimals*." By building better detectors, we will be better equipped to look for those "future truths." Such is the nature of any experimental science.

What future detectors?

Ongoing efforts to build more sensitive gravitational-wave detectors on Earth include the Einstein Telescope [3], Cosmic Explorer [4, 5] and NEMO [6], among others. The reach of these detectors can be extended to lower frequencies through detectors on the Moon [7] and in space: besides LISA [8], TianQin [9] and Taiji [10], there are proposals for follow-up space missions on a longer timescale [11–13]. There are also major experimental efforts to develop detectors based on atom interferometry [14–17], and ideas to extend the range of observable gravitational waves to high frequencies [18]. Each of these future detectors will observe different astrophysical sources and probe different physics.

What are we going to learn about theories beyond general relativity, or about general relativity itself, from these observations? Here I will focus on the science achievable with the Einstein Telescope, Cosmic Explorer and LISA – detectors for which the main astrophysical sources (inspiraling compact binaries) are relatively well understood. The science case for these detectors is clearly much broader and stronger than just tests of general relativity (see e.g. [5, 8, 19–21]). For single events, the increased sensitivity of ground-based detectors implies higher probability of detecting "exotic" binary merger events with unexpected parameters; better constraints on the nuclear equation of state; more accurate studies of binary dynamics, including spin precession and eccentricity; and a broad range of applications to (and of) multimessenger astronomy [4]. Detecting hundreds or thousands of events per day, these observatories will drastically improve our understanding of compact binary populations and formation channels. In particular, they will probe sources in the high-redshift universe, such as Population III stars, primordial black holes, and intermediate-mass black holes. One of the Holy Grails of future interferometers is the detection of stochastic backgrounds of astrophysical and cosmological origin. When viewed in this broader context, testing general relativity is only a cherry on top of the cake – but of course, the observation of a real violation of general relativity (as opposed to one of the many possible false violations due to noise systematics, waveform systematics or astrophysics [22]) could lead to a paradigm shift in physics, and for this reason alone it is well worth pursuing.

Pushing the boundaries: inspiral radiation

A flexible, theory-agnostic framework to explore the theoretical physics implications of binary merger observations, particularly in the inspiral phase, is the so-called parametrized post-Einsteinian formalism [23–26]. The idea is that any specific theory will induce modifications in the waveform phasing starting at some post-Newtonian order. If those modifications are consistent with zero, they can be turned into bounds on specific classes of theories. Perkins et al. [27] used astrophysical models to estimate how these bounds will improve in the future. The conclusions are affected by astrophysical uncertainties on compact binary population, and they also depend on the sensitivity upgrades of the instruments. For theory-agnostic bounds, ground-based observations of stellar-mass black holes and LISA observations of massive black holes can each lead to improvements ranging between 2 and 4 orders of magnitude with respect to present constraints. Multiband observations of sources inspiraling in the LISA band and merging in the band of ground-based interferometers can yield improvements between 1 and 6 orders of magnitude.

These predictions must be taken with a grain of salt for two main reasons. The first, and most important, is that they ignore systematics due to detector noise, waveform modeling and astrophysics [22]. Our understanding of waveforms within general relativity must improve by orders of magnitude if we want to perform tests of gravity at high signal-to-noise ratios. In fact, some events display apparent violations of general relativity due either to waveform systematics or to data-quality issues even at current sensitivities [28]. The second reason is that any predictions based on the population observed so far are conservative, in the sense that a single "special" event can be sufficient to kill many theories in one stroke. We already have two excellent examples of this: (i) the simultaneous detection of gravitational and electromagnetic waves in the binary neutron star event GW170817 ruled out broad classes of theories that predict modifications in the propagation properties of gravitational waves [28, 30]; (ii) the recent observation of the merger of a neutron star with a compact object in the lower mass-gap (GW200115_042309) placed stringent bounds on high-order curvature theories with a mass-dependent coupling [33].

Pushing the boundaries: ringdown radiation

The LIGO-Virgo-KAGRA collaboration routinely performs "black hole spectroscopy" tests of the Kerr nature of a merger remnant [34–39] on all detected events [40, 41]. There are claims that an overtone of the dominant ($\ell = m = 2$) component of the radiation may be present in GW150914 [42] and that higher multipole modes can be seen in GW190521 [43], but these conclusions depend on subtle assumptions about waveform modeling and data analysis. The evidence for a second mode in the data is affected by uncertainties in the ringdown starting time, choices of sampling rate, noise modeling, and data analysis methods (e.g., whether the analysis is performed in the time or frequency domain, or whether one considers only the ringdown portion of the waveform or models the whole inspiral-merger-ringdown signal) [43, 45, 46, 48, 50].

From a theoretical point of view, it is important to perform an agnostic analysis of numerical relativity waveforms that takes into account all of the relevant physics (including a proper choice of the BMS frame, the early transient, late-time tails, spherical-spheroidal mode mixing, counterrotating and nonlinear modes) and not just a simple superposition of linear modes [51]. High overtones do not contribute much to the remnant's mass and spin estimates, and (unlike the fundamental mode) they can be spectrally unstable [51, 53–55]. A quasinormal mode analysis within linear perturbation theory can only be trusted $\sim 10\,M$ after the waveform peak, where the ringdown signal-to-noise ratio in current detectors is low. Many of these issues will be resolved in the future, when "golden" events will have ringdown signal-to-noise ratios in the hundreds for next-generation ground-based detectors, and in the thousands for LISA [56–60]. However, even at such high signal-to-noise ratios there is a compromise between the complexity of the signal model and the information that can be extracted because of Occam penalties [61].

Can ringdown radiation be used to look for smoking guns of modified gravity?

Finding an answer to this question is technically challenging. It is well known that the separability of the Teukolsky equation relies on the Petrov Type D of the Kerr metric and on the existence of "hidden symmetries" (i.e., of a Killing tensor) [62]. These special properties are lost, in general, in modified theories of gravity: it is difficult (if not impossible) to find closed-form analytical expressions for the background metric, the perturbation equations are usually not separable, and the equations of motion are often of higher order. Despite these difficulties, there has been remarkable progress in quasinormal mode calculations in modified theories of gravity using three main approaches:

(1) Metric perturbations (similar to the classic Regge-Wheeler/Zerilli approach) have been used to compute quasinormal modes for slowly rotating black holes in specific theories (such as Einstein-scalar-Gauss-Bonnet and dynamical Chern Simons gravity). This approach is limited to low spins, but it can deal with large couplings [63–66].

(2) Generalized Teukolsky equations can be derived under the assumption of small coupling. This technique can be used to compute shifts in the quasinormal mode frequencies that are linear in the coupling using either variants of Leaver's method [67] or eigenvalue perturbation techniques. While the technique is based on a small-rotation expansion, it has been pushed (in some cases) up to order a^{18} in the small rotation parameter a [68–75].

(3) Spectral methods can be used to compute quasinormal mode frequencies of black holes in modified gravity for arbitrary coupling and (in principle, but not in practice) for arbitrarily large rotation parameters [76–78].

If modified gravity effects can be treated perturbatively, one can incorporate theory-specific predictions within a "parametrized ringdown" formalism [79, 80] that has recently been extended to the rotating case [81]. Parametrized ringdown templates (similar to the "parametrized post-Einsteinian" formalism) can also be implemented in data analysis [82, 83]. This idea can be used to estimate how well future ringdown detections can constrain modified gravity theories. A recent study considers agnostic (null) tests, as well as theory-specific tests for two quadratic gravity theories (Einstein-scalar-Gauss-Bonnet and dynamical Chern-Simons gravity) and various classes of effective field theories [84]. It finds that high-order terms in the slow-rotation expansion are necessary for robust inference of hypothetical corrections to general relativity. However, even when high-order expansions are available, ringdown observations alone may not be sufficient to measure deviations from the Kerr spectrum for theories with dimensionful coupling constants. This is because the constraints are dominated by "light" black hole merger remnants (where curvature-dependent corrections are largest), but even for next-generation ground-based detectors, only a few of these events have sufficiently high signal-to-noise ratio in the ringdown.

Light ring tests are complementary to ringdown tests

The well-known connection between light ring geodesics and ringdown physics [85] means that experiments probing light rings, such as the Event Horizon Telescope [85, 86, 88] and its next-generation follow-ups (ngEHT [88] and the Black Hole Explorer [89]) probe similar physics and yield complementary tests [90, 91]. Much work is needed to explore the complementarity of these experiments and gravitational wave detectors in testing general relativity and black hole dynamics.

"Null tests" of general relativity in the strong gravity regime: nonlinear quasinormal modes and gravitational memory

The high signal-to-noise ratio of future detectors holds promise to better test the nonlinear nature of gravity in at least two ways:
Nonlinear quasinormal modes are present in binary black hole waveforms [92–94] and potentially detectable: optimistic astrophysical scenarios predict thousands of LISA binary black hole merger events with a detectable quadratic mode [95]. The ratio between the nonlinear mode amplitudes and the linear amplitudes driving them can be computed within Einstein's theory (see e.g. [96–111]), and used as an additional "null" test of the nonlinear nature of general relativity.
Gravitational wave memory effects arise from non-oscillatory components of gravitational wave signals. They are predictions of general relativity in the nonlinear regime that have close connections to the asymptotic properties of isolated gravitating systems, and can shed light on the infrared structure of gravity [112, 113]. The largest of

these effects from binary black hole mergers are the “displacement” and “spin” memories – a change in the relative separation of two initially comoving observers due to a burst of gravitational waves, and a portion of the change in relative separation of observers with initial relative velocity, respectively. Both of these effects are small, but by combining data from multiple events, the displacement memory could be detected by a LIGO-Virgo-KAGRA network operating at the (current) O4 sensitivity for 1.5 years, and then operating at the O5 sensitivity for an additional year. By contrast, a next-generation interferometer such as Cosmic Explorer could detect the displacement memory for *individual* loud events, and detect the spin memory in a population within 2 years of observation time [114].

Should we care about testing general relativity?

The answer is obviously “yes” if it turns out that general relativity is indeed modified *“in the sixth place of decimals,”* but I would argue that we should care anyway. Some of the biggest mysteries in modern physics (dark matter, dark energy, the information paradox, strong cosmic censorship) are related to the behavior of gravity at large scales and near black holes. Any progress in understanding the gravitational interaction can only lead us closer to unveiling those mysteries.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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