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Topologically nontrivial B-fields on nodal Calabi–Yau three-folds and hybrid GLSM

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Our goal in this paper is to give a brief survey of recent developments in the study of M-theory on singular Calabi–Yau three-folds, topological strings in the presence of a flat but topologically nontrivial B-field and their relationship to hybrid phases of certain Gauged Linear Sigma Models (GLSM).

Keywords: Topological strings; M-theory; noncommutative resolutions; Gopakumar–Vafa invariants.

1. Introduction

We consider projective Calabi–Yau threefolds X that have isolated nodal singularities $S \subset X$ such that the exceptional curves in a small resolution $\pi : \widehat{X} \to X$ are torsion in homology. For simplicity, we will assume that

$$H_2(\widehat{X}, \mathbb{Z}) \simeq \mathbb{Z}^{b_2(X)} \times \mathbb{Z}_r \tag{1}$$

and that $X \backslash S$ is smooth, but the results easily generalize. The small resolution is then a complex manifold, but since the exceptional curves are holomorphic, it cannot be Kähler. However, the nodes can always be deformed away and the complex

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structure deformations of X that preserve the nodes can be identified with a sublocus of the complex structure moduli space of a smooth Calabi–Yau manifold $\tilde{X}$, which we assume is Kähler.

Since $\widehat{X}$ is non-Kähler, it cannot be used directly as a stable string background. Nevertheless, it was argued that the geometry of $\widehat{X}$ determines the physics of M-theory and Type IIA string theory on the singular Calabi–Yau X in the following way[1–4,a]:

- The gauge symmetry of the five-dimensional effective theory obtained from compactifying M-theory on X is

$$G = \mathrm{Hom}(H_2(\widehat{X}, \mathbb{Z}), U(1)). \tag{2}$$

  The charge lattice can thus be identified with $H_2(\widehat{X}, \mathbb{Z})$. Assuming (1), this implies that the effective theory has a gauged $\mathbb{Z}_r$ symmetry under which the particles that arise from $M2$-branes wrapping the exceptional curves carry nontrivial charge.
- Flat B-fields in Type IIA compactifications on X are in one to one correspondence with flat B-fields on $\widehat{X}$. A choice of flat B-field on $\widehat{X}$ in turn corresponds to a pair (ω_B, α) with $\omega_B \in H^2(\widehat{X}, \mathbb{R})/H^2(\widehat{X}, \mathbb{Z})$ and

$$\alpha \in \mathrm{Br}(\widehat{X}) \simeq \mathrm{Tors}\ H_2(\widehat{X}, \mathbb{Z}), \tag{3}$$

  where α measures the global topology of the B-field.[b] The B-field topology is dual to a $\mathbb{Z}_r$-Wilson line in the M-theory compactification on $X \times S^1$.

We use (X, α) to refer to the Calabi–Yau X together with a choice of B-field topology.

The torsional exceptional curves only measure the topological part of the B-field. If the holonomy of the B-field along those curves is nontrivial, the B-field "stabilizes" the corresponding node in X, meaning that the Type IIA worldsheet theory is expected to be nonsingular.[5]

Assuming that all of the nodes are stabilized in this sense, the regularity of the string worldsheet theory together with the dynamics of topological B-branes in the presence of a topologically nontrivial B-field then motivates the following mathematical conjecture.

Conjecture 1.1. For each choice $\alpha \in \mathrm{Br}(\widehat{X})$ there is a sheaf of Azumaya algebras $\mathcal{B}$ on $\widehat{X}$ representing α such that $R\pi_*\mathcal{B}$ is a crepant categorical resolution of X, in the sense of Ref. 7, and there is a twisted derived equivalence $D^b(\widehat{X}, \alpha) \simeq D^b(X, R\pi_*\mathcal{B})$.

[a] The proposal is inspired by and builds on Ref. 5, where the large resolution has been used that is Kähler but not Calabi–Yau. While a comparison of the two approaches is beyond the scope of this review, we refer to Refs. 1 and 2 for further details and references.

[b] We will make the implicit assumption that the Brauer group $\mathrm{Br}(\widehat{X})$ of $\widehat{X}$ is equivalent to the cohomological Brauer group $\mathrm{Br}'(\widehat{X})$, a fact that is only proven in general when X is smooth projective.[6] For a discussion of this issue and a (very) brief introduction to the Brauer group, we refer to Subsec. 4.4 of Ref. 2.

Since the Azumaya algebras are noncommutative, one can refer to such a categorical resolution as a noncommutative resolution of X. The derived category physically describes the topological B-branes and therefore the noncommutative resolution can be interpreted as the open string perspective on (X, α).

The discrete gauge symmetry that M-theory develops on X can be straightforwardly included in the physical definition of Gopakumar–Vafa (GV) invariants from Refs. 8 and 9. This leads to a proposal for *torsion refined GV invariants* associated to X, that are encoded in the topological string A-model partition functions on (X, α) for $\alpha \in \mathrm{Br}(\widehat{X})$.[1,2,4] As we will review below, the mathematical definition of GV invariants as a "Lefschetz refinement" of Donaldson–Thomas invariants[10] together with Conjecture 1.1 then connects these invariants to the enumerative geometry of noncommutative resolutions of X.[c]

Examples of this behavior have been studied in the context of torus fibered Calabi–Yau threefolds in Refs. 1–3 and for Calabi–Yau threefolds that are double covers of Fano threefolds with the ramification locus being a symmetric determinantal surface.[2,4,11–13] In the latter case, the connection to noncommutative resolutions has been made particularly precise for symmetric determinantal surfaces defined by 4×4 matrices (the length 4 case in the terminology that will be introduced in Sec. 2). In this case, $\mathcal{B}$ can be chosen so that $R\pi_*\mathcal{B}$ is quasi-isomorphic to a sheaf of (even parts) of Clifford algebras $\mathcal{B}_0$ on the base of the double cover that has first appeared in the context of homological projective duality.[14]

It is these "Clifford type" noncommutative resolutions, or, from the closed string perspective, the corresponding nodal Calabi–Yau varieties with a topologically nontrivial B-field, which can be realized as spaces of vacua of hybrid phases of certain Gauged Linear Sigma Models (GLSM) and that will be the focus of our survey.

2. Symmetric Determinantal Double Solids

The family of geometries that we are going to consider has been introduced in Ref. 4 and is indexed by vectors $\mathbf{d} \in \mathbb{N}_{\geq 0}^{2n}$, for some $n \in \mathbb{N}_{>0}$, which we refer to as *decompositions*. Before we construct the geometries themselves, let us briefly introduce some related definitions.

We refer to $|\mathbf{d}| = d_1 + \cdots + d_{2n}$ as the *degree* of the decomposition. For reasons that will become clear later, we require that $\mathbf{d}$ is either *even* or *odd*, meaning that all of the entries are either even or odd. Note that the degree of an even or odd decomposition is always even. We say that an even/odd decomposition $\mathbf{d}$ is *normalized* if the entries are in decreasing order and $\mathbf{d}$ contains at most one zero and we define its *length* $l_{\mathbf{d}}$ to be the number of nonzero entries. We will use exponents to indicate repeated entries, such that e.g. $(5, 1, 1, 1) = (5, 1^3)$.

[c] We continue to use the terminology "noncommutative resolution" as in Conjecture 1.1 and the subsequent discussion. See also Ref. 2 for additional discussion of this terminology.

Given a decomposition $\mathbf{d} \in \mathbb{N}_{\geq 0}^{2n}$ of degree $d \in 2\mathbb{N}$, we choose a symmetric matrix

$$A_{\mathbf{d}} \in M_{2n}(\mathbb{C}[x_1, \ldots, x_4]), \tag{4}$$

with entries $A_{i,j}$ that are generic homogeneous polynomials of degree $(d_i + d_j)/2$. We then identify $x_1, \ldots, x_4$ with homogeneous coordinates on $\mathbb{P}^3$ and define

$$X_{\mathbf{d}} = \{y^2 = \det A_{\mathbf{d}}(x_1, \ldots, x_4)\} \subset \mathbb{P}^4(1^4, d/2). \tag{5}$$

This is a double cover of $\mathbb{P}^3$ that is ramified over the determinantal surface

$$S_{\mathbf{d}} = \{\det A_{\mathbf{d}} = 0\} \subset \mathbb{P}^3. \tag{6}$$

Requiring that $\mathbf{d}$ is either even or odd ensures that $S_{\mathbf{d}}$ is irreducible. The double cover $X_{\mathbf{d}}$ then has $n_{\mathbf{d}}$ isolated nodes $S_{\mathbf{d},1} = \{\text{corank} A_{\mathbf{d}} = 2\}$, with

$$n_{\mathbf{d}} = (e_1(\mathbf{d})e_2(\mathbf{d}) - e_3(\mathbf{d}))/2, \tag{7}$$

in terms of the elementary symmetric polynomials e_i, $i = 1, 2, 3$, and is smooth everywhere else.

The following proposition has been proven using conifold transitions in Ref. 4.

Proposition 2.1. *Let $\mathbf{d} \in \mathbb{N}^{2n}$ be an even or odd decomposition of degree $d \in 2\mathbb{N}$ with $0 < d \leq 8$. The corresponding double covers $X_{\mathbf{d}}$ fall into three different classes, depending on the length $l_{\mathbf{d}}$:*

(i) $l_{\mathbf{d}} = 1$*: The branch locus $S_{\mathbf{d}}$ is smooth and $X_{(d)}$ is a generic hypersurface of degree d in $\mathbb{P}^4(1^4, d/2)$ with $H_2(X_{(d)}) = \mathbb{Z}$.*

(ii) $l_{\mathbf{d}} = 2$*: $X_{\mathbf{d}}$ has isolated nodal singularities but admits a Kähler small resolution $\widehat{X_{\mathbf{d}}}$ with $H_2(\widehat{X_{\mathbf{d}}}) \simeq \mathbb{Z}^2$.*

(iii) $l_{\mathbf{d}} \geq 3$*: $X_{\mathbf{d}}$ has isolated nodal singularities and does not admit a Kähler small resolution. In those cases the exceptional curves in a small non-Kähler resolution $\widehat{X_{\mathbf{d}}}$ are 2-torsion and $H_2(\widehat{X_{\mathbf{d}}}) \simeq \mathbb{Z} \times \mathbb{Z}_2$.*

In the following, we will focus on decompositions $\mathbf{d}$ of degree $d = 8$, so that $X_{\mathbf{d}}$ is a nodal Calabi–Yau threefold. Since our interest lies in cases where the exceptional curves in $\widehat{X}_{\mathbf{d}}$ are torsion, we then consider the six inequivalent choices

$$\mathbf{d} \in \{(5, 1^3), (4, 2^2, 0), (3^2, 1^2), (3, 1^5), (2^4), (1^8)\}, \tag{8}$$

so that $H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2$[d] and the respective number of nodes in $X_{\mathbf{d}}$ is given by

$$\begin{gathered} n_{(5,1^3)} = 64, \quad n_{(4,2^2,0)} = 72, \quad n_{(3^2,1^2)} = 76, \\ n_{(3,1^5)} = 80, \quad n_{(2^4)} = 80, \quad n_{(1^8)} = 84. \end{gathered} \tag{9}$$

Adapting the language from Ref. 15, we refer to the geometries $X_{\mathbf{d}}$ as *determinantal octic double solids*. Note that all of these geometries can be seen as corresponding to a

[d] Let us point out that the isomorphism is not canonical and we assume that a consistent choice has been made for the remainder of this paper.

subslice in the complex structure moduli space of the smooth generic octic $X_{(8)}$ with $H_2(X_{(8)}, \mathbb{Z}) = \mathbb{Z}$.

Let us briefly sketch the idea behind the proof of Proposition 2.1 from Ref. 4 because it will be closely related to the physics of M-theory on $X_{\mathbf{d}}$. First write $A_{\mathbf{d}}$ as

$$A_{\mathbf{d}} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{matrix} \}n+1, \\ \}n-1 \end{matrix} \tag{10}$$

and define

$$X^{\mathrm{r}}_{\mathbf{d}} = \{y^2 = \det \overset{\mathrm{r}}{\underset{\mathbf{d}}{A}}(x_1, \ldots, x_4)\} \subset \mathbb{P}^4(1^4, 4), \quad A^{\mathrm{r}}_{\mathbf{d}} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}. \tag{11}$$

One can show that $X^{\mathrm{r}}_{\mathbf{d}}$ still has $n_{\mathbf{d}}$ isolated nodes $S^{\mathrm{r}}_{\mathbf{d},1} = \{\mathrm{corank} A^{\mathrm{r}}_{\mathbf{d}} = 2\}$, which are just deformations of the nodes $S_{\mathbf{d},1}$ in $X_{\mathbf{d}}$, but additional isolated nodes appear at $S^{\mathrm{r}}_{\mathbf{d},2} = \{\mathrm{corank} B = 1\}$. However, unlike $X_{\mathbf{d}}$ the double cover $X^{\mathrm{r}}_{\mathbf{d}}$ always admits a Kähler small resolution $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ and $H_2(\widehat{X}^{\mathrm{r}}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}$. The exceptional curves in $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ that resolve a node in $S^{\mathrm{r}}_{\mathbf{d},1}$ are all homologous to each other and the same holds for the exceptional curves resolving nodes in $S^{\mathrm{r}}_{\mathbf{d},2}$. Denoting the respective classes by C_1, C_2 one also finds that $2C_1 = C_2$. The Mayer–Vietoris sequence then relates the homology of $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ to that of a small resolution $\widehat{X}_{\mathbf{d}}$ of $X_{\mathbf{d}}$ and completes the proof of Proposition 2.1.

We can also interpret the transition from a physical perspective in terms of the corresponding M-theory compactifications. The effective theory associated to $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ has a $G = U(1) \times U(1)$ gauge symmetry and corresponds to a generic point on the Coulomb branch. Moreover, $M2$-branes that wrap curves in the class C_i lead to particles with charge $(0, i)$ for $i = 1, 2$. Going to the boundary of the Coulomb branch where these particles become massless brings us to the compactification on $X^{\mathrm{r}}_{\mathbf{d}}$. The deformation to $X_{\mathbf{d}}$ then removes the nodes that are resolved by curves in the class C_2 and therefore amounts to giving a vacuum expectation value to scalar fields with charge $(0, 2)$. This breaks the gauge symmetry from $U(1) \times U(1)$ to $U(1) \times \mathbb{Z}_2$. The same conifold transitions used in proving Proposition 2.1 therefore lead to the following result about M-theory on $X_{\mathbf{d}}$:

Claim 2.1. *For decompositions* $\mathbf{d}$ *of degree* $|\mathbf{d}| = 8$ *and length* $l_{\mathbf{d}} \geq 3$, *the five-dimensional effective theory from M-theory compactified on* $X_{\mathbf{d}}$ *has a gauge symmetry* $G \simeq U(1) \times \mathbb{Z}_2$.

We have therefore explicitly verified (2) in this class of examples.

3. Noncommutative Resolutions from Hybrid GLSM

For each $X_{\mathbf{d}}$ we will now construct a family of GLSMs that exhibits the so-called hybrid phase which, as was argued in Refs. 2 and 4, flows to the worldsheet theory of Type IIA strings that propagate on $X_{\mathbf{d}}$ in the presence of a flat but topologically nontrivial B-field.

Given a normalized decomposition $\mathbf{d} \in \mathbb{N}_{\geq 0}^{2n}$ of degree $d = 8$ and a corresponding symmetric matrix $A_{\mathbf{d}}$, we let $q = \gcd(2, d_i)$ and construct the GLSM as follows. The gauge group is

$$G = \begin{cases} U(1) \times \mathbb{Z}_2 & q = 2, \\ U(1) & q = 1 \end{cases} \tag{12}$$

and we include chiral fields $P_{i=1,\dots,2n}$, $X_{j=1,\dots,4}$. We will refer to the vacuum expectation values of the corresponding scalar fields as p_i and x_j. The charges of the fields under the gauge symmetry and under a vector R-symmetry $U(1)_V$ are

	P_1	$\dots$	P_{2n}	$X_{j=1,\dots,4}$
$U(1)$	$-d_1/q$	$\dots$	$-d_{2n}/q$	$2/q$
$\mathbb{Z}_2$	$-$	$\dots$	$-$	$+$
$U(1)_V$	1	$\dots$	1	0

, (13)

and the superpotential takes the form

$$W = \mathbf{P}^T A_{\mathbf{d}}(X_1, \dots, X_4)\mathbf{P}. \tag{14}$$

We denote the FayetIliopoulos parameter and the theta angle associated to the $U(1) \subset G$ gauge symmetry, respectively, by r, θ and define the complexified FI-parameter $t = \theta/(2\pi) + ir$.

Note that if $q = 1$ then the $\mathbb{Z}_2$ symmetry in (13) is embedded in the $U(1)$. In general, the $\mathbb{Z}_2$ Gauge symmetry ensures the charge integrality condition, meaning that every Gauge invariant operator has a $U(1)_V$ R-charge that agrees with its statistics modulo 2. The $U(1)$ gauge charges of the chiral fields sum to zero so that the GLSM satisfies the Calabi–Yau condition and the axial R-symmetry is also unbroken.

The GLSM associated to $\mathbf{d} = (1^8)$ has first been discussed in Ref. 16, while the relation to noncommutative resolutions was observed in Refs. 11, 12 and 17 and further studied in Refs. 2 and 4. The case $\mathbf{d} = (2^4)$ has previously been discussed in Refs. 4, 13 and 18. More recently, GLSMs of this type and their relation to noncommutative resolutions have also been discussed in Ref. 19.

Let us now describe the relationship to $X_{\mathbf{d}}$ first from the closed string and then from the open string perspective. The relevant hybrid phase appears in the region $r \gg 0$ of the FI-parameter space. The D-term equation $2(|x_1|^2 + \cdots + |x_4|^2)/q - \sum_i d_i |p_i|^2/q = r$ then implies that the x_i cannot vanish simultaneously. The F-term equations $\partial_{p_i} W = \partial_{x_j} W = 0$ further require that $p_1 = \cdots = p_{2n} = 0$ so that after identifying gauge equivalent vacua the $x_{=1,\dots,4}$ can be interpreted as homogeneous coordinates on $\mathbb{P}^3$. Adiabatically, over a point in $\mathbb{P}^3$, the fields P_i flow to a $\mathbb{Z}_2$ orbifold of a Landau–Ginzburg model with quadratic superpotential (14).

Over a generic point in $\mathbb{P}^3$, the mass matrix of the Landau–Ginzburg fiber has full rank and therefore all of the fields can be integrated out so that the $\mathbb{Z}_2$-orbifold action becomes ineffective. The phenomenon of decomposition, first described in Ref. 20 with a recent introduction being,[21] then implies that the resulting theory is equivalent to two copies of the theory without the orbifold and therefore exhibits two vacua instead of one. This changes over the points in $\{\det A_{\mathbf{d}} = 0\}$ where the rank of the mass matrix drops and the two vacua fall together. As a result, the vacuum manifold in the hybrid phase of the GLSM takes the form of the double cover $X_{\mathbf{d}}$ of $\mathbb{P}^3$.

However, while the Calabi–Yau $X_{\mathbf{d}}$ is singular one can check that the GLSM is actually regular. To see this, note that the F-term equations

$$\partial_{p_i} W = [A_{\mathbf{d}}(x_1, \ldots, x_4)\mathbf{p}]_i = 0, \quad \partial_{x_j} W = \mathbf{p}^T \partial_{x_i} A_{\mathbf{d}}(x_1, \ldots, x_4)\mathbf{p} = 0, \tag{15}$$

force $\mathbf{p}$ to be a zero eigenvector of $A_{\mathbf{d}}(x_1, \ldots, x_4)$ that satisfies four quadratic equations. Since for a generic choice of $A_{\mathbf{d}}$ the space of zero eigenvectors is at most two-dimensional over $\mathbf{x} \in \mathbb{P}^3$ this forces $\mathbf{p}$ to vanish and the space of vacua has no noncompact directions.[e]

The resolution to this apparent mismatch between the singularity of the underlying geometry and the regularity of the GLSM is the following claim.

Claim 3.1. *The infrared theory associated to the hybrid phase is the worldsheet theory of strings propagating on $X_{\mathbf{d}}$ in the presence of a flat topologically nontrivial B-field.*

This has been argued for the case $\mathbf{d} = (1^8)$ in Ref. 2 and was demonstrated explicitly in Subsec. 5.2 of Ref. 4 by studying the relationship between the GLSM (13), the GLSM associated to the nonlinear sigma model on $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ and the corresponding limit in the stringy Kähler moduli space of the latter. We will review the discussion from Subsec. 5.2 of Ref. 4 in Sec. 5.

From the open string perspective, it has been argued in Refs. 11, 12 and 17 that the string background associated to the hybrid phase for $\mathbf{d} = (1^8)$ can be interpreted as a noncommutative, or categorical, resolution of $X_{(1^8)}$ in terms of a sheaf of even parts of Clifford algebras on $\mathbb{P}^3$. The argument generalizes to other choices of $\mathbf{d}$ and is based on the result from Ref. 22 that the 0-branes of a Landau–Ginzburg model with quadratic superpotential can be described as modules over the Clifford algebra that is associated to the quadratic form. Taking into account the $\mathbb{Z}_2$-orbifold leads one to consider only the even part of the Clifford algebra and after fibering the construction over the base of the hybrid model one obtains a sheaf of even parts of Clifford algebras $\mathcal{B}_0$ on $\mathbb{P}^3$. This sheaf has previously been constructed by Kuznetsov in the context of homological projective duality[14,23] and is conjecturally describing the so-called crepant categorical resolution of $X_{\mathbf{d}}$.[7] We will connect this to Claim 3.1 in Sec. 6 but first discuss the topologically nontrivial B-fields on $X_{\mathbf{d}}$.

4. Flat *B*-Fields with Nontrivial Topology

In addition to the metric, a Type II string compactification on a Calabi–Yau X also requires a choice of B-field. Our focus in this paper is on singular Calabi–Yau

[e]The fact that the equations lead to sufficiently many independent constraints, while not completely obvious, can be verified for random choices of coefficients by calculating the Groebner basis of the ideal $I = \langle \partial_{p_i} W, \partial_{x_j} W, \prod_i (p_i - 1), \prod_j (x_j - 1)\rangle \subset \mathbb{C}[x_{1,\ldots,4}, p_{1,\ldots,2n}]$ to see that $I = \langle 1 \rangle$.

threefolds X with isolated nodal singularities that are resolved by torsion curves in a non-Kähler small resolution $\widehat{X}$. In Refs. 1, 2 and 4, building on Ref. 5, it was argued that the following claim.

Claim 4.1. *flat B-fields on the singular Calabi–Yau X can be described by corresponding B-fields on any small non-Kähler resolution $\widehat{X}$.*

We actually do not have a good description of B-fields on the singular Calabi–Yau X itself, so this statement can also be seen as a provisional definition. The idea is then that whatever the correct framework for describing flat B-fields directly on X turns out to be, we expect that choosing a B-field on $\widehat{X}$ and then shrinking the exceptional curves is a well-defined procedure that leads to a one-to-one correspondence between flat B-fields on $\widehat{X}$ and on X.

While the B-field on a smooth Calabi–Yau Y is often described as an antisymmetric two-form field, it should actually be thought of as a connection on a Gerbe in the sense developed in Ref. 24, see, e.g. Refs. 25 and also Refs. 26 and 27 for an introduction to this point of view. It is in this sense a higher analogue of the usual one-form gauge field, which is a connection on a line bundle.

To a one-form gauge field one can associate the first Chern class $c_1(L) \in H^2(Y,\mathbb{Z})$ of the associated line bundle L and locally this is just the curvature of the connection. Similarly, the B-field determines a characteristic class of the Gerbe, the so-called Dixmier–Douady class, which takes values in $H^3(Y,\mathbb{Z})$. At the level of the differential form representative of the B-field $B \in \Omega^2(Y)$, the latter is just the usual H-flux

$$[H]_{\mathrm{dR}} = dB, \tag{16}$$

where $[H]_{\mathrm{dR}}$ denotes the image of H in de Rham cohomology. However, if $H^3(Y,\mathbb{Z})$ contains torsion then the Gerbe can be topologically nontrivial even if the B-field is flat, i.e. $[H]_{\mathrm{dR}} = 0$.

To see this, one first notes that from a sigma model perspective the B-field assigns phases to curves and the cohomology class naturally takes values in

$$H^2(Y,U(1)) \simeq \mathrm{Hom}(H_2(Y,\mathbb{Z}),U(1)), \tag{17}$$

as was pointed out, e.g. in Refs. 5 and 28. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ then leads to the long exact sequence

$$\ldots \to H^2(Y,\mathbb{Z}) \to H^2(Y,\mathbb{R}) \to H^2(Y,U(1)) \xrightarrow{\beta} H^3(Y,\mathbb{Z}) \to \cdots, \tag{18}$$

and the Dixmier–Douady class is the image of the cohomology class of the B-field under the Bockstein map β. In particular, the cohomology classes of flat B-fields are in one-to-one correspondence with tuples (ω_B, α) with $\omega_B \in H^2(Y,\mathbb{R})/H^2(Y,\mathbb{Z})$ and $\alpha \in \mathrm{Tors}H^3(Y,\mathbb{Z})$.

The universal coefficient theorem implies that

$$\mathrm{Tors}H^3(Y,\mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{Tors}H_2(Y,\mathbb{Z}),U(1)). \tag{19}$$

Given a curve class $C \in H_2(Y, \mathbb{Z})$, we can write the corresponding holonomy of a flat B-field $B \in H^2(Y, U(1))$ which is represented by (ω_B, α) as

$$\langle B, C\rangle = \alpha(C)\exp\left(2\pi i \int_C \omega_B\right). \tag{20}$$

Recall now that from the perspective of *M*-theory on $Y \times S^1$, the class ω_B parametrizes the Wilson lines of $U(1)$ gauge fields along the circle. If the five-dimensional effective theory exhibits a discrete Gauge symmetry, then one also has to specify a choice of discrete Wilson line.

Taking X again to be our nodal projective Calabi–Yau three-fold, we arrive at the following claim.

Claim 4.2. *The choice of discrete Wilson line in the M-theory compactification on* $X \times S^1$ *is dual to the topology of the flat B-field in the Type IIA compactification on the same Calabi–Yau.*

Let us now apply this discussion to the symmetric determinantal double solids $X_{\mathbf{d}}$ introduced in Sec. 2. Again we focus on the choices (8) for the decomposition $\mathbf{d} \in \mathbb{N}^{2n}$. One claim from Refs. 1, 2 and 4 is that we can describe the B-field on $X_{\mathbf{d}}$ by working with any non-Kähler small resolution $\widehat{X}_{\mathbf{d}}$. We then have an isomorphism $\text{Tors}H^3(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z}_2$ and can describe the topology of the flat B-field as $\alpha = [k]$ for $k = 0, 1$. Given also a class $b \in H^2(\widehat{X}_{\mathbf{d}}, \mathbb{R}) \simeq \mathbb{R}$, the holonomy of a curve C that represents the class $(d, [p]) \in H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2$ with $p = 0, 1$ is given by

$$\langle B, C\rangle = (-1)^{pk} e^{2\pi i d b}. \tag{21}$$

We denote the singular Calabi–Yau $X_{\mathbf{d}}$ together with a choice of B-field topology $[k] \in \mathbb{Z}_2$ by $(X_{\mathbf{d}}, [k])$. Note that the integral of a two-form over torsion curves vanishes and therefore the holonomies of the exceptional curves in $\widehat{X}_{\mathbf{d}}$ (which all represent the class $(0, [1]) \in H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z})$) are only sensitive to the topology of the flat B-field.

An important point is that a flat B-field with nontrivial topology stabilizes the nodes that are resolved by torsional exceptional curves in the sense of Ref. 5.

For the singular Calabi–Yau $X_{\mathbf{d}} = (X_{\mathbf{d}}, [0])$ itself, the B-field holonomy of the exceptional curves in $\widehat{X}_{\mathbf{d}}$ is trivial and can be deformed to the generic smooth octic $X_{(8)}$, removing all of the nodes in the process. The number of complex structure deformations of $(X_{\mathbf{d}}, [0])$ is therefore $h^{2,1}(X_{(8)}) = 149$. On the other hand, for $(X_{\mathbf{d}}, [1])$ the B-field holonomy along all of the exceptional curves is -1. As a result, the nodes are "frozen" by the B-field and the number of complex structure deformations of $(X_{\mathbf{d}}, [1])$ is $149 - n_{\mathbf{d}}$, where $n_{\mathbf{d}}$ is the number of nodes given in (9).

Recall that from the perspective of strings propagating on a Calabi–Yau, a conifold singularity arises when not only the volume of the corresponding exceptional curve vanishes but also the B-field holonomy along the curve is trivial. Therefore, even though $X_{\mathbf{d}}$ contains nodal singularities, $(X_{\mathbf{d}}, [1])$ is a smooth string background and the worldsheet theory is expected to be regular.

5. Conifold Transitions with B-Fields and Exoflop Phases

We will now motivate Claim 3.1 that the GLSM$_{\mathbf{d}}$ in the phase $r \gg 0$ flows to a worldsheet theory of strings propagating on $X_{\mathbf{d}}$ in the presence of a flat but topologically nontrivial B-field. To this end we consider the GLSM associated to $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ and connect it to GLSM$_{\mathbf{d}}$ via an exoflop phase. We focus on the case $\mathbf{d} = (2^4)$ but the discussion easily generalizes to all $\mathbf{d}$ in (8).

From the explicit description of the small resolution $\widehat{X}^{\mathrm{r}}_{(2^4)}$ of $X^{\mathrm{r}}_{(2^4)}$ in Ref. 4, we see that the associated GLSM has gauge symmetry $G = U(1)_1 \times U(1)_2$ and nine chiral fields $P_{1,2}$, $U_{1,2,3}$, $X_{1,2,3,4}$ with gauge charges and $U(1)_V$ R-charges given by

$$\begin{array}{c|cccc|c} & P_1 & P_2 & U_{1,2,3} & X_{1,\dots,4} & \mathrm{FI} \\ \hline U(1)_1 & -2 & -1 & 1 & 0 & r_1 \\ U(1)_2 & -2 & -2 & 0 & 1 & r_2 \\ U(1)_V & 2 & 2 & 0 & 0 & \end{array}, \tag{22}$$

where r_1, r_2 denote the Fayet–Iliopoulous parameters. We denote the corresponding scalar fields by $p_{1,2}$, $u_{1,2,3}$, $x_{1,2,3,4}$ and the theta angles by θ_1, θ_2 and then define the complexified FI-theta parameters $z_k = \exp(-2\pi r_k + i\theta_k)$, $k = 1, 2$. The superpotential takes the following form:

$$W = \Phi^T \begin{pmatrix} P_1 A(X) & B(X)^T \\ B(X) & 0 \end{pmatrix} \mathbf{\Phi}, \tag{23}$$

where $\mathbf{\Phi} = (U_1, U_2, U_3, P_2)$ and the matrices A, B are the components of $A^{\mathrm{r}}_{(2^4)}$ from (11), of respective dimensions 3×3 and 1×3, with entries that are quadratic polynomials in $X_{1,\dots,4}$. The geometric phase in which the GLSM flows to a nonlinear sigma model on $\widehat{X}^{\mathrm{r}}_{(2^4)}$ corresponds to the region $r_1, r_2 \gg 0$.

Let us instead consider the phase $r_1 \ll 0, r_2 \gg 0$. The D-term equations read

$$\begin{aligned} -2|p_1|^2 - |p_2|^2 + |u_1|^2 + \cdots + |u_3|^2 &= r_1, \\ -2|p_1|^2 - 2|p_2|^2 + |x_1|^2 + \cdots + |x_4|^2 &= r_2 \end{aligned} \tag{24}$$

and we see that the deleted set takes the following form:

$$\{p_1 = p_2 = 0\} \cup \{x_1 = \cdots = x_4 = 0\}. \tag{25}$$

The expectation values of $x_1, \dots, x_4$ can again be interpreted as homogeneous coordinates on $\mathbb{P}^3$ but, since the geometry $\widehat{X}^{\mathrm{r}}_{(2^4)}$ is smooth and p_1, p_2 are not allowed to vanish simultaneously, we now need to have $\mathbf{u} = 0$ in order to solve the F-term equations.

The F-term equations now also imply that $\frac{\partial W}{\partial u_i} = p_2 B_i = 0$. For points of $\mathbb{P}^3$ where $B \neq 0$ this implies that $p_2 = 0$ and therefore $|p_1| = \sqrt{-r_1/2}$, which breaks $U(1)_1$ to a $\mathbb{Z}_2$ under which $U_{1,\dots,3}$ and P_2 have odd charge. Adiabatically, over points $\mathbf{x} \in \mathbb{P}^3$ where B has full rank, we therefore find that the low energy dynamics precisely match those of the GLSM$_{\mathbf{d}}$ for $\mathbf{d} = (2^4)$ in the phase $r \gg 0$ and for the special choice of superpotential $W' = \mathbf{P}^T A^{\mathrm{r}}_{(2^4)}(X_1, \dots, X_4)\mathbf{P}$.

On the other hand, at the points $\mathbf{x} \in \mathbb{P}^3$ where $B = 0$ the values of p_1, p_2 are unconstrained and span a $\mathbb{P}(2,1) \simeq \mathbb{P}^1$ with volume proportional to $-r_1$. Since this $\mathbb{P}^1$ is not contained in the toric ambient space of $\widehat{X}^{\mathrm{r}}_{(2^4)}$ but instead parametrized by fields that are, from the toric perspective, nongeometric, we find the so-called exoflop phase.[29–31]

It is known that ordinary conifold transitions between smooth Calabi–Yau threefolds often pass through such an exoflop phase and the geometry with smaller $h^{1,1}$ is then obtained by ignoring the exoflopped $\mathbb{P}^1$'s and deforming away the resulting singularities, see, e.g. Ref. 31. Assuming that the same procedure is valid in this case as well, we therefore conclude that the string vacua associated to the phase $r \gg 0$ of $\mathrm{GLSM}_{\mathbf{d}}$ are connected to those associated to the GLSM (22) by a conifold transition that takes place in the limit $r_1 \ll 0, r_2 \gg 0$.

At the level of the underlying geometries, this conifold transition connects $\widehat{X}^{\mathrm{r}}_{\mathbf{d}}$ and $X_{\mathbf{d}}$. However, using either a localization calculation or mirror symmetry one finds that the quantum volume t_1 of the curves in the class $C_1 \in H_2(\widehat{X}^{\mathrm{r}}_{\mathbf{d}}, \mathbb{Z})$ is related to the FI-theta parameters z_1, z_2 via

$$t_1 = \frac{1}{2\pi i} \log\left(\frac{1 - \sqrt{1-4z_1} - 2z_1}{2z_1}\right) + \mathcal{O}(z_2), \tag{26}$$

such that $\lim_{r_1 \to -\infty} t_1 = 1/2$. Recall that the quantum volume, or complexified volume, of a curve takes the form $t_1 = b + iv$, where $v \in \mathbb{R}$ is the quantum corrected volume and $b = \frac{1}{2\pi i}\log B \in \mathbb{R}/\mathbb{Z}$ is the phase of the B-field holonomy. The quantum volume of the curves in the class C_2 in the limit $r_1 \to -\infty$ is $2t_1 = 1 \sim 0$ and the nodes $S^{\mathrm{r}}_{\mathbf{d},2}$ can be deformed away. On the other hand, the curves in the class C_1 have (real) volume $\operatorname{Im} t_1 = 0$ but a nontrivial B-field holonomy of $\operatorname{Re} t_1 = 1/2$ and the nodes $S^{\mathrm{r}}_{\mathbf{d},1}$ are therefore "frozen" due to the presence of the B-field.

This leads us to conclude that the infrared theory in the phase $r \gg 0$ of $\mathrm{GLSM}_{\mathbf{d}}$ corresponds to the worldsheet theory of strings propagating on $\widehat{X}_{\mathbf{d}}$ with the quantum volume of the exceptional curves that resolve the nodes $S_{\mathbf{d},1}$ being $1/2$. In the light of the discussion from Sec. 4, this can be naturally interpreted in terms of a flat but topologically nontrivial B-field. Since the volume of the exceptional curves is zero we can equivalently interpret this as strings propagating on $X_{\mathbf{d}}$ with a corresponding B-field and therefore arrive at Claim 3.1.

6. *D*-Branes in Nontrivial *B*-Field Backgrounds

It is known that topological D-branes in the presence of a flat topologically nontrivial B-field are described in terms of twisted sheaves Subsec. 5.3 of Ref. 32. We explain what this means in our context.

We consider a flat B-field $B \in H^2(\widehat{X}_{\mathbf{d}}, U(1))$. The image α of B in $H^2(\widehat{X}_{\mathbf{d}}, \mathcal{O}^*_{\widehat{X}_{\mathbf{d}}})$ can be represented by an $\mathcal{O}^*_{\widehat{X}_{\mathbf{d}}}$-valued cocycle α_{ijk} for an open cover $\{U_i\}$ of $\widehat{X}_{\mathbf{d}}$. A twisted sheaf is a collection of coherent sheaves F_i on U_i and isomorphisms $\phi_{ij} : (F_i)|_{U_i \cap U_j} \to (F_j)|_{U_i \cap U_j}$ satisfying $\phi_{ki}\phi_{jk}\phi_{ij} = \alpha_{ijk}$. The category of branes is then the derived category $D^b(\widehat{X}_{\mathbf{d}}, \alpha)$ of twisted sheaves. The notation acknowledges that the derived category of twisted sheaves only depends on α up to equivalence and not on the specific cocyle representing α.

In certain cases the twisted derived category $D^b(\widehat{X}, \alpha)$ on a small non-Kähler resolution is equivalent to the ordinary derived category $D^b(Y)$ of another smooth Kähler Calabi–Yau three-fold Y. One example is $\widehat{X}_{\mathbf{d}}$ with $\mathbf{d} = (1^8)$ in which case Y is a complete intersection of four quadrics in $\mathbb{P}^7$. This suggests that even though $\widehat{X}$ is not Kähler, the twisted derived category $D^b(\widehat{X}, \alpha)$ "behaves" like the category of topological B-branes on an ordinary smooth Kähler Calabi–Yau background. This leads us to expect more generally that $D^b(\widehat{X}, \alpha)$ is identical to the category of topological B-branes on the singular Calabi–Yau itself in the presence of the topologically nontrivial B-field.

One interpretation of this phenomenon would be that the worldsheet theory of strings on $(\widehat{X}, \alpha)$ is related via renormalization group flow to the CFT that describes strings on (X, α). Since going from complexes of coherent sheaves to the derived category can physically be interpreted as identifying configurations of branes that flow to the same configuration in the infrared, see e.g. Sec. 3 of Ref. 33, it is then natural that $D^b(\widehat{X}, \alpha)$ provides a good description of topological B-branes on (X, α). In the following, we will therefore talk about branes on $(\widehat{X}_{\mathbf{d}}, [1])$ even though we actually compactify Type IIA string theory on $(X_{\mathbf{d}}, [1])$.

It was shown in Ref. 34 that a $D6$-brane in the presence of a topologically nontrivial B-field determines a sheaf of Azumaya algebras on the Calabi–Yau, i.e. a locally free sheaf of matrix algebras. It is natural from the viewpoint of physics to conjecture the existence of a $D6$-brane on $(X_{\mathbf{d}}, [1])$. This would imply the existence of a global twisted vector bundle $\mathcal{E}$ on $\widehat{X}_{\mathbf{d}}$, i.e. each $\mathcal{E}_i$ is a vector bundle on U_i. The sheaf of Azumaya algebras on $\widehat{X}_{\mathbf{d}}$ is then the (ordinary) sheaf $\mathcal{B} = \underline{\mathrm{End}}_{\mathcal{O}_{\widehat{X}_{\mathbf{d}}}}(\mathcal{E})$. In this situation, the sheaf $\mathcal{B}$ is expected to satisfy the conditions of Conjecture 1.1.

From the viewpoint of mathematics, the existence of a sheaf $\mathcal{B}$ of Azumaya algebra associated to the class α follows from the assumption made in Sec. 1 that the Brauer group and the cohomological Brauer group are equal. It follows readily from the definitions that $D^b(\widehat{X}_{\mathbf{d}}, \alpha) = D^b(\widehat{X}_{\mathbf{d}}, \mathcal{B})$, so that the category of branes can be identified with $D^b(\widehat{X}_{\mathbf{d}}, \mathcal{B})$.

The sheaf $\mathcal{B}$ of Azumaya algebras has been proven to exist in the length 4 case when α is the nontrivial Brauer class.[23] Letting $\rho : X_{\mathbf{d}} \to \mathbb{P}^3$ be the double cover, it is also shown that $\mathcal{B}_0 = \rho_*(\pi_*(\mathcal{B}))$ is a sheaf on $\mathbb{P}^3$ of even parts of Clifford algebras, and that the category of branes $D^b(\widehat{X}_{\mathbf{d}}, \mathcal{B})$ can be identified with $D^b(\mathbb{P}^3, \mathcal{B}_0)$. See Ref. 4 for further discussion of the role of sheaves of $\mathcal{B}_0$-modules in the context of this paper.

In the case $\mathbf{d} = (1^8)$, the category of branes $D^b(\widehat{X}_{\mathbf{d}}, \alpha)$ is equivalent to the ordinary derived category of a complete intersection Z of quadrics in $\mathbb{P}^7$.[35] This connection was exploited in Ref. 2, where a second MUM point of the moduli space of the mirror of Z was used to set up the B-model calculations employed to compute the torsion refined invariants.

7. Torsion Refined Gopakumar–Vafa Invariants from Physics

The physical definition of GV invariants from Refs. 8 and 9 only relies on the fact that the effective theory is a five-dimensional supergravity with eight supercharges. This is still true when X is a projective Calabi–Yau three-fold with isolated nodes and was used in Ref. 1 to propose a torsion refinement of the GV invariants of the smooth deformation.

To be concrete, let us consider again M-theory on the determinantal octic double solids $X_{\mathbf{d}}$ with $|\mathbf{d}| = 8$ and $l_{\mathbf{d}} \geq 3$. The massive little group of the five-dimensional theory is

$$\text{Spin}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R \tag{27}$$

and we found that the gauge group is always $G = U(1) \times \mathbb{Z}_2$ while the charge lattice can be identified with $H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}_2$ for any small resolution $\widehat{X}_{\mathbf{d}}$ of $X_{\mathbf{d}}$. We let $N_{j_L,j_R}^{d,p} \in \mathbb{N}$ be the number of BPS-particles that carry charge $(d,p) \in \mathbb{Z} \times \mathbb{Z}_2$ and that transform in the representation

$$\left[\left(0, \frac{1}{2}\right) \oplus 2(0,0)\right] \otimes (j_L, j_R), \tag{28}$$

of the little group (27). The torsion refined GV invariants $n_g^{d,p} \in \mathbb{Z}$ are then defined via the usual trace over j_R,

$$\sum_{g\geq 0} I_g n_g^{d,p} = \sum_{j_R \in \frac{1}{2}\mathbb{N}} (-1)^{2j_R}(2j_R+1) N_{j_L,j_R}^{d,p}[j_L], \quad I_g = \left(2[0] \oplus \left[\frac{1}{2}\right]\right)^g. \tag{29}$$

As discussed in Sec. 2, smoothing $X_{\mathbf{d}}$ to the Calabi–Yau threefold $X_{(8)}$ can physically be interpreted as a Higgs transitions where scalar fields with charge $(d,p) = (0,1)$ get a nontrivial vacuum expectation value, thus breaking the gauge symmetry from $U(1) \times \mathbb{Z}_2$ to $U(1)$. This implies that the GV invariants n_g^d of $X_{(8)}$ are related to the torsion refined GV invariants of $X_{\mathbf{d}}$ via

$$n_g^d = n_g^{d,0} + n_g^{d,1}. \tag{30}$$

In other words, the torsion refined invariants resolve the charge under a discrete gauge symmetry that M-theory develops on a special subslice of the complex structure moduli space of a smooth Calabi–Yau. Note that we have left the dependence of the torsion refined invariants on $X_{\mathbf{d}}$ implicit and each $X_{\mathbf{d}}$ corresponds to a different subslice in the complex structure moduli space of $X_{(8)}$ and a different torsion refinement of the corresponding GV invariants.

Let us now consider the M-theory compactification on $X_{\mathbf{d}} \times S^1$. Recall that the Wilson lines of the five-dimensional gauge fields along the circle parametrize the B-field holonomies along curves in the Calabi–Yau. They combine with the Kähler modulus into the complexified Kähler parameter t that parametrizes the Coulomb branch of the four-dimensional effective theory.

As discussed in Sec. 4, circle compactifications of the five-dimensional theory with different choices of discrete bundles, i.e. discrete Wilson lines, are dual to Type IIA compactifications on $X_{\mathbf{d}}$ with different topologies $[k] \in \mathrm{Tors}H^3(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq \mathbb{Z}_2$ of the flat B-field. Generalizing again the results from Refs. 8 and 9, the Gopakumar–Vafa expansion of the A-model topological string partition function on $(X_{\mathbf{d}}, [1])$ then naturally takes the following form:

$$Z_{\mathrm{top.}}(t,[k]) = \sum_{g\geq 0}\sum_{d\geq 1}\sum_{p=0,1}\sum_{m\geq 1}\frac{n_g^{d,p}}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2g-2}(-1)^{kmp}e^{2\pi i m d t}. \tag{31}$$

Note that $Z_{\mathrm{top.}}(t,[0])$ is identical to the topological string partition function on $X_{(8)}$ which can also be seen as a consequence of (30) and invariance of $Z_{\mathrm{top.}}$ under complex structure deformations.

We will discuss in Sec. 10 how a localization calculation in the GLSM from Sec. 3 can be used to obtain the periods of the Calabi–Yau that is mirror to $(X_{\mathbf{d}}, [1])$ and sketch how one can calculate the topological string free energies, at least up to some maximal genus, in order to extract the torsion refined invariants.

8. Torsion Refined Gopakumar–Vafa Invariants from Geometry

We begin this section by briefly reviewing the mathematical definition of GV invariants given in Ref. 10. This definition incorporates the physics definition of GV invariants outlined in Sec. 7 into a precise mathematical framework. An exposition of these ideas for physicists, in the context of ordinary GV invariants without the torsion refinement, appears in Sec. 3 of Ref. 36.

Let Y be a (smooth) Calabi–Yau threefold and let $\beta \in H_2(Y,\mathbb{Z})$ be a curve class. Fixing a unit of $D0$-brane charge, the semistable $D2$-$D0$ branes of $D2$-brane charge determined by β are identified at large radius with the moduli space $\mathcal{M}_\beta(Y)$ of stable sheaves F on Y with $[F] = \beta$ and $\chi(F) = 1$. The one-cycles (curves with multiplicities) supporting these sheaves are parametrized by the Chow variety $\mathrm{Chow}_\beta(Y)$, and there is a Hilbert–Chow map

$$\pi_\beta : \mathcal{M}_\beta(Y) \to \mathrm{Chow}_\beta(Y), \tag{32}$$

taking a sheaf to its support cycle. Since $\mathcal{M}_\beta(Y)$ is singular in general, we cannot use ordinary cohomology to quantize the moduli space of branes. However, perverse sheaves and perverse cohomology are well adapted to singular spaces. The moduli space $\mathcal{M}_\beta(Y)$ always supports perverse sheaves of vanishing cycles ϕ which are locally determined by a holomorphic Chern–Simons functional. Globally, ϕ depends on the mathematical notion of an oriented d-critical locus.[37]

Using perverse cohomology sheaves ${}^p\mathcal{H}^*$, the GV invariants are defined in terms of ϕ by

$$\sum_{g\geq 0} n_g^\beta (y^{1/2}+y^{-1/2})^{2g} = \sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(R\pi_{\beta *}\phi))y^i. \tag{33}$$

This identity is an expression for the $\mathrm{SU}(2)_L$ character of (29).

It is proven that these n_g^β are independent of the choice of orientation satisfying an additional condition, called the *Calabi–Yau condition* in Ref. 10. The existence of an orientation satisfying the Calabi–Yau condition is still conjectural.

If $\mathcal{M}_\beta(Y)$ is a moduli space of sheaves supported on curves of genus g and $\mathrm{Chow}_\beta(Y)$ is smooth, then the calculation simplifies and we get $n_\beta^g = (-1)^{\dim(\mathrm{Chow}_\beta(Y))} e(\mathrm{Chow}_\beta(Y))$.[38,39]

We can now describe our proposal for modifying the above definition in order to mathematically define the torsion refined invariants. While this developing theory is not yet at the same level of rigor as the theory of ordinary GV invariants, it leads to well-defined geometric computations which always agree with the physical invariants whenever the geometric computation can be completed.

Let $\mathcal{E}$ be the twisted sheaf describing the $D6$-brane as in Sec. 6. Then to any $F^\bullet \in D^b(\widehat{X}_{\mathbf{d}})$ we can associate the twisted object $F^\bullet \otimes \mathcal{E} \in D^b(\widehat{X}_{\mathbf{d}}, \alpha)$, so that D-branes in the absence of a B-field can be used to construct D-branes in our non-trivial B-field background.

Furthermore, any two small resolutions of $X_{\mathbf{d}}$ are derived equivalent,[40] so the derived category of any small resolution also maps to the category of branes $D^b(\widehat{X}_{\mathbf{d}})$. We therefore see that sheaves on any small resolution lead to twisted sheaves on a fixed $\widehat{X}_{\mathbf{d}}$ and can contribute to the torsion refined invariants.

Given two small resolutions $\widehat{X}_{\mathbf{d}}$ and $\widehat{X}'_{\mathbf{d}}$ of $X_{\mathbf{d}}$, we have a canonical identification $H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z}) \simeq H_2(\widehat{X}'_{\mathbf{d}}, \mathbb{Z})$ where the classes of the exceptional curves are identified. So we fix any small resolution $\widehat{X}_{\mathbf{d}}$ and let $\beta \in H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z})$. We then let $\mathcal{M}_\beta(X_{\mathbf{d}})$ denote the moduli space of stable sheaves F on any small resolution of $X_{\mathbf{d}}$ with $[F] = \beta$ and $\chi(F) = 1$. We then have a map

$$\pi : \mathcal{M}_\beta(X_{\mathbf{d}}) \to \mathrm{Chow}_\beta(X_{\mathbf{d}}), \tag{34}$$

analogous to (32). Assuming the existence of a perverse sheaf of vanishing cycles ϕ satisfying the Calabi–Yau condition, with good properties, we can then define the torsion refined invariants by adapting (33) to (34).

These ideas are discussed in more detail in Ref. 2. In particular, it was also conjectured in Ref. 2 that the torsion refined GV invariants can be defined in terms of a Bridgeland stability condition on $D^b(\widehat{X}_{\mathbf{d}}, \alpha)$ at large radius. This may not seem possible at first glance since there are sheaves supported on the exceptional curves (of any small resolution) which would have vanishing central charge at large radius in contradiction to the existence of a Bridgeland stability condition. However, the topologically nontrivial B-field shifts the central charge away from zero, simultaneously removing the contradiction and requiring the exceptional curves of all small resolutions to contribute to the torsion refined GV invariants.

9. Enumerative Geometry

We quickly review the description of $H_2(\widehat{X}_{\mathbf{d}}, \mathbb{Z})$ from Refs. 2 and 4 and then compute some torsion refined invariants using geometry.

As noted in Proposition 2.1, we have isomorphisms

$$H_2(\widehat{X}_{\mathbf{d}},\mathbb{Z}) \simeq \mathbb{Z}\times\mathbb{Z}_2. \tag{35}$$

Since $\mathbb{Z}\times\mathbb{Z}_2$ has a nontrivial automorphism, these isomorphisms (35) are not canonical. We fix one such isomorphism and denote curve classes β by $(d,p)\in\mathbb{Z}\times\mathbb{Z}_2$. The degree d of a curve class β is canonical, independent of the choice of isomorphism (35), but the torsion class $p\in\mathbb{Z}_2$ is canonical only if d is even.

The nontrivial element of the torsion subgroup $\mathbb{Z}_2$ is represented by any of the exceptional curves $C_i\simeq\mathbb{P}^1$ over the conifold points p_i.

Consider $\tilde{\rho}=\rho\circ\pi:\widehat{X}_{\mathbf{d}}\to\mathbb{P}^3$ obtained by composing the small resolution with the double cover. Letting $\ell\subset\mathbb{P}^3$ be a line, then $\tilde{\rho}^{-1}(\ell)$ has degree 2 and parity $p_{\mathbf{d}}$ equal to the parity of $\mathbf{d}$.[4] These curves are double covers of ℓ branched over 8 points and so have $g=3$. The moduli space of these curves is the Grassmannian $G(2,4)$ of lines in $\mathbb{P}^3$. Since $G(2,4)$ is smooth, even-dimensional, and has Euler characteristic 6, we get

$$n_3^{2,p_{\mathbf{d}}}=6,\quad n_3^{2,p_{\mathbf{d}}-1}=0, \tag{36}$$

in agreement with B-model calculations.

Degree 1 curves C are only slightly more subtle. First, $\tilde{\rho}(C)$ must be a line ℓ, so that $C\subset\tilde{\rho}^{-1}(\ell)$. By degree considerations, $\rho^{-1}(\ell)$ must be a union $C\cup C'$ of two degree 1 curves. If $\mathbf{d}$ is an odd decomposition, then since $\tilde{\rho}^{-1}(\ell)$ has class $(2,1)$, necessarily one of the curves C,C' has class $(1,0)$ and the other has class $(1,1)$ (having fixed an arbitrary isomorphism (35) to assign a torsion class to each curve). Each of the curves C,C' is isomorphic to ℓ and so has $g=0$, thus $n_0^{1,0}=n_0^{1,1}$, and this number is equal to the number of lines in $\mathbb{P}^3$ for which the double cover splits into two components. This number was computed in the 19th century to be 14752,[41] so that

$$n_0^{1,0}=n_0^{1,1}=14752, \tag{37}$$

if $\mathbf{d}$ is odd. If $\mathbf{d}$ is even, the same reasoning only tells us that

$$n_0^{1,0}+n_0^{1,1}=29504. \tag{38}$$

These results all agree with B-model calculations.

As our last example, we continue discussing degree 2 curves $C\subset\widehat{X}_{\mathbf{d}}$ and the invariants $n_g^{2,p}$ for $g\geq 2$. Necessarily $\tilde{\rho}(C)\subset\mathbb{P}^3$ has degree 1 or 2. If $\tilde{\rho}(C)$ has degree 2, then $\tilde{\rho}|_C$ is an isomorphism of C onto a degree 2 curve, so C has genus zero and these curves do not contribute to $n_g^{2,p}$ for $g\geq 2$. If $\tilde{\rho}(C)$ is a line ℓ, then necessarily $C\subset\tilde{\rho}^{-1}(\ell)$, which as we have seen has class $(2,p_{\mathbf{d}})$.

We therefore get two contributions to the torsion refined GV invariants $n_2^{2,p}$. First, from the $g=2$ contribution to sheaves on the genus 3 curves $\tilde{\rho}^{-1}(\ell)$ obtained by the expansion (33), and second from these genus 2 curves obtained from lines ℓ containing a conifold by removing the exceptional curve from $\tilde{\rho}^{-1}(\ell)$. The lines containing any conifold p_i are parametrized by a $\mathbb{P}^2$, with Euler characteristic 3, and there are two such families of curves C, one for each of the two small resolutions of p_i. We therefore get a contribution to $n_2^{2,p_{\mathbf{d}}-1}$ of $2 \cdot 3 \cdot n_{\mathbf{d}} = 6n_{\mathbf{d}}$. Combining with the contribution of the genus 3 curves computed in Ref. 2, we find complete agreement with the B-model. This example shows that it was essential to consider all small resolutions when mathematically defining the torsion refined GV invariants. Additional examples are given in Refs. 2 and 4.

10. Localization and Mirror Symmetry

We will now discuss how the topological string free energies on $(X_{\mathbf{d}}, [1])$ can be calculated using mirror symmetry, at least up to some genus. It was argued in Ref. 4 that the mirror of $(X_{\mathbf{d}}, [1])$ can be obtained using the techniques developed in Ref. 42. However, for the purpose of calculating the free energies we only need information about the periods of the Calabi–Yau and these can be obtained directly from the GLSM.[4]

The sphere partition function $Z_{S^2}(v, \bar{v})$ of the GLSM can be evaluated using the localization calculation from Refs. 43 and 44 and is related to the Kähler potential $K(v, \bar{v})$ on the complexified Kähler moduli space via[45,46]

$$Z_{S^2}(v, \bar{v}) = e^{-K(v,\bar{v})}. \tag{39}$$

Using the relation to the fundamental period of the mirror Calabi–Yau $e^{-K(z,\bar{z})} \sim \varpi_0(z) + \mathcal{O}(\log(z), \bar{z})$, one obtains[4]

$$\varpi_0(z) = \sum_{m\geq 0} z^{qm} \left[\prod_{i=1}^{k} \frac{\Gamma(1+\frac{2d_i}{q}m)}{\Gamma(1+\frac{d_i}{q}m)}\right] \frac{1}{\Gamma(1+\frac{2}{q}m)^4}, \tag{40}$$

where a change of coordinates has been performed such that the mirror map takes the form $t(z) = \frac{1}{2\pi i}\log(z) + \mathcal{O}(z)$.

From the fundamental period (40) one can then determine a complete basis of periods $\mathbf{\Pi}(z)$ such that $\mathbf{\Pi}(z(t)) = \varpi_0(1, t, \partial_t F_0, 2F_0 - t\partial_t F_0)$ in terms of the genus zero free energy

$$F_0(t) = -\frac{1}{6}\kappa t^3 + \cdots + \frac{\zeta(3)}{(2\pi i)^3}\left(\frac{\chi}{2} + \frac{7}{4}n_{\mathbf{d}}\right) + \mathcal{O}(e^{2\pi i t}), \tag{41}$$

where $\kappa = 2$ and $\chi = -296$ are, respectively, the triple intersection number and the Euler characteristic of the smooth deformation $X_{(8)}$ of $X_{\mathbf{d}}$.[f]

The correction of $7/4n_{\mathbf{d}}$ to the usual factor $\chi/2$ of the constant term in (41) was derived in Ref. 2 by identifying the local geometry around each node in the presence of the topologically nontrivial B-field with the noncommutative conifold from Ref. 47. For the free energies $\mathcal{F}_g(t, \bar{t})$ at genus $g \geq 2$ the resulting formula for the

[f]Let us point out that this basis does not lead to integral monodromies as is explained in Subsec. 5.5 of Ref. 2 and Sec. 7 of Ref. 4.

Table 1. Some torsion refined Gopakumar–Vafa invariants for $X_{(5,1^3)}$.

$n_g^{d,0}$	$d=1$	2	3	4	5
$g=0$	**14752**	64444512	711860273440	11596529531321056	233938237312624658400
1	**0**	20480	10732175296	902645866490432	50712027457008177856
2	**0**	**384**	−8275872	6249796276400	2700746768622436448
3	**0**	**0**	−**8**8512	−87425677776	10292236849965248
4	**0**	**0**	**0**	198020184	−337281112359424
5	**0**	**0**	**0**	150666	6031964134528
6	**0**	**0**	**0**	**2232**	−43153905216
7	**0**	**0**	**0**	**24**	18764544
8	**0**	**0**	**0**	**0**	**177024**
9	**0**	**0**	**0**	**0**	**0**
$n_g^{d,1}$	$d=1$	2	3	4	5
$g=0$	**14752**	64390400	711860273440	11596526493472256	233938237312624658400
1	**0**	20832	10732175296	902646226215424	50712027457008177856
2	**0**	**480**	−8275872	6249871001344	2700746768622436448
3	**0**	**6**	**−88512**	−87433826048	10292236849965248
4	**0**	**0**	**0**	198195616	−337281112359424
5	**0**	**0**	**0**	150784	6031964134528
6	**0**	**0**	**0**	**1920**	−43153905216
7	**0**	**0**	**0**	**0**	18764544
8	**0**	**0**	**0**	**0**	**177024**
9	**0**	**0**	**0**	**0**	**0**

constant map contributions is

$$\mathcal{F}^{\text{const.}}_{g\geq 2}(t,\bar{t}) = \frac{(-1)^{g-1}B_{2g}B_{2g-2}}{2g(2g-2)[(2g-2)!]}\left(\frac{\chi+2n_{\mathbf{d}}}{2}+(1-2^{2g-2})n_{\mathbf{d}}\right), \tag{42}$$

in terms of the Bernoulli numbers B_n.

The higher genus free energies can be obtained by integrating the so-called holomorphic anomaly equations,[48,49] following Ref. 50 with the modifications due to the B-field discussed in Ref. 2. Fixing the holomorphic ambiguity that arises at each integration step requires sufficiently strong boundary conditions in the moduli space and/or vanishing conditions on the invariants. The direct integration for $X_{(8)}$, and therefore also $(X_{\mathbf{d}},[0])$, has recently been carried out in Ref. 51 up to genus $g=48$. On the other hand, the procedure has been carried out for $(X_{\mathbf{d}},[1])$ with $\mathbf{d}=(1^8)$ up to genus $g=32$ in Ref. 2 and for $\mathbf{d}=(5,1^3)$, $\mathbf{d}=(2^4)$ and $\mathbf{d}=(4,2^2,0)$, respectively, up to genus $g=25$, $g=14$ and $g=9$ in Ref. 4.

Some of the torsion refined invariants associated to $\mathbf{d}=(5,1^3)$ are shown in Table 1. Invariants that are checked by the enumerative calculations from Refs. 2 and 4, which we partly reviewed in Sec. 9, are highlighted in blue.

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