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A note on coherent states for Virasoro orbits

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There are two sets of orbits of the Virasoro group which admit a Kähler structure. We consider the construction of coherent states for the orbit $\widehat{\text{diff}\, S^1}/U(1)$ which furnishes unitary representations of the group. The procedure is analogous to geometric quantization using a holomorphic polarization. We also give an explicit formula for the Kähler potential for this orbit and comment on normalization of the coherent states. We further explore some of the properties of these states, including the definition of symbols corresponding to operators and their star products. Some comments which touch upon the possibility of applying this to gravity in $(2+1)$ dimensions are also given.

Keywords: Coherent states, Virasoro algebra, geometric quantization, Kähler geometry

1. Introduction

It has been well known for a long time that one can construct coherent states which realize unitary representations of a Lie group via the method of geometric quantization.[1–8] For this one considers coadjoint orbits of the Lie group G which are of the form G/H for a suitable subgroup H such that the coset space has a Kähler structure. For the general case, one chooses H to be the maximal torus in G, with the Kähler two-form given by $-\sum_1^r w_i \text{Tr}(h_i g^{-1}dg \wedge g^{-1}dg)$ where g denotes a general group element in the fundamental representation (viewed as a matrix), h_i are the generators of the maximal torus in a suitable basis and $(w_1, w_2, \ldots, w_r)$ defines the highest weight state of some unitary representation of G, and r denotes the rank of the group. The result of the quantization will be a Hilbert space $\mathcal{H}$ corresponding to the carrier space of the representation with the highest weight indicated. One can assign wave functions to the states of this Hilbert space; they obey a holomorphicity condition depending on the Kähler potential. Orthonormality conditions, symbols corresponding to operators on $\mathcal{H}$, star products, the diagonal coherent state representation, etc. can then be defined in a straightforward way.

In this paper, we consider a similar construction of coherent states, star products, etc. for the Virasoro group, which will be identified as the centrally extended version of diffeomorphisms of a circle, denoted $\widehat{\text{diff}\, S^1}$. This problem is clearly of some intrinsic mathematical interest, but there are several motivating factors from physics as well. After all, the Virasoro algebra is one of the foundational ingredients for the formulation of string theory. Partly motivated by this, coadjoint orbits of $\widehat{\text{diff}\, S^1}$ were classified and some of their properties analyzed many years ago.[9,10] Another context in which $\widehat{\text{diff}\, S^1}$ emerges is $(2+1)$-dimensional gravity. The action for this theory (with a cosmological constant) is given as the difference of two $SL(2,\mathbb{R})$ Chern-Simons actions, with the connection forms A_L, A_R given as combinations of the frame fields and spin connection. Witten's analysis of the partition function of this theory shows that the inclusion of the BTZ black holes will require Virasoro representations in the relevant sum over states.[11,12] The construction of the coherent states is useful for such analyses. Further, the semiclassical limit of this analysis corresponds to taking the central charge c to be large, a limit which is suitable for a star-product expansion for observables.

In the case of finite dimensional Lie groups, the Hilbert space $\mathcal{H}$ can be used as a model for a description of the noncommutative version of the manifold G/H. Coherent states are then useful in defining symbols corresponding to operators on $\mathcal{H}$ and the star products give the (noncommutative) algebra of functions on G/H. Given the appearance of Virasoro representations in $(2+1)$-dimensional gravity, if we envisage a noncommutative antecedent for gravity, then coherent states for the Virasoro algebra become important in defining symbols and star products and obtaining a continuous manifold description in the large c limit.

In the case of $(2+1)$-dimensional gravity on asymptotically anti-de Sitter space, the asymptotic symmetries also lead to a Virasoro algebra.[13] Some of the issues of gravity may thus be cast in terms of holography or the AdS/CFT correspondence, which is a different facet of string theory. Combined with the observations in the previous paragraphs, this suggests a way to bring together ideas of string theory and/or gravity and noncommutative geometry. The coherent states for the Virasoro group will also be central to any such attempt.

Among the coadjoint orbits of the Virasoro group, there are two which admit Kähler structures and hence are amenable to defining coherent states satisfying appropriate holomorphicity conditions. These orbits correspond to $\widehat{\text{diff}\, S^1}/S^1$ and $\widehat{\text{diff}\, S^1}/SL(2,\mathbb{R})$. The Kähler potentials for these cases are characterized by two numbers, the central charge c, which must be viewed as the eigenvalue of the central element of the algebra, and h, which may be viewed as the eigenvalue of the generator L_0 on the highest weight state of the representation. For most purposes, the quantization of $\widehat{\text{diff}\, S^1}/SL(2,\mathbb{R})$ may be considered as a special case of the quantization of $\widehat{\text{diff}\, S^1}/S^1$, so we will focus on the latter space. Some of the other orbits are relevant for representations which contain null vectors and can lead to unitary representations for $c < 1$. We will not discuss these here.

Not surprisingly, there have been many discussions in the literature which touch upon various aspects of coherent states for the Virasoro group. Many of the papers we have cited contain some implicit statements about such states. Representations of the Virasoro algebra as differential operators on functions of a suitable set of variables occur in the context of the KP hierarchy and the so-called string equation.[7] A different, but related, representation has been used in discussions of the stochastic Loewner equation.[15] These do not directly lead to coherent states. The group manifold approach to quantization, discussed in Ref. 16, is a generalization of the idea of the coadjoint orbit quantization and as such is closer to our discussion. This approach starts with the formal group law and imposes polarization conditions on functions on the group to obtain irreducible representations. In Ref. 16, the composition laws for the Virasoro group parameters, including the central parameter relevant to the central extension, have been constructed. Our approach is more traditional, but clearly there are some points of overlap; we will briefly comment on this later. For us holomorphicity is important as it provides a simple way to construct a suitable star product. (The Kontsevich formula for star products does not need coherent states *per se*, but it does need more information about the symplectic structure on the space.[17] In turn, this would require an analysis similar to what we do in this paper, so it does not seem like our analysis can be evaded.) Standard coherent states for the bosonic operators obeying the Heisenberg algebra in the mode expansion for the target space coordinate in string theory have also been considered in the literature.[18] While these can be useful for certain applications, the states we are concerned with are not these; we are interested in directly quantizing the Virasoro orbits.

This paper is organized as follows. In Sec. 2, we will briefly discuss coherent states for $SL(2,\mathbb{R})/U(1)$. The results here are well-known, but it helps to set the stage for a similar analysis for the Virasoro case, which is taken up in Sec. 3. The key steps are the following. We first construct an expression for the Kähler potential for the orbit of interest. This will give a functional version of the coherent states and it will also inform the issues of normalization discussed later. We then define an operator U and show that it is possible to choose coordinates on the Virasoro orbit such that $U^{-1}dU$ can be split into holomorphic and antiholomorphic forms corresponding to the Virasoro generator L_{-n} and its conjugate. This naturally leads to a set of wave functions associated with the states of the Verma module and which obey a certain holomorphicity condition. The normalization of these wave functions will involve the Kähler potential. The normalization integral with the appropriate measure of integration is then discussed. Symbols and star products are considered in Sec. 4. In the limit of large central charge, the star products reduce to those on $SL(2,\mathbb{R})/U(1)$, i.e. on a noncommutative version of AdS_2. The paper concludes with a short discussion in Sec. 5. There is also an Appendix A which gives details of some of the assertions made in Sec. 3.

2. Coherent States for $SL(2,\mathbb{R})/U(1)$

In this section, we will briefly consider the construction of coherent states for $SL(2,\mathbb{R})$. While these results are well known, our presentation will help to highlight certain points which can help to clarify the lines of argument employed later for the Virasoro case.

The generators of the Lie algebra can be taken as L_0, $L_{\pm 1}$ with the commutation rules

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0. \tag{1}$$

The highest weight state is defined by

$$L_1|0\rangle = 0, \quad L_0|0\rangle = h|0\rangle. \tag{2}$$

We will consider representations with $h > \frac{1}{2}$ which are the simplest for exemplifying the arguments used for the Virasoro case. Other states are obtained by the action of powers of L_{-1} on $|0\rangle$. The normalized states are given by

$$|n\rangle = \frac{1}{\sqrt{N(n)}} L_{-1}^n |0\rangle, \quad N(n) = \frac{\Gamma(n+1)\Gamma(2h+n)}{\Gamma(2h)}, \tag{3}$$

where $\Gamma(u)$ is Eulerian gamma function for the argument u. The state $|n\rangle$ corresponds to a value of L_0 equal to $h+n$, so the representation we are considering is bounded below. Now we introduce the unitary operator[a]

$$U = \exp(wL_{-1} - \bar{w}L_1), \quad U^\dagger = \exp(\bar{w}L_1 - wL_{-1}), \tag{4}$$

where w, $\bar{w}$ are functions of some complex coordinates s, $\bar{s}$ which parametrize $SL(2,\mathbb{R})/U(1)$. We now define the coherent states by the wave functions

$$\Psi_n = \langle 0|U^\dagger|n\rangle = \frac{1}{\sqrt{N(n)}} \langle 0|U^\dagger L_{-1}^n|0\rangle. \tag{5}$$

(A similar definition will be used later for the Virasoro case.)

For the case of $SL(2,\mathbb{R})$, a general element of the group can be written in the 2×2 matrix representation as

$$g = \frac{1}{\sqrt{1-\bar{s}s}} \begin{pmatrix} 1 & is \\ -i\bar{s} & 1 \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}, \quad |s| < 1. \tag{6}$$

The correspondence of the Lie algebra elements in this representation is given by $L_0 = \frac{1}{2}\sigma_3$, $L_1 = \frac{i}{2}(\sigma_1 - i\sigma_2)$, $L_{-1} = \frac{i}{2}(\sigma_1 + i\sigma_2)$, in terms of the standard Pauli matrices. The group parameters s, $\bar{s}$ provide local coordinates (for a patch around

[a] The coordinates w, $\bar{w}$ correspond to what we later call w_1, $\bar{w}_1$ in the context of the Virasoro algebra; we drop the subscripts for this section to avoid clutter.

$s = 0$) for the coset space $SL(2,\mathbb{R})/U(1)$. For this parametrization, the standard Kähler one-form $\mathrm{Tr}(\sigma_3 g^{-1}dg)$ will contain a term $d\varphi$, which will drop out in the two-form $d(\mathrm{Tr}(\sigma_3 g^{-1}dg))$. So, for the quantization of the orbit corresponding to the chosen h-value on $SL(2,\mathbb{R}/U(1)$, we can drop φ on the local coordinate patch. The parametrization also shows a singularity at $|s| = 1$; this is integrable for $h > \frac{1}{2}$, and will not affect results below.

For the parametrization in (6), for a unitary representation of g as U, we get

$$\begin{aligned} dU^\dagger U^{\dagger -1} &= \left(\frac{s d\bar{s} - \bar{s} ds}{1 - \bar{s}s} L_0 + \frac{d\bar{s}}{1 - \bar{s}s} L_1 - \frac{ds}{1 - \bar{s}s} L_{-1} \right), \\ U^{\dagger -1} dU^\dagger &= \left(- \frac{s d\bar{s} - \bar{s} ds}{1 - \bar{s}s} L_0 + \frac{d\bar{s}}{1 - \bar{s}s} L_1 - \frac{ds}{1 - \bar{s}s} L_{-1} \right). \end{aligned} \tag{7}$$

From the first of these relations, and using (2), we get the holomorphicity conditions for the coherent states as

$$\left(\frac{\partial}{\partial s} + \frac{h\bar{s}}{1 - \bar{s}s} \right) \Psi_n = 0. \tag{8}$$

The second equation in (7) also gives

$$\left(\frac{\partial}{\partial \bar{s}} + \frac{hs}{1 - \bar{s}s} \right) \Psi_0 = 0. \tag{9}$$

For Ψ_0 we can solve these equations to find $\langle 0|U^\dagger|0\rangle = (1 - \bar{s}s)^h$. The possible arbitrary multiplicative constant is set to one, since $U^\dagger = 1$ at $s = 0$. For the higher states, we can develop a recursion rule using (8) as follows.

$$\begin{aligned} \Psi_n &= \frac{1}{\sqrt{N(n)}} \langle 0|U^\dagger L_{-1} L_{-1}^{n-1}|0\rangle \\ &= \frac{1}{\sqrt{N(n)}} \langle 0| \left(-(1 - \bar{s}s) \frac{\partial U^\dagger}{\partial s} + U^\dagger \bar{s} L_0 \right) L_{-1}^{n-1} |0\rangle \\ &= \sqrt{\frac{N(n-1)}{N(n)}} \left(-(1 - \bar{s}s) \frac{\partial}{\partial s} + \bar{s}(h + n - 1) \right) \Psi_{n-1}. \end{aligned} \tag{10}$$

This can be solved to write

$$\Psi_n = \sqrt{\frac{\Gamma(2h + n)}{\Gamma(n+1)\Gamma(2h)}} \bar{s}^n (1 - \bar{s}s)^h. \tag{11}$$

The Kähler potential may be identified from the symplectic form or from the holomorphicity condition written as $(\partial_s + \frac{1}{2}\partial_s K)\Psi_n = 0$. We can also identify K from the relation $\langle 0|U^{-1}dU|0\rangle = \frac{1}{2}(\partial K - \bar{\partial} K)$. The Kähler potential and the symplectic form can be worked out as

$$K = -2h \log(1 - \bar{s}s), \quad \omega = i\partial\bar{\partial}K = 2hi \frac{d\bar{s}ds}{(1 - \bar{s}s)^2}. \tag{12}$$

We will use a slightly different normalization for the phase volume defined by ω (with a prefactor $2h - 1$ rather than $2h$) which makes the Ψ_n orthonormal. It is easy to verify that

$$(2h-1)\int \frac{d^2s}{\pi(1-\bar{s}s)^2}\,\Psi_n^*\Psi_m = \delta_{nm}. \tag{13}$$

The range of integration is over the disk $|s| \leq 1$; the singularity at $|s| = 1$ is integrable for $h > \frac{1}{2}$, so the inner product is well-defined.

In Eqs. (8), (11) and (13), we have reproduced the standard and well-known results for coherent states on $SL(2,\mathbb{R})/U(1)$. For $SL(2,\mathbb{R})$ we have the advantage of a parametrization for the coset space given as in (6); for the Virasoro case, we have to rely on a power series expansion for w, $\bar{w}$. We will determine the expressions for w, $\bar{w}$ as a series in s, $\bar{s}$ relying on the Kähler property and requiring that the coefficient of L_{-1} be a holomorphic one-form. We will briefly illustrate the strategy here. Note that, using (4), we can write

$$\begin{aligned} U^\dagger dU &= \int_0^1 d\alpha\; e^{-\alpha(wL_{-1}-\bar{w}L_1)}(dwL_{-1} - d\bar{w}L_1)e^{\alpha(wL_{-1}-\bar{w}L_1)} \\ &\approx \left[dw\left(1+\frac{1}{3}\bar{w}w\right) - d\bar{w}\frac{1}{3}w^2\right]L_{-1} - \left[d\bar{w}\left(1+\frac{1}{3}\bar{w}w\right) - dw\frac{1}{3}\bar{w}^2\right]L_1 \\ &\quad + (\bar{w}dw - wd\bar{w})L_0 + \cdots . \end{aligned} \tag{14}$$

We take w to be given in terms of the local coordinates as

$$w = s + w^{(2)}(s,\bar{s}) + w^{(3)}(s,\bar{s}) + \cdots . \tag{15}$$

Setting the coefficient of $d\bar{s}L_{-1}$ to zero we find $w^{(2)} = s^2\bar{s}/3$. Using this, we can simplify $dU^\dagger U = -U^\dagger dU$ as

$$dU^\dagger U = d\bar{s}(1+\bar{s}s)L_1 - ds(1+\bar{s}s)L_{-1} + (sd\bar{s} - \bar{s}ds)L_0 + \cdots . \tag{16}$$

This matches with the expression in (7) to first order in the expansion in powers of $\bar{s}s$. We have checked that the process can be continued by developing w as a series in s, $\bar{s}$ and requiring that the coefficient of L_{-1} is a holomorphic one-form, to reproduce the results in (7). We will use a similar strategy for the Virasoro group, developing a series expansion for w_n, $\bar{w}_n$, as worked out in Appendix A. Since the procedure is exactly parallel, the results we obtain in that case will revert to the discussion in the present section when restricted to the coordinates w_1, $\bar{w}_1$, and to $L_0, L_{\pm 1}$, setting $w_n, \bar{w}_n = 0$, $n \geq 2$.

In the case of $SL(2,\mathbb{R})$, one can also directly use the Baker–Campbell–Hausdorff (BCH) formula

$$\begin{aligned} e^{wL_{-1}-\bar{w}L_1} &= e^{sL_{-1}}e^{\log(1-\bar{s}s)L_0}e^{-\bar{s}L_1}, \\ s &= \frac{w}{|w|}\tanh|w|. \end{aligned} \tag{17}$$

Since the BCH formula is determined by the commutator algebra of L_0, $L_{\pm1}$, the matrix representation $L_0 = \sigma_3/2$, $L_{\pm1} = i(\sigma_1 \pm i\sigma_2)/2$ can be used to verify (17). Writing w in terms of s we get $w = (s/|s|)\text{arctanh}|s| \approx s + s^2\bar{s}/3 + \cdots$, in agreement with the series expansion in (15). Further, by taking the expectation value of the first equation in (17) for the highest weight state $|0\rangle$, we find

$$\langle 0|U^\dagger|0\rangle = \langle 0|e^{-sL_{-1}} e^{\log(1-\bar{s}s)L_0} e^{-\bar{s}L_1}|0\rangle = e^{h\log(1-\bar{s}s)} \equiv e^{-K/2}, \tag{18}$$

which agrees with the expression for the Kähler potential in (12).

Finally, we also note that $K \to \infty$ at the boundary of the region of integration, which is $|s| \to 1$. This corresponds to $|w| \to \infty$ if we use the coordinates $w, \bar{w}$.

3. Construction of the States for the Virasoro Case

As mentioned in the Introduction, unitary representations of the Virasoro group, for $c > 1$, are obtained by quantizing $\widehat{\mathrm{diff}\, S^1}/S^1$ and $\widehat{\mathrm{diff}\, S^1}/SL(2,\mathbb{R})$. Further, these orbits have a Kähler structure.[9,10] We will focus on $\widehat{\mathrm{diff}\, S^1}/S^1$ for reasons mentioned earlier. This manifold can be described by complex coordinates s_k and $\bar{s}_k$, where k takes integer values from 1 to infinity. The basic generators of the Virasoro algebra are L_n, $n \in \mathbb{Z}$, with the commutator algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\,\mathbb{1}. \tag{19}$$

Here $\mathbb{1}$ is the identity operator; the central charge may be viewed as the eigenvalue of a central charge operator $\hat{c}$ which is proportional to the identity. This is necessary to view the algebra (19) as having a closed Lie algebra structure. Thus, there is a slight abuse of notation in writing (19) directly in terms of the eigenvalue; this will be immaterial for what we want to do. We are interested in highest weight representations, with the highest weight state obeying

$$L_0|0\rangle = h|0\rangle, \quad L_n|0\rangle = 0, \quad n \geq 1. \tag{20}$$

Further, we will be interested in the case of $c > 1$ and $h > 0$, so that we will not have null vectors in the associated Verma module. From now on we will use the notation L_0, L_n, L_{-n}, with $n \geq 1$ explicitly distinguishing the three sets of operators. The subgroup S^1 in $\widehat{\mathrm{diff}\, S^1}/S^1$ is generated by the action of L_0, with L_n, L_{-n} forming the translation operators on the coset $\widehat{\mathrm{diff}\, S^1}/S^1$.

3.1. *The Kähler potential*

Some considerations on the nature of the Kähler potential will be useful before embarking on constructing explicit formulae for the coherent states. An expression for the symplectic two-form relevant for $\widehat{\mathrm{diff}\, S^1}/SL(2,\mathbb{R})$ was given by Stanford and Witten in Ref. 19 as[b]

$$\omega = \frac{1}{4\pi}\int_0^{2\pi} d\tau \left[\frac{c}{12}\frac{\delta\phi'}{\phi'}\frac{\partial}{\partial\tau}\left(\frac{\delta\phi'}{\phi'}\right) - \frac{c}{12}\delta\phi\delta\phi'\right]. \tag{21}$$

[b] While we use the expression as given in Ref. 19, we note that it is also related to expressions given in Refs. 20 and 21, in addition to Refs. 9 and 10.

Here $\phi(\tau)$ may be considered as a field on the circle parametrized by $0 \leq \tau \leq 2\pi$. The prime denotes differentiation with respect to τ, Explicitly, ϕ is of the form

$$\begin{aligned}\phi(\tau) &= \tau + \varphi + \bar{\varphi} + \chi + \bar{\chi},\\ \varphi = &= s_1 e^{i\tau}, \quad \bar{\varphi} = \bar{s}_1 e^{-i\tau},\\ \chi &= \sum_2^\infty s_n e^{in\tau}, \quad \bar{\chi} = \sum_2^\infty \bar{s}_n e^{-in\tau}.\end{aligned} \tag{22}$$

Further, δ in (21) denotes exterior derivative on the space of the fields ϕ and wedge products for the δ's is understood. The two-form ω has zero modes corresponding to $L_0, L_\pm 1$, i.e. for $n = 0, \pm 1$. This is in accordance with the fact that ω is the symplectic form for $\widehat{\text{diff } S^1}/SL(2,\mathbb{R})$. The zero modes, as argued in Ref. 19, correspond to the vector fields

$$V_n = \int d\tau e^{in\phi(\tau)} \frac{\delta}{\delta\phi(\tau)}, \quad n = 0, \pm 1. \tag{23}$$

The zero modes can be removed by a suitable gauge-fixing condition to obtain an ω which is invertible. In Ref. 19, this was chosen as $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi''(0) = 0$. For us, it will be convenient to make a different choice. Under the action of V_n, with parameters $\beta_0, \beta_{\pm 1}$, the field ϕ transforms as

$$\begin{aligned}\phi &\to \phi + \beta_0 + \beta_{+1} e^{i\phi} + \beta_{-1} e^{-i\phi}\\ &= \tau + \varphi + \bar{\varphi} + \beta_0 + \beta_{+1} e^{i\tau} e^{i(\varphi+\bar{\varphi})} + \beta_{-1} e^{-i\tau} e^{-i(\varphi+\bar{\varphi})}\\ &\approx \tau + \varphi + \bar{\varphi} + \beta_0 + \beta_{+1} e^{i\tau} + \beta_{-1} e^{-i\tau} + \cdots,\end{aligned} \tag{24}$$

where, in the last line, we have expanded out $e^{\pm i(\varphi+\bar{\varphi})}$. This shows that the coordinates s_1, $\bar{s}_1$ shift by β_{+1}, β_{-1} for a neighborhood around $s_n = 0$. Even though the full transformation will be nonlinear, this shows that a suitable gauge-fixing for $V_{\pm 1}$ is to set $s_1 = \bar{s}_1 = 0$. The gauge-fixing for V_0 is taken care of by setting $\phi = \tau$ for $s_n = 0$. Thus, a gauge-fixed version of ω is given by the same expression as in (21), but with

$$\phi = \tau + \chi + \bar{\chi}, \tag{25}$$

omitting the s_1, $\bar{s}_1$ terms.

We now note that $\widehat{\text{diff } S^1}/U(1)$ can be regarded as a bundle over $\text{diff} S^1/SL(2,\mathbb{R})$ with $SL(2,\mathbb{R}/U(1)$ as the fiber. (This is a refinement of regarding $\text{diff} S^1$ as an $SL(2,\mathbb{R})$-bundle over $\text{diff} S^1/SL(2,\mathbb{R})$.) The gauge-fixing thus provides a local section of $\widehat{\text{diff } S^1}/U(1)$ in the form $\widehat{\text{diff } S^1}/U(1) \sim \text{diff} S^1/SL(2,\mathbb{R}) \times SL(2,\mathbb{R}/U(1)$. Therefore, for the symplectic form on $\widehat{\text{diff } S^1}/U(1)$, we have to modify ω to take care of the term corresponding to $SL(2,\mathbb{R})/(U(1)$. A suitable modification is given by

$$\omega = \frac{1}{4\pi} \int_0^{2\pi} d\tau \left[\frac{c}{12} \frac{\delta\phi'}{\phi'} \frac{\partial}{\partial\tau} \left(\frac{\delta\phi'}{\phi'} \right) + \left(2h - \frac{c}{12} \right) \delta\phi\delta\phi' - 4h \frac{\delta\bar{\varphi}'\delta\varphi}{(1 - i\bar{\varphi}'\varphi)^2} \right]. \tag{26}$$

We have made two changes compared to (21). We have added the contribution due to s_1, $\bar{s}_1$ which is the part due to $SL(2,\mathbb{R})/(U(1)$. Secondly, recall that the

n^3-term in the central term of the algebra (19) is the cohomologically nontrivial term. This corresponds to the first term in ω. The term linear in n, which corresponds to the integral of $\delta\phi\delta\phi'$, can be modified by changing the value of L_0. Accordingly, to take account of the h-dependence for the modes s_n, $\bar{s}_n$, $n \geq 2$, we have added a term proportional to $2h\delta\phi\delta\phi'$.

If we substitute from (25) into (26), ω will have terms of the form $\delta\chi'\delta\chi''$ and its conjugate, i.e. of the $(2,0)$ and $(0,2)$ types of differential forms. We can separate out the terms which correspond to the $(1,1)$-type as

$$\begin{aligned} \omega &= \omega_{\rm K} + F, \\ \omega_{\rm K} &= \frac{1}{4\pi}\int_0^{2\pi} d\tau \left\{ \frac{c}{12}\left[\frac{\delta\bar{\chi}'\delta\chi'' - \delta\bar{\chi}''\delta\chi'}{(1+\chi'+\bar{\chi}')^2} + 2\frac{(\bar{\chi}''-\chi'')\delta\bar{\chi}'\delta\chi'}{(1+\chi'+\bar{\chi}')^3}\right] \right. \\ &\quad \left. + \left(4h - \frac{c}{6}\right)\delta\bar{\chi}\delta\chi' - 4h\frac{\delta\bar{\varphi}'\delta\varphi}{(1-i\bar{\varphi}'\varphi)^2}\right\}, \end{aligned} \tag{27}$$

$$F = \frac{1}{4\pi}\int_0^{2\pi} d\tau \left\{ \frac{c}{12}\left[\frac{\delta\chi'\delta\chi'' + \delta\bar{\chi}'\delta\bar{\chi}''}{(1+\chi'+\bar{\chi}')^2} - 2\frac{(\bar{\chi}''-\chi'')\delta\bar{\chi}'\delta\chi'}{(1+\chi'+\bar{\chi}')^3}\right]\right\}. \tag{28}$$

It is also easy to see that we can write $F = dA = (\delta + \bar{\delta})A$, where

$$A = -\frac{1}{4\pi}\frac{c}{12}\int_0^{2\pi} d\tau \frac{\chi''\delta\chi' + \bar{\chi}''\delta\bar{\chi}'}{(1+\chi'+\bar{\chi}')^2}. \tag{29}$$

The two-form F in (27) can be removed by a coordinate transformation. This is essentially the idea of Moser's lemma. Even though we have an infinite number of modes in χ, $\bar{\chi}$, Moser's argument should apply since ω is invertible. We define $\omega_t = \omega_{\rm K} + tF$. The condition that $\omega_{t+\epsilon} - \omega_t$ is given by an infinitesimal coordinate transformation by the vector field X_t becomes

$$\frac{d\omega_t}{dt} = \mathcal{L}_{X_t}\omega_t = d(i_{X_t}\omega_t), \tag{30}$$

where $\mathcal{L}$ denotes the Lie derivative. This equation reduces to

$$d(A - i_{X_t}\omega_t) = 0. \tag{31}$$

In terms of the components corresponding to s_n, $\bar{s}_n$, the solution is given by

$$X_t^k = (\omega_t)^{kl}A_l. \tag{32}$$

There are no zero modes for ω_t, so the inverse should exist, albeit it is an infinite dimensional matrix. This shows how the required coordinate transformation can be constructed. Effectively, we can connect $\omega_{\rm K}$ (which is ω_t at $t=0$) to ω (which is ω_t at $t=1$) by a series of coordinate transformations. (A series expansion around s_n, $\bar{s}_n = 0$ may be needed for constructing the explicit formulae for the required coordinate transformation.) The key point is that ω in (27) is thus symplectomorphic to $\omega_{\rm K}$ which is of the $(1,1)$-type. We can therefore choose $\omega_{\rm K}$ as the symplectic form for our problem.

It is now easy to work out the Kähler potential for $\omega_{\rm K}$. It is given by

$$K = \frac{1}{4\pi}\int_0^{2\pi} d\tau \left\{ \frac{c}{12}\left[-i\frac{(\bar\chi'' - \chi'')}{(1+\chi'+\bar\chi')}\right] + \left(4h - \frac{c}{6}\right)(-i\bar\chi\chi') - 4h\log(1 - i\bar\varphi'\varphi)\right\}$$
$$= \frac{1}{4\pi}\int_0^{2\pi} d\tau \left\{ \frac{c}{12}\left[\frac{i}{(1+\chi'+\bar\chi')}\left(\frac{\bar\chi''\chi'}{(1+\bar\chi')} - \frac{\chi''\bar\chi'}{(1+\chi')}\right)\right] + \left(4h - \frac{c}{6}\right)(-i\bar\chi\chi') - 4h\log(1 - i\bar\varphi'\varphi)\right\}. \tag{33}$$

In passing to the second line of this equation, we used the fact that the integral of expressions which are purely holomorphic in the fields, i.e. depending only on χ (or antiholomorphic involving only $\bar\chi$) vanish upon τ-integration. Using (33), it is straightforward to check that $\omega_{\rm K} = i\bar\delta\delta K$. The expansion of this expression for small $s_n, \bar s_n$ will also agree with the alternate derivation of K, as will be clear from the discussion in Subsec. 3.2, Eq. (42). Also, if we restrict to just the modes s_1, $\bar s_1$, corresponding to $SL(2,\mathbb{R})$, it coincides with the expression for K in (12). If we restrict to s_n, $\bar s_n$ for fixed choice of n, corresponding to the subgroup defined by L_0, $L_{\pm n}$, the resulting expression is not of the same form as for s_1, $\bar s_1$. A further coordinate transformation will be needed to bring it to the form of the Kähler potential for $SL(2,\mathbb{R})/U(1)$ with $s_1, \bar s_1 \to s_n, \bar s_n$.

Once we have the explicit form of the Kähler potential, one can construct coherent states as is usually done for any Kähler manifold. In the functional form, these are given by

$$\Psi[\bar\varphi] = \mathcal{N} e^{-\frac{1}{2}K}\Phi[\bar\varphi, \bar\chi]. \tag{34}$$

We will now construct another expression for the coherent states in terms of the states of the Verma module, the final result being (51). We expect that (34) will coincide with (51), up to a possible coordinate transformation, or, equivalently, a redefinition of fields, once it is projected to the states of the Verma module.

3.2. *The operator U and the choice of complex coordinates*

While (34) is formally a definition of the coherent states, a more explicit formula in terms of the states of the Verma module, similar to the construction of the $SL(2,\mathbb{R})/U(1)$ states in Sec. 2, will also be useful. As a first step towards this, we define a unitary operator U by

$$U = \exp\left(\sum_{n=1}^{\infty} \bar w_n L_n - w_n L_{-n}\right). \tag{35}$$

We will be considering unitary highest weight representations, so U will be unitary. In (35), w_n and $\bar w_n$ are functions of the coordinates $s_k, \bar s_k$. Generally, from the commutation rules, we can see that we can write

$$U^{-1}dU = \sum_n (\bar{\mathcal{E}}^n L_n - \mathcal{E}^n L_{-n}) + (\mathcal{E}^0 - \bar{\mathcal{E}}^0)L_0 + (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1}, \tag{36}$$

where the coefficients $\mathcal{E}^n$, $\mathcal{E}^0$, $\mathcal{E}$ and their conjugates are one-forms on the coset space, with components which are functions of the coordinates. In other words, we can write

$$\mathcal{E}^n = \sum_k (\mathcal{E}^n_k ds_k + \mathcal{E}^n_{\bar{k}} d\bar{s}_k), \quad \bar{\mathcal{E}}^n = \sum_k (\bar{\mathcal{E}}^{\,n}_{\,k} ds_k + \bar{\mathcal{E}}^{\,n}_{\,\bar{k}} d\bar{s}_k). \tag{37}$$

As for the coefficients of L_0 and $\mathbb{1}$, we take $\mathcal{E}^0$, $\mathcal{E}$ to be holomorphic one-forms, with $\bar{\mathcal{E}}^0$, $\bar{\mathcal{E}}$ being the conjugates.

Our construction of the coherent states will follow the steps we outlined for the case of $SL(2,\mathbb{R})$. We will first show that $\mathcal{E}^n$ can be chosen to be a holomorphic one-form and $\bar{\mathcal{E}}^n$ as an antiholomorphic one-form. We can then define coherent states which will obey an antiholomorphicity condition. We will then set up and analyze the normalization of the wave functions for the coherent states. This will give us the ingredients for the symbols and star products taken up in the next section.

The first key result we will need is that it is possible to choose w_n, $\bar{w}_n$ as functions of the coordinates in such a way that $\sum_k \mathcal{E}^n_{\bar{k}} d\bar{s}_k = 0$ and $\sum_k \bar{\mathcal{E}}^{\,n}_{\,k} ds_k = 0$.[c] In other words, $\mathcal{E}^n$ is a holomorphic one-form and $\bar{\mathcal{E}}^n$ is an antiholomorphic one-form,

$$\mathcal{E}^n = \sum_k \mathcal{E}^n_k ds_k, \quad \bar{\mathcal{E}}^n = \sum_k \bar{\mathcal{E}}^{\,n}_{\,\bar{k}} d\bar{s}_k, \quad \mathcal{E}^n_{\bar{k}} = \bar{\mathcal{E}}^{\,n}_{\,k} = 0. \tag{38}$$

This result can be established by expansion around the origin and then using homogeneity of the orbit. The key point is that we can consider w_n and $\bar{w}_n$ to be defined by a power series expansion in terms of the coordinates s_k, $\bar{s}_k$, the coefficients of the expansion can be fixed by requiring $\mathcal{E}^n_{\bar{k}} d\bar{s}_k = 0$ and $\bar{\mathcal{E}}^{\,n}_{\,k} ds_k = 0$. To the quadratic order in the coordinates, we find

$$\bar{w}_n = \bar{s}_n - \frac{1}{2}\sum_m (n+2m)\bar{s}_{n+m} s_m + \cdots \tag{39}$$

with w_n given by the complex conjugate. (The details of the required calculations are given in Appendix A.) Thus, for a small neighborhood around the origin in the chosen coordinates we can see that we do obtain the holomorphicity conditions (38). To this order, we then find

$$(\mathcal{E}^0 - \bar{\mathcal{E}}^0)L_0 + (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1} = -\frac{1}{2}\sum_n (s_n d\bar{s}_n - \bar{s}_n ds_n)\Big[2nL_0 + \frac{c}{12}(n^3 - n)\Big] + \cdots. \tag{40}$$

If we evaluate this on the highest weight state, we find

$$\begin{aligned} \big((\mathcal{E}^0 - \bar{\mathcal{E}}^0)L_0 + (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1}\big)|0\rangle &= \frac{1}{2}\sum_k \left(\frac{\partial \mathcal{W}}{\partial s_k} ds_k - \frac{\partial \overline{\mathcal{W}}}{\partial \bar{s}_k} d\bar{s}_k\right)|0\rangle \\ &= \frac{1}{2}(\partial \mathcal{W} - \bar{\partial}\overline{\mathcal{W}})|0\rangle. \end{aligned} \tag{41}$$

[c] An issue with notation: To distinguish the holomorphic and antiholomorphic pieces of the one-forms $\mathcal{E}^n$, $\bar{\mathcal{E}}^n$, we use subscripts with an overbar. Once we argue that $\mathcal{E}^n_{\bar{k}}$ and $\bar{\mathcal{E}}^{\,n}_{\,k}$ are zero, we will drop this distinction to avoid the notational clutter of writing $\bar{s}^{\bar{k}}$, etc.

Here ∂ and $\bar{\partial}$ are the $(1,0)$ and $(0,1)$ components of the exterior derivative. We show in Appendix A that $\mathcal{E}^0 = \frac{1}{2}\partial W^0$, $\mathcal{E} = \frac{1}{2}\partial W$ for some functions W^0 and W and $\mathcal{W} = W^0 h + W$ in (41). As we will see from the Kähler two-form given later, the Kähler potenial is given by $K = \frac{1}{2}(\mathcal{W} + \overline{\mathcal{W}})$. To the order we have evaluated these functions in (40),

$$K = \frac{1}{2}(\mathcal{W} + \bar{\mathcal{W}}) \approx \sum_n s_n \bar{s}_n \Big[2nh + \frac{c}{12}(n^3 - n)\Big] + \cdots . \tag{42}$$

The fact that the Kähler potential K is characterized by two numbers h and c has been observed and commented on before.[9,10] In Appendix A, we give the expansion of $\mathcal{E}^n$ and $\bar{\mathcal{E}}^n$ to the next order in the coordinates.

To go beyond the small neighborhood around the origin, we will use the homogeneity of the coset space. Assume that we have obtained the result (38) in some neighborhood of the origin. We then consider translations of U as UV, where V is of the form

$$V = \exp\left(\sum_{n=1}^{\infty} \bar{\xi}_n L_n - \xi_n L_{-n}\right) \tag{43}$$

for infinitesimal ξ_n, $\bar{\xi}_n$. Writing out $(UV)^{-1}d(UV)$ to first order in ξ_n, $\bar{\xi}_n$, the holomorphicity conditions (38) become differential equations for these quantities. The integrability conditions for these equations are satisfied by virtue of the Maurer–Cartan identities for $U^{-1}dU$. Therefore, one can extend the neighborhood where the holomorphicity conditions are obtained. The detailed calculations supporting these statements are given in Appendix A.

3.3. *Coherent states*

We will now move to the next step in the construction of the coherent states. Starting from the highest weight state $|0\rangle$, we can define the states in the Verma module of the form

$$|\{\tilde{n}\}\rangle = \cdots L_{-3}^{n_3} L_{-2}^{n_2} L_{-1}^{n_1}|0\rangle, \tag{44}$$

where we can use the Virasoro algebra to order the L_{-n}'s in increasing level number to the left. (We use $\{\tilde{n}\}$ with a tilde over n to denote the unnormalized states; for the normalized states, given below, we will write $\{n\}$.) The matrix of inner products for the (unnormalized) states of the Verma module is given as

$$\mathcal{M}_{\{n\},\{m\}} = \langle\{\tilde{n}\}|\{\tilde{m}\}| = \langle 0|L_1^{n_1} L_2^{n_2} L_3^{n_3} \cdots L_{-3}^{m_3} L_{-2}^{m_2} L_{-1}^{m_1}|0\rangle. \tag{45}$$

This has a block-diagonal form, with the inner product between states of different level number being zero. The properly normalized states are then

$$|\{n\}\rangle = \sum_{\{m\}} (\mathcal{M}^{-\frac{1}{2}})_{\{n\},\{m\}} \cdots L_{-3}^{m_3} L_{-2}^{m_2} L_{-1}^{m_1}|0\rangle. \tag{46}$$

We can now define the coherent state wave function corresponding to the state $|\{n\}\rangle$ by

$$\Psi_{\{n\}} = \langle 0|U^\dagger|\{n\}\rangle e^{i\Theta}, \tag{47}$$

where Θ is a phase to be specified shortly. Using $U^\dagger dU = -dU^\dagger U$ and (36), write[d]

$$dU^\dagger = \left[\sum_{n,k} -\bar{\mathcal{E}}^{\,n}_{\,k} d\bar{s}_k L_n + \mathcal{E}^{\,n}_{\,k} ds_k L_{-n} - \big((\mathcal{E}^0 - \bar{\mathcal{E}}^0)L_0 - (\mathcal{E} - \bar{\mathcal{E}})\mathbb{1}\big)\right] U^\dagger. \tag{48}$$

This shows that the wave functions (47) obey the antiholomorphicity condition

$$\left(\frac{\partial}{\partial s_k} + \frac{1}{2}\frac{\partial \mathcal{W}}{\partial s_k} + i\frac{\partial \Theta}{\partial s_k}\right)\Psi_{\{n\}} = 0. \tag{49}$$

We now choose $\Theta = i(\mathcal{W} - \bar{\mathcal{W}})/2$, so that this equation becomes

$$\left(\frac{\partial}{\partial s_k} + \frac{1}{2}\frac{\partial K}{\partial s_k}\right)\Psi_{\{n\}} = 0, \tag{50}$$

where $K = \frac{1}{2}(\mathcal{W} + \bar{\mathcal{W}})$ is the Kähler potential. This equation can be solved for the coherent state wave functions to write

$$\Psi_{\{n\}} = \langle 0|U^\dagger|\{n\}\rangle e^{i\Theta} = e^{-\frac{1}{2}K}\Phi_{\{n\}}(\bar{s}), \tag{51}$$

where $\Phi_{\{n\}}(\bar{s})$ depend only on the antiholomorphic coordinates $\bar{s}_k$.

3.4. *The Kähler two-form and the phase volume*

Our next step is to write down the Kähler two-form and the Kähler potential in terms of U, for the purpose of setting up the normalization integrals for the states (51).

Going back to (36), note that it defines a left-invariant one-form $U^{-1}dU$, since $(V_L U)^{-1} d(V_L U) = U^{-1}dU$ for constant V_L. Further,

$$\langle 0|U^{-1}dU|0\rangle = \frac{1}{2}(\partial\mathcal{W} - \bar{\partial}\,\overline{\mathcal{W}}). \tag{52}$$

[d] We expect that the one-form $U^\dagger dU$ is closely related to a similar quantity in reference 9, which develops and uses the group manifold approach to quantization of the Virasoro group. In this approach, the group law is first obtained recursively starting from the commutation rules. Left and right group actions can then be obtained on functions on the group manifold. The resulting representations are reducible in general. Subsidiary conditions, the polarization conditions, are chosen to obtain irreducible representations. The approach is more general than, but closely related to, the standard geometric quantization. From the group law, the so-called quantization one-form is constructed, Eq. (3.4) of that paper. Holomorphicity is not *a priori* important, since one is using the group law. However, the two cases of "complete polarization" considered in Ref. 16 do reduce to the two orbits with Kähler structure, $\widehat{\text{diff}\, S^1}/U(1)$ and $\widehat{\text{diff}\, S^1}/SL(2,\mathbb{R})$. We expect that there is a transformation of the coordinates l_k used in Ref. 16 to our $s_k, \bar{s}_k$, by which the one-form in (3.4) of that paper reduces to our result (48), up to a total differential. For us holomorphicity is important in developing the star product, so we have chosen to separate the coordinates into holomorphic and antiholomorphic ones from the beginning so that we have holomorphicity for the one-forms $\mathcal{E}$.

This leads to a left-invariant two-form, which is the Kähler two-form and which can serve as the symplectic structure of interest, given by

$$\omega = id\langle 0|U^{-1}dU|0\rangle = \frac{i}{2}\bar{\partial}\partial(\mathcal{W}+\overline{\mathcal{W}}) = i\bar{\partial}\partial K. \tag{53}$$

We also note that the existence of a left-invariant symplectic structure has been emphasized by Witten.[9] Since $d(U^{-1}dU) = -U^{1-}dUU^{-1}dU$, we can use (48) to obtain another useful expression for ω,

$$\begin{aligned}
\omega =& -i\langle 0|U^{1-}dUU^{-1}dU|0\rangle \\
=& -i\sum_{n,k}\langle 0|\big[\big(\bar{\mathcal{E}}^n L_n - \mathcal{E}^n L_{-n} + (\mathcal{E}^0-\bar{\mathcal{E}}^0)L_0 + (\mathcal{E}-\bar{\mathcal{E}})\mathbb{1}\big) \\
&\wedge\Big(\bar{\mathcal{E}}^k L_k - \mathcal{E}^k L_{-k} + (\mathcal{E}^0-\bar{\mathcal{E}}^0)L_0 + (\mathcal{E}-\bar{\mathcal{E}})\mathbb{1}\Big)\Big]|0\rangle \\
=& \, i\sum_{n,k}\bar{\mathcal{E}}^n\wedge\mathcal{E}^k\langle 0|L_nL_{-k}|0\rangle \\
=& \, i\sum_{n}\bar{\mathcal{E}}^n\wedge\mathcal{E}^n\Big(2nh+\frac{c}{12}(n^3-n)\Big) \equiv i\Omega_{kl}d\bar{s}_k\wedge ds_l,
\end{aligned} \tag{54}$$

$$\Omega_{kl} = \sum_n \bar{\mathcal{E}}^n_k\mathcal{E}^n_l\Big(2nh+\frac{c}{12}(n^3-n)\Big).$$

The Kähler one-form $\mathcal{A}$ corresponding to this is given by (52) as

$$\mathcal{A} = \frac{i}{2}(\partial\mathcal{W}-\bar{\partial}\overline{\mathcal{W}}) = \frac{i}{2}(\partial K-\bar{\partial}K) + d(\Theta/2). \tag{55}$$

The antiholomorphicity condition (50) may be viewed as the polarization condition for geometric quantization of ω in (53) or (54). The exact term in $\mathcal{A}$ is removed by a phase transformation of the wave functions, as we have already done in defining Ψ_n in (47).

The Kähler two-form defines a phase-space volume $d\mu$ or an integration measure which is invariant under constant left translations of U. This may be written as

$$d\mu = (\det\Omega)\prod_k d\bar{s}_k ds_k. \tag{56}$$

Since there are an infinite number of coordinates, any integration carried out using this must be understood in a regularized sense, as defined over a finite set of modes, taking the limit of the total number of modes becoming infinite at the end. The determinant of Ω must be understood in a similar regularized way.

3.5. *Normalization of wave functions*

With the understanding of the left-invariant measure as given above, we can now show that the coherent states (47) can be normalized. We will first show the reasoning for the orthonormality of the wave functions, assuming the existence of the relevant integrals. The latter issue will be taken up subsequently. We start by considering the normalization integral

$$\begin{aligned} N_{\{n\}\{m\}} &= \int d\mu(U)\ \Psi^*_{\{n\}}(U)\Psi_{\{m\}}(U) = \int d\mu(U)\langle\{n\}|U|0\rangle\langle 0|U^\dagger|\{m\}\rangle \\ &\equiv \langle\{n\}|N|\{m\}\rangle, \end{aligned} \tag{57}$$

where N denotes the operator

$$N = \int d\mu(U) U|0\rangle\langle 0|U^\dagger. \tag{58}$$

Complex conjugation of this equation shows that $N_{\{n\}\{m\}}$ is a hermitian matrix or it can be viewed as a hermitian operator on the states (46) of the Verma module. Further, by translational invariance of the integration measure we have

$$\begin{aligned} N_{\{n\}\{m\}} &= \int d\mu(V_L U)\langle\{n\}|V_L U|0\rangle\langle 0|U^\dagger V_L^\dagger|\{m\}\rangle \\ &= \int d\mu(U)\langle\{n\}|V_L U|0\rangle\langle 0|U^\dagger V_L^\dagger|\{m\}\rangle. \end{aligned} \tag{59}$$

We take V_L to be of the form

$$V_L = \exp\left[\sum_k \bar{\theta}_k L_k - \theta_k L_{-k}\right], \tag{60}$$

where θ_k, $\bar{\theta}_k$ are infinitesimal parameters which are constant, i.e. independent of s_l, $\bar{s}_l$. To linear order in θ_k, Eq. (59) then leads to

$$\int d\mu(U)\left[\langle\{n\}|L_{-k}U|0\rangle\langle 0|U^\dagger|\{m\}\rangle - \langle\{n\}|U|0\rangle\langle 0|U^\dagger L_{-k}|\{m\}\rangle\right] = 0. \tag{61}$$

Since we can write $\langle\{n\}|L_{-k} = (L_{-k})_{\{n\}\{m\}}\langle\{m\}|$, this translates to $[L_{-k}, N] = 0$. Similarly, from the coefficient of $\bar{\theta}_{\bar{n}}$, we also get $[L_k, N] = 0$. It is also easy to see that $[L_0, N] = 0$. Since N commutes with all L_k for all states of the form (46), and since these states form a basis, we can write $N_{\{n\}\{m\}} = \Lambda\delta_{\{n\}\{m\}}$, where Λ is a constant independent of the state labels $\{n\}$, $\{m\}$. But Λ can depend on c and h which characterize the representation. This constant Λ can be absorbed into the definition of the measure $d\mu$; we will do this from now on, so that we have the result

$$\int d\mu(U)\Psi^*_{\{n\}}(U)\Psi_{\{m\}}(U) = \delta_{\{n\}\{m\}}. \tag{62}$$

Given the definition of the wave functions (47), this can also be viewed as the completeness relation

$$\int d\mu(U) U|0\rangle\langle 0|U^\dagger = \mathbb{1}. \tag{63}$$

We now come to the question of whether Λ is finite, i.e. whether the normalization integrals exist. From the discussion given above, Λ is given by the integral of e^{-K},

$$\Lambda = \int d\mu \langle 0|U|0\rangle\langle 0|U^\dagger|0\rangle = \int d\mu e^{-K}. \tag{64}$$

There are two issues one needs to address here. There are an infinite number of coordinates, so a suitable regularization has to be used, as is the case for the functional integral in any field theory. We have to assume the existence of such a regularization to make the question well-defined. The simplest possibility would be to truncate the integral to a finite number of variables, say N, taking the limit $N \to \infty$ after the expectation values of operators have been evaluated. Even after truncation to a finite set of modes, there is still the question of whether the integration over each coordinate is convergent. In other words, does the factor e^{-K} provide sufficient damping for large magnitudes for the coordinates? We will now present three sets of arguments pointing out features of K relevant to these questions.

First of all, we note that in the $SL(2,\mathbb{R})$ case, the integral of e^{-K} does exist, with the restriction $h > \frac{1}{2}$, as indicated in Sec. 2. In particular, $K \to \infty, e^{-K} \to 0$ as $|w| \to \infty$ (which is equivalent to the limit $|s| \to 1$). Therefore, as the first step towards analyzing the asymptotic behavior of K, we can use $SL(2,\mathbb{R})$ subalgebras, defined by $L_0, L_{\pm m}$, for fixed m. If we consider all $(w_n, \bar{w}_n)$ to be zero except for one pair, say, $(w_m, \bar{w}_m)$, then K will reduce to the Kähler potential for $SL(2,\mathbb{R})$ and hence we get $K \to \infty$, $e^{-K} \to 0$ as $|w_m| \to \infty$. Thus, for every plane in the orbit space, $K \to \infty$ for large $|w|$'s. This property also holds if we take $|w_m| \to \infty$ holding all other w's fixed and finite, but not necessarily zero. Towards this, note that U involves the combination $\bar{w}_n L_n - w_n L_{-n}$. We then write

$$\sum \bar{w}_n L_n = \sqrt{2}\bar{w}_2 \mathcal{L}_2, \quad \mathcal{L}_2 = \frac{1}{\sqrt{2}}\left(L_2 + \sum_{n\neq 2} \frac{\bar{w}_n L_n}{\bar{w}_2}\right). \tag{65}$$

This is for the case where we plan to take $|w_2| \to \infty$ keeping other w's fixed. The Virasoro algebra leads to the commutation rules

$$\begin{aligned} [\mathcal{L}_2, \mathcal{L}_{-2}] &= 2\mathcal{L}_0 + \mathbb{X}, \\ [\mathcal{L}_0, \mathcal{L}_2] &= -2\mathcal{L}_2 - \mathbb{Y}, \\ [\mathcal{L}_0, \mathcal{L}_{-2}] &= 2\mathcal{L}_{-2} + \mathbb{Y}^\dagger, \end{aligned} \tag{66}$$

where $\mathcal{L}_0 = L_0 + (c/8)$ and

$$\begin{aligned} \mathbb{X} &= \frac{1}{2}\sum_{n\neq 2}\left[\frac{(n+2)w_n}{w_2} L_{2-n} + \frac{(n+2)\bar{w}_n}{\bar{w}_2} L_{n-2} + \frac{c}{12}\frac{\bar{w}_n w_n}{\bar{w}_2 w_2}(n^3 - n)\right] \\ &\quad + \frac{1}{2}\sum_{n,m\neq 2} \frac{(n+m)\bar{w}_n w_m}{\bar{w}_2 w_2} L_{n-m} \\ \mathbb{Y} &= \sum_{n\neq 2} \frac{(n+2)\bar{w}_n}{\sqrt{2}\bar{w}_2} L_n. \end{aligned} \tag{67}$$

The terms $\mathbb{X}$, $\mathbb{Y}$ vanish as $|w_2| \to \infty$ keeping other coordinates finite, leading to an $SL(2,\mathbb{R})$ subalgebra of $\mathcal{L}_0$, $\mathcal{L}_{\pm 2}$. Note also that we can write

$$U = \exp\left(\sum_n \bar{w}_n L_n - w_n L_{-n}\right) = \exp\left(\sqrt{2}(\bar{w}_2 \mathcal{L}_2 - w_2 \mathcal{L}_{-2})\right). \tag{68}$$

In calculating $\langle 0|U|0\rangle$, $\langle 0|U^\dagger|0\rangle$, we use the Baker–Campbell–Hausdorff formula to rearrange $U^\dagger$ as

$$U^\dagger = e^{f_n L_{-n}} e^{-\frac{1}{2}(W^0 L_0 + W)} e^{\bar{f}_n L_n}. \tag{69}$$

This rearrangement only uses the commutation rules. Since the structure constants in the commutation rules (66) become those of $SL(2,\mathbb{R})$ as $|w_2| \to \infty$, keeping all other w_n's fixed, we get the result

$$K = K_{SL(2,\mathbb{R})}(\sqrt{2}w_2, \sqrt{2}\bar{w}_2) + \cdots, \tag{70}$$

where the ellipsis indicates terms which vanish in the limit. It then follows that e^{-K} vanishes in the limit $|w_2| \to \infty$, keeping all other w_n's fixed. A similar argument holds for any $|w_m| \to \infty$ keeping all other w's fixed.

Our second observation relates to the behavior of K under scaling, i.e. how it behaves as we go to large $|w_n|$ uniformly for all n. We can see that a common scaling up of the w's will increase K. Writing $U = e^{iC}$ in terms of the hermitian operator $C = -i\sum_n (\bar{w}_n L_n - w_n L_{-n})$, we find

$$\begin{aligned}\exp\left[-K((1+\epsilon)w, (1+\epsilon)\bar{w})\right] &= \langle 0|e^{-i(1+\epsilon)C}|0\rangle\langle 0|e^{i(1+\epsilon)C}|0\rangle \\ &= \exp\left[-K(w,\bar{w})\right] - i\epsilon\langle 0|e^{-iC}C|0\rangle\langle 0|e^{iC}|0\rangle \\ &\quad + i\epsilon\langle 0|e^{iC}C|0\rangle\langle 0|e^{-iC}|0\rangle + \cdots.\end{aligned} \tag{71}$$

This leads to

$$\begin{aligned}\sum_n \left[w_n \frac{\partial}{\partial w_n} + \bar{w}_n \frac{\partial}{\partial \bar{w}_n}\right] K &= 2e^K \left[\langle 0|\cos C|0\rangle\langle 0|C \sin C|0\rangle\right. \\ &\quad \left.- \langle 0|\sin C|0\rangle\langle 0|C\cos C|0\rangle\right],\end{aligned} \tag{72}$$

where, for brevity, we use $\langle \cos C\rangle = \langle 0|\cos C|0\rangle$, $\langle C\sin C\rangle = \langle 0|C\sin C|0\rangle$, etc. Since C is hermitian, we can diagonalize it. If $|\alpha\rangle$ denote the states which diagonalize C, with eigenvalues c_α, we can write

$$\begin{aligned}\langle\cos C\rangle\langle C\sin C\rangle - \langle \sin C\rangle\langle C\cos C\rangle &= \sum p_\alpha p_\beta c_\beta [\cos c_\alpha \sin c_\beta - \sin c_\alpha \cos c_\beta] \\ &= -\sum p_\alpha p_\beta c_\beta \sin(c_\alpha - c_\beta) \\ &= \frac{1}{2}\sum p_\alpha p_\beta (c_\alpha - c_\beta)\sin(c_\alpha - c_\beta),\end{aligned} \tag{73}$$

where $p_\alpha = |\langle 0|\alpha\rangle|^2$. This shows that the right-hand side of (72) is always positive. Therefore, K will continually increase with λ under scaling $(w_n, \bar{w}_n) \to (\lambda w_n, \lambda \bar{w}_n)$,

$\lambda > 0$. The limits of integration for the w's should be defined by the vanishing of e^{-K} in (64), and we see that, with the two properties of K given above, we can expect convergence for integration over each mode.

Our third observation is about regularizing the integration over the infinite number of modes. As mentioned before, a regularization truncating to a finite number of coordinates, say N, is needed, with the limit $N \to \infty$ eventually. Ultimately, this will require treating K as if it is the action for a quantum field theory. Regularization by truncation to a finite number of modes is familiar from field theory. This is also somewhat similar to what is done in Ref. 19, which presents the calculation of the integral $\int d\mu\, e^{-H}$, where $d\mu$ is the symplectic measure on $\widehat{\text{diff}\, S^1}/SL(2,\mathbb{R})$ and H is the Hamiltonian for translations on S^1, i.e. for the action of L_0. The result up to two loops is obtained; regularization is implicit as in standard field theory calculations. It is also argued that a similar result holds for $\widehat{\text{diff}\, S^1}/U(1)$, which is the orbit we are interested in. For the present situation, it is not the integral of e^{-H} we need, but note that we do have a field theoretic way of understanding the normalization integral (64). The action is not defined by the Hamiltonian corresponding to the action of L_0, but rather it is given by the Kähler potential K. From (33) we see that K can indeed be viewed as an action for a complex field χ. In particular, if we consider the large c limit, then we can scale $\chi \to \chi/\sqrt{c}$ for all the modes s_n, $\bar{s}_n$, $n \geq 2$. The set of terms in K which are proportional to c do not involve s_1, $\bar{s}_1$, so this scaling does not affect those modes. For large c, we then get

$$\begin{aligned} K &\approx \frac{1}{4\pi} \int d\tau \left[\frac{i}{12} (\bar{\chi}''\chi' - \chi''\bar{\chi}') + \frac{i}{6} \bar{\chi}\chi' - 4h \log(1 - \bar{s}_1 s_1) \right] \\ &= \frac{1}{12} \sum_n (n^3 - n) \bar{s}_n s_n - 2h \log(1 - \bar{s}_1 s_1). \end{aligned} \tag{74}$$

We get a Gaussian integral for s_n, $n \neq 1$ and the $SL(2,\mathbb{R})$ integral for the s_1 mode. The determinant arising from the Gaussian term can be regularized as is done in any field theory, for example, using a cutoff or using the ζ-function. We see that the integral does exist in the sense of quantum field theory, at least for large c (and for $h > \frac{1}{2}$). The expression given above suggests that K may be viewed as a deformation of the Fubini-Study potential. It is related to the central extension. A different choice of coordinates which can make this a little more transparent would involve embedding $\widehat{\text{diff}\, S^1}/U(1)$ in the Siegel disk, see Eq. (31) of Ref. 21.

Another observation of relevance would be that, just based on algebraic considerations, the states of the Verma module do have a finite norm, and this, in our language, is related to to the integral in (57). Further, Virasoro representations occur in the partition function for $(2+1)$-dimensional gravity.[11,12] Defining such partition functions is, in an indirect way, equivalent to the existence of Λ.

We also note that, bypassing the normalization integral in (57), it may be possible to obtain a resolution of identity by suitable restrictions, either by going to a quotient space,[22] or by considering a family of coadjoint orbits,[23] or by considering a subset of coherent states.[20,e]

4. Symbols and Star Products

We are now in a position to define the symbols associated to an operator and the corresponding star products.[f] We consider operators AB, etc. acting on the states (46). We may also view them as matrices with elements of the form $A_{\{n\}\{m\}}$, $B_{\{n\}\{m\}}$, etc. The symbols corresponding to these operators will be defined as

$$\begin{aligned}(A) &= \sum_{\{n\},\{m\}} \langle 0|U^\dagger|\{n\}\rangle A_{\{n\}\{m\}}\langle\{m\}|U|0\rangle = \langle 0|U^\dagger A U|0\rangle,\\ (B) &= \langle 0|U^\dagger B U|0\rangle.\end{aligned} \tag{75}$$

The symbol corresponding to a product of two operators takes the form

$$(AB) = \langle 0|U^\dagger A B U|0\rangle. \tag{76}$$

This can be written in terms of the symbols of the individual operators and their derivatives which will constitute the star product. For this, we first note that the completeness relation for the states (44) takes the form

$$\sum_{\{n\},\{m\}} |\{\tilde{n}\}\rangle (\mathcal{M}^{-1})_{\{n\},\{m\}}\langle\{\tilde{m}\}| = \mathbb{1}. \tag{77}$$

We can now rewrite the symbol of the operator product in (76) by using the completeness relation as

$$\begin{aligned}(AB) &= \langle 0|U^\dagger A U \mathbb{1} U^\dagger B U|0\rangle\\ &= \sum_{\{n\},\{m\}} \langle 0|U^\dagger A U|\{\tilde{n}\}\rangle(\mathcal{M}^{-1})_{\{n\},\{m\}}\langle\{\tilde{m}\}|U^\dagger B U|0\rangle\\ &= \sum_{\{n\},\{m\}} \langle 0|U^\dagger A U \cdots L_{-2}^{n_2}L_{-1}^{n_1}|0\rangle(\mathcal{M}^{-1})_{\{n\},\{m\}}\langle 0|L_1^{m_1}L_2^{m_2}\cdots U^\dagger B U|0\rangle\\ &= \langle 0|U^\dagger A U|0\rangle\langle 0|U^\dagger B U|0\rangle + \langle 0|U^\dagger A U L_{-1}|0\rangle\,\frac{1}{2h}\,\langle 0|L_1 U^\dagger B U|0\rangle + \cdots\\ &\equiv (A) * (B).\end{aligned} \tag{78}$$

Terms of the form $\langle 0|U^\dagger A U L_{-k} L_{-n}|0\rangle$ can be simplified using the following relations which are a consequence of Eqs. (36) and (41):

$$\begin{aligned}\frac{\partial U}{\partial s_k} &= -\sum_n \mathcal{E}_k^n U L_{-n} + U(\mathcal{E}_k^0 L_0 + \mathcal{E}_k \mathbb{1}),\\ \frac{\partial U^\dagger}{\partial s_k} &= \sum_n \mathcal{E}_k^n L_{-n} U^\dagger - (\mathcal{E}_k^0 L_0 + \mathcal{E}_k\mathbb{1})U^\dagger,\end{aligned} \tag{79}$$

[e] I thank a reviewer of an early version of this paper for pointing this out and bringing the relevant papers to my attention.
[f] In the terminology often used in the context of Berezin–Toeplitz quantization,[25] these are covariant symbols.

$$
\begin{aligned}
\frac{\partial U}{\partial \bar{s}_k} &= \sum_n \bar{\mathcal{E}}^{\,n}_{\,k} U L_n - U(\bar{\mathcal{E}}^{\,0}_{\,k} L_0 + \bar{\mathcal{E}}_k \mathbb{1}),\\
\frac{\partial U^\dagger}{\partial \bar{s}_k} &= -\sum_n \bar{\mathcal{E}}^{\,n}_{\,k} L_n U^\dagger + (\bar{\mathcal{E}}^{\,0}_{\,k} L_0 + \bar{\mathcal{E}}_k \mathbb{1}) U^\dagger.
\end{aligned} \tag{80}
$$

We now define covariant derivatives $\mathcal{D}_n$, $\bar{\mathcal{D}}_n$ by

$$
\begin{aligned}
\mathcal{D}_n &= -\sum_k (\mathcal{E}^{-1})^{\,k}_{\,n} \left[\frac{\partial}{\partial s_k} - \ell_0 \mathcal{E}^{\,0}_{\,k}\right],\\
\bar{\mathcal{D}}_n &= -\sum_k (\bar{\mathcal{E}}^{-1})^{\,k}_{\,n} \left[\frac{\partial}{\partial \bar{s}_k} + \ell_0 \bar{\mathcal{E}}^{\,0}_{\,k}\right],
\end{aligned} \tag{81}
$$

where ℓ_0 denotes the eigenvalue of L_0 for the expression on which these covariant derivatives act; i.e. it is the level number of the expression to the right of these derivatives. Using (79) and (80), it is easy to check that we can write

$$
\begin{aligned}
\mathcal{D}_{\{n\}}(A) &\equiv \cdots \mathcal{D}_{n_2} \mathcal{D}_{n_1}(A) = \langle 0 | U^\dagger A U \cdots L_{-n_2} L_{-n_1} | 0 \rangle,\\
\bar{\mathcal{D}}_{\{n\}}(B) &\equiv \cdots \bar{\mathcal{D}}_{n_2} \bar{\mathcal{D}}_{n_1}(B) = \langle 0 | L_{n_1} L_{n_2} \cdots U^\dagger B U | 0 \rangle.
\end{aligned} \tag{82}
$$

The star product can thus be rewritten as

$$
(A) * (B) = \sum_{\{n\},\{m\}} \mathcal{D}_{\{n\}}(A) (\mathcal{M}^{-1})_{\{n\},\{m\}} \bar{\mathcal{D}}_{\{m\}}(B). \tag{83}
$$

The symbols themselves have a value of zero for ℓ_0; we also have $\ell_0 = \pm n$ for $\mathcal{D}_n$ and $\bar{\mathcal{D}}_n$, respectively.

The second term on the right-hand side in the expansion in (78), which involves only $L_\pm 1$, comes from the $SL(2,\mathbb{R})$ subalgebra. In fact, as argued below, if we consider just the states generated by powers of L_{-1}, the star product in (83) will become the star product for $SL(2,\mathbb{R})$ states defined in Sec. 2.

It is useful to work out the next term which is at level 2. The matrix of inner products and its inverse are given by

$$
\begin{aligned}
&\mathcal{M} = \begin{bmatrix} 8h^2 + 4h & 6h \\ 6h & 4h + \frac{1}{2}c \end{bmatrix}, \quad \mathcal{M}^{-1} = \frac{1}{\det \mathcal{M}} \begin{bmatrix} 4h + \frac{1}{2}c & -6h \\ -6h & 8h^2 + 4h \end{bmatrix},\\
&\det \mathcal{M} = 2h(2h+1)(c-1) + 2h(4h-1)^2.
\end{aligned} \tag{84}
$$

ar Here the matrix elements refer to the states $|1\rangle = L^{\,2}_{-1}|0\rangle$, $|2\rangle = L_{-2}|0\rangle$. Explicitly, to this order, we get

$$
\begin{aligned}
(A) * (B) = {} & (A)(B) + \frac{1}{2h} (\mathcal{E}^{-1})^{\,k}_{\,1} \frac{\partial (A)}{\partial s_k} (\bar{\mathcal{E}}^{-1})^{\,k'}_{\,1} \frac{\partial (B)}{\partial \bar{s}_{k'}}\\
&+ \frac{4h + \frac{1}{2}c}{\det \mathcal{M}} \left[\left((\mathcal{E}^{-1})^{\,k}_{\,1} \frac{\partial}{\partial s_k} - (\mathcal{E}^{-1})^{\,k}_{\,1} \mathcal{E}^{\,0}_{\,k} \right) (\mathcal{E}^{-1})^{\,l}_{\,1} \frac{\partial (A)}{\partial s_l} \right.\\
&\left. \times \left((\bar{\mathcal{E}}^{-1})^{\,k'}_{\,1} \frac{\partial}{\partial \bar{s}_{k'}} - (\bar{\mathcal{E}}^{-1})^{\,k'}_{\,1} \bar{\mathcal{E}}^{\,0}_{\,k'} \right) (\bar{\mathcal{E}}^{-1})^{\,l'}_{\,1} \frac{\partial (B)}{\partial \bar{s}_{l'}} \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{6h}{\det\mathcal{M}}\Bigg[\left((\mathcal{E}^{-1})_1^{\,k}\frac{\partial}{\partial s_k}-(\mathcal{E}^{-1})_1^{\,k}\mathcal{E}_{\,k}^{0}\right)(\mathcal{E}^{-1})_1^{\,l}\frac{\partial(A)}{\partial s_l}(\bar{\mathcal{E}}^{-1})_2^{\,l'}\frac{\partial(B)}{\partial \bar{s}_{l'}}\\
&+(\mathcal{E}^{-1})_2^{\,k}\frac{\partial(A)}{\partial s_k}\left((\bar{\mathcal{E}}^{-1})_1^{\,k'}\frac{\partial}{\partial \bar{s}_{k'}}-(\bar{\mathcal{E}}^{-1})_1^{\,k'}\bar{\mathcal{E}}_{\,k'}^{0}\right)(\mathcal{E}^{-1})_1^{\,l'}\frac{\partial(B)}{\partial \bar{s}_{l'}}\Bigg]\\
&+\ \frac{8h^2+4h}{\det\mathcal{M}}\left[(\mathcal{E}^{-1})_2^{\,k}\frac{\partial(A)}{\partial s_k}(\bar{\mathcal{E}}^{-1})_2^{\,l'}\frac{\partial(B)}{\partial \bar{s}_{l'}}\right]+\cdots. \qquad (85)
\end{aligned}
$$

There is summation over k, k', l, l' in various expressions in this equation.

If we take the large c limit at fixed h, the matrix $\mathcal{M}^{-1}$ reduces to its $(1,1)$-component $\mathcal{M}_{11}^{-1} \approx (8h^2+4h)^{-1}$ with all other elements zero. The corresponding term, along with the terms with first derivatives of (A) and (B) given in the first line, give the first two terms of the star-product for the orbit $SL(2,\mathbb{R})/U(1)$ labeled by the highest weight $L_0 = h$.

It is easy to see that this property is obtained for higher terms as well. For this, consider the matrix $\mathcal{M}$ at level n. The element $\mathcal{M}_{11}$ arises from $L_{-1}^{n}|0\rangle$, so this term is exactly what it is for $SL(2,\mathbb{R})$. Further, terms arising from $L_{-2}L_{-1}^{n-2}|0\rangle$, $L_{-3}L_{-1}^{n-3}|0\rangle$, $L_{-3}L_{-2}L_{-1}^{n-5}|0\rangle$, etc. will all have powers of c, since we get the central terms for $\langle 0|L_k L_{-k}|0\rangle$. The principal minor or cofactor corresponding to $\mathcal{M}_{11}$ thus dominates the determinant as $c \to \infty$, and is order c^{n-1}, while the cofactors corresponding to the other elements will have smaller powers of c. As a result, in the inverse of $\mathcal{M}$, the element $(\mathcal{M}^{-1})_{11}$ dominates as $c \to \infty$, with the limiting value $1/\mathcal{M}_{11}$, while other elements tend to zero. Since $1/\mathcal{M}_{11}$ corresponds to $SL(2,\mathbb{R})$, we get the result that the star product for the Virasoro group characterized by h, c becomes the star product for the $SL(2,\mathbb{R})/U(1)$ orbit labeled by h; i.e.

$$
(A) * (B)]_{\mathrm{Virasoro},h,c} \to (A) * (B)]_{SL(2,\mathbb{R}),h} \quad \text{as } c \to \infty. \qquad (86)
$$

This reduction also conforms to what is expected from the large c behavior discussed in connection with (74). In the context of this result, it may be interesting to recall that the $c \to \infty$ limit is important in the context of semiclassical limits of partition functions for $(2+1)$-dimensional gravity.[4,5] The coset $SL(2,\mathbb{R})/U(1)$ with the star product in (86), is the noncommutative version of AdS_2. For more on this space, see Ref. 26.

5. Discussion

We have worked out the construction of coherent states for the Virasoro group using a class of orbits with $c > 1$, $h > \frac{1}{2}$. The basic results are in Sec. 3. The wave functions for the coherent states are given in (47). They satisfy a certain antiholomorphicity property, as is evident from (51). We also give an explicit formula for the Kähler potential K in terms of complex fields defined on the circle. We consider some of the key properties of K and also discuss the normalization integral for the wave functions as the partition function for the one-dimensional field theory corresponding to this K.

In the case of the well-known coherent states on orbits of compact groups as well as the historically first case of the oscillator, one can define reproducing kernels via the overlap of coherent state wave functions. It would be interesting to explore the properties of such kernels in the Virasoro case.

Also, we have restricted ourselves to $c > 1$, since we were partly motivated by possible applications to $(2+1)$-dimensional gravity. However, the discrete set of unitary $c < 1$ representations are also very interesting from a physics point of view, since they are relevant to the so-called minimal models in CFT. Coherent states for such cases are also worth exploring, but are obviously beyond the scope of this work.

Turning to the more physical side of things, our results can be viewed as furnishing a noncommutative version of the infinite dimensional Kähler space $\widehat{\text{diff}\, S^1}/S^1$. The star product (83) gives the noncommutative algebra for functions. As mentioned in the Introduction, this analysis was partially motivated by potential application to $(2+1)$-dimensional gravity. In the spirit of noncommutative geometry, one can use a Hilbert space of states to model the spatial manifold.[27,28] We have recently argued for elaborating this framework, with a doubling of the Hilbert space as in thermofield dynamics, with the gauge fields for gravity, i.e. the frame fields and the spin connection, coupling to the two Hilbert spaces in parity-conjugate ways.[29] In this framework, it is possible to obtain the Einstein–Hilbert action for gravity in $(2+1)$ dimensions (with a nonzero cosmological constant), upon taking the commutative limit. The explicit calculations were done using quantization of the orbits $SU(2)/U(1)$ or $SL(2,\mathbb{R})/U(1)$ to model the corresponding noncommutative spaces. However, since the partition function for gravity naturally involves representations of the Virasoro algebra due to the contributions from black holes,[11,12] one can ask whether it is possible to extend the analysis and use the carrier space of the representation as the Hilbert space of interest modeling the noncommutative space. The coherent states discussed here provide a way to define symbols and star products for such a formulation. We should expect that the commutative limit, which is the large (c, h) limit, will then lead to the Einstein–Hilbert action.

Our results may also be interpreted in terms of a mock quantum Hall system. We have analyzed the quantum Hall problem in arbitrary dimensions in a series of papers and argued that the lowest Landau level of such systems model the corresponding noncommutative spaces.[30] Effective actions, edge states, etc. were analyzed in such cases. The present discussion may be viewed as another example of this, now applied to an infinite-dimensional case.

As noted before, the Kähler potential defines an interesting one-dimensional field theory in its own right, given by

$$\begin{aligned} Z &= \int d\mu e^{-K} \\ K &= \frac{1}{4\pi}\int_0^{2\pi} d\tau \Big\{ \frac{c}{12}\left[\frac{i}{(1+\chi'+\bar{\chi}')}\left(\frac{\bar{\chi}''\chi'}{(1+\bar{\chi}')} - \frac{\chi''\bar{\chi}'}{(1+\chi')}\right)\right] \\ &\quad + \left(4h - \frac{c}{6}\right)(-i\bar{\chi}\chi') - 4h\log(1 - i\bar{\varphi}'\varphi)\Big\}. \end{aligned} \tag{87}$$

This should prove to be an interesting theory since it is closely tied to the Virasoro group. We propose to investigate this further in future.

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Appendix A

In this appendix, we will go over some of the details of the results mentioned in Sec. 3.

Analysis near the origin

The quantities w_n, $\bar{w}_n$ in U will be chosen as functions of the coordinates s_n, $\bar{s}_n$, i.e.

$$\bar{w}_k = \bar{s}_k + \bar{w}_k^{(2)} + \bar{w}_k^{(3)} + \cdots , \tag{A.1}$$

where $\bar{w}_k^{(r)}$ is of order r in powers of s_n, $\bar{s}_n$. Our strategy is to write down $U^{-1}dU$ and choose these functions such that $\mathcal{E}^n$ is a holomorphic differential and $\bar{\mathcal{E}}^n$ is an anti-holomorphic differential. For an operator X, a real number α, and for a variation of X, we have the identity

$$\begin{aligned}\frac{\partial}{\partial\alpha}[e^{-\alpha X}\delta[e^{\alpha X}]] &= e^{-\alpha X}\delta\left[\frac{\partial}{\partial\alpha}e^{\alpha X}\right] + \frac{\partial}{\partial\alpha}(e^{-\alpha X})\delta[e^{\alpha X}]\\ &= e^{-\alpha X}\delta[Xe^{\alpha X}] - e^{-\alpha X}X\delta[e^{\alpha X}]\\ &= e^{-\alpha X}\delta X e^{\alpha X}.\end{aligned} \tag{A.2}$$

Integrating over α from zero to one, we get

$$e^{-X}\delta[e^{X}] = \int_0^1 d\alpha e^{-\alpha X}\delta X e^{\alpha X}. \tag{A.3}$$

Using this equation, from the definition of U in (35), we can write

$$\begin{aligned}U^{-1}dU &= \sum_n \int_0^1 d\alpha e^{-\alpha X}(d\bar{w}_n L_n - dw_n L_{-n})e^{\alpha X}\\ X &= \sum_m (\bar{w}_m L_m - w_m L_{-m}).\end{aligned} \tag{A.4}$$

Expanding the exponential, we can write out the first two terms as

$$\text{Term } 1 = \sum d\bar{w}_n L_n - dw_n L_{-n},$$

$$\begin{aligned}\text{Term 2} &= -\frac{1}{2!}\sum_n [X, d\bar{w}_n L_n - dw_n L_{-n}]\\ &= -\frac{1}{2!}\sum_{m,n} \big[(m-n)\bar{w}_m d\bar{w}_n L_{m+n} - (m-n) w_m dw_n L_{-m-n}\\ &\quad + (m+n)(w_n d\bar{w}_m - \bar{w}_m dw_n) L_{m-n} + \frac{c}{12}(n^3-n)(w_n d\bar{w}_n - \bar{w}_n dw_n)\big].\end{aligned} \tag{A.5}$$

The L_{m-n} can have terms of the form L_k and L_{-k}, $k > 0$, as well as L_0 terms. Separating these out we find

$$\begin{aligned}\text{Term 2} &= -\frac{1}{2!}\sum_n (w_n d\bar{w}_n - \bar{w}_n dw_n)\Big(2nL_0 + \frac{c}{12}(n^3-n)\mathbb{1}\Big)\\ &\quad -\frac{1}{2!}\sum_k [\bar{C}^{(2)}_k L_k - C^{(2)}_k L_{-k}],\\ \bar{C}^{(2)}_k &= \sum_{n=1}^{k-1}(k-2n)\bar{w}_{k-n} d\bar{w}_n \Theta(k-3)\\ &\quad -\sum_n (k+2n)(\bar{w}_{k+n} dw_n - w_n d\bar{w}_{k+n})\Theta(k-1)\end{aligned} \tag{A.6}$$

with $C^{(2)}_k$ being the complex conjugate of $\bar{C}^{(2)}_k$. Also $\Theta(k-a) = 1$ for $k \geq a$ and zero otherwise. From the expression for $\bar{C}^{(2)}_k$, we see that there is one term in the coefficient of L_k which has dw_n. We rewrite this term using

$$\sum (k+2n)\bar{w}_{k+n} dw_n = d\Big[\sum (k+2n)\bar{w}_{k+n} w_n\Big] - \sum (k+2n) w_n d\bar{w}_{k+n}. \tag{A.7}$$

Thus, we find

$$\begin{aligned}\text{Coefficient of } L_k &= d\Big[\bar{w}_k + \frac{1}{2}\sum (k+2n)\bar{w}_{k+n} w_n\Big] - \frac{1}{2}\sum_1^{k-1}(k-2n)\bar{w}_{k-n} d\bar{w}_n \Theta(k-3)\\ &\quad -\sum_n (k+2n) w_n d\bar{w}_{k+n}\Theta(k-1).\end{aligned} \tag{A.8}$$

We now define w_k, $\bar{w}_k$ as functions of complex coordinates s_k, $\bar{s}_k$, with an expansion around the origin as

$$\bar{w}_k = \bar{s}_k - \frac{1}{2}\sum_n (k+2n)\bar{s}_{k+n} s_n + \bar{w}^{(3)}_k + \cdots, \tag{A.9}$$

where $\bar{w}^{(3)}_k$ denote terms which are cubic in $s_n, \bar{s}_n$. The coefficient of L_k now becomes

$$\begin{aligned}\text{Coefficient of } L_k &= d\bar{s}_k - \frac{1}{2}\sum_1^{k-1}(k-2n)\bar{s}_{k-n} d\bar{s}_n \Theta(k-3)\\ &\quad - \sum (k+2n) s_k d\bar{s}_{k+n}\Theta(k-1) + \cdots\\ &\equiv d\bar{s}_k - \frac{1}{2}\sum_l D^{(2)}_{kl} d\bar{s}_l + \cdots.\end{aligned} \tag{A.10}$$

We see that, to this order, there are no ds_n-terms in the coefficient of L_k, $k > 0$; i.e. it is an antiholomorphic one-form. To the same approximation, the coefficient of the L_0 and central terms become

$$(L_0, \mathbb{1}) - \text{terms} = -\frac{1}{2!}\sum_n (s_n d\bar{s}_n - \bar{s}_n ds_n)\Big(2nL_0 + \frac{c}{12}(n^3 - n)\mathbb{1}\Big). \tag{A.11}$$

The next term in the expansion of (A.4), corresponding to the double commutator of X with $\sum d\bar{w}_n L_n - dw_n L_{-n}$ is of the form

$$\text{Term 3} = \frac{1}{3!}\Big[\sum (w_n \bar{C}^{(2)}_n - \bar{w}_n C^{(2)}_n)\Big[2nL_0 + \frac{c}{12}(n^3 - n)\Big] + \sum \bar{C}^{(3)}_k L_k - C^{(3)}_k L_{-k}\Big]. \tag{A.12}$$

We can use $w_n \approx s_n$, $\bar{w}_n \approx \bar{s}_n$ in working out the cubic terms in this expression, to get expressions valid to the third order in s_n, $\bar{s}_n$. To this order, $\bar{C}^{(3)}_k$ is given by

$$\begin{aligned}\bar{C}^{(3)}_k \approx \sum_n 2kn(s_n d\bar{s}_n - \bar{s}_n ds_n)\bar{s}_k + \sum \delta_{k,m+n}(m-n)\bar{s}_m \bar{C}^{(2)}_n \\ + \sum \delta_{k,m-n}(m+n)[s_n \bar{C}^{(2)}_m - \bar{s}_m C^{(2)}_n].\end{aligned} \tag{A.13}$$

In this expression, a term like $\bar{s}_k \bar{s}_n ds_n$ (which is a holomorphic form and hence not what we want) can be removed by a term like $\bar{s}_k \bar{s}_n s_n$ in the expression for $\bar{w}^{(3)}_k$. The simplification of the expression for $\bar{C}^{(3)}_k$ is straightforward but long. With some rearrangements of terms, it can be brought to the form

$$\bar{C}^{(3)}_k = \sum_l D^{(3)}_{k,l} d\bar{s}_l - d\chi^{(3)}_k - \frac{3}{2}\sum \delta_{k,r-s-n}(r+s+n)(n-s)\bar{s}_m s_n ds_s. \tag{A.14}$$

The expression for $D^{(3)}_{k,l} d\bar{s}_l$ is long and not important for our argument (since it is an antiholomorphic form anyway), but we give it here for the sake of completeness,

$$\begin{aligned}\sum_l D^{(3)}_{k,l} d\bar{s}_l = \sum \big[& s_n \bar{s}_k d\bar{s}_n + 2kn\bar{s}_n s_n ds_k + \delta_{k,m+n}\delta_{n,r+s}(m-n)(r-s)\bar{s}_m \bar{s}_r d\bar{s}_s \\ & + 2\delta_{k,m+n}\delta_{n,r-s}(m-n)(r+s)\bar{s}_m s_s ds_r \\ & + \delta_{k,m+n}\delta_{n,r-s}(m-n)(r+s)\bar{s}_r s_s d\bar{s}_m \\ & + \delta_{k,m-n}\delta_{m,r+s}(m+n)(r-s)s_n \bar{s}_r d\bar{s}_s \\ & + \delta_{k,m-n}\delta_{m,r-s}(m+n)(r+s)s_n s_s d\bar{s}_r \\ & + \frac{1}{2}\delta_{k,r-s-n}\sigma(r,s,n)s_n s_s d\bar{s}_r + 2\delta_{k,m-n}\delta_{n,r-s}(m+n)(r+s)\bar{s}_m s_r d\bar{s}_s \\ & + \delta_{k,m-n}\delta_{n,r-s}(m+n)(r+s)\bar{s}_s s_r d\bar{s}_m \big].\end{aligned} \tag{A.15}$$

Also χ_k is given by

$$\begin{aligned}\chi^{(3)}_k = \sum \Big[& 2kn\bar{s}_n \bar{s}_k s_n + \delta_{k,m+n}\delta_{n,r-s}(m-n)(r+s)\bar{s}_m \bar{s}_r s_s \\ & + \frac{1}{2}\sigma(r,s,n)\delta_{k,r-s-n}\bar{s}_m s_n s_s + \delta_{k,m-n}\delta_{n,r-s}(m+n)(r+s)\bar{s}_m \bar{s}_s s_r\Big].\end{aligned} \tag{A.16}$$

In (A.15) and (A.16), $\sigma(r,s,n)$ is given by

$$\sigma(r,s,n) = \frac{1}{2}(2r^2 - s^2 - n^2 + r(n+s) + 2ns). \tag{A.17}$$

This is symmetric in n, s. In Term 3, $\chi_k^{(3)}$ can be removed by a suitable choice of $\bar{w}_k^{(3)}$, making the coefficient of L_k to be an antiholomorphic form, except for the last term in (A.14). We also note that there are cubic terms arising from the use of $\bar{w}_k^{(2)}$, as in (A.10), in Term 2. For this, we rewrite $\bar{C}_k^{(2)}$ as

$$\bar{C}_k^{(2)} = \sum_l D_{kl}^{(2)} d\bar{s}_l - d\chi_k^{(2)} + \tilde{\bar{C}}_k^{(2)},$$

$$\begin{aligned}\tilde{\bar{C}}_k^{(2)} = -\frac{1}{2}\Big[&\sum[\delta_{k,m+n}\delta_{n,r-s}(m-n)(r+s)\bar{s}_m(d\bar{s}_r s_s + \bar{s}_r ds_s)] \\ &+ \sum(m+n)(r+s)[\delta_{k,m-n}\delta_{m,r-s}s_n(d\bar{s}_r s_s + \bar{s}_r ds_s) + \delta_{k,m-n}\delta_{n,r-s}\bar{s}_r s_s d\bar{s}_m \\ &- \delta_{k,m-n}\delta_{n,r-s}\bar{s}_m(d\bar{s}_r s_s + \bar{s}_r ds_s) - \delta_{k,m-n}\delta_{m,r-s}\bar{s}_r s_s ds_n]\Big].\end{aligned} \tag{A.18}$$

In this expression, terms of the form $\bar{s}\bar{s}ds$ can be written as $d[\bar{s}\bar{s}s] - d\bar{s}\bar{s}s - \bar{s}d\bar{s}s$; the total derivative adds to the expression for $\chi_k^{(3)}$ and is removed by choice of $\bar{w}_k^{(3)}$. What is left will be an antiholomorphic form. This does not work for the two terms in (A.18) which have the $\bar{s}sds$ combination. These potentially problematic terms can be simplified as

$$\begin{aligned}\text{Problematic terms in } \tilde{\bar{C}}_k^{(2)} &= -\frac{1}{2}\sum \delta_{k,r-s-n}(r-s+n)(r+s)\bar{s}_r(s_n ds_s - s_s ds_n) \\ &= -\frac{1}{2}\sum \delta_{k,r-s-n}(n+s+r+(n-s)\bar{s}_r s_n ds_s,\end{aligned} \tag{A.19}$$

where, in the second line, we have used the antisymmetry of the first term in n, s to simplify the result. Comparing the contribution of the last term in (A.14) to (1/3!) $\bar{C}_k^{(3)}$ and the contribution of (A.19) to $-(1/2!)\bar{C}_k^{(2)}$, we see that they cancel out exactly. After removal of $\chi_k^{(3)}$ terms via choice of $\bar{w}_k^{(3)}$, we see that what is left of $\bar{C}_k^{(3)}$ is an antiholomorphic form.

The coefficient of L_k can thus be written as

$$\text{Coefficient of } L_k = \bar{\mathcal{E}}^k = d\bar{s}_{\bar{k}} - \frac{1}{2}\sum_l D_{kl}^{(2)} d\bar{s}_l + \frac{1}{3!}\sum_l D^{(3)} d\bar{s}_l + \cdots. \tag{A.20}$$

We have thus verified, in an expansion around the origin to cubic order in the coordinates, that there is a choice of $w_n, \bar{w}_n$ as a function of the coordinates $s_n, \bar{s}_{\bar{n}}$ for which $\bar{\mathcal{E}}^n$ is an antiholomorphic one-form.

Maurer–Cartan relations

The analysis given above for small values of s_k, $\bar{s}_k$ shows that one can choose $\bar{\mathcal{E}}^n$ to be an antiholomorphic one-form in an infinitesimal neighborhood of the origin. Our aim is now to extend this to larger and larger regions by a sequence of translations, $U \to UV$, where V is as in (43). For this, we will also need to use the Maurer–Cartan identity for U, so we will first work this out. From (36), assuming that we have already obtained the holomorphicity properties for $\mathcal{E}^n$ and $\bar{\mathcal{E}}^n$, we can write

$$\begin{aligned} U^\dagger \partial U &= -\sum_n \mathcal{E}^n L_{-n} + \mathcal{E}^0 L_0 + \mathcal{E}\mathbb{1}, \\ U^\dagger \bar{\partial} U &= \sum_n \bar{\mathcal{E}}^n L_n - \bar{\mathcal{E}}^0 L_0 - \bar{\mathcal{E}}\mathbb{1}. \end{aligned} \tag{A.21}$$

Taking the holomorphic exterior derivative of the first of these equations, we get one of the Maurer–Cartan identities as

$$\sum \left[-\partial\mathcal{E}^n L_{-n} + \partial\mathcal{E}^0 L_0 + \partial\mathcal{E}\mathbb{1} + \frac{1}{2}(m-n)\mathcal{E}^n \wedge \mathcal{E}^m L_{-n-m} - n\mathcal{E}^0 \wedge \mathcal{E}^n L_{-n} \right] = 0. \tag{A.22}$$

This yields three sets of relations corresponding to the coefficients of L_{-n}, L_0 and $\mathbb{1}$. These are

$$-(\partial\mathcal{E}^n + n\mathcal{E}^0 \wedge \mathcal{E}^n) + \frac{1}{2}\sum_{r,s}(s-r)\delta_{r+s,n}\mathcal{E}^r \wedge \mathcal{E}^s = 0, \tag{A.23}$$

$$\partial\mathcal{E}^0 = 0, \quad \partial\mathcal{E} = 0. \tag{A.24}$$

The last two relations tell us that we can write

$$\mathcal{E}^0 = \frac{1}{2}\partial W^0, \quad \mathcal{E} = \frac{1}{2}\partial W. \tag{A.25}$$

Note that the Kähler potential is related to these as $K = W^0 h + W$. We can now use these expressions for $\mathcal{E}^0$, $\mathcal{E}$ to write (A.23) as

$$-\partial\tilde{\mathcal{E}}^n + \frac{1}{2}\sum_{r,s}(s-r)\delta_{r+s,n}\tilde{\mathcal{E}}^r \wedge \tilde{\mathcal{E}}^s = 0, \tag{A.26}$$

where $\tilde{\mathcal{E}}^n = \mathcal{E}^n \exp(nW^0/2)$. This is the key identity we will need for extending the previous result.

Extending the result by use of translational invariance

Now consider defining $\mathcal{E}$'s and $\bar{\mathcal{E}}$'s after translation by V. To first order in $\sum \bar{\xi}_n L_n - \xi_n L_{-n}$, this leads to

$$\begin{aligned} (UV)^{-1}d(UV) &= V^{-1}dV + V^{-1}(U^{-1}dU)V \\ &\approx U^{-1}dU + \sum \left[d\bar{\xi}_n L_n - d\xi_n L_{-n} - [\bar{\xi}_n L_n - \xi_n L_{-n}, U^{-1}dU] \right]. \end{aligned} \tag{A.27}$$

We want to show that the coefficient of L_n, $n > 0$ is an antiholomorphic one-form; i.e. one can choose $\bar{\xi}_n$ such that the holomorphic differential part vanishes. The condition for this, upon using (36) and evaluating the commutator term, becomes

$$\sum [\partial\bar{\xi}_n L_n + \bar{\xi}^n \mathcal{E}^m (m+n) L_{n-m} - \bar{\xi}^n \mathcal{E}^0 n L_n] = 0 \tag{A.28}$$

with $n > m$. Rewriting this by isolating the coefficient of L_k, we get

$$\partial\bar{\xi}_k - k\mathcal{E}^0\bar{\xi}_k + \sum(2m+k)\bar{\xi}_{m+k}\mathcal{E}^m = 0. \tag{A.29}$$

Defining $\tilde{\bar{\xi}}_k = \bar{\xi}_k \exp(-kW^0/2)$, we can further write these conditions as

$$\partial\tilde{\bar{\xi}}_k + \sum_m (2m+k)\tilde{\bar{\xi}}_{m+k}\tilde{\mathcal{E}}^m = 0. \tag{A.30}$$

These are to be regarded as a set of equations which can be solved for $\bar{\xi}_k$. However, there are integrability conditions for these equations. They correspond to taking another holomorphic exterior derivative of (A.30), and upon using (A.30) again, become

$$\sum(2m+k)[\tilde{\bar{\xi}}_{m+k}\partial\tilde{\mathcal{E}}^m - \sum_r (2r+m+k)\tilde{\bar{\xi}}_{m+k+r}\tilde{\mathcal{E}}^r \wedge \tilde{\mathcal{E}}^m] = 0. \tag{A.31}$$

Because of the wedge product, we can antisymmetrize the coefficient of $\tilde{\mathcal{E}}^r \wedge \tilde{\mathcal{E}}^m$ in r, m. For this, we can use

$$\frac{1}{2}[(2m+k)(2s+m+k) - (m \leftrightarrow s)] = \frac{1}{2}(m-s)[2(m+s)+k]. \tag{A.32}$$

Further, we take $m \to n$ in the first term and $m \to s$, $m + r \to n$ in the second term. Equation (A.31) can then be written as

$$\sum(2n+k)\tilde{\bar{\xi}}_{n+k}\left[\partial\tilde{\mathcal{E}}^n - \frac{1}{2}\sum_{r,s}(s-r)\delta_{r+s,n}\tilde{\mathcal{E}}^r \wedge \tilde{\mathcal{E}}^s\right] = 0. \tag{A.33}$$

These are obviously satisfied as a result of the Maurer–Cartan identity (A.26).

What we have shown is that if we have U with a choice of w_n, $\bar{w}_n$ as functions of s_k, $\bar{s}_k$ for which $\mathcal{E}^n$ is a holomorphic one-form and $\bar{\mathcal{E}}^n$ is an antiholomorphic one-form, then we can find ξ_n, $\bar{\xi}_n$ such that $(UV)^{-1}d(UV)$ will have a holomorphic one-form as the coefficient of L_{-n} and an antiholomorphic one-form as the coefficient of L_n. This result, combined with the previous result that this property can be obtained in an infinitesimal neighborhood of the origin, as shown by explicit power series expansion, shows we can find coordinates such that $\mathcal{E}^n$ is a $(1,0)$-form and $\bar{\mathcal{E}}^n$ is a $(0,1)$-form.

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