



Well-posedness of nonlinear flows on manifolds of bounded geometry

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Abstract

We present straightforward conditions which ensure that a strongly elliptic linear operator L generates an analytic semigroup on Hölder spaces on an arbitrary complete manifold of bounded geometry. This is done by establishing the equivalent property that L is ‘sectorial,’ a condition that specifies the decay of the resolvent $(\lambda I - L)^{-1}$ as λ diverges from the Hölder spectrum of L . A key step is that we prove existence of this resolvent if λ is sufficiently large using a geometric microlocal version of the semiclassical pseudodifferential calculus. The properties of L and e^{-tL} we obtain can then be used to prove well-posedness of a wide class of nonlinear flows. We illustrate this by proving well-posedness on Hölder spaces of the flow associated with the ambient obstruction tensor on complete manifolds of bounded geometry. This new result for a higher-order flow on a noncompact manifold exhibits the broader applicability of our technique.

Keywords Manifolds of bounded geometry · Analytic semigroups · Sectorial operators · Semiclassical resolvent · Geometric heat flows · Bach flow

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1 Introduction

Methods from semigroup theory provide an elegant abstract method to establish well-posedness, stability and convergence results for large classes of evolutionary partial differential equations, such as those which govern geometric heat flows. For such purposes, the most useful semigroups are those which admit a holomorphic continuation in the ‘time’ variable. There is a nice characterization of these *analytic semigroups* in terms of their infinitesimal generators. More specifically, we write the semigroup as e^{-tA} , where A is a closed linear operator acting on a certain Banach space X . A classical theorem states that the function $t \mapsto e^{-tA}$ from $\mathbb{R}^+$ to the space $\mathcal{L}(X)$ of bounded linear operators on X admits a holomorphic extension to a neighborhood of $\mathbb{R}^+$ in $\mathbb{C}$ if and only if the operator A satisfies a property known as *sectoriality*, which as indicated below, involves restrictions on the spectrum of A , along with an estimate on its resolvent $(\lambda I - A)^{-1}$. We refer to [17, Chapters 5 and 6] for an elementary introduction to sectoriality and analytic semigroups.

In this work, we consider a broad class of geometric operators defined on sections of vector bundles over complete Riemannian manifolds with bounded geometry and show that they are indeed sectorial. This is the content of our main result—Theorem A—which is proved in Sect. 3 using tools from microlocal analysis. However, the main consequence of our work is that if the operators corresponding to geometric heat flows on manifolds of bounded geometry act on the naturally associated (little) Hölder spaces, and if their symbols satisfy readily verified algebraic conditions, then the initial value problems for those flows are well posed, and have good stability and convergence properties.

More precisely, in applications to PDE, the generator A is a (typically elliptic) differential operator and the sectoriality of such operators is known in a variety of settings. Our first goal in this paper is to prove the sectoriality estimate for strongly elliptic differential operators which satisfy a certain uniformity property, acting between sections of vector bundles over a complete Riemannian manifold of bounded geometry. We characterize such operators as *admissible*. The Banach spaces on which we let these operators act are little Hölder spaces; our emphasis on these is because of their use in applications to nonlinear problems. The immediate examples of such operators are generalized Laplacians, i.e., operators of the form $\nabla^* \nabla + \mathcal{R}$, where $\mathcal{R}$ is an endomorphism usually constructed from the curvature tensor of the underlying metric and its covariant derivatives. However, the method of proof extends naturally to allow us to prove this estimate for more general higher-order operators as well.

Our second goal is to apply this sectoriality to deduce stability estimates for nonlinear parabolic evolution equations on these manifolds, acting on little Hölder spaces. We are particularly interested in geometric curvature flows, e.g., the Ricci flow, the mean curvature flow, and certain relatively unexplored higher-order flows such as the one associated with the ambient obstruction tensor (see Sect. 5 for a description of this). As discussed below, these flows typically require some sort of gauge fixing in order to become suitably parabolic. The applications in this paper illustrate how one can easily establish well-posedness of quite general flows, on spaces that are not necessarily compact, using this sectoriality property. In a subsequent paper [8], we describe an application in which sectoriality is a key part of the proof of a longtime stability and convergence result.

Sectoriality for admissible operators on manifolds on spaces of uniformly bounded geometry has, in fact, been treated previously, notably by H. Amann and his collaborators; see for example [2, 21]. Those techniques are spread out over several papers, are considerably more abstract, and, from a geometric point of view, are perhaps less accessible. Our goal here is

to provide a straightforward approach which is more convenient for geometric applications, cf. Sect. 5.

Let us briefly recall the functional analytic setting in more detail. The fundamental idea in applying semigroup theory to a partial differential equation is to recast the problem as an ordinary differential equation with values in some Banach space. Let X be a complex Banach space and D a dense linear subspace. Consider the X -valued autonomous ordinary differential equation

$$\frac{du}{dt} = F(u(t)), \quad (1.1)$$

where $u : [0, T) \rightarrow X$ is a C^1 mapping. Here $F : D \rightarrow X$ is a (nonlinear) Fréchet differentiable map satisfying certain structural assumptions. The most important of these is that the linearization $L = DF_u$ of F is a sectorial operator on X . We now explain the sectoriality hypothesis.

The resolvent set of a closed linear operator $L : X \rightarrow X$ is the subset $\text{res}_X(L) \subset \mathbb{C}$ consisting of all numbers λ such that $(\lambda I - L) : D \rightarrow X$ has an inverse which is a bounded operator on X . L is called *sectorial* on X if it satisfies:

- S1. The resolvent set $\text{res}_X(L)$ contains a sector of opening angle $2\theta < \pi$ which contains a left half-plane $\text{Re } \lambda \leq \omega$ for some $\omega \in \mathbb{R}$; i.e., there exists $\theta \in (0, \pi/2)$ such that

$$\text{res}_X(L) \supset S_{\omega, \theta} := \{\lambda \in \mathbb{C} \setminus \{\omega\} : |\arg(\omega - \lambda)| > \theta\}, \quad \text{and}$$

- S2. There exists a constant $C > 0$ so that for all $\lambda \in S_{\omega, \theta}$, we have

$$\|(\lambda I - L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda - \omega|}.$$

These conditions on L turn out to be equivalent to the analyticity of the semigroup e^{-tL} , and from this a wealth of well-posedness and regularity results are available. In particular, we may then find suitable function spaces for the associated linear homogeneous problem to Eq. (1.1) using maximal regularity theory, yielding existence and uniqueness results via contraction mapping arguments. Many nonlinear parabolic partial differential operators can be cast into this framework; see [31] for a thorough account of this general theory. Of particular importance is that spectral stability analysis of L yields stability results for the nonlinear problem in some cases.

As for the geometric applications, we recall that a complete Riemannian manifold (M, g) is defined to have bounded geometry (of a certain order) if its injectivity radius is bounded from below and there is a uniform bound for the norm of the curvature tensor and its covariant derivatives up to a corresponding order. This is equivalent to uniform control of the coefficients of the metric and its inverse in an atlas of normal coordinate balls of uniform radius. This class of spaces includes compact Riemannian manifolds, of course, but also complete noncompact manifolds which are asymptotically Euclidean, conic, cylindrical or hyperbolic, respectively, or more generally which are asymptotically modeled on other noncompact symmetric or homogeneous spaces.

On any such space (M, g) , we consider elliptic differential operators, acting between sections of vector bundles, which satisfy uniformity conditions on their coefficients in these local uniform coordinate charts. The most obvious examples are operators determined directly from the metric g , for example, generalized Laplace-type operators

$$L = \nabla^* \nabla + \mathcal{R},$$

acting on sections of some tensor bundle E over M . Here ∇ is the induced covariant derivative on this bundle, ∇^* its adjoint, and $\mathcal{R}$ a symmetric endomorphism built out of tensor products of contractions of the curvature tensor and its covariant derivatives. Many aspects of the mapping properties of L on $L^2(M, dV_g)$ can be deduced from Hilbert space techniques. However, it is often more convenient for nonlinear geometric problems to consider this operator acting on weighted Hölder spaces instead. Consider a general weighted Hölder space $X = \mathfrak{w} C^{k,\alpha}(M, g)$, where $\mathfrak{w}$ is a (strictly positive) weight function. The assumptions imposed on $\mathfrak{w}$ are specified in Definition 2.12. More generally, we consider L to be a strongly elliptic operator which satisfies certain uniformity conditions specified in Definition 2.8, acting on weighted Hölder sections of some vector bundle. Note that by replacing L with $\mathfrak{w}^{-1} L \mathfrak{w}$ we may as well consider L as acting simply on $C^{k,\alpha}(M, g)$. The conditions on $\mathfrak{w}$ are precisely the ones necessary for this conjugated operator to satisfy the same uniformity hypothesis.

The first main result of this paper states that any uniform strongly elliptic (that is, *admissible*) operator is *sectorial*. See Definition 2.9 for the precise statement of admissibility.

Theorem A *Let (M^n, g) be a complete Riemannian manifold of bounded geometry of order $\ell + \alpha' > m + k + \alpha$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 < \alpha < \alpha' < 1$, and suppose that L is an admissible operator of order m which acts on the little Hölder space $X = C^{k,\alpha}(M, g)$ of sections of some vector bundle E over M . Then L is sectorial on X .*

We briefly enumerate the main ideas in the proof. We begin with the observation that the sectoriality estimate is equivalent to a uniform estimate for the associated *semiclassical* operator $\zeta I - \varepsilon^m L$, where $\varepsilon = |\lambda|^{-1/m}$ and $\zeta = \lambda/|\lambda|$. So the first step is to show that this operator is invertible on X if ε is sufficiently small. This is deduced by constructing an approximation for the inverse of this operator, which is called the *semiclassical resolvent*. This involves a detour into the methods of geometric microlocal analysis, and the construction itself is sketched in some detail in Sect. 4 in order for the paper to be as self-contained as possible. This geometric microlocal analytic construction has appeared implicitly in the literature, but does not seem to appear explicitly in a readily available form elsewhere. Having established the existence of $(\zeta I - \varepsilon^m L)^{-1}$ as a bounded operator on $X = C^{0,\alpha}(M, g)$ for each $\varepsilon > 0$ that is sufficiently small, we need to establish uniformity of its norm in ε . This is argued by contradiction: We show using a number of rescaling and blowup arguments that the failure of uniformity of this estimate would lead to various impossible conclusions. A key feature of this argument is that we parlay the (essentially tautological) uniform estimates of this operator acting on *semiclassical* Hölder spaces (see Sect. 2) to uniformity for the action of this operator on standard Hölder spaces. The passage from sectoriality on $C^{0,\alpha}$ to sectoriality on $C^{k,\alpha}$ is a trivial extension.

The key motivation for all this work is its application to proving well-posedness of geometric flows on complete noncompact manifolds. The admissibility property in Theorem A is a straightforward algebraic criterion for proving sectoriality, which then gives access to powerful semigroup theory results. For example, we obtain the following theorem; see [31]:

Theorem B *Let (M^n, g) be a complete Riemannian manifold of bounded geometry of order $k + m + \alpha'$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 < \alpha < \alpha' < 1$; let $F : C^{m+k,\alpha}(M, g) \rightarrow C^{k,\alpha}(M, g)$ be a continuous and Fréchet differentiable map of order m such that the graph norm of DF_u is equivalent to the norm of $D = C^{m+k,\alpha}(M, g)$. Further, we assume that for any $u \in D$ there exists an open set containing u and a constant $c > 0$ such that $\|DF_u(v) - DF_u(w)\|_{L(D,X)} \leq c\|v - w\|_D$, for v, w in a sufficiently small ball about u in D .*

If DF_u is admissible for all u in some open subset $\mathcal{O} \subset D$ containing u_0 , then

- (1) (Short-time existence, uniqueness) There exists a maximal time $T > 0$ so that the initial value problem

$$\begin{aligned}\frac{du}{dt} &= F(u(t)) \\ u(0) &= u_0\end{aligned}\tag{1.2}$$

has a unique solution $u : [0, T) \rightarrow \mathcal{C}^{k,\alpha}(M, g)$.

- (2) (Continuous dependence) For any $0 < \tau < T$, there exist constants $r, C > 0$ depending on u_0 and τ such that if $v_0 \in B_r(u_0) \subset \mathcal{O}$, then the solution $v(t)$ of (1.2) with initial value v_0 exists until at least time τ , and

$$\|v(t) - u(t)\|_{\mathcal{C}^{m+k,\alpha}} \leq C \|v_0 - u_0\|_{\mathcal{C}^{m+k,\alpha}}, \text{ for all } t \in [0, \tau].$$

There are many possible applications of this work. Consider the ambient obstruction tensor $\mathcal{O}_n$, developed by Fefferman and Graham [23]. As we review in Sect. 5, this is a conformally invariant tensor that involves n derivatives of the metric. A flow for the metric involving the ambient obstruction tensor was developed by the first author and Helliwell in [5, 6]. Due to the higher-order nature of the system of equations for this flow, the usual technique of using an exhaustion and maximum principles to prove existence are not easy to apply. To our knowledge, our method is the only well-posed result to date for this flow on a noncompact manifold. Our main result is:

Theorem C *Let (M^n, g_0) be an even-dimensional complete Riemannian manifold of bounded geometry of order $l + \alpha'$, and where $0 < \alpha < \alpha' < 1$. Then there exists $T > 0$ and a family of unique Riemannian metrics*

$$g : [0, T) \rightarrow \mathcal{C}^{n,\alpha}(M, \text{Sym}^2(T^*M))$$

solving the ambient obstruction flow

$$\begin{cases} \partial_t g = \mathcal{O}_n(g) + c_n(-1)^{\frac{n}{2}} \left((-\Delta)^{\frac{n}{2}-1} S \right) g \\ g(0) = g_0, \end{cases}\tag{1.3}$$

where $c_n = (2^{n/2-1}(\frac{n}{2}-2)!(n-2)(n-1))^{-1}$, S is the scalar curvature of g , and $\text{Sym}^2(T^*M)$ is the bundle of symmetric 2-tensors.

The remainder of this paper is structured as follows. In Sect. 2, we describe the analytic and geometric background. After discussing sectoriality, we define manifolds of bounded geometry and define the operators of interest. We discuss pointed limits of manifolds of bounded geometry and prove a relationship between the resolvent set of an operator and its limiting operators under this construction, Proposition 2.11, that may be of independent interest. In Sect. 3, we explain the reduction of sectoriality to semiclassical estimates and prove Theorem A. In this section, we presume the existence of uniform bounds for the semiclassical resolvent of an admissible operator, and we give a detailed construction of this resolvent in Sect. 4 using the techniques of geometric microlocal analysis. Finally in Sect. 5 we apply our results to obtain Theorem B, and conclude with the proof of the well-posed result for the ambient obstruction flow, Theorem C.

2 Background

2.1 Sectoriality

We begin by defining what it means for a closed unbounded operator acting on a Banach space to be sectorial. The abstract notion of sectoriality, and its precise relationship with the theory of analytic semigroups, is classical and can be found, for example, in [43, Chapter IX]. The monographs [1, 31] contain applications of sectoriality to the study of evolution equations, and the papers [7, 27] focus on its specific application to Ricci flow.

Let X be a complex Banach space and $\mathcal{L}(X)$ the space of bounded linear operators on X ; we denote the operator norm by $\|\cdot\|_{\mathcal{L}(X)}$. Suppose that L is a closed *unbounded* linear operator on X which has dense domain $D \hookrightarrow X$. The *resolvent set* of L , $\text{res}_X(L)$, is the set of $\lambda \in \mathbb{C}$ for which the resolvent operator

$$R_L(\lambda) := (\lambda I - L)^{-1}$$

lies in $\mathcal{L}(X)$. The range of $R_L(\lambda)$ is the domain D . The *spectrum* of L , denoted $\text{spec}_X(L)$, is the complement $\mathbb{C} \setminus \text{res}_X(L)$. As defined above:

Definition 2.1 A closed unbounded linear operator $L : X \rightarrow X$ with domain D is *sectorial* in X if:

- S1. The resolvent set $\text{res}_X(L)$ contains a sector of opening angle $2\theta < \pi$ which contains a left half-plane $\text{Re } \lambda \leq \omega$ for some $\omega \in \mathbb{R}$; i.e., there exists $\theta \in (0, \pi/2)$ such that

$$\text{res}_X(L) \supset S_{\omega, \theta} := \{\lambda \in \mathbb{C} \setminus \{\omega\} : |\arg(\omega - \lambda)| > \theta\}, \quad \text{and}$$

- S2. There exists a constant $C > 0$ so that

$$\|R_L(\lambda)\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda - \omega|} \quad \text{for all } \lambda \in S_{\omega, \theta}. \quad (2.1)$$

We often simply say that L is sectorial if the space X is understood.

Remark 2.2 We adopt the convention that the spectrum of L lies in a sector with an acute opening angle and is strictly contained in a *right* half-plane $\text{Re } \lambda \geq \omega$. In the applications below, L is a differential operator with the leading part equal to some power of an iterated Laplacian Δ^k , and our convention then agrees with the one where the L^2 spectrum of Δ lies in the positive half-line. Note that our convention is different from our earlier work [7] and the monograph [31].

Sectoriality is equivalent to an apparently weaker condition:

Lemma 2.3 (Proposition 2.1.11 of [31]) *Let X be a complex Banach space, and $L : X \rightarrow X$ a closed linear operator with dense domain D such that $\text{res}_X(L)$ contains a closed half-plane $\{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \omega\}$, for some $\omega \in \mathbb{R}$. If there exists a constant $C > 0$ such that*

$$\|(\lambda - L)^{-1}\|_{\mathcal{L}(X)} \leq C, \quad (2.2)$$

for all λ in this half-plane, then L is sectorial.

Proof By (2.2), $\|R_L(\omega + i\mu)\|_{\mathcal{L}(X)} \leq \frac{C}{|\omega + i\mu|}$, so if $|\lambda - (\omega + i\mu)| \leq |\omega + i\mu|/2C$, then

$$\begin{aligned} (\lambda I - L) &= ((\omega + i\mu)I - L) + (\lambda - (\omega + i\mu))I \\ &= ((\omega + i\mu)I - L) (I + (\lambda - (\omega + i\mu)) R_L(\omega + i\mu)). \end{aligned}$$

The second factor on the right is of the form $I + A$ where $\|A\| \leq 1/2$, and the first factor on the right is invertible by hypothesis, so their product is invertible and hence $\lambda \in \text{res}_X(L)$. Since the radii of the balls around $\omega + i\mu$ on which the resolvent is defined grow asymptotically linearly in μ , the union of the original half-plane together with these balls contains a sector $S_{\omega, \theta}$ for some $\theta \in (0, \pi/2)$. The estimate (2.1) follows. $\square$

2.2 Manifolds of bounded geometry

As stated in Introduction, we consider the sectoriality of a general class of *admissible* differential operators of even order $m = 2m'$, $m' \in \mathbb{N}$, acting between Hölder spaces, $L : \mathcal{C}^{m+k, \alpha}(M, g) \rightarrow \mathcal{C}^{k, \alpha}(M, g)$, where (M, g) is a complete manifold with bounded geometry of order at least $m + k + \alpha'$ for some $\alpha' \in (\alpha, 1)$. We may consider any such L to be an *unbounded* operator on $\mathcal{C}^{k, \alpha}(M, g)$.

In this paper, we work exclusively with the ‘little’ Hölder spaces, which by definition are the closures of $\mathcal{C}^\infty$ in the corresponding Hölder norms. (Here $\mathcal{C}^\infty$ is the space of smooth functions on M which are uniformly bounded along with all of their covariant derivatives.) For a given k, α , this is a closed subspace of the full Hölder space, but if M is noncompact, it is not separable. In fact, we never use separability below, only the fact that elements of these little Hölder spaces can be approximated in the global Hölder norm by smooth functions. As an aside, we could obtain a separable space by taking the closure of $\mathcal{C}_0^\infty$ in the Hölder norm; this is, however, a much smaller space, with all elements decaying at infinity. In any case, we use only the larger space taking the closure of $\mathcal{C}^\infty$; to lighten the notational burden, we denote this by the same symbol $\mathcal{C}^{k, \alpha}$, with the understanding that we never use the full Hölder spaces in this paper.

In this section, we begin with a description of manifolds of bounded geometry, and direct the reader to Sects. 2.3 and 2.4 for more detail on the operators and function spaces which appear below. Briefly, key examples of the operators we consider are elliptic operators arising naturally in geometric analysis of the form

$$L = (\nabla^* \nabla)^{m'} + \text{lower-order terms}$$

where ∇ is the covariant derivative acting on sections of some Hermitian vector bundle V over M , and where the lower-order terms involve the curvature tensor of the underlying metric. More generally, we also consider such operators acting between weighted (little) Hölder spaces:

$$L : \mathfrak{w} \mathcal{C}^{m+k, \alpha}(M, g) \longrightarrow \mathfrak{w} \mathcal{C}^{k, \alpha}(M, g), \quad (2.3)$$

where $\mathfrak{w}$ is a weight function satisfying certain uniformity hypotheses; see Definition 2.12. The mapping (2.3) is equivalent to

$$(\mathfrak{w})^{-1} L \mathfrak{w} = (\nabla^* \nabla)^{m'} + S : \mathcal{C}^{m+k, \alpha}(M, g) \longrightarrow \mathcal{C}^{k, \alpha}(M, g),$$

where S is an operator of order $m - 1$ which includes both the conjugate of the lower-order terms in L and also $(\mathfrak{w})^{-1}[(\nabla^* \nabla)^{m'}, \mathfrak{w}]$.

Let us proceed by recalling the definition of a manifold of bounded geometry:

Definition 2.4 A complete Riemannian manifold (M, g) has *bounded geometry of order* $\ell + \alpha'$, where $\ell \in \mathbb{N}_0$ and $0 \leq \alpha' < 1$, if:

- (a) There exists a radius $r_0 > 0$ such that for every $q \in M$, the exponential map $\exp_q : \{v \in T_q M : |v| < r_0\} \rightarrow B_{r_0}(q)$ is a diffeomorphism; i.e., the injectivity radius at q is greater than r_0 ;
- (b) For every $q \in M$, the components of the pulled back metric $\exp_q^* g$ are bounded in $C^{\ell, \alpha'}$ and the components of the matrix inverse of $\exp_q^* g$ are bounded in C^0 on $\{v \in T_q M : |v| < r_0\}$, where the bounds are independent of q , and hence uniform over M .

Remark 2.5 We have denoted the fractional part of this uniformity order by α' to distinguish it from the α index in the Hölder spaces we are using. We need the extra room given by the inequality $\alpha' > \alpha$ in taking limits using the Arzela–Ascoli Theorem.

It is often easier to check an intrinsic version of condition (b). As discussed in [20], for example, if $\ell \geq 1$ and $\alpha' = 0$, then (b) is implied by

$$(b') \sup_{j \leq \ell} |\nabla^j \text{Riem}| \leq C_\ell \text{ for some constant } C_\ell.$$

There is very likely to be a straightforward extension of this result using Hölder norms instead of the simpler C^k norms, but we have not checked this and do not pursue this further here.

Any compact Riemannian manifold has bounded geometry of order equal to the regularity class of the metric. There are many other natural examples of manifolds with bounded geometry. We list some familiar classes:

- (i) Asymptotically Euclidean or asymptotically conic manifolds,
- (ii) Manifolds with asymptotically cylindrical ends,
- (iii) Asymptotically (real) hyperbolic manifolds,
- (iv) Asymptotically complex hyperbolic manifolds,
- (v) Any symmetric space $M = G/K$ of noncompact type, with invariant metric g , or indeed, any perturbation $g = g_0 + h$ of the symmetric metric g_0 , where $|\nabla^j h|_{g_0} \leq C_j$ for $j \leq \ell$ (where ℓ is the desired order of uniform geometry),
- (vi) Any infinite cover (M, g) of a compact manifold (M_0, g_0) .

Let us briefly discuss each of these classes. Before doing so, we observe that if (M, g) has bounded geometry of order $\ell + \alpha'$ and if $\tilde{g} = g + h$, where $|h|_g \leq 1 - \epsilon$ for some $\epsilon \in (0, 1)$ (so that $\tilde{g}$ is boundedly equivalent to g) and the $C^{\ell, \alpha'}$ norms of the components of h are controlled in g normal coordinate charts, then $\tilde{g}$ also has bounded geometry of order $\ell + \alpha'$. This means that we may describe these various classes of spaces in their simplest model forms. Bounded geometry then follows for any metrics which are perturbations of these models in the above sense. We are particularly interested in perturbations which decay to the appropriate model metrics in a suitable sense at infinity, and shall mention the rate of decay in each of these cases.

Asymptotically Euclidean and asymptotically conic metrics. A Riemannian manifold (M^n, g) is called *conic at infinity* if there exists a compact Riemannian manifold (Y, h_0) of dimension $n - 1$, a compact set K in M and a diffeomorphism from $M \setminus K$ to $[r_0, \infty) \times Y$, such that

$$g = dr^2 + r^2 h_0.$$

More generally, (M, g) is called *asymptotically conic* (AC) if it can be written as the sum of a metric which is conic at infinity and an extra term k which satisfies $|\nabla^j k|_g \leq C r^{-\beta-j}$ for some $\beta > 0$ and for $0 \leq j \leq \ell$, and $[\nabla^\ell k]_{0, \alpha'} \leq C r^{-\beta-\ell-\alpha'}$.

In the following examples, we shall simply state a decay rate, e.g., $r^{-\beta}$, but with corresponding decay rates on the derivatives implicit.

An AC space is called *asymptotically Euclidean* (AE) if (Y, h_0) is isometric to the standard sphere. Elliptic theory on this class of spaces has been thoroughly studied for several decades; see [9] for a survey of results from a ‘classical’ perspective, and [40] for another approach which appears frequently below. There are important generalizations of AE and AC spaces that arise in various geometric settings, including the classes of *quasi-asymptotically conic* (QAC) manifolds [18], and certain of the four-dimensional ‘gravitational instantons’ (of types ALE/F/G/H), along with their higher-dimensional generalizations [12].

Asymptotically cylindrical metrics. (M^n, g) is cylindrical at infinity if outside some compact set it is isometric to a product cylinder $(a, \infty) \times Y^{n-1}$ with metric $dt^2 + h_0$. Setting $r = e^{-t}$ we arrive at the equivalent form

$$g = \frac{dr^2}{r^2} + h_0,$$

which is conformal to the exact conic metric $dr^2 + r^2 h_0$. This is useful for translating results from one setting to the other. The allowable perturbations in this setting decay like $e^{-\beta t}$ as $t \rightarrow \infty$, or equivalently, like r^β as $r \rightarrow 0$.

Asymptotically hyperbolic metrics. Next, suppose that M is a compact Riemannian manifold with boundary. Fix a smooth boundary defining function ρ on M ; i.e., $\rho \geq 0$ on M , $\rho^{-1}(0) = \partial M$, and $d\rho \neq 0$ at the boundary. Fix also a metric h_0 on ∂M . The class of ‘exact’ asymptotically hyperbolic (AH) metrics consists of metrics taking the form

$$g = \frac{d\rho^2 + h_0}{\rho^2}$$

near $\rho = 0$. The metric $\bar{g} = \rho^2 g$ is called a conformal compactification of g . Allowable perturbations in this case are tensors which decay like ρ^μ for some $\mu > 0$.

This geometry mimics that of the Poincaré ball model of hyperbolic space, where

$$g = \frac{4|dz|^2}{(1 - |z|^2)^2}.$$

Thus, $\rho = (1 - |z|^2)/2$ and the Euclidean metric $|dz|^2$ equals a particular choice of conformal compactification $\bar{g}$. Since ρ is not canonically defined in terms of g , only the conformal class of $\bar{g}$ (and in particular, $\bar{g}|_{T\partial M}$) is intrinsic to g .

To see that an AH space has bounded geometry, one calculates that the sectional curvatures of any AH metric tend to -1 and covariant derivatives of the curvature tensor tend to 0, all as $\rho \rightarrow 0$. If $q \in M$ and $\rho(q) = \epsilon$, then the $\bar{g}$ ball $B_{\epsilon/2}(q)$ has an inradius and a diameter that are uniformly bounded away from both 0 and ∞ , and the restriction of g to any such ball converges to a hyperbolic metric on a ball of nonzero radius. We refer the reader to [29] for more details on this.

2.2.1 Asymptotically complex hyperbolic metrics

One generalization of this last example is to the class of asymptotically complex hyperbolic manifolds. There are various ways to define these spaces; we refer to [10] for one approach and a more extended discussion than the one below. Proceeding as in the AH case, let M be a compact Riemannian manifold with boundary, of even real dimension $2n$, and ρ a boundary defining function. Suppose that η is a contact form on ∂M , i.e., η is a 1-form such that $\eta \wedge (d\eta)^{n-1}$ is everywhere nonvanishing. Let T denote the Reeb vector field on ∂M ,

i.e., the unique vector field such that $\eta(T) \equiv 1$ and $d\eta(T, \cdot) \equiv 0$. Finally, choose a set of smooth independent vector fields $X_1, \dots, X_{2n-2}$ which span the kernel of η in $T\partial M$, all on ∂M . Let $\eta, \omega_1, \dots, \omega_{2n-2}$ be the coframe dual to $T, X_1, \dots, X_{2n-2}$. Fixing a product decomposition of a collar neighborhood of ∂M in M , we say that g is (exact) asymptotically complex hyperbolic (ACH) if

$$g = \frac{d\rho^2 + \sum \omega_j^2}{\rho^2} + \frac{\eta^2}{\rho^4}$$

in that neighborhood. (Here $\omega_j^2 = \omega_j \otimes \omega_j$ and $\eta^2 = \eta \otimes \eta$.) The difference from the AH case is that the metric blows up faster in the α^2 direction. A CR (Cauchy–Riemann) structure involves not only the hyperplane bundle $\ker \eta$ but also an endomorphism J on this subbundle which satisfies $J^2 = -I$; however, this almost complex structure is not relevant to these metric asymptotics. An allowable perturbation k again decays like some ρ^μ as $\rho \rightarrow 0$.

This mimics a standard representation of the complex hyperbolic metric on the unit ball in $\mathbb{C}^n$ (with holomorphic sectional curvature -4),

$$g = \frac{g_{\text{Euc}}}{1-r^2} + \frac{r^2(dr^2 + (Jdr)^2)}{(1-r^2)^2},$$

where g_{Euc} is the Euclidean metric, $r = |z|$, and $z \in \mathbb{C}^n$. The monograph [10] explains the relationship of this construction to CR geometry of the boundary. Bounded geometry of ACH metrics can be proved similarly to the AH case. The ‘cubes’ of approximate radius 1 have (approximate) dimensions ϵ in the ∂_ρ and X_j directions and ϵ^2 in the T direction.

There are further generalizations to classes of exact and asymptotically quaternion hyperbolic metrics, and (asymptotically) octonion hyperbolic planes. These involve generalizing the contact structures used to define the ACH metric; see [10].

2.2.2 Other examples

A Riemannian symmetric space $M = G/K$ of noncompact type, or more generally a Riemannian homogeneous space $M = G/H$ with invariant metric, again has bounded geometry. The definitions are a bit more intricate, and we point the reader to one of the many standard references on the subject, [28], at least for the symmetric space case.

Infinite covers of compact Riemannian manifolds, where the metric is obtained by pullback via the covering map, are an interesting class of spaces. General results about the L^2 -resolvent of even the scalar Laplacian on such manifolds are almost nonexistent, except if the covering group is ‘small’ (e.g., amenable). Perhaps surprisingly, we are able to carry out the analysis below for the resolvents of admissible operators on $C^{0,\alpha}$, not only on these classes of spaces, but even on general manifolds of bounded geometry.

2.2.3 Pointed limits of manifolds of bounded geometry

To conclude this subsection, we recall an important construction in the category of manifolds with bounded geometry which shows that this class of spaces is complete in a certain sense. Let (M, g) have bounded geometry of order $\ell + \alpha'$ and consider the sequence of pointed spaces (M, g, p_j) where p_j is a sequence of points in M which diverges to infinity. Then there is a complete Riemannian manifold $(M_\infty, g_\infty, p_\infty)$, which is the pointed Gromov–Hausdorff limit of the sequence (M, g, p_j) and has bounded geometry. More specifically, for any $R > 0$, the g -ball of radius R around p_j in M converges, at least up to passing

to a subsequence, as a Riemannian manifold, to the g_∞ ball of radius R around p_∞ . This convergence can be shown to occur in $C^{\ell, \alpha''}$ for any $0 < \alpha'' < \alpha'$.

This is a standard fact in the ‘convergence theory’ of Riemannian manifolds. This type of construction is originally due to Cheeger, and admits many generalizations. A version that encompasses the above particular statement appears as Theorem 11.36 in Petersen’s book [41]. (We are grateful to Guofang Wei for pointing out this reference.)

As a brief sketch of how this is proved, one first shows that some subsequence of the (M, g, p_j) converges in the Gromov–Hausdorff topology to $(M_\infty, g_\infty, p_\infty)$; this topology is, of course, quite weak and in fact one initially only obtains its metric space structure. Further analysis shows that this convergence happens in a much stronger topology. Indeed, using the bounds on the metric tensor in normal coordinates, we may extract a subsequence of metrics on each ball $B_g(p_j, c)$ which converges in $C^{\ell, \alpha''}$. The curvature bounds imply that any ball of larger radius R can be covered by a controlled number of balls of radius r_0 . By a successive diagonalization argument, the metric tensor converges on each one of these. The lower bound on the injectivity radius is used to show that there is no collapsing in the limit.

Here are two examples of this sort of convergence. If (M, g) has an asymptotically cylindrical end, and if p_j diverges along this cylindrical end, then the corresponding limit space M_∞ is the Riemannian product cylinder $\mathbb{R} \times Y$. Similarly, if (M, g) is asymptotically hyperbolic, and p_j diverges to some point $\bar{p}$ on the boundary of the conformal compactification of M , then M_∞ is a copy of hyperbolic space $\mathbb{H}^n$. These illustrate that the limit space can ‘lose’ a lot of the topology of the original manifold M .

2.3 Function spaces

The definition of bounded geometry of order $\ell + \alpha'$ relies on the Hölder norms in uniform local coordinate charts, and the standard local Euclidean definition can be used. To define Hölder spaces globally on M we need to say a bit more.

Fix a manifold (M, g) with bounded geometry of order $\ell + \alpha'$. We define the Hölder spaces $C^{\kappa, \alpha}(M, g)$ for any $0 \leq \kappa \leq \ell$ and $0 < \alpha < \alpha' < 1$. Introduce the $C^{\kappa, \alpha}$ norm

$$\|u\|_{\kappa, \alpha} := \sum_{j=0}^{\kappa} \sup |\nabla^j u|_g + \sup_B \sup_{\substack{x, y \in B \\ x \neq y}} \frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha}.$$

The first supremum in the final term on the right is over all geodesic balls $B \subset M$ of radius r_0 (as given in the definition of bounded geometry). We assume that the tensor bundles over each such B are trivialized, for example using the exponential map based at the centers of these balls. If we are considering sections of some other vector bundle $V \rightarrow M$, we assume the existence of a uniform set of local trivializations, relative to some ‘uniform’ cover of M by balls $B_{r_0}(q_j)$. The details are straightforward and left to the reader.

As noted in Introduction, in this paper we use the ‘little’ Hölder spaces, $C^{\kappa, \alpha}(M, g)$ exclusively; by definition, these are the completion of C^∞ with respect to the above Hölder norms.

We recall the useful fact that an equivalent norm is obtained by taking the supremum over all $x \neq y$ in M in the final Hölder seminorm, rather than just over $x, y \in B$; in other words, we claim that

$$\sup_{\substack{x, y \in M \\ x \neq y}} \frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha} \leq C \|u\|_{\kappa, \alpha}$$

for some fixed $C > 0$. If $\text{dist}_g(x, y) < r_0$, this is obvious, while if $\text{dist}_g(x, y) \geq r_0$, then

$$\frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha} \leq 2r_0^{-\alpha} \sup |\nabla^\kappa u|_g.$$

It is also clear that for each B ,

$$\sup_{\substack{x, y \in B \\ x \neq y}} \frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha} \leq \sup_{\substack{x, y \in M \\ x \neq y}} \frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha}.$$

Hence, taking the supremum over all balls B on the left, we conclude that we may define the $C^{\kappa, \alpha}$ seminorm either as we have done initially, or else by replacing the final term in that definition with one where the supremum is taken over all distinct $x, y \in M$.

It is clear that it makes sense to consider the spaces $C^{\kappa, \alpha}$ on a manifold of bounded geometry of order $\ell + \alpha'$ only if $\kappa \leq \ell$ and if $\kappa = \ell$ then $\alpha \leq \alpha'$. We assume the strict inequality $\alpha < \alpha'$ because, as described in the last subsection, the pointed limit of spaces of order $\ell + \alpha'$ may only have order $\ell + \alpha''$ for any $\alpha'' < \alpha'$. (Actually, by a standard real analysis argument, the limiting space does have order $\ell + \alpha'$ but the convergence only takes place in the weaker norm, which may be important at certain points.)

We also define the family of *semiclassical* Hölder spaces $C_\varepsilon^{\kappa, \alpha}(M, g)$, where ε is a parameter in $(0, 1]$. The name comes from their natural association with families of operators undergoing semiclassical degeneration, which is described below. These spaces appear in a fundamental way in the arguments of Sect. 3. For any given $\kappa \leq \ell$ and α (as usual with $\alpha \leq \alpha'$ if $\kappa = \ell$), this family of spaces contains all functions u such that

$$\|u\|_{\kappa, \alpha, \varepsilon} := \sum_{j=0}^{\kappa} \varepsilon^j \sup |\nabla^j u|_g + \varepsilon^{\kappa + \alpha} \sup_B \sup_{\substack{x, y \in B \\ x \neq y}} \frac{|\nabla^\kappa u(x) - \nabla^\kappa u(y)|}{\text{dist}_g(x, y)^\alpha} < \infty.$$

In other words, every derivative is accompanied by a power of ε and the α Hölder seminorm has a factor of ε^α .

In fact, these semiclassical spaces are simply the ordinary Hölder spaces associated with the family of rescaled metrics $g_\varepsilon := \varepsilon^{-2}g$:

$$C_\varepsilon^{\kappa, \alpha}(M, g) \cong C^{\kappa, \alpha}(M, g_\varepsilon).$$

Clearly each g_ε has bounded geometry of order $\ell + \alpha'$, and the bounds are uniform as $\varepsilon \rightarrow 0$.

Remark 2.6 From these last comments, it is clear that in the definition of the above semiclassical Hölder seminorm, we may take the supremum either over all balls B of radius 1 or alternately of radius ε with respect to g .

2.4 Admissible differential operators

We prove our main sectoriality estimate for any linear differential operator L which is strongly elliptic, and satisfies an additional uniformity condition.

We begin by recalling that the principal symbol of L of order m is a smooth function $\sigma_m(L)(x, \xi)$ on T^*M which restricts to be a homogeneous polynomial of order m (matrix-valued if L acts between bundles) on each fiber T_x^*M . There are various intrinsic ways to define this principal symbol, but the most familiar depends on coordinate choices: the principal symbol is obtained by dropping all terms of order less than m and then replacing each derivative ∂_x^α , $|\alpha| = m$, by the monomial $(i\xi)^\alpha$. (The factor of i is customary because

of the relationship of this symbol with the Fourier transform.) It is straightforward to check that this is well defined in the sense that it defines a smooth function on the cotangent bundle.

Definition 2.7 We say that a linear operator L is *strongly elliptic* if $\sigma_m(L)(x, \xi)$ has numerical range (or spectrum, if it is a matrix) contained in a sector in the right half-plane:

$$\text{spec } \sigma_m(L)(x, \xi) \subset \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta' < \pi/2\}$$

for all $(x, \xi) \in T^*M$.

It is straightforward to check that if L is strongly elliptic, then its order m is even. If L is symmetric and has real-valued coefficients, then $\sigma_m(L)(x, \xi)$ is real-valued. For example, if

$$L = (\nabla^* \nabla)^{m/2} + \text{lower-order terms},$$

then $\sigma_m(L)(x, \xi) = |\xi|^m$ (or $|\xi|^m$ times the identity matrix), which clearly satisfies this condition.

Definition 2.8 L is *uniform of order $\ell + \alpha'$* (relative to a metric g with bounded geometry of order $\ell + \alpha'$ on a manifold M) if the following two conditions are satisfied:

- (i) the pullback by $\exp_q$ of the operator L has coefficients bounded in $\mathcal{C}^{\ell, \alpha'}$ in each ball $B_{r_0}(q)$, with r_0 as in Definition 2.4;
- (ii) there exists a closed cone Γ strictly contained in $\{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\} \cup \{0\}$ such that if $\zeta \in \mathbb{C} \setminus \Gamma$, then the endomorphism $\zeta I - \sigma_m(L)(x, \xi)$ is invertible, with the inverse satisfying

$$\|(\zeta I - \sigma_m(L)(x, \xi))^{-1}\| \leq C(1 + |\xi|)^{-m}$$

for some fixed C (depending on ζ) for all (x, ξ) in the cotangent bundle of M .

Definition 2.9 A linear differential operator L is *admissible* if it is both strongly elliptic and uniform of order $\ell + \alpha'$.

It is important for our purposes that admissibility is preserved under passage to a limiting space:

Lemma 2.10 Suppose that (M, g) has bounded geometry of order $\ell + \alpha'$, and let p_j be a diverging sequence of points in M . Let L be an admissible differential operator on M , as described above. If (M, g, p_j) converges to some limiting space $(M_\infty, g_\infty, p_\infty)$ in $\mathcal{C}^{\ell, \alpha''}$, $0 < \alpha'' < \alpha'$, then (some subsequence of) the restrictions of the operator L to balls $B_R(p_j)$ converges in $\mathcal{C}^{\ell, \alpha''}$, as both $j \rightarrow \infty$ and $R \rightarrow \infty$ to an operator L_∞ on this limiting space, and any such limiting operator L_∞ obtained in this way is admissible on its space of definition.

Proof By a diagonalization process using Arzela–Ascoli, it is clear that L induces a limiting operator L_∞ on M_∞ , and that L_∞ is again strongly elliptic. (Its coefficients are only in $\mathcal{C}^{\ell, \alpha''}$ for any $\alpha'' < \alpha'$.) $\square$

There is an additional important relationship between L and its limiting operators:

Proposition 2.11 There is an inclusion

$$\bigcap \text{res}_{\mathcal{C}^{k, \alpha}}(L_\infty) \supset \text{res}_{\mathcal{C}^{k, \alpha}}(L),$$

or equivalently,

$$\bigcup \operatorname{spec}_{\mathcal{C}^{k,\alpha}}(L_\infty) \subset \operatorname{spec}_{\mathcal{C}^{k,\alpha}}(L),$$

In both cases, the intersection or union is over all possible limiting spaces M_∞ and model operators L_∞ .

Proof Using this second formulation, suppose that $\lambda \in \operatorname{spec}_{\mathcal{C}^{k,\alpha}}(L_\infty)$ for some limit L_∞ ; i.e., $\lambda I - L_\infty$ is not boundedly invertible on $\mathcal{C}^{k,\alpha}(M_\infty, g_\infty)$. In the following considerations, note that the natural domain of this unbounded map is $\mathcal{C}^{k+m,\alpha}(M_\infty, g_\infty)$. There are three ways that invertibility might fail:

- (a) there exists a nontrivial function $u \in \mathcal{C}^{m+k,\alpha}$ such that $L_\infty u = \lambda u$;
- (b) the range of $\lambda I - L_\infty$ is dense in $\mathcal{C}^{k,\alpha}$, but not closed;
- (c) the closure of the range of $\lambda I - L_\infty$ is equal to some proper closed subspace of $\mathcal{C}^{k,\alpha}$.

We show that each of these three possibilities is incompatible with the assumption that $\lambda \notin \operatorname{spec}_{\mathcal{C}^{k,\alpha}}(L)$. In the first case, suppose that u is a $\mathcal{C}^{m+k,\alpha}$ solution of this limiting equation. Choose a sequence of radii $R_i \rightarrow \infty$, and let χ_i be a sequence of smooth cutoff functions on M_∞ such that

$$\chi_i = \begin{cases} 1 & \text{on } B_{R_i/2}(p_\infty) \\ 0 & \text{outside } B_{R_i}(p_\infty) \end{cases},$$

$0 \leq \chi_i \leq 1$ everywhere, and

$$|\nabla^q \chi_i| \leq C/R_i^q$$

for $q \leq m$ and with C independent of i . For fixed i , the ball $B_{R_i}(p_\infty)$ in M_∞ is a limit as $j \rightarrow \infty$ of balls $B_{R_i}(p_j)$ in M . Using this, we may transplant $\chi_i u$ to $B_{R_i}(p_j)$ and then extend it to equal 0 on the rest of M .

We now compute that

$$(\lambda I - L)(\chi_i u) := h_i = \chi_i(\lambda I - L_\infty)u + \chi_i(L_\infty - L)u + [L, \chi_i]u.$$

Clearly, there exist constants c_1, c_2 such that $0 < c_1 \leq \|\chi_i u\|_{k,\alpha} \leq c_2$, uniformly in i , and using the limiting properties in this construction, $\|h_i\|_{k,\alpha} \rightarrow 0$. Since the $\chi_i u$ span an infinite-dimensional space, we conclude that $(\lambda I - L)$ does not have a closed range on $\mathcal{C}^{k,\alpha}$, contrary to the choice of λ .

Next suppose that $(\lambda I - L_\infty)$ has a dense but nonclosed range, so it does not have a bounded inverse. There exists a sequence u_i on M_∞ with infinite-dimensional span such that $\|u_i\|_{k,\alpha} = 1$ and $\|(\lambda I - L_\infty)u_i\|_{k,\alpha} \rightarrow 0$. Precisely the same transplantation argument used above shows that $c_1 \leq \|\chi_i u_i\|_{k,\alpha} \leq c_2$ and $\|(\lambda I - L)(\chi_i u_i)\|_{k,\alpha} \rightarrow 0$ on M , which is once again a contradiction.

Finally, suppose that the range of $(\lambda I - L_\infty)$ is equal to or at least dense in some proper closed subspace. Then the dual operator $(\bar{\lambda} I - L_\infty^*)$ has a nontrivial nullspace in $(\mathcal{C}^{k,\alpha})^*$. This dual space is distributional, of course, but since L_∞^* is elliptic, any element v of its nullspace is again as regular as the coefficients of the operator and the metric allow, and this element must be bounded as well (or else it would be easy to find some $\phi \in \mathcal{C}^{k,\alpha}$ such that $\langle v, \phi \rangle$ is undefined). We are then in the situation of the first case, once we observe that the dual operator L^* is admissible.

This completes the proof. $\square$

2.5 Examples of admissible operators

The main examples of admissible operators that we have in mind are generalized Laplacians on Riemannian manifolds with bounded geometry, or more generally, operators of the form $(\nabla^*\nabla)^{m/2} + S$, where S is an operator of order $m - 1$ usually closely associated with the metric g . We begin with the second-order case.

By definition, a generalized Laplacian is an operator of the form

$$\nabla^*\nabla + \mathcal{K},$$

acting on sections of some tensor bundle E over M (or slightly more generally, a twisted spin bundle—the key feature is that its connection is induced from the Levi-Civita connection for g .) Here ∇^* is the adjoint of the covariant derivative with respect to the natural inner product on each fiber of E , and with respect to the volume form dV_g . The term $\mathcal{K}$ is a symmetric endomorphism of E obtained via contractions of sums of tensor products of the curvature tensor and its covariant derivatives.

The following is a list of standard examples of such operators:

- (a) The scalar Laplacian $\Delta_g = \nabla^*\nabla$ acts on the trivial rank 1 bundle; slightly more generally, we also consider the Hodge Laplace operator $\Delta_{g,k} = d\delta + \delta d$ acting on sections of the bundle of exterior p -forms, $p = 0, \dots, n$. The original Weitzenböck formula states that

$$\Delta_{g,p} = \nabla^*\nabla + \mathcal{K}_p,$$

where $\mathcal{K}_p$ is an endomorphism of $\bigwedge^p M$ constructed from the curvature tensor. As an example, for $p = 1$, $\mathcal{K}_1 = \text{Ric}$, considered as a symmetric endomorphism on 1-forms.

- (b) Next, consider the Lichnerowicz Laplacian

$$\nabla^*\nabla + 2(\text{Ric} - \text{Riem})$$

where Ric is the Ricci tensor and Riem the full curvature tensor. These act as symmetric endomorphisms on symmetric 2-tensors via

$$\begin{aligned} h_{ij} &\mapsto (\text{Ric}(h))_{ij} = \frac{1}{2} \left(\text{Ric}_{ik} h_j^k + \text{Ric}_{jk} h_i^k \right), \\ h_{ij} &\mapsto (\text{Riem}(h))_{ij} = R_{ipjq} h^{pq}. \end{aligned}$$

- (c) Generalizing (a) in a different way is the conformal Laplacian

$$\nabla^*\nabla + \frac{n-2}{4(n-1)} R_g,$$

acting on scalar functions, where R_g is the scalar curvature of the metric.

Each of the operators in the above list is symmetric, i.e.,

$$\langle Lu, v \rangle = \langle u, Lv \rangle \quad \text{for } u, v \in \mathcal{C}_0^\infty(M).$$

A classical theorem due to Chernoff [14] states that because g is complete, each of these has a unique self-adjoint extension as an unbounded operator on $L^2(M, dV_g)$. Self-adjointness guarantees that the L^2 spectrum lies on the real line. If g has bounded geometry of high enough order, the pointwise norm of the endomorphism $\mathcal{K}$ is uniformly bounded. Since $\nabla^*\nabla \geq 0$, we deduce that

$$\nabla^*\nabla + \mathcal{K} \geq -C \implies \text{spec}_{L^2}(\nabla^*\nabla + \mathcal{K}) \subset [-C, \infty).$$

There are many interesting higher-order elliptic operators associated with the metric g . The most well known are the higher-order ‘GJMS operators,’ which generalize the conformal Laplacian (see [24]). The GJMS operator P_m of order m is a conformally covariant operator which is simply equal to $(\nabla^* \nabla)^{m/2}$ if g is the flat Euclidean metric, but in general has a (complicated) set of lower-order terms involving the curvature tensor. This operator exists only for $m \leq n$ if n is even, and for all m if n is odd. A (very thoroughly studied) example is the Paneitz operator

$$P_4 = \Delta^2 - \delta((n-2)J - 4V)d + (n-4)Q,$$

where V is the so-called Schouten tensor of the metric g , J is its trace, Q is an associated scalar quantity called the Q -curvature, and d and δ are the exterior derivative and its formal adjoint. It is known [26] that these operators are symmetric, and the same reasoning as above implies that the L^2 spectrum lies in a half-line $[-C, \infty)$.

Notice that we have said nothing about the spectrum of any of these operators on $C^{0,\alpha}(M, g)$, and in general the relationship between the $C^{0,\alpha}$ and the L^2 spectrum may be quite difficult to understand. In many geometric problems, however, we actually wish to study the action of L not on $C^{0,\alpha}$ itself, but between some *weighted* Hölder spaces:

$$L : \mathfrak{w} C^{m,\alpha}(M, g) \longrightarrow \mathfrak{w} C^{0,\alpha}(M, g), \quad (2.4)$$

where, by definition, $\mathfrak{w}$ is a strictly positive C^∞ (or $C^{\ell,\alpha'}$) function and

$$\mathfrak{w} C^{k,\alpha}(M, g) = \{u = \mathfrak{w} v : v \in C^{k,\alpha}(M, g)\}.$$

Observe that the mapping in (2.4) is equivalent to

$$L_{\mathfrak{w}} := \mathfrak{w}^{-1} L \mathfrak{w} : C^{m,\alpha}(M, g) \longrightarrow C^{0,\alpha}(M, g). \quad (2.5)$$

Observe also that since $\lambda I - L_{\mathfrak{w}} = \mathfrak{w}^{-1}(\lambda I - L)\mathfrak{w}$, the spectra of (2.4) and (2.5) are the same.

In most of the examples of manifolds of bounded geometry we have given, it is possible to prove that there exist weight functions $\mathfrak{w}$ for which the mappings in (2.4) [or equivalently (2.5)] is Fredholm, i.e., has closed range and finite-dimensional kernel and cokernel. It is sometimes possible to choose $\mathfrak{w}$ so that $\mathfrak{w} C^{0,\alpha}(M, g) \subset L^2$, and if this is the case, and if $f \in \mathfrak{w} C^{0,\alpha}(M, g)$, then so long as $\lambda \notin \text{spec}_{L^2}(L)$, there exists a function $u \in L^2$ with $(\lambda I - L)u = f$. Local elliptic regularity implies that $u \in C^{m,\alpha}$ on each ball $B_{r_0}(p)$. If we could then somehow show that $u \in \mathfrak{w} C^{m,\alpha}$, we would have obtained information about $\text{spec}_{C^{0,\alpha}}(L_{\mathfrak{w}})$.

Before doing so, we list some necessary assumptions on these weight functions.

Definition 2.12 A weight function $\mathfrak{w}$ is *uniform with respect to L* if:

- (a) the mappings in (2.4) and (2.5) are Fredholm;
- (b) the conjugated operator $L_{\mathfrak{w}}$ is admissible with respect to the metric g .

It is clear that $L_{\mathfrak{w}}$ is strongly elliptic if L is, since they have the same leading-order terms. The uniformity of the coefficients of $L_{\mathfrak{w}}$ imposes a strong condition on the weight function and its derivatives. For example, if L is the scalar Laplacian, then

$$\Delta_{\mathfrak{w}} = \Delta + \mathfrak{w}^{-1}[\Delta, \mathfrak{w}] = \Delta + 2 \frac{\nabla \mathfrak{w}}{\mathfrak{w}} \cdot \nabla + \frac{\Delta \mathfrak{w}}{\mathfrak{w}}$$

involves first and second derivatives of $\mathfrak{w}$. Thus, all these lower-order terms must be uniform in the sense we have described above.

2.6 The resolvent on asymptotically hyperbolic spaces

We conclude this section with a description of one particular setting, namely the class of asymptotically hyperbolic spaces, where there is a somewhat more direct path to understanding the $C^{0,\alpha}$ spectrum of generalized Laplacians.

Suppose that (M, g) is asymptotically hyperbolic, as described in Sect. 2.2, and let $L = \nabla^* \nabla + \mathcal{K}$. Choose coordinates (x, y) on M near the boundary of M , where x is a boundary defining function and y is a local coordinate on the boundary extended to this collar neighborhood. Using this, we identify the collar neighborhood with $[0, 1) \times \partial M$. It is known that we can choose these functions in such a way that

$$g = \frac{dx^2 + h(x, y)}{x^2},$$

where $x \mapsto h(x, \cdot)$ is a family of tensors on ∂M in this collar neighborhood decomposition and $h(0, y)$ is any prescribed metric representing the conformal class on ∂M associated with g .

We have already noted that the L^2 spectrum of L lies in some half-line $[-C, \infty)$; hence, $\text{res}_{L^2}(L) \supset \mathbb{C} \setminus [-C, \infty)$. Let R_L denote both the resolvent of L as an abstract operator, but also its Schwartz kernel, which is a distribution on $M \times M$. This distribution, $R_L(\lambda; x, y, \tilde{x}, \tilde{y})$, is singular along the diagonal, where $x = \tilde{x}$ and $y = \tilde{y}$, and has additional singularities along the boundaries where $x \rightarrow 0$ or $\tilde{x} \rightarrow 0$. The nature of these singularities can be understood in a detailed way using the methods of geometric microlocal analysis. We refer to [34] for the construction of this distribution.

The question we wish to consider is whether there exists a range of μ for which

$$R_L : x^\mu C^{k,\alpha}(M, g) \longrightarrow x^\mu C^{k,\alpha}(M, g), \quad (2.6)$$

is bounded if $\text{Re } \lambda \leq -C$, or equivalently, whether the conjugated Schwartz kernel

$$x^{-\mu} R_L(\lambda; x, y, \tilde{x}, \tilde{y})(\tilde{x})^\mu \quad (2.7)$$

acting by convolution induces a bounded mapping on $C^{0,\alpha}$.

This is true for μ lying in a certain interval determined by the so-called indicial roots of L . To describe this more carefully, recall that an indicial root of L is a number $\gamma \in \mathbb{C}$ such that

$$L(x^\gamma u(x, y)) = \mathcal{O}(x^{\gamma+1}),$$

where u is any function smooth up to ∂M . (We consider only the scalar case for notational simplicity.) It is not hard to see that this can only happen if there is some leading-order cancelation, which depends only on some algebraic condition determined by γ and the values at $x = 0$ of certain of the coefficients of L . For a second-order scalar operator, this algebraic condition is a quadratic polynomial in γ , and hence, there are two indicial roots. For a higher-order operator or system, there are more. In this second-order setting where L is assumed to be symmetric on L^2 , these indicial roots take the form

$$\gamma^\pm = \frac{n-1}{2} \pm \zeta_0$$

for some ζ_0 which is either real and nonnegative or else purely imaginary. We can define the indicial roots of $\lambda I - L$ in the same way, and write these as

$$\gamma^\pm(\lambda) = \frac{n-1}{2} \pm \zeta_0(\lambda).$$

If λ is real and sufficiently negative, then $\zeta_0(\lambda) > 0$. There exists some $C_0 \in \mathbb{R}$ such that if $\lambda > C_0$ then $\zeta_0(\lambda)$ is purely imaginary. If $\lambda \in \mathbb{C} \setminus [C_0, \infty)$, then $\operatorname{Re} \zeta_0(\lambda) > 0$ and tends to infinity as the distance from λ to $[C_0, \infty)$ gets larger. One consequence of this is that any $\lambda > C_0$ lies in the continuous L^2 -spectrum of L ; see [36].

Fix the half-plane $\operatorname{Re} \lambda \leq -B$, and define

$$\delta = \inf_{\operatorname{Re} \lambda \leq -B} \operatorname{Re} \zeta_0(\lambda).$$

Note then, by the remarks immediately above, we can increase δ by increasing B along the real axis.

We need a key structural theorem from [34] about the pointwise behavior of the Schwartz kernel of $R_L(\lambda)$:

Proposition 2.13 *The distribution $R_L(\lambda; x, y, \tilde{x}, \tilde{y})$ satisfies*

$$\begin{aligned} |R_L(\lambda; x, y, \tilde{x}, \tilde{y})| &\leq Cx^{\frac{n-1}{2}+\delta}, \quad x \rightarrow 0, \quad \tilde{x} \geq c > 0, \\ |R_L(\lambda; x, y, \tilde{x}, \tilde{y})| &\leq C\tilde{x}^{\frac{n-1}{2}+\delta}, \quad \tilde{x} \rightarrow 0, \quad x \geq c > 0. \end{aligned}$$

There is a considerably sharper structural theorem which also describes the precise behavior of R_L as $x, \tilde{x} \rightarrow 0$, but for the present purposes we do not need this.

We now state and prove a basic result on the spectrum of L acting on weighted Hölder spaces.

Theorem 2.14 *Let (M^n, g) be an asymptotically hyperbolic space, and L be a generalized Laplacian acting on some tensor bundle over M . For every $\delta > 0$, there exists $\omega = \omega(\delta) > 0$ so that if*

$$\mu \in \left(\frac{n-1}{2} - \delta, \frac{n-1}{2} + \delta \right), \quad (2.8)$$

then $\operatorname{res}_{x^\mu C^{0,\alpha}}(L) \supset \{\lambda : \operatorname{Re} \lambda \leq -\omega\}$.

Proof For simplicity, we confine our discussion to the scalar case.

Given a choice of $\delta > 0$, then by the above defining remarks concerning ζ_0 , we need only increase the value of ω to ensure that the interval in Eq. (2.8) is nonempty. Hence, by Proposition 2.13, the conjugated kernel (2.7) decays at least at the rate $\frac{n-1}{2} + \delta + \mu$ as $\tilde{x} \rightarrow 0$ and at least like $\frac{n-1}{2} + \delta - \mu$ as $x \rightarrow 0$. We have thus reduced the problem to understanding whether the mapping from $C^{0,\alpha}(M, g)$ to itself defined by the kernel

$$(\tilde{x}/x)^\mu R_L(\lambda; x, y, \tilde{x}, \tilde{y})$$

is bounded, where this kernel acts by

$$u(x, y) \mapsto \int (\tilde{x}/x)^\mu R_L(\lambda; x, y, \tilde{x}, \tilde{y}) u(\tilde{x}, \tilde{y}) \frac{d\tilde{x}d\tilde{y}}{\tilde{x}^n}.$$

(The singular measure is uniformly equivalent to the L^2 measure for g .) In order for this integral to converge as $\tilde{x} \rightarrow 0$, it is necessary that the product of factors in the integrand (including the singular Jacobian factor) is bounded by $\tilde{x}^{-1+\epsilon}$ for some $\epsilon > 0$. However, by assumption, u does not necessarily decay, which explains the necessity of the condition

$$\frac{n-1}{2} + \delta + \mu - n > -1 \Leftrightarrow \mu > \frac{n-1}{2} - \delta.$$

On the other hand, the rate of growth (or decay) of the output is determined by the behavior of this conjugated kernel in two regions: the first as both $x, \tilde{x} \rightarrow 0$ and the second as $x \rightarrow 0, \tilde{x} > 0$. We refer to [34, Theorem 3.27] for the precise explanation. The kernel is bounded in this first regime and is bounded by $x^{(n-1)/2+\delta-\mu}$ in the second, which explains the requirement that

$$\mu < \frac{n-1}{2} + \delta.$$

The case of equality must be omitted here because in that special case the solution might behave like $\log x$ as $x \rightarrow 0$.

Altogether then, we have argued that

$$(\tilde{x}/x)^\mu R_L(\lambda; x, y, \tilde{x}, \tilde{y}) : \mathcal{C}^{0,0}(M, g) \longrightarrow \mathcal{C}^{0,0}(M, g)$$

is bounded if and only if

$$\frac{n-1}{2} - \delta < \mu < \frac{n-1}{2} + \delta. \quad (2.9)$$

Boundedness on Hölder spaces now follows readily from elliptic estimates. $\square$

We have described the asymptotically hyperbolic case in some detail, but note that it is possible to prove similar results in the asymptotically conic, cylindrical, complex hyperbolic and symmetric or homogeneous cases described above. There is no such argument (to our knowledge) for infinite covers of compact manifolds.

This material is included to indicate that the spectral hypothesis of admissibility can be verified in various ways. In Sect. 4, we describe a different sort of parametrix construction which turns out to be sufficient for our purposes. It works for arbitrary manifolds of bounded geometry, but only if λ has sufficiently large negative real part. In many ways, that parametrix construction is simpler than the one needed in the AH case, but is perhaps less familiar.

3 Sectoriality of admissible operators on Hölder spaces

We now turn to the proof of the main sectoriality theorem, Theorem A.

The first key observation is that if L is an elliptic differential operator of order m , then the sectoriality of L is equivalent to a certain estimate for the *semiclassical* resolvent of L , which means the following. Define $\varepsilon = |\lambda|^{-1/m}$ and $\zeta = \lambda/|\lambda|$. We then rewrite the operator that appears in Lemma 2.3 characterizing sectoriality as

$$\lambda(\lambda I - L)^{-1} = \frac{\lambda}{|\lambda|} ((\lambda/|\lambda|)I - |\lambda|^{-1}L)^{-1} = \zeta(\zeta I - \varepsilon^m L)^{-1}.$$

Disregarding the harmless unit prefactor ζ , the operator $(\zeta I - \varepsilon^m L)^{-1}$ is called the semiclassical resolvent of L . Since $\operatorname{Re} \lambda \leq \omega$, and we may assume that $\omega < 0$, we only need consider such ζ with $\operatorname{Re} \zeta < 0$. Altogether then, an equivalent formulation of the sectoriality of L is that

$$(\zeta - \varepsilon^m L)^{-1} : \mathcal{C}^{k,\alpha}(M, g) \longrightarrow \mathcal{C}^{k,\alpha}(M, g) \quad (3.1)$$

exists and has a norm which is uniformly bounded independently of $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$, and ζ with $|\zeta| = 1, \operatorname{Re} \zeta \leq -c < 0$. In other words,

sectoriality is equivalent to the uniform boundedness of the *semiclassical* resolvent on regular, i.e., *nonsemiclassical*, function spaces.

Part of our assertion is that this resolvent exists as a bounded operator provided ε is sufficiently small, i.e., $\lambda I - L$ is invertible on $C^{k,\alpha}(M, g)$ if $\operatorname{Re} \lambda$ is sufficiently negative.

As explained at the end of this section, the case $k > 0$ follows from the case $k = 0$, so we assume that $X = C^{0,\alpha}$ until the final part of this section.

We prove this equivalence in a series of steps, outlined here and carried out in the rest of this section. In the first step, we show that $(\zeta I - \varepsilon^m L)^{-1}$ exists as a bounded operator for each sufficiently small ε and for every ζ with $\operatorname{Re} \zeta < 0$, but with no claim about uniformity. Here it is irrelevant whether standard or semiclassical Hölder spaces (as defined in Sect. 2.3) are used since they are equivalent for each fixed ε . This is a ‘perturbative result’ and follows from the existence of a semiclassical parametrix. We state this result carefully in the next proposition and for the reader’s convenience sketch this parametrix construction using geometric microlocal analytic techniques in Sect. 4; this methodology is explained there. Next we recall the ‘easy’ semiclassical elliptic Schauder estimate associated with any strongly elliptic semiclassical family $\zeta I - \varepsilon^m L$. This is equivalent to the uniform boundedness of the inverse of this semiclassical operator between *semiclassical* Hölder spaces. The main step of the whole proof is to upgrade this to an estimate between standard Hölder spaces $C^{0,\alpha}$ which is independent of $\varepsilon > 0$. This is done by establishing uniform bounds for this semiclassical resolvent as a map $C^j \rightarrow C^j$ for $j = 0, 1$, and applying interpolation.

Proposition 3.1 *Let L be an admissible operator of order m on a manifold (M, g) of bounded geometry of order $\ell + \alpha'$. Then the unbounded operator $\zeta I - \varepsilon^m L : C^{0,\alpha}_\varepsilon(M, g) \rightarrow C^{0,\alpha}_\varepsilon(M, g)$ has a bounded inverse for each sufficiently small $\varepsilon > 0$ and for each ζ with $|\zeta| = 1$ and $\operatorname{Re} \zeta < 0$, where $m < \ell$ and $\alpha < \alpha'$.*

Proof This proof uses the machinery of microlocal analysis extensively. A detailed introduction to these methods is provided in Sect. 4. As is carefully defined and explained in that section, there exists a parametrix G_ε for $\zeta I - \varepsilon^m L$ which is an element of order $-m$ in the semiclassical uniform pseudodifferential calculus $\Psi^{*,*}_{\text{sc-unif}}(M, g)$ (see Definition 4.5). Thus, for each ε , G_ε is an approximate inverse for $\zeta I - \varepsilon^m L$; its discrepancy from being an exact inverse is captured by the ‘residual operators’ $Q_{j,\varepsilon} \in \Psi^{-\infty,\infty}_{\text{sc-unif}}(M, g)$, $j = 1, 2$, via the identities

$$(\zeta I - \varepsilon^m L)G_\varepsilon = I - Q_{1,\varepsilon}, \quad G_\varepsilon(\zeta I - \varepsilon^m L) = I - Q_{2,\varepsilon}.$$

Each $Q_{j,\varepsilon}$ is a smoothing operator (this is the meaning of the first superscript $-\infty$) with Schwartz kernel supported in some fixed neighborhood of the diagonal $\{(z, \tilde{z}) : \operatorname{dist}_g(z, \tilde{z}) \leq C\}$, and vanishing to all orders as $\varepsilon \searrow 0$ (which is the meaning of the second superscript, $+\infty$).

In more detail, G_ε and the $Q_{j,\varepsilon}$ are one-parameter families of operators, with Schwartz kernels $G(\varepsilon, z, \tilde{z})$ and $Q_j(\varepsilon, z, \tilde{z})$, $z, \tilde{z} \in M$. For each $\varepsilon > 0$, $G(\varepsilon, z, \tilde{z})$ is an ordinary pseudodifferential operator of order $-m$ which is a parametrix for $\zeta I - \varepsilon^m L$, and the Q_j are the smoothing error terms. These Schwartz kernels vary smoothly in ε for $\varepsilon > 0$. The important new feature is their behavior as $\varepsilon \rightarrow 0$. First, if $z \neq \tilde{z}$, then $G(\varepsilon, z, \tilde{z})$ and $Q_j(\varepsilon, z, \tilde{z})$ decay faster than ε^N for any N . This convergence is uniform in any region where $\operatorname{dist}_g(z, \tilde{z}) \geq c'' > 0$ for any $c'' > 0$. The behavior of $G(\varepsilon, \cdot, \cdot)$ near the diagonal as $\varepsilon \rightarrow 0$ requires a bit more work to describe; this is done in Sect. 4. On the other hand, for $j = 1, 2$, $Q_j(\varepsilon, z, \tilde{z}) \in C^\infty([0, \varepsilon_0) \times M^2)$, these kernels decay rapidly along with all derivatives as

$\varepsilon \rightarrow 0$, uniformly on $M \times M$. This construction works assuming that $\operatorname{Re} \zeta < 0$, but the rate of decay (which is actually exponential) diminishes as $\operatorname{Re} \zeta \rightarrow 0$.

It is straightforward to deduce from this structure that $\|Q_{j,\varepsilon}\|_{\mathcal{L}(C^{0,\alpha})} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, both $(I - Q_{1,\varepsilon})^{-1}$ and $(I - Q_{2,\varepsilon})^{-1}$ exist as bounded operators on $C^{0,\alpha}$ for any fixed sufficiently small $\varepsilon > 0$. Thus, we can write

$$(\zeta I - \varepsilon^m L)^{-1} = G_\varepsilon (I - Q_{1,\varepsilon})^{-1} = (I - Q_{2,\varepsilon})^{-1} G_\varepsilon;$$

this proves that $(\lambda I - L)^{-1}$ exists for any λ with $\operatorname{Re} \lambda$ sufficiently negative. $\square$

In the next step, we continue in this same semiclassical vein:

Proposition 3.2 (Semiclassical elliptic estimate) *There exists a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $u \in C^{2,\alpha}_\varepsilon$ and ζ with $|\zeta| = 1$ and $\operatorname{Re} \zeta \leq c' < 0$,*

$$\|u\|_{m,\alpha,\varepsilon} \leq C \left(\|(\zeta I - \varepsilon^m L)u\|_{0,\alpha,\varepsilon} + \sup_M |u| \right). \quad (3.2)$$

Proof As already hinted in the definition given above of semiclassical Hölder spaces, we show that (3.2) is simply the ‘standard’ Schauder estimate relative to the metric $g_\varepsilon = \varepsilon^{-2}g$.

If z is a normal coordinate for g in a ball $B(r_0)$ of radius r_0 about any point, then $w = z/\varepsilon$ is a normal coordinate for the metric $g_\varepsilon = \varepsilon^{-2}g$ on a ball of radius r_0/ε . Indeed,

$$g = g_{ab}(z)dz^a dz^b = (\delta_{ab} + E_{ab}(z))dz^a dz^b, \quad |E|_g = \mathcal{O}(|z|^2);$$

hence,

$$g_\varepsilon = (\varepsilon^{-2}\delta_{ab} + \varepsilon^{-2}E_{ab}(\varepsilon w))dz^a dz^b = (\delta_{ab} + E_{ab}(\varepsilon w))dw^a dw^b.$$

Using the hypothesis of bounded geometry, this shows that the coefficients and derivatives of g_ε are uniformly controlled in the rescaled normal coordinates. Denoting this dilation operator on a given ball by S_ε , then the admissibility hypothesis shows that the rescaled operator $\varepsilon^m S_\varepsilon^* L$ is strongly elliptic on each such ball.

The standard local elliptic estimate on any geodesic ball B for g_ε states that if B' is the ball of half the radius and same center, then there exists a constant C such that for all $u \in C^{m,\alpha}(g_\varepsilon)$,

$$\|u\|_{B',m,\alpha,g_\varepsilon} \leq C \left(\|(\zeta I - \varepsilon^m L)u\|_{B,0,\alpha,g_\varepsilon} + \sup_B |u| \right).$$

Taking the supremum of the right-hand side over all balls B of fixed radius (provided by bounded geometry), and then taking the supremum over the corresponding balls B' on the left yields the global estimate. Crucially, the constant C is independent of $\varepsilon \leq 1$.

The proof is now completed by observing that the usual Hölder norm for $C^{m,\alpha}(g_\varepsilon)$ is precisely the same as the semiclassical Hölder norm for $C^{m,\alpha}_\varepsilon(g)$. $\square$

We now establish the C^0 version of the sectoriality estimate.

Proposition 3.3 *There is a constant $C > 0$ such that*

$$\|u\|_0 \leq C \|(\zeta I - \varepsilon^m L)u\|_0$$

for all unit ζ with $\operatorname{Re} \zeta \leq c' < 0$, $\varepsilon \in (0, 1]$ and $u \in C^{m,\alpha}(M, g)$.

Proof If this assertion were false, then there would exist sequences ζ_j , ε_j , and $u_j \in \mathcal{C}^{m,\alpha}$ such that

$$\|u_j\|_0 > j \|(\zeta_j - \varepsilon_j^m L)u_j\|_0. \quad (3.3)$$

Replace u_j by $v_j = u_j / \|u_j\|_0$ and set $f_j = (\zeta_j - \varepsilon_j^m L)v_j$, so that $\|v_j\|_0 = 1$ and $\|f_j\|_0 \leq 1/j \rightarrow 0$. Passing to a subsequence, we assume that $\zeta_j \rightarrow \zeta_*$.

Next, choose a point $p_j \in M$ where $|v_j(p_j)| > 1/2$. By virtue of the bounded geometry of (M, g) , the restriction of the metric g and the coefficients of the operator L (expressed in normal coordinates) on the balls $B_j := B_{r_0}(p_j)$ are bounded in $\mathcal{C}^{m,\alpha}$, independently of j .

There are two cases to consider. First, suppose that $\varepsilon_j \rightarrow \varepsilon_* > 0$. If the p_j remain in a compact set of M , then it is straightforward to extract a limit $v \in \mathcal{C}^{0,\alpha}$ of the sequence v_j which is not identically zero, and which satisfies $(\zeta_* I - \varepsilon_*^m L)v = 0$. This is impossible given our hypothesis that $\varepsilon_*^{-m} \zeta_*$ does not lie in the spectrum.

Now we turn to the case where the sequence p_j diverges. Per the discussion in ‘Pointed limits of manifolds of bounded geometry’ in Sect. 2.2, choose a subsequence so that (M, g, p_j) converges to a limiting space $(M_\infty, g_\infty, p_\infty)$ as pointed Riemannian spaces in the $\mathcal{C}^{\ell,\alpha''}$ topology, and so that L converges in this construction to an operator L_∞ on M_∞ . By Lemma 2.10, L_∞ is admissible. Since $1/2 \leq |v_j(p_j)| \leq 1 = \sup |v_j|$, it follows as above that some subsequence of these functions converges to a nontrivial limiting function v_∞ on M_∞ which satisfies $(\zeta_* I - \varepsilon_*^m L_\infty)v_\infty = 0$. Clearly $|v_\infty| \leq 1$ everywhere, and by local Schauder estimates there exists a constant C such that $1 \leq \|v_\infty\|_{0,\alpha} \leq C$.

We may now employ the same transplantation argument as in the proof of Lemma 2.10 to show that the existence of this solution v_∞ contradicts the fact that $\varepsilon_*^{-2} \zeta_*$ does not lie in the spectrum of L . The only minor modification from the proof is that we now write

$$\begin{aligned} (\zeta_j I - \varepsilon_j^m L)(\chi_i v_\infty) &:= h_i = \chi_i (\zeta_* I - \varepsilon_*^m L_\infty) v_\infty \\ &\quad + (\zeta_j - \zeta_*) \chi_i v_\infty - \chi_i (\varepsilon_j^m - \varepsilon_*^m) L_\infty v_\infty - \chi_i \varepsilon_j^m (L - L_\infty) v_\infty \\ &\quad - \varepsilon_j^m [L, \chi_i] v_\infty; \end{aligned}$$

but this tends to 0 in norm, as above. Thus, we reach a contradiction in this case too.

We have now reduced to the case where $\varepsilon_j \rightarrow 0$. If z is a normal coordinate in B_j , then as discussed above, $w = z/\varepsilon_j$ is a normal coordinate for the metric $g_j = \varepsilon_j^{-2} g$ on a ball of radius r_0/ε_j . Indeed,

$$g_j = (\delta_{ab} + E_{ab}(\varepsilon_j w)) dw^a dw^b, \quad E_{ab}(\varepsilon_j w) = \mathcal{O}(\varepsilon_j^2 C^2) \quad \text{for } |w| \leq C.$$

Thus, g_j converges uniformly to the Euclidean metric on any compact subset of $\mathbb{R}^n$.

A similar computation shows that if L_j denotes the operator L expressed in these rescaled coordinates, then

$$\varepsilon_j^m L_j \rightarrow L_E,$$

a constant coefficient operator on $\mathbb{R}^n$; this limit is uniform on any compact subset of $\mathbb{R}^n$. Observe that any term in L involving a derivative of order less than m tends to 0 in this limit. In fact, using multi-index notation,

$$\text{if } L = \sum_{|J| \leq m} a_J(z) \partial_z^J, \text{ then } L_E = \sum_{|J|=m} a_J(0) \partial_w^J.$$

(In the language developed in Sect. 4, L_E is the constant coefficient operator associated with the semiclassical symbol $\sigma_m^{\text{sc}}(L)(z, \xi)$ of L at $z = 0$; we refer to that section for more on this.)

Now, passing to a subsequence, we may assume that $\zeta_j \rightarrow \zeta_*$ and that $v_j \rightarrow v$ in $C_{\text{loc}}^m(\mathbb{R}^n)$. Clearly $|v| \leq 1$; furthermore, $1/2 \leq |v_j(0)| \leq 1$, so the function v is nontrivial. It also must satisfy

$$(\zeta_* - L_E)v = 0$$

on all of $\mathbb{R}^n$. The existence of such a bounded solution is easily ruled out by Fourier analysis. Indeed, if we regard v as a tempered distribution, then taking the Fourier transform transforms this equation to

$$(\zeta_* - \sigma_m(L)(0, \xi))\hat{v}(\xi) = 0.$$

Using the definition of strong ellipticity, and the fact that $\text{Re } \zeta_* \leq 0$, $\zeta_* \neq 0$, the factor $(\zeta_* - \sigma_m(L)(0, \xi))$ is invertible for all $\xi \in \mathbb{R}^n$; hence, $\hat{v} = 0$, and so $v = 0$ as asserted. We have arrived at a final contradiction and have thus established the C^0 bound. $\square$

The (nearly) final step in the proof of Theorem A is to establish the corresponding C^1 sectoriality estimate, by reducing it to the C^0 estimate.

Proposition 3.4 *For any fixed $c' < 0$ that is sufficiently small, there exists a constant C such that for all unit ζ with $\text{Re } \zeta \leq c' < 0$ and $\varepsilon \in (0, 1]$,*

$$\|u\|_1 \leq C\|(\zeta - \varepsilon^m L)u\|_1 \quad (3.4)$$

for all $u \in C^{m+1, \alpha}(M, g)$.

Proof We begin by differentiating both sides of the equation $(\zeta - \varepsilon^m L)u = f$, and then commuting derivatives to obtain

$$(\zeta - \varepsilon^m L)(\nabla u) = \nabla f - \varepsilon^m [\nabla, L]u.$$

The C^0 bound in the previous proposition implies that

$$\|\nabla u\|_0 \leq C\|\nabla f\|_0 + C\varepsilon^m \|[\nabla, L]u\|_0.$$

However, $[\nabla, L]$ is a differential operator of order m with uniformly C^1 coefficients, and the semiclassical estimate in Lemma 3.2 shows that for an appropriate constant C ,

$$\varepsilon^m \|[\nabla, L]u\|_0 \leq C\|u\|_{m, \alpha, \varepsilon} \leq C' \left(\|f\|_{0, \alpha, \varepsilon} + \sup_M |u| \right) \leq C'\|f\|_1.$$

We have used the C^0 estimate from the previous Proposition in this last inequality to estimate $\sup_M |u| \leq \sup_M |f|$.

This completes the proof of Proposition 3.4. $\square$

The proof of the sectoriality estimate on $C^{0, \alpha}$ is attained by applying an interpolation argument. The above two results show that there exist constants C_0, C_1 such that

$$\|(\zeta - \varepsilon^m L)^{-1} f\|_{\ell} \leq C_{\ell} \|f\|_{\ell}, \quad \ell = 0, 1, \quad (3.5)$$

for all $f \in C^{\infty}(M, g)$ and for all unit ζ with $\text{Re } \zeta \leq c' < 0$, $\varepsilon \in (0, 1]$. The little Hölder space $C^{0, \alpha}(M, g)$ is identified with the real interpolation space $(C^0(M, g), C^1(M, g))_{\alpha}$; see for example [31, Chapter 1]. From estimate (3.5), we conclude that

$$\|(\zeta - \varepsilon^m L)^{-1} f\|_{0, \alpha} \leq C_0^{1-\alpha} C_1^{\alpha} \|f\|_{0, \alpha}$$

for all unit ζ with $\operatorname{Re} \zeta \leq c' < 0$, $\varepsilon \in (0, 1]$ and $u \in C^\infty(M, g)$, and hence in little Hölder spaces by the density of C^∞ in the little Hölder spaces. This is the sectoriality estimate on $\mathcal{C}^{0,\alpha}$.

We conclude this section by showing how sectoriality on $\mathcal{C}^{k,\alpha}$, $k \geq 1$, follows from the case $k = 0$.

Corollary 3.5 *If (M, g) has bounded geometry of order $k + m + \alpha'$ for some $0 < \alpha' < 1$, and L is an admissible operator with coefficients uniform of order $k + \alpha'$, then L is sectorial on $\mathcal{C}^{k,\alpha}(M, g)$.*

Proof We have proved the uniform boundedness of the norm of $\lambda(\lambda I - L)^{-1} : \mathcal{C}^{0,\alpha} \rightarrow \mathcal{C}^{0,\alpha}$. Suppose that $f \in \mathcal{C}^{k,\alpha}(M, g)$, and write $u = u_\lambda = \lambda(\lambda I - L)^{-1}f$. The $\mathcal{C}^{0,\alpha}$ sectoriality estimate we have just proved shows that $\|u_\lambda\|_{0,\alpha} \leq C\|f\|_{0,\alpha}$ uniformly in λ . Furthermore, by local elliptic regularity, $u_\lambda \in C_{\text{loc}}^{k+m,\alpha}$, and if B is any ball of radius $\frac{1}{2}r_0$ and B' the ball of radius r_0 with the same center, then

$$\|u_\lambda\|_{B,k+m,\alpha} \leq C(\|f\|_{B',k,\alpha} + \|u_\lambda\|_{B',0}) \leq C(\|f\|_{B',k,\alpha} + \|u_\lambda\|_{B',0,\alpha}).$$

Taking the supremum over all such balls, we obtain

$$\|u_\lambda\|_{k+m,\alpha} \leq C(\|f\|_{k,\alpha} + \|u_\lambda\|_{0,\alpha}) \leq C(\|f\|_{k,\alpha} + \|f\|_{0,\alpha}) = C'\|f\|_{k,\alpha},$$

which is the sectoriality estimate on $\mathcal{C}^{k,\alpha}(M, g)$. $\square$

This concludes the proof of Theorem A.

4 The semiclassical parametrix construction

We now provide details about the construction of the semiclassical parametrix for the family of operators $(\zeta I - \varepsilon^m L)$. There are two novel features in our presentation. The first is a minor one: the semiclassical calculus is best documented in the setting of operators on $\mathbb{R}^n$, cf. [19, 32], or in more modern expositions, on certain other special manifolds [44]. It adapts easily to the setting of manifolds of bounded geometry. The second is that, unlike the approaches in those citations, we carry out this construction using geometric microlocal analysis. This particular application of that theory has been observed by Melrose and partially developed in his lecture notes [39]. We review this method to keep this paper relatively self-contained.

Strictly speaking, the techniques here tacitly assume that both the operator L and the Riemannian manifold (M, g) are C^∞ . Therefore, we shall assume that this is the case, i.e., that all data are smooth, until near the end of this section. Only there do we show how to obtain the main conclusion of this section, namely the existence of the semiclassical resolvent, if L and g only have finite regularity.

4.1 The semiclassical double space

A family of operators $A = A_\varepsilon$ is called a *semiclassical family of pseudodifferential operators* if each A_ε is a pseudodifferential operator in some standard calculus of such operators on M (see below). The Schwartz kernel K_{A_ε} of each A_ε is a distribution on $M \times M$ which has a classical conormal, or polyhomogeneous, singularity along the diagonal $\text{diag} \subset M^2$ (see Definition 4.2 for the precise definitions of these terms). As $\varepsilon \searrow 0$, the distribution K_{A_ε} , and in particular its singularity along the diagonal, must degenerate somehow. The ‘geometric’

microlocal way to describe this involves regarding this family of Schwartz kernels as a single distribution $K_A(\varepsilon, z, \tilde{z})$ on the augmented double space $[0, \varepsilon_0)_\varepsilon \times M^2$. As such, it has a singularity along the family of diagonals $(0, \varepsilon_0) \times \text{diag}$, and as we now describe, an extra singularity along $\{0\} \times \text{diag}$. Our goal is to describe this extra singularity.

To do this, we pass to a slightly larger space obtained by taking the real blowup of the submanifold $\{0\} \times \text{diag}$ in $[0, \varepsilon_0) \times M^2$. This process—which should be carefully distinguished not only from the process of ‘complex blowup’ which is common in algebraic geometry, but also from the use of the phrase blowup associated with various sorts of rescaling arguments in PDE—amounts to introducing a singular cylindrical coordinate system around this submanifold and ‘adding’ the points where ‘ $r = 0$ ’ as a new boundary hypersurface. Assuming that $z, \tilde{z}$ are identical coordinates on the two copies of M , we introduce polar coordinates $r = |(\varepsilon, z - \tilde{z})| \geq 0$, $\omega = (\varepsilon, z - \tilde{z})/r \in S_+^n$, and declare this new *semiclassical double space*, which we write as M_{sc}^2 , to be the manifold with corners on which $(r, \omega, \tilde{z})$ is a nonsingular smooth coordinate system. Note that $M_{\text{sc}}^2 \setminus \{r = 0\}$ is canonically isomorphic to $[0, \varepsilon_0) \times M^2 \setminus (\{0\} \times \text{diag})$, but each point on this diagonal at $\varepsilon = 0$ is replaced by its inward-pointing spherical normal bundle at that point. The entire hypersurface $\{r = 0\}$ is called the *front face* and denoted by ff . It is the total space of a fibration over $\{0\} \times \text{diag}$, with each fiber a closed hemisphere S_+^n . The points of ff correspond to directions of approach to $\{0\} \times \text{diag}$.

Although the polar coordinates constitute a nonsingular coordinate system, it is usually more convenient to use projective coordinates. Therefore, we introduce

$$\varepsilon, w = (z - \tilde{z})/\varepsilon, \tilde{z}$$

as a coordinate system on M_{sc}^2 away from the ‘original’ face $\{\varepsilon = 0\} \times (M^2 \setminus \text{diag})$. These are defined and are smooth up to points in the interior of ff , but are undefined at $\varepsilon = 0$ away from the diagonal. In their region of definition, ε serves as a defining function for ff . Using these, the interior of each hemisphere fiber in ff is identified with $\mathbb{R}^n$, with linear coordinate w . In fact, this projective identification of each fiber of ff with $\mathbb{R}^n$ is *well defined up to linear transformations*. That is, if z' is any other choice of local coordinates, and $\tilde{z}'$ is the same coordinate system on the second copy of M , then $w' = (z' - \tilde{z}')/\varepsilon = Aw + \mathcal{O}(\varepsilon)$, for some $A \in \text{GL}_n$, and hence $w' = Aw$ at $\varepsilon = 0$, where A is a matrix (that may depend smoothly on $\tilde{z}$, i.e., vary with the hemisphere fibers).

Let $\pi : M_{\text{sc}}^2 \rightarrow [0, \varepsilon_0)$ be the composition of the blowdown $M_{\text{sc}}^2 \rightarrow [0, \varepsilon_0) \times M^2$ followed by projection to the first factor. Each level set $\pi^{-1}(\varepsilon)$, $\varepsilon > 0$, is a copy of M^2 . The preimage $\pi^{-1}(0)$ is the union of two manifolds with boundary: namely, the manifold with boundary obtained by blowing up M^2 along its diagonal, and the closure of ff , which is a bundle of closed hemispheres. The intersection of these two hypersurfaces is naturally identified with the spherical normal bundle of the diagonal in M^2 .

4.2 Lifts of semiclassical differential operators

The blowup construction in this particular setting is motivated by a simple computation: consider the lift of $\zeta I - \varepsilon^m L$ first to the left factor of M in M^2 (i.e., differentiating in z rather than $\tilde{z}$), then to $[0, \varepsilon_0) \times M^2$ and finally to M_{sc}^2 . To compute this lift, first observe that

$$\varepsilon \partial_{z_j} = \partial_{w_j};$$

hence, writing $L = \sum_{|\beta|} a_\beta(z) \partial_z^\beta$ using that $z = \tilde{z} + \varepsilon w$, we arrive at

$$\varepsilon^m \sum_{|\beta| \leq m} a_\beta(z) \partial_z^\beta = \sum_{|\beta|=m} a_\beta(\tilde{z}) \partial_w^\beta + \sum_{|\beta| \leq m} c_\beta(\tilde{z}, w, \varepsilon) \varepsilon^{m-|\beta|} \partial_w^\beta.$$

The coefficients $c_\beta(\tilde{z}, w, \varepsilon)$ depend smoothly on all three variables, and arise from the Taylor expansions in ε of the functions $a_\beta(\tilde{z} + \varepsilon w)$. The first set of terms on the right in this equality are the terms of order 0 in this Taylor expansion if $|\beta| = m$; these contain no positive powers of ε . On the other hand, the coefficient functions with $|\beta| = m$ in the second sum on the right arise only from the higher-order terms of the Taylor expansions of $a_\beta(\tilde{z} + \varepsilon w)$ for the same β , and hence, these vanish at least like ε . The key point in this calculation is that the lift of $\varepsilon^m L$ is the sum of a homogeneous operator of order m with coefficients which depend only on $\tilde{z}$, and hence has *constant coefficients* on each fiber of ff at $\varepsilon = 0$, and a remainder term which is a differential operator of order m with coefficients depending smoothly on ε , w and $\tilde{z}$, and which vanishes at $\varepsilon = 0$. Using this, we now write

$$\zeta I - \varepsilon^m L = (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) + (\text{error term vanishing at } \varepsilon = 0).$$

Here, by definition, the *semiclassical symbol* of L , $\sigma_m^{\text{sc}}(L)(\partial_w)$, is the constant coefficient operator which appears in the above computation. A standard microlocal notation is to set $D_{w_j} = (1/i)\partial_{w_j}$; under Fourier transform, this corresponds to the linear variable ξ_j . This semiclassical symbol is closely related to the principal symbol of L by

$$\sigma_m(\sigma_m^{\text{sc}}(L)((1/i)\partial_w)) = \sigma_m(L)(\tilde{z}, \xi).$$

(Since σ_m^{sc} is homogeneous and m is even, the factor of i reduces to an even more harmless factor of ± 1 .)

The semiclassical symbol is well defined up to a linear change of coordinates in w . It also depends smoothly on $\tilde{z}$, but the dependence is only parametric, and we frequently drop it from the notation below.

The key point in all that follows below is that there exists a distribution $H_\zeta(w) = \varepsilon^{-n} \bar{H}_\zeta(w)$ (which depends smoothly on $\tilde{z}$) such that

$$(\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) \bar{H}_\zeta(w) = \bar{H}_\zeta(w) (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) = \delta(w).$$

As we now discuss, this is a consequence of the strong ellipticity of L . We explain this and develop some further properties of H_ζ , in the next subsection. The appearance of the slightly odd-looking factor of ε^{-n} is explained there too. The full parametrix construction itself is simply a perturbative construction which uses this distribution as the leading term of a formal series in ε which is constructed to be a ‘formal’ inverse of $\zeta I - \varepsilon^m L$. This is then readily converted to a good parametrix with rapidly decaying error terms, and then to an actual inverse if ε is small.

4.3 Green function for the model problem

Here we establish the existence and certain properties of $\bar{H}_\zeta(w)$.

Lemma 4.1 *There exists a distribution $\bar{H}_\zeta(w)$ on $\mathbb{R}^n$ such that $\bar{H}_\zeta(w - \tilde{w})$ is the Schwartz kernel of a translation-invariant pseudodifferential operator on $\mathbb{R}^n$ of order $-m$ and which has the following two properties. First*

$$(\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) \circ \bar{H}_\zeta = \bar{H}_\zeta \circ (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) = \delta(w). \quad (4.1)$$

In addition, $\overline{H}_\zeta$ depends smoothly on $\tilde{z}$, and satisfies $|\overline{H}_\zeta(w)| \leq C e^{-\delta|w|}$ where $\delta > 0$ provided $\operatorname{Re} \zeta < 0$.

Proof Strong ellipticity of L implies that the function $\xi \mapsto (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1}$ is $\mathcal{C}^\infty$ and polynomially bounded, hence is an element of $\mathcal{S}'$, the space of tempered distributions. As such, we may take its inverse Fourier transform and define

$$\overline{H}_\zeta(w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iw \cdot \xi} (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi.$$

This too is an element of $\mathcal{S}'$. This integral converges if $m > n$, but if this is not the case, we can make sense of it as the distributional limit

$$\lim_{\delta \rightarrow 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iw \cdot \xi} \chi(\delta \xi) (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi,$$

where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ equals 1 in some neighborhood of the origin. In other words, pairing this expression with a Schwartz function $\phi(w)$, we note that the limit in δ can be taken in the classical sense on the right side of

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{iw \cdot \xi} \chi(\delta \xi) (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi \right) \phi(w) dw \\ &= \int_{\mathbb{R}^n} \hat{\phi}(-\xi) \chi(\delta \xi) (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi. \end{aligned}$$

Using similar standard distributional manipulations, we can also justify that

$$(\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) \overline{H}_\zeta(w) = \overline{H}_\zeta(w) (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iw \cdot \xi} d\xi = I.$$

We recall also that since $(\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1}$ is $\mathcal{C}^\infty$, its Fourier transform decays rapidly. Indeed, interpreted again as Fourier transforms of tempered distributions, this follows from the (distributional) identity

$$\begin{aligned} \overline{H}_\zeta(w) &= |w|^{-2\ell} (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\Delta_\xi^\ell e^{iw \cdot \xi} \right) (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi \\ &= |w|^{-2\ell} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iw \cdot \xi} \Delta_\xi^\ell (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1} d\xi \end{aligned}$$

for any $\ell > 0$, and the fact that the integral is classically convergent if $-n > -m - 2\ell$, i.e., $\ell > (n-m)/2$. Next, apply any derivative ∂_w^β to both sides of this equality, where β is a multi-index, and choose $\ell > (n-m+|\beta|)/2$ so that the integral is still absolutely convergent, to deduce that $\overline{H}_\zeta(w)$ is smooth away from $w = 0$. For an even more refined statement, apply $w^\alpha \partial_w^\beta$ where $|\alpha| = |\beta|$. Passing through the Fourier transform, this becomes a constant multiple of $\partial_\xi^\alpha \xi^\beta$ acting on the exponential, which integrates by parts to an expression where $\xi^\beta (-\partial_\xi)^\alpha$ acts on $\Delta_\xi^\ell (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1}$. The resulting integral is once again convergent if $\ell > (n-m)/2$. This shows that $\overline{H}_\zeta(w)$ has ‘stable regularity’ with respect to repeated differentiation by the vector fields $w_i \partial_{w_j}$, a property which is known as conormality with respect to the origin $\{w = 0\}$. By a further analysis, which is left to the reader, the expansion of $\Delta_\xi^\ell (\zeta I - \sigma_m^{\text{sc}}(L)(\xi))^{-1}$ as $|\xi| \rightarrow \infty$ is transformed to an expansion in powers of w as $w \rightarrow 0$, which is the assertion that $\overline{H}_\zeta(w)$ is polyhomogeneous at $w = 0$.

We now explain the assertion about exponential decay. The integrand is a smooth function of ξ . By the uniformity of L as a function of $\tilde{z}$, there exists some $\delta' > 0$ sufficiently small

depending on $\operatorname{Re} \zeta < 0$ so that if $\eta \in \mathbb{R}^n$ and $|\eta| < \delta'$, then the deformed map $\xi \mapsto (\zeta I - \sigma_m^{\text{sc}}(L)(\xi + i\eta))^{-1}$ remains smooth in ξ and decays as $\xi \rightarrow \infty$ at the same rate as the undeformed function.

Using these observations and a similar renormalization scheme to make the integral convergent, we can deform the contour of integration from $\mathbb{R}^n$ to $\mathbb{R}^n + i\eta$ and write

$$\overline{H}_\zeta(w) = e^{-w \cdot \eta} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i w \cdot \xi} (\zeta I - \sigma_m^{\text{sc}}(L)(\xi + i\eta))^{-1} d\xi.$$

The integral is defined as above, using the same observations and calculations as above, and is bounded as $w \rightarrow \infty$, but is now accompanied by the prefactor $e^{-w \cdot \eta}$ which decays exponentially in any cone properly contained in the half-space $w \cdot \eta \geq 0$. Since η can point in any direction, this shows that $\overline{H}_\zeta(w)$ decays exponentially, at some rate depending on $\operatorname{Re} \zeta < 0$, as $|w| \rightarrow \infty$. This solution is smooth in the parameters ζ and $\bar{\zeta}$. This establishes the lemma. $\square$

Using these results, it is now straightforward to see that the convolution $\overline{H}_\zeta \star f(w)$ is exponentially decreasing in w if $f \in C_0^\infty(\mathbb{R}^n)$.

4.4 Manifolds with corners, blowups and polyhomogeneity

To get further into the details, we first recall the following terminology and notions. All of these are described more carefully in [34]. First, suppose that X is a manifold with corners. This means that near any point $q \in X$ there is a local coordinate system $(x_1, \dots, x_k, y_1, \dots, y_{\ell-k})$, $\ell = \dim X$, such that each $x_i \geq 0$ and each $y_i \in (-\epsilon, \epsilon)$, and $(x, y) = (0, 0)$ at q . We say that q then lies on a corner of codimension k . We next define the space $\mathcal{V}_b(X)$ of b -vector fields on X to consist of all smooth vector fields which are unconstrained in the interior of X and which are tangent to all boundary faces. In terms of the above coordinates, any $V \in \mathcal{V}_b(X)$ is, near q , a linear combination, with smooth coefficients, of the basic vector fields $x_i \partial_{x_j}$, $i, j = 1, \dots, k$ and ∂_{y_i} , $i = 1, \dots, \ell - k$.

Definition 4.2 A distribution u on X is said to be *conormal* to ∂X if there exists some fixed function space E such that

$$V_1 \dots V_N u \in E, \quad \text{for all } V_i \in \mathcal{V}_b(X) \text{ and all } N \in \mathbb{N}.$$

Typically we let E be a weighted L^2 or L^∞ space. Thus, if u is conormal, then u is C^∞ on the interior of X , but may have singular behavior along the boundary. These singularities are, however, ‘tangentially smooth.’

Definition 4.3 A distribution u on X is said to be *polyhomogeneous* at ∂X if it is conormal, and in addition has a classical expansion at each boundary face and product-type expansions at the corners. Thus, near a codimension one face $x = 0$, for example, with y a coordinate system along that face,

$$u(x, y) \sim \sum_{\operatorname{Re} \gamma_j \rightarrow \infty} \sum_{k=0}^{N_j} a_{jk}(y) x^{\gamma_j} (\log x)^k,$$

for some discrete set of exponents γ_j , where each coefficient a_{jk} is smooth. Near each corner, where $x_1, \dots, x_k = 0$, u has an expansion involving various (possibly nonintegral) powers of each x_j and, for each monomial in the series, additional factors which are positive integer powers of each $\log x_i$.

These expansions hold for all derivatives of u as well, in the sense that any derivative of u has an expansion where the summands are the corresponding derivatives of each term in the series for u . These series are asymptotic, but (usually) not convergent.

The space of all such polyhomogeneous distributions is denoted $\mathcal{A}_{\text{phg}}(X, \partial X)$. We may easily extend this to define polyhomogeneous sections of vector bundles over X as well.

A submanifold $Z \subset X$ is called a *p-submanifold* if near any $q \in Z$ there is a set of coordinates for X as above such that in some neighborhood of q , $Z = \{x_1 = \dots = x_r = 0, y_1 = \dots = y_s = 0\}$. Thus, locally near q , X is the product of (some small neighborhood of) Z with another manifold with corners; the p stands for ‘product.’ If Z is such a p -submanifold, then we may define the blowup of X around Z , denoted $[X; Z]$, to be the union of $X \setminus Z$ and the inward-pointing spherical normal bundle of Z . Thus, $[X; Z]$ is a new manifold with corners, with a new boundary hypersurface created by this blowup. As in our special case of the semiclassical double space defined above, we may ‘construct’ this blowup by taking cylindrical coordinates around Z in X , say (r, ω, z) , where $z \in Z$, ω is a spherical normal vector and $r \geq 0$, and ‘adding the $r = 0$ face.’ We denote the new ‘front face’ of this blowup by $\text{ff}[X; Z]$.

4.5 Pseudodifferential operators via their Schwartz kernels

Let M be a manifold (possibly open, but without boundary). We may define the blowup $[M^2, \text{diag}]$. This has a front face, ff , which is the spherical normal bundle of diag in M^2 .

Definition 4.4 A pseudodifferential operator $A : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$ on M is a linear operator for which the Schwartz kernel K_A of A is the sum of an element of $\mathcal{A}_{\text{phg}}([M^2, \text{diag}], \text{ff})$ and a distribution which is conormal and supported along ff .

This is a purely intrinsic and ‘geometric’ definition of the space of pseudodifferential operators. It is not easy to work with computationally, however, and it is customary to use other definitions using oscillatory integrals, cf. [42] for example, which are better suited for proving such things as showing that the composition of two pseudodifferential operators is again pseudodifferential.

The inclusion in this definition of the extra terms which are supported on the front face may seem a technical annoyance, but it is worth pointing out that the Schwartz kernel of the identity operator, $\delta(z - \tilde{z})$, has this property. In fact, if P is any differential operator (with smooth coefficients) on M , then the Schwartz kernel of P is equal to P (acting in the z variable) applied to $\delta(z - \tilde{z})$, and hence is again supported on this front face.

4.6 Semiclassical pseudodifferential operators

We are in position to define semiclassical families of pseudodifferential operators on M . The semiclassical double space M_{sc}^2 has a distinguished submanifold diag_{sc} , which we call the lifted diagonal. It is the closure in M_{sc}^2 of $(0, \varepsilon_0) \times \text{diag}$. This closure intersects the front face of M_{sc}^2 at a submanifold which contains a single point $\{w = 0\}$ on each hemisphere fiber. Clearly diag_{sc} is a p -submanifold of M_{sc}^2 , so we may pass to the blowup $[M_{\text{sc}}^2, \text{diag}_{\text{sc}}]$. This has three boundary hypersurfaces: the original face at $\varepsilon = 0$ away from the diagonal, the front face ff obtained by blowing up the diagonal at $\varepsilon = 0$, and the new front face obtained in this final blowup.

We say that A_ε is a *semiclassical pseudodifferential operator* if its Schwartz kernel $K_A(\varepsilon, z, \tilde{z})$ lifts to $[M_{\text{sc}}^2, \text{diag}_{\text{sc}}]$ to be the sum of a distribution polyhomogeneous at all

boundaries of this manifold with corners and a conormal distribution supported on the new front face, and if K_A vanishes faster than any power of ε along the original face. It makes sense to restrict K_A to each level set $\pi^{-1}(\varepsilon)$ for $\varepsilon > 0$, and on each of these we must obtain the Schwartz kernel of a pseudodifferential operator. This definition now imposes precise constraints on how these Schwartz kernels degenerate as $\varepsilon \rightarrow 0$. In particular, if we change to coordinates $\varepsilon, w, \tilde{z}$ on M_{sc}^2 , then the ‘pseudodifferential singularity’ occurs at $w = 0$ for each ε , and at $\varepsilon = 0$ there is an expansion $K_A \sim \sum \varepsilon^{\gamma_j} K_A^{(j)}(w, \tilde{z})$ where each coefficient is a pseudodifferential operator on each hemisphere fiber of ff (and as above, $\{\gamma_j\}$ is a discrete set of exponents with real parts tending to infinity).

In fact, we are only interested in Schwartz kernels for which the expansion as $\varepsilon \rightarrow 0$ is of the form

$$K_A(\varepsilon, w, \tilde{z}) \sim \sum_{j=-n}^{\infty} \varepsilon^j K_{A,j}(w, \tilde{z}).$$

The reason for starting this series at $-n$ is as follows. As described above, the lift of $\varepsilon^m L$ (as a differential operator, not its Schwartz kernel) is an operator with coefficients smooth up to ff ; in particular, it has a series expansion as $\varepsilon \rightarrow 0$ with leading term $\sigma_m^{\text{sc}}(L)(\partial_w)$. On the other hand, the identity operator has Schwartz kernel

$$\delta(z - \tilde{z}) = \delta(\varepsilon w) = \varepsilon^{-n} \delta(w).$$

Thus, if we expand each factor in the equation

$$(\zeta I - \varepsilon^m L)G(\varepsilon, w, \tilde{z}) = \varepsilon^{-n} \delta(w) \quad (4.2)$$

in powers of ε , and continue to think of the first factor on the left as a differential operator instead of a Schwartz kernel, then at least formally we expect G to have a series expansion involving the powers ε^j for $j \geq -n$.

In any case, this illustrates how the introduction of the space M_{sc}^2 provides a setting where it is possible to ‘see’ the transition from the ordinary pseudodifferential operators on each $\pi^{-1}(\varepsilon)$ to the limiting model problem on $\mathbb{R}^n$.

Accordingly, we now define a class of semiclassical pseudodifferential operators whose expansions at ff involve only integer powers:

Definition 4.5 If (M, g) is a manifold of bounded geometry, then $\Psi_{\text{sc-unif}}^{k,\ell}(M, g)$ consists of those semiclassical pseudodifferential operators on M which have pseudodifferential order k on each level set $\pi^{-1}(\varepsilon)$ and which have a series expansion in ε at ff with only integer powers, with initial term ε^ℓ .

The subscript ‘unif’ indicates that we restrict further to allow only kernels which are supported in some neighborhood $\text{dist}_g(z, \tilde{z}) \leq C$ of $\text{diag}_{\text{sc}} \cup \text{ff}$, where the constant C may depend on the operator.

4.7 Parametrix construction

We commence with the parametrix construction. Our goal is to find an element $G \in \Psi_{\text{sc-unif}}^{-m,-n}(M, g)$ such that, for each ζ with $|\zeta| = 1$ and $\text{Re } \zeta < 0$,

$$(\zeta I - \varepsilon^m L)G = I - Q_1, \quad G(\zeta I - \varepsilon^m L) = I - Q_2,$$

where $Q_1, Q_2 \in \Psi_{\text{sc-unif}}^{-\infty,\infty}(M, g)$. As explained above, we do this ‘formally,’ i.e., in a Taylor series, at ff , and then take a Borel sum of the resulting formal expansion.

More carefully, returning to (4.2), expand each factor into its formal series expansion:

$$\left((\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) + \sum_{j=1}^{\infty} \varepsilon^j E_j \right) \sum_{k=-n}^{\infty} \varepsilon^k G_k(w, \tilde{z}) = \varepsilon^{-n} \delta(w).$$

Here the E_j are the differential operators (of order $\leq m$) arising in the Taylor expansion of $\zeta I - \varepsilon^m L$. Carrying out the composition and collecting like powers of ε , we obtain a sequence of equations

$$\begin{aligned} (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) G_{-n}(w, \tilde{z}) &= \delta(w) \\ (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) G_{-n+k}(w, \tilde{z}) &= F_k(w, \tilde{z}), \end{aligned}$$

where each $F_k = \sum_{i=1}^k E_i G_{-n+k-i}$ is an ‘error term.’ This is a sequence of equations on $\mathbb{R}_w^n$, where the various terms all depend smoothly on $\tilde{z}$. These equations indicate that we must set

$$G_{-n}(w) = \overline{H}_\zeta(w), \quad \text{and} \quad G_{-n+k}(w) = \overline{H}_\zeta(w) \star F_k(w) \quad k \geq 1$$

for each $\tilde{z}$. Since the equations with $k \geq 1$ involve constant coefficients in w , they are solved by convolving with the fundamental solution $\overline{H}_\zeta(w)$.

Since each $E_i G_{-n+k-i}$ has pseudodifferential order at most 0 and $\overline{H}_\zeta$ has order $-m$, each F_k is the Schwartz kernel of a translation-invariant (in w) pseudodifferential operator of order no more than $-m$. We may solve this equation inductively, given the properties of $\overline{H}_\zeta$ established in Lemma 4.1, but then are faced with showing that the Borel sum of this series of singular kernels still has the correct behavior. There is a slightly easier way to proceed which allows us to work more directly with $\mathcal{C}^\infty$ kernels. Namely, we first find a pseudodifferential operator $\tilde{G}(\varepsilon, \cdot, \cdot)$ on each level set $\pi^{-1}(\varepsilon)$ which solves

$$(\zeta I - \varepsilon^m L) \tilde{G}(\varepsilon) = \varepsilon^{-n} \delta(w) + \varepsilon^{-n} \mathcal{R}(\varepsilon, w, \tilde{z}),$$

where $\mathcal{R}$ is smooth in all variables, $\varepsilon, w, \tilde{z}$. This involves carrying out the complete parametrix construction for the nondegenerate operator on each level set in the standard pseudodifferential calculus, but carrying along ε as a smooth parameter. To compensate for the factors ε^{-n} on the right, we choose $\tilde{G} \in \Psi_{\text{sc-unif}}^{-m, -n}$. Said differently, this is the nondegenerate elliptic parametrix construction (with parametrix cut off to have support in a neighborhood of the diagonal), carried out smoothly in the parameter ε .

We now need to find additional terms in the parametrix which cancel off the full Taylor series in ε of the remainder term $\varepsilon^{-n} \mathcal{R}$. This involves inductively solving a sequence of equations

$$\begin{aligned} (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) \tilde{G}_{-n}(w, \tilde{z}) &= \mathcal{R}_0(w, \tilde{z}) \\ (\zeta I - \sigma_m^{\text{sc}}(L)(\partial_w)) \tilde{G}_{-n+k}(w, \tilde{z}) &= \sum_{i=1}^k E_i \mathcal{R}_{-n+k-i}, \end{aligned}$$

where $\mathcal{R} \sim \sum \varepsilon^k \mathcal{R}_k$. The advantage is that the right-hand sides are all smooth and we can assume that they are compactly supported in, say, $\{|w| \leq 1\}$ for all $\tilde{z}$. The solutions are given by

$$\tilde{G}_{-n} = \overline{H}_\zeta \star \mathcal{R}_0(w), \quad \tilde{G}_{-n+k} = \overline{H}_\zeta \star \left(\sum_{i=1}^k E_i \mathcal{R}_{-n+k-i} \right) (w).$$

4.8 Conclusion of parametrix construction

We have constructed both $\tilde{G}$ and the sequence of smooth, exponentially decaying terms $\tilde{G}_{-n+k}$, $k \geq 0$. An important fact is that given any such sequence, it is possible to construct a function $\tilde{G}'(w)$ which is smooth in the interior of M_{sc}^2 , decays to all orders at the original face, and has its expansion in powers of ε at ff of the form

$$\tilde{G}'(w) \sim \sum_{k=0}^{\infty} \varepsilon^{-n+k} \tilde{G}_k(w).$$

This is the Borel sum of this series and is an element of $\Psi_{sc-unif}^{-\infty, -n}$.

Our final semiclassical parametrix is now defined by

$$G(\varepsilon, w, \tilde{z}) = \tilde{G}(\varepsilon, w, \tilde{z}) + \tilde{G}'(\varepsilon, w, \tilde{z}) \in \Psi_{sc-unif}^{-m, -n}.$$

4.9 Boundedness properties

We conclude this section by sketching the proof of boundedness properties of elements of $\Psi_{sc-unif}^{*,*}(M)$ on the semiclassical Hölder spaces $C_{sc}^{k,\alpha}(M, g)$, as defined in Sect. 2. First we define the family of spaces

$$\varepsilon^\lambda C_\varepsilon^{k,\alpha} = \{u(\varepsilon, z) = \varepsilon^\lambda \tilde{u}(\varepsilon, z) \text{ where } \tilde{u} \in C^\infty([0, \varepsilon_0]; C_\varepsilon^{k,\alpha}(M, g))\}.$$

In other words, an element of this space can be represented by a formal series $u \sim \sum_{j \geq 0} \varepsilon^{\lambda+j} u_j$ with each $u_j \in C_\varepsilon^{k,\alpha}$.

Proposition 4.6 *If $A \in \Psi_{sc-unif}^{\kappa,\mu}$ for some $\kappa \in \mathbb{Z}$, then*

$$A : \varepsilon^\lambda C_\varepsilon^{k,\alpha} \longrightarrow \varepsilon^{\lambda+\mu+n} C_\varepsilon^{k-\kappa,\alpha}$$

is bounded provided $\kappa \leq k$.

Remark 4.7 We can extend this result to allow operators of nonintegral order, for example using a standard interpolation result, but we omit this here since it is not needed.

Proof The first observation is that if $\kappa \leq k$, and $P = \sum_{|\alpha| \leq \kappa} a_\alpha(z) \varepsilon^{|\alpha|} \partial_z^\alpha$ is a semiclassical differential operator with coefficients which are uniformly bounded in C^∞ , then directly from the definition,

$$P : C_\varepsilon^{k,\alpha}(M, g) \longrightarrow C_\varepsilon^{k-\kappa,\alpha}(M, g) \quad (4.3)$$

is bounded. To relate this to a pseudodifferential boundedness theorem, we have noted that the Schwartz kernel of P on M_{sc}^2 is a distribution supported along diag_{sc} , namely

$$K_P(\varepsilon, w, \tilde{z}) = \varepsilon^{-n} \sum_{|\alpha| \leq \kappa} a_\alpha(\tilde{z} + \varepsilon w) (\partial_w^\alpha \delta(w)) \in \Psi_{sc-unif}^{\kappa, -n}.$$

Thus, (4.3) is a special case of the boundedness in Proposition 4.6, with $\lambda = 0$.

Recalling the definition of a semiclassical pseudodifferential operator, Definition 4.4, we prove the boundedness property for kernels in $\Psi_{sc-unif}^{-\infty, -n}(M, g)$ supported away from the diagonal, and another for kernels in $A \in \Psi_{sc-unif}^{\kappa, -n}(M, g)$ supported near the diagonal.

First suppose that $A \in \Psi_{\text{sc-unif}}^{-\infty, -n}(M, g)$ has its Schwartz kernel K_A supported away from diag_{sc} in M_{sc}^2 ; to be specific, suppose $\text{supp}(K_A) \subset \{\varepsilon \leq \text{dist}(z, \tilde{z}) \leq C\}$, or equivalently, $1 \leq |w| \leq C/\varepsilon$. We then have that in projective coordinates $(z, \tilde{w})$, $\tilde{w} = (\tilde{z} - z)/\varepsilon$,

$$\begin{aligned} \left| \int_M K_A(\varepsilon, z, \tilde{z}) u(\tilde{z}) dV_g(\tilde{z}) \right| &\leq C \|u\|_\infty \int_{1 \leq |\tilde{w}| \leq C/\varepsilon} |K_A(\varepsilon, z, \tilde{w})| \varepsilon^n dV(\tilde{w}) \\ &\leq C \|u\|_\infty \int_{1 \leq |\tilde{w}| \leq C/\varepsilon} (1 + |\tilde{w}|)^{-N} d\tilde{w} \leq C' \|u\|_\infty < \infty \end{aligned}$$

since K_A blows up like ε^{-n} and $\varepsilon^n |K_A|$ is Schwartz in $\tilde{w}$, uniformly as $\varepsilon \searrow 0$. Furthermore, any semiclassical derivative $\varepsilon^{|\alpha|} \partial_z^\alpha$ applied to K_A yields a kernel of the same form. This proves that if $A \in \Psi_{\text{sc-unif}}^{-\infty, -n}$ has its kernel with this special support property, then

$$A : C_\varepsilon^{k, \alpha}(M, g) \longrightarrow C_\varepsilon^{r, \alpha}(M, g)$$

for any $r \in \mathbb{N}$.

Now let $A \in \Psi_{\text{sc-unif}}^{k, -n}(M, g)$ have Schwartz kernel supported in the region $|w| \leq 1$, and as before, assume $\kappa \leq k$. We do not need to assume that κ is an integer, but then must change the Hölder indices accordingly and interpret the borderline cases where $\kappa = k - i + \alpha$, $i \in \mathbb{N}$, in terms of Zygmund spaces. We can then proceed by a combination of a rescaling argument and invoking the fact that an ordinary pseudodifferential operator of order κ in a ball of size 2 induces a map $C_0^{k, \alpha}(B_1(0)) \rightarrow C^{k-\kappa, \alpha}(B_1(0))$, between *ordinary* Hölder spaces (and where elements of the domain space are compactly supported in $B_1(0)$). Indeed, given any $u \in C_\varepsilon^{k, \alpha}$, choose a locally finite cover $B_{2\varepsilon}(q_j)$ of bounded covering multiplicity and a partition of unity χ_j for which there are uniform bounds on its semiclassical derivatives up to order k , and such that $\chi_j = 1$ on $B_\varepsilon(q_j)$. Then $u = \sum \chi_j u$ and since only at most some fixed number of the $A(\chi_j u)$ have support at any one point, it suffices to estimate the norms of each of these summands. Recalling Remark 2.6, we may compute the semiclassical Hölder norms by taking the supremum over balls of radius ε (or any fixed multiple of ε). Thus, rescaling the coordinates of each summand by a factor of $1/\varepsilon$, we reduce to the action of a standard pseudodifferential operator of order κ on a standard $C^{k, \alpha}$ function on a ball of fixed radius, where the result is well known.

Finally, if $A \in \Psi_{\text{sc-unif}}^{k, \mu}(M, g)$, then it is clear from all of the above, and the basic definitions, that the assertion of Proposition 4.6 holds. $\square$

4.10 Operators with finite regularity coefficients

As explained at the beginning of this section, the geometric microlocal techniques used here require the smoothness of both the metric g and the coefficients of L . Using that assumption, we have shown that the inverse $(\zeta I - \varepsilon^m L)^{-1}$ exists and is bounded on $C^{k, \alpha}$ if ε is sufficiently small. We now extend this to operators and metrics of lower regularity.

Proposition 4.8 *Suppose that (M, g) has bounded geometry of order $\ell + \alpha'$ where $\ell \geq k + m$ and $\alpha' \in (\alpha, 1)$, and that L is an admissible operator of order m with coefficients uniformly bounded in $C^{k, \alpha'}$. Then there exists $\varepsilon_0 > 0$ such that $G_{\zeta, \varepsilon} := (\zeta I - \varepsilon^m L)^{-1}$ exists as a bounded operator on $C^{k, \alpha}$ if $0 < \varepsilon < \varepsilon_0$.*

While it is possible to carry out some version of the parametrix construction under these regularity hypotheses, this would take extra work, particularly if the regularity order k is small. Thus, we prove this another way, using techniques close to those in Sect. 3.

Proof Choose an approximating sequence of metrics $g^{(i)}$ and operators $L^{(i)}$, which are all smooth, but so that $g^{(i)} \rightarrow g$ in $\mathcal{C}^{\ell, \alpha'}$ and the coefficients of $L^{(i)}$ converge to those of L in $\mathcal{C}^{k, \alpha'}$. This can be done by selecting a locally finite open cover of M by normal coordinate balls for g and using a mollifier in each such ball.

Although the norms on the spaces $\mathcal{C}^{j, \beta}$ depend on the metric, it is clear that we may define these all relative to any fixed smooth metric. In fact, we assume that the metric g is fixed (and smooth) for simplicity, since the way it enters the argument below is minor.

As has been shown above in this section, for each i , and for $0 < \varepsilon < \varepsilon_0^{(i)}$, there exists a bounded inverse

$$G_{\varepsilon, \zeta}^{(i)} = \left(\zeta I - \varepsilon^m L^{(i)} \right)^{-1} : \mathcal{C}^{k, \alpha} \longrightarrow \mathcal{C}^{k, \alpha}.$$

Denote the operator norm of this inverse by $A_{\varepsilon, \zeta, i}$. There are two main points we must address. The first is that there exists ε_0 where $\varepsilon_0^{(i)} \geq \varepsilon_0 > 0$ for all i , and the second is that the operator norms $A_{\varepsilon, \zeta, i}$ are bounded for each ε and ζ as $i \rightarrow \infty$.

Suppose that there exist sequences $\varepsilon^{(i)} \rightarrow 0$ and $\zeta^{(i)}$ such that

$$P_i := \left(\zeta^{(i)} I - (\varepsilon^{(i)})^m L^{(i)} \right)$$

does not have a bounded inverse. This failure occurs for one of three reasons: either P_i has its nullspace in $\mathcal{C}^{k, \alpha}$, or its range is dense but not closed, or the closure of its range has positive codimension. All of this is just as in Proposition 2.11, and the proofs to rule out each of these cases are similar to the proof of that Proposition, as well as the arguments of Sect. 3. For that reason, we shall be brief.

In the first of these cases, there exists a sequence $u^{(i)} \in \mathcal{C}^{k, \alpha}$ such that $\|u^{(i)}\|_{k, \alpha} = 1$ and $P_i u^{(i)} = 0$. We may extract a limit u of this sequence by rescaling around a point $q_i \in M$ where $|u^{(i)}(q_i)| \geq 1/2$. Since $\varepsilon^{(i)} \rightarrow 0$, this limiting function u lies in $\mathcal{C}^{k, \alpha}(\mathbb{R}^n)$ and satisfies $(\zeta I - L_E)u = 0$, where ζ is a limit of (a subsequence of the) $\zeta^{(i)}$, and L_E is the constant coefficient strongly elliptic operator arising in this rescaling process. As shown in the proof of Proposition 3.3, by the strong ellipticity of L (and hence L_E), there are no nontrivial solutions of this equation.

Next, if the range of P_i is dense, there exist sequences $u^{(i)}, f^{(i)} \in \mathcal{C}^{k, \alpha}$ such that $P_i u^{(i)} = f^{(i)}$, with $\|u^{(i)}\|_{k, \alpha} = 1$ and $\|f^{(i)}\|_{k, \alpha} \rightarrow 0$. Just as in the above paragraph, there is a nontrivial limit $u \in \mathcal{C}^{k, \alpha}(\mathbb{R}^n)$ such that $(\zeta I - L_E)u = 0$, which is impossible.

Finally, if the closure of the range of P_i is a proper subspace, then we may apply the same type of argument to the sequence of distributions $v^{(i)} \in (\mathcal{C}^{k, \alpha})^*$ which satisfy $P_i^* v^{(i)} = 0$. To do this, we must note that since $v^{(i)}$ satisfies this elliptic equation, it lies in $\mathcal{C}^{k+m, \alpha}$. There is a limiting function v which satisfies $(\zeta I - L_E^*)v = 0$, which again cannot happen. This proves that the inverses $G_{\varepsilon, \zeta}^{(i)}$ all exist for ε lying in some fixed interval $(0, \varepsilon_0)$.

Now fix any ε in this interval, and any ζ , and suppose that the norms $A_i = \|G_{\varepsilon, \zeta}^{(i)}\|_{\mathcal{L}(\mathcal{C}^{k, \alpha})}$ are unbounded as $i \rightarrow \infty$. This implies that there is a sequence $f_i \in \mathcal{C}^{k, \alpha}$ such that

$$\|G_{\varepsilon, \zeta}^{(i)} f_i\|_{k, \alpha} \geq \frac{1}{2} A_i \|f_i\|_{k, \alpha}.$$

Writing $u_i = G_{\varepsilon, \zeta}^{(i)} f_i$, this is the same as

$$\|u_i\|_{k, \alpha} \geq \frac{1}{2} A_i \left\| \left(\zeta I - \varepsilon^m L^{(i)} \right) u_i \right\|_{k, \alpha}.$$

Normalizing so that $\|u_i\|_{k,\alpha} = 1$ for all i , then $\|f_i\|_{k,\alpha} \leq 2/A_i \rightarrow 0$. Passing to a limit, but recalling that ε is fixed, there exists a limiting function in $C^{k+m,\alpha}$, defined either on M or on one of its limiting spaces M_∞ , such that $(\zeta I - \varepsilon^m L)u = 0$ or $(\zeta I - \varepsilon^m L_\infty)u_\infty = 0$.

The proof is complete once we show that these last possibilities cannot occur. Let us focus on the first, since the second is essentially the same. The point is simply that $\zeta I - \varepsilon^m L$ cannot have a nonvanishing nullspace for arbitrarily small values of ε . Indeed, if this operator were to have a nonvanishing nullspace in $C^{k,\alpha}$ for some sequence $\varepsilon_i \rightarrow 0$, then the same rescaling argument as we have done several times already would yield a limiting function u in $C^{k,\alpha}$ on $\mathbb{R}^n$ such that $(\zeta I - L_E)u = 0$, which we know is impossible. $\square$

Remark 4.9 It is perhaps worth emphasizing the flow of logic in this argument. We first show that for every one of the approximating operators $L^{(i)}$, the operator $\zeta I - \varepsilon^m L^{(i)}$ is invertible for $\varepsilon < \varepsilon_0$ where ε_0 does not depend on i . However, it may be necessary to restrict to a slightly smaller interval $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$ in order to guarantee that $\zeta I - \varepsilon^m L$ does not have nonvanishing nullspace.

The main point is that it is only the constant coefficient operators $\zeta I - L_E$ which we can check specifically do not have a nonvanishing nullspace. These operators only appear as limits as $\varepsilon \rightarrow 0$, and the argument to rule out a nonvanishing nullspace if ε is sufficiently small, while it is not quantitative, is insensitive to the regularity of the coefficients of L .

5 Applications

In this section, we present an application of the results proved in this paper. We begin by stating a general theorem which establishes short-time existence, uniqueness and continuous dependence on initial conditions for a large class of geometric flows on manifolds with bounded geometry. We then illustrate its application by obtaining a new result concerning short-time existence and uniqueness of the higher-order ‘ambient obstruction flow’ on open manifolds with bounded geometry.

We are now ready to prove Theorem B.

Proof of Theorem B By Theorem A, the linearization DF_u is sectorial on $C^{k,\alpha}(M, g)$ for all $u \in \mathcal{O}$. The result therefore follows from Theorem 8.1.1 and Corollary 8.1.2 in [31]. $\square$

5.1 Ambient obstruction flows

In their study of conformal invariants of a compact manifold endowed with a conformal structure, Fefferman and Graham introduce the ambient obstruction tensor, [23]. If $n = 2\ell$ is even, the ambient obstruction tensor $\mathcal{O}_n$ on a manifold (M^n, g) is a conformally covariant, trace-free, divergence-free symmetric 2-tensor associated with the metric g . Its expression involves $n - 2$ derivatives of the Ricci tensor. In the particular case of $n = 4$, the obstruction tensor $\mathcal{O}_4$ coincides with the Bach tensor

$$B_{ij} = P_{ij,k}{}^k - P_{ik,j}{}^k - P^{kl} W_{kijl}, \quad (5.1)$$

where $P_{ij} = \frac{1}{2}(Rc_{ij} - \frac{S}{6}g_{ij})$ is the Schouten tensor, W_{ijkl} is the Weyl tensor, and S is the scalar curvature of g . We refer to [23], where the importance of the Bach tensor to conformal geometry is explained.

We now study a flow associated with this ambient obstruction tensor. If this flow exists and converges as $t \rightarrow \infty$, the limit must be ‘obstruction-flat,’ a condition describing a natural

class of canonical metrics in higher dimensions. On compact manifolds, the well-posedness and uniqueness of solutions to this flow is the topic of the two papers [5, 6] by the first author and Helliwell. As an application of the methods of the present paper, we generalize these results to the setting of complete manifolds of bounded geometry. We describe the flow briefly here and refer the reader to [5, 6] for more detail.

The obstruction flow itself, namely $\partial_t g = \mathcal{O}_n(g)$, is degenerate because of both the underlying conformal covariance and its diffeomorphism invariance. To counter the first of these, we introduce the modified obstruction flow

$$\begin{cases} \partial_t g = \mathcal{O}_n(g) + c_n(-1)^{\frac{n}{2}} \left((-\Delta)^{\frac{n}{2}-1} S \right) g \\ g(0) = g_0, \end{cases} \quad (5.2)$$

where

$$c_n = \frac{1}{2^{n/2-1} \left(\frac{n}{2} - 2 \right)! (n-2)(n-1)}. \quad (5.3)$$

In 4 dimensions this is the modified Bach flow

$$\begin{cases} \partial_t g = B(g) - \frac{1}{12} (\Delta S) g \\ g(0) = g_0. \end{cases} \quad (5.4)$$

This modification breaks the conformal gauge in the sense that stationary points of this modified flow are obstruction-flat metrics with harmonic scalar curvature. The scalar curvature term induces a normalization within a conformal class. The proof of this uses the fact that $\mathcal{O}_n$ is trace-free.

The invariance under diffeomorphisms can be handled using a version of DeTurck's method, much as for the Ricci flow (see Chapter 2, Section 6 of [16]). We now describe this method in the present setting. Fix a background metric $\tilde{g}$; then any smooth one-parameter family of metrics $g(t)$ defines a time-dependent vector field

$$V(t) = \sum V_k(t, z) \partial_{z_k}, \quad \text{where} \quad V^k(t, z) := g^{pq}(t) \left(\Gamma(g(t))_{pq}^k - \Gamma(\tilde{g})_{pq}^k \right) \quad (5.5)$$

using the Christoffel symbols Γ of the indicated metrics. From this, we define the vector field

$$U = c_n(n-1)(-1)^{\frac{n}{2}-1} (-\Delta)^{\frac{n}{2}-1} V + \frac{c_n(n-2)(-1)^{\frac{n}{2}}}{2} (-\Delta)^{\frac{n}{2}-2} \nabla S, \quad (5.6)$$

and finally the obstruction-DeTurck flow

$$\begin{cases} \partial_t g = \mathcal{O}_n(g) + c_n(-1)^{\frac{n}{2}} \left((-\Delta)^{\frac{n}{2}-1} S \right) g + L_U g \\ g(0) = g_0. \end{cases} \quad (5.7)$$

As usual, one must show that solutions of this gauged flow lead to solutions of the original (modified) flow (5.2). To this end, given a solution $g(t)$ to (5.7), solve the ODE

$$\begin{cases} \frac{d}{dt} \phi_t = -U \circ \phi_t \\ \phi_0 = \text{id}, \end{cases} \quad (5.8)$$

and let ϕ_t be the one-parameter family of diffeomorphisms generated by $-U$. The fact that $\tilde{g}$ and $g(t)$ have bounded geometry implies that ϕ_t exists at least for t in some small interval around 0. A short calculation then shows that $\bar{g}(t) = \phi_t^* g(t)$ solves (5.2).

Uniqueness of solutions to the gauged flow (5.7) follows directly from the semigroup method that we invoke below. Uniqueness of solutions to the ungauged flow (5.2) requires

more work. This is explained carefully in [6], but the main ideas are as follows. Given a particular solution $\bar{g}(t)$ to (5.2) and a choice of reference metric $(M, \tilde{g})$, one may again use semigroup techniques to solve a higher-order analogue of the harmonic map heat flow equation for a family of diffeomorphisms ϕ_t from $(M, \bar{g}(t))$ to $(M, \tilde{g})$. This equation is chosen so that the pullback $g(t) = (\phi_t^{-1})^* \bar{g}(t)$ solves (5.7) with reference metric $\tilde{g}$ and V and U in (5.5), (5.6). The various uniqueness statements then imply that $\bar{g}$ is uniquely determined.

We now discuss further the existence and uniqueness statements. Taking the reference metric $\tilde{g}$ equal to the initial metric, i.e., $\tilde{g} = g_0$, we define

$$F(g) := \mathcal{O}_n(g) + c_n(-1)^{\frac{n}{2}} \left((-\Delta)^{\frac{n}{2}-1} S \right) g + L_U g. \quad (5.9)$$

As proved in [5],

$$DF_{g_0} = \frac{d}{ds} F(g_0 + sh) \Big|_{s=0} = (-1)^{\frac{n}{2}-1} A_{g_0} h + \mathcal{P} \left(\partial^{n-1} h, \partial^n g_0, g_0^{-1}, \partial^n \tilde{g}, \tilde{g}^{-1} \right), \quad (5.10)$$

where the leading term

$$(A_{g_0} h)_{jk} := g_0^{r_1 s_1} g_0^{r_2 s_2} \dots g_0^{r_{n/2} s_{n/2}} \partial_{r_1} \partial_{s_1} \dots \partial_{r_{n/2}} \partial_{s_{n/2}} h_{jk} \quad (5.11)$$

is an operator of order n and $\mathcal{P}$ is a polynomial expression in the input tensors and their derivatives of appropriate order. Note that A_{g_0} is the leading term in $\Delta^{n/2}$ and is strongly elliptic.

We may now prove Theorem C.

Proof of Theorem C Set $D = C^{n,\alpha} \subset X = C^{0,\alpha}$, and note that by construction of little Hölder spaces, D is dense in X . Take F as in Eq. (5.9), with background metric g_0 . As explained in [5], the tensors appearing in F are polynomial natural tensors—natural tensors are those such that the pullback of the tensor by any diffeomorphism is the tensor for the pullback of the metric—of order n that can be locally expressed as linear combinations of contractions of the metric, its inverse and coordinate derivatives of the metric up to order n . As such, F is continuous and Fréchet differentiable at g_0 and the linearization DF_{g_0} satisfies a local Lipschitz estimate. As remarked above, Eq. (5.11) shows that DF_{g_0} is strongly elliptic, and so by Theorem A, DF_{g_0} is admissible and thus sectorial on X . We can choose a neighborhood $\mathcal{O} \subset D$ of g_0 so that $\|DF_g - DF_{g_0}\|_{\mathcal{L}(D,X)}$ is small for every $g \in \mathcal{O}$. Therefore, writing $DF_g = DF_{g_0} + (DF_g - DF_{g_0})$, then DF_g is sectorial for each $g \in \mathcal{O}$ by Proposition 2.4.2 in [31], since it is a small perturbation of a sectorial operator.

Hence, by Theorem B, there is a short-time solution to the obstruction-DeTurck flow.

As explained above, Eq. (5.8) can then be solved to obtain the family of diffeomorphisms ϕ_t , and we then deduce that $\bar{g}(t) = \phi_t^* g(t)$ is a short-time solution to the obstruction flow with initial condition g_0 .

To argue uniqueness, suppose that $\bar{g}_i(t)$, $i = 1, 2$ are two solutions to (5.2) with the same initial condition g_0 . Again choose the reference metric $\tilde{g} = g_0$. Following Section 5.2 of [6], for each i we set

$$E(\phi_i) := (-1)^{n/2} c \Delta_{\bar{g}_i, g}^{n/2} \phi_i + \mathcal{P}(\phi_i),$$

where $\Delta_{\bar{g}_i, g}$ is the Laplacian associated with the ‘map covariant derivative’ for the identity map $(M, \bar{g}_i(t)) \rightarrow (M, \tilde{g})$, as described in [6], and where $\mathcal{P}$ is a nonlinear differential

operator of order $n - 1$ acting on ϕ . Combining this with the ODE for ϕ itself, we arrive at the strictly parabolic initial value problem

$$\partial_t \phi_i = E(\phi_i), \quad (\phi_i)(0) = \text{id}. \quad (5.12)$$

Taking advantage of the explicit coordinate expression for E in [6], and using the bounded geometry of M with respect to either of the metrics $g(t)$ or $\bar{g}(t)$ (valid in some fixed time interval), we see that DE_{id} is an admissible operator, and hence, Theorem B may be applied to (5.12) to conclude that this initial value problem has a unique solution on some short time interval that remains a diffeomorphism. The remainder of the argument proceeds exactly as in Section 5.3 of [6]. $\square$

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