Fractional quantum Hall effect in higher dimensions

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Generalizing from previous work on the integer quantum Hall effect, we construct the effective action for the analog of Laughlin states for the fractional quantum Hall effect in higher dimensions. The formalism is a generalization of the parton picture used in two spatial dimensions, the crucial ingredient being the cancellation of anomalies for the gauge fields binding the partons together. Some subtleties which exist even in two dimensions are pointed out. The effective action is obtained from a combination of the Dolbeault and Dirac index theorems. We also present expressions for some transport coefficients such as Hall conductivity and Hall viscosity for the fractional states.

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I. INTRODUCTION

The phenomenon of the quantum Hall effect (QHE) hardly needs any stress on its importance as it has been the topic of intense investigations, both theoretically and experimentally, over the last several decades [1]. While most of the research has focused on two dimensions, already several years ago, the enticing mathematical structure of QHE prompted suggestions on generalizations to higher dimensions. Even though these seemed to be mathematical curiosities initially, it is interesting that QHE in higher dimensions may in fact be experimentally realizable using the idea of synthetic dimensions [2,3]. The initial proposal for higher dimensional QHE considered space as a 4-sphere S^4 [4]. Shortly after this, QHE on complex manifolds of arbitrary dimensions were analyzed. The explicit solution of the Landau problem, the construction of integer quantum Hall states, the analysis of the edge excitations, etc. were carried out leading to a uniform extension to all higher even dimensions [5–7], see also [8,9]. A specific case of odd dimensions was also investigated [10]. To a large extent, the problem is defined by topological considerations. Since the lowest Landau level obeys a certain holomorphicity condition, the Dolbeault index theorem can be used to analyze many features of the phenomenon [11]. It is then possible to show that a Chern-Simons action associated to the Dolbeault index density describes the bulk dynamics of a QHE droplet of fermions (for integer filling fractions), including fluctuations of gauge and gravitational fields, as well as Abelian and non-Abelian background magnetic fields [12]. Needless to say, the specialization of this general effective action to $2+1$ dimensions agrees with explicit derivations based on wave functions carried out by many authors [13–17]. Higher dimensions also allow for an enlarged set of transport coefficients. Some of these were recently worked out in [18], where it was also explained how the band structures of the electrons could be incorporated in the effective action.

All the higher dimensional generalizations considered so far have been for the integer QHE. In $2+1$ dimensions, we also have a wealth of information regarding the fractional QHE, including the wave functions for many of the experimentally realized states (such as the Laughlin, Jain, and Moore-Read states), effective actions for the bulk and boundary dynamics of a droplet of fermions, various transport coefficients, etc. In this paper, we consider the question of how quantum Hall states can be defined in higher dimensions for fractional filling, in particular higher dimensional analogs of the $\nu = 1/m$ Laughlin states in $2+1$ dimensions, where m is an odd integer. This is a natural next step for QHE and can be particularly relevant in the light of potential experimental realizations in higher dimensions. In $2+1$ dimensions, a variety of methods exist to map the fundamental excitations, namely electrons, to composite particles whose known states map to fractional excitations for the electrons. One standard approach is that

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of flux attachment; see Ref. [19] for pedagogical introductions to the topic of flux attachment, and [20,21] for reviews of aspects of flux attachment applied to various contemporary issues. A complementary approach available in $2+1$ dimensions is that of the parton construction of the fundamental electrons [22,23].

While flux attachment is very natural in two spatial dimensions, so far we have not been able to find a workable extension to higher dimensions. Therefore in this paper we will consider the generalization of the parton picture. In this scenario, the fundamental fermion, i.e., the electron is viewed as a composite particle made of m partons, one parton each of m species. If ψ denotes the electron field, and q_i, $i = 1, 2, \ldots, m$, denote the parton fields, then this is equivalent to the statement

$$\psi \sim q_1 q_2 \cdots q_m. \tag{1}$$

The partons themselves are taken to be fermions of charge e/m and m is taken to be an odd positive integer, i.e., $m = 2l + 1$, $l = 0, 1, \ldots$, to obtain fermionic statistics for the composite particle. (We hasten to add that the partons themselves are not viewed as physical entities, rather this construction is a convenient mathematical device that encodes some of the multiparticle strong coupling effects, in a way that is not yet fully understood.) One then constructs a state where the partons, in the external magnetic field, form an integer quantum Hall state, say, with filling fraction equal to 1, i.e., $\nu = 1$, for simplicity. In terms of the electrons this state may be viewed as a state of filling fraction $\nu = 1/m$. For this strategy to be consistent, the occupation number in the given $\nu = 1$ state for each species of partons should be the same, i.e., $n_1 = n_2 = \cdots = n_m \equiv n$, so that we have n electrons, with one parton of each kind for each electron. The equality of the n_is is enforced by use of a set of $U(1)$ gauge fields, which we will refer to as the b fields in this paper. The same fields can also be taken to be the agency binding the partons to form the electron as in (1).

The use of the b fields to bind the partons and the use of the equations of motion for time components of the b fields to obtain the equality of the n_is show clearly that they are dynamical fields, unlike the external magnetic field or geometrical characteristics such as the metric and spin connection of the manifold. In a functional integral approach, they are therefore to be integrated out. This can be formally carried out in $2+1$ dimensions, where, because the action is a Chern-Simons 3-form and quadratic in the Abelian b fields, the integration can be done in closed form, leading to a framing anomaly [16]. In the resulting effective action, this is equivalent to a gravitational Chern-Simons term, and hence to an additional gravitational anomaly for the edge modes of a quantum Hall droplet. The generalization of the framing anomaly calculation to higher dimensions is not straightforward, but, as we shall now argue, there is an alternate way of viewing the procedure of integrating out the b fields.

If we consider a quantum Hall droplet, then the bulk Chern-Simons action is not gauge invariant. The complete effective action is gauge invariant because there are edge modes on the boundary of the droplet which cancel the variation of the bulk terms under gauge transformation of the electromagnetic field A and the spin connection ω. In the case of integer QHE this cancellation helps to identify the nature of the edge excitations. When we consider partons coupled to the b fields, again, the bulk action is not gauge invariant. This will lead to nonzero terms on the edge under the gauge transformation of the b fields as well. While physical edge excitations, as before, can cancel the terms due to the gauge variation of A, ω, physical excitations should not carry b-charges since the partons and the b-fields constitute only a theoretical trick to incorporate certain nonperturbative effects. Therefore, we consider a set of auxiliary fields, which we will refer to as the spectator fields (using the terminology of 't Hooft), which have an anomaly on the edge that can cancel the variation of the parton bulk action under the gauge transformation of the b fields. (As we shall see shortly, the spectator fields will be chiral spinors.) With this "anomaly cancellation," the dynamics of b fields is made consistent with no leftover Chern-Simons (CS)-type terms for the b fields and one can integrate out the b fields without worrying about a framing anomaly. However, the spectator fields can contribute to the gravitational anomaly and this indeed captures the effect of framing anomaly calculations. This strategy of anomaly cancellation provides a way to bypass the intricacies of deriving the framing anomaly even for $2+1$ dimensions. Further, it can be generalized to higher dimensions.

We emphasize that the spectator fields are defined on the boundary. But we can associate a bulk action with the spectators, since their anomaly (which is on the edge), via the standard descent procedure, can be expressed as the gauge variation of a bulk Chern-Simons term. This bulk term is a convenient way to encode the anomaly of the spectator fields. The bulk action due to the partons plus the bulk term associated with the spectator fields will then give the bulk effective action for the fractional quantum Hall state of interest, but the cancellation of anomalies is really implemented on the boundary of the droplet.

Our strategy for generalization of fractional QHE to higher dimensions will thus be as follows. We will consider the electron or the fundamental fermion to be made of m partons. These partons will be coupled to a set of $U(1)$ b fields. The parton fields obey a holomorphicity condition since we restrict them to a particular Landau level. Therefore we can use the Dolbeault index theorem to obtain the bulk action of the partons, along the lines of our earlier work [12].

The spectator fields do not couple to the electromagnetic field and so do not fall into Landau levels. Therefore, the anomaly for such fields must arise as in standard field theory. We will take the spectators to be chiral spinor fields as they are the ones which can generate a gauge anomaly.[1] The anomaly due to the spectators can then be calculated using the Dirac index theorem.

We emphasize that there is a distinction in the use of the index theorem for the partons and the spectators. The Dolbeault index theorem leads to the bulk action for the partons because they obey a holomorphicity condition. The Dirac index theorem for the spectators is used to calculate a gauge anomaly.

Once the anomaly cancellation has been ensured, the remaining Chern-Simons terms, involving the external magnetic field and the spin connections, can be assembled to give the effective action for the fractional quantum Hall state. This is basically made of the leftover terms in the parton bulk action plus the bulk action which encodes the anomaly due to the spectators. Transport coefficients such as the Hall conductivity, the Hall viscosity, etc. can then be read off this effective action.

In the next section we will give a brief summary of how the Dolbeault index theorem can be used to derive the bulk effective action for integer QHE. This is to help set the stage for the subsequent analysis. We also indicate how the anomaly due to the spectator fields can be identified. In Sec. III we will consider the Laughlin states (of filling fraction $\nu = 1/m$) in $2+1$ dimensions. Primarily this is to show how our modified parton picture recovers known results in $2+1$ dimensions. However, it will also highlight some subtleties in the application of the parton picture, even in $2+1$ dimensions, to states with other values for the filling fraction. Further it will lay out the necessary mathematical steps for generalization to higher dimensions and the subtleties to be aware of. In Sec. IV we will consider the construction of Laughlin-type states in $4+1$ dimensions. Anomaly cancellation for the dynamical b fields will again be the guiding principle. The effective action, valid for generic four-dimensional spatial manifolds, is worked out, see (56), (57). The electromagnetic current and the energy-momentum tensor can then be read off, they are given in (58), (61), (62). It is then straightforward to write down the Hall conductivity and Hall viscosity, which are the primary response functions of interest. A key point worthy of remark is that the leading term of the Hall conductivity has a factor $1/m^2$ in four dimensions, rather than $1/m$ as in two dimensions. We analyze fractional quantum Hall effect (FQHE) on the complex manifold $S^2 \times S^2$, as a special case of our general result, and comment on the dimensional reduction of the $(4+1)$-dimensional results to $2+1$ dimensions. Section V gives a brief recapitulation of the key steps in our analysis as well as the key results. We conclude with a number of remarks on possible extensions of the present results.

Before we embark on the details of our construction, we note that there have been other attempts at generalizing the fractional quantum Hall effect to higher dimensions [25]. The effective action in $2+1$ dimensions can be obtained from a set of Chern-Simons actions involving 1-form fields, put together using a so-called K matrix. Elimination of some of the fields via their equations of motion leads to the effective action in terms of the electromagnetic field. In higher dimensions, a similar possibility is to consider Chern-Simons-like actions again, but using higher forms as the basic fields. This can then lead to an effective action for fractional QHE. However, in this approach the basic constituents will be extended objects of suitable dimension coupling to the higher form fields, while, in our case, the basic constituents are particles coupling to the usual electromagnetic field. Thus the two generalizations are *a priori* different; it would indeed be interesting to see if there is any way to relate them.

II. THE EFFECTIVE ACTION FROM AN INDEX THEOREM

In this section we give a brief resume of the derivation of the bulk effective action from the index theorem for integer QHE in arbitrary even spatial dimensions [12]. We consider the case where the spatial manifold is a complex manifold and thus we will be using the Dolbeault index theorem which is the relevant one for such cases. However, once we obtain the effective action, it can be extended to include small but arbitrary perturbations of the metric and spin connection which do not necessarily preserve the complex structure. This will be relevant for deriving the Hall viscosity, for example.

The single particle Hamiltonian is of the form $-\frac{1}{2}D_i\bar{D}_{\bar{i}}$, where D_i and $\bar{D}_{\bar{i}}$ are the holomorphic and antiholomorphic covariant derivatives. These include both gauge and gravitational fields; for the present work, the gauge fields will include the b fields mentioned in the introduction as well as the external electromagnetic vector potential. Ignoring the spin of the fermion (which can be the electron or the parton), its wave function is a complex function of the coordinates. For the lowest Landau level the wave functions obey the holomorphicity condition

$$\bar{D}_{\bar{i}}\Phi = 0. \tag{2}$$

The number of normalizable solutions to this equation is given by the index theorem for the twisted Dolbeault complex as

[1]A chiral boson in two dimensions can generate an anomaly, but, via fermionization, this is equivalent to a chiral spinor. A self-dual field in 4k + 2 dimensions can generate gravitational anomalies, but this is not relevant for the present paper since we consider $(2+1)$ and $(4+1)$-dimensional cases only [24]. In two dimensions, a self-dual field is again equivalent to a chiral boson.

$$\mathrm{Index}(\bar{D}) = \int_M \mathcal{I} = \int_M \mathrm{td}(T_c M) \wedge \mathrm{ch}(V), \qquad (3)$$

where $\mathrm{td}(T_c M)$ is the Todd class on the complex tangent space of the spatial manifold M and $\mathrm{ch}(V)$ is the Chern character of the relevant vector bundle [11]. The formula (3) is rather cryptic, so we will give a brief explanation of how it can be used to construct the effective action.

For a real manifold of dimension equal to $2k$, the holonomy group, which is the group corresponding to parallel transport, is $SO(2k)$. This means that the spin connections and curvatures take values in the Lie algebra of $SO(2k)$. However, for a complex manifold, we only allow coordinate transformations which preserve the complex structure, so that the notion of holomorphicity is preserved. This restricts the holonomy group to $U(k) \subset SO(2k)$. The frame fields, viewed as 1-forms, separate into holomorphic forms and antiholomorphic forms. These are combinations of the real ones given by the complex structure. A corresponding set of combinations can be made for the tangent space, which is, after all, dual to the forms. This leads to $T_c M$. The Todd class in (3) is given in terms of the curvature 2-form for $T_c M$. For any vector bundle with curvature $\mathcal{F}$, the Chern classes are defined by[2]

$$\det\left(1 + \frac{i\mathcal{F}}{2\pi} t\right) = \sum_i c_i t^i. \qquad (4)$$

One can then give an expansion of the Todd class in terms of the Chern classes as [11]

$$\mathrm{td} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 + \frac{1}{720}(-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) + \cdots. \qquad (5)$$

A general expression for the Todd class valid for all dimensions (and hence including higher rank differential forms) is given in [11,24]. It is best expressed in terms of what is known as the splitting principle. We do not quote it here, since, for the present purpose, Eq. (5) will suffice.

The field $\mathcal{F}$ in (4) is given by the curvature 2-form R for $T_c K$. Explicit formulas for the first few Chern classes are then as follows:

$$\begin{aligned}
c_1(T_c K) &= \mathrm{Tr}\frac{iR}{2\pi},\\
c_2(T_c K) &= \frac{1}{2}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2 - \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right],\\
c_3(T_c K) &= \frac{1}{3!}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^3 - 3\mathrm{Tr}\frac{iR}{2\pi}\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 + 2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^3\right],\\
c_4(T_c K) &= \frac{1}{4!}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^4 - 6\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 + 8\mathrm{Tr}\frac{iR}{2\pi}\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^3 + 3\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 - 6\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^4\right].
\end{aligned} \qquad (6)$$

Since the curvatures R take values in the Lie algebra of $U(k)$, the traces in the above formulas are over this algebra. Explicitly, we can write $iR = d\omega^0 \mathbb{1} + R^a t_a$, where 1 and t_a form a Hermitian basis for the $U(k)$ algebra and t_a are normalized so that $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. ω^0 is the Abelian part of the $U(k)$ spin connection.

The Chern character $\mathrm{ch}(V)$ in the index formula (3) involves the gauge fields with the trace defined over the representations of the gauge group to which the matter fields belong. It is given by

$$\mathrm{ch}(V) = \mathrm{Tr}(e^{i\mathcal{F}/2\pi}) = \dim V + \mathrm{Tr}\frac{i\mathcal{F}}{2\pi} + \frac{1}{2!}\mathrm{Tr}\frac{i\mathcal{F} \wedge i\mathcal{F}}{(2\pi)^2} + \cdots, \qquad (7)$$

where $\dim V$ is the dimension of the bundle V, i.e., the number of matter fields. $\mathcal{F}$ is the gauge field strength, it includes the external magnetic field and the b fields in the present case. (We will also be considering only Abelian gauge fields in this paper, for simplicity, although the methodology is applicable to non-Abelian fields as well.)

Since the Dolbeault index gives the degeneracy of the lowest Landau level (LLL), it is also the charge of the $\nu = 1$ state if we assign unit charge to the particles. Taking the charge density of the LLL as J_0, we may then identify

$$\frac{\delta S_{\mathrm{eff}}}{\delta A_0} = J_0 = \text{Index density}, \qquad (8)$$

where the "index density" in the above expression is the integrand of (3). We can then "integrate up" from this formula to identify most of the terms in S_{eff}. This was the strategy used in [12]. The leading term in S_{eff} will be a Chern-Simons term $CS(A)$ whose variational derivative with respect to A_0 is the index density. There can be subleading terms which correspond to dipole and higher multipole terms in J_0, which can lead to terms involving derivatives of the fields in the effective action; such terms are nontopological in nature. We will focus here on the

[2]We start with connections and curvatures in an anti-Hermitian basis since they are natural allowing us to write $F = dA + AA$, etc. This leads to some factors of i in various expressions at this stage, since the Hermitian fields are $i\mathcal{F}$, iR. Later we will move to a Hermitian basis.

topological terms. We have also discussed in [12] how purely gravitational terms can be added to this procedure. The end result is the following.

We start from the term in the Dolbeault index density in (3) corresponding to the $(2k+2)$-form, say, $\mathcal{I}_{2k+2}$. We can then obtain an associated Chern-Simons form $(\Lambda)_{2k+1}$ by writing

$$\mathcal{I}_{2k+2} = \frac{1}{2\pi} d(\Lambda)_{2k+1}. \tag{9}$$

The effective action is then given by the integral of $(\Lambda)_{2k+1}$ over the manifold $M \times \mathbb{R}$, where M is the $2k$-dimensional spatial manifold and $\mathbb{R}$ denotes the time direction. Written out, this is of the form

$$S_{\rm eff} = \int \Lambda_{2k+1} = \int \Big[{\rm td}(T_c M) \wedge \sum_p (CS)_{2p+1}(A) \Big]_{2k+1} + 2\pi \int \Omega^{\rm grav}_{2k+1}. \tag{10}$$

The integrand on the right-hand side of this equation identifies the explicit form of Λ_{2k+1}. Here $(CS)_{2p+1}(A)$ is the Chern-Simons term associated with just the gauge part and is defined by

$$\frac{1}{2\pi} d(CS)_{2p+1} = \frac{1}{(p+1)!} {\rm Tr} \left(\frac{i\mathcal{F}}{2\pi} \right)^{p+1}. \tag{11}$$

One should expand the terms in the square brackets in (10) in powers of curvatures and F and pick out the term corresponding to the $(2k+1)$-form. The subscript $2k+1$ for the square brackets is meant to signify this. The purely gravitational term $\Omega^{\rm grav}_{2k+1}$ in (10) is defined by

$$[{\rm td}(T_c M)]_{2k+2} = d\Omega^{\rm grav}_{2k+1}. \tag{12}$$

This method of constructing the effective action can be extended to include higher Landau levels for some special cases. It is equivalent to considering fields of nonzero spin in the lowest Landau level. However, since we do not need it for this article, we refer the interested reader to [12] for details.

Turning to the spectator fields, note that anomalies in $2k$ dimensions are obtained from the index density (or index polynomial) in $(2k+2)$ dimensions via the descent procedure. (See Ref. [24] for a general discussion of the descent procedure.) For the spectator fields, which are chiral spinors, we will need the Dirac index. This is given by

$$\text{Dirac Index} = \int \hat{A}(M) \wedge {\rm ch}(V) = \int \hat{A}(M) \wedge {\rm Tr}(e^{\frac{i\mathcal{F}}{2\pi}}). \tag{13}$$

Here $\hat{A}(M)$ is the $\hat{A}$ genus which has the expansion

$$\hat{A}(M) = 1 - \frac{1}{24}(c_1^2 - 2c_2) + \cdots, \tag{14}$$

where the ellipsis denotes higher forms. Also the gauge field will be just made of the b fields, so that $i\mathcal{F} = Q^{(n)} db^{(n)}$, $Q^{(n)}$ being the charge for coupling to $b^{(n)}$ for the spinor of interest.

III. THE PARTON CONSTRUCTION IN 2+1 DIMENSIONS AND ANOMALY CANCELLATION

We now turn to the construction of the Laughlin states of filling fraction $\nu = 1/m$ in $2+1$ dimensions using the parton picture. We consider m parton fields q_i, $i = 1, 2, \ldots, m$. The partons are coupled to the external electromagnetic field with charge $1/m$; we absorb e into the gauge field A_μ. In addition, the partons couple to a set of b fields. We will need at least $(m-1)$ $U(1)$ gauge fields to ensure the equality of all the parton occupation numbers. It is convenient to choose the charges for their coupling to the partons as proportional to the set of diagonal matrices $h^{(n)}$, $n = 1, 2, \ldots, (m-1)$, in the fundamental $m \times m$ matrix representation of $SU(m)$. In particular the $h^{(n)}$ matrices can be written as

$$h^{(n)} = \frac{1}{\sqrt{2n(n+1)}} {\rm diag}\{\underbrace{1, 1, \ldots, 1}_{n}, -n, 0, \ldots, 0\}, \quad n = 1, 2, \ldots, (m-1). \tag{15}$$

The matrices $h^{(n)}$ are traceless and are normalized as ${\rm Tr}(h^{(n)} h^{(n')}) = \frac{1}{2}\delta^{nn'}$. Although we use this language, we do not have an $SU(m)$ gauge theory, we are only using the diagonal matrices, i.e., the Cartan subalgebra, so the gauge group for the b fields is $U(1)^{m-1}$. [A generalization to using an $SU(m)$ gauge theory is possible, but not particularly useful in this context.]

Turning to the effective action, note that in $2+1$ dimensions, we need the term $\mathcal{I}_4$ corresponding to the 4-form in the Dolbeault index density from (3) with

$$i\mathcal{F} = \frac{dA}{m} \mathbb{1} + \sum_{n=1}^{m-1} db^{(n)} h^{(n)}. \tag{16}$$

It is straightforward to see that $\mathcal{I}_4$ is given by

$$\begin{aligned}\mathcal{I}_4 &= \frac{1}{2!} {\rm Tr} \frac{i\mathcal{F} \wedge i\mathcal{F}}{(2\pi)^2} + \frac{c_1}{2} {\rm Tr} \frac{i\mathcal{F}}{(2\pi)} + \frac{c_1^2 + c_2}{12} \dim V, \\ &= \frac{1}{2\pi} \left[\frac{1}{4\pi m} dA dA + \frac{1}{4\pi} dA d\omega + \frac{m}{24\pi} d\omega d\omega + \frac{1}{8\pi} \sum_{n=1}^{m-1} db^{(n)} db^{(n)} \right].\end{aligned} \tag{17}$$

We have used c_1, c_2 from (6) with the simplifications appropriate to $2+1$ dimensions, i.e., $c_2 = 0$, $c_1 = d\omega^0/2\pi \equiv d\omega/2\pi$. The contribution to the bulk effective action from the partons is thus

$$S_{\text{eff}}(q) = \int \frac{1}{4\pi}\left[\frac{AdA}{m} + Ad\omega\right] + \frac{m}{24\pi}\omega d\omega + \frac{1}{8\pi}\sum_{n=1}^{m-1} b^{(n)}db^{(n)}. \tag{18}$$

The last term is what gives rise to the anomaly for the b fields on the boundary of the droplet.

A. A solution for anomaly cancellation

We now turn to the cancellation of the b anomaly. In $(2+1)$ dimensions the direct integration of the Chern-Simons action for the b fields leads to the so-called framing anomaly [16]. Effectively, this introduces an additional gravitational Chern-Simons term. As mentioned in the Introduction, we shall follow a different strategy. Integration over the b fields is straightforward, as in any gauge theory without gauge anomalies, if we cancel out the anomaly for the b fields. This can be done by introducing a set of auxiliary fields coupled to b fields but not to the electromagnetic fields. This method of anomaly cancellation has the advantage of being generalizable to higher dimensions where integrating out higher Chern-Simons forms for the b fields in a nontrivial gravitational background is not straightforward.

We will first give a particular solution and then discuss in what sense this would be the minimal solution. Consider $(m-1)$ spectator spinors denoted by χ_i, $i = 1, 2, \ldots, (m-1)$, which are of right-handed chirality (i.e., of opposite chirality compared to the partons) and couple only to the b fields. They have the same charges as the partons for the $b^{(n)}$ gauge fields, $n = 1, 2, \ldots, (m-2)$ and a charge $Q/\sqrt{2m(m-1)}$ for $b^{(m-1)}$. The χ fields do not couple to the electromagnetic field. Therefore the corresponding $\mathcal{F}$ in (13) is of the form

$$i\mathcal{F} = \sum_{n=1}^{m-2} db^{(n)}\tilde{h}^{(n)} + \frac{Q db^{(m-1)}}{\sqrt{2m(m-1)}}\mathbb{1}, \tag{19}$$

where $\tilde{h}$ are traceless $(m-1)\times(m-1)$ matrices, such that $\text{Tr}(\tilde{h}^{(n)}\tilde{h}^{(n')}) = \frac{1}{2}\delta^{nn'}$. They are as in (15) with n ranging from 1 to $(m-2)$. [In other words they are identical to $h^{(n)}$ with the range of n truncated at $(m-2)$.] Using this, we find that the four-dimensional Dirac index density for this case is

$$\text{Dirac } \mathcal{I}_4 = \frac{1}{2(2\pi)^2}\left[\sum_{n=1}^{m-2}\frac{1}{2}db^{(n)}db^{(n)} + \frac{Q^2}{2m}db^{(m-1)}db^{(m-1)}\right] - \frac{(m-1)}{24}\frac{d\omega d\omega}{(2\pi)^2} \tag{20}$$

and the corresponding effective action for the χ fields is

$$S_{\text{eff}}(\chi) = -\int\left[\frac{1}{8\pi}\sum_{n=1}^{m-2} b^{(n)}db^{(n)} + \frac{Q^2}{8\pi m}b^{(m-1)}db^{(m-1)}\right] + (m-1)\int\frac{\omega d\omega}{48\pi}. \tag{21}$$

The last term in this expression is the purely gravitational contribution from $\hat{A}(M)$.

Notice that the anomalous terms for $b^{(n)}$ for $n = 1, 2, \ldots, (m-2)$, cancels out between (18) and (21). In order to cancel the anomalous term for $b^{(m-1)}$ we introduce two additional spinors, one of left chirality denoted by χ' and one of right chirality denoted by χ''. These fields couple to $b^{(m-1)}$ and gravity. Their $b^{(m-1)}$ charges will be $\alpha/\sqrt{2m(m-1)}$ for χ' and $\beta/\sqrt{2m(m-1)}$ for χ''. Since we have one of each chirality, the gravitational contribution from $\hat{A}(M)$ will cancel out and we find

$$S_{\text{eff}}(\chi', \chi'') = \int \frac{\alpha^2 - \beta^2}{2m(m-1)}\frac{1}{4\pi}b^{(m-1)}db^{(m-1)}. \tag{22}$$

Combining terms (18), (21), and (22) we find the anomaly cancellation condition

$$\frac{1}{8\pi}b^{(m-1)}db^{(m-1)}\left[1 - \frac{Q^2}{m} + \frac{\alpha^2}{m(m-1)} - \frac{\beta^2}{m(m-1)}\right] = 0. \tag{23}$$

This has the solution

$$Q = \pm(2l+1), \qquad \alpha = \pm 2l(l+1), \qquad \beta = \pm 2l^2, \qquad m = 2l+1, \tag{24}$$

where we write $m = 2l+1$, since it is an odd integer.

The choice of a particular sign for α, β in this solution can be motivated by consideration of possible composite fields. The electron or the physical fermion is made up of the partons, so it can be represented by the composite field

$$\psi \sim q_1 q_2 \cdots q_m. \tag{25}$$

This field ψ has the correct electric charge, and zero charge for all the b fields, i.e., for $b^{(n)}$, $n = 1, 2, \ldots, (m-1)$. The b fields could bind the spectator fields as well; for example, one could make a composite from all the χs of the form

$$\Xi \sim \mathcal{C}_{r_1 r_2 \cdots r_{m-1}}\bar{\chi}^{r_1}\bar{\chi}^{r_2}\cdots\bar{\chi}^{r_{m-1}}\chi'_s\bar{\chi}''^s \tag{26}$$

for a suitable choice of coefficients $\mathcal{C}_{r_1 r_2 \cdots r_{m-1}}$. This field has zero charge for all $b^{(n)}$, $n = 1, 2, \ldots, (m-2)$. The charge carried by Ξ for the $b^{(m-1)}$ field is

$$Q^{\Xi}_{m-1} = \frac{\alpha - \beta - Q(m-1)}{\sqrt{2m(m-1)}}. \tag{27}$$

If we require this to be zero as well, then a consistent choice of signs in the solution (24) is

$$Q = 2l + 1, \qquad \alpha = 2l(l+1), \qquad \beta = -2l^2. \tag{28}$$

The spectator fields are thus χ_i, χ', χ'' with the charges for coupling to $b^{(m-1)}$ as given in this equation.

We can now collect all the remaining terms in the effective action from (18), (21), and (22) as

$$\begin{aligned} S_{\text{eff}} &= \int \frac{1}{4\pi}\left[\frac{AdA}{m} + Ad\omega + \left(\frac{m}{4} - \frac{1}{12}\right)\omega d\omega\right], \\ &= \int \frac{1}{4\pi m}\left(A + \frac{1}{2}m\omega\right) d\left(A + \frac{1}{2}m\omega\right) - \frac{1}{48\pi}\int \omega d\omega. \end{aligned} \tag{29}$$

This result agrees with what has been calculated for the $\nu = 1/m$ Laughlin states [16,17]. Notice that the Hall conductivity (defined by the functional derivative of S_{eff} with respect to A_i) has the expected $\nu = 1/m$ behavior. The Wen-Zee term is the same as for the $\nu = 1$ case. The gravitational term is often split as shown in the second line of (29), with the last term of the second line referred to as the "pure gravitational anomaly" or even just the "gravitational anomaly." Our calculation shows the precise mathematical sense in which this distinction is to be made. As shown in [26], the Dolbeault index can be written as

$$\text{Index}(\bar{D}) = \int_M \text{td}(T_c M) \wedge \text{ch}(V) = \int_M \hat{A}(M) \wedge \text{Tr}(e^{\frac{i\mathcal{F}}{2\pi} + \frac{iR}{4\pi}}). \tag{30}$$

Since

$$\frac{i\mathcal{F}}{2\pi} + \frac{iR}{4\pi} = \frac{1}{m} d\left(A + \frac{1}{2}m\omega\right), \tag{31}$$

the Chern character shows how the combination of the gauge fields and the spin connection as $(A + \frac{1}{2}m\omega)$ arises in a natural way. For the partons we get m times the contribution from the $\hat{A}$ genus, while the χ fields give $-(m-1)$ times the same contribution, leading to the last term in the second line of (29). Thus the "pure gravitational anomaly" can be understood as the contribution of the $\hat{A}$ genus.

An alternative solution for anomaly cancellation would be to take m Dirac spinors for the spectators with b charges which are identical to the charges of the m partons. This would obviously cancel the b anomalies but would give a different value for the purely gravitational term in the final effective action. In particular, the action will not agree with the action for the integer QHE when we set $m = 1$. Recall that the action for the integer case does not require spectators (or partons) and is unambiguously determined by the Dolbeault index density. Requiring that the anomaly cancellation should be consistent with that result (upon setting $m = 1$) rules out this alternative cancellation solution.

B. Transport coefficients

We can now read off the transport coefficients, the Hall conductivity and the Hall viscosity, from the effective action (29). The variations of the effective action with respect to the electromagnetic field A and the metric g_{ml} give the current J^i and the energy-momentum tensor T^{ml} as

$$\delta S_{\text{eff}} = \int d^{2k+1}x \sqrt{\det g}\left[J^i \delta A_i - \frac{1}{2} T^{ml} \delta g_{ml}\right]. \tag{32}$$

In the present case, the Hall current is given by

$$J^i = \epsilon^{ij}\left[\frac{E_j}{2\pi m} + \frac{R_{j0}}{4\pi}\right], \tag{33}$$

where $E_i = F_{i0}$. The Hall conductivity is $\sigma_H = \frac{\nu}{2\pi}$. An interesting feature of (33), which is also valid in all higher dimensions [18], is that a Hall current can be generated from time variation of the metric even if there is no external electric field applied to the system.

In order to identify the Hall viscosity we have to derive the energy-momentum tensor from the effective action (29) and further identify the term involving the time derivative of the metric. A detailed calculation for the Hall viscosity in two and four dimensions for $\nu = 1$ was done in [18]. It is straightforward to modify those results for $m \neq 1$. We then find

$$\begin{aligned} T^{ml} &= \frac{1}{8\pi\sqrt{\det g}}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk}) \\ &\quad \times \left\{\left[\frac{B}{2} + \left(\frac{m}{4} - \frac{1}{12}\right)\left(\frac{R}{2} - \nabla^2\right)\right]\dot{g}_{ki} \right. \\ &\quad \left. + \frac{1}{2}\left(\frac{m}{4} - \frac{1}{12}\right)\nabla_i \nabla_k (g^{rn}\dot{g}_{rn})\right\}, \end{aligned} \tag{34}$$

where R is the Ricci scalar curvature and the magnetic field B given by

$$F_{ij} = \epsilon_{ij} B \sqrt{\det g}. \tag{35}$$

Comparing (34) with the expression of the energy-momentum tensor in terms of the Hall viscosity, we see that we can write

$$\sqrt{\det g}T^{ml} = \frac{1}{2}\eta_H (g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\dot{g}_{ki} + \frac{1}{2}\eta_H^{(2)}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\nabla_i\nabla_k(g^{rn}\dot{g}_{rn}), \quad (36)$$

where the coefficients η_H and $\eta_H^{(2)}$ can be read off as

$$\eta_H = \frac{1}{4\pi}\left[\frac{B}{2} + \left(\frac{m}{4} - \frac{1}{12}\right)\left(\frac{R}{2} + \vec{k}^2\right)\right],$$
$$\eta_H^{(2)} = \frac{1}{8\pi}\left(\frac{m}{4} - \frac{1}{12}\right). \quad (37)$$

We will close this subsection with a few clarifying remarks. Although we have focused on the effective action and anomaly cancellation, it is important to consider the occupation numbers for the partons. In the case of a two-dimensional spatial manifold, this will be given by the integral of the 2-form from the index polynomial (which will also be the variation of the action with respect to A_0).

The occupation number for the fully filled LLL for each of the partons then takes the form

$$n_i = \int\left[\frac{F}{2\pi m} + \frac{R}{4\pi}\right] + \int (h^{(n)})_i \frac{db^{(n)}}{2\pi}, \quad (38)$$

where we have separated out the part depending on the b fields. Notice that this means that $b^{(1)}$ couples to q_1 and q_2, with charges $\pm\frac{1}{2}$, but not to the other partons. Similarly, $b^{(2)}$ couples to q_1, q_2, and q_3 (with charges $\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, -\frac{2}{\sqrt{12}}$), but not to others, etc. For equality of the n_i, we therefore need to require that

$$\int db^{(n)} = 0 \quad (39)$$

for all n. This does not in any way come into the question of the anomaly cancellation since (39) does not imply $\int b^{(n)} db^{(n)} = 0$.

Our second remark relates to the nature of the composite field Ξ in (26). While the spectator fields are introduced for anomaly cancellation, composites like Ξ have zero electric charge. So there is no measurable response for them from changes in external electromagnetic fields. They do respond to gravitational perturbations, but this is already captured by the effective action. Thus, beyond effects derivable from the effective action in (29), composites of the spectator fields such as Ξ are more or less irrelevant.

C. A concern for other values of ν

Finally, we point out a difficulty with the parton picture for states with filling fractions different from $\nu = 1/m$ when one deals with curved manifolds. This can be illustrated by a simple example, say, for $\nu = 2/5$. The usual parton picture for this uses three partons, with charge $2/5$ for q_1, q_2 and charge $1/5$ for q_3. The partons q_1, q_2 fill the LLL, $\nu = 1$, while q_3 fills the LLL and the first excited level, $\nu = 2$. With the b-charge assignments as in (15), (16), we find [12,15]

$$\begin{aligned} S_{\rm eff}(q) = \frac{1}{4\pi}\Bigg\{&\left(\frac{2}{5}A + \frac{b^{(1)}}{2} + \frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right)d\left(\frac{2}{5}A + \frac{b^{(1)}}{2} + \frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right) - \frac{1}{12}\omega d\omega \\ &+ \left(\frac{2}{5}A - \frac{b^{(1)}}{2} + \frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right)d\left(\frac{2}{5}A - \frac{b^{(1)}}{2} + \frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right) - \frac{1}{12}\omega d\omega \\ &+ \left(\frac{1}{5}A - 2\frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right)d\left(\frac{1}{5}A - 2\frac{b^{(2)}}{\sqrt{12}} + \frac{\omega}{2}\right) - \frac{1}{12}\omega d\omega \\ &+ \left(\frac{1}{5}A - 2\frac{b^{(2)}}{\sqrt{12}} + \frac{3\omega}{2}\right)d\left(\frac{1}{5}A - 2\frac{b^{(2)}}{\sqrt{12}} + \frac{3\omega}{2}\right) - \frac{1}{12}\omega d\omega\Bigg\} \\ = \frac{1}{4\pi}\Bigg\{&\frac{2}{5}AdA + \frac{8}{5}Ad\omega + \frac{8}{3}\omega d\omega - \frac{6}{\sqrt{12}}b^{(2)}d\omega + \frac{1}{2}b^{(1)}db^{(1)} + \frac{5}{6}b^{(2)}db^{(2)}\Bigg\}. \end{aligned} \quad (40)$$

The AdA and $Ad\omega$ terms agree with the corresponding terms in the effective action for $\nu = 2/5$ in [16]. There is a discrepancy for the $\omega d\omega$ term, but there will be similar terms from the χ fields after the anomaly cancellation, so this is not yet important. However a problem with this choice of partons is already apparent at this stage, before we even consider anomaly cancellation. For the number of partons of each kind, we find

$$n_1 = \frac{1}{2\pi}\int\left[\frac{2}{5}dA + \frac{1}{2}d\omega + \frac{db^{(1)}}{2} + \frac{db^{(2)}}{\sqrt{12}}\right],$$
$$n_2 = \frac{1}{2\pi}\int\left[\frac{2}{5}dA + \frac{1}{2}d\omega - \frac{db^{(1)}}{2} + \frac{db^{(2)}}{\sqrt{12}}\right],$$
$$n_3 = \frac{1}{2\pi}\int\left[\frac{2}{5}dA + \frac{4}{2}d\omega - 4\frac{db^{(2)}}{\sqrt{12}}\right]. \quad (41)$$

These are given by varying the action (40) with respect to $(2/5)A_0$ for q_1, q_2 and with respect to $(1/5)A_0$ for q_3. Alternatively, these can be obtained directly from the two-dimensional index theorem.

As should be clear from (41), for curved spaces, such as the sphere for example, the occupation numbers for the partons will not be equal to each other, which will be problematic in writing down the many-body electron wave function in terms of partons. This is true for all cases where $\int d\omega \neq 0$, although we can still use the action (40) for manifolds with small perturbations of the metric around a background with $\int d\omega = 0$. Another possibility is to choose nontrivial backgrounds with $db \neq 0$ so as to match the numbers in (41). We do not pursue these issues in detail here, our aim is to point out that there are subtleties in extending a parton picture to manifolds of nontrivial geometry and topology. The case of $\nu = 1/m$ discussed above is not affected by this issue.

IV. THE PARTON CONSTRUCTION IN 4+1 DIMENSIONS

A. Anomaly cancellation and the effective action

The construction for the fractional states in $4+1$ dimensions is fairly straightforward since the method is clear from the $(2+1)$-dimensional case worked out in detail in the last section. Towards this, we again consider m partons, each of charge $1/m$, and coupled to b fields with charges as given in (15). Each type of parton will be in a $\nu = 1$ state.

The 6-form Dolbeault index density can be identified from (3), where $i\mathcal{F} = (dA/m)\mathbb{1} + \sum_{n=1}^{m-1} db^{(n)} h^{(n)}$

$$\mathcal{I}_6 = \frac{1}{3!}\mathrm{Tr}\frac{i\mathcal{F}\wedge i\mathcal{F}\wedge i\mathcal{F}}{(2\pi)^3} + \frac{c_1}{4}\mathrm{Tr}\frac{i\mathcal{F}\wedge i\mathcal{F}}{(2\pi)^2} + \frac{c_1^2+c_2}{12}\mathrm{Tr}\frac{i\mathcal{F}}{2\pi} + \frac{c_1 c_2}{24}\dim V. \tag{42}$$

The corresponding effective action for the partons is then

$$\begin{aligned} S_{\mathrm{eff}}(q) = {}& \frac{1}{24\pi^2 m^2}AF^2 + \frac{c_1}{8\pi m}AF + \frac{c_1^2+c_2}{12}A + \frac{m}{192\pi^2}\mathrm{Tr}\omega[(\mathrm{Tr}d\omega)^2 - \mathrm{Tr}(RR)] \\ & + \left[\frac{1}{16\pi^2 m}F + \frac{c_1}{16\pi}\right]\sum_{n=1}^{m-1} b^{(n)}db^{(n)} + \frac{1}{24\pi^2}\sum_{n,n',n''=1}^{m-1} b^{(n)}db^{(n')}db^{(n'')}\mathrm{Tr}(h^{(n)}h^{(n')}h^{(n'')}). \end{aligned} \tag{43}$$

For ease of working out the anomaly cancellation, it is useful to separate out the term involving the $b^{(m-1)}$ field in the last term of this expression. We find

$$\begin{aligned} & \frac{1}{24\pi^2}\sum_{n,n',n''=1}^{m-1} b^{(n)}db^{(n')}db^{(n'')}\mathrm{Tr}(h^{(n)}h^{(n')}h^{(n'')}) \\ & = \frac{1}{24\pi^2}\Bigg\{\sum_{n,n',n''}^{m-2} b^{(n)}db^{(n')}db^{(n'')}\mathrm{Tr}(h^{(n)}h^{(n')}h^{(n'')}) \\ & \quad + \frac{3}{2}\frac{b^{(m-1)}}{\sqrt{2(m^2-m)}}\sum_{n=1}^{m-2} b^{(n)}db^{(n)} - m(m-1)(m-2)\frac{b^{(m-1)}db^{(m-1)}db^{(m-1)}}{(2(m^2-m))^{\frac{3}{2}}}\Bigg\}. \end{aligned} \tag{44}$$

Turning to the cancellation of anomalies, we consider $(m-1)$ right-chiral spinors χ with b charges as in (19) for $n = 1, 2, \ldots, (m-2)$. As in the two-dimensional case the χ spinors carry a charge of $Q/\sqrt{2(m^2-m)}$ for coupling to $b^{(m-1)}$ and they do not couple to electromagnetism (i.e., have zero electromagnetic charge).

The relevant Dirac index density in six dimensions is

$$\text{Dirac index} = \frac{1}{3!}\mathrm{Tr}\frac{i\mathcal{F}\wedge i\mathcal{F}\wedge i\mathcal{F}}{(2\pi)^3} - \frac{c_1^2 - 2c_2}{24}\mathrm{Tr}\frac{i\mathcal{F}}{2\pi}, \tag{45}$$

where $i\mathcal{F} = \sum_{n=1}^{m-2} db^{(n)}\tilde{h}^{(n)} + \frac{Q}{\sqrt{2m(m-1)}}db^{(m-1)}\mathbb{1}$ and $\tilde{h}$ are traceless $(m-1)\times(m-1)$ matrices, such that $\mathrm{Tr}(\tilde{h}^{(n)}\tilde{h}^{(n')}) = \frac{1}{2}\delta^{nn'}$, as in (19).

The corresponding effective action for χ is

$$S(\chi) = -\frac{1}{24\pi^2}\left\{\sum_{n,n',n''}^{m-2} b^{(n)} db^{(n')} db^{(n'')} \mathrm{Tr}(h^{(n)} h^{(n')} h^{(n'')}) + \frac{3Q}{2}\frac{b^{(m-1)}}{\sqrt{2(m^2-m)}}\sum_{n=1}^{m-2} db^{(n)} db^{(n)} + (m-1)Q^3 \frac{b^{(m-1)} db^{(m-1)} db^{(m-1)}}{(2(m^2-m))^{\frac{3}{2}}}\right\} + Q\frac{m-1}{24}(c_1^2 - 2c_2)\frac{b^{(m-1)}}{\sqrt{2(m^2-m)}}. \tag{46}$$

We will also introduce N left-chiral spinors, denoted by χ', coupled to $b^{(m-1)}$ with charge $\frac{\alpha}{\sqrt{2m(m-1)}}$ and M right-chiral spinors χ'' coupled to $b^{(m-1)}$ with charge $\frac{\beta}{\sqrt{2m(m-1)}}$. The contribution of these will be

$$S(\chi') = \frac{1}{24\pi^2} N\alpha^3 \frac{b^{(m-1)} db^{(m-1)} db^{(m-1)}}{(2(m^2-m))^{\frac{3}{2}}} - \frac{1}{24}(c_1^2 - 2c_2) N\alpha \frac{b^{(m-1)}}{\sqrt{2(m^2-m)}},$$
$$S(\chi'') = -\frac{1}{24\pi^2} M\beta^3 \frac{b^{(m-1)} db^{(m-1)} db^{(m-1)}}{(2(m^2-m))^{\frac{3}{2}}} + \frac{1}{24}(c_1^2 - 2c_2) M\beta \frac{b^{(m-1)}}{\sqrt{2(m^2-m)}}. \tag{47}$$

Combining (43)–(47), we see that the leading anomaly, the cubic b term, cancels if

$$Q - 1 = 0$$
$$-m(m-1)(m-2) + N\alpha^3 - (m-1)Q^3 - M\beta^3 = 0. \tag{48}$$

Further the linear b term, as well as the total $b^{(m-1)}$ charge, cancels if

$$N\alpha - (m-1)Q - M\beta = 0. \tag{49}$$

We will consider two possible solutions to (48), (49), given by

$$\text{I.}\quad Q = 1, \qquad N = 0, \qquad M = 1, \qquad \beta = -(m-1). \tag{50}$$

$$\text{II.}\quad Q = 1, \qquad N = 1, \qquad M = 1, \qquad \alpha = 0, \qquad \beta = -(m-1). \tag{51}$$

For the first solution, the m spectator fields χ, χ'' have b charges which are identical to the b charges for the partons (and there is no χ' field). The second solution is essentially what we had in $2+1$ dimensions, with one χ' and one χ'' field in addition to $(m-1)$ χs. Notice that, as far as b anomalies are concerned, these two solutions have the same content, since $\alpha = 0$ for the second solution. In principle, we could also add an equal number of left and right chirality fields whose anomalies will cancel against each other, but (50), (51) represent the minimal choice.

As in the case of two dimensions, the electron will correspond to a composite field

$$\psi \sim q_1 q_2 \cdots q_m. \tag{52}$$

There can also be composites made of the spectator fields, but, for reasons given after (39), these are not important for us.

Adding the action for the partons (43) and the action for the right chiral spinors χ, χ'' (46), (47) we find an effective action where the leading anomalous term, the cubic-b term, is canceled. (Notice that χ' with $\alpha = 0$ does not contribute to the action.) The resulting effective action takes the form

$$S_{\text{eff}} = \frac{1}{24\pi^2 m^2} AF^2 + \frac{c_1}{8\pi m} AF + \frac{c_1^2 + c_2}{12} A + \frac{m}{192\pi^2} \mathrm{Tr}\omega[(\mathrm{Tr}d\omega)^2 - \mathrm{Tr}(RR)] + \left[\frac{1}{16\pi^2 m} F + \frac{c_1}{16\pi}\right] \sum_{n=1}^{m-1} b^{(n)} db^{(n)}. \tag{53}$$

The expression above is the bulk effective action for fractional Hall states in $4+1$ dimensions. The expression still involves the b fields. Unlike in $2+1$ dimensions, we cannot resort to known framing-anomaly-type results to integrate them out. However, we can still eliminate them as follows.

The final term in the effective action can be rewritten as

$$\left[\frac{1}{16\pi^2 m}F+\frac{c_1}{16\pi}\right]\sum_{n=1}^{m-1}b^{(n)}db^{(n)}=\left[\frac{1}{16\pi^2}\left(\frac{A}{m}+\frac{\mathrm{Tr}\omega}{2}\right)\sum db^{(n)}db^{(n)}\right]+d\left[\frac{1}{16\pi^2}\left(\frac{A}{m}+\frac{\mathrm{Tr}\omega}{2}\right)\sum b^{(n)}db^{(n)}\right]. \quad (54)$$

The second term on the right-hand side is a total derivative and integrates to a term on the boundary. Since it is a local term involving the gauge fields on the boundary, it can be removed by a choice of regularization when the spectator fields are integrated out. The bulk term is thus invariant under gauge transformations of the b fields, so there are no more obstructions to integrating out the b fields. Notice also that A and ω are coupled to a current which is conserved in the bulk. In fact, the first term on the right hand side of (54) can be written as

$$\frac{1}{16\pi^2}\left(\frac{A}{m}+\frac{\mathrm{Tr}\omega}{2}\right)\sum db^{(n)}db^{(n)}=\left(\frac{A_\mu}{m}+\frac{\mathrm{Tr}\omega_\mu}{2}\right)\mathcal{J}^\mu, \quad (55)$$

where $\mathcal{J}^\mu$ is the dual of $\frac{1}{16\pi^2}\sum db^{(n)}db^{(n)}$. By construction $\mathcal{J}^\mu$ is a conserved current. When the b fields are integrated out, this term will produce powers of A and ω coupled to correlators of the current $\mathcal{J}^\mu$. In general, such correlators are not topological but since $\mathcal{J}^\mu$ is conserved, they will have appropriate transversality properties so that the result will involve only dAs and $d\omega$s. Further, the expectation value of just one power of the current should be zero, indicating that the monopole moment of the charge should be zero on average. Thus we expect that these nontopological terms are of the dipole or higher multipole nature. Setting aside these nontopological terms, the effective action is thus reduced to

$$S_{\mathrm{eff}}=\frac{1}{24\pi^2 m^2}AF^2+\frac{c_1}{8\pi m}AF+\frac{c_1^2+c_2}{12}A+\frac{m}{192\pi^2}\mathrm{Tr}\omega[(\mathrm{Tr}d\omega)^2-\mathrm{Tr}(RR)]. \quad (56)$$

Using (6) we can rewrite (56) as

$$S_{\mathrm{eff}}=\frac{1}{(2\pi)^2}\int\left\{\frac{1}{3!m^2}(A+m\omega^0)[d(A+m\omega^0)]^2-\frac{1}{12}(A+m\omega^0)\left[(d\omega^0)^2+\frac{1}{4}R^a\wedge R^a\right]\right\}, \quad (57)$$

where, for a complex manifold with holonomy $U(k)$, we can write R in terms of the $U(1)$ and $SU(k)$ components as $R=d\omega^0\mathbb{1}+R^a t_a$.

B. Transport coefficients

We now turn to the transport coefficients. For $m=1$, the $(4+1)$-dimensional action (57) describing the bulk dynamics of the integer QHE and its corresponding transport coefficients were studied in detail in [18]. Here we quote those results appropriately modified for the $m\neq 1$ case. The Hall current takes the form

$$J^i=\frac{1}{2}\frac{1}{(2\pi)^2}\epsilon^{ijkl}\frac{E_j}{m^2}\left(F_{kl}+m\frac{\mathrm{Tr}R_{kl}}{2}\right), \quad (58)$$

where we have neglected terms involving time derivatives of the metric. The Hall conductivity, defined by the term proportional to the electric field $E_j=F_{j0}$, can be identified as

$$\sigma_{\mathrm{H}}^{ij}=\frac{1}{8\pi m^2}\epsilon^{ijkl}\left(F_{kl}+m\frac{\mathrm{Tr}R_{kl}}{2}\right). \quad (59)$$

In order to identify the Hall viscosity one has to obtain the energy-momentum tensor. The coefficient of the term proportional to the time derivative of the metric in the expression for T_{ij} will give the Hall viscosity. Following the calculation done in [18] and extending it to the case where $m\neq 1$, we find that the energy momentum tensor derived from (56) involves two terms,

$$T^{ml}=T_1^{ml}+T_2^{ml}, \quad (60)$$

where

$$T_1^{ml}=\frac{1}{8(2\pi)^2}(g^{mn}(J^0)^{lj}+g^{ln}(J^0)^{mj})\dot{g}_{nj}K^0+\cdots,$$

$$d^4x\sqrt{\det g}K^0=\left[\frac{1}{2m}dAdA+dAd\omega^0+\frac{m}{2}d\omega^0d\omega^0+\frac{m}{48}R^{\alpha\beta}R^{\beta\alpha}\right], \quad (61)$$

where $\omega^0=\frac{1}{4}\epsilon^{\alpha\beta}\omega^{\alpha\beta}$, $R^{\alpha\beta}R^{\beta\alpha}=-4R^0R^0-R^aR^a$ and

$$T_2^{ml}=-\frac{1}{96(2\pi)^2}(g^{mn}(R_{rs})^{lj}+g^{ln}(R_{rs})^{mj})\dot{g}_{nj}\partial_p(A_q+m\omega_q^0)\times\frac{\epsilon^{rspq}}{\sqrt{\det g}}+\cdots. \quad (62)$$

The ellipsis in (61) and (62) refers to momentum-dependent terms of the form $\partial\dot{g}$. The antisymmetric tensor $(J^0)^{lj}$ in (61) is defined in terms of the inverse frame fields $e^{-1l\alpha}$ and the antisymmetric tensor $\epsilon^{\alpha\beta}$

$$(J^0)^{lj} = e^{-1l\alpha}e^{-1j\beta}\epsilon^{\alpha\beta}. \tag{63}$$

The antisymmetric tensor $\epsilon^{\alpha\beta}$ is defined so that $\epsilon^{12} = \epsilon^{34} = 1$; $\epsilon^{13} = \epsilon^{24} = 0$.

Expressions (61) and (62) give the momentum-independent terms of the Hall viscosity in 4D. As should be clear from these expressions, the tensorial structure in the case of a curved manifold is rather involved. However, there is simplification for zero curvature. In the flat limit, the 4D complex manifold decomposes into $\mathbb{C}\times\mathbb{C}$, corresponding to the planes (1, 2) and (3, 4) where there is a constant magnetic field B_1, B_2 for each plane. Since the curvature terms vanish in this limit the contribution from T_2^{ml} is zero. Further, we can write $(J^0)^{lj}\to\epsilon^{ij}$ and the contribution from T_1^{ml} is of the form

$$T^{ml} = \frac{1}{8(2\pi)^2}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\frac{B_1B_2}{m}\dot{g}_{ki}. \tag{64}$$

Comparing with (36) we find that the Hall viscosity in this limit is

$$\eta_H = \frac{1}{4m}\frac{B_1B_2}{(2\pi)^2}. \tag{65}$$

Notice that the leading term in the Hall conductivity in (59) behaves as $1/m^2$ while the leading term of the Hall viscosity in (65) behaves as $1/m$.

C. FQHE on $S^2\times S^2$ and dimensional reduction

An interesting special case to consider is the complex manifold $S^2\times S^2$. In that case the curvature and spin connection decompose in terms of the appropriate quantities on each sphere, i.e.,

$$\begin{pmatrix} d\omega_1 & 0 \\ 0 & d\omega_2 \end{pmatrix} = d\omega^0\mathbb{1} + R^a t_a, \tag{66}$$

where $d\omega^0 = \frac{1}{2}(d\omega_1 + d\omega_2)$ and $R^3 = d\omega_1 - d\omega_2$, $R^1 = R^2 = 0$.

Given the above expressions we find that for $S^2\times S^2$

$$c_1 = \frac{\mathrm{Tr}R}{2\pi} = \frac{d\omega_1 + d\omega_2}{2\pi},$$
$$c_2 = \frac{(\mathrm{Tr}R)^2 - \mathrm{Tr}R\wedge R}{2(2\pi)^2} = \frac{d\omega_1 d\omega_2}{(2\pi)^2}. \tag{67}$$

It is interesting to notice that, if we assume that the electromagnetic interactions reside only on the first sphere, the $(4+1)$D bulk action (56) for $m=1$ (noninteracting case) dimensionally reduces to the $(2+1)$D bulk action (29) for $m=1$. This is obtained by integrating over the second sphere using

$$\int_{S^2}\frac{d\omega_2}{2\pi} = 2. \tag{68}$$

Similarly for $m\neq 1$, if we assume that the gauge fields such as A and bs reside only on the first sphere, the $(4+1)$D parton effective action (43) (partons are at $\nu = 1$) dimensionally reduces to the $(2+1)$D parton effective action (18). The dimensional reduction however does not go through at the level of the $m\neq 1$ total effective actions (57) and (29) since the gravitational contribution of the chiral edge spinors is very different in $(2+1)$D and $(4+1)$D. This is understandable since interparticle (interparton) interactions are important for FQHE, in particular between the two spheres in the present case of $S^2\times S^2$, so a naive reduction to FQHE on one of the spheres is not to be expected.

One can further calculate the corresponding transport coefficients on $S^2\times S^2$. The Hall conductivity is as in (59). Regarding the Hall viscosity and keeping only the momentum-independent terms, we find the following contributions from the energy momentum tensors T_1^{12} and T_2^{12},

$$\eta_{H,1} = \frac{1}{4(2\pi)^2}\left[\frac{B_1B_2}{m} + \frac{1}{4}(B_1R_2 + B_2R_1) + \frac{m}{16}R_1R_2\right],$$
$$\eta_{H,2} = -\frac{1}{4(2\pi)^2}\frac{1}{24}\left(R_1B_2 + \frac{m}{2}R_1R_2\right), \tag{69}$$

where $R_{1,2}$ are Ricci scalars. The calculation of the Hall viscosity from the T_1^{34} and T_2^{34} will give similar expressions with $R_1\leftrightarrow R_2$.

The total contribution for the Hall viscosity from $(T_1^{12} + T_2^{12})$ is

$$\eta_H = \frac{1}{4(2\pi)^2}\left[\frac{B_1B_2}{m} + \frac{1}{4}(B_1R_2 + B_2R_1) - \frac{1}{24}B_2R_1 + \frac{m}{24}R_1R_2\right]. \tag{70}$$

For $m=1$, it is straightforward to check that the limit where $B_2 = 0$ and $\int R_2 = 8\pi$ indeed produces the $(2+1)$D expression for the Hall viscosity for $m=1$, Eq. (37), confirming the dimensional reduction situation for $m=1$ mentioned earlier.

V. CONCLUDING REMARKS

It is useful to recapitulate briefly the arguments and results of this paper, since a number of necessary but ancillary comments were made along the way and the main thread of logic may not have been easy to follow. The basic idea is to generalize the parton picture which has been used to construct fractional quantum Hall states in two spatial dimensions. The electron is viewed as a composite particle made of several partons with auxiliary gauge fields (the b fields) binding them together. The partons are in quantum Hall states of integer filling fraction; this state, viewed in terms of the electron, is a fractional quantum Hall state. The action for the b fields involves terms of the Chern-Simons type in $2+1$ dimensions. One can integrate them out to get a gravitational CS term, the action for the so-called framing anomaly. While this is fairly straightforward in $2+1$ dimensions, a similar procedure in higher dimensions would lead to higher CS forms and this leads to an impasse since integrating out CS theories in higher dimensions is still not well understood. However, we notice that one can introduce a set of auxiliary fields to cancel out any gauge anomaly for the b fields on the boundary of a quantum Hall droplet, thereby eliminating CS type (potentially anomaly-generating) terms in the bulk. We showed that this does lead to the same results in $2+1$ dimensions, same as integrating out the b fields and as obtained in various explicit calculations. Anomaly cancellation thus constitutes an alternate formulation of the key idea of the parton picture and this is indeed generalizable to higher dimensions.

We worked out the parton picture in $4+1$ dimensions in Sec. IV. The effective action for the partons can be obtained as the Chern-Simons term corresponding to the twisted Dolbeault index density in $2k+2$ dimensions, for QHE in $2k+1$ dimensions. The reason for the use of this index is the same as for the integer QHE, namely, because of the holomorphicity condition for the fields in the lowest Landau level. The spectator fields are chiral spinors. Their anomaly can be obtained by the standard descent procedure starting with the Dirac index density in $2k+2$ dimensions. The resulting effective actions are given in (56) and (57).

The Hall current is given by deriving this action with respect to the electromagnetic field. Likewise, the variation of the action with respect to the metric gives the energy-momentum tensor. From the current and the energy-momentum tensor, we obtained the Hall conductivity and the Hall viscosity, given in (59) and (65). The m dependence for the leading term for the Hall conductivity is $1/m^2$ while it is $1/m$ for Hall viscosity.

A related point worthy of comment is about the case of QHE on S^4 studied by Hu and Zhang in [4]. The authors argue that the filling fraction for the corresponding Laughlin-type wave functions is $1/m^3$, based on the analysis of the degeneracy and the spectrum of the lowest Landau level. While this is different from the $1/m^2$ behavior of the Hall current for a complex 4D manifold, it is easy to see how this arises. We have shown in [5] that QHE on S^4 with $SU(2)$ magnetic background can be understood in terms of QHE on $\mathbb{CP}^3$ with an Abelian magnetic field. This arises from the fact that $\mathbb{CP}^3$ is an S^2 bundle over S^4. The filling fraction of the corresponding Laughlin-type wave functions on $\mathbb{CP}^3$ is also $\nu = 1/m^3$. In terms of the effective action approach, the leading term of the CS action for the electromagnetic field A will be proportional to $1/m^3$ for $\mathbb{CP}^3$; this is in accordance with [4].

In fact we expect that for a general complex manifold of dimension $2k$ for which we can construct an effective action for Laughlin type states as described in this paper, the leading term for the Hall conductivity will be proportional to $\nu = \frac{1}{m^k}$ while the Hall viscosity will scale as $\frac{1}{m^{k-1}}$, up to curvature corrections. This is because the dominant term in the effective action for the Hall current is the CS term $\int A(dA/m)^k$, while the dominant term for the derivation of the Hall conductivity is the next order term of the form $\int A(dA/m)^{k-1}R$, involving one power of the curvature and hence one less power of A.

We have only considered states which are the higher dimensional analogs of the Laughlin states, i.e., of the $\nu = 1/m^k$ type. For other fractional values of ν, ensuring that each species of partons has the same degeneracy (over the integer Landau levels they fill) is nontrivial even in $2+1$ dimensions for spaces of nontrivial topology, as argued at the end of Sec. III.

Another point worth emphasizing is about the use of the spectator fields. It is important that the picture of the electron as a composite particle made of partons is to be viewed only as a theoretical technique highlighting certain nonperturbative features of the mutually interacting electrons in a magnetic field. While it is useful and seems to work well in $2+1$ dimensions, the ultimate reason for its success is still unclear. In reality, the only physical particle involved is just the electron. So while response functions derivable from the effective action are to be viewed as physical, possible composites of the spectator fields are to be viewed as artifacts of the technique. For this reason, we do not think the excitations of Ξ in (26), or similar fields in $4+1$ dimensions, are to be viewed as physical.

Obviously, the generalization of the present work to arbitrary even dimensions and to non-Abelian gauge field backgrounds will be very interesting. In envisaging such prospects, we note that the issue of anomaly cancellation becomes more involved. In $4+1$ dimensions, in (53), we encountered a term corresponding to a mixed gauge-gravitational anomaly. In even higher dimensions, there are additional terms corresponding to mixed gauge-gravitational anomalies generated by the index density. A consistent anomaly cancellation scenario incorporating these elements is beyond the scope of this first attempt, but nevertheless it remains a worthwhile avenue to explore.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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