An Overview of Almost Minimizers of Bernoulli-Type Functionals

Mariana Smit Vega Garcia

Abstract In David and Toro (Calc Var 54(1):455–524, 2015) and David et al. (Adv Math 350:1109–1192, 2019), the authors studied almost minimizers for functionals of the type first studied by Alt and Caffarelli (J Reine Angew Math325:105–144, 1981) and Alt, Caffarelli and Friedman (Trans Am Math Soc 282:431–461, 1984). In this chapter, we study the regularity of almost minimizers to energy functionals with variable coefficients (as opposed to Alt and Caffarelli (J Reine Angew Math325:105–144, 1981), Alt, Caffarelli and Friedman (Trans Am Math Soc 282:431–461, 1984), David et al. (Adv Math 350:1109–1192, 2019), and David and Toro (Calc Var 54(1):455–524, 2015) that deal only with the "Laplacian" setting). We prove Lipschitz regularity up to, and across, the free boundary, fully generalizing the results of David and Toro (Calc Var 54(1):455–524, 2015) to the variable coefficient setting.

Keywords Almost minimizer · Free boundary problem · Alt–Caffarelli functional · Alt–Caffarelli–Friedman

2010 Mathematics Subject Classification Primary 35R35

1 Introduction

In this chapter, we provide an overview of almost minimizers of Bernoulli-type functionals, with a focus on the results addressing the variable coefficient setting in [10]. We start Sect. 2 with a history of the minimizing problems that inspired the

M.S.V.G. has been partially supported by the NSF grant DMS 2054282.

M. Smit Vega Garcia (✉)
Department of Mathematics, Western Washington University, Bellingham, WA, USA
e-mail: Mariana.SmitVegaGarcia@wwu.edu

T. Mengesha, A. J. Salgado (eds.), *A^3N^2M: Approximation, Applications, and Analysis of Nonlocal, Nonlinear Models*, The IMA Volumes in Mathematics and its Applications 165, https://doi.org/10.1007/978-3-031-34089-5_8

study of almost minimizers, following the ground-breaking papers of Alt–Caffarelli and Alt–Caffarelli–Friedman, [3, 4]: given a domain $\Omega \subset \mathbb{R}^n$ and functions $q_{\pm} \in L^{\infty}(\Omega)$, one minimizes

$$J(u) = \int_{\Omega} \Big(|\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u\le 0\}}(x) \Big)\, dx$$

among all $u \in \{u \in L^1_{\mathrm{loc}}(\Omega) \ : \ \nabla u \in L^2(\Omega)\}$ with $u = u_0$ on $\partial\Omega$, for a given u_0. An overview of such minimizers and important results concerning them is described in Sect. 2.1.

In Sect. 2.2, we introduce almost minimizers, first in the context of the Laplacian, as considered by Anzellotti in [1], and then in the context of Bernoulli-type functionals. We also mention related literature concerning almost minimizers to other energy functionals and quasi-minimizers. We discuss in more detail the papers of De Silva–Savin [14, 15], along with the work of David–Toro and David–Engelstein–Toro [9, 17], which more directly relate to the work [10].

In Sect. 2.3, we introduce and motivate almost minimizers to a variable coefficient version of the Bernoulli-type functional previously described: one considers almost minimizers to

$$J_{\Omega}(u) = \int_{\Omega} \Big(\langle A(x)\nabla u(x), \nabla u(x)\rangle + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x) \Big)\, dx,$$

where $A(x) = (a_{ij}(x))$ with $a_{ij} \in C^{0,\alpha}(\Omega)$, $A(x)$ symmetric, uniformly elliptic, and $q_{\pm} \in L^{\infty}(\Omega)$. In Sect. 2.3 we discuss related literature, in particular the papers [25], which considers almost minimizers to a related variable coefficient energy, [30], which assumes a priori Lipschitz regularity of almost minimizers to prove free boundary regularity, [24], where the notion of ω-almost minimizers is adapted to the framework of multiple-valued functions in the sense of Almgren, and [31], where the author considers vectorial "quasi-minimizers."

In Sect. 3, we introduce our precise notions of almost minimizers in the variable coefficient setting. In Sect. 3.1, we address basic facts regarding the change of coordinates that are used throughout [10]; in Sect. 3.2, we address the connection between the "multiplicative" almost minimizers used in [9, 17, 25] and the "additive" almost minimizers introduced in [10]. In Sect. 4, we briefly discuss how the continuity of almost minimizers is obtained in [10], and in Sect. 5, we discuss the $C^{1,\beta}$ regularity of almost minimizers in $\{u > 0\}$ and $\{u < 0\}$. In Sect. 6, we discuss the bulk of the technical results needed to obtain local Lipschitz regularity for both the one-phase and two-phase problems. In Sect. 7, we discuss the first main result of [10]: the local Lipschitz continuity of almost minimizers of the one-phase problem. A sketch of the proof is provided. In Sect. 8, we discuss an analogue of the Alt–Caffarelli–Friedman monotonicity formula for variable coefficient almost minimizers. Finally, in Sect. 9, we provide a sketch of the proof of the second main result of [10], the local Lipschitz continuity for two-phase almost minimizers.

2 History

2.1 Minimizers

Before we discuss almost minimizers of Bernoulli-type functionals, let us first introduce the minimizing problems from which they arose. Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, Lipschitz domain, $q_{\pm} \in L^{\infty}(\Omega)$, and define the following set of admissible functions:

$$K(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \ : \ \nabla u \in L^2(\Omega) \right\}.$$

In [3] and [4], the authors considered the problem of minimizing

$$J(u) = \int_{\Omega} \left(|\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u\leq 0\}}(x) \right) dx$$

among all $u \in K(\Omega)$ with $u = u_0$ on $\partial\Omega$, where $u_0 \in K(\Omega)$. When $q_- \equiv 0$ and $u_0 \geq 0$, one obtains a so-called *one-phase* problem; otherwise, one obtains a *two-phase* problem.

One-Phase Problem The paper [3] proved very important results, opening the way to the study of more general Bernoulli-type functionals. First, minimizers of the one-phase problem were proved to exist. Moreover, if u is such a minimizer, then $u \geq 0$, u is subharmonic in Ω, and $\Delta u = 0$ in $\{u > 0\}$. In terms of optimal regularity, it was proved in [3] that $u \in C^{0,1}_{\mathrm{loc}}(\Omega)$. It was also shown that if $\exists c_+ > 0$ such that $q_+ \geq c_+$, then given $x \in \{u > 0\}$,

$$\frac{u(x)}{\mathrm{dist}(x, \partial\{u > 0\})} \approx 1.$$

Furthermore, $\{u > 0\} \cap \Omega$ is a set of locally finite perimeter, and $\partial\{u > 0\} \cap \Omega$ is $(n-1)$-rectifiable.

Two-Phase Problem The ground-breaking paper [4] introduced several important ideas. First, the authors proved that minimizers of the two-phase problem exist. Moreover, it was shown that if u is a minimizer of the two-phase problem, then $u^{\pm}$ are subharmonic in Ω and $\Delta u = 0$ in $\{u > 0\} \cup \{u < 0\}$. As in the case of the one-phase problem, minimizers of the two phase have the optimal regularity $C^{0,1}_{\mathrm{loc}}(\Omega)$, and if there exist $c_{\pm} > 0$ such that $q_{\pm} \geq c_{\pm}$, then for any $x \in \{u^{\pm} > 0\}$,

$$\frac{u^{\pm}(x)}{\mathrm{dist}(x, \partial\{u^{\pm} > 0\})} \approx 1.$$

Furthermore, the positive and negative phases $\{u^{\pm} > 0\} \cap \Omega$ are of locally finite perimeter.

Free Boundary In the case of the one-phase problem, the free boundary is defined as $\Gamma = \partial\{u > 0\}$. In terms of its regularity, one can show (see [3, 4, 7, 22, 32]) that if $q_+ \in C^{0,\gamma}(\Omega)$ and $q_+ \geq c_+ > 0$, then:

- If $n = 2, 3, 4$: Γ is a $C^{1,\beta}$ $(n-1)$-dimensional submanifold.
- If $n \geq 5$, then $\Gamma = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is a $C^{1,\beta}$ $(n-1)$-dimensional submanifold, and $\mathcal{S}$ is a closed set of Hausdorff dimension less than or equal to $n-5$.

In the case of the two-phase problem, the free boundary is defined as $\Gamma = \partial\{u > 0\} \cup \partial\{u < 0\}$. In this case, one can show (see [3, 4, 7, 22, 32]) that:

- If $n = 2, 3, 4$: Γ is a $C^{1,\beta}_{\text{loc}}$ $(n-1)$-dimensional submanifold.
- If $n \geq 5$, then $\Gamma = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is a $C^{1,\beta}_{\text{loc}}$ $(n-1)$-dimensional submanifold, and $\mathcal{S}$ is a closed set of Hausdorff dimension less than or equal to $n-5$.

Moreover, a very interesting result De Silva–Jerison [12] shows the existence of a non-smooth minimizer for J in $\mathbb{R}^7$ such that Γ is a cone. Consequently, $\mathcal{S} \neq \emptyset$ when $n \geq 7$.

Intimately connected to the energy functionals studied by Alt and Caffarelli [3], Alt et al. [4] is [13]. Here, the authors consider local minimizers of

$$\int_\Omega \left(|\nabla u|^2 + q_+^2 \chi_{\{u>0\}} + q_-^2 \chi_{\{u<0\}}\right) dx.$$

We say that $x_0 \in \Gamma$ is a *two-phase point* if $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\} \cap \Omega$. The collection of all two-phase free boundary points is called the two-phase free boundary.

Assuming $x_0 \in \Gamma$ is a two-phase point, the authors of [13] proved that $\exists r_0 > 0$ such that

$$\partial\{u > 0\} \cap B(x_0, r_0), \qquad \partial\{u < 0\} \cap B(x_0, r_0)$$

are $C^{1,\eta}$ graphs for some $\eta > 0$.

Moreover, $\partial\{u > 0\} \cap \Omega = \text{Reg}(\partial\{u > 0\}) \cup \text{Sing}(\partial\{u > 0\})$, where:

- $\text{Reg}(\partial\{u > 0\})$ is a relatively open subset of $\partial\{u > 0\} \cap \Omega$ and is locally $C^{1,\eta}$ for some $\eta > 0$. Also, the two-phase free boundary is regular.
- $\text{Sing}(\partial\{u > 0\})$ is a closed subset of $\partial\{u > 0\} \cap \Omega$ of Hausdorff dimension at most $n-5$. Also, $\exists n^* \in [5, 7]$ such that:
  - If $n < n^*$, then $\text{Sing}(\partial\{u > 0\}) = \emptyset$.
  - If $n = n^*$, then $\text{Sing}(\partial\{u > 0\})$ is locally finite in Ω.
  - If $n > n^*$, then $\text{Sing}(\partial\{u > 0\})$ is a closed $(n-n^*)$-rectifiable subset of $\partial\{u > 0\} \cap \Omega$, with locally finite $\mathcal{H}^{n-n^*}$ measure.

2.2 Almost Minimizers

Having discussed motivating minimizing problems, we start our discussion of almost minimizers with the simplest type of almost minimizers, considered in the context of the Laplacian. For this, we need to first introduce local minimizers.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given $u \in W^{1,2}(\Omega)$, consider its Dirichlet energy

$$J_\Omega(u) = \int_\Omega |\nabla u(x)|^2 dx.$$

Definition 2.1 We say $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of J_Ω if for any ball $B(x_0, r) \Subset \Omega$, we have

$$J_{B(x_0,r)}(u) \leq J_{B(x_0,r)}(v) \tag{2.1}$$

for any $v \in u + W^{1,2}_0(B(x_0, r))$.

Notice that local minimality implies the Laplacian is equal to zero.

In [2], Anzellotti introduced the concept of *almost minimizers* for energy functionals. Intuitively, a function is an almost minimizer for J_Ω if certain types of error that depend on r are allowed to be present in (2.1). To formalize this, let us specify what we require of the error:

Definition 2.2 Given $r_0 > 0$, a function $\omega : (0, r_0) \to [0, \infty)$ is a modulus of continuity, or gauge function, if $\omega(r)$ is monotone non-decreasing and $\omega(0+) = \lim_{r \to 0+} \omega(r) = 0$.

Definition 2.3 Let $\Omega \subset \mathbb{R}^n$ be an open set. Given $r_0 > 0$ and a gauge function $\omega(r)$ defined on $(0, r_0)$, we say $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a multiplicative almost minimizer, or ω-multiplicative almost minimizer for the functional J_Ω, if for any ball $B(x_0, r) \Subset \Omega$ with $0 < r < r_0$, we have

$$J_{B(x_0,r)}(u) \leq (1 + \omega(r)) J_{B(x_0,r)}(v) \quad \text{for any } v \in u + W^{1,2}_0(B(x_0, r)). \tag{2.2}$$

Heuristically, the energy of u on $B(x_0, r)$ might not be minimal among all competitors $v \in u + W^{1,2}_0(B(x_0, r))$, but *almost minimal*.

Almost minimizers have been receiving increasing attention, due to three important facts. First, they can be viewed as perturbations of minimizers: they are natural to consider once the presence of noise or lower order terms in a problem is taken into account. Second, minimizers with certain constraints, for example, with fixed volume, or solutions of the obstacle problem, can be realized as almost minimizers of unconstrained problems [2]. Third, the study of almost minimizers requires an analysis from a different point of view, leading to the development of different techniques that also lead to results about minimizers.

Almost minimizers of energy functionals of the type

$$\int_{\Omega} f(x, u, Du)dx$$

were studied in [16, 18, 23, 29]. Related notions were also studied in the context of geometric measure theory in [1, 5, 6, 24]. It is also worth mentioning that the concept of almost minimizers is related to that of quasi-minimizers, introduced in [26, 27], see also the quasi-minimizers from [31].

For energy functionals exhibiting free boundaries, almost minimizers have been addressed only more recently in [9, 10, 14, 15, 17, 25, 30], and in the case of obstacle-type problems, in [19] (for the thin obstacle problem), [20] (for the fractional obstacle problem), and [21] (for the variable coefficient thin obstacle problem).

We briefly describe the results of [9, 14, 15, 17], as this will enlighten their differences to [10], which we will describe in more detail in the rest of this chapter.

In [14], the authors considered thin one-phase almost minimizers, that is, almost minimizers of the energy

$$E_{\Omega}(u) = \int_{\Omega} |\nabla u(x)|^2 dx + \mathcal{H}^n(\{(x, 0) \in \Omega \, : \, u(x, 0) > 0\}).$$

The authors prove optimal $C^{0,\frac{1}{2}}_{\mathrm{loc}}(\Omega)$ regularity of almost minimizers, along with partial regularity of the free boundary. More precisely, they show that the free boundary $\Gamma(u) = \partial\{u > 0\}$ is $C^{1,\alpha}$ regular outside a closed singular set of Hausdorff dimension $n - 3$. This allows them to prove the $C^{1,\alpha}$ regularity of Lipschitz free boundaries.

In [15], the authors addressed almost minimizers of the one-phase free boundary problem given by the Alt–Caffarelli functional. That is, they studied almost minimizers of the energy

$$J_{\Omega}(u) = \int_{\Omega} \Big(|\nabla u(x)|^2 + \chi_{\{u>0\}}(x) \Big) \, dx \qquad \text{for } u \geq 0.$$

Their first main theorem proves the optimal $C^{0,1}_{\mathrm{loc}}(\Omega)$ Lipschitz regularity of almost minimizers. Regarding the free boundary $\Gamma(u) = \partial\{u > 0\}$, the authors prove an improvement of flatness result that leads them to showing the free boundary is locally $C^{1,\alpha}$ regular outside a closed singular set of Hausdorff dimension $n - 5$, also allowing them to prove the $C^{1,\alpha}$ regularity of Lipschitz free boundaries.

In the case of [17] and [9], the authors considered the following energy:

$$J_{\Omega}(u) = \int_{\Omega} \Big(|\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}} \Big) \, dx,$$

where $q_{\pm} \in L^{\infty}(\Omega)$. In [17], the authors proved optimal regularity $C^{0,1}_{\mathrm{loc}}(\Omega)$ of almost minimizers. To understand the challenges involved in the proof of optimal regularity for almost minimizers, it is important to notice that in [3, 4], the following facts were key ingredients in the proof of the optimal $C^{0,1}_{\mathrm{loc}}(\Omega)$ regularity for minimizers:

(a) If u is a minimizer, then $u^{\pm}$ are subharmonic in Ω.
(b) If u is a minimizer, then u is harmonic in $\{u^{\pm} > 0\}$.
(c) In the case of the two-phase problem, one uses the Alt–Caffarelli monotonicity formula. This formula says that the functional

$$\Phi(r) = \frac{1}{r^4}\left(\int_{B_r(x)} \frac{|\nabla u^+(y)|^2}{|x-y|^{n-2}} dy\right)\left(\int_{B_r(x)} \frac{|\nabla u^-(y)|^2}{|x-y|^{n-2}} dy\right)$$

is monotone non-decreasing as a function of r.

In contrast, when working with almost minimizers, one does not have (a), (b), or (c), as almost minimizers do not solve partial differential equations as minimizers do. To bypass this challenge, the use of good comparison functions is required. As the goal is to prove $C^{0,1}_{\mathrm{loc}}(\Omega)$ regularity of almost minimizers, it suffices to control, in a careful way, $\omega(x,s) = \left(⨍_{B(x,s)} |\nabla u|^2\right)^{\frac{1}{2}}$ for $s \in (0,r)$ if $B(x,r) \subset \Omega$.

Through comparisons with harmonic replacements and an iteration scheme, the authors proved in [17] the optimal regularity for almost minimizers of the one-phase problem. In the case of the two-phase problem, more was needed. First, the authors of [17] analyzed the interplay between

$$m(x,r) = \frac{1}{r} ⨍_{\partial B(x,r)} u, \quad n(x,r) = \frac{1}{r} ⨍_{\partial B(x,r)} |u|, \quad \omega(x,r) = \left(⨍_{B(x,r)} |\nabla u|^2\right)^{\frac{1}{2}}. \tag{2.3}$$

This analysis, together with an almost Alt–Caffarelli–Friedman monotonicity formula, allowed them to prove the optimal regularity of almost minimizers of the two-phase problem. More precisely, the almost Alt–Caffarelli–Friedman monotonicity formula proved in [17] states that if u is an almost minimizer for J in Ω, then $\exists \delta > 0$ such that for $K \Subset \Omega$, $\exists r_K > 0$, $C_K > 0$ such that for all $x \in \Gamma(u) \cap K$, for all $0 < s < r < r_K$,

$$\Phi(s) \le \Phi(r) + C_K r^{\delta},$$

where $\Phi(r)$ is defined as in [4].

In terms of the free boundary of almost minimizers, in [17], the authors first proved a non-degeneracy result. Assuming u is an almost minimizer of the one-phase problem, $q_+ \in L^{\infty}(\Omega) \cap C(\Omega)$ and $q_+ \ge c_+ > 0$, [17] proved that there exists $\eta > 0$ so that if $x_0 \in \Gamma$ and $B(x_0, 2r_0) \subset \Omega$, then for $0 < r < r_0$,

$$\frac{1}{r} ⨍_{\partial B(x_0,r)} u^+ \ge \eta$$

and $u(x) \geq \frac{\eta}{4}\text{dist}(x, \partial\{u > 0\})$, for $x \in B(x_0, r_0) \cap \{u > 0\}$. Under the same assumptions, [17] also show $\{u > 0\} \subset \Omega$ is "locally" NTA. Moreover, if $x_0 \in \Gamma$ with $B(x_0, 2r_0) \subset \Omega$, then there exists an Ahlfors regular measure μ_0 supported in $B(x_0, r_0) \cap \Gamma$. Furthermore, Γ is $(n-1)$-uniformly rectifiable, and $\{u > 0\} \cap \Omega$ is a set of locally finite perimeter.

With respect to the two-phase problem, assume $q_\pm \in L^\infty(\Omega) \cap C(\overline{\Omega})$ and $q_\pm \geq c_0 > 0$. In [9], the authors proved that Γ is locally Ahlfors regular and uniformly rectifiable. In the case of the one-phase problem, [9] assumed $q_+ \in L^\infty \cap C^\gamma(\Omega)$ and $q_+ \geq c_+ > 0$. They proved that $\Gamma = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is a $C^{1,\beta}$ $(n-1)$-dimensional submanifold and $\mathcal{S}$ is a closed set with $\mathcal{H}^{n-1}(\mathcal{S}) = 0$, following the result for minimizers. Furthermore, $\mathcal{S} = \emptyset$ when $n = 2, 3, 4$. In terms of the dimension of the singular set, let k^* be the smallest natural number such that there exists a stable one-homogeneous globally defined minimizer $u : \mathbb{R}^{k^*} \to \mathbb{R}$ that is not the half-plane solution. From the work of Caffarelli–Jerison–Kenig, Jerison–Savin, and De Silva–Jerison, one knows that $4 < k^* \leq 7$. Let u be an almost minimizer of J^+ in Ω, with $q_+ \in L^\infty \cap C^\gamma(\Omega)$ and $q_+ \geq c_+ > 0$. David et al. [9] proved that if $s > n - k^*$, then $\mathcal{H}^s(\Gamma^+ \setminus \mathcal{R}) = 0$.

We also mention the recent preprint [11], where the authors construct a family of minimizers to an Alt–Caffarelli–Friedman-type functional whose free boundaries contain branch points in the strict interior of the domain. They also give an example showing that branch points in the free boundary of almost minimizers of the same functional can have very little structure.

2.3 *Almost Minimizers with Variable Coefficients*

In contrast to the energy considered in [9, 15, 17], and [10] dealt with a variable coefficient version of this energy:

$$J_\Omega(u) = \int_\Omega \Big(\langle A(x)\nabla u(x), \nabla u(x)\rangle + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x) \Big)\, dx,$$

where $A(x) = (a_{ij}(x))$ with $a_{ij} \in C^{0,\alpha}(\Omega)$, $A(x)$ symmetric, uniformly elliptic, and $q_\pm \in L^\infty(\Omega)$. The point of this generalization is to allow anisotropic energies that depend mildly on the point of the domain, so that in particular the classes of minimizers considered in [10] are essentially invariant by $C^{1+\alpha}$ diffeomorphisms.

Almost minimizers to functionals of Alt–Caffarelli or Alt–Caffarelli–Friedman type with variable coefficients arise naturally in measure-penalized minimization problems for Dirichlet eigenvalues of elliptic operators (e.g., the Laplace–Beltrami operator on a manifold; see [28] for a treatment of the analogous measure-constrained problem).

In the following sections, we describe the developments in [10]. These address the regularity of almost minimizers to energy functionals with variable coefficients (as opposed to [3, 4, 9, 15, 17], which deal only with the "Laplacian" setting). First, we discuss the connection of [10] to other literature.

Variable coefficient problems have been studied before: Caffarelli, in [8], proved regularity for solutions to a more general free boundary problem. De Queiroz and Tavares, in [25], provided the first results for almost minimizers with variable coefficients: the authors proved regularity away from the free boundary for almost minimizers to the same functionals considered in [10] (they considered a slightly broader class of functionals, of which the functionals from [10] are a limiting case). More precisely, the authors prove in [17] regularity away from the free boundary $\Gamma(u) = \partial\{u > 0\} \cup \{u < 0\}$ for almost minimizers of

$$J_\Omega(u) = \int_\Omega \Big(\langle A(x)\nabla u(x), \nabla u(x)\rangle + q_+(x)(u^+(x))^\gamma + q_-(x)(u^-(x))^\gamma \Big)\, dx,$$

where $A(x) = (a_{ij}(x))$ with $a_{ij} \in C^{0,\alpha}_{\text{loc}}(\Omega)$, $A(x)$ is symmetric, uniformly elliptic, and $q_\pm \in L^\infty(\Omega)$ for $0 \le \gamma \le 1$ and $q_\pm \ge q_0 > 0$. The work of [10] differs from that of [25] in two ways: first, the definition of almost minimizing in [10] is, a priori, broader than that considered in [9, 14, 15, 17, 19, 25] (for more discussion, see Sect. 3.2 below). Second, and more significantly, [10] prove Lipschitz regularity up to, and across, the free boundary, in contrast to [25], thus fully generalizing the results of [17] to the variable coefficient setting.

Another connected paper is [30]. There, the authors assume a priori Lipschitz regularity of almost minimizers to the functionals studied in [10]. With this assumption, they prove $C^{1,\alpha}$ regularity of the free boundary, in dimension two, for almost minimizers of the constrained one-phase Alt–Caffarelli and the two-phase Alt–Caffarelli–Friedman functionals for an energy with variable coefficients. The paper [10] shows (as alluded to in their paper) that the a priori Lipschitz assumption is redundant. Note that the class of almost minimizers considered in [30] is equivalent to the one considered in [10].

Besides including the notion of almost minimizers from [9, 14, 17, 25], or [15], the definition of almost minimizers from [10] also connects to the work of [24]. There, the authors extend the notion of ω-minimizers introduced by Anzellotti in [1], to the framework of multiple-valued functions in the sense of Almgren, and prove Hölder regularity of Dirichlet multiple-valued (c, α)-almost minimizers.

Finally, in [31], the author studies the regularity of shape optimizers for variable coefficient divergence form elliptic operators (e.g., the domain of area one that minimizes the sum of the first k Dirichlet eigenvalues of a given operator). In particular, Trey adapts the approach of [17] to prove results of similar flavor to the ones from [10], but for vectorial "quasi-minimizers" of (3.2), with the additional property that they are solutions of a divergence form elliptic PDE with right-hand side. The results from [10] neither imply nor are implied by those of [31, Theorem 1.2], due to the different notions of "minimization" used and to the presence of an underlying PDE in [31].

3 Preliminaries

As in [10], consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and study the regularity of almost minimizers of the functional

$$J(u) = \int_\Omega \big\langle A(x)\nabla u(x), \nabla u(x)\big\rangle + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x), \tag{3.1}$$

where $q_+, q_- \in L^\infty(\Omega)$ are bounded real-valued functions and $A \in C^{0,\alpha}(\Omega; \mathbb{R}^{n\times n})$ is a Hölder continuous function with values in symmetric, uniformly positive definite matrices. Let $0 < \lambda \leq \Lambda < \infty$ be such that $\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2$ for all $x \in \Omega$.

We will also consider the situation where $u \geq 0$ and $q_- \equiv 0$, and

$$J^+(u) = \int_\Omega \langle A\nabla u, \nabla u\rangle + q_+^2(x)\chi_{\{u>0\}}, \tag{3.2}$$

where q_+ and A are as above.

Definition 3.1 (Definition 1 of Almost Minimizers, with Balls) Set

$$K_{\text{loc}}(\Omega) = \Big\{u \in L^1_{\text{loc}}(\Omega) \ : \forall\ B(x,r) \Subset \Omega,\ \nabla u \in L^2(B(x,r))\Big\}, \tag{3.3}$$

$$K^+_{\text{loc}}(\Omega) = \big\{u \in K_{\text{loc}}(\Omega) \ : u(x) \geq 0 \text{ almost everywhere on } \Omega\big\}, \tag{3.4}$$

and let constants $\kappa \in (0, +\infty)$ and $\alpha \in (0, 1]$ be given.

We say that u is a (κ, α)-almost minimizer for J_B^+ in Ω if $u \in K^+_{\text{loc}}(\Omega)$ and $\forall B(x,r) \Subset \Omega$ and $\forall v \in L^1(B(x,r))$ such that $\nabla v \in L^2(B(x,r))$ and $v = u$ on $\partial B(x,r)$,

$$J^+_{B,x,r}(u) \leq J^+_{B,x,r}(v) + \kappa r^{n+\alpha}, \tag{3.5}$$

where

$$J^+_{B,x,r}(v) = \int_{B(x,r)} \Big(\big\langle A(x)\nabla v(x), \nabla v(x)\big\rangle + q_+^2(x)\ \chi_{\{v>0\}}(x)\Big)\, dx. \tag{3.6}$$

Similarly, we say that u is a (κ, α)-almost minimizer for J_B in Ω if $u \in K_{\text{loc}}(\Omega)$ and $\forall B(x,r) \Subset \Omega$ and $\forall v \in L^1(B(x,r))$ such that $\nabla v \in L^2(B(x,r))$ and $v = u$ on $\partial B(x,r)$,

$$J_{B,x,r}(u) \leq J_{B,x,r}(v) + \kappa r^{n+\alpha}, \tag{3.7}$$

where

$$J_{B,x,r}(v)=\int_{B(x,r)}\Big(\big\langle A(x)\nabla v(x),\,\nabla v(x)\big\rangle + q_+^2(x)\chi_{\{v>0\}}(x)+q_-^2(x)\chi_{\{v<0\}}(x)\Big)\,dx. \tag{3.8}$$

This definition differs from the one found in [17] (or [25]), even when $A = I$ (see Sect. 3.2). Let us already comment that the definition given by (3.7) is more general than that of [17].

When working with variable coefficients, it is very convenient to work with a definition of almost minimizers that considers ellipsoids instead of balls. For this effect, we define

$$T_x(y) = A^{-1/2}(x)(y-x)+x, \qquad T_x^{-1}(y) = A^{1/2}(x)(y-x)+x,$$
$$E_x(x,r) = T_x^{-1}(B(x,r)). \tag{3.9}$$

Note that

$$E_x(x,r) \subset B(x,\Lambda^{1/2}r) \qquad \text{and} \qquad B(x,r)\subset E_x(x,\lambda^{-1/2}r). \tag{3.10}$$

For completeness, we include here what it means to be an almost minimizer with respect to ellipsoids.

Definition 3.2 (Definition 2 of Almost Minimizers, with Ellipsoids) Let

$$K_{\rm loc}(\Omega,E) = \Big\{u\in L^1_{\rm loc}(\Omega)\,:\,\nabla u\in L^2(E_x(x,r)) \text{ for } \overline{E}_x(x,r)\subset\Omega\Big\} \tag{3.11}$$

and

$$K^+_{\rm loc}(\Omega,E) = \big\{u\in K_{\rm loc}(\Omega,E)\,:\,u(x)\geq 0 \text{ almost everywhere on } \Omega\big\}. \tag{3.12}$$

We say that u is a (κ,α)-almost minimizer for J_E^+ in Ω if $u\in K^+_{\rm loc}(\Omega,E)$ and

$$J^+_{E,x,r}(u)\leq J^+_{E,x,r}(v)+\kappa r^{n+\alpha} \tag{3.13}$$

for every ellipsoid such that $E_x(x,r)\Subset\Omega$ and every $v\in L^1(E_x(x,r))$ such that $\nabla v\in L^2(E_x(x,r))$ and $v=u$ on $\partial E_x(x,r)$, where

$$J^+_{E,x,r}(v) = \int_{E_x(x,r)}\Big(\big\langle A(x)\nabla v(x),\,\nabla v(x)\big\rangle + q_+^2(x)\chi_{\{v>0\}}(x)\Big)\,dx. \tag{3.14}$$

Similarly, we say that u is a (κ,α)-almost minimizer for J_E in Ω if $u\in K_{\rm loc}(\Omega,E)$ and

$$J_{E,x,r}(u)\leq J_{E,x,r}(v)+\kappa r^{n+\alpha} \tag{3.15}$$

for every ellipsoid with $E_x(x,r) \Subset \Omega$ and every $v \in L^1(E_x(x,r))$ such that $\nabla v \in L^2(E_x(x,r))$ and $v = u$ on $\partial E_x(x,r)$, where

$$J_{E,x,r}(v) = \int_{E_x(x,r)} \Big(\big\langle A(x)\nabla v(x), \nabla v(x) \big\rangle + q_+^2(x)\chi_{\{v>0\}}(x) + q_-^2(x)\chi_{\{v<0\}}(x) \Big). \tag{3.16}$$

Remark 3.1 The following summarizes the relationship between Definitions 3.1 and 3.2:

- When $A = I$, both definitions coincide.
- For a general matrix A, if u is a (κ, α)-almost minimizer for J_B in Ω according to Definition 3.1, then it satisfies (3.13) in Definition 3.2 (with constant $\Lambda^{(n+\alpha)/2}\kappa$ and exponent α) whenever x and r are such that $\overline{B}(x, \Lambda^{1/2}r) \subset \Omega$.
- If u is a (κ, α)-almost minimizer for J_E in Ω according to Definition 3.2, then it satisfies (3.5) in Definition 3.1 (with constant $\lambda^{-(n+\alpha)1/2}\kappa$ and exponent α) whenever x and r are such that $\overline{B}(x, \Lambda^{1/2}\lambda^{-1/2}r) \subset \Omega$.

Given that we are mostly interested in the regularity of almost minimizers away from $\partial\Omega$, these definitions are essentially equivalent. Bearing this in mind, we work with almost minimizers according to Definition 3.2, recalling that such functions satisfy (3.5) when $\overline{B}(x, \Lambda^{1/2}\lambda^{-1/2}r) \subset \Omega$. We will most often not write "(κ, α)-almost minimizer," but only "almost minimizer," and we will drop the subscripts B and E from the energy functional.

Notation We write $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $\partial B(x,r) = \{y \in \mathbb{R}^n : |y - x| = r\}$. We will consider $A \in C^{0,\alpha}(\Omega; \mathbb{R}^{n\times n})$ a Hölder continuous function with values in symmetric, uniformly positive definite matrices, and $0 < \lambda \le \Lambda < \infty$ such that $\lambda|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2$ for all $x \in \Omega$. Additionally, $q_\pm \in L^\infty(\Omega)$ will be bounded real-valued functions. We also refer to

$$\begin{aligned} T_x(y) &= A^{-1/2}(x)(y - x) + x, \quad T_x^{-1}(y) = A^{1/2}(x)(y - x) + x, \\ E_x(x,r) &= T_x^{-1}(B(x,r)). \end{aligned} \tag{3.17}$$

Moreover, we will write

$$\begin{aligned} u_x(y) &= u(T_x^{-1}(y)), \quad (q_x)_\pm(y) = q_\pm(T_x^{-1}(y)), \\ A_x(y) &= A^{-1/2}(x)A(T_x^{-1}(y))A^{-1/2}(x). \end{aligned} \tag{3.18}$$

Notice that $T_x(x) = x$ and $A_x(x) = I$.

3.1 Coordinate Changes

Compared to [17] and [9], the proofs of [10] use two new ingredients: the good invariance properties of our notion with respect to bijective affine transformations, and the fact that the slow variations of A allow us to make approximations by freezing the coefficients. We take care of the first part in this subsection.

Many of our proofs use the affine mapping T_x to transform our almost minimizer u into another one u_x, which corresponds to a new matrix function $A_x(y)$ that coincides with the identity at x. In this subsection, we check that our notion of almost minimizer behaves well under bijective affine transformations. Our second definition, with ellipsoids, is more adapted to this.

Lemma 3.1 *Let u be a (κ, α)-almost minimizer for J_E (or J_E^+) in $\Omega \subset \mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an injective affine mapping, and denote by S the linear map corresponding to T (i.e., $Tx = Sx + z_0$ for some $z_0 \in \mathbb{R}^n$). Also let $0 < a \le b < +\infty$ be such that $a|\xi| \le |S\xi| \le b|\xi|$ for $\xi \in \mathbb{R}^n$. Then define functions u_T, $q_{T,+}$, $q_{T,-}$ on $\Omega_T = T(\Omega)$ by*

$$u_T(y) = u(T^{-1}(y)) \text{ and } q_{T,\pm}(y) = q_\pm(T^{-1}(y)) \text{ for } y \in \Omega_T, \tag{3.19}$$

and a matrix-valued function A_T by

$$A_T(y) = SA(T^{-1}y)S^t \text{ for } y \in \Omega_T, \tag{3.20}$$

where S^t is the transposed matrix of S.

Then u_T is a $(\tilde{\kappa}, \alpha)$-almost minimizer of $J_{E,T}$ (or $J_{E,T}^+$) in Ω_T, according to Definition 3.2, where $J_{E,S}$ (or $J_{E,S}^+$) is defined in terms of A_T and the $q_{T,\pm}$, i.e.,

$$J_{E,S}(v) = \int \langle A_T(y)\nabla v(y), \nabla v(y)\rangle + q_{T,+}^2(y)\chi_{\{v>0\}}(y) + q_{T,-}^2(y)\chi_{\{v<0\}}(y)\, dy, \tag{3.21}$$

and $\tilde{\kappa} = \kappa |\det T|$.

This lemma says that under an affine change of variables, almost minimizers are transformed to almost minimizers for a modified functional. Its proof (see [10]) shows why Definition 3.2 is natural. Lemma 3.1 is applied almost exclusively in the following circumstances: let $x \in \Omega$, and take $S = A^{-1/2}(x)$. In this case, $T(y) = x + S(y - x)$, we recognize the affine mapping T_x from (3.17), and then $u_T = u_x$ and $A_T(y) = A_x(y)$ (from (3.18)). The advantage is that $A_T(x_0) = I$, and simpler competitors can be used in the definition of almost minimizer.

3.2 "Additive" Almost Minimizers

Let us now address the differences between the definition of almost minimizers used in [10] under (3.7) or (3.15) and the definition of an almost minimizer in [17] (similarly used in [9, 14, 25]). Recall that when $A = I$, being an almost minimizer for J_E is equivalent to being an almost minimizer for J_B, and that in [17] (with $A = I$), u was an almost minimizer for J_E if, instead of satisfying (3.15) for all admissible v, it satisfied

$$J_{E,x,r}(u) \leq (1 + \kappa r^{\alpha}) J_{E,x,r}(v) \tag{3.22}$$

(and similarly for J_E^+). Here A is variable, and we stick to J_E (J_B would work the same way). Let almost minimizers in the sense of (3.15) be **additive almost minimizers**, whereas almost minimizers in the sense of (3.22) are **multiplicative** almost minimizers.

In [10], we prove results for additive minimizers, first showing that multiplicative almost minimizers are also additive almost minimizers. To conclude this, we first need to show that multiplicative almost minimizers, in the variable coefficient setting, obey a certain decay property. This is done through the next Lemma (see the proof in [10]). With this result in hand, we showed in [10] that every multiplicative almost minimizer is actually an additive almost minimizer, reducing our study to the case of additive minimizers, see Lemma 3.3 below.

Lemma 3.2 *Let u be a multiplicative almost minimizer for J_E in Ω. Then $\exists C > 0$ such that if $x \in \Omega$ and $r > 0$ are such that $\overline{E}_x(x, r) \subset \Omega$, then for $0 < s \leq r$,*

$$\left(\fint_{E_x(x,s)} |\nabla u|^2 \right)^{1/2} \leq C \left(\fint_{E_x(x,r)} |\nabla u|^2 \right)^{1/2} + C \log(r/s). \tag{3.23}$$

Lemma 3.3 *Let u be a multiplicative almost minimizer of J_E in Ω with constant κ and exponent α, and let $\widetilde{\Omega} \subset\subset \Omega$ be an open subset of Ω whose closure is a compact subset of Ω. Then u is an additive almost minimizer of J_E in $\widetilde{\Omega}$, with exponent $\alpha/2$ and a constant $\widetilde{\kappa}$ that depends on the constants for J, u, and $\widetilde{\Omega}$.*

Given this result, we restrict ourselves to working with additive almost minimizers and refer to them simply as almost minimizers.

4 Continuity of Almost Minimizers

Given the equivalence between almost minimizers of J_B and J_E, we omit the subscript. Inspired by the ideas of [17], we first prove in [10] the continuity of almost minimizers for J and J^+:

Theorem 4.1 *Almost minimizers of J are continuous in Ω. Moreover, if u is an almost minimizer for J and $\overline{B}(x_0, 2r_0) \subset \Omega$, then there exists a constant $C > 0$ such that for $x, y \in B(x_0, r_0)$*

$$|u(x) - u(y)| \leq C|x - y| \left(1 + \log\left(\frac{2r_0}{|x - y|}\right)\right). \tag{4.1}$$

A simple consequence of Theorem 4.1 is:

Corollary 4.1 *If u is an almost minimizer for J, then for each compact $K \subset \Omega$, there exists a constant $C_K > 0$ such that for $x, y \in K$,*

$$|u(x) - u(y)| \leq C_K|x - y| \left(1 + \left|\log \frac{1}{|x - y|}\right|\right). \tag{4.2}$$

5 Almost Minimizers are $C^{1,\beta}$ in $\{u > 0\}$ and in $\{u < 0\}$

While our final goal is to prove Lipschitz regularity also across the free boundary, we first prove in [10] Lipschitz bounds away from the free boundary. Note that since u is continuous, $\{u > 0\}$ and $\{u < 0\}$ are open sets.

Theorem 5.1 *Let u be an almost minimizer for J (or J^+) in Ω. Then u is locally Lipschitz in $\{u > 0\}$ and in $\{u < 0\}$.*

This is done by estimating the energy of the almost minimizer u in comparison to that of the harmonic replacements of u on $B(x, r)$, first under the assumption that $A(x) = I$, together with careful iterations. Finally, a careful analysis of how to address the case $A(x) \neq I$, which involves applying the previous case to u_x (defined in 3.18), completes the proof.

We then improve Theorem 5.1 and prove that u is $C^{1,\beta}$ away from the free boundary. Initially, we wanted bounds on averages of $|\nabla u|^2$, and now we want to be more precise and control the variations of ∇u. Our main tool is a (more careful) comparison with the harmonic approximation $(u_x)_r^*$, where u_x was defined in 3.18 and $(u_x)_r^*$ denotes the harmonic function on $B(x, r)$ with boundary data u_x.

Theorem 5.2 *Let u be an almost minimizer for J in Ω and set $\beta = \frac{\alpha}{n+2+\alpha}$. Then u is of class $C^{1,\beta}$ locally in $\{u > 0\}$ and in $\{u < 0\}$.*

6 Estimates Toward Lipschitz Continuity

In this section, we discuss some technical results needed to obtain local Lipschitz regularity for both the one-phase and two-phase problems.

Define the quantities

$$b(x,r) = \fint_{\partial B(x,r)} u_x \quad \text{and} \quad b^+(x,r) = \fint_{\partial B(x,r)} |u_x|, \tag{6.1}$$

where we recall that $u_x = u \circ T_x^{-1}$ and T_x is the affine mapping from (3.17). We sometimes write $b(u_x, x, r)$ and $b^+(u_x, x, r)$ to stress the dependence on u_x.

One has to distinguish two types of pairs (x, r), for which we have to use different estimates. For constants $\tau \in (0, 10^{-2})$, $C_0 \geq 1$, $C_1 \geq 3$, and $r_0 > 0$, we study the class $\mathcal{G}(\tau, C_0, C_1, r_0)$ of pairs $(x, r) \in \Omega \times (0, r_0]$ such that

$$E_x(x, 2r) \subset \Omega, \tag{6.2}$$

$$C_0 \tau^{-n} (1 + r^\alpha \omega(u_x, x, r)^2)^{1/2} \leq r^{-1} |b(x,r)| \tag{6.3}$$

(recall the definition of ω from (2.3)) and

$$b^+(x,r) \leq C_1 |b(x,r)|. \tag{6.4}$$

We force $r \leq r_0$ to have uniform estimates. We end up choosing τ very small, so (6.3) says that the quantity $r^{-1}|b(x,r)|$ is as large as we want. This quantity has the same dimensionality of the expected variation of u on $B(x,r)$. In addition, (6.4) says that b accounts for a significant part of b^+, which measures the average size of $|u|$. We mostly expect this to happen only far from the free boundary, and the next lemmas go in that direction.

One of the challenges of the variable coefficient setting stems from the fact that one has to consider different centers x and at the same time also consider various ellipsoids $E_z(z, \rho)$, with z near x, with different orientations. Set

$$k = \frac{1}{6} \lambda^{1/2} \Lambda^{-1/2}, \tag{6.5}$$

which we choose like this so that

$$E_z(z, kr) \subset B(z, \Lambda^{1/2} kr) \subset E_x(x, r/2) \text{ whenever } x \in \Omega \text{ and } z \in E_x(x, r/3). \tag{6.6}$$

We start with a self-improvement lemma.

Lemma 6.1 *Assume u is an almost minimizer for J in Ω. For each choice of constants $C_1 \geq 3$ and r_0, there is a constant $\tau_1 \in (0, 10^{-2})$ (which depends only on $n, \kappa, \alpha, r_0, C_1, \lambda$ and Λ), such that if $(x, r) \in \mathcal{G}(\tau, C_0, C_1, r_0)$ for some* *choice of* $\tau \in (0, \tau_1)$ *and* $C_0 \geq 1$*, then for each* $z \in E_x(x, \tau r/3)$*, we can find* $\rho_z \in (\tau kr/2, \tau kr)$ *such that* $(z, \rho_z) \in \mathcal{G}(\tau, 10C_0, 3, r_0)$*. Here* k *is defined as in* (6.5) *and satisfies* (6.6)*.*

Lemma 6.2 *Let* u, x, r *satisfy the hypothesis of Lemma 6.1; in particular* $(x, r) \in \mathcal{G}(\tau, C_0, C_1, r_0)$ *for some* $C_0 \geq 1$*,* $C_1 \geq 3$*, and* $\tau \leq \tau_1$*. Recall that* $b(x, r) \neq 0$ *by* (6.3)*. If* $b(x, r) > 0$*, then*

$$u \geq 0 \text{ on } E_x(x, \tau r/3) \text{ and } u > 0 \text{ almost everywhere on } E_x(x, \tau r/3). \tag{6.7}$$

Similarly, if $b(x, r) < 0$*, then*

$$u \leq 0 \text{ on } E_x(x, \tau r/3) \text{ and } u < 0 \text{ almost everywhere on } E_x(x, \tau r/3). \tag{6.8}$$

For the next lemma, we use Lemma 6.2 to get some regularity for u near a point x such that $(x, r) \in \mathcal{G}(\tau, C_0, C_1, r_0)$, with the same method as for the local regularity of u away from the free boundary.

Lemma 6.3 *There exist constants* $k_1 \in (0, k/2)$*, depending only on* λ *and* Λ*, and* $\tau_2 \in (0, \tau_1)$*, with* τ_1 *as in Lemmas 6.1 and 6.2 with the following properties. Let* u *be an almost minimizer for* J *in* Ω*. Let* $(x, r) \in \mathcal{G}(\tau, C_0, C_1, r_0)$ *for some* $\tau \in (0, \tau_2)$ *and* $C_0 \geq 1$*. Then for* $z \in B(x, \tau r/10)$ *and* $s \in (0, k_1 \tau r)$*,*

$$\omega(u, z, s) \leq C\left(\tau^{-\frac{n}{2}} \omega(u_x, x, r) + r^{\frac{\alpha}{2}}\right), \tag{6.9}$$

and for $y, z \in B(x, \tau r/10)$*,*

$$|u(y) - u(z)| \leq C\left(\tau^{-\frac{n}{2}} \omega(u_x, x, r) + r^{\frac{\alpha}{2}}\right) |y - z|. \tag{6.10}$$

Here $C = C(n, \kappa, \alpha, \lambda, \Lambda, r_0)$*. Finally, there is a constant* $C(\tau, r)$ *depending on* $n, \kappa, \alpha, r_0, \tau, r, \lambda, \Lambda$*, such that*

$$|\nabla u(y) - \nabla u(z)| \leq C(\tau, r)(\omega(u_x, x, r) + 1)|y - z|^{\beta}, \tag{6.11}$$

for any $y, z \in B(x, \tau r/10)$*, where as before* $\beta = \frac{\alpha}{n+2+\alpha}$*.*

The next lemma allows us to conclude that $(x, \rho) \in \mathcal{G}(\tau, C_0, 3, r_0)$, provided we have certain estimates on $b(u_x, x, r)$, and $\omega(u_x, x, r)$.

Lemma 6.4 *Let* u *be an almost minimizer for* J *in* Ω*. There exists* $K_2 = K_2(\lambda, \Lambda) \geq 2$ *such that for each choice of* $\gamma \in (0, 1)$*,* $\tau > 0$*, and* $C_0 \geq 1$*, we can find* r_0*,* η *small and* $K \geq 1$ *with the following property: if* $x \in \Omega$ *and* $r > 0$ *are such that* $0 < r \leq r_0$*,* $B(x, K_2 r) \subset \Omega$ *and*

$$|b(u_x, x, r)| \geq \gamma r(1 + \omega(u_x, x, r)), \tag{6.12}$$

and

$$\omega(u_x, x, r) \geq K, \tag{6.13}$$

then there exists $\rho \in \left(\frac{\eta r}{2}, \eta r\right)$ *such that* $(x, \rho) \in \mathcal{G}(\tau, C_0, 3, r_0)$.

7 Local Lipschitz Regularity for One-phase Almost Minimizers

The last intermediate result needed to obtain optimal regularity is the following:

Lemma 7.1 *Let* u *be an almost minimizer for* J^+ *in* Ω. *Let* $\theta \in (0, 1/2)$. *There exist* $\gamma > 0$, $K_1 > 1$, $\beta \in (0, 1)$, *and* $r_1 > 0$ *such that if* $x \in \Omega$ *and* $0 < r \leq r_1$ *are such that* $B(x, r) \subset \Omega_x$,

$$b(u_x, x, r) \leq \gamma r(1 + \omega(u_x, x, r)), \tag{7.1}$$

and

$$\omega(u_x, x, r) \geq K_1, \tag{7.2}$$

then

$$\omega(u_x, x, \theta r) \leq \beta\omega(u_x, x, r). \tag{7.3}$$

We are ready to combine all of the previous lemmas to obtain our main result for almost minimizers of the one-phase problem:

Theorem 7.1 *Let* u *be an almost minimizer for* J^+ *in* Ω. *Then* u *is locally Lipschitz in* Ω.

We want to show that there exist $r_2 > 0$ and $C_2 \geq 1$ (depending on $n, \kappa, \alpha, \lambda, \Lambda$) such that for each choice of $x_0 \in \Omega$ and $r_0 > 0$ such that $r_0 \leq r_2$ and $B(x_0, K_2 r_0) \subset \Omega$, where K_2 is as in Lemma 6.4,

$$|u(x) - u(y)| \leq C_2(\omega(u_{x_0}, x_0, 2r_0) + 1)|x - y| \text{ for } x, y \in B(x_0, r_0). \tag{7.4}$$

Sketch of the Proof Let (x, r) be such that $B(x, K_2 r) \subset \Omega$. We want to use the different lemmas above to find a pair (x, ρ) that allows us to control u. Pick $\theta = 1/3$, and let β, γ, K_1, r_1 be as in Lemma 7.1.

Let $\tau = \tau_2/2$, where $\tau_2 \in (0, \tau_1)$, and τ_1 is the constant that we get in Lemma 6.1 applied with $C_1 = 3$ and $r_0 = r_1$. Here τ_2 is the corresponding constant that appears in Lemma 6.3. Let now r_0, η, K be as in Lemma 6.4 applied to $C_0 = 10$, and to τ and γ as above. From Lemma 6.4, we get a small r_γ. Set

$$K_3 \geq \max(K_1, K), \quad \text{and} \quad r_2 \leq \min(r_1, r_\gamma). \tag{7.5}$$

Let $r \leq r_2$. The proof of the theorem then relies on the analysis of three cases:

Case 1:

$$\begin{cases} \omega(u_x, x, r) \geq K_3 \\ b(u_x, x, r) \geq \gamma r(1 + \omega(u_x, x, r)) \end{cases}. \tag{7.6}$$

Case 2:

$$\begin{cases} \omega(u_x, x, r) \geq K_3 \\ b(u_x, x, r) < \gamma r(1 + \omega(u_x, x, r)) \end{cases}. \tag{7.7}$$

Case 3:

$$\omega(u_x, x, r) < K_3. \tag{7.8}$$

Case 1 ends up yielding additional regularity, as by Lemma 6.3 we know that u is $C^{1,\beta}$ in a neighborhood of x.

In the two remaining cases, we set

$$r_k = \theta^k r = 3^{-k} r, \;\; k \geq 0.$$

Our task is to control $\omega(u_x, x, r_k)$. If the pair (x, r_k) ever satisfies (7.6), we denote k_{stop} the smallest integer such that (x, r_k) satisfies (7.6) (notice that $k \geq 1$, since we are not in Case 1). Otherwise, set $k_{\text{stop}} = \infty$.

Let $k < k_{\text{stop}}$ be given. If (x, r_k) satisfies (7.7), we can apply Lemma 7.1 to it. Therefore,

$$\omega(u_x, x, r_{k+1}) \leq \beta \omega(u_x, x, r_k). \tag{7.9}$$

Otherwise, (x, r_k) satisfies (7.8) (since $k < k_{\text{stop}}$). Then

$$\omega(u_x, x, r_{k+1}) = \left(\fint_{B(x, r_{k+1})} |\nabla u_x|^2 \right)^{1/2} \leq 3^{\frac{n}{2}} \omega(u_x, x, r_k) \leq 3^{\frac{n}{2}} K_3. \tag{7.10}$$

By (7.9) and (7.10), we obtain that for $0 \leq k \leq k_{\text{stop}}$,

$$\omega(u_x, x, r_k) \leq \max\left(\beta^k \omega(u_x, x, r), 3^{\frac{n}{2}} K_3 \right).$$

If $k_{\text{stop}} = \infty$, this implies that $\limsup_{k \to \infty} \omega(u_x, x, r_k) \leq 3^{\frac{n}{2}} K_3$. In particular, if x is a Lebesgue point of ∇u_x (hence a Lebesgue point for ∇u), $|\nabla u_x(x)| \leq 3^{n/2} K_3$.

This implies

$$|\nabla u(x)| \le C3^{n/2}K_3. \tag{7.11}$$

If $k_{\text{stop}} < \infty$, we apply our argument from Case 1 to the pair $(x, r_{k_{\text{stop}}})$ and get that u is $C^{1,\beta}$ in a neighborhood of x. We end up concluding

$$|\nabla u(x)| \le C'\omega(u_x, x, r) + C', \tag{7.12}$$

where C' depends on $n, \kappa, \alpha, \lambda, \Lambda$. We actually still have (7.12) in Case 1. Since (7.11) is better than (7.12), we proved that if $r \le r_2$, (7.12) holds for almost every $x \in \Omega$ with $B(x, K_2 r) \subset \Omega$.

Now let $x_0 \in \Omega$ and $r_0 < r_2$ be such that $B(x_0, K_2 r_0) \subset \Omega$. For almost every $x \in B(x_0, r_0)$, (7.12) holds with $r = r_0/2$ (so that $B(x, K_2 r) \subset B(x_0, K_2 r_0)$) and so

$$|\nabla u(x)| \le C'\omega(u_x, x, r) + C' \le 2^{n/2}C'\omega(u_x, x_0, 2r_0) + C'. \tag{7.13}$$

Since we already know that u is in the Sobolev space $W^{1,2}_{\text{loc}}(B(x_0, r_0))$, we deduce from (7.13) that u is Lipschitz in $B(x_0, r_0)$ and (7.4) holds, proving Theorem 7.1.

8 Almost Monotonicity

In this section, we describe an analogue of the Alt–Caffarelli–Friedman [4] monotonicity formula for variable coefficient almost minimizers, which we obtained in [10]. Recall, for the remainder of this section, the notation $f^{\pm} = \max\{\pm f, 0\}$. In [4], it was shown that the quantity

$$\begin{aligned} \Phi(f, y, r) &\equiv \frac{1}{r^4}\left(\int_{B(y,r)} \frac{|\nabla f^+|^2}{|z-y|^{n-2}}dz\right)\left(\int_{B(y,r)} \frac{|\nabla f^-|^2}{|z-y|^{n-2}}dz\right) \\ &\equiv \frac{1}{r^4}\Phi_+(f, y, r)\Phi_-(f, y, r) \end{aligned} \tag{8.1}$$

is monotone increasing in r as long as $f(y) = 0$ and f is harmonic. While we cannot expect to get the same monotonicity, we proved an almost-monotonicity result in the style of [17].

We first proved the following estimate in [10].

Lemma 8.1 *Let u be an almost minimizer for J in Ω, and assume that $B(x, 2r) \subset \Omega$, where x is such that $A(x) = I$. Let $\varphi \in W^{1,2}(\Omega) \cap C(\Omega)$ be such that $\varphi(y) \ge 0$ everywhere, $\varphi(y) = 0$ on $\Omega \backslash B(x, r)$, and let $\lambda \in \mathbb{R}$ be such that*

$$|\lambda\varphi(y)| < 1, \text{ on } \Omega. \tag{8.2}$$

Then, for each choice of sign, $\pm$*,*

$$0 \le Cr^\alpha J_{x,r}(u) + Cr^{\alpha+n} + 2\lambda\left[\int_{B(x,r)} \varphi|\nabla u^\pm|^2 + \int_{B(x,r)} u^\pm\left\langle \nabla u^\pm, \nabla\varphi\right\rangle\right]$$
$$+ \lambda^2\left[\int_{B(x,r)} \varphi^2|\nabla u^\pm|^2 + (u^\pm)^2|\nabla\varphi|^2 + 2\varphi u^\pm\left\langle\nabla u^\pm, \nabla\varphi\right\rangle\right], \tag{8.3}$$

where $C < \infty$ *is a constant that depends only on* $\kappa, n, \Lambda, \lambda$ *and the* $C^{0,\alpha}$ *norm of* A.

The following are the variable coefficient analogues of Lemmas 6.2, 6.3, and 6.4 in [17]. The proofs in [17] use Lemma 6.1, the continuity of almost minimizers, and the logarithmic growth of $\omega(x, r)$. In particular, the proofs go through virtually unchanged for almost minimizers with variable coefficients.

Lemma 8.2 *Still assume that* $n \ge 3$*. Let* u *be an almost minimizer for* J *in* Ω *and assume that* $B(x_0, 4r_0) \subset \Omega$ *and that* $u(x_0) = 0$ *and* $A(x_0) = I$*. Then, for* $0 < r < \min(1, r_0)$ *and for each choice of sign,* $\pm$*,*

$$\left|\frac{c_n}{r^2}\Phi_\pm(u, x_0, r) - \frac{1}{n(n-2)}\fint_{B(x_0,r)} |\nabla u^\pm|^2 - \frac{1}{2}\fint_{\partial B(x_0,r)} \left(\frac{u^\pm}{r}\right)^2\right|$$
$$\le Cr^{\frac{\alpha}{n+1}}\left(1 + \fint_{B(x_0,\tilde{C}r_0)} |\nabla u^2| + \log^2(r_0/r) + \log^2(1/r)\right). \tag{8.4}$$

Again, $c_n = (n(n-2)\omega_n)^{-1}$ *and* $C > 0$ *depending only on* $n, \Lambda, \lambda, \|A\|_{C^{0,\alpha}}$ *and the almost-minimizing constants of* u.

Lemma 8.3 *Let* u *be an almost minimizer for* J *in* Ω*, and assume that* $B(x_0, 4r_0) \subset \Omega$ *with* $u(x_0) = 0$ *and* $A(x_0) = I$*. For* $0 < r < \frac{1}{2}\min(1, r_0)$*, set* $t \equiv t(r) \equiv \left(1 - \frac{r^{\alpha/4}}{10}\right)r$*. Then for* $0 < r < \min(1/2, r_0)$ *and each choice of sign,* $\pm$*,*

$$\left|\fint_{t(r)}^r \left(\int_{B(x_0,s)} |\nabla u^\pm(y)|^2 dy\right) ds - \fint_{t(r)}^r \left(\int_{\partial B(x_0,s)} u^\pm \frac{\partial u^\pm}{\partial n}\right) ds\right|$$
$$\le Cr^{n+\alpha/4}\left(1 + \fint_{B(x_0,\tilde{C}r_0)} |\nabla u|^2 + \log^2\frac{r_0}{r}\right). \tag{8.5}$$

Here, $\partial u^\pm/\partial n$ *denotes the radial derivative of* $u^\pm$ *and* $C > 0$ *depend only on* $\|q_\pm\|_\infty, n, \Lambda, \lambda, \|A\|_{C^{0,\alpha}}$*, and the almost-minimization constants.*

Finally, we obtain an Alt–Caffarelli–Friedman almost-monotonicity type result.

Theorem 8.1 *Let u be an almost minimizer for J in Ω and let δ be such that $0 < \delta < \alpha/4(n+1)$. Let $B(x_0, 4r_0) \subset \Omega$ with $u(x_0) = 0$ and $A(x_0) = I$. Then there exists $C > 0$, depending on the usual parameters such that for $0 < s < r < \frac{1}{2}\min(1, r_0)$,*

$$\Phi(u, x_0, s) \leq \Phi(u, x_0, r) + C(x_0, r_0)r^{\delta}, \tag{8.6}$$

where

$$C(x_0, r_0) \equiv C + C\left(\fint_{B(x_0, 2r_0)} |\nabla u|^2\right)^2 + C((\log r_0)_+)^4. \tag{8.7}$$

9 Local Lipschitz Continuity for Two-phase Almost Minimizers

The proof of two-phase Lipschitz continuity follows the same blueprint as the one-phase case. We start with Lemma 9.1 that is an analogue of Lemma 7.1. However, the proof of Lemma 9.1 is a bit more involved as it requires the use of the two-phase almost-monotonicity formula, (8.6), to control oscillations. We state the appropriate version of that lemma here.

Lemma 9.1 *Let u be an almost minimizer for J in Ω and let $B_0 \equiv B(x_0, \lambda^{-1/2}r_0) \subset \Omega$ be given. Let $\theta \in (0, 1/3)$ and $\beta \in (0, 1)$. Then there exists $\gamma > 0$, $K_1 > 1$ and $r_1 > 0$ (which may depend on θ and β) such that if $x \in B(x_0, r_0)$ and $0 < r \leq r_1$ satisfy*

$$u_x(y) = 0 \text{ for some } y \in B(x, 2r/3), \tag{9.1}$$

$$|b(u_x, x, r)| \leq \gamma r(1 + \omega(u_x, x, r)), \text{ and} \tag{9.2}$$

$$\omega(u_x, x, r) \geq K_1. \tag{9.3}$$

Then,

$$\omega(u_x, x, \theta r) \leq \beta\omega(u_x, x, r). \tag{9.4}$$

We are now ready to prove our main result.

Theorem 9.1 *Let u be an almost minimizer for J in Ω. Then u is locally Lipschitz in Ω.*

The goal is actually to show a more precise estimate: that there exist $r_2 > 0$ and $C_2 \geq 1$ (depending on $n, \kappa, \alpha, \lambda, \Lambda$) such that for each choice of $x_0 \in \Omega$ and $r_0 > 0$ such that $r_0 \leq r_2$ and $B(x_0, K_2 r_0) \subset \Omega$ (with K_2 as in Lemma 6.4), then

$$|u(x) - u(y)| \leq C_2(\omega(u_{x_0}, x_0, 2r_0) + 1)|x - y| \text{ for } x, y \in B(x_0, r_0). \tag{9.5}$$

Idea of the Proof Let (x, r) be such that $B(x, K_2 r) \subset \Omega$. We want to use the different lemmas above to find a pair (x, ρ) that allows us to control u. Pick $\theta = 1/3, \beta = 1/2$ (smaller values would work as well), and let γ, K_1, r_1 be as in Lemma 9.1.

Pick $\tau = \tau_2/2$, where $\tau_2 \in (0, \tau_1)$, where τ_1 is the constant that we get in Lemma 6.1 applied with $C_1 = 3$ and $r_0 = r_1$. Here τ_2 is the corresponding constant that appears in Lemma 6.3. Let now r_0, η, K be as in Lemma 6.4 applied to $C_0 = 10$, and to τ and γ as above. From Lemma 6.4, we get a small r_γ. Set

$$K_3 \geq \max(K_1, K), \quad \text{and} \quad r_2 \leq \min(r_1, r_\gamma). \tag{9.6}$$

Let $r \leq r_2$. In the case of the two-phase problem, one has to consider four cases:

Case 0:

$$u_x(z) \neq 0, \qquad \forall z \in B(x, 2r/3). \tag{9.7}$$

Case 1: $u_x(z) = 0$ for some $z \in B(x, 2r/3)$ and

$$\begin{cases} \omega(u_x, x, r) \geq K_3 \\ b(u_x, x, r) \geq \gamma r(1 + \omega(u_x, x, r)). \end{cases} \tag{9.8}$$

Case 2: $u_x(z) = 0$ for some $z \in B(x, 2r/3)$ and

$$\begin{cases} \omega(u_x, x, r) \geq K_3 \\ b(u_x, x, r) < \gamma r(1 + \omega(u_x, x, r)). \end{cases} \tag{9.9}$$

Case 3: $u_x(z) = 0$ for some $z \in B(x, 2r/3)$ and

$$\omega(u_x, x, r) < K_3. \tag{9.10}$$

The proof now follows the same general strategy of that of Theorem 7.1.

Acknowledgments We thank an anonymous referee for providing valuable feedback on a preliminary version of the paper.

References

1. L. Ambrosio, Corso introduttivo alla teoria geometrica della misura ed alle superfici minime, Appunti dei Corsi Tenuti da Docenti della Scuola. [Notes of Courses Given by Teachers at the School], Scuola Normale Superiore, Pisa, 1997 (Italian).
2. G. Anzellotti, On the $C^{1,\alpha}$ regularity of ω-minima of quadratic functionals, *Boll. Unione Mat. Ital., VI. Ser., C, Anal. Funz. Appl.* **2** (1983), 195–212.
3. H. W. Alt & L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* **325** (1981), 105–144.
4. H. W. Alt, L. A. Caffarelli & A. Friedman, Variational problems with two phases and their free boundaries, *Trans. Amer. Math. Soc.* **282** (1984), 431–461.
5. F. J. Almgren Jr., Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. Amer. Math. Soc.* 4 (1976), no. 165.
6. E. Bombieri, Regularity theory for almost minimal currents, *Arch. Rational Mech.* Anal. 78 (1982), no. 2, 99–130.
7. L.A. Caffarelli, D. Jerison & C. Kenig C, Global energy minimizers for free boundary problems and full regularity in three dimension, *Contemp. Math.*, 350, Amer. Math. Soc., Providence, RI (2004), 83–97.
8. L. A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. Part III: existence theory, compactness, and dependence on X. *Ann. del. Sc. Norm. Sup. di Pisa* **15** (1988) 583–602.
9. G. David, M. Engelstein & T. Toro, Free Boundary Regularity for almost-minimizers, *Adv. Math*, 350, (2019), 1109–1192.
10. G. David, M. Engelstein, M. Smit Vega Garcia & T. Toro, Regularity for almost-minimizers of variable coefficient Bernoulli-type functionals. *Math. Z.* 299, 2131–2169 (2021).
11. G. David, M. Engelstein, M. Smit Vega Garcia & T. Toro, Branch Points for (Almost-)Minimizers of Two-Phase Free Boundary Problems. arXiv:2204.05367 (2022).
12. D. De Silva & D. Jerison, A Singular Energy Minimizing Free Boundary, *J. Reine Angew. Math.,* 635 (2009), 1–22.
13. G. De Philippis, L. Spolaor & B. Velichkov, Regularity of the free boundary for the two-phase Bernoulli problem, *Invent. Math.* 225, 347–394 (2021).
14. D. De Silva & Ovidiu Savin, Thin one-phase almost minimizers. *Nonlinear Anal.* 193 (2020), 111507, 23 pp.
15. D. De Silva & O. Savin, Almost minimizers of the one-phase free boundary problem. *Comm. Partial Differential Equations* 45 (2020), no. 8, 913–930.
16. F. Duzaar, A. Gastel & J. F. Grotowski, Partial regularity for almost minimizers of quasi-convex integrals. *SIAM J. Math. Anal.* 32 (2000), no. 3, 665–687.
17. G. David & T. Toro, Regularity of almost minimizers with free boundary, *Calc. Var.* 54, no. 1, 455–524 (2015).
18. L. Esposito & G. Mingione, A regularity theorem for ω-minimizers of integral functionals, *Rend. Mat. Appl.* (7) 19 (1999), no. 1, 17–44.
19. S. Jeon & A. Petrosyan, Almost minimizers for the thin obstacle problem. *Calc. Var.* 60, 124 (2021).
20. S. Jeon & A. Petrosyan, Almost minimizers for certain fractional variational problems. *Algebra i Analiz*, 32 (2020), no. 4, 166–199.
21. S. Jeon, A. Petrosyan & M. Smit Vega Garcia, Almost minimizers for the thin obstacle problem with variable coefficients, arXiv:2007.07349.

22. D. Jerison & O. Savin, Some remarks on stability of cones for the one-phase free boundary problem. *Geom. Funct. Anal.* 25 (2015), no. 4, 1240–1257.
23. L. Esposito, F. Leonetti & G. Mingione, Sharp regularity for functionals with (p,q) growth. *J. Differential Equations* 204 (2004), no. 1, 5–55.
24. J. Goblet & W. Zhu, Regularity of Dirichlet nearly minimizing multiple-valued functions, J. Geom. Anal. 18 (2008), no. 3, 765–794.
25. O. S. de Queiroz & L. S. Tavares, Almost minimizers for semilinear free boundary problems with variable coefficients, *Mathematische Nachrichten.* **291** (2018), 1486–1501.
26. M. Giaquinta & E. Giusti, On the regularity of the minima of variational integrals, *Acta Math.* 148 (1982), 31–46.
27. M. Giaquinta & E. Giusti, Quasiminima, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1 (1984), no. 2, 79–107.
28. J. Lamboley & P. Sicbaldi, Existence and regularity of Faber Krahn minimizers in a Riemannian manifold, preprint, arXiv: 1907.08159.
29. G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, *Appl. Math.* 51 (2006), no. 4, 355–426.
30. L. Spolaor, B. Trey & B. Velichkov, Free boundary regularity for a multiphase shape optimization problem, preprint, arXiv: 1810.06963.
31. B. Trey, Lipschitz continuity of the eigenfunctions on optimal sets for functionals with variable coefficients, preprint, arXiv: 1909.12597.
32. G. Weiss, Partial regularity for a minimum problem with free boundary, *Journal Geom. Anal.* 9 (1999), 317–326.