Units of twisted group rings and their correlations to classical group rings ☆

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A B S T R A C T

This paper is centred around the classical problem of extracting properties of a finite group G from the ring isomorphism class of its integral group ring $\mathbb{Z}G$. This problem is considered via describing the unit group $\mathcal{U}(\mathbb{Z}G)$ generically for a finite group. Since the '90s' several well known generic constructions of units are known to generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ if $\mathbb{Q}G$ does not have so-called exceptional simple epimorphic images, e.g. $M_2(\mathbb{Q})$. However it remained a major open problem to find a *generic* construction under the presence of the latter type of simple images. In this article we obtain such generic construction of units. Moreover, this new construction also exhibits new properties, such as providing generically free subgroups of large rank. As an application we answer positively for several classes of groups recent conjectures on the rank and the periodic elements of the abelianisation $\mathcal{U}(\mathbb{Z}G)^{ab}$. To obtain all this, we investigate the group ring $R\Gamma$ of an extension Γ of some normal subgroup N by a group G, over a domain R. More precisely, we obtain a direct sum decomposition of the (twisted) group algebra of

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Γ over the fraction field F of R in terms of various twisted group rings of G over finite extensions of F. Furthermore, concrete information on the kernel and cokernel of the associated projections is obtained. Along the way we also launch the investigations of the unit group of twisted group rings and of $\mathcal{U}(R\Gamma)$ via twisted group rings.

Contents

1. Introduction

This paper contributes to the study of a finite group Γ via its representation theory over a number field[1] F and its ring of integers R. The overarching question is which group theoretical invariants of Γ are determined by the R-algebra $R\Gamma$, or in other words by the regular $R\Gamma$-module. From the vast literature, see for example [26,27,36,51,54,55], one can somehow distil an approach in two steps to that. Firstly one considers the decomposition of the semisimple group algebra $F\Gamma$ given by Wedderburn-Artin's theorem:

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[1] Most of the results will in fact hold in the generality where F is any field with $\mathrm{char}(F) \nmid |\Gamma|$.

$$F\Gamma \cong \mathrm{M}_{n_1}(D_1) \oplus \cdots \oplus \mathrm{M}_{n_q}(D_q),$$

where D_i are finite dimensional division F-algebras. If one now chooses an order $\mathcal{O}_i$ in each D_i, then we obtain two orders in $F\Gamma$, namely $R\Gamma$ and $\bigoplus_{i=1}^{q} \mathrm{M}_{n_i}(\mathcal{O}_i)$. Because F is a number field, in particular R is a 'nice' Dedekind domain,[2] it is well known that the two orders share many properties. In particular their unit groups have a common subgroup of finite index (see for example [26,54]). The aim of the first step is to obtain as much information as possible on the F-character degrees n_i and the form of the division algebras D_i. Consequently, it is to be expected that the number theoretical properties of the number field F will play a special role here.

Recall that there is a bijection between the matrix components above, the absolutely irreducible F-characters of Γ and the primitive central idempotents of $F\Gamma$ (a set denoted $\mathrm{PCI}(F\Gamma)$). For a survey on the construction of primitive central idempotents and a description of its associated matrix components we refer to [26] and for some of the most recent progress on this topic to [7,6,5]. A setback of making the switch from $R\Gamma$ to $\bigoplus_{i=1}^{q} \mathrm{M}_{n_i}(\mathcal{O}_i)$ is that one is somehow replacing the group Γ by the larger group $\prod_{e \in \mathrm{PCI}(F\Gamma)} \Gamma e$. Due to this, one loses information on 'ties'. The role of the second step consists then to focus on the given group Γ. This might be via representation theoretical methods or group theoretical ones. To clarify the latter, we now set $F = \mathbb{Q}$ and so $R = \mathbb{Z}$. Then it is known [24] that the group isomorphism type of the unit group $\mathcal{U}(\mathbb{Z}\Gamma)$ and the ring isomorphism type of $\mathbb{Z}\Gamma$ contain the same information.[3] The gain of this is that $\mathcal{U}(\mathbb{Z}\Gamma)$ is an arithmetic subgroup in some linear reductive algebraic group (in particular it is a finitely presented group), allowing the use of classical but strong methods in algebraic groups or geometric group theory.

From this unit group point of view and using [34], the first step describes a full list of invariants determining the commensurability class of $\mathcal{U}(\mathbb{Z}\Gamma)$, whereas the second step aims to filter till its isomorphism class. A usual approach to the latter consists of constructing an 'interesting' torsion-free subgroup N of finite index in $\mathcal{U}(\mathbb{Z}\Gamma)$. On one hand such N would (ideally) be normal and the associated finite quotient would reflect properties of the finite subgroups of $\mathcal{U}(\mathbb{Z}\Gamma)$. On the other hand, importantly, the construction of N needs to be generic in the sense that it does not require knowing the isomorphism type of the group basis Γ.

In this article we will contribute to both steps in a novel way and it can be summarized as follows:

(1) In the first step we make a shift in the traditional philosophy by regrouping simple components into certain twisted group algebras which arise by viewing Γ as a non-trivial extension. Section 3 till Section 5 is devoted to constructing in general

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[2] The property that the unit group of two orders in a common semisimple algebra are commensurable requires that R/I is finite for every non-zero ideal I of R (e.g. see [26, Lemma 4.6.9.(3)]). The use of 'nice' refers to this extra property.

[3] Thus $\mathbb{Z}\Gamma \cong \mathbb{Z}H$ if and only if $\mathcal{U}(\mathbb{Z}\Gamma) \cong \mathcal{U}(\mathbb{Z}H)$. It is folklore that it holds more generally for any Γ-adapted coefficient ring R, i.e. where all prime divisors of $|\Gamma|$ are not divisible in $\mathcal{U}(R)$. In short, this follows in that case from the linear independence of finite subgroups of $\mathcal{U}(RG)$.

such decomposition and giving down-to-earth descriptions of the projections and their
(co)kernels. Below we will give an overview of those results.

(2) Classically, the main generic constructions of units in $\mathcal{U}(\mathbb{Z}\Gamma)$ are Bass and bicyclic
units. Using the solutions to the subgroup congruence problem for $\mathrm{SL}_n(D)$, with $n \geq 2$,
it is known since the mid '90s [28] that if $\mathbb{Q}\Gamma$ has no simple 2×2-components[4] and
the only simple 1×1 components are commutative, then the group generated by the
Bass and (generalized) bicyclic units is of finite index. In particular the subgroup $\mathcal{B}$
generated by them could serve as N in the explanation above. Therefore, a major open
problem in group rings of the last decennia has been to find generic constructions if
2×2 components are present. In Section 10 we give the first such generic construction,
which we call H-units. Moreover these elements can also contribute to other components,
hence even without such problematic components, when combined with $\mathcal{B}$ it may yield
a 'stronger N'. As an illustration thereof we answer in the positive for various infinite
families of groups conjectures on the rank and the torsion of the abelianisation $\mathcal{U}(\mathbb{Z}\Gamma)^{ab}$.
This is possible due to the nice properties of H-units, e.g. they yield free groups of
large ranks. The proof of all this builds on the work for step (1) and Section 6 till
Section 8.

We will now explain the main results of this article in more detail, starting with the
contributions to (classical) questions in (untwisted) group rings.

A new generic construction of units and their contribution to the structure of $\mathcal{U}(\mathbb{Z}G)$.

For any groups $W \leq H$ and K and any group homomorphism $f : H \to K$ it is
easily observed that $[K : f(W)]$ is finite exactly when $[H : \langle \ker(f), W \rangle]$ is finite. For
the unit group of $\mathbb{Z}G$, a useful incarnation thereof is for f the norm map nr of $\mathbb{Z}G$
which maps an element on a tuple recording the reduced norm of each projection onto a
simple component of $\mathbb{Q}G$. By definition $\mathrm{SL}_1(\mathbb{Z}G) = \ker(nr)$, see (24) and (25) for precise
definitions. Doing so, see [26, Proposition 5.5.1], one has that

$$\langle \mathrm{SL}_1(\mathbb{Z}G), \mathcal{U}(\mathcal{Z}(\mathbb{Z}G)) \rangle \text{ is of finite index in } \mathcal{U}(\mathbb{Z}G).$$

Another important incarnation of the above is for f the mapping of $\mathcal{U}(\mathbb{Z}G)$, via $GL_1(\mathbb{Z}G)$,
into its Whitehead group $K_1(\mathbb{Z}G) := \mathrm{GL}(\mathbb{Z}G)^{ab}$. This enables one to replace $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$
by any subgroup of $\mathcal{U}(\mathbb{Z}G)$ that maps to a finite index subgroup of $K_1(\mathbb{Z}G)$. Thanks to
a theorem of Bass and Milnor, an example of such a group working for any finite group
is given by the group generated by the so-called Bass units. For background we refer to
[26, Section 1].

In conclusion, the problem to generically construct a finite index subgroup of $\mathcal{U}(\mathbb{Z}G)$
is reduced to $\mathrm{SL}_1(\mathbb{Z}G)$, i.e. the elements of norm 1. For any tuple (g, h) with $g \notin N_G(\langle h \rangle)$
one can construct the elements $1 + (1-h)g \sum_{j=1}^{o(h)} h^j$ and $1 + (\sum_{j=1}^{o(h)} h^j)g(1-h)$ which

are unipotent units and hence in $\mathrm{SL}_1(\mathbb{Z}G)$. These elements are called *bicyclic units* and the group they generate is denoted $\mathrm{Bic}(G)$. A theorem of Jespers and Leal [28] says that $\mathrm{Bic}(G)$ is of finite index in $\mathrm{SL}_1(\mathbb{Z}G)$ under some restrictions on the simple components $\mathrm{M}_n(D)$ of $\mathbb{Q}G$. For $n \geq 2$ the restriction is that there is no $\mathrm{M}_2(D)$ with $D \in \{\mathbb{Q}, \mathbb{Q}(\sqrt{-d}), \left(\frac{-a,-b}{\mathbb{Q}}\right)\}$, for $a, b, d \in \mathbb{Z}_{>0}$ and $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ denotes a quaternion algebra. Therefore these $\mathrm{M}_2(D)$ are called *exceptional* of type (II) (see paragraphs behind Corollary 6.7 for complete definition).

In Section 10 we produce a new generic construction of elements in $\mathrm{SL}_1(\mathbb{Z}G)$. For any triple (g, h, Q) with $g, h \in G, Q \leq G$ satisfying the two conditions in Definition 10.5 we construct a group $\mathcal{H}(g, h, Q)$. The elements therein are called[5] H-units and interestingly the generators are usually not unipotent. For every quadruple of numbers $(x_1, x_2, y_1, y_2) \in \mathbb{N}^4$ satisfying the equations (46) there is an associated H-unit denoted $v_{(x_1, x_2, y_1, y_2)}$. The free group of rank n we denote by F_n.

Theorem A *(Theorem 10.6). Let (g, h, Q) be a triple as in Definition 10.5. Then,*

(1) $\mathcal{H}(g, h, Q)$ *is a finitely generated subgroup of* $\mathrm{SL}_1(\mathbb{Z}G)$ *and* $v^{-1}_{(x_1, x_2, y_1, y_2)} = v_{(-x_1, -x_2, y_2, y_1)}$,
(2) $\mathcal{H}(g, h, Q) \neq 1$ *if and only if* $[g, h] \notin \langle g \rangle Q$.

Moreover, for $\mathcal{H}(g, h, Q) \neq 1$,

(3) *if $o(gQ)|Q| = 2$, then $\mathcal{H}(g, h, Q) \cong F_3 \times C_2$, and*
(4) *if $o(gQ)|Q| > 2$, then $\mathcal{H}(g, h, Q) \cong F_n$ with $n = 1 + \frac{(o(gQ)|Q|)^3}{6} \prod_p (1 - \frac{1}{p^2})$, where the product runs over the prime divisors p of $o(gQ)|Q|$.*

More precisely, see Theorem 10.8, we give a concrete description of $\mathcal{H}(g, h, Q)$ and not only of its isomorphism type. Nevertheless, the structure of the group $\langle \mathcal{H}(g_i, h_i, Q_i) \mid i \in I \rangle$ generated by the H-units corresponding to several triples (g_i, h_i, Q_i) is still mysterious to us. There are several natural questions, in particular Question 10.9: when it is a direct product of the groups $\mathcal{H}(g_i, h_i, Q_i)$?

From Section 10.3 on we consider the case that $3 \nmid |G|$ and $\mathbb{Q}G$ has exceptional components of the type $\mathrm{M}_2(\mathbb{Q})$. As a first application of Theorem A we show that the Jespers-Leal theorem can be extended to include the difficult case $\mathrm{M}_2(\mathbb{Q})$. This is recorded in Remark 10.12 and follows from the proof of Theorem B, resolving in this case the problem of constructing generically a finite index subgroup. By $\mathcal{B}(G)$ we denote the subgroup generated by the Bass and bicyclic unit and by $\mathcal{H}(G)$ the subgroup generated by $\mathcal{H}(g, h, Q)$ for any admissible triple (g, h, Q).

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[5] As explained in Remark 10.10, their name refers to the crucial role of the second cohomology group, and in particular twisted group rings, to both discover the elements and the proof of the subsequent theorem.

Theorem B *(Theorem 10.11). Let G be a 2-group such that the only exceptional components of $\mathbb{Q}G$ are of the form $\mathrm{M}_2(\mathbb{Q})$, then $\langle \mathrm{Bic}(G), \mathcal{H}(G) \rangle$ is of finite index in $\mathrm{SL}_1(\mathbb{Z}G)$. Consequently, $\langle \mathcal{B}(G), \mathcal{H}(G) \rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$.*

We have chosen to focus on $\mathrm{M}_2(\mathbb{Q})$ as it is by far the most frequent exceptional component, cf. [2, Appendix A], and such a component naturally yields triples (g, h, Q) as in Definition 10.5. However, the statement of Theorem B sometimes also holds in the presence of other exceptional components, but we have not tried to pursue this line of investigations.

Our *second application* is about the abelianisation $\mathcal{U}(\mathbb{Z}G)^{ab} \cong \mathbb{Z}^n \times T$, with T a finite abelian group. The number n is called the rank of the abelianisation of $\mathcal{U}(\mathbb{Z}G)$. Recall that $\mathcal{U}(\mathbb{Z}G) = \pm V(\mathbb{Z}G)$ with $V(\mathbb{Z}G)$ the group of invertible elements with augmentation one. Recently the following questions got some attention:

(R1) Is the rank, as abelian group, of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ and $\mathcal{U}(\mathbb{Z}G)^{ab}$ equal? In particular if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite, is $\mathcal{U}(\mathbb{Z}G)^{ab}$ also finite? (See [2, Question 7.8 and Proposition 7.9].)

(P) Let p be a prime. If $V(\mathbb{Z}G)^{ab}$ contains an element of order p, does G^{ab} also contain an element of order p? (See [4, page 2].)

Question (P) is one of three questions formulated by Bächle, Maheshwary and Margolis [4] and the labelling refers to theirs. A stronger version of (P) was also formulated in [4], namely that $\exp(V(\mathbb{Z}G)^{ab}) = \exp(G^{ab})$. Currently, the only infinite family for which (P) has been proven are the dihedral groups D_{2p} with p prime [4, Theorem C].

We consider the 'most degenerate' classes of 2-groups, namely those where all the matrix components are exceptional and of the form $\mathrm{M}_2(\mathbb{Q})$. Such groups have been classified by Jespers, Leal and del Río [30,25], namely $G \cong K \times C_2^n$ with K a finite group that is a member of seven (infinite) families of groups $\mathcal{G}_1, \ldots, \mathcal{G}_7$, recalled in Section 10.3. However our proofs do not require the precise classification. In the next result we denote by $cl_\Gamma(H)$ the normal closure of some subgroup H in the larger group Γ. This result shows that in presence of exceptional components there are some natural obstructions towards conjectures (R1) and (P).

Theorem C *(Theorem 10.16 and Corollary 10.14). Let $G = K \times C_2^n$, with K a group in $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_7$, and π the natural epimorphism of $\mathcal{U}(\mathbb{Z}G)$ onto $\mathcal{U}(\mathbb{Z}G)^{ab}$. Then*

$$\mathrm{rank}\, \mathcal{U}(\mathbb{Z}\mathrm{G})^{ab} = \mathrm{rank}\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})) + \mathrm{rank}\, \pi(\langle \mathcal{H}(\mathrm{G})_{un} \rangle)$$

where $\mathcal{H}(G)_{un} = \{ x \in \mathcal{H}(G) \mid x \text{ is unipotent} \}$. Furthermore,

$$\exp(V(\mathbb{Z}G)^{ab}) = \mathrm{lcm} \left(\exp(G^{ab}), \exp\left(\frac{V(\mathbb{Z}G)}{\mathcal{Z}(V(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), \pm G \rangle)} \right)^{ab} \right).$$

Moreover,

(1) *The H-units $\mathcal{H}(G)$ are of finite index in $\mathrm{SL}_1(\mathbb{Z}G)$ despite that $\mathrm{Bic}(G)$ might be of infinite index.*
(2) *If G satisfies (54), then $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\,cl_{\mathcal{U}(\mathbb{Z}G)}(\langle\mathrm{Bic}(G),\pm G\rangle)$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$ and both (R1) and (P) have a positive answer.*

Recall that $\mathrm{span}_{\mathbb{Q}}\{ge \mid g \in G\} = \mathrm{M}_n(D)$ for e a primitive central idempotent of $\mathbb{Q}G$ and $(\mathbb{Q}G)e \cong \mathrm{M}_n(D)$. Condition (54) appearing in part (2) of Theorem C is about whether the generators of the finite group Ge have pre-images in G of the same order. If this is the case, then part (2) tells that the obstructions (i.e. the second factor in each formula) vanish.

Alternatively (54) can be interpreted as a condition on the proportions $o(g)/o(ge)$ for $g \in G$. Inspired by Theorem C, in upcoming work by the first author, it will be proven that if these proportions are 'large' then both obstructions are non-trivial and hence (54) is a non-artificial condition.

As a final application, we recover in Corollary 10.14 a result of del Río and Ruiz [12, Theorem 1.1] saying that $M := \prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} \mathrm{SL}_1(\mathbb{Z}G) \cap (1 - e + \mathbb{Q}Ge)$ is the largest direct product of free groups in $\mathcal{U}(\mathbb{Z}G)$. However, using H-units, our proof is uniform, i.e. we do not use the classification for G, and yield more explicit generators. Namely, $M = \langle \mathcal{H}(g_i, h_i, Q_i) \rangle \cong F_n^q$ for some triples (g_i, h_i, Q_i), $1 \leq i \leq q$ and n explicit. Furthermore, in some cases the bicyclic units and H-units together yield a normal complement of G in $V(\mathbb{Z}G)$. As formulated in Question 10.15, it would be interesting to investigate this phenomenon further.

Decomposition in twisted group rings as valuable substitute for the Wedderburn-Artin decomposition. The statements and proofs of Theorem B and Theorem C are in the framework of untwisted group rings. However, interestingly, they heavily depend on Theorem A whose proof crucially requires twisted group rings. Recall that F is a number field and R is its ring of integers. The starting observation is that an exceptional component of $F[\Gamma]$ corresponds to an irreducible F-representation of Γ, say φ. Furthermore a representation allows in a natural way to view Γ as an extension

$$1 \to N \to \Gamma \xrightarrow{\lambda} G \to 1 \tag{1}$$

where $N = \ker(\varphi)$ and $G = \mathrm{Im}(\varphi)$. Corresponding to this is an algebra decomposition $F\Gamma \cong FG \oplus FG(1 - \widehat{N})$ where $\widehat{N}$ is some central idempotent. The second observation is that the summand FG somehow originates from the trivial representation of N. In Section 3 we show how the irreducible representations of N can be used to concretely decompose $F\Gamma$ in terms of certain twisted group rings and crossed products of the smaller group G.

Before going into details, we first recall the definition of a twisted group ring. For a 2-cocycle $\alpha \in Z^2(G, R^*)$, where G acts trivially on R^*, the twisted group ring $R^\alpha[G]$

of G over R with respect to α is the free R-module with basis $\{u_g\}_{g \in G}$ where the multiplication is defined via

$$u_g u_h = \alpha(g,h) u_{gh} \quad \text{for all} \ \ g,h \in G$$

and any u_g commutes with the elements of R. Note that the ring structure of $R^\alpha[G]$ depends only on the cohomology class $[\alpha] \in H^2(G, R^*)$ of α and not on the particular 2-cocycle. Of importance is that there is a 1-1 correspondence between α-projective representations of G and $R^\alpha[G]$-modules. As such projective representations, although not explicitly used, are recurrent objects behind the scenes (see [15, §2.1] for detailed explanation).

In order to keep the introduction notationally lighter, we will restrict ourselves to abelian extensions (i.e. N is abelian in (1)). However from Section 2 till Section 5 we will work with general extensions. When N is abelian, the extension (1) corresponds[6] to a cohomology class $[\alpha] \in H^2_\sigma(G, N)$ where σ is the action of G on N. Via σ the group G acts on the set $\mathrm{Lin}(N, F)$ of linear F-characters of N and we denote the orbit space by $\mathrm{Lin}(N, F)/G$. Now, for a G-invariant linear character χ of N over F, the transgression of χ with respect to α is a 2-cocycle $T_\alpha(\chi) \in Z^2(G, F^*)$ which is defined by $T_\alpha(\chi)(g,h) := \chi(\alpha(g,h))$. Via so-called inflation one can extend a cocycle of G to one of Γ, see Section 2 for details.

Theorem D (*Theorem 3.2 and Proposition 3.1*). *With notations as above, N abelian and $[\beta] \in H^2(\Gamma, F^*)$ inflated from G. We have that*

$$F^\beta[\Gamma] \cong \bigoplus_{[\chi] \in \mathrm{Lin}(N,F)/G} (F(\chi) E_\chi) * G \tag{2}$$

for some concrete idempotents E_χ and explicit skewing and twisting of the crossed product $(F(\chi) E_\chi) * G$. In particular, if $\chi \in \mathrm{Lin}(N, F)^G$ is a G-invariant character, then[7]

$$(F(\chi) E_\chi) * G \cong F(\chi)^{\beta.T_\alpha(\chi)}[G].$$

In particular we recover the case where N is central, obtained in [47, Theorem 5.3] by Margolis and Schnabel. However even in that case our methods give a new, more explicit, proof. It is interesting to pause a second on the classical case when β is trivial, i.e. $F^\beta[\Gamma] = F\Gamma$ is a group ring. The above theorem then tells that even if one is solely interested in non-twisted group rings, one should still study twisted group rings over finite extensions of the chosen number field F. In particular, at this point one has another point of view on the first step alluded to at the start of the introduction.

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[6] The necessary background is briefly introduced in Section 2.
[7] With $\beta.T_\alpha(\chi)$ is meant the 2-cocycle of G with values in $F(\chi)^*$ defined pointwise, i.e. $(\beta.T_\alpha(\chi))(g,h) = \beta(g,h).T_\alpha(\chi)(g,h) = \beta(g,h).\chi(\alpha(g,h))$.

Important for the applications later on is that the decomposition (2) is not simply an abstract one. Among others, the projections p_χ onto the direct summands have a down-to-earth description. For this we need to fix a section $\mu : G \to \Gamma$ of λ in (1). For example, for $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$, Proposition 3.3 says that the projection p_χ viewed over R agrees with the ring epimorphism

$$\Psi_{\chi,\beta} : R^\beta[\Gamma] \to R[\chi]^{\beta.T_\alpha(\chi)}[G] : r\, u_{n.\mu(g)} \mapsto r\chi(n)v_g$$

where the sets $\{u_h \mid h \in \Gamma\}$ and $\{v_g \mid g \in G\}$ are the bases of the mentioned twisted group rings.

This ring morphism induces a group morphism

$$\widetilde{\Psi}_{\chi,\beta} : \mathcal{U}(R^\beta[\Gamma]) \to \mathcal{U}(R[\chi]^{\beta.T_\alpha(\chi)}[G]).$$

Another reason that (2) is a reasonable alternative for Wedderburn-Artin's decomposition is that the kernel and cokernel can be worked with.

Theorem E *(Theorem 5.3 and Theorem 5.5). Let Γ be some extension as in (1), $[\beta] \in H^2(\Gamma, F^*)$ inflated from G and $\chi \in \mathrm{Lin}(N, F)^G$. Also let R be an order in the number field F. Then*[8]*:*

(1) $\mathrm{coker}(\widetilde{\Psi}_{\chi,\beta})$ *is finite.*
(2) *If N is central, then $\{torsion\ units\ in\ \mathrm{ker}(\widetilde{\Psi}_{\chi,\beta})\} = \{\chi(a)^{-1}a \mid a \in N\}$.*

Moreover, we obtain conditions for $\mathrm{ker}(\widetilde{\Psi}_{\chi,\beta})$ to be finite and some computational reduction for determining $|\mathrm{coker}(\widetilde{\Psi}_{\chi,\beta})|$.

Along the way we obtain a version for twisted group algebras of certain classical theorems of Higman and Berman-Higman (see for example [26, Proposition 1.5.1 and Theorem 1.5.6]). More precisely, in Theorem 4.1 we describe when the unit group of a twisted group ring is finite and in Theorem 5.6 we show that torsion units must have trace zero. Also, in Proposition 4.5 we answer a question of Margolis and Schnabel [47, Remark 3.2.], in case of a torsion 2-cocycle, on when $F^\alpha[G] \cong F^{\alpha^j}[G]$.

Units in twisted group rings and a full description in the elementary abelian case

By Theorem D, describing the unit group of twisted group rings is interlaced with the classical problem of describing $\mathcal{U}(R\Gamma)$. In Section 6 we launch the investigations of generic constructions of units in twisted group rings and investigate generators of $\mathrm{coker}(\widetilde{\Psi}_{\chi,\beta})$.

More precisely, we consider $\mathcal{U}(R^\gamma[G])$ where R is the ring of integers in a cyclotomic field $F = \mathbb{Q}(\zeta_n)$, with ζ_n some primitive root of unity, and $[\gamma] \in H^2(G, R^*)$. Firstly,

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[8] Again, in order to avoid more notations in the introduction, some parts of the statements are left vague.

we construct in Definition 6.9 a class of units $\mathcal{H}_\gamma(G)$ which truly makes use of the twisting $\gamma \in Z^2(G, R^*)$. These elements can be thought of as some deformations of the classical bicyclic units in non-twisted group rings. An intriguing and crucial feature of these elements is that the generators live in $\mathrm{coker}\left(\widetilde{\Psi}_\chi : \mathcal{U}(R\Gamma) \to \mathcal{U}(R^{T_\alpha(\chi)}[G])\right)$, see Proposition 6.12 and Question 6.13.

In other words, the generators of $\mathcal{H}_\gamma(G)$ are intrinsic to twisted group rings. In fact, the newly constructed H-units arose as the pullback along the transgression map Ψ_χ of words in the generators of $\mathcal{H}_\gamma(G)$ for G an elementary abelian 2-group. The difficulty hereby is two-fold: (i) the inverse under Ψ_χ of a unit is usually not a unit due to $1 \neq \ker(\widetilde{\Psi}_\chi)$; (ii) the generators of $\mathcal{H}_\gamma(G)$ are not attained. In fact the length of the words depends on $|\mathrm{coker}(\widetilde{\Psi}_\chi)|$ which on its turn depends on $N = \ker(\lambda : \Gamma \twoheadrightarrow G)$.

An important step for investigating units in twisted group rings is Theorem 6.3 and Corollary 6.7 saying that $\langle \mathrm{Bic}(G), \mathcal{H}_\gamma(G) \rangle$ contains enough elementary matrices of each simple component. This generalizes the much used theorem of Jespers and Leal [29].

Finally, in Sections 8 and 9 we apply all the above machinery to study the case that $\Gamma = G \times C_2^m$ for some m and some $[\gamma] \in H^2(\Gamma, \mathbb{Z}^*)$ inflated from $[\delta] \in H^2(G, \mathbb{Z}^*)$. Note that the number of simple components of the group algebra increases exponentially with m, which for the investigations of unit groups makes these extensions more subtle as it first looks likes. In Section 7 we give a description of $\mathcal{U}(\mathbb{Z}^\gamma[G \times C_2^m])$ in terms of $\mathcal{U}(\mathbb{Z}^\delta[G])$, see Theorem 7.3. As an application, we are able to deduce the following result. Recall that G is said to have a normal complement in $\mathcal{U}(\mathbb{Z}[G])$ if $\mathcal{U}(\mathbb{Z}[G]) = N \rtimes (\pm G)$ for some normal subgroup N. If a torsion free complement N exists then the integral isomorphism problem has a positive answer for G (see for example [54, Proposition (30.4)]). Also recall that G satisfies the Higman subgroup property if every finite subgroup $H \leq V(\mathbb{Z}G)$ is isomorphic to a subgroup of G (this property was asked for the first time in Graham Higman's thesis).

Theorem F *(Corollary 7.4). Let G be a finite group and $[\gamma] \in H^2(G \times C_2^m, \mathbb{Z}^*)$ inflated from a cohomology class $[\delta] \in H^2(G, \mathbb{Z}^*)$. If G has a (torsion-free) complement in $\mathcal{U}(\mathbb{Z}^\delta[G])$ or satisfies the Higman subgroup property, then the same holds for $G \times C_2^m$ and $\mathcal{U}(\mathbb{Z}^\gamma[G \times C_2^m])$.*

Subsequentely, in Section 8, all the protagonists are computed explicitly in the case that $\Gamma = D_8 \times C_2^m$ (where D_8 denotes the dihedral group of order 8) and $G = C_2^{m+2}$. More precisely, we consider D_8 as some extension $[\alpha] \in H^2(C_2 \times C_2, C_2)$ and look at the projections $\widetilde{\Psi}_\chi : \mathcal{U}(\mathbb{Z}[D_8 \times C_2^m]) \to \mathcal{U}(\mathbb{Z}^{T_\alpha(\chi)}[C_2^{m+2}])$. Together with the general description of $\ker(\widetilde{\Psi}_\chi)$ and Theorem F we are able to pullback a precise description of $\mathcal{U}(\mathbb{Z}^{T_\alpha(\chi)}[C_2^{m+2}])$ obtained in Proposition 8.3. All together, the main achievement here is an unexpected uniform description of $\mathcal{U}(\mathbb{Z}[D_8 \times C_2^m])$ for all m. The case that $m = 0, 1, 2$ has been dealt with in earlier papers [22,23,40,42].

Along the article we formulated several questions we believe to be of interest. Some we would like to attract the attention to are Question 4.3, Question 6.5, Question 7.6 and Conjecture 5.7.

In conclusion, (a subclass of the) H-units arose by applying all the machinery above to an extension of $C_2 \times C_2$. These units might extend the bicyclic units with infinite index and are particularly useful in presence of an exceptional component of the form $M_2(\mathbb{Q})$. However we have not yet enough generic constructions that, without any condition on the 2×2 simple components, give a finite index subgroup for any finite group. Nevertheless, extending other groups in [2, Appendix A], the methods developed in this article should yield other generic constructions. Besides, this work can also simply be seen as an invitation to the study of units in twisted group rings as a problem of independent interest.

Notational conventions.

(1) All rings, denoted R, will be assumed unital and associative.
(2) $\mathcal{U}(R)$ denotes the unit group of R. If $\mathcal{U}(R)$ appears as the coefficients of a cohomology group, i.e. $H^j(G, \mathcal{U}(R))$, we will instead write R^* (thus $H^j(G, R^*)$).
(3) If $f : R \to S$ is a ring homomorphism, then we denote the induced map on the unit groups by $\widetilde{f} : \mathcal{U}(R) \to \mathcal{U}(S)$.
(4) Except stated otherwise, with 'an order R' we will mean a $\mathbb{Z}$-order.
(5) We will use the convention $g^h = h^{-1}gh$ and $\mathrm{conj}(h)(g) = g^h$ for conjugation.
(6) A commutator is $[g, h] = g^{-1}h^{-1}gh$.
(7) $cl_\Gamma(H)$ is the normal closure of some subgroup H in the larger group Γ.

Acknowledgment

2. Cohomological preliminaries

We now recall the required minimum on group cohomology for which details can be found in the book [10]. Given a finite group G with an action $\varphi : G \to \mathrm{Aut}(A)$ on an abelian group A, extensions[9]

$$1 \to A \to \Gamma \xrightarrow{\lambda} G \to 1 \tag{3}$$

of A by G, whose induced action is φ, are parametrised by the second group cohomology group $H^2_\varphi(G, A)$. If the action of G on A is trivial, in which case we write $H^2(G, A)$, it is well known that the cohomology group parametrizes the central extensions (i.e. $\mathcal{Z}(A) \subseteq \Gamma$). Given a commutative ring R and a 2-cocycle $\alpha \in Z^2(G, R^*)$ one can define the *twisted group ring* $R^\alpha[G]$ whose basis we will denote by $\{u_g \mid g \in G\}$. Up to R-algebra isomorphism, $R^\alpha[G]$ does not depend on α but only on the cohomology class $[\alpha] \in H^2(G, R^*)$. A 2-cocycle α is said to be *normalized* if u_1 is the identity element of $R^\alpha[G]$, i.e. if $\alpha(1, g) = 1 = \alpha(g, 1)$ for all $g \in G$. Notice also that then the ring $R = Ru_1$ is central in the twisted group ring $R^\alpha[G]$.

Example 2.1. A cohomology class $[\alpha] \in H^2(G, \mathbb{Z}^*)$ over $\mathbb{Z}$ corresponds to a central extension

$$[\alpha] : \quad 1 \to C_2 \to \Gamma \to G \to 1. \tag{4}$$

As a result, for an abelian group G generated by $\{g_1, g_2, \ldots, g_r\}$, $[\alpha] \in H^2(G, \mathbb{Z}^*)$ is determined by the values in $\{\pm 1\}$ of

$$u_{g_i}^{\circ(g_i)}, \ 1 \leq i \leq r, \text{ and the commutators } [u_{g_i}, u_{g_j}], \ 1 \leq i < j \leq r, \tag{5}$$

in the corresponding twisted group ring $\mathbb{Z}^\alpha[G]$.

Let Q be a normal subgroup of G, let M be an abelian group which we equip with a *trivial* G-action (i.e. M is a trivial G-module) and let $\pi : G \to G/Q$ be the quotient map. Then, for any $\delta \in Z^i(G/Q, M) \subset \mathrm{Map}(G/Q, M)$ we can define $\gamma \in Z^i(G, M)$ by

$$\gamma(x_1, \ldots, x_i) = \delta(\pi(x_1), \ldots, \pi(x_i)).$$

The map from $Z^i(G/Q, M)$ to $Z^i(G, M)$ sending δ to γ induces a map

$$\mathrm{Inf} : H^i(G/Q, M) \to H^i(G, M)$$

which is called the *inflation map*. Sometimes we will want to emphasize Q and use the notation Inf_Q. Notice also, that for any subgroup H of G, there is a natural restriction

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[9] Thus A is normal and $G \cong \Gamma/A$. In other words (3) is a short exact sequence.

map from $H^i(G, M)$ to[10] $H^i(H, M)$. In case of M a trivial G-module, the Lyndon-Hochschild-Serre spectral sequence yields a very concrete exact sequence between the lower cohomology groups.

Lemma 2.2 *(Inflation-restriction exact sequence). Let Q be a normal subgroup of G and M a trivial G-module. Then one has an exact sequence*

$$0 \to H^1(G/Q, M) \overset{\text{Inf}}{\to} H^1(G, M) \overset{\text{Res}}{\to} H^1(Q, M)^{G/Q} \overset{d}{\to} H^2(G/Q, M) \overset{\text{Inf}}{\to} H^2(G, M)$$

where the first connecting map d is the transgression map and $H^1(\cdot, \cdot) = \text{Hom}(\cdot, \cdot)$.

In case of the abelian extension (3), i.e. taking Γ and A (as respectively G and Q) the first transgression map is a morphism from $\text{Hom}(A, M)^G$ to $H^2(G, M)$ whose definition we now recall. For $\chi \in \text{Hom}(A, M)^G$ one can define a 2-cocycle $\text{Tra}(\chi) \in Z^2(G, M)$ via

$$\text{Tra}(\chi)(g_1, g_2) = \chi(\alpha(g_1, g_2))$$

for any $g_1, g_2 \in G$. The cohomology class $[\text{Tra}(\chi)]$ does not depend on the choice of $\alpha \in [\alpha]$.

Definition 2.3. With the above notations, the map

$$\text{Tra}_\alpha : \text{Hom}(A, M)^G \to H^2(G, M) : \chi \mapsto T_\alpha(\chi) := [\text{Tra}(\chi)]$$

is called the (first) *transgression map* associated to α.

Using notations as in (3), let Q be a normal subgroup of Γ such that $A \cap Q = 1$. Then $Q \cong \lambda(Q)$ and from now on *we will implicitly identify the both* and speak about G/Q.

Remark 2.4. Notice that $G/\lambda(Q) \cong G/Q$ and $AQ/Q \cong A/A \cap Q = A$. Hence, if $\alpha = \text{Inf}_Q(\gamma)$ then Γ/Q is the central extension

$$1 \to A \to \Gamma/Q \overset{\lambda}{\to} G/Q \to 1, \tag{6}$$

corresponding to $[\gamma]$.

In Section 5.2 we will consider various such normal subgroups Q and the (co)kernel of the transgression map associated to Γ/Q. Therefore we also write Tra_Q for the transgression map associated to the abelian extension (6). When the extension or Q are clear from the context we simply write Tra.

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[10] If $H = Q$ is normal, then $\text{Im}(\text{Res})$ is easily seen to be contained in the subgroup $H^i(Q, M)^{G/Q}$ of G/Q-invariant cocycles.

We record a corollary of $\alpha(g, 1) = \alpha(1, 1) = \alpha(1, g)$ for all $g \in \Gamma$, entailed by the 2-cocycle condition, which will repeatedly be used without further notice.

Corollary 2.5. *Let G be a group, Q a normal subgroup and suppose that $[\alpha] \in H^2(G, R^*)$ is inflated from a cohomology class $[\gamma] \in H^2(G/Q, R^*)$. Then, for $g \in G$ and $x \in Q$, u_g and u_x commute in $R^\alpha[G]$ whenever g and x commute in G. In particular, u_a is central in $R^\alpha[G]$ for any $a \in \mathcal{Z}(Q)$.*

Finally, we mention a lemma that might be known (however, to our knowledge, does not appear in the literature) and which will be useful in Proposition 4.5.

Lemma 2.6. *Let G be a finite group, let F be a field and let $\alpha \in Z^2(G, F^*)$ be of finite order. Assume that m is a number relatively prime to the order of $[\alpha] \in H^2(G, F^*)$. Then there exists a cocycle $\beta \in [\alpha]$ such that m is relatively prime with the order of β.*

Proof. Consider the greatest common divisor of m and the order of α, which we denote by d and it exists as $o(\alpha)$ is assumed to be finite. Denote $o(\alpha) = d \cdot \ell$ with $(d, \ell) = 1$. Since the order of α is finite, notice that $H = \langle \text{Im}(\alpha) \rangle$ is a finite subgroup of F^* and hence it is cyclic. Therefore H can be decomposed as $H = H_1 \times H_2$ where $|H_2| = \ell$ and $|H_1| = d$. Hence α admits a natural decomposition as $\alpha = \alpha_{H_1} \times \alpha_{H_2} \in Z^2(H_1, F^*) \times Z^2(H_2, F^*)$. Since m is relatively prime to the order of $[\alpha]$ and $\exp(H^2(H_1, F^*)) \mid |H_1| = d$, $[\alpha_{H_1}]$ is cohomological trivial. Therefore there exists $\lambda : G \rightarrow H_1$ such that $\alpha_{H_1}(g, h) = \lambda(g)\lambda(h)\lambda(gh)^{-1}$ for any $g, h \in G$. Consequently, $\alpha(g, h) = \lambda(g)\lambda(h)\lambda(gh)^{-1}\alpha_{H_2}(g, h)$ that is α is cohomologous to α_{H_2} and clearly m is relatively prime to the order of α_{H_2}. $\square$

Standing assumptions: In the sequel of the paper all cohomology groups will be with respect to a trivial action, except stated otherwise. Also, given a cohomology class $[\alpha]$, we always assume a chosen representative α which we assume to be normalized.

3. Decomposition of twisted group algebras for general extensions and transgression

In this section, given a central extension as in (3), we will recover the decomposition [47, Theorem 5.3.] of the twisted group algebra $F^\beta[\Gamma]$ for any $[\beta] \in H^2(\Gamma, F^*)$ inflated from G and F a field with $\text{char}(F) \nmid |\Gamma|$. Note that [47, Theorem 5.3.] was a generalisation of [35, Theorem 3.2.9.]. Our aim is to use other methods than [35,47] which allow to work with general extensions (i.e. with A not necessarily central or even abelian) and to interpret in Section 3.2 the projection maps as a kind of transgression morphisms. The most general structural result will be Proposition 3.1, but Theorem 3.2 which focuses on abelian extensions will be more precise. These results prove Theorem D from the introduction.

3.1. Concrete decomposition

Consider an arbitrary short exact sequence

$$1 \to N \to \Gamma \xrightarrow{\lambda} G \to 1. \tag{7}$$

Fix also a section μ of λ. In particular $\alpha(g, h) := \mu(g)\mu(h)\mu(gh)^{-1} \in N$ for all $g, h \in G$ and conjugation with $\mu(g)$ gives an outer automorphism of N:

$$\sigma : G \to \mathrm{Out}(N) := \mathrm{Aut}(N)/\mathrm{Inn}(N) : g \mapsto \mathrm{conj}(\mu(g)^{-1}). \tag{8}$$

It is well known that the associativity of Γ means that $\alpha(g, h)\alpha(gh, k) = \sigma_g(\alpha(h, k))\alpha(g, hk)$.

Let F *be any field with* $\mathrm{char}(F) \nmid |N|$ and $[\beta] \in \mathrm{im}(\mathrm{Inf} : H^2(G, F^*) \to H^2(\Gamma, F^*))$. Note that

$$F^\beta[\Gamma] \cong (FN) * G = \sum_{g \in G} (FN)v_g \tag{9}$$

is a crossed product of G over FN with the following operations[11]:

$$v_g v_h = \beta(g, h)\alpha(g, h)v_{gh} \qquad \text{(twisting)}$$

and, for $x \in FN$,

$$v_g x = \sigma_g(x)v_g. \qquad \text{(skewing)}$$

Before giving a more precise description, recall that there is a bijective correspondence between $\mathrm{PCI}(FN)$, the set of the primitive central idempotents of FN, and the simple FN-modules up to isomorphism. Moreover, via Galois descent, $\mathrm{PCI}(FN)$ can be computed from the set of the complex irreducible representations $\mathrm{Irr}(N, \mathbb{C})$. Also recall that G, interpreted as Γ/N, acts on $\mathrm{PCI}(FN)$. The orbit space will be denoted $\mathrm{PCI}(FN)/G$.

Let K be a splitting field of N containing F and let $\mathrm{Irr}(N, F)$ be a set of K-characters of N containing the character of exactly one composition factor of $V \otimes_F K$ for each simple FN-module V. Recall that the choice of the composition factor does not affect the character by [26, Theorem 3.3.1].

By definition, one has a bijection between $\mathrm{Irr}(N, F)$ and the simple FN-modules up to isomorphism, and thus with $\mathrm{PCI}(FN)$. Denote by e_χ the unique primitive central idempotent of FN corresponding to $\chi \in \mathrm{Irr}(N, F)$ (see Section 3.3 in [26]). Note that G also acts on the irreducible K-characters of N via $\chi^g(n) := \chi(\sigma_g(n))$. The bijection can be chosen such that $e_\chi^g := e_{\chi^g}$ and hence G acts on $\mathrm{Irr}(N, F)$.

---

[11] In $F^\beta[\Gamma]$ an arbitrary F-basis element has the form $u_a u_{\mu(g)}$ for some $a \in N, g \in G$. We are identifying it with av_g where a is considered as a coefficient in $F[N]$. Implicitly we are extending the action σ of G on N to FN.

Next, define

$$\mathrm{Lin}(N, F) := \{\chi \in \mathrm{Irr}(N, F) \mid \chi(1) = 1\}.$$

It is easily verified that

$$\chi \in \mathrm{Lin}(N, F) \text{ if and only if } FNe_\chi \text{ is commutative.}$$

Moreover, for $\chi \in \mathrm{Lin}(N, F)$ one has that

$$ne_\chi = \chi(n)e_\chi \quad \text{and} \quad FNe_\chi \cong F(\chi), \tag{10}$$

where $F(\chi)$ is the smallest field containing F and $\mathrm{im}(\chi)$.

Proposition 3.1. *With notation as above, we have the following:*

$$F^\beta[\Gamma] \cong \bigoplus_{[\chi] \in \mathrm{Irr}(N,F)/G} \left((FNE_\chi) * G\right),$$

where $E_\chi := \sum_{\chi^g \in [\chi]} e_\chi^g$ *and the skewing of* $(FNE_\chi) * G$ *is given by* σ *and the twisting by*[12] $E_\chi.(\beta \cdot \alpha)$*. Moreover, if* [13] $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$*,*

$$(FNE_\chi) * G \cong F(\chi)^{\beta.T_\alpha(\chi)}[G]. \tag{11}$$

Proof. Since $[\beta] \in \mathrm{im}(\mathrm{Inf}_N)$ we have that $F^{\mathrm{Res}(\beta)}[N] = FN = \bigoplus_{\chi \in \mathrm{Irr}(N,F)} FNe_\chi$. In general the orthogonal idempotents e_χ are not central in $F^\beta[\Gamma]$, however the element E_χ (being the sum of the orbit under conjugation by G) is. These elements are again two-by-two orthogonal idempotents, hence (9) can be rewritten as $F^\beta[\Gamma] = \bigoplus_{[\chi] \in \mathrm{Irr}(N,F)/G}(FNE_\chi) * G$. Since E_χ is central we see that indeed only the twisting changes and in the way asserted. This proves the first part of the result.

Next note that $\chi \in \mathrm{Irr}(N, F)^{\Gamma/N}$ being an invariant character exactly means that the orbit of e_χ is a singleton and hence $E_\chi = e_\chi$. Hence if moreover $\chi \in \mathrm{Lin}(N, F)$, then by (10) $FNE_\chi \cong F(\chi)$. Also, from the earlier reformulation of the associativity of Γ, it readily follows that $\alpha_\chi := \mathrm{Tra}_\alpha(\chi)$ satisfies the 2-cocycle condition

$$\alpha_\chi(g, h)\alpha_\chi(gh, k) = \alpha_\chi(h, k)\alpha_\chi(g, hk).$$

In particular, $(FNE_\chi) * G$ is a twisted group algebra where, by the above, the twisting is indeed $E_\chi.\beta \cdot \alpha = \beta \cdot \mathrm{Tra}_\alpha(\chi)$. $\square$

---

[12] The notation needs clarification: we implicitly consider both β and α as having image in FN and hence $(\beta \cdot \alpha)(g, h) := \beta(g, h).\alpha(g, h)$. Subsequently, $E_\chi.(\beta \cdot \alpha)$ also multiplies the result with the central idempotent E_χ.

[13] We denote by $\mathrm{Lin}(N, F)^{\Gamma/N}$ the set of Γ/N-invariant linear characters.

In order to obtain a more insightful decomposition we *now assume that N is abelian.* In this case the extension (7) corresponds to the cohomology class[14] $[\alpha] \in H^2_\sigma(G, N)$ and $\mathrm{Irr}(N, F) \subseteq \mathrm{Hom}(N, \overline{F}^*)$ where $\overline{F}$ denotes the algebraic closure of F. Moreover, since by assumption $\mathrm{char}(F) \nmid |N|$, the theorem of Perlis and Walker [26, Theorem 3.3.6.] tells us that

$$FN = \bigoplus_{d \,|\, |N|} F(\zeta_d)^{\oplus a_d}, \tag{12}$$

where $a_d = k_d \frac{[\mathbb{Q}[\zeta_d]:\mathbb{Q}]}{[F[\zeta_d]:F]}$, with k_d the number of cyclic subgroups of N of order d and ζ_d a primitive d-th root of unity. Proposition 3.1 now readily translates into the following:

Theorem 3.2. *With notations as above we have that:*

(1) *If N is abelian, then*

$$F^\beta[\Gamma] \cong \bigoplus_{[\chi] \in \mathrm{Lin}(N,F)/G} \Big(\bigoplus_{\chi\prime \in [\chi]} F(\chi\prime) e_{\chi\prime} * G \Big) = \bigoplus_{[\chi] \in \mathrm{Lin}(N,F)/G} (F(\chi) E_\chi) * G$$

with the skewing of $(F(\chi) E_\chi) * G$ given by σ and the twisting by $E_\chi.(\beta \cdot \alpha)$.

(2) *If N is central,[15] then*

$$F^\beta[\Gamma] \cong \bigoplus_{\chi \in \mathrm{Lin}(N,F)} F(\chi)^{\beta.T_\alpha(\chi)}[G].$$

Proof. The first part is a direct application of Proposition 3.1 and remarking that now $\mathrm{Lin}(N, F) = \mathrm{Irr}(N, F)$ and $F(\chi) = F(\chi\prime)$ for two conjugated characters χ and $\chi\prime$. When N is central then $\mathrm{Irr}(N, F) = \mathrm{Lin}(N, F)^{\Gamma/N}$ and $e_\chi^g = e_\chi$. Therefore the second part follows from the first and the last assertion in Proposition 3.1. $\square$

3.2. Transgression and natural morphisms as projections

Consider again the general extension (7). The projection in Proposition 3.1 of $F^\beta[\Gamma]$ on $FN E_\chi * G$, denoted p_χ, is given by multiplying with the central idempotent E_χ. If $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$ then this idempotent E_χ and, in particular, the isomorphism (11) can be made explicit and, moreover, we recover some classical constructions (see Example 3.4). More generally, let R be a *domain* and F its field of fractions. For $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$ we need its R-linear extension

$$\chi_R : RN \to R[\chi] : \sum_{a \in N} r_a a \mapsto \sum_{a \in N} r_a \chi(a)$$

---

[14] Since N is not necessarily central we might thus have a non-trivial action on the coefficients given by σ.
[15] As mentioned earlier, this case was already obtained in [47, Theorem 5.3].

which is an R-algebra map whose kernel we denote

$$I_\chi := \ker(\chi_R). \tag{13}$$

Proposition 3.3. *Let Γ, N, G, F and μ be as in Section 3.1, $[\beta] \in \mathrm{im}(\mathrm{Inf} : H^2(G, R^*) \to H^2(\Gamma, R^*))$ and $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$. Then, the projection p_χ restricted to $R^\beta[\Gamma]$ agrees with the map defined by (for $n \in N$ and $g \in G$)*

$$\Psi_{\chi,\beta} : R^\beta[\Gamma] \to R[\chi]^{\beta.T(\chi)}[G] : r\, u_{n.\mu(g)} \mapsto r\chi(n)v_g, \tag{14}$$

which is a ring epimorphism with $\ker(\Psi_{\chi,\beta}) = R^\beta[\Gamma]I_\chi = R^\beta[\Gamma](1 - e_\chi) \cap R^\beta[\Gamma]$. In particular,

$$R^\beta[\Gamma]/R^\beta[\Gamma]I_\chi \cong R^{\beta.T(\chi)}[G].$$

Moreover, if $\mathrm{im}(\chi) \subseteq R^*$, then I_χ is a free R-module with R-basis $\{u_a - \chi(a)u_1 \mid 1 \neq a \in A\}$.

When β is trivial we will simply write Ψ_χ. If R is a field and N central then most of the previous result is known[16] (e.g. if β is trivial in [35, Theorem 3.2.8.] and for $\beta \in \mathrm{im}(\mathrm{Inf}_N)$ in [47, Proposition 5.2.]).

Example 3.4.

- If χ is the trivial character, i.e. $\chi(n) = 1$ for all $n \in N$, then χ_R is the augmentation map of RN. In that case, writing $[\beta] = \mathrm{Inf}([\gamma])$,

$$\Psi_{\chi,\beta} : R^\beta[\Gamma] \to R^\gamma[G] : r\, u_{n.\mu(g)} \mapsto r\, v_g$$

  is the so-called natural homomorphism with respect to N (i.e. induced from the canonical map from Γ to G, denoted ω_N. If β is trivial it also is called the relative augmentation with respect to N). We will denote this specific case by $\omega_{\beta,N}$.
- If Γ is a central extension, i.e. $N \subseteq \mathcal{Z}(\Gamma)$, then $\mathrm{Lin}(N, F)^{\Gamma/N} = \mathrm{Irr}(N, F)$ which is a subgroup of $\mathrm{Hom}(N, \overline{F}^*)$ and $\Psi_{\chi,\beta}$ is the ring morphism associated to the transgression map Tra as in [47, Section 5]. Therefore we call the morphism (14) the *generalized transgression morphism*.

The reader may wish now to look at Example 5.8 where we will give an example of a group of order 16, along with its decomposition in twisted group algebras and the associated generalized transgression maps.

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[16] If χ is the trivial character and N arbitrary, then the statement has to be compared with [35, Lemma 3.2.12].

Proof of Proposition 3.3. A direct calculation verifies that $\Psi_{\chi,\beta}$ is multiplicative when χ is an invariant character. Concretely, $(u_{n_1}u_{\mu(g_1)}).(u_{n_2}u_{\mu(g_2)}) = \beta(g_1, g_2)u_{n_1}u_{\sigma_{g_1}(n_2)} \times u_{\mu(g_1g_2)}$ which is mapped by $\Psi_{\chi,\beta}$ to $\beta(g_1, g_2)\chi(n_1\sigma_{g_1}(n_2))v_{g_1g_2} = \beta(g_1, g_2)\chi(n_1)\chi(n_2) \times v_{g_1g_2}$. The latter is $\Psi_{\chi,\beta}(u_{n_1}u_{\mu(g_1)}).\Psi_{\chi,\beta}(u_{n_2}u_{\mu(g_2)})$, as needed. Now it is clear that it is an epimorphism.

Next, because $\chi \in \mathrm{Lin}(N, F)^{\Gamma/N}$, multiplying with the idempotent E_χ, i.e. the value of the projection, was already explicitly mentioned in (10). By comparing we see that $\Psi_{\chi,\beta}$ indeed agrees with p_χ.

Concerning the kernel, it follows from Proposition 3.1 that $\ker(\Psi_{\chi,\beta}) = R^\beta[\Gamma](1-e_\chi) \cap R^\beta[\Gamma]$. For the other description, since $\Psi_{\chi,\beta}|_{RN} = \chi_R$, we already have that $R^\beta[\Gamma]I_\chi \subseteq \ker(\Psi_{\chi,\beta})$. Also, an element y of $R^\beta[\Gamma]$ can be uniquely written as $y = \sum_{g \in G} y_g u_{\mu(g)}$, with each $y_g \in RN$. Since the elements v_g are $R[\chi]$-linearly independent, the concrete form of $\Psi_{\chi,\beta}$ implies that $y \in \ker(\Psi_{\chi,\beta})$ if and only if each $y_g \in \ker(\Psi_{\chi,\beta})$, or equivalently each $y_g \in I_\chi$. Consequently, $\ker(\Psi_{\chi,\beta}) = \sum_{g \in G} I_\chi u_{\mu(g)} \subseteq R^\beta[\Gamma]I_\chi$, as asserted.

Finally, suppose $\mathrm{im}(\chi)$ is contained in R^* and denote the free R-module generated by the set $\{u_a - \chi(a)u_1 \mid a \in N\}$ by M. Clearly M is contained in the kernel I_χ of $\chi_R : RN \to R$. Conversely, if $x = \sum_{a \in N} r_a u_a \in \ker(\chi_R)$, then $\sum_{a \in N} r_a \chi(a)u_1 = 0$ and hence $x = x - \sum_{a \in N} r_a \chi(a)u_1 \in M$. It follows that $I_\chi = M$. $\square$

In Section 5.1 we will be in the setting of Remark 2.4 where χ is a (linear) character of the abelian subgroup A, but where once we will be working with Γ and once with some Γ/Q for $A \cap Q = 1$. In order to distinguish, we will sometimes write

$$\Psi_{\chi,Q} : R^\beta[\Gamma/Q] \to R^{\beta.T(\chi)}[G/Q] \tag{15}$$

and hence assume $[\beta]$ is understood from the context.

3.3. Refined decomposition in case of an α-representation group of G

Often in this article we will be concentrating on a fixed 2-cocycle $\alpha \in Z^2(G, F^*)$ of G. In that case it is useful to consider the following group which will be recurrent:

$$G_\alpha := \{\alpha(a,b)u_c \mid a, b, c \in G\} = \langle u_g \mid g \in G \rangle \leq \mathcal{U}(F^\alpha[G]). \tag{16}$$

If α is of finite order,[17] then every element in G_α is of the form $\zeta^i u_g$ with ζ a $o(\alpha)$-primitive root of unity. Hence it is a central extension of $\langle \zeta \rangle$ by G with $\lambda : G_\alpha \to G : \zeta^i u_g \mapsto g$. Considering the canonical section $\mu(g) = u_g$ one has that $\alpha(g,h) = \mu(g)\mu(h)\mu(gh)^{-1}$. Thus $\mathrm{Hom}(\langle \zeta \rangle, F^*) = \langle \chi \rangle$ with $\chi(\zeta) = \zeta$ and $\alpha = \mathrm{Tra}(\chi)$. Therefore we may now apply Theorem 3.2 to recover [35, Proposition 3.3.8.].

---

[17] Since G is finite we know that the cohomology class $[\alpha]$ has finite order dividing $|G|$, however there has not to be a representative of finite order. For example the cohomology class of $C_2 = \langle x \rangle$ over $\mathbb{Q}$ defined by $u_x^2 = 2$ has order 2 but any 2-cocycle representative has infinite order. Such a representative however exists when the values are in a $|G|$-divisible group.

Corollary 3.5. *Let G be a finite group, F a number field and $\alpha \in Z^2(G, F^*)$ a cocycle of finite order. Then*

$$F[G_\alpha] \cong \bigoplus_{i=0}^{o(\alpha)-1} F^{\alpha^i}[G].$$

Note that the isomorphism class of G_α depends on the chosen cocycle. It turns out that even for cohomologous cocycles of the same finite order the associated groups might be non isomorphic. The following example was communicated to us by Y. Ginosar.

Example 3.6. Denote

$$G = C_2 \times C_2 = \langle x \rangle \times \langle y \rangle$$

and let $F = \mathbb{Q}(i)$. Next, we wish to define two cocycles $\alpha, \alpha' \in Z^2(G, F^*)$. We will take cocycles which are normalized, that is for any $g \in G$

$$\alpha(g, e_G) = 1 = \alpha(e_G, g), \quad \alpha'(g, e_G) = 1 = \alpha'(e_G, g).$$

Further, define

$$\alpha(x, x) = \alpha(x, y) = \alpha(x, xy) = 1$$
$$\alpha(y, y) = 1, \ \alpha(y, x) = \alpha(y, xy) = -1$$
$$\alpha(xy, x) = 1, \ \alpha(xy, y) = \alpha(xy, xy) = -1$$

and also

$$\alpha'(x, x) = -1, \quad \alpha'(x, y) = 1, \quad \alpha'(x, xy) = -1$$
$$\alpha'(y, y) = -1, \quad \alpha'(y, x) = -1, \quad \alpha'(y, xy) = 1$$
$$\alpha'(xy, x) = 1, \quad \alpha'(xy, y) = -1, \quad \alpha'(xy, xy) = -1.$$

Now choose a basis $\{u_g\}_{g \in G}$ for $F^\alpha G$ and a basis $\{v_g\}_{g \in G}$ for $F^{\alpha'} G$. Notice that both cocycles $\alpha, \alpha' \in Z^2(G, F^*)$ are of order 2 and also notice that the cohomology classes which corresponds to these cocycles are

$$[\alpha]: \quad u_x^2 = 1, \quad u_y^2 = 1, \quad [u_x, u_y] = -1$$

and

$$[\alpha']: \quad v_x^2 = -1, \quad v_y^2 = -1, \quad [v_x, v_y] = -1.$$

It follows that

$$G_\alpha = \langle u_g \mid g \in G \rangle \cong D_8 \quad \text{and} \quad G_{\alpha'} = \langle v_g \mid g \in G \rangle \cong Q_8$$

and therefore $G_\alpha^* \not\cong G_{\alpha'}^*$. The crucial point is that these cocycles are cohomologous over[18] $F = \mathbb{Q}(i)$ but not over $\mathbb{Q}$.

4. When is the unit group of a twisted group ring finite?

Let F be a number field, R a $\mathbb{Z}$-order in F and $\alpha \in Z^2(G, R^*)$ a *fixed* (normalized) 2-cocycle. We refer to [26, Section 4.6] for the necessary background on orders. In this section we determine when the unit group $\mathcal{U}(R^\alpha[G])$ is finite.

Generally speaking, [26, Corollary 5.5.8.], if $\mathcal{O}$ a is $\mathbb{Z}$-order in a finite dimensional semisimple $\mathbb{Q}$-algebra A, then $\mathcal{U}(\mathcal{O})$ is finite if and only if every simple component of A is either $\mathbb{Q}$, a quadratic imaginary field extension of $\mathbb{Q}$ or a totally definite quaternion algebra over $\mathbb{Q}$. Recall that a totally definite quaternion algebra over $\mathbb{Q}$ is a 4-dimensional $\mathbb{Q}$-algebra with basis $1, i, j, k$ so that $ij = k = -ji$, $i^2, j^2 \in \mathbb{Q}$ and $i^2 < 0$, $j^2 < 0$. We will denote this algebra by $\left(\frac{u,v}{\mathbb{Q}}\right)$ where $i^2 = u$ and $j^2 = v$.

As a by-product, if $\mathcal{U}(R^\alpha[G])$ is finite and $x \in \mathcal{U}(R^\alpha[G])$ is of finite order, say n, then n must[19] divide 4 or 6. Consequently, the exponent of G_α is a divisor of 4 or 6 if $\mathcal{U}(R^\alpha[G])$ is finite. Moreover, also $\mathcal{U}(R)$ would need to be finite and hence by the above $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ with $d > 0$. In the former case $R = \mathbb{Z}$ is the only order in F. We obtain the following characterisation, generalizing a classical result of Higman [26, Theorem 1.5.6] for untwisted group rings.

Theorem 4.1. *Let G be a finite group, F be a number field, R a $\mathbb{Z}$-order in F and $\alpha \in Z^2(G, R^*)$ non-trivial normalized cocycle. Then the following are equivalent:*

(1) $\mathcal{U}(R^\alpha[G])$ *is finite,*
(2) $\mathcal{U}(R[G_\alpha])$ *is finite,*
(3) $\mathcal{U}(R^{\alpha^i}[G])$ *is finite for all i.*

One of the above holds if and only if one of the following conditions is satisfied:

  (i) G_α *is an abelian group of exponent 4 and $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$,*
 (ii) G_α *is an abelian group of exponent 3 and $F = \mathbb{Q}(\sqrt{-3})$,*
(iii) G_α *is an abelian group of exponent 6 and $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-3})$,*
(iv) G_α *is a non-abelian Hamiltonian[20] 2-group and $F = \mathbb{Q}$.*

---

[18] The coboundary here is a map $f : G \to \mathbb{Q}(i)$ determined by $f(x) = i = f(y)$, $f(xy) = -1$ and $f(1) = 1$. Indeed $\alpha'(g_1, g_2) = f(g_1)f(g_2)f(g_1 g_2)^{-1}\alpha(g_1, g_2)$ for all $g_1, g_2 \in G$.

[19] The F-subalgebra $F[x]$ of $F^\alpha G$ is a commutative semisimple subalgebra and hence a direct sum of cyclotomic extensions of $\mathbb{Q}(\xi_m)$, with ξ_m a root of unity of order m, a divisor of n. Moreover, by the above, $[\mathbb{Q}(\xi_m) : \mathbb{Q}] \leq 2$ and one of the m must be equal to n. Now the Dirichlet unit theorem yields the claim as a $\mathbb{Z}$-order in $\mathbb{Q}(\xi_n)$ is finite exactly when n divides 4 and 6.

[20] A group is called *Hamiltonian* if every subgroup is normal.

Recall that Higman's result says that $\mathcal{U}(\mathbb{Z}G)$ is finite if and only if G is abelian of exponent dividing 4 or 6 or $G \cong Q_8 \times C_2^n$ for some $n \in \mathbb{N}$. To put into perspective, it is good to recall Baer-Dedekind's classification theorem [52, Theorem 1.8.5.] which says that G is Hamiltonian (i.e. all subgroups are normal) if and only if G is abelian or $G \cong Q_8 \times C_2^n \times A$ with A an odd order abelian group. Therefore, Higman's result says that $\mathcal{U}(\mathbb{Z}G)$ is finite exactly for the Hamiltonian groups with exponent dividing 4 or 6. Note that the list of possibilities stated above for a non-trivial cocycle is more restrictive than for a trivial cocycle, which is the reason that we excluded the trivial case in Theorem 4.1

Also note that the group G_α is abelian exactly when the cocycle α is symmetric (i.e. $\alpha(g, h) = \alpha(h, g)$ for all $h, g \in G$) and G is abelian. In Corollary 4.4 below we will give a concrete interpretation in terms of G and α for case (iv).

Proof. Clearly in any of the cases (1)-(3) $\langle \text{im}(\alpha) \rangle$ is a finitely generated torsion subgroup of $R^* \subset F^*$ and hence it is a finite cyclic group. In particular $o(\alpha) < \infty$ which allows to apply Corollary 3.5:

$$F[G_\alpha] \cong \bigoplus_{i=0}^{o(\alpha)-1} F^{\alpha^i}[G].$$

Hence the equivalence between (2) and (3) is a consequence thereof and commensurability of the unit group of $\mathbb{Z}$-orders in $F[G_\alpha]$ (see [26, Lemma 4.6.9]). Also any order in $F^{\alpha^i}[G]$ is a direct summand of an order in $F[G_\alpha]$ and hence (2) implies (1). The main bulk of the proof goes into proving that (1) implies (2). More precisely, we will show that G_α and F are of the form (i)-(iv). In those cases one can directly see that $\mathcal{U}(R[G_\alpha])$ is finite (e.g. this can be deduced from [26, Theorem 1.5.6.], handling the case $R = \mathbb{Z}$, together with Dirichlet unit theorem [26, Theorem 5.2.4.]).

Suppose $\mathcal{U}(R^\alpha[G])$ is finite and hence, as noticed at the start of this section, $\exp(G_\alpha) \mid 4$ or 6 and $F^\alpha[G]$ is a direct sum of (certain) division F-algebras. In particular, it contains no non-zero nilpotent elements.

Claim 1. *If $u_k \in G_\alpha$, with $k \in G$ such that $o(u_k) = o(k)$ then $\langle u_k \rangle$ is normal in G_α. Moreover, this condition holds when $o(u_k)$ is square-free and if $o(u_k) = 2$ then $u_k \in \mathcal{Z}(G_\alpha)$.*

The proof of this claim will be carried out inside $F^\alpha[G]$ (and not $F[G_\alpha]$). Let $\widetilde{u_k} := \sum_{i=0}^{o(u_k)-1} u_k^i$ and note that $u_k \widetilde{u_k} = \widetilde{u_k}$. Hence for any $t \in G$ the element $(1 - u_k)u_t\widetilde{u_k} \in R^\alpha[G]$ is a nilpotent element. By an earlier remark the nilpotent element is zero and thus $u_t^{-1}u_k u_t \widetilde{u_k} = \widetilde{u_k}$. Consider now $F^\alpha[G]$ with its canonical G-grading. Then $\deg(u_k) = k$ and $\deg(u_t^{-1}u_k u_t) = t^{-1}kt$. Since $o(u_k) = o(k)$, all summands of $\widetilde{u_k}$ have different degrees (in particular $\widetilde{u_k} \neq 0$) and thus a degree argument shows that $u_t^{-1}u_k u_t = u_k^j$ for some j.

Consequently, $\langle u_k \rangle \triangleleft G_\alpha$, as desired. Now suppose $o(u_k)$ is square-free. It is well known[21] that if $o(\alpha)$ is finite then $u_k^{o(k)}$ is a root of unity in R^* of order dividing $o(k)$. Hence if $o(u_k)$ is square-free, it is easily seen that $u_k^{o(k)} = 1$. The last part follows from the rest.

Claim 2. *Either (I) G_α is abelian of exponent dividing 4 or 6, (II) $G_\alpha \cong Q_8 \times E$ with E an elementary abelian 2-group.*

Note that by Baer-Dedekind's classification theorem [52, Theorem 1.8.5.] these cases are exactly those Hamiltonian groups with exponent dividing 4 or 6. Due to claim 1 it remains to prove that the generators of order 4 are also normal. For this recall that [58, Theorem 11.5.12] the unit group of a maximal order in a quaternion algebra $\left(\frac{-a, -b}{\mathbb{Q}} \right)$, with $a, b \in \mathbb{N}$, is cyclic except for the maximal order in $\left(\frac{-1, -1}{\mathbb{Q}} \right)$ and $\left(\frac{-1, -3}{\mathbb{Q}} \right)$. In those cases the unit group is $SL(2, 3) = Q_8 \rtimes C_3$, resp. $Dic_3 := C_3 \rtimes C_4$. Now consider $G_\alpha \leq \prod_{e \in \mathrm{PCI}(F^\alpha[G])} G_\alpha e$. Since also $\exp(G_\alpha e) \mid 4, 6$ we see that $G_\alpha e$ is either cyclic or a subgroup of Q_8 or D_6. Thus one can check that any element of order 4 in the direct product generates a normal therein and in particular the same holds for G_α. This finishes the proof of the second claim.

We will now study both cases in more detail. To start, as already used, we know that $\mathcal{U}(R)$ is finite and hence that $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ with $d > 0$. It remains to restrict the possible values of d and exclude exponent 2.

Case (I): Suppose G_α is abelian. To start remark $\exp(G_\alpha) = 2$ is not possible. Indeed, otherwise $1 = (u_g u_h)^2 = \alpha(g, h) u_g^2 u_h^2 = \alpha(g, h)$ for all $g, h \in G$, in contradiction with the assumption that α is non-trivial. By computations of the same type, $\exp(G_\alpha) = 3$ and $F = \mathbb{Q}$ is also impossible for non-trivial α. The restriction on F for the other cases will follow from the fact that all simple components have degree at most 2 over $\mathbb{Q}$. Indeed, suppose $\exp(G_\alpha) = 4$ and write $F = \mathbb{Q}(\sqrt{-d})$, with d a square free non-negative integer. Then $F^\alpha[G]$ has a simple component $\mathbb{Q}(\sqrt{-d}, \sqrt{-1}) = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ which is of degree at most 2 if and only if $d = 0$, or 1. If the exponent is 3 or 6, then there is a simple component $\mathbb{Q}(\sqrt{-d}, \zeta_3)$ which would have degree larger than 2 if $d \neq 0, 3$.

Case (II): Finally consider the case that $G_\alpha \cong Q_8 \times E$ is an Hamiltonian 2-group. Since $F^\alpha[G]$ has a simple component $\left(\frac{-1, -1}{F} \right)$ which needs to be totally definite (in particular F is totally real), we indeed get that $F = \mathbb{Q}$. $\quad \square$

Remark 4.2. The condition on the coefficient ring R can be generalized further. Namely, let F be a global field and S be a finite set of places of F containing the archimedian ones. Denote by $\mathcal{O}_S$ the ring of S-integers of F, which is well known to be a Dedekind domain with finite quotients. Therefore, any $\mathcal{O}_S$-order R is commensurable [26, Lemma

---

[21] More generally, if g_1 and g_2 commute, then $[u_{g_1}, u_{g_2}] := u_{g_1}^{-1} u_{g_2}^{-1} u_{g_1} u_{g_2} = \alpha(g_1, g_2)$. Since $u_g^{o(g)}$ is central in $R^\alpha[G]$, it follows that $1 = [u_{g_1}^{o(g_1)}, u_{g_2}] = \alpha(g_1, g_2)^{o(g_1)}$. In particular $\alpha(k^i, k)$ is a $o(k^i)$-root of unity in R^* and so also a $o(k)$-root of unity. Since $u_k^{o(k)} = \prod_{i=1}^{o(k)-1} \alpha(k^i, k) u_1$ and R is commutative we obtain the claim.

4.6.9] with $\mathcal{O}_S$. Since $|\mathcal{U}(\mathcal{O}_S)|$ is finite only for $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$, with $d > 0$, and $S = \{\infty\}$ (e.g. see [11, Theorem 3.24.]) we see that the conclusion of Theorem 4.1 also holds for such R.

Theorem 4.1 and its proof could be considered as a first contribution about the interplay of torsion units and nilpotent elements between $R^\alpha[G]$ and $R[G_\alpha]$. A satisfactory answer to the following general question would very useful:

Question 4.3. Is there a concrete connection between the torsion and nilpotent elements of $R^\alpha[G]$ and $R[G_\alpha]$? In particular, when (in terms of G, R and α) does $R^\alpha[G]$ not have nonzero nilpotent elements?

Using Theorem 4.1 one can give an especially precise characterisation when $\mathcal{U}(R^\alpha[G])$ is a finite non-abelian group in terms of G and α. For this recall that any non-abelian Hamiltonian 2-group G is isomorphic to $Q_8 \times C_2^n$ for some n. In other words, it can be written as a stem[22] extension

$$1 \to C_2 \to G \to C_2^{n+2} \to 1.$$

On the other hand, any non-trivial cohomology class $[\alpha] \in H^2(G, \mathbb{Z}^*)$ corresponds to a central extension of G by C_2. Therefore, from Theorem 4.1 (iv) we deduce

Corollary 4.4. *Let G be a finite group, F be a number field, R an order in F and $[\alpha] \in H^2(G, R^*)$ a non-trivial cohomology class. Then $\mathcal{U}(R^\alpha[G])$ is a finite non-abelian group if and only if the following conditions are satisfied*

(1) $R = \mathbb{Z}$,
(2) G *is an elementary abelian 2-group of rank at least 2,*
(3) $[\alpha]$ *is inflated from a cohomology class $[\gamma] \in H^2(C_2 \times C_2, C_2)$ which is determined by $u_x^2 = u_y^2 = [u_x, u_y] = -1$ where x, y are generators of $C_2 \times C_2$ and -1 is the generator of C_2.*

Notice that the cohomology class $[\gamma] \in H^2(C_2 \times C_2, C_2)$ above corresponds to Q_8.

To finish this section, we would like to come back on the proof of Theorem 4.1 which unfortunately is quite indirect. Indeed the implication from (1) to (2) goes by classifying all the possibilities for G_α and F and then noticing that in all these cases $\mathcal{U}(R[G_\alpha])$ is finite. A more natural approach would have been to construct for all j an isogeny between $\mathcal{U}(R^\alpha[G])$ and $\mathcal{U}(R^{\alpha^j}[G])$. Such a map can however only come from ring (epi)morphism

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[22] An extension is called *stem* if the base normal group is contained in $G' \cap \mathcal{Z}(G)$. In particular it is central.

when $\gcd(j, n) = 1$. The statement answers a question of Margolis and Schnabel[23] [47, remark 3.2.] in case $[\alpha]$ has a cocycle representative of finite order.

Proposition 4.5. *Let G be a finite group, F a field of characteristic zero and $\alpha \in Z^2(G, F^*)$. If $F^\alpha[G] \cong F^{\alpha^j}[G]$, isomorphic as rings, then $\gcd(j, o([\alpha])) = 1$. If α has finite order and F is a number field, then converse also holds.*

Proof. First suppose that $F^\alpha[G] \cong F^{\alpha^j}[G]$. Note that since G is finite, $\langle \mathrm{im}(\alpha) \rangle$ is a finitely generated abelian subgroup of F^*. In particular it lies in some countable subfield, say $L \subseteq F$. Moreover, with abuse of notation, $F^\alpha[G] \cong F \otimes_L L^\alpha[G]$ and one can restrict the given isomorphism restricts to a ring isomorphism $L^\alpha[G] \cong L^{\alpha^j}[G]$. Now, since L is countable it can be embedded in $\mathbb{C}$ and we can view α as having values in $\mathbb{C}^*$. Consequently, by tensoring with $\mathbb{C} \otimes_L -$ we see that $\mathbb{C}^\alpha[G] \cong \mathbb{C}^{\alpha^j}[G]$ as rings. Therefore we can apply [47, Theorem 3.1.] saying that $\gcd(j, o([\alpha])) = 1$.

Conversely, let $n = o([\alpha])$ and let $j \in \mathbb{Z}_{\geq 0}$ such that $\gcd(j, n) = 1$. Since cohomologous cocycles $\alpha, \beta \in Z^2(G, F^*)$ admit isomorphic twisted group rings $F^\alpha G \cong F^\beta G$ we may assume, by Lemma 2.6, that $\gcd(j, o(\alpha)) = 1$. Hence we have an isomorphism $\sigma_j : \mathbb{Q}(\zeta_{o(\alpha)}) \to \mathbb{Q}(\zeta_{o(\alpha)}^j)$ mapping $\zeta_{o(\alpha)}$ to its j-th power. With this at hand, define $\psi : \mathbb{Q}(\zeta_{o(\alpha)})^\alpha[G] \to \mathbb{Q}(\zeta_{o(\alpha)})^{\alpha^j}[G]$ by $\psi(\sum a_g u_g) = \sum \sigma_j(a_g) v_g$. Note that for all $g, h \in G$:

$$\psi(u_g u_h) = \psi(\alpha(g,h) u_{gh}) = \alpha(g,h)^j v_{gh} = v_g v_h = \psi(u_g)\psi(u_h).$$

Consequently, ψ is a ring epimorphism and hence isomorphism. Now note that F contains a $\zeta_{o(\alpha)}$-root of unity. Therefore an extension of scalars with field F now finishes the proof. □

5. Correlations between $R[\Gamma]$ and $R^\alpha[G]$ - a unit group point of view

Throughout this section *we fix an extension*

$$1 \to N \to \Gamma \xrightarrow{\lambda} G \to 1. \tag{17}$$

As in (7), fix also a section μ of λ and define

$$\alpha(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}.$$

In particular when N is abelian we will always choose this α as the normalized representative of the cohomology class $[\alpha]$ corresponding (17). We *will always assume* that the underlying field F is such that $\mathrm{char}(F) \nmid |\Gamma|$ and R is some order in F.

We wish to compare $\mathcal{U}(R[\Gamma])$ and $\mathcal{U}(R^\alpha[G])$ with the aim to pullback results from the smaller group G, but in the twisted context, to the larger group Γ. For this we study the kernel and cokernel of the generalized transgression Ψ_χ from Proposition 3.3. Also in this section we will work more generally with $R^\beta[\Gamma]$ for some $[\beta] \in \mathrm{im}(\mathrm{Inf}_N)$.

Notation. Recall that if $f : R \to S$ is a ring homomorphism, then we denote the induced map on the unit groups by $\widetilde{f} : \mathcal{U}(R) \to \mathcal{U}(S)$.

5.1. On the cokernel of the generalized transgression map

In the sequel, finiteness of cokernels of group morphisms will always follow from the following somehow folklore lemma.

Lemma 5.1. *Let[24] R be a Dedekind domain with field of fractions F such that R/I is finite for all $0 \neq I \lhd R$ and let A and B be semisimple F-algebras. Furthermore consider R-orders $\mathcal{O}_A$ in A and $\mathcal{O}_B$ in B. If there exists a R-algebra epimorphism $\pi : \mathcal{O}_A \to \mathcal{O}_B$, then $\mathrm{coker}(\widetilde{\pi})$ is finite.*

Proof. By definition $\mathcal{O}_A$ contains an F-basis of A and similarly for $\mathcal{O}_B$. Therefore we can extend F-linearly π to an F-algebra epimorphism $\widehat{\pi} : A \to B$. Formally $\widehat{\pi} = id_F \otimes_R \pi$ and we identify $A = F\mathcal{O}_A$ with $F \otimes_R \mathcal{O}_A$. Due to the semisimplicity of A, there exists a central idempotent e in A such that $\widehat{\pi}|_{Ae} : Ae \to B$ is an isomorphism. In other words, when decomposing $A = Ae \oplus A(1-e)$ we can consider $\widehat{\pi}$ as projection on the first component. To obtain the desired statement, consider the R-orders $\mathcal{O}_A \subseteq \mathcal{O}_Ae \oplus \mathcal{O}_A(1-e)$ in A. Due to the conditions on R there exists a non-zero element $r \in R$ such that $r(\mathcal{O}_Ae \oplus \mathcal{O}_A(1-e)) \subseteq \mathcal{O}_A$ and see [26, Lemma 4.6.9.],

$$[\mathcal{U}(\mathcal{O}_Ae) \times \mathcal{U}(\mathcal{O}_A(1-e)) : \mathcal{U}(\mathcal{O}_A)] \leq [\mathcal{O}_A : r(\mathcal{O}_Ae \oplus \mathcal{O}_A(1-e))] < \infty.$$

Consequently also its epimorphic image $\mathrm{Im}(\widetilde{\pi}) \cong \mathcal{U}(\mathcal{O}_A)e$ is of finite index in $\mathcal{U}(\mathcal{O}_Ae) \cong \mathcal{U}(\mathcal{O}_B)$ (with upper bound the above number), as desired. $\square$

Note that the proof in fact gives a method to obtain an upper bound on $|\mathrm{coker}(\widetilde{\pi})|$ which however depends on the element $r \in R$ obtained which is not explicit.

Question 5.2. What is an explicit, and generic, upper bound on $|\mathrm{coker}(\widetilde{\pi})|$?

Using Proposition 3.3 a first useful incarnation of Lemma 5.1 is with $\widetilde{\Psi}_\chi$ for some $\chi \in \mathrm{Lin}(\Gamma, R)^{\Gamma/N}$. However, in contrast to the kernel and despite it to be finite, in general

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[24] The prototypical example of such a ring is the ring of integers in a number field. The condition is the minimal needed to use classical methods with R-orders, e.g. that two R-orders have commensurable unit groups.

a concrete description (or even generators) of the cokernel is out of reach. Instead, we will focus on comparing the cokernel of $\widetilde{\Psi}_{\chi,Q}$, defined in (15), for certain types of 'nice' normal subgroups Q of Γ. The restrictions on Q will be as in Remark 2.4 and be such that we have the following[25] commutative diagram:

$$
\begin{array}{ccc}
\mathcal{U}(R^\beta[\Gamma]) & \xrightarrow{\ \widetilde{\Psi}_\chi\ } & \mathcal{U}(R^{\beta.T(\chi)}[G]) \\
{\scriptstyle \widetilde{\omega}_{Q,\beta}}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\omega}_{Q,\beta.T(\chi)}} \\
\mathcal{U}(R^\beta[\Gamma/Q]) & \xrightarrow[\ \widetilde{\Psi}_{\chi,Q}\ ]{} & \mathcal{U}(R^{\beta.T(\chi)}(\chi)[G/Q])
\end{array}
\tag{18}
$$

Note that (co)restricting $\widetilde{\Psi}_\chi$ yields a morphism

$$
\widetilde{\Psi}_{ker} : \mathcal{U}\left(1 + R^\beta[\Gamma].I_Q\right) \longrightarrow \mathcal{U}\left(1 + R^{\beta.T(\chi)}[G].I_Q\right) : 1 + x \mapsto 1 + \Psi_\chi(x)
$$

between $\ker(\widetilde{\omega}_N)$ and $\ker(\widetilde{\omega}_{N,\alpha})$ (cf. Proposition 3.3 and (13) for definition I_Q).

Theorem 5.3. *Let Γ be the extension (17), $[\beta] \in \operatorname{im}(\operatorname{Inf}_N)$, $\chi \in \operatorname{Lin}(N,F)^{\Gamma/N}$ and R a Dedekind domain such that R/I is finite for all $0 \neq I \triangleleft R$ and $\operatorname{char}(R) \nmid \Gamma$. Then,*

(1) $\operatorname{coker}(\widetilde{\Psi}_{\chi,\beta})$ *is finite,*
(2) *for any normal subgroup Q of Γ such that $Q \cap N = 1$, $\alpha(x,y) = 1$ for all $x,y \in Q$, $[\beta] \in \operatorname{im}(\operatorname{Inf}_Q)$ and $\widetilde{\omega}_Q$ and $\widetilde{\omega}_{Q,\beta.T(\chi)}$ are surjective, then*

$$
|\operatorname{coker}(\widetilde{\Psi}_{ker})| . |\operatorname{coker}(\widetilde{\Psi}_{\chi,Q})| = |\operatorname{coker}(\widetilde{\Psi}_{\chi,\beta})|.
$$

If Q has a complement in Γ (which will be our setting in the later sections), then the maps $\widetilde{\omega}_Q$ and $\widetilde{\omega}_{Q,\beta.T(\chi)}$ are surjective and the inflation condition on α will also be satisfied. The surjectivity of the augmentation is also the case when R is an Artinian ring [8, Lemma 3.4.].

Proof of Theorem 5.3. The first statement follows by combining Proposition 3.3 and Lemma 5.1. The second statement will follow from a diagram chasing argument starting from diagram (18). For notation simplicity we will write $\widetilde{\omega}$ for $\widetilde{\omega}_{Q,\beta}$ and $\widetilde{\omega}_\chi$ for $\widetilde{\omega}_{Q,\beta.T(\chi)}$.

We complete the rows to an exact sequence by adding their kernels and cokernels with the canonical embedding and projection denoted by i_1, π_1 (resp. i_Q, π_Q). Due to the commutativity of (18), $\pi_Q \circ \widetilde{\omega}_{Q\chi}$ induces a map F_{Cok} between the cokernels. All together we obtain the following diagram where all squares commute:

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[25] In the right column one should be careful with the notation $T(\chi)$. More precisely in the right upper corner $T(\chi) = [\chi \circ \alpha] \in H^2(G, R^*)$. In the right lower corner $T(\chi) = [\chi \circ \gamma] \in H^2(G/Q, R^*)$ with $\alpha = \operatorname{Inf}(\gamma)$ as in Equation (6). In particular $\operatorname{Tra}_\alpha(\chi) = \operatorname{Inf}_N(\operatorname{Tra}_{\gamma,N}(\chi))$ and so the going down arrows exist.

$$\ker(\widetilde{\Psi}_\chi) \overset{i_1}{\hookrightarrow} \mathcal{U}(R^\beta[\Gamma]) \xrightarrow{\ \widetilde{\Psi}_\chi\ } \mathcal{U}(R^{\beta.T(\chi)}[G]) \overset{\pi_1}{\twoheadrightarrow} \operatorname{coker}(\widetilde{\Psi}_\chi)$$

$$\downarrow{\scriptstyle \pi \circ \widetilde{\omega}} \qquad\qquad \downarrow{\scriptstyle \widetilde{\omega}_\chi} \qquad\qquad \downarrow{\scriptstyle F_{Cok}}$$

$$\frac{\mathcal{U}(R^\beta[\Gamma/Q])}{\ker(\widetilde{\Psi}_{\chi,Q})} \xrightarrow{\ \widetilde{\Psi}_{\chi,Q}\ } \mathcal{U}(R^{\beta.T(\chi)}[G/Q]) \xrightarrow{\ \pi_Q\ } \operatorname{coker}(\widetilde{\Psi}_{\chi,Q})$$

where π is the canonical epimorphism from $\mathcal{U}(R^\beta[\Gamma/Q])$ to $\frac{\mathcal{U}(R^\beta[\Gamma/Q])}{\ker(\widetilde{\Psi}_{\chi,N})}$. The Snake lemma applied to the diagram above now yields an exact sequence:

$$\ker(\pi \circ \widetilde{\omega}) \xrightarrow{\ \widetilde{\Psi}_{ker}\ } \ker(\widetilde{\omega}_\chi) \rightarrow \ker(F_{Cok}) \overset{d}{\rightarrow} \operatorname{coker}(\pi \circ \widetilde{\omega}) \rightarrow \operatorname{coker}(\widetilde{\omega}_\chi) \twoheadrightarrow \operatorname{coker}(F_{Cok})$$

with all the morphisms being the straightforward ones and d the connecting morphism.

Now assume that $\widetilde{\omega}$ and $\widetilde{\omega}_\chi$ are surjective. Then $\operatorname{coker}(\pi \circ \widetilde{\omega})$ is trivial and hence, going through the exact sequence above via isomorphism theorems, $\ker(F_{Cok}) \cong \operatorname{coker}(\widetilde{\Psi}_{ker})$. Also, $\operatorname{coker}(F_{Cok})$ is trivial thus all together we obtain the desired statement. $\quad\square$

5.2. On the kernel of the generalized transgression map

Unlike $\operatorname{coker}(\widetilde{\Psi}_\chi)$, the kernel of $\widetilde{\Psi}_\chi$ will usually be infinite as already seen with augmentation maps.

Example 5.4. When $N = \Gamma$ and χ the trivial character, then $\widetilde{\Psi}_\chi$ is simply the usual augmentation map from $\mathcal{U}(\mathbb{Z}[\Gamma])$ to $\mathcal{U}(\mathbb{Z}) = \{\pm 1\}$. We now see that $[\mathcal{U}(\mathbb{Z}[\Gamma]) : \ker(\widetilde{\Psi}_\chi)] = 2$ and therefore the kernel is infinite when $\mathcal{U}(\mathbb{Z}[\Gamma])$ is infinite which, by a theorem of Higman [26, Th. 1.5.6.], exactly happens when Γ is not an Hamiltonian 2-group or an abelian group of exponent dividing 4 or 6.

The advantage of the kernel is that thanks to Proposition 3.3 and Theorem 3.2 one has a significant amount of information about its elements. Combined with Theorem 4.1 one can describe finiteness of $\ker(\widetilde{\Psi}_\chi)$, in case of a central extension. Namely, $\ker(\widetilde{\Psi}_\chi)$ is finite if and only if $\mathcal{U}(R(\chi\prime)^{\beta.T(\chi\prime)}[G])$ is finite for all $\chi\prime \in \operatorname{Lin}(N, F)$ different of the given χ. Recall that $R(\chi\prime)$ denotes the smallest ring containing R and the values of $\operatorname{Im}(\chi\prime)$. Below we translate this to easily verifiable necessary conditions on G, N and χ

Theorem 5.5. *Suppose that the extension (17) is central (i.e. $N \subseteq \mathcal{Z}(\Gamma)$), $[\beta] \in \operatorname{im}(\operatorname{Inf}_N)$, $\chi \in \operatorname{Hom}(N, R^*)$ and R the ring of integers in a number field*[26] *F. Then,*

(1) *{torsion units in $\ker(\widetilde{\Psi}_{\chi,\beta})$} = $\{\chi(a)^{-1}a \mid a \in N\}$,*

---

[26] This restriction on the coefficient ring is only due to the restriction in Theorem 5.6. With more work, as in [53, Theorem III.1], one could probably take any domain of characteristic 0 such that no prime divisor of $|G|$ is divisible in R^*.

(2) *if* $\ker(\widetilde{\Psi}_\chi)$ *is finite then* $\mathcal{U}(R^\beta[\Gamma])$ *is finite or one of the following holds:*

- $N \cong C_p$ *for* $p \geq 5$ *prime,* $\Gamma \cong N \rtimes G, \chi \neq \omega_N$ *and* $\mathcal{U}(R^{\mathrm{Res}(\beta)}[G])$ *is finite,*
- $N \cong C_9, F = \mathbb{Q}, \chi$ *the faithful character and* G_β *abelian with* $\exp(G_\beta) \mid 6$
- $N \cong C_8, F = \mathbb{Q}, \chi$ *faithful and* G_β *abelian with* $\exp(G_\beta) \mid 4$,
- $\exp(N) \mid 4, 6$ *and if* N *is non-cyclic then* $\mathrm{lcm}(\exp(G), \exp(N)) \mid 4, 6$

To prove the second part we will need the next theorem which is a generalisation of the classical result of Berman and Higman on torsion units (see [36, Theorem 2.3.] or [53, Theorem III.1]). The proof is similar to the one given by Karpilovsky in the untwisted case. For completeness sake and convenience of the reader we include a proof.

Theorem 5.6. *Let* G *be a finite group,* R *the ring of integers in a number field* F *and* $[\beta] \in H^2(G, R^*)$. *If* $x = \sum_{g \in G} a_g u_g$ *is a torsion unit in* $R^\beta[G]$ *such that* $a_1 \neq 0$, *then* $x \in \mathcal{U}(R)$.

Proof. The proof is along the same lines as [36, Theorem 2.3.]. Let $u = \sum_{g \in G} r_g u_g$ be a torsion unit in $R^\beta[G]$, say of order m. Assume $r_1 \neq 0$. Let $n = |G|$ and consider the left regular representation $\rho : F^\beta[G] \to M_n(F)$. Then $\rho(u)$ is of finite order m (note that ρ is injective) and thus it is diagonalisable over the algebraic closure of F in $\mathbb{C}$ (recall that F is a number field). Its eigenvalues $\zeta_1, \cdots, \zeta_n$ are roots of unity, each of order a divisor of m. We get that $\zeta_1 + \cdots + \zeta_n = \sum_{g \in G} r_g \mathrm{tr}(\rho(u_g)) = nr_1$. Thus $|nr_1| = |\zeta_1 + \cdots + \zeta_n| \leq n$. Moreover, $|nr_1| = n$ if and only if $\zeta_1 = \zeta_2 = \ldots = \zeta_n$. If this holds then the diagonalisation of $\rho(u)$ is a multiple of the identity matrix, hence $\rho(u)$ is a central, i.e. $\rho(u) \in F$. Because ρ is injective this yields that $u \in F \cap R^\beta[G] = R$, as desired.

So, it remains to show that $|nr_1| = n$. Assume, the contrary, i.e. suppose $|nr_1| < n$. Let ϵ be a primitive m-th root of unity. Note that $r_1 \in \mathbb{Q}(\epsilon) \subseteq \mathbb{C}$. Since also $r_1 \in R$ and thus r_1 is an algebraic integer, it follows that r_1 is in the ring of integers of $\mathbb{Q}(\epsilon)$.

We claim that, for every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$, we have $|\sigma(nr_1)| < n$. Indeed, note that complex conjugation and elements of $\mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$ commute and thus if $n = |\sigma(nr_1)| = \sigma(nr_1)\overline{\sigma(nr_1)} = \sigma(nr_1)\sigma(\overline{nr_1})$ then also $n = \sigma^{-1}(\sigma(nr_1)\sigma(\overline{nr_1})) = |nr_1| < 1$. This proves the claim.

Next, the assumption says that $|r_1| < 1$ and from the claim we get that $|\sigma(r_1)| < 1$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$. It follows that the norm $0 \neq |N_{\mathrm{Gal}(\mathbb{Q}((\epsilon)/\mathbb{Q})}(r_1)| = |\prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})} \sigma(r_1)| = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})} |\sigma(r_1)| < 1$. However, this yields a contradiction as $N_{\mathrm{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})}(r_1)$ is an algebraic integer (see for example Proposition 4.1.8 in [26]) and thus belongs to $\mathbb{Z}$; so it can not be strictly between 0 and 1. □

It would be interesting to have a proof of Theorem 5.6 which makes a reduction to the well-known Berman-Higman theorem for group rings. So, this is for example an instance where understanding Question 4.3 would help. We now proceed to the proof of the main theorem

Proof of Theorem 5.5. The first part follows from a classical trick in untwisted group rings. Namely, first note that $\{\chi(a)^{-1}a \mid a \in N\}$ is indeed contained in $\ker(\widetilde{\Psi}_{\chi,\beta})$. Conversely, take a torsion unit $u \in \ker(\widetilde{\Psi}_{\chi,\beta})$. Because of Proposition 3.3, $\ker(\widetilde{\Psi}_{\chi,\beta}) = \mathcal{U}(1 + R^\beta[\Gamma]I_\chi)$ and thus $u = 1 + \sum_{g \in G} y_g u_{\mu(g)}$, with $y_g \in I_\chi = \mathrm{span}_R\{u_a - \chi(a)u_1 \mid a \in N\}$. In particular, if there would be no $1 \neq a \in N$ so that u_a is in $\mathrm{supp}(u)$, then the coefficient of u_1 in the expression of $u \in R^\beta[\Gamma]$ is equal to 1 and hence, by Theorem 5.6, $u \in \mathcal{U}(R)$. As $u \in \ker(\widetilde{\Psi}_{\chi,\beta})$ this implies $u = 1$.

Now assume there exists a non-trivial $a \in N \cap \mathrm{supp}(u)$ and consider the element $z := u.u_{a^{-1}}\chi(a)$. Note that, due to the centrality of N, the element z again is a torsion unit in $\ker(\widetilde{\Psi}_{\chi,\beta})$. However now $1 \in \mathrm{supp}(z)$ and therefore, by applying Theorem 5.6, $z = 1$. In other words, $u = \chi(a)^{-1}a$ as needed.

For the second part, we will *assume* that we are not in the trivial case that $\mathcal{U}(R^\beta[\Gamma])$ is finite. Then, by Theorem 3.2, we see that $\ker(\widetilde{\Psi}_\chi)$ is finite if and only if $\mathcal{U}(R(\chi\prime)^{\beta.T(\chi\prime)}[G])$ is finite for all $\chi \neq \chi\prime \in \mathrm{Lin}(N, F)$ but $\mathcal{U}(R(\chi)^{\beta.T(\chi)}[G])$ is infinite. The rest of the proof will reinterpret all those conditions simultaneously using Theorem 4.1. To start, $\mathcal{U}(R(\chi\prime))$ must also be finite for all $\chi \neq \chi\prime$. In particular $\mathcal{U}(R)$ needs to be finite and hence $F = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ with $d > 0$. The exact restrictions are recorded by the following claim.

Claim. *If $\mathcal{U}(RN)$ is finite then: (i) $\exp(N) = 2$ or (ii) $\exp(N) = 4$; $F = \mathbb{Q}, \mathbb{Q}(i)$ or (iii) $\exp(N) = 3, 6$; $F = \mathbb{Q}, \mathbb{Q}(\sqrt{-3})$.*

If $\mathcal{U}(RN)$ is infinite then: (iv) $N \cong C_p$ for a prime $p \geq 5$ or (v) $N \cong C_8, C_9$; $F = \mathbb{Q}$.

Decompose $FN \subseteq \mathcal{Z}(F^\beta[\Gamma])$ as in (12), i.e. using the theorem of Perlis and Walker: $FN = \bigoplus_{d||N|} F(\zeta_d)^{\oplus a_d}$, then there has to be at most one ζ_d for which the unit group of the ring of integers in $F(\zeta_d)$, denoted R_d, is infinite. Moreover, for this value we need $a_d = 1$. Note that this property is inhered by subgroups of N. Hence, if $N = N_1 \times N_2$, a direct product of two non-trivial subgroups, then, as $FN = FN_1 \otimes_F FN_2$, we get that all simple epimorphic images of both group algebras FN_1 and FN_2 must be such that the unit group of their respective ring of integers is finite; because otherwise we would have at least two distinct d and d' so that $F(\zeta_d)$ and $F(\zeta_{d'})$ are simple epimorphic images of FN and the unit groups $\mathcal{U}(R_d)$ and $\mathcal{U}(R_{d'})$ are infinite. Hence, if N is decomposable as a direct product then $\mathcal{U}(RN)$ is finite and thus, by Theorem 4.1, the exponent of N is a divisor of 4 or 6. The restrictions on F follow in the same way as at the end of the proof of Theorem 4.1, i.e. by determining when $\mathbb{Q}(\sqrt{-d}, \zeta_{\exp(N)})$ is at most of degree 2 over $\mathbb{Q}$.

On the other hand, if N is indecomposable and its exponent is not a divisor of 4 or 6, then N is cyclic and $N = p^n$ for some prime p. If $n > 1$ then FN has $F(\zeta_p)$ as a summand and R_p is finite only if $p = 2, 3$ and $F = \mathbb{Q}, \mathbb{Q}(\sqrt{-d})$ with $d > 0$. Thus either $n = 1$ or $p = 2, 3$. In the latter case, as R_{p^n} is finite only if $p^n = 2, 3, 4$ we obtain the claim about the form of N. The according restrictions on F can be proven with a similar argument as when $\exp(N) \mid 4, 6$.

Claim. *If N is cyclic, then χ is the unique faithful character. Moreover if $N = C_p$, then Γ is a split extension.*

If N is cyclic, then there is a bijection between the irreducible characters and the divisors d of $|N|$, say χ_d corresponds. Since we always assume that $\mathcal{U}(R^\beta[\Gamma])$ is infinite we know that there is a unique component of $\bigoplus_{d||N|} R[\chi_d]^{\beta.T(\chi_d)}[G]^{\oplus a_d}$ with infinite unit group. Theorem 4.1 implies that if $\mathcal{U}(R[\chi_d]^{\beta.T(\chi_d)}[G]^{\oplus a_d})$ is infinite, then also for a multiple of d. Hence indeed the only infinite component is $d = n$, i.e. the unique faithful character. If $|N|$ is odd, then by the previous claim $N = C_9$ or C_p with $p \geq 5$ prime. In both cases $\mathcal{U}(R^{\mathrm{Res}(\beta)}[G])$ is finite and hence $\exp(G_\beta) \mid 4, 6$. Therefore $\alpha(g, h)$ needs to divide both $lcm(o(g), o(h))$ and $p \geq 5$ which is only possible if $\alpha(g, h) = 1$ for all $g, h \in G$. Hence the extension is split if $N = C_p$.

We are now able to finish to obtain the desired restrictions in the case that $\mathcal{U}(\mathbb{Z}N)$ is infinite, i.e. $N = C_p, C_9, C_8$ by the first claim. For C_p the desired statement follows by the above claims and the fact that there are only two components in this case (hence the faithful one is the one different from the trivial representation, i.e. ω_N). For C_8 and C_9 it remains to prove the desired value on $\exp(G_\beta)$. For this note that $\chi = \omega_N$ yields $\exp(G_\beta) \mid 4, 6$ by Theorem 4.1. If $\exp(G) = 4$, then $\mathbb{Q}(\zeta_3)^\gamma[G]$ is infinite for any 2-cocycle by Theorem 4.1. Similarly for $\mathbb{Q}(i)^\gamma[G]$ when $3 \mid \exp(G_\beta)$. This finishes the $\mathcal{U}(\mathbb{Z}N)$ infinite case.

Finally suppose that $\mathcal{U}(\mathbb{Z}N)$ is finite. In this case $\exp(N) \mid 4, 6$ and F is of the form as in the first claim. Also $\exp(G) \mid \exp(G_{\beta T(\chi)}) \mid 4, 6$ for various characters χ of N. If N is cyclic, then the only restriction is that $\mathcal{U}(R^\beta[G])$ is finite. So suppose that N is not cyclic. If $\exp(N) = 4$ and $3 \mid |G|$, then there are several components having an element of order 12 which is not possible. Hence in that case $\exp(|G|)|4$. Similarly if $3 \mid \exp(N)$ and $4 \mid |G|$. Thus we indeed obtain that $lcm(\exp(G), \exp(N)) \mid 4, 6$ if N is non-cyclic. □

For applications it would be useful to also understand the torsion units in $\ker(\widetilde{\Psi}_{\chi,\beta})$ in case that the (17) is abelian (i.e. N is abelian but not necessarily central). The first part of Theorem 5.5 could be read as saying that a torsion unit in $\ker(\widetilde{\Psi}_{\chi,\beta})$ must be in $\mathbb{Z}[N]$. Indeed because N is abelian one can then use Theorem 5.6 to conclude that the torsion units are trivial and hence of the form $\chi(a)^{-1}a$. So we expect the following to be true.

Conjecture 5.7. *If the extension (17) is abelian, then $\{torsion\ units\ in\ \ker(\widetilde{\Psi}_{\chi,\beta})\} \subseteq \mathcal{U}(\mathbb{Z}[N])$. In particular all torsion units in the kernel are trivial.*

To finish this section, we illustrate the concepts of Section 3 and Section 5 on an example.

Example 5.8. Consider $\Gamma = C_4 \rtimes C_4 := \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1} \rangle$ which can be viewed in the following way as a central extension

$$1 \to \langle y_1 : y_1^2 = 1 \rangle \times \langle y_2 : y_2^2 = 1 \rangle \longrightarrow \Gamma \xrightarrow{\lambda} \langle x_1 : x_1^2 = 1 \rangle \times \langle x_2 : x_2^2 = 1 \rangle \to 1 \qquad (19)$$

where in fact $y_1 = a^2$ and $y_2 = b^2$. The epimorphism $\lambda : \Gamma \to C_2 \times C_2$ is determined by $\lambda(a) = x_1$ and $\lambda(b) = x_2$. Furthermore, we choose as a section $\mu : \langle x_1, x_2 \rangle \to \Gamma$ defined by

$$\mu(1) = 1, \mu(x_1) = a, \mu(x_2) = b, \mu(x_1 x_2) = ab$$

and consider the induced normalized cocycle $\gamma \in Z^2(\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle)$ (i.e. $\gamma(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}$). Explicitly:

$$\gamma(x_1, x_1) = y_1, \gamma(x_1, x_2) = 1, \gamma(x_1, x_1 x_2) = y_1$$
$$\gamma(x_2, x_1) = y_1, \gamma(x_2, x_2) = y_2, \gamma(x_2, x_1 x_2) = y_1 y_2$$
$$\gamma(x_1 x_2, x_1) = 1, \gamma(x_1 x_2, x_2) = y_2, \gamma(x_1 x_2, x_1 x_2) = y_2.$$

The corresponding class $[\gamma] \in H^2(\Gamma, C_2 \times C_2)$ is determined by

$$[x_1, x_2] = y_1, x_1^2 = y_1 \text{ and } x_2^2 = y_2.$$

Now consider the irreducible characters of $C_2 \times C_2 = \langle y_1, y_2 \rangle$, say $\mathrm{Irr}(\langle y_1, y_2 \rangle) = \{1, \chi_1, \chi_2, \chi_3\}$ with $\chi_1(y_i) = -1$, $\chi_2(y_1) = -1 = -\chi_2(y_2)$ and $\chi_3(y_1) = 1 = -\chi_3(y_2)$. Then using Theorem 3.2 we obtain that

$$\mathbb{Q}\Gamma \cong \mathbb{Q}[C_2 \times C_2] \oplus \bigoplus_{i=1}^{3} \mathbb{Q}^{T_\gamma(\chi_i)}[C_2 \times C_2].$$

It is easily seen that

$$\mathbb{Q}^{T_\gamma(\chi_1)}[C_2 \times C_2] \cong \left(\frac{-1, -1}{\mathbb{Q}} \right), \ \mathbb{Q}^{T_\gamma(\chi_2)}[C_2 \times C_2]$$
$$\cong \mathrm{M}_2(\mathbb{Q}) \text{ and } \mathbb{Q}^{T_\gamma(\chi_3)}[C_2 \times C_2] \cong 2\mathbb{Q}(i).$$

Therefore

$$\widetilde{\Psi}_{\chi_2} : \mathcal{U}(\mathbb{Z}\Gamma) \to \mathcal{U}(\mathbb{Z}^{T_\gamma(\chi_2)}[C_2 \times C_2])$$

is the only projection with infinite codomain. Moreover, by Theorem 5.5 the kernel is finite and is given by

$$\ker(\widetilde{\Psi}_{\chi_2}) = \langle -a^2, b^2 \rangle \cong C_2 \times C_2.$$

Also $\mathrm{coker}(\widetilde{\Psi}_{\chi_2})$ is finite by Theorem 5.3 and a precise description will follow from the methods in the upcoming sections.

6. Deforming bicyclic units via twisting and their contribution

In this section we will introduce a generic construction of units in twisted group rings (of finite groups) which generalizes the bicyclic units. They differ by a factor determined by the twisting. We will show in Section 6.2 that these units play a role analogous to the one of elementary matrices inside the special linear group, building on a generalisation of the Jespers-Leal theorem obtained in Theorem 6.3. Thereafter we return to the context of the previous section and investigate their contribution to $\mathrm{coker}(\Psi_\chi)$.

6.1. Generalized bicyclic units versus elementary matrices

Before specifying to twisted group rings, we wish to write down a general 'S-order version' of an important '$\mathbb{Z}$-order result' of Jespers and Leal [29] on bicyclic units which was only informally known to some experts. For this let B be a finite dimensional semisimple $\mathbb{Q}$-algebra, S a finite set of places of $\mathbb{Q}$ such that $\{\infty\} \subseteq S$ and $\mathbb{Z}_S$ the ring of S-integers in $\mathbb{Q}$. Further let A be a finite dimensional semisimple B-algebra and R an $\mathbb{Z}_S$-order in B.

Example 6.1. Let $S = \{\infty, (p_1), \ldots, (p_l)\}$ with p_i a prime number. Then $\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_l}]$ are the S-integers in $\mathbb{Q}$. If we now consider some root of unity ζ, then $R = \mathbb{Z}_S[\zeta] = \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_l}, \zeta]$ is a possible $\mathbb{Z}_S$-order in $B = \mathbb{Q}(\zeta)$. For this choice a first main example is $A = \mathbb{Q}(\zeta)^\alpha[G]$ for some $[\alpha] \in H^2(G, \mathbb{Q}(\zeta)^*)$ and a finite group G. Another frequent example is $A = M_n(D)$ with D a finite dimensional division $\mathbb{Q}$-algebra.

Let $\mathcal{A} = \{g_1, \ldots, g_n\}$ be a B-basis of A and $\{x_1, \ldots, x_q\}$ a $\mathbb{Z}_S$-basis of R. For every idempotent $f \in A$, there exists a minimal integer $n_f \in \mathbb{Z}$ such that $n_f f \in \mathrm{span}_R(\mathcal{A})$. Associated to these, one can now consider the elements

$$b(m_i x_i g, f) = 1 + n_f^2(1 - f)m_i x_i g f \text{ and } b(f, m_i x_i g) = 1 + n_f^2 f m_i x_i g(1 - f)$$

with $m_i \in \mathbb{Z}_S$ and $g \in \mathcal{A}$. As usual, these are units because $(1 - f)m_i x_i g f$ has square zero.

Definition 6.2. Let $\mathcal{F}$ be a set of idempotents in A. Then the elements in

$$\{b(m_i x_i g, f), b(f, m_i x_i g) \mid f \in \mathcal{F}, 1 \le i \le q, g \in \mathcal{A}, m_i \in \mathbb{Z}_S\}$$

are called the *generalized bicyclic units* corresponding to $\mathcal{F}$. The group generated by these is denoted by $\mathrm{GBic}^{\mathcal{F}}(A, R)$.

Classically only the generators with $m_i = 1$ are called generalized bicyclic units. The values m_i are added in order to still obtain all elements of $R\mathcal{A}$ in the middle between $(1 - f)$ and f (when taking the group generated by). Note that $R\mathcal{A}$ is an $\mathbb{Z}_S$-order in A.

We can now formulate the general version of [29] saying that generalized bicyclic units should be considered as the analogue of elementary matrices. The proof of [29] shows that Jespers and Leal handled the case $S = \{\infty\}$, i.e. of $\mathbb{Z}$-orders, and with A an (untwisted) group ring.

Theorem 6.3. *Let $e \in \mathrm{PCI}(A)$ such that $Ae \cong \mathrm{M}_n(D)$ with $n \geq 2$. Let $\mathcal{O}$ be a $\mathbb{Z}_S$-order in D. If f is an idempotent in A such that $fe \notin \mathcal{Z}(Ae)$, then there exists a non-zero $y \in \mathbb{Z}$ such that*

$$1 - e + E_n(y\mathcal{O}) \subseteq GBic^{\{f\}}(A, R)$$

Recall that, a set of matrix units in a simple algebra Ae is a set of elements $\{E_{i,j} : 1 \leq j \leq n\}$ in $\mathrm{SL}_1(Ae)$, where n is the reduced degree of Ae, such that $\sum_{i=1}^{n} E_{i,i} = 1$ and $E_{i,j}E_{p,q} = \delta_{j,p}E_{i,q}$. Concretely, fixing the isomorphism $Ae \cong \mathrm{M}_n(D)$ the element $E_{i,j}$ can be identified with the matrix having 1 in the (i,j) entry and zero elsewhere. For an ideal J in $\mathcal{O}$ we denote by $E_n(J)$ the subgroup of $SL_n(\mathcal{O})$ generated by the elements $e + rE_{i,j}$ for $i \neq j$ and $r \in J$.

Proof of Theorem 6.3. Multiplying generalized bicyclic units one sees that

$$\{1 + n_f^2 f\alpha(1-f), 1 + n_f^2(1-f)\alpha f \mid \alpha \in R\mathcal{A}\} \subseteq \mathrm{GBic}^{\{f\}}(A, R).$$

Following [26, Lemma 11.2.4]) one can decompose $fe = E_{1,1} + \cdots + E_{l,l}$ for some $0 < l < n$ and $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ a set of matrix units. In case $1 \leq i \leq l$ and $l + 1 \leq j \leq n$ one readily sees that by using $E_{i,j}E_{p,q} = \delta_{j,p}E_{i,q}$ the products can be rewritten as follows:

$$f\mathcal{O}E_{i,j}(1-f)e = \mathcal{O}E_{i,j} \text{ and } (1-f)\mathcal{O}E_{j,i}fe = \mathcal{O}E_{j,i}$$

for all $1 \leq i \leq l$ and $l + 1 \leq j \leq n$. Since $\mathcal{O}$ is an S-order it has a finite $\mathbb{Z}_S$-basis, which we denote $\mathcal{B}_\mathcal{O} = \{b_1, \ldots, b_k\}$. Now, for any i, j, [26, Lemma 4.6.9] yields a smallest number n_{ij} such that $(1 + bE_{i,j})^{n_{ij}} = 1 + n_{ij}bE_{i,j} \in R\mathcal{A}e$ for all $b \in \mathcal{B}_\mathcal{O}$. Consequently, if we also consider the smallest value $n_e \in \mathbb{Z}$ such that $n_e e \in R\mathcal{A}$, then

$$1 + n_f^2 n_e n_{ij} \mathcal{O}E_{i,j} \subset \mathrm{GBic}^{\{f\}}(A, R)$$

for all $1 \leq i \leq l$ and $l + 1 \leq j \leq n$. Similarly, for these i and j one has that $1 + n_f^2 n_e n_{ji} \mathcal{O}E_{j,i}$ is a subset of $\mathrm{GBic}^{\{f\}}(A, R)$. The other indices can now be reached by taking the appropriate commutators. For example if $1 \leq i, j \leq l, i \neq j, x \in \mathbb{Z}$ and $\alpha \in \mathcal{O}$ then

$$1 + x^2 \alpha E_{i,j} = (1 + x\alpha E_{i,l+1}, 1 + xE_{l+1,j})$$

and using the $E_{j,i}$ one also reaches the $l + 1 \leq i, j \leq n$. Appropriate choices of x now yield the desired result. $\quad\square$

Recall that the literature on the subgroup congruence problem yields that the difference between $E_n(y\mathcal{O})$ and the congruence subgroup of level y of $\mathrm{SL}_n(\mathcal{O})$ increases with y. Moreover if $n = 2$ and D is a division algebra containing an order with finitely many units,[27] then $[\mathrm{SL}_n(\mathcal{O}), E_n(y\mathcal{O})] = \infty$ starting already from rather small values of y. Therefore, for applications an answer to the following would be incredibly valuable.

Question 6.4. Suppose $n = 2$ and D is a division algebra containing an order with finitely many units. What is

(1) a tight upper bound for the scalar y of Theorem 6.3 in terms of the starting data?
(2) a tight upper bound for the scalar y in terms of the proportions $o(g)/o(ge)$ for $g \in G$?

Note that the proof above shows that one obtains the following value for $y \in \mathbb{Z}$:

$$y = \left(n_e\, n_f^2\, \mathrm{gcm}\{n_{ij}, n_{ji} : 1 \le i \le l, l+1 \le j \le n\} \right)^2. \tag{20}$$

Moreover if $n = 2$, then $x = \sqrt{y}$ suffices as a multiple. A limitation of the value in (20) is that the decomposition $fe = E_{1,1} + \cdots + E_{l,l}$ and the numbers n_{ij} are not explicit from the starting data in Theorem 6.3. A value for n_{ij} can be deduced by following the steps in the proofs of [26, Lemma 4.6.6 & 4.6.9], however the value would require to know too much of the isomorphism type of G and hence to weak for practical use. Nevertheless, according to [26, Lemma 4.6.9] there exists a $0 \ne r \in \mathbb{Z}_S$ such that $\mathrm{M}_n(r\mathcal{O}) \subseteq R\mathcal{A}e$ and that its additive index yields a 'uniform looking' upper-bound:

$$y = \left(n_e\, n_f^2\, [R\mathcal{A}e : \mathrm{M}_n(r\mathcal{O})] \right)^2. \tag{21}$$

Unfortunately in practice the additive index above is hard to compute.

6.2. Deforming bicyclic units - case of twisted group rings

From now on we consider the setting where $S = \{\infty\}$, i.e. $\mathbb{Z}$-orders, and $A = \mathbb{Q}(\zeta_n)^\alpha[G]$ where ζ_n is some primitive n-th root and $[\alpha] \in H^2(G, \mathbb{Z}[\zeta_n]^*) \subseteq H^2(G, \mathbb{Q}(\zeta_n)^*)$ arbitrary. The importance of this case for the study of $\mathbb{Q}[\Gamma]$ for Γ some central extension is highlighted by Theorem 3.2 and Proposition 3.3.

Idempotents from trivial units

Let $\alpha \in Z^2(G, \mathbb{Z}[\zeta_n]^*)$ be a 2-cocycle of finite order with $o(\alpha) \mid n$. Then, for $0 \le j < n$ we can partition G into the following sets,

$$\mathcal{G}_j^\alpha = \{g \in G \mid u_g^{o(g)} = \zeta_n^j\}. \tag{22}$$

---

[27] A division algebra contains an order with finite unit group if and only if $D = \mathbb{Q}(\sqrt{-d})$ or $\left(\frac{u,v}{\mathbb{Q}} \right)$ with $d \ge 0$ and $u, b < 0$, see [2, Theorem 2.10.].

The observation is now that if $g \in \mathcal{G}_0^\alpha = \{g \in G \mid o(u_g) = o(g)\}$, then $\widehat{u_g} = \frac{1}{o(g)}\widetilde{u_g}$ with

$$\widetilde{u_g} := u_1 + u_g + \ldots + u_g^{o(u_g)-1} \tag{23}$$

is again a non-trivial idempotent in $\mathbb{Q}(\zeta_n)^\alpha[G]$ if $g \neq 1$. Consequently, $\widetilde{u_g}(o(g) - \widetilde{u_g}) = 0$. If $g \in \mathcal{G}_j^\alpha$ for a non-zero j, then[28] $\widetilde{u_g} = 0$. In that case, one could instead take the hat $\widehat{(\cdot)}$ of $u_g^{n/gcd(j,n)}$ which would be an idempotent but in practice this construction will not be useful. Hence:

Question 6.5. Is there a generic way to produce a non-trivial idempotent from $g \in \mathcal{G}_j^\alpha$ for a non-zero j?

Remark 6.6. It is well known (see e.g. [35, Theorem 2.3.1]) that for a cyclic group $C_n = \langle g \rangle$ the second cohomology group over a commutative ring R is isomorphic to $R^*/(R^*)^n$. Due to this, the value of $u_g^{o(g)}$ is not uniquely determined by the cohomology class $[\alpha]$. For example, the class $[\alpha] \in H^2(C_n, \mathbb{C}^*)$ determined by the value $\lambda = u_g^{o(g)}$ is trivial for any $0 \neq \lambda \in \mathbb{C}$. Therefore the sets $\mathcal{G}_j^\alpha$ are unfortunately dependant of the chosen cocycle. However, one could define in an independent way the sets $\mathcal{G}_0$ and $\mathcal{G}_{\neq 0}$ where the latter would be all $g \in G$ such that $u_g^{o(g)}$ is not in the class of 1 in $R^*/(R^*)^n$. In particular for $R = \mathbb{Z}$ the situation simplifies and the value $u_g^{o(g)} \in \{\pm 1\}$ is uniquely determined by the class $[\alpha]$.

Despite the absence of an answer to Question 6.5, one can prove that under natural conditions the set $\mathcal{G}_0^\alpha$ yields enough idempotents to apply Theorem 6.3. More precisely that for $\mathcal{F} = \{\widehat{u_g} = \frac{1}{o(g)}\widetilde{u_g} \mid g \in \mathcal{G}_0^\alpha\}$ the group $\mathrm{GBic}^{\mathcal{F}}(\mathbb{Q}(\zeta_n)^\alpha[G], \mathbb{Z}[\zeta_n])$ contains sufficiently many elementary matrices.

Corollary 6.7. *Let G be a finite group and $\alpha \in Z^2(G, \mathbb{Z}[\zeta_n]^*)$ of finite order such that G_α has no fixed-point free non-abelian images.[29] Then for any $e \in \mathrm{PCI}(\mathbb{Q}(\zeta_n)^\alpha[G])$ such that $\mathbb{Q}(\zeta_n)^\alpha[G]e \cong M_m(D)$ with $m \geq 2$ we have that*

$$1 - e + E_m(y\mathcal{O}) \subseteq GBic^{\mathcal{F}}(\mathbb{Q}(\zeta_n)^\alpha[G], \mathbb{Z}[\zeta_n])$$

with $\mathcal{O}$ an order in D and some $y \in \mathbb{Z}$. In particular, if $\mathbb{Q}(\zeta_n)^\alpha[G]$ has no exceptional components then $GBic^{\mathcal{F}}(\mathbb{Q}(\zeta_n)^\alpha[G], \mathbb{Z}[\zeta_n])$ is of finite index in $\mathrm{SL}_1(\mathbb{Z}[\zeta_n]^\alpha[G])$.

---

[28] Going in the sum till the usual upper bound $o(g) - 1$ will not yield an idempotent.

[29] A finite group is called fixed point free if it has an irreducible complex representation ρ such that 1 is not an eigenvalue of $\rho(g)$ for all $1 \neq g \in G$. Such groups are exactly the Frobenius complements [26, Proposition 11.4.6.] and hence those that are subgroups of D^* for some finite dimensional division algebra D.

Recall that for a subring R of a semisimple algebra $A = \prod_{i=1}^{q} \mathrm{M}_{n_i}(D_i)$, h_i the projection on the i-th component and E_i a splitting field of $K_i := \mathcal{Z}(D_i)$ one defines

$$nr(r) = (\mathrm{Rnr}_{\mathrm{M}_{n_1}(D_1)/K_1}(h_1(r)), \ldots, \mathrm{Rnr}_{\mathrm{M}_{n_q}(D_q)/K_q}(h_q(r))) \tag{24}$$

with $\mathrm{Rnr}_{\mathrm{M}_{n_i}(D_i)/K_i}(h_i(r)) := \det(1_{E_i} \otimes_{K_i} h_i(r))$ the *reduced norm over* K_i and

$$\mathrm{SL}_1(R) = \ker(nr) = \{r \in \mathcal{U}(R) \mid \forall i : \mathrm{Rnr}_{\mathrm{M}_{n_i}(D_i)/K_i}(h_i(r)) = 1\} \tag{25}$$

the (multiplicative) group of reduced norm 1 elements. Also recall that a simple quotient of $\mathbb{Q}(\zeta_n)^{\alpha}[G]$ is called an *exceptional component* if it is either (I) a non-commutative division algebra different from a totally definite quaternion algebra or (II) of the form $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or $\mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right)$ with $a, b > 0$ and $d \in \mathbb{N}$. The division algebras appearing in these matrix algebras are exactly those having an order with finite unit group [2, Theorem 2.10.].

With the conditions assumed in Corollary 6.7, the proof of [26, Theorem 11.3.2] also works in this setting and therefore we will omit the details. In short, that G_α has no fixed point free images guarantees that for every $e \in \mathrm{PCI}(\mathbb{Q}(\zeta_n)^{\alpha}[G])$ there exists $g \in \mathcal{G}_0^\alpha$ so that $\widehat{u_g}e$ is a non-central idempotent. If $e_C \in \mathrm{PCI}(\mathbb{C}^{\alpha}[G]e)$ is such that $(\mathbb{C}^{\alpha}[G]e)e_C$ is non-commutative, then the element g is chosen among those for which $\rho(g)$ has eigenvalue one, where $\rho : \mathbb{C} \otimes_{\mathbb{Q}(\zeta_n)} \mathbb{Q}(\zeta_n)[G_\alpha] \to (\mathbb{C} \otimes_{\mathbb{Q}(\zeta_n)} \mathbb{Q}(\zeta_n)^{\alpha}[G]e)e_C$ is the complex representation factorising through the projection of $\mathbb{Q}(\zeta_n)[G_\alpha]$ on $\mathbb{Q}(\zeta_n)^{\alpha}[G]$ in Corollary 3.5. The last part follows from the known results on the subgroup congruence problem, e.g. see [26, Theorem 11.2.3.].

Convention. For an integral twisted group ring $\mathbb{Z}^{\alpha}G$, for any element of $g \in G$ of odd order, the restriction of $[\alpha]$ to $\langle g \rangle$ is trivial. Therefore we *may and will always from now on choose a normalized representative* α *such that* $u_g^{o(g)} = 1$ *in* $\mathbb{Z}^{\alpha}G$ *for all* g *of odd order.*

Deforming a bicyclic unit: construction

We will use the G-module structure of the twisted group ring $R^{\alpha}[G]$ described in [15, §3.2]

$$g(u_x) = u_g u_x u_g^{-1} = \alpha(g, x)\alpha(gxg^{-1}, g)^{-1} u_{gxg^{-1}}.$$

For every $x \in G$, the restriction of the conjugation representation from G to the centralizer $C_G(x)$ admits a 1-dimensional invariant subspace spanned by u_x. By the above this ordinary 1-dimensional representation $\chi_x : C_G(x) \to R^*$ is given by

$$\chi_x(g) = [u_g, u_x] = \alpha(g, x)\alpha(x, g)^{-1}.$$

For $R = \mathbb{Z}$, for any $x \in G$ we can partition $C_G(x)$ into two disjoint sets

$$C_x^+ = \{g \in C_G(x) \mid \chi_x(g) = 1\}, \quad C_x^- = \{g \in C_G(x) \mid \chi_x(g) = -1\}.$$

We remark that if $C_x^- = \emptyset$ then x (and its conjugacy class) is *called α-regular* and the set of all α-regular conjugacy classes forms a basis for the centre of $\mathbb{Z}^\alpha G$ (see [49]). An extreme case for that is when α is trivial. In this case $C_x^- = \emptyset$ for any $x \in G$.

Example 6.8. For $g, h \in G$ and $\alpha \in Z^2(G, R^*)$, a straightforward calculation yields that in $R^\alpha G$ if $[u_g, u_h] = \zeta \in R^*$ is a root of unity of order k, then k is a divisor of the greatest common divisor of the orders of g and h in G. Hence, for $\alpha \in Z^2(G, \mathbb{Z}^*)$, all elements of G of odd order are α-regular.

Using this we now introduce a new type of units.

Definition 6.9. With the above notation, the elements of the set

$$\{u_1 + y\widetilde{u_g}; \, u_1 + \widetilde{u_g}y \mid g \in \mathcal{G}_0^\alpha, y \in \mathbb{Z}^\alpha[G], \operatorname{supp}(y) \subseteq C_g^-\}$$

are called *H-units*. The subgroup of $\mathcal{U}(\mathbb{Z}^\alpha[G])$ generated by these is denoted $\mathcal{H}_\alpha(G)$.

At first it might seem surprising that the above elements are invertible in $\mathbb{Z}^\alpha[G]$. This is because the square of $y\widetilde{u_g}$ and $\widetilde{u_g}y$ are zero. Hence they are unipotent units with inverse $u_1 - y\widetilde{u_g}$, resp. $u_1 - \widetilde{u_g}y$. One way to see that those terms are indeed square zero, is by noticing that $y\widetilde{u_g} = (o(g) - \widetilde{u_g})\frac{y}{o(g)}\widetilde{u_g}$. This follows from the fact that $u_g^j u_h \widetilde{u_g} = (-1)^j u_h \widetilde{u_g}$ for $h \in C_g^-$ and hence when $o(g)$ is even[30] and $g \in \mathcal{G}_0^\alpha$

$$
\begin{aligned}
(o(g) - \widetilde{u_g})u_h\widetilde{u_g} &= u_h \left(o(g) - \sum_{j=0}^{o(g)-1} (-1)^j u_g^j \right) \widetilde{u_g} \\
&= u_h \left(o(g) - \sum_{j=0}^{o(g)-1} (-1)^j \right) \widetilde{u_g} \\
&= o(g)u_h\widetilde{u_g}.
\end{aligned}
\tag{26}
$$

Analogously $\widetilde{u_g}u_h(o(g) - \widetilde{u_g}) = o(g)\widetilde{u_g}u_h$. Thus the *H*-units are indeed elements of $\mathcal{U}(\mathbb{Z}^\alpha[G])$. Furthermore, for $\mathcal{F} = \{\widehat{u_g} = \frac{1}{o(g)}\widetilde{u_g} \mid g \in \mathcal{G}_0^\alpha\}$, every *H*-unit has an appropriate power that is in $\operatorname{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z})$. However, as will be pointed out in Remark 8.4, due to the factor $\frac{y}{o(g)}$ in general

$$\mathcal{H}_\alpha(G) \nsubseteq \operatorname{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z}). \tag{27}$$

---

[30] If $o(g)$ is odd, the last sum in (26) equals $o(g) - 1$ and hence in general the *H*-unit will not be an integral combination. However, as remarked in Section 6.2, elements of odd order are α-regular (i.e. $C_g^- = \emptyset$) and thus one can only take $y = 0$. In particular we don't need to specify the order of the elements from which we built the *H*-units in order to be a subset of $\mathbb{Z}^\alpha[G]$.

A concrete example will be given in Proposition 8.2. That $\langle \text{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z}), \mathcal{H}_\alpha(G) \rangle$ is not necessarily obtained by bicyclic units, even not up to commensurability, is truly its raison d'être and hence will be crucial for the applications.

Example. If α is a trivial cocycle, then $\mathcal{G}_j^\alpha = \emptyset = C_g^-$ for any $j \neq 0$, $g \in G$ and $\mathcal{G}_0^\alpha = G$. Therefore, in that case $\mathbb{Q}^\alpha[G] = \mathbb{Q}[G]$ and $\mathcal{H}_\alpha(G) = \{u_1\}$. In particular $\langle \text{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z}), \mathcal{H}_\alpha(G) \rangle = \text{Bic}(G)$, i.e. we recover exactly the classical bicyclic units.

Remark 6.10.

(1) The elements $u_1 + (o(g) - \widetilde{u_g}) \frac{u_h}{o(g)} \widetilde{u_g}$ and $u_1 + \widetilde{u_g} \frac{u_h}{o(g)} (o(g) - \widetilde{u_g})$ are in fact instances of a common more general construction of units. Indeed, consider $\alpha \in Z^2(G, \mathbb{Z}[\zeta_n]^*)$ with $o(\alpha) \mid n$ and let ξ be a $o(g)$-root of unity inside $\mathbb{Z}[\zeta_n]$. Then $u_1 + u_h \widetilde{\xi u_g}$ is an invertible element of $\mathbb{Z}[\zeta_n]^\alpha G$ with inverse $u_1 - u_h \widetilde{\xi u_g}$. Note that the choice $\xi = -1$ yields $u_1 + u_h \widetilde{(-u_g)} = u_1 + \widetilde{u_g} u_h = u_1 + \widetilde{u_g} \frac{u_h}{o(g)} (o(g) - \widetilde{u_g})$, as desired. As we will see in Section 10.1, an essential ingredient is that $[\widetilde{u_g}, u_h] = 0$.

(2) The H in the name H-unit refers to the cohomology group $H^2(G, \mathbb{Z}^*)$. Following (26), the elements in Definition 6.9 look like roots of bicyclic units. However these elements and those in the previous point fit in a general new generic construction of units which will be developed in Section 10. There the role of the H^2, i.e. splitting into extensions, will become more apparent. In Section 10 the units will in fact be called primitive H-units (as they are the smallest of their kind).

(3) Using the twisting as above one could instead have considered a deformed version of the classical bicyclic units, i.e. for any $x \in \mathbb{Z}^\alpha[G]$ all the elements of the form $1 + (o(g) - \widetilde{u_g})(x + \frac{y}{o(g)})\widetilde{u_g}$ are units. Such elements include both H-units and $\text{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z})$. However, as hopefully Section 10 will convince, it seems to be better to not think in terms of bicyclic units.

Thanks to (27), one can apply Corollary 6.7 to show that $\langle \text{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z}), \mathcal{H}_\alpha(G) \rangle$ contains sufficiently many elementary matrices. It follows from the proof of Corollary 6.7 that the value $y \in \mathbb{Z}$ is the one yielded by Theorem 6.3. However by adding the H-units, depending on the order of the twisted elements and the number of α-irregular elements, one can aim to decrease the value of y.

Question 6.11. What is a formula for the smallest $y \in \mathbb{N}$ such that $1 - e + E_m(y\mathcal{O}) \leq \langle \text{GBic}^{\mathcal{F}}(\mathbb{Q}^\alpha[G], \mathbb{Z}), \mathcal{H}_\alpha(G) \rangle$?

6.3. On H-units in cokernel of the transgression map

We now return to the setting of Section 3.2. More precisely we consider the central extension (3) corresponding to $[\alpha] \in H^2(G, A)$. We also fix a normalized representant $\alpha \in Z^2(G, A)$ of the form $\alpha(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}$ for a section μ of $\lambda : \Gamma \twoheadrightarrow G$.

The H-units from the previous section will now allow to construct elements in coker($\widetilde{\Psi}_\chi$). For this take elements $g \in \mathcal{G}_0^{\mathrm{Tra}_\alpha(\chi)}$ and $h \in C_g^-$. Note that the existence of such an element h indirectly assumes that g is $\mathrm{Tra}_\alpha(\chi)$-irregular. In particular $\mathrm{Tra}_\alpha(\chi)$ must be a non-trivial cocycle and hence $\chi \neq \omega_A$. Furthermore we will also consider a central subgroup Q of Γ such that $A \cap Q = 1$. Now with every such triple (g, h, Q) we consider the set

$$\mathcal{H}_{g,h,Q}^{min} := \{1 + z u_h \widetilde{u_g} \in \mathbb{Z}^{\mathrm{Tra}_\alpha(\chi)}[G] \mid z \in \{1 - u_q \: : \: q \in Q \text{ or } u_q = 0\}\}. \qquad (28)$$

Proposition 6.12. *Let $\chi \in \mathrm{Hom}(A, \mathbb{Z}^*), g \in \mathcal{G}_0^{\mathrm{Tra}_\alpha(\chi)}, h \in C_g^-$ and Q a central subgroup of Γ such that $A \cap Q = 1$. Suppose $\mathcal{U}(\mathbb{Z}G)$ is finite, then the following hold*

(1) *if $q \in Q \setminus \langle \mu(g) \rangle$ or $u_q = 0$, then $1 + (1 - u_q) u_h \widetilde{u_g} \notin \mathrm{im}\left(\widetilde{\Psi}_\chi \Big|_{\mathcal{U}(\mathbb{Z}[\Gamma])} \right)$*

(2) *if $Q \cap \langle \mu(g) \rangle = 1 = A \cap \langle \mu(g) \rangle$, then $\langle \mathcal{H}_{g,h,Q}^{min} \rangle \cong C_2^{|Q|}$ as subgroup of $\mathrm{coker}\left(\widetilde{\Psi}_\chi \Big|_{\mathcal{U}(\mathbb{Z}[\Gamma])} \right)$.*

Keeping in mind that $\{1 - q \mid 1 \neq q \in Q\}$ is a $\mathbb{Z}$-basis of $\ker(\omega_Q)$, a direct computation yields that

$$\langle \mathcal{H}_{g,h,Q}^{min} \rangle = \{1 + y u_h \widetilde{u_g} \in \mathbb{Z}^{\mathrm{Tra}_\alpha(\chi)}[G] \mid y \in \ker(\omega_Q) \cup \{1\}\}.$$

In particular, when $Q \cap \langle u_g \rangle = 1$ then $\langle \mathcal{H}_{g,h,Q}^{min} \rangle$ can be viewed as subgroup of coker($\widetilde{\Psi}_{ker}$). Note also that $qh \in C_g^-$ for all $q \in Q$ and hence $\langle \mathcal{H}_{g,h,Q}^{min} \rangle \leq \mathcal{H}_\alpha(G)$.

Proof of Proposition 6.12. Due to the centrality of $z = 1 - u_q$ it follows that $1 + z u_h \widetilde{u_g}$ is trivial if and only if $u_q \in \langle u_g \rangle$ (in particular for $z = 1$ it is never trivial). On turn this is equivalent to $q \in Q \setminus \langle \mu(g) \rangle$ (or q trivial) thanks to the assumption $o(g) = o(u_g)$. From now we assume that this is the case and let y be a sum of different such elements ($1 - u_q$ with $q \in Q \setminus \langle \mu(g) \rangle$ or 1). Then the unit $1 + (o(g) - \widetilde{u_g}) y \frac{u_h}{o(g)} \widetilde{u_g}$ is also non-trivial. We claim that more generally all such elements are not in the image.

An element in $\Psi_\chi^{-1}(1 + y u_h \widetilde{u_g})$ is of the form $\tau := 1 + y \mu(h)(1 + \mu(g) + \cdots + \mu(g)^{o(g)-1}) + x$ for some $x \in \ker(\Psi_\chi)$ and due to the non-triviality $\tau \neq 1$. We need to prove that $\tau \notin \mathcal{U}(\mathbb{Z}\Gamma)$ for any x. Assume otherwise and start by decomposing x according to Proposition 3.3:

$$x = \sum_{a \in \ker(\chi)} x_a(a - 1) + \sum_{a \in A \setminus \ker(\chi)} x_a(a + 1)$$

with $x_a = t_a + \sum_{1 \neq w \in G} t_{w,a} \mu(w) \in \mathbb{Z}[\Gamma]$.

Therefore, since $\mu(g)^{o(g)} \in A$,

$$\omega_A(\tau) = 1 + yh\widetilde{g} + 2 \sum_{a \in A \setminus \ker(\chi)} x_a.$$

Note that $yh\widetilde{g} \neq 0$ and in fact all its coefficients are ± 1 due to all the assumptions. Furthermore, $\omega_A(\tau)$ is a torsion unit since $\mathcal{U}(\mathbb{Z}G)$ is finite. Since the coefficient of the identity element of G is $1 + 2\sum_{a \in A \setminus \ker(\chi)} t_a$, which is non-zero, Theorem 5.6 yields that $\omega_A(\tau) = \pm 1$. However $\omega_A(\tau) = 1 + yh\widetilde{g} \mod 2 \neq 1$. This yields the desired contradiction and finishes the proof of the claim and in particular of (1).

For the second part, assume that $Q \cap \langle \mu(g) \rangle = 1 = A \cap \langle \mu(g) \rangle$. Note that $g \in \mathcal{G}_0^{\text{Tra}_\alpha(\chi)}$ implies that $\mu(g)^{o(g)} \in \ker(\chi)$. The stronger condition $A \cap \langle \mu(g) \rangle = 1$, equivalently $\mu(g)^{o(g)} = 1$, is assumed to have that $\Psi_\chi(\widetilde{\mu(g)}) = \widetilde{u_g}$. Denote $b_z := 1 + zu_h\widetilde{u_g}$. Note that $b_{z_1}b_{z_2} = 1 + (z_1 + z_2)u_h\widetilde{u_g}$. Moreover, all the b_z commute and in fact $\langle b_{1-q} \mid 1 \neq q \in Q \rangle$ is isomorphic to the additive group $\ker(\omega_Q)$. By the general claim every $b_{z_1} \ldots b_{z_l}$, for different $z_i \in \{1, 1 - u_q\}$, is not in the image. Thus it remains to prove that the square of every b_z is attained. For this rewrite

$$\begin{aligned} b_{1-q}^2 &= 1 + 2(u_h - u_{qh})\widetilde{u_g} \\ &= (1 + 2u_h\widetilde{u_g})(1 + 2u_{qh}\widetilde{u_g})^{-1} \\ &= \widetilde{\Psi_\chi(u_1)}\,\widetilde{\Psi_\chi(u_2^{-1})} \end{aligned}$$

where $u_1 = 1 + (1 - [\mu(g), \mu(h)])\mu(h)\widetilde{\mu(g)}$ and $u_2 = 1 + (1 - [\mu(g), \mu(qh)])\mu(qh)\widetilde{\mu(g)}$. Note that $[\mu(g), \mu(h)], [\mu(g), \mu(qh)] \in A \setminus \ker(\chi)$ because $h, qh \in C_g^-$. That u_1 and u_2 are units can be seen by rewriting them as bicyclic units, e.g. $u_1 = 1 + (1 - \mu(g))\mu(h)\widetilde{\mu(g)}$. $\square$

We expect that H-units contribute to $\text{coker}(\Psi_\chi)$ in full generality, in the way they do in Proposition 6.12.

Question 6.13. Is the conclusion of Proposition 6.12 also valid without the condition that $\mathcal{U}(\mathbb{Z}G)$ is finite?

Most likely, in general other type of elements can be contained in $\text{coker}(\widetilde{\Psi_\chi})$. However in Section 8 we will see that for Γ an extension of C_2 with an elementary abelian 2-group $G = C_2^n$ that these elements will generate the full cokernel. Consequently, in that case $\text{coker}(\widetilde{\Psi_{ker}}) = \ker(F_{Cok}) \cong C_2^{n-2}$. It would be interesting to when this happens.

Remark 6.14. With exactly the same proof, the statement of Proposition 6.12 also holds for the H-units $1 + \widetilde{u_g}u_h z$. Hence Question 6.13 can also be formulated for these elements. In fact it is also an interesting question to understand the image of $\langle b_1 := 1 + u_h\widetilde{u_g}, b_2 := 1 + \widetilde{u_g}u_h \rangle$. In case $o(u_h) = o(u_g) = o(g) = 2$, then a direct computation shows that $b_1 b_2^{-1} \in \text{im}\left(\widetilde{\Psi_\chi}\Big|_{\mathcal{U}(\mathbb{Z}[\Gamma])}\right)$. Hence $\langle b_1, b_2 \rangle \cong C_2$ as subgroup of $\text{coker}\left(\widetilde{\Psi_\chi}\Big|_{\mathcal{U}(\mathbb{Z}[\Gamma])}\right)$. It is however very likely that this is an order 2 phenomena.

7. Method to describe $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])$

Let G be a finite group and

$$C_2^n = \langle x_1 \rangle \times \cdots \times \langle x_n \rangle$$

an elementary abelian 2-group. In this section we will consider a, potentially trivial, 2-cohomology class $[\alpha]$ in the image of the inflation map

$$\text{Inf} : H^2(G, \mathbb{Z}^*) \to H^2(G \times C_2^n, \mathbb{Z}^*). \tag{29}$$

In particular, by Corollary 2.5, the subgroup $C_2^n = \langle u_{x_i} \rangle$ is central in $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])$. For such cocycles we will now present a method to describe $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])$ whenever $\mathcal{U}(\mathbb{Z}^\alpha[G])$ is known.

Convention. C_2^i denotes the concrete subgroup $\langle x_1, \ldots, x_i \rangle$. In particular when we write $\mathbb{Z}^\alpha[G \times C_2^i]$ we truly mean the subring of $\mathbb{Z}^\alpha[G \times C_2^n]$ generated by $G \times C_2^i$.

To start, consider for every $1 \le i \le n$ the natural projection

$$\psi_i : \mathbb{Z}^\alpha[G \times C_2^i] \to \mathbb{Z}^\alpha[G \times C_2^{i-1}]$$

induced by mapping x_i to 1, which is a ring morphism.[31] Note that, due to the centrality of x_i, $\mathbb{Z}^\alpha[G \times C_2^i] = \left(\mathbb{Z}^\alpha[G \times C_2^{i-1}] \right)[\langle x_i \rangle]$ and therefore ψ_i is globally defined by $\psi_i(u + v x_i) = u + v$ for $u, v \in \mathbb{Z}^\alpha[G \times C_2^{i-1}]$. By definition ψ_i induces an epimorphism

$$\widetilde{\psi}_i : \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^i]) \to \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^{i-1}]).$$

Accordingly, we obtain the following splitting.

Lemma 7.1. *Let $K_i = \ker(\widetilde{\psi}_i)$. Then, $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^i]) = K_i \rtimes \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^{i-1}])$. Consequently,*

$$\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n]) \cong K_n \rtimes \left(K_{n-1} \rtimes \left(\cdots K_2 \rtimes (K_1 \rtimes \mathcal{U}(\mathbb{Z}^\alpha G)) \right) \right).$$

Proof. Since $\widetilde{\psi}_i$ is surjective we obtain an extension

$$1 \to K_i \hookrightarrow \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^i]) \xrightarrow{\widetilde{\psi}_i} \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^{i-1}]) \to 1.$$

Clearly, this extension is split by the identity map. The second part now follows by iteration. $\square$

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[31] This is a ring-morphism due to the centrality of x_i. In particular for arbitrary 2-cocycles this method does not work.

In view of Lemma 7.1 we will now concentrate on describing all the kernels K_i. To this end we define the groups

$$U_i = \{1 + 2^i u \mid u \in \mathbb{Z}^\alpha G\} \cap \mathcal{U}(\mathbb{Z}^\alpha G).$$

Our goal is to show that K_i can be built from isomorphic copies of the groups $U_1, \ldots, U_i$ and the unit group $\mathcal{U}(\mathbb{Z}^\alpha G)$. Note that, by Theorem 5.6, U_i is torsion-free when $i \geq 2$ and the only torsion elements in U_1 are ± 1.

As a tool we need to consider more generally the groups

$$U_{k,j} = \{1 + 2^k u \mid u \in \mathbb{Z}^\alpha[G \times C_2^j]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^j]).$$

Note that $U_{k,0} = U_k$ and $U_{k_1,j_1} \leq U_{k_2,j_2}$ for $k_2 \leq k_1$ and $j_1 \leq j_2$. Next to these, the projections

$$\varphi_i : \mathbb{Z}^\alpha[G \times C_2^i] \to \mathbb{Z}^\alpha[G \times C_2^{i-1}] \text{ with } x_i \mapsto -1$$

will also be instrumental. The induced epimorphism on the unit groups is denoted by $\widetilde{\varphi}_i$.

Lemma 7.2. *For all $1 \leq k, j$, with notations as above, we have that*

- $K_j \cong \widetilde{\varphi}_j(K_j) = U_{1,j-1}$,
- $U_{k,j} \cong U_{k+1,j-1} \rtimes U_{k,j-1}$.

Proof. By definition,

$$K_j = \{1 + u(1 - x_j) \mid u \in \mathbb{Z}^\alpha[G \times C_2^{j-1}]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^j]).$$

Using the centrality of x_j, one gets that $\mathbb{Z}^\alpha[G \times C_2^j] = \left(\mathbb{Z}^\alpha[G \times C_2^{j-1}] \right) \langle x_j \rangle$ and thus one immediately obtains that $\widetilde{\varphi}_j$ is injective on K_j. Consequently, $K_j \cong \widetilde{\varphi}_j(K_j)$ and more explicitly

$$\widetilde{\varphi}_j(K_j) = \{1 + 2u \mid u \in \mathbb{Z}^\alpha[G \times C_2^{j-1}]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^{j-1}]) = U_{1,j-1}.$$

Note that $\widetilde{\varphi}_j(K_j)$ contains all the units above because $\mathbb{Z}^\alpha[G \times C_2^{j-1}]$ is truly meant as the subring of $\mathbb{Z}^\alpha[G \times C_2^j]$.

For the second statement, remark that $\widetilde{\psi}_j$ is the identity map on $U_{k,i}$ for all $0 \leq i < j$ and all k. Also, $\widetilde{\psi}_j(U_{k,j}) = \widetilde{\psi}_j(U_{k,j-1})$ where $U_{k,j-1}$ is the subgroup $U_{k,j} \cap \mathcal{U}(\mathbb{Z}^\alpha[G \times \langle x_1, \ldots, x_{j-1} \rangle])$. Combined we obtain the internal splitting

$$U_{k,j} = \ker(\widetilde{\psi}_j \big|_{U_{k,j}}) \rtimes U_{k,j-1}. \tag{30}$$

Next, since $\ker(\widetilde{\psi}_j\big|_{U_{k,j}}) = \{1 + 2^k u(1 - x_j) \mid u \in \mathbb{Z}^\alpha[G \times C_2^{j-1}]\} \cap U_{k,j}$ the map $\widetilde{\varphi}_j$ is

injective on it and so $\widetilde{\varphi}_j\left(\ker(\widetilde{\psi}_j\big|_{U_{k,j}})\right) \cong U_{k+1,j-1}$, finishing the proof. $\quad\square$

An iterative process of the previous lemma decreases the second index of $U_{k,j}$, with the cost of producing extra complements. Hence it now readily follows that we can reduce to the groups U_i. For the applications to classical group rings in Section 10 we however need an explicit internal splitting of K_i. As indicated in the proof of Lemma 7.2 such a splitting is available for $U_{k,j}$. Inspired by this we define for every tuple $\vec{j} = (j_1, \ldots, j_{i-1}) \in \{0, 1\}^{i-1}$ the following:

$$K_{\vec{j}} := \{1 + u(1 - x_1)^{j_1} \cdots (1 - x_{i-1})^{j_{i-1}}(1 - x_i) \mid u \in \mathbb{Z}^\alpha[G]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^i]). \quad (31)$$

Using that $gx_l(1 - x_l) = -g(1 - x_l)$, it is easily verified that $K_{\vec{j}}$ is a normal subgroup of $\mathcal{U}(\mathbb{Z}^\alpha[G \times \langle x_l : j_l \neq 0 \rangle])$. In general however it won't be normal in K_i.

Theorem 7.3. *Let G be a finite group, $[\alpha] \in \mathrm{Im}(\mathrm{Inf})$ as in (29). Then, for all $i \geq 2$,*

$$K_i := \overrightarrow{\underset{\vec{j} \in \{0,1\}^{i-1}}{\rtimes}} K_{\vec{j}} \tag{32}$$

where the internal semi-direct product is with respect to a specific ordering on $\{0, 1\}^{i-1}$, and $K_{\vec{j}} \cong U_{1+\sum_{t=1}^{i-1} j_t}$ for $\vec{j} = (j_1, \ldots, j_{i-1}) \in \{0, 1\}^{i-1}$. Furthermore,

$$\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n]) = K_n \rtimes \left(K_{n-1} \rtimes \left(\cdots K_2 \rtimes (K_1 \rtimes \mathcal{U}(\mathbb{Z}^\alpha G)) \right) \right)$$
$$= \left(\left((N_n \rtimes N_{n-1}) \rtimes \cdots \right) \rtimes N_1 \right) \rtimes \left(\mathcal{U}(\mathbb{Z}^\alpha[G]) \times \langle x_1, \ldots, x_n \rangle \right)$$

for some torsion-free normal subgroup N_i of K_i such that $K_i \cong N_i \rtimes \langle x_i \rangle$.

Remark. The ordering in (32) is deducible from the proof. In particular, in terms of the U_i the decomposition of K_i can be defined recursively as follows: $K_1 \cong U_1$. For $i \geq 2$, $K_i \cong \mathcal{U}_i \rtimes K_{i-1}$ with $\mathcal{U}_2 := U_2$ and $\mathcal{U}_i := \mathcal{U}_{i-1}[1] \rtimes \mathcal{U}_{i-1}$ where $\mathcal{U}_{i-1}[1]$ looks like $\mathcal{U}_{i-1}$ but with all indices increased by one. For example, $K_2 \cong U_2 \rtimes K_1$, $K_3 \cong (U_3 \rtimes U_2) \rtimes K_2$, $K_4 \cong ((U_4 \rtimes U_3) \rtimes (U_3 \rtimes U_2)) \rtimes K_3$ and

$$K_5 \cong \left(\left((U_5 \rtimes U_4) \rtimes (U_4 \rtimes U_3) \right) \rtimes \left((U_4 \rtimes U_3) \rtimes (U_3 \rtimes U_2) \right) \right) \rtimes K_4.$$

Remark. For certain class of nilpotent groups G, Jespers, Leal and del Río give in [30, Proposition 5] a description of $\mathcal{U}(\mathbb{Z}G)$ in terms of some subnormal series. The 2-groups in their class are of the form $H \times C_2^n$ for some n and $H \leq G$. In that case their subnormal series coincides with the one in Theorem 7.3.

Proof of Theorem 7.3. The description of K_i follows from an iterative use of Lemma 7.2 or rather the explicit form in (30). In fact we will prove more generally that

$$U_{k,i} = \overrightarrow{\underset{\vec{j}\in\{0,1\}^i}{\rtimes}} U_{k,\vec{j}}$$

where $U_{k,\vec{j}} = \{1 + 2^k u(1-x_1)^{j_1}\cdots(1-x_i)^{j_i} \mid u \in \mathbb{Z}^\alpha[G]\}\cap\mathcal{U}(\mathbb{Z}^\alpha[G\times C_2^i])$. To start one uses

$$K_i \cong \widetilde{\varphi}_i(K_i) \cong U_{1,i-1} \cong U_{2,i-1}\rtimes U_{1,i-2} \cong U_{2,i-1}\rtimes K_{i-1}.$$

Explicitly the copy of $U_{2,i-1}$ in K_i is given by $\{1 + u(1-x_{i-1})(1-x_i)\in K_i \mid u \in \mathbb{Z}[G\times\langle x_1,\ldots,x_{i-2}\rangle]\}$ and the copy of K_{i-1} is $\{1 + v(1-x_i)\in K_i \mid v \in \mathbb{Z}[G\times\langle x_1,\ldots,x_{i-2}\rangle]\}$. In terms of $U_{1,i-1}$ the $1-x_i$ is identified with a factor 2 via $\widetilde{\varphi}$. Now, if $i = 2$ the procedure finishes here and yields the decomposition (32). Next, for an arbitrary i one case use now induction on the copy of K_{i-1} inside K_i. This yields all the terms $K_{\vec{j}}$ such that $j(i-1) = 0$ (i.e. no x_{i-1} in the support). The other terms will appear by applying induction to $U_{2,i-1}$ (in the recursive process this corresponds to applying the second part of Lemma 7.2 to $U_{2,i-1}$).

The first equality of the second part was proven in Lemma 7.1. Furthermore, it is easy to see that the brackets can be rearranged to $\mathcal{U}(\mathbb{Z}^\alpha[G\times C_2^n]) = \Big(\big((K_n\rtimes K_{n-1})\rtimes\cdots\big)\rtimes K_1\Big)\rtimes\mathcal{U}(\mathbb{Z}^\alpha G)$. Hence due to the centrality of the subgroup C_2^n, it remains to prove the existence of such N_i that is normal in K_{i-1}. For this notice that (32) yields only one subgroup isomorphic to U_1, namely $K_{\vec{0}} = \{1 + (1-x_i)u \mid u \in \mathbb{Z}^\alpha[G]\}\cap\mathcal{U}(\mathbb{Z}^\alpha[G\times\langle x_i\rangle])$. All the other $K_{\vec{j}}$ will be isomorphic to U_l with $2\leq l\leq i$ and hence are torsion-free. Now, under the isomorphism $K_{\vec{0}}\cong U_1$, the subgroup $\langle x_i\rangle$ corresponds to $\{\pm 1\}$ which is the only torsion in U_1. Therefore $K_{\vec{0}}\cong F_i\times\langle x_i\rangle$ for some torsion-free subgroup F_i. Taking F_i together with all the $K_{\vec{j}}$ with $\vec{j}\neq\vec{0}$ we obtain the desired torsion-free subgroup N_i of K_i such that $K_i\cong N_i\times\langle x_i\rangle$. $\square$

The decomposition obtained allows to transfer properties of $\mathcal{U}(\mathbb{Z}^\alpha G)$ to $\mathcal{U}(\mathbb{Z}^\alpha[G\times C_2^n])$. Earlier interesting results in that line can be found in [3,17,41,18].

Corollary 7.4. *Let G be a finite group, $[\alpha]\in\mathrm{Im}(\mathrm{Inf})$ as in (29). If $\mathcal{U}(\mathbb{Z}^\alpha G)$ satisfies one of the following properties:*

(1) *$\pm G$ has a normal (torsion-free) complement in $\mathcal{U}(\mathbb{Z}^\alpha[G])$*
(2) *it is commensurable with a direct product of free-by-free groups*
(3) *has a non-trivial amalgam decomposition*
(4) *G satisfies (HSP)[32]*

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[32] (HSP) stands for the Higman subgroup property, i.e. any finite subgroup in $V(\mathbb{Z}^\alpha G)$ is isomorphic to a subgroup in G. See also [45].

then the same holds for $G \times C_2^n$ and $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])$.

In the case of untwisted[33] group rings it was proven in [18] that the first Zassenhaus conjecture (ZC1) is preserved. It would be interesting to prove so for twisted group rings. Especially since the counterexample to (ZC1) of Eisele and Margolis [14], positive instances of the Zassenhaus conjectures are of special interest. For a recent survey see [46].

Proof of Corollary 7.4. The first part directly follows from the last decomposition in Theorem 7.3, since $\big((N_n \rtimes N_{n-1}) \rtimes \cdots\big) \rtimes N_1$ is torsion-free.

Next, for the second statement, let A be a finite dimensional semisimple $\mathbb{Q}$-algebra and $\mathcal{O}$ an order in A. The desired statement is a direct consequence of the fact that '$\mathcal{U}(\mathcal{O})$ commensurable with a direct product of free-by-free groups' is fully determined by the type of the simple components of the algebra A, as follows from [33, Theorem 2.1]. For instance, [33, Lemma 3.1. & Proposition 3.3.] (or [27]) yields that[34] the property holds if and only if the simple quotients of A are either a field, a totally definite quaternion algebra or a matrix algebra $\mathrm{M}_2(D)$ with D in a list of certain quadratic imaginary extensions of $\mathbb{Q}$ or certain quaternion algebras over totally real number fields. However if $[\alpha]$ is in the image of (29), then

$$\mathbb{Q}^\alpha[G \times C_2^n] \cong \mathbb{Q}^\alpha[G] \otimes_\mathbb{Q} \mathbb{Q}[C_2^n] \cong \underbrace{\mathbb{Q}^\alpha[G] \oplus \cdots \oplus \mathbb{Q}^\alpha[G]}_{2^n - times}.$$

Thus the isomorphism types of simple quotients of $\mathbb{Q}^\alpha[G \times C_2^n]$ are the same as for $\mathbb{Q}^\alpha[G]$.

Concerning the third statement, note that, due to Theorem 7.3, $\mathcal{U}(\mathbb{Z}^\alpha[G])$ is an epimorphic image of $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])$ and hence its amalgam decomposition can be lifted along the epimorphism.

For the fourth statement, first notice that by induction we may assume that $n = 1$. Hence, with the above notation, $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2]) = K_1 \rtimes \mathcal{U}(\mathbb{Z}^\alpha[G])$ with $K_1 \cong N_1 \times \langle x_1 \rangle$ where N_1 is a torsion-free normal subgroup. For (SIP), assume now that H is a finite subgroup of $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2])$. As N_1 is torsion-free, $H \cong H/(H \cap N_1)$ and so we may consider H as a subgroup of $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2])/N_1 \cong \mathcal{U}(\mathbb{Z}^\alpha[G]) \times C_2$. Consequently, there exist finite index sets I and J and $t_i, k_j \in \mathcal{U}(\mathbb{Z}^\alpha[G])$ such that $H/(H \cap N_1) = \{t_i, x_1 k_j \mid i \in I, \, j \in J\}$. Now consider the larger group $T = \langle t_i, k_j, x_1 \mid i \in I, \, j \in J \rangle$. Using that x_1 is a central element of order 2, it is directly checked that T is a (finitely generated) torsion group. Since it is also linear, T is in fact finite. By assumption $\langle t_i, k_j \mid i \in I, \, j \in J \rangle$ is

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[33] The classical Zassenhaus conjectures are stated for group rings, however also literally makes sense for twisted group rings. For example (ZC3) would be that all finite $H \leq U(\mathbb{Z}^\alpha[G])$ are conjugated over $\mathbb{Q}^\alpha[G]$ to a subgroup of G.

[34] To conclude this it is used that free groups are exactly the groups with cohomological dimension one [56,57]. Due to this free-by-free groups have virtual cohomological dimension at most two which is the content of [33, Section 3].

isomorphic to a subgroup of G and hence T to one of $G \times C_2$. Consequently, also the smaller group $H \cong H/(H \cap N_1)$ also as needed. $\quad\square$

Remark 7.5. The proof of (2) in Corollary 7.4 in fact shows that any group theoretical property $\mathcal{P}$ that can be read off the Wedderburn-Artin components (i.e. the simple quotients) is inherited. Other examples of such properties are: Kazhdan's property (T), property FAb and HFA (see [1,2]). In general, good candidates for such properties are the ones that are constant on commensurability classes.

Following Corollary 7.4 the property to have a (torsion-free) normal complement is preserved, however a concrete description thereof seems difficult. For example, in [40] it was proven that no normal complement of the trivial units in $\mathcal{U}(\mathbb{Z}[D_8 \times C_2 \times C_2])$ is generated by bicyclic units, although a normal complement in $\mathcal{U}(\mathbb{Z}[D_8 \times C_2])$ is $\mathrm{Bic}(D_8 \times C_2)$ [23]. For $D_8 \times C_2 \times C_2$ it is unknown whether bicyclic units nevertheless form a subgroup of finite index. Recently Bächle, Maheshwary and Margolis have proven [4, Theorem A] that rank B^{ab} =rank $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ where B is the group generated by bicyclic and Bass units in Γ, B^{ab} denotes the abelianisation of B and the rank is as finitely generated abelian group. As a consequence, they deduced that if the bicyclic units are of finite index[35] in $\mathrm{SL}_1(\mathbb{Z}\Gamma)$ then rank $\mathcal{U}(\mathbb{Z}\Gamma)^{ab}$ =rank $\mathcal{Z}(\mathcal{U}(\mathbb{Z}\Gamma))$. One can read this result as a new method to detect whether the bicyclic units are of infinite index in $\mathrm{SL}_1(\mathbb{Z}\Gamma)$, which happens if rank $\mathcal{U}(\mathbb{Z}\Gamma)^{ab} \gneq$ rank $\mathcal{Z}(\mathcal{U}(\mathbb{Z}\Gamma))$. This motivates the following question.

Question 7.6. With notations as above, what is the connection between rank $\mathcal{U}(\mathbb{Z}^\alpha G)^{ab}$ and rank $\mathcal{U}(\mathbb{Z}^\alpha[G \times C_2^n])^{ab}$?

8. Full description for elementary abelian 2-groups

Let G be an elementary abelian 2-group of rank $n + 2$,

$$G = C_2 \times C_2 \times \ldots \times C_2 = \langle g \rangle \times \langle h \rangle \times \langle x_1 \rangle \times \ldots \times \langle x_n \rangle, \tag{33}$$

and consider the cohomology class $[\alpha] \in H^2(G, C_2)$ determined by the values

$$[u_g, u_h] = -1, \quad u_g^2 = u_h^2 = u_{x_i}^2 = [u_g, u_{x_i}] = [u_h, u_{x_i}] = [u_{x_i}, u_{x_j}] = 1, \tag{34}$$

for any $1 \le i \ne j \le n$. We can think of this $[\alpha]$ as the cohomology class determining the group $\Gamma \cong D_8 \times C_2^n$ and the twisted group ring $\mathbb{Z}^\alpha G$. Recall our standing convention of choosing a normalized 2-cocycle representant of $[\alpha]$.

To start, we handle the case that $n = 0$, i.e. we describe $\mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle])$.

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[35] This is equivalent to say that the bicyclic units together with the Bass units are of finite index in $\mathcal{U}(\mathbb{Z}\Gamma)$ due to a Theorem of Bass and Milnor, see [26, Theorem 11.1.2. & Prop. 9.5.11.].

8.1. The starting case $C_2 \times C_2$

Denote by $\widetilde{D}$ the subalgebra of $M_2(\mathbb{Z})$ consisting of the elements

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid a \equiv_2 d,\ b \equiv_2 c \right\}.$$

It easily can be verified that

$$\widetilde{D} = \left\{ \begin{pmatrix} m+n & k+r \\ k-r & m-n \end{pmatrix} \in M_2(\mathbb{Z}) \mid m, n, k, r \in \mathbb{Z} \right\}. \tag{35}$$

Consider now the $\mathbb{Z}$-linear map $\phi : \mathbb{Z}^\alpha[C_2 \times C_2] \to \widetilde{D}$ defined by

$$u_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u_h \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_{gh} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{36}$$

which can easily be seen to be a ring morphism.

Proposition 8.1. *With notations as above, $\mathbb{Z}^\alpha[C_2 \times C_2] \cong \widetilde{D}$ as rings.*

Proof. The map ϕ is surjective because

$$A = \phi\left(mu_1 + nu_g + ku_h + ru_{gh}\right),$$

for an arbitrary element $A = \begin{pmatrix} m+n & k+r \\ k-r & m-n \end{pmatrix}$ of $\widetilde{D}$ (see (35)). Injectivity follows from a linear independence argument. $\square$

Next we will obtain a presentation of the unit group $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$. Already in this small example the role of H-units is decisive, as they will yield a normal complement for the trivial units. As expanded at the end of this section, see Remark 8.4, the generalized bicyclic units together with the trivial units are not sufficient to generate the full unit group. Furthermore, without them we would not be able to handle later on, even not up to commensurability, the elementary abelian 2-group case.

Proposition 8.2. *Let α be the 2-cocycle determined by the values $(1, 1, -1)$ on $(u_g^2, u_h^2, [u_g, u_h])$. Then*

$$\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2]) = F_2 \rtimes D_8$$

where $D_8 = \langle u_g, u_h \rangle$ and $F_2 = \langle v, w \rangle$ is a free group of rank 2 generated by the H-units $v = u_1 + u_h - u_{gh}$ and $w = u_1 + u_h + u_{gh}$. The action of D_8 on F_2 is defined by $u_h^{-1} v u_h = w,\ u_h^{-1} w u_h = v$ and $u_g^{-1} x u_g = x^{-1}$ for $x = v, w$. Furthermore,

$$\mathrm{SL}_1(\mathbb{Z}^\alpha[C_2 \times C_2]) = \mathcal{H}_\alpha(C_2 \times C_2) = \langle v, w, u_{gh} \rangle.$$

In particular, it is a subgroup of index 2 in $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$. Also, $\phi(\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2]))$ has index 3 in $\mathrm{GL}_2(\mathbb{Z})$.

Recall that $\mathrm{SL}_1(\mathbb{Z}^\alpha[C_2 \times C_2])$ denotes the group of reduced norm 1 elements, see (25).

Proof. The elements v and w are examples of H-units, see Definition 6.9 and (26). Moreover, their respective image under ϕ is equal to

$$\phi(v) = \phi(1 + u_h - u_{gh}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and

$$\phi(w) = \phi(1 + u_h + u_{gh}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Thus the images $\{\phi(v), \phi(w)\}$ generate a well-known free group of rank 2, sometimes called the Sanov subgroup, which is of index 24 in $\mathrm{GL}_2(\mathbb{Z})$ (see e.g. [39, Example 2]). Therefore, $\{v, w\}$ generates a free group of rank 2 in $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$, say of index r. Denote this subgroup by H and denote by $T\left(\tilde{D}\right) = \pm\{u_1, u_g, u_h, u_{gh}\}$ the trivial units of $\tilde{D}$. It is easy to verify that $T\left(\tilde{D}\right) \cong D_8$, the dihedral group of order 8, and moreover for any $x \neq y \in \phi^{-1}(T(\tilde{D}))$, the cosets $xH \neq yH$. Consequently, $[U(\mathbb{Z}^\alpha[C_2 \times C_2]) : H] \geq 8$. As can be seen with the matrices $\phi(u_h)$ and $1 + E_{12}$, the subgroup $\phi(\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])) = \mathcal{U}(\tilde{D})$ is not normal in $GL_2(\mathbb{Z})$ and therefore its index is at least 3. Since

$$24 = [GL_2(\mathbb{Z}) : H] = [GL_2(\mathbb{Z}) : \mathcal{U}(\tilde{D})].[\phi(\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])) : H] \geq 3 \cdot 8,$$

we conclude that $[\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2]) : H] = 8$. A simple calculation shows that

$$u_h^{-1} v u_h = w, \quad u_g^{-1} v u_g = v^{-1}, \quad u_{gh}^{-1} v u_{gh} = w^{-1} \text{ and } u_g^{-1} w u_g = w^{-1}.$$

Therefore H is normal in $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$, and consequently

$$\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2]) = H \rtimes \phi^{-1}(T(\tilde{D})) \cong F_2 \rtimes D_8.$$

The second part of the statement can be checked explicitly. To start, note that $\mathcal{G}_0^\alpha = \{u_h, u_g, u_1\}, C_g^- = \{u_h, u_{gh}\}$ and $C_h^- = \{u_g, u_{gh}\}$. So in total one obtains 8 non-trivial H-units that generate $\mathcal{H}_\alpha(C_2 \times C_2)$. However they all can be expressed in terms of v, w, u_{gh}. Indeed $1 + u_{gh}\widetilde{u_g} = 1 - u_h u_g \widetilde{u_g} = w^{-1}$ and $1 + u_g \widetilde{u_h} = 1 + u_g + u_g u_h = u_{gh} v$. The others are similar and, in particular, this shows that $\mathcal{H}_\alpha(C_2 \times C_2) = \langle v, w, u_{gh} \rangle$.

Finally, recall that $u_{gh}^2 = -1$ and thus $[D_8 : \langle u_{gh} \rangle] = 2$. In particular, from the description of $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$ obtained earlier, we now see that $\mathcal{H}_\alpha(C_2 \times C_2)$ is a subgroup of index 2 in $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$. Besides, because H-units are unipotent, $\mathcal{H}_\alpha(C_2 \times C_2)$ is a subgroup of $\mathrm{SL}_1(\mathbb{Z}^\alpha[C_2 \times C_2])$. Since the latter does not contain u_h, as seen from the matrix representation, it is a proper subgroup of $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$ and hence indeed is equal to $\mathcal{H}_\alpha(C_2 \times C_2)$. That $\phi(\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2]))$ has index 3 in $\mathrm{GL}_2(\mathbb{Z})$ follows from the fact that $\langle \phi(v), \phi(w) \rangle$ has index 24 in $\mathrm{GL}_2(\mathbb{Z})$ and $\phi(\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])) = \langle \phi(v), \phi(w) \rangle \rtimes D_8$. $\square$

The above proof shows the importance of having understanding on subgroup of small index in $\mathrm{SL}_2(\mathbb{Z})$.

8.2. The general case

Now we consider the general case of $G = \langle g, h \rangle \times C_2^n$. More precisely, we will follow the method outlined in Section 7 and therefore describe the isomorphism type of the groups $U_1, \ldots, U_n$. Combined with Theorem 7.3 this would reduce a full description of $\mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle \times C_2^n])$ to describing the actions in the semi-direct products. *We use freely the objects and notations introduced in Section 7.*

Proposition 8.3. *The groups U_i satisfy the following:*

- $[\mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle]) : U_1] = 8$,
- $[U_i : U_{i+1}] = 8$ *for all $i \geq 1$,*
- $U_1 \cong F_3 \times C_2$,
- *If $i \geq 2$, then U_i is a free group of rank $n_i = 1 + 8^{i-1}$.*

Proof. Recall that, by definition, $U_i = \{1 + 2^i u \mid u \in \mathbb{Z}^\alpha[\langle g, h \rangle]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle])$. In particular, elements of U_1 are of the form $1 + 2u$ for $u \in \mathbb{Z}^\alpha[\langle g, h \rangle]$ and hence the only trivial units of $\mathbb{Z}^\alpha[\langle g, h \rangle]$ contained in U_1 are ± 1. In fact, using the notations of the proof of Proposition 8.2, one has that $A = \langle v^2, w^2, wv^{-1}, -1 \rangle \subseteq U_1$. Indeed, recall that $v = 1 + u_h - u_{gh}$ and $w = 1 + u_h + u_{gh}$, hence a simple calculation yields that

$$v^2 = 1 + 2u_h - 2u_{gh} = 1 + 2(u_h - u_{gh}), \quad w^2 = 1 + 2u_h + 2u_{gh} = 1 + 2(u_h + u_{gh}).$$

Additionally, $v^{-1} = 1 - u_h + u_{gh}$ and hence

$$wv^{-1} = -1 - 2u_g + 2u_{gh} = 1 + 2(-1 - u_g + u_{gh}).$$

We conclude that indeed $A \subseteq U_1$. However v cannot be written in the form $1 + 2u$ (which already can be seen mod 2), thus $A \leq U_1 \leq \langle v, w \rangle \times \langle -1 \rangle$. Note that $[\langle v, w \rangle : \langle v^2, w^2, vw^{-1} \rangle] = 2$ as the product of any two generators of $\langle v, w \rangle$ is in $\langle v^2, w^2, vw^{-1} \rangle$. Thus $A = U_1$. Now recall the Nielsen-Schreier formula that says that a subgroup of

index t in a free group of rank e is a free group of rank $1 + t(e-1)$. Consequently, since $\langle v, w \rangle \cong F_2$ by Proposition 8.2, we obtain all together that

$$U_1 = \langle v^2, w^2, wv^{-1} \rangle \times \langle -1 \rangle \cong F_3 \times C_2. \tag{37}$$

Now comparing it to the generators of $\mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle])$ we see that $[\mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle]) : U_1] = 8$.

Next, for the statements concerning the U_i, we use Proposition 8.1 and that $\phi(\mathbb{Z}^\alpha[\langle g, h \rangle])$ is an order contained in the maximal order $\mathrm{M}_2(\mathbb{Z})$ and thus, by a well-known fact, that an element is invertible in the former if and only if it is in the latter. Hence, we obtain that

$$U_i \cong \phi(U_i) = \{ \begin{pmatrix} 1 + 2^i a_{11} & 2^i a_{12} \\ 2^i a_{21} & 1 + 2^i a_{22} \end{pmatrix} \mid a_{11} \equiv_2 a_{22}, \, a_{12} \equiv_2 a_{21} \} \cap \mathrm{GL}_2(\mathbb{Z}). \tag{38}$$

Further remark that $U_1 \leq \mathrm{SL}_1(\mathbb{Z}^\alpha[C_2 \times C_2])$ and hence $\phi(U_i) \leq \phi(U_1) \leq \mathrm{SL}_2(\mathbb{Z})$.

Consider now the principal congruence subgroup $\Gamma(2^i)$ of level 2^i in $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(2^i) = \ker(\pi_{2^i}) : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2^i\mathbb{Z}).$$

More concretely, $\Gamma(2^i) = (1 + 2^i \mathrm{M}_2(\mathbb{Z})) \cap \mathrm{SL}_2(\mathbb{Z})$. Note that $\Gamma(2^{i+1}) \leq \phi(U_i) \leq \Gamma(2^i)$. The second part of the statement will be a consequence of the following:

Claim: $[\Gamma(2^i) : \phi(U_i)] = 2$ and $[\phi(U_i) : \Gamma(2^{i+1})] = 4$ for all $i \geq 1$

To start, recall the following well-known formula (e.g. see[36] [50, p 146]):

$$[\mathrm{SL}_2(\mathbb{Z}) : \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))] = n^3 \prod_{p \mid n} (1 - \frac{1}{p^2}), \tag{39}$$

where the product runs over all the prime divisors p of n. Consequently, $[\Gamma(2^i) : \Gamma(2^{i+1})] = \frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(2^{i+1})]}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(2^i)]} = 8$ for all $i \geq 1$. Further note that $\begin{pmatrix} 1 & 2^i \\ 0 & 1 \end{pmatrix} \in \Gamma(2^i) \setminus \phi(U_i)$ for all i. Therefore, if we prove that $A.B \in \phi(U_i)$ for any pair of matrices $A = \begin{pmatrix} 1 + 2^i a_{11} & 2^i a_{12} \\ 2^i a_{21} & 1 + 2^i a_{22} \end{pmatrix}, B = \begin{pmatrix} 1 + 2^i b_{11} & 2^i b_{12} \\ 2^i b_{21} & 1 + 2^i b_{22} \end{pmatrix} \in \Gamma(2^i) \setminus \phi(U_i)$, then $[\Gamma(2^i) : \phi(U_i)] = 2$, which is the first part of the claim.[37] Subsequently, the second part follows from this and our value of the index $[\Gamma(2^i) : \Gamma(2^{i+1})]$. A direct computation shows that $AB \in \phi(U_i)$ if and only if $a_{11} + a_{22} \equiv_2 b_{11} + b_{22}$ and $a_{12} + a_{21} \equiv_2 b_{12} + b_{21}$. However

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[36] In this reference the formula for $[\mathrm{SL}_2(\mathbb{Z}) : \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z}))]$ is given. If $n > 2$ this differs by a factor 2^{-1} with difference coming from the central matrix -1.

[37] A concrete matrix in $\phi(U_i)$ but not in $\Gamma(2^{i+1})$ is $\begin{pmatrix} 1 + 2^i & 2^i \\ -2^i & 1 - 2^i \end{pmatrix}$.

it also is easy to prove that A having determinant 1 already implies that $a_{11} \equiv_2 a_{22}$ (analogously for B). Hence if $A \in \Gamma(2^i) \setminus \phi(U_i)$ then $a_{12} + a_{21}$ is odd and the same for $b_{12} + b_{21}$. All together the conditions for $AB \in \phi(U_i)$ are always satisfied, as needed. The second part of the statement now follows from the claim as follows:

$$[U_i : U_{i+1}] = [\phi(U_i) : \phi(U_{i+1})] = [\phi(U_i) : \Gamma(2^{i+1})] \cdot [\Gamma(2^{i+1}) : \phi(U_{i+1})] = 8.$$

Finally we prove the last part of the result. By the above, U_2 is a torsion-free subgroup of index 8 in $U_1 = \langle v^2, w^2, wv^{-1} \rangle \times \langle -1 \rangle$. Note that an element $-1.(1 + 2u)$ with $u \in \mathbb{Z}^\alpha[\langle g, h \rangle]$ can not be of the form $1 + 4v$ with $v \in \mathbb{Z}^\alpha[\langle g, h \rangle]$, as seen by working modulo 4. Thus U_2 is a subgroup of index 4 in $F_3 = \langle v^2, w^2, wv^{-1} \rangle$. Thus the Nielsen-Schreier formula yields that $U_2 \cong F_{1+4\cdot2}$. Since $U_i \leq U_{i-1}$ also U_i is a free group whose rank is computed with a recursive use of Nielsen-Schreier's formula. $\quad\square$

Remark 8.4. Consider $\mathcal{F} = \{\widehat{u_g} = \frac{1}{o(g)} \widetilde{u_g} \mid g \in \mathcal{G}_0^\alpha\}$. The computations in the previous proof show that v^2, w^2, wv^{-1} are in the group $\mathrm{GBic}^\mathcal{F}(\mathbb{Q}^\alpha[\langle g, h \rangle], \mathbb{Z})$ generated by the (generalized) bicyclic units (see Definition 6.2). Actually these two groups are equal (as can for example be checked by expressing the generating bicyclic units in terms of v^2, w^2, wv^{-1}). Hence $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$ is an example where

$$\mathrm{GBic}^\mathcal{F}(\mathbb{Q}^\alpha[G], \mathbb{Z}) \lneq \mathcal{H}_\alpha(G).$$

That the difference is still of finite index heavily relies on the fact that the generators of $\mathcal{H}_\alpha(G)$ are all formed from elements of order 2 (which allowed wv^{-1} to be a generalized bicyclic unit). In larger examples however the H-units can be an infinite index overgroup, as will be shown in Section 10.3.

9. Description of $\mathcal{U}(\mathbb{Z}[D_8 \times C_2^n])$

Let G be the group $C_2^{n+2} = \langle g, h, x_1, \ldots, x_n \rangle$ as in (33) and

$$\Gamma \cong D_8 \times C_2 \times \ldots \times C_2 = \langle a, b \rangle \times \langle y_1 \rangle \times \ldots \times \langle y_n \rangle,$$

where $D_8 = \langle a, b \mid a^4 = 1 = b^2, a^b = a^{-1} \rangle$. We consider the epimorphism $\lambda : D_8 \times C_2^n \to G$ determined by $\lambda(a) = gh, \lambda(b) = g, \lambda(y_i) = x_i$. With the associated canonical choice of section $\mu : G \to D_8 \times C_2^n$, one directly checks that the associated 2-cocycle $\alpha(s, t) = \mu(s)\mu(t)\mu(st)^{-1}$ is exactly the one determined by (34). In other words, the central extension Γ associated with the $[\alpha] \in H^2(G, C_2)$ from Section 8 is isomorphic to $D_8 \times C_2^n$.

Taking now $\chi \in \mathrm{Hom}(\langle a^2 \rangle, \mathbb{Z}^*)$ with $\chi(a^2) = -1$ we see that $\Psi_\chi : \mathbb{Z}[\Gamma] \to \mathbb{Z}^{\mathrm{Tra}(\chi)}[G]$ is nothing else than the ring epimorphism.

$$\psi : \mathbb{Z}[D_8 \times C_2^n] \to \mathbb{Z}^\alpha[G] \tag{40}$$

defined by

$$a \mapsto u_{gh}, \quad b \mapsto u_g, \quad x_i \mapsto u_{y_i}.$$

In this section we will pullback the description of $\mathcal{U}(\mathbb{Z}^\alpha[G])$, obtained in Section 8, by using the methods from Section 5.2 in order to obtain a description of $\mathcal{U}(\mathbb{Z}[D_8 \times C_2^n])$. First we invoke Theorem 5.5 and Theorem 5.3 to give a precise description of the (co)kernel of the (co)restricted morphism $\widetilde{\psi}_{res} : \mathcal{U}(\mathbb{Z}[D_8]) \to \mathcal{U}(\mathbb{Z}^{\mathrm{Res}(\alpha)}[\langle g, h \rangle]) : a \mapsto u_{gh}, b \mapsto u_g$.

Lemma 9.1. *With notations as above we have that*

(i) $\ker(\widetilde{\psi}) = \langle -a^2 \rangle \cong C_2$,
(ii) $|\operatorname{coker}(\widetilde{\psi}_{res})| = 2$.

Proof. The description of the kernel follows from Theorem 5.5 which says that the kernel is finite and $\ker(\widetilde{\psi}) = \{1, -a^2\}$, as desired.

In order to describe $\operatorname{coker}(\widetilde{\psi}_{res})$, recall that by Proposition 8.2

$$\mathcal{U}(\mathbb{Z}^{\mathrm{Res}(\alpha)}[C_2 \times C_2]) = \langle v, w \rangle \rtimes D_8$$

with $v = u_1 + u_h \widetilde{u_g}$ and $w = u_1 + \widetilde{u_g} u_h$. As was claimed (without providing details) in Remark 6.14, in this case the product of any two generators of $\mathcal{U}(\mathbb{Z}^{\mathrm{Res}(\alpha)}[\langle g, h \rangle])$ belongs to $\operatorname{Im}(\widetilde{\psi}_{res})$. Furthermore Proposition 6.12 asserts that v is a non-trivial element in $\operatorname{coker}(\widetilde{\psi}_{res})$, hence $\widetilde{\psi}_{res}$ is not surjective and thus $|\operatorname{coker}(\widetilde{\psi}_{res})| = 2$.

To prove the above stated claim on the product of generators, note first that the trivial units are attained. So we need to consider now the torsion-free part. For this consider the bicyclic units $b_1 = 1 + (1 - b)ab(1 + b)$ and $b_2 = 1 + (1 + b)ab(1 - b)$. Recalling (26) we see that $\psi(b_1) = v^2$ and $\psi(b_2) = w^2$. A direct computation also gives that

$$wv^{-1} = -u_1 - 2u_g - 2u_h u_g = -1(u_1 + (1 + u_h)u_g(1 - u_h))$$

Therefore if we consider the bicyclic unit $b_3 = 1 - (1 - ab)b(1 + ab)$, we also notice that $\psi(-b_3) = wv^{-1}$. Thus the claim follows and it finishes the proof. $\square$

Lemma 9.1 combined with Proposition 8.3 and Theorem 7.3 yield the description we were looking for.

Proposition 9.2. *The following hold:*

(1) $\mathcal{U}(\mathbb{Z}[D_8]) \cong F_3 \rtimes (\pm D_8)$ *with F_3 generated by bicyclic units,*
(2) $U_1 \cong F_9 \times C_2$,
(3) $U_i \cong F_{n_i}$ *with $n_i = 1 + 8^i$ for $i \geq 2$,*

(4) $\pm D_8 \times C_2^n$ *has a torsion-free normal complement in* $\mathcal{U}(\mathbb{Z}[D_8 \times C_2^n])$.

This result for example yields that $\mathcal{U}(\mathbb{Z}[D_8 \times C_2]) \cong (F_9 \rtimes F_3) \rtimes (\pm D_8 \times C_2)$. Such a decomposition was already obtained in [32, Theorem 5].

Proof. To start note that $\psi(\pm D_8) = \langle g, h \rangle \cong D_8$. Also it is not hard to see that the elements b_1, b_2, b_3 (used in the proof of Lemma 9.1) are in fact generators for $\langle \mathrm{Bic}(D_8) \rangle$, the group generated by the bicyclic units of D_8 (e.g. see [26, part (3) of proof Example 1.5.4]). Hence we already know that

$$\langle \mathrm{Bic}(D_8), a^2 \rangle \cong \langle \mathrm{Bic}(D_8) \rangle \times \langle a^2 \rangle \cong \pm 1. \langle v^2, w^2, wv^{-1} \rangle = U_1(\mathbb{Z}^\alpha[C_2 \times C_2])$$

with $U_i(\mathbb{Z}^\alpha[C_2 \times C_2]) := \{1 + 2^i u \mid u \in \mathbb{Z}^\alpha[\langle g, h \rangle]\} \cap \mathcal{U}(\mathbb{Z}^\alpha[\langle g, h \rangle])$ and where the second isomorphism is given by $\widetilde{\psi}_{res}$. The latter is isomorphic to $F_3 \times C_2$ by Proposition 8.3.

Now, via Proposition 8.3 and Lemma 9.1 one infers that $\ker(\widetilde{\psi}) \subset \langle \mathrm{Bic}(D_8), \pm D_8 \rangle$ and that $\widetilde{\psi}(\langle \mathrm{Bic}(D_8), \pm D_8 \rangle)$ is of index two in $\mathcal{U}(\mathbb{Z}^\alpha[C_2 \times C_2])$, thus equals $\mathrm{im}(\widetilde{\psi})$. Consequently, the first part follows and we have given a new proof of the well known fact that $\mathcal{U}(\mathbb{Z}[D_8]) = \langle \mathrm{Bic}(D_8) \rangle \rtimes (\pm D_8)$. In particular all elements in $\langle \mathrm{Bic}(D_8) \rangle$ are of the form $1 + (1 - a^2)u$ with $u \in \mathbb{Z}[D_8]$. This in particular implies that for all i

$$U_i(\mathbb{Z}D_8) = \{1 + 2^i(1 - a^2)u \mid u \in \mathbb{Z}[D_8]\} \cap \mathcal{U}(\mathbb{Z}[D_8]) \tag{41}$$

Next, due to the above, we see that for each i the torsion-free elements of $U_i(\mathbb{Z}D_8)$ are in $\langle \mathrm{Bic}(D_8) \rangle$. In particular, for each $i \geq 2$ this means that $U_i(\mathbb{Z}D_8) \leq \langle \mathrm{Bic}(D_8) \rangle$ and $\widetilde{\psi}$ is injective on them by Lemma 9.1. Using the description (41) we moreover have that $\widetilde{\psi}(U_i(\mathbb{Z}D_8)) = U_{i+1}(\mathbb{Z}^\alpha[C_2 \times C_2])$. In summary, $\widetilde{\psi}$ induces an isomorphism

$$U_i(\mathbb{Z}D_8) \cong U_{i+1}(\mathbb{Z}^\alpha[C_2 \times C_2]) \tag{42}$$

for all $i \geq 2$. If $i = 1$ then by Theorem 7.3 the group $U_1(\mathbb{Z}D_8) \cong \{\pm 1\} \times N_1$ with N_1 torsion-free. By the above $N_1 \leq \langle \mathrm{Bic}(D_8) \rangle$ and concretely $N_1 = \{1 + 2(1 - a^2)u \mid u \in \mathbb{Z}[D_8]\}$. So via $\widetilde{\psi}$ we have the isomorphism $N_1 \cong U_2(\mathbb{Z}^\alpha[C_2 \times C_2])$. Therefore the second and third statement now follows from Proposition 8.3. The last statement follows from the first and Corollary 7.4. $\square$

As mentioned after Remark 7.5, it is known that the group generated by the bicyclic units form a normal complement in $\mathcal{U}(\mathbb{Z}[D_8 \times C_2^n])$ if $n \leq 1$, but for $n = 2$ they do not form a normal complement as shown in [40]. In fact, it is expected that if $n \geq 2$ the bicyclic units are even of infinite index. It would be especially instructive to describe a minimal set of generators (in a generic way) of the torsion-free normal complement mentioned Proposition 9.2.

To finish this section, we would like to record the exact size and generators of $\mathrm{coker}(\widetilde{\psi})$.

Lemma 9.3. *With notations as above we have that*

(1) $|\operatorname{coker}(\widetilde{\psi})| = [\mathcal{U}(\mathbb{Z}^{\alpha}[G]) : \operatorname{Im}(\widetilde{\psi})] = 2^{n+3} - 4n - 6$,
(2) $\operatorname{coker}(\widetilde{\psi})$ *is generated by H-units.*

Proof. By Theorem 7.3, each of the unit groups $\mathcal{U}(\mathbb{Z}[\Gamma])$ and $\mathcal{U}(\mathbb{Z}^{\alpha}[G])$ is determined by respective groups $K_{\vec{j}}$. To distinguish, we respectively write $K_{\vec{j}}^{(i)}(\Gamma)$ and $K_{\vec{j}}^{(i)}(G)$ for $\vec{j} \in \{0,1\}^{i-1}$. The methods in Section 7 imply that

$$|\operatorname{coker}(\widetilde{\psi})| = |\operatorname{coker}(\widetilde{\psi}_{res})| + \sum_{i=1}^{n} \sum_{\vec{j} \in \{0,1\}^{i-1}} [K_{\vec{j}}^{(i)}(G) : \widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))].$$

Following Lemma 9.1, $|\operatorname{coker}(\widetilde{\psi}_{res})| = 2$. Next, to compute $[K_{\vec{j}}^{(i)}(G) : \widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))]$ we use (see Theorem 7.3) the isomorphism $K_{\vec{j}}^{(i)}(G) \cong U_{1+\sum_{t=1}^{i-1} j(t)}$ which we will denote by θ (where we abuse notations as it in fact depends on i and $\vec{j}$). For notational simplicity denote $\sum \vec{j} := \sum_{t=1}^{i-1} j(t)$. Note that $\theta(\widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))) \cong \widetilde{\psi}_{res}(\theta(K_{\vec{j}}^{(i)}(\Gamma)))$. Therefore, if $\vec{j} \neq \vec{0}$, using (42) and Proposition 8.3 yields

$$[K_{\vec{j}}^{(i)}(G) : \widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))] = [\theta\left(K_{\vec{j}}^{(i)}(G)\right) : \theta\left(\widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))\right)] = [U_{1+\sum \vec{j}} : U_{2+\sum \vec{j}}] = 8.$$

If $\vec{j} = \vec{0}$, then both $K_{\vec{j}}^{(i)}(G)$ and $K_{\vec{j}}^{(i)}(\Gamma)$ contain a copy of C_2 (namely $\langle u_{y_i} \rangle$, resp. $\langle x_i \rangle$) and $\widetilde{\psi}$ induces an isomorphism on these. Thus in this the same passage through the above isomorphisms yields that $[K_{\vec{j}}^{(i)}(G) : \widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))] = [\langle v, w, vw^{-1} \rangle : U_2] = 4$. So all together

$$|\operatorname{coker}(\widetilde{\psi})| = 2 + 4n + 8 \sum_{i=1}^{n} (2^{i-1} - 1) = 2 + 4n - 8n + 8(2^n - 1) = 2^{n+3} - 4n - 6.$$

For the second part of the statement, using the elements in $\mathcal{H}_{\alpha}(C_2 \times C_2)$ we will give explicit generators of the quotient $K_{\vec{j}}^{(i)}(G)/\widetilde{\psi}(K_{\vec{j}}^{(i)}(\Gamma))$. First recall that by Proposition 8.3 $U_{1+\sum \vec{j}}/U_{2+\sum \vec{j}}$ is of order 8 if $\vec{j} \neq \vec{0}$ and 4 else. Now consider the set

$$\{v^{2^{1+\Sigma \vec{j}}}, (vw^{-1})^{2^{\Sigma \vec{j}}}, v^{2^{\Sigma \vec{j}}} w^{-2^{\Sigma \vec{j}}}\}. \tag{43}$$

Note that the second and third generators are equal if $\vec{j} = \vec{0}$. Computing their image under the isomorphism ϕ from the proof of Proposition 8.3, and also of all their double products, one deduces that they generate an elementary abelian 2-group of order 8 if $\vec{j} \neq \vec{0}$ and of order 4 else. In particular, for all $\vec{j}$ their images generate $\phi(U_{1+\sum \vec{j}})/\phi(U_{2+\sum \vec{j}})$ and so the set itself generates $U_{1+\sum \vec{j}}/U_{2+\sum \vec{j}}$. It now remains to construct three H-units

in $K_{\vec{j}}^{(i)}(G)$ which map under θ to the elements in (43). For notation ease we denote $(1 - u_x)^{\vec{j}} = (1 - u_{x_1})^{j(1)} \dots (1 - u_{x_{i-1}})^{j(i-1)}$. We claim that the desired units are:

$$z_{\vec{j},1}^{(i)} = 1 + (1 - u_{x_i})(1 - u_x)^{\vec{j}} u_h \widetilde{u_g},$$

$$z_{\vec{j},2}^{(i)} = 1 - (1 - u_{x_i})(1 - u_x)^{\vec{j}} \widetilde{u_h} u_g,$$

$$z_{\vec{j},3}^{(i)} = \left(1 + (1 - u_x)^{\vec{j}} u_h \widetilde{u_g}\right) \left(1 - (1 - u_x)^{\vec{j}} u_h \widetilde{(u_{x_i} u_g)}\right).$$

Writing out the third element results in

$$z_{\vec{j},3}^{(i)} = 1 + (1 - u_x)^{\vec{j}} u_h u_g (1 - u_{x_i}) - 2^{\Sigma\vec{j}} (1 - u_x)^{\vec{j}} (1 - u_g)(1 - u_{x_i}).$$

Thus we see that $\{z_{\vec{j},1}^{(i)}, z_{\vec{j},2}^{(i)}, z_{\vec{j},3}^{(i)}\} \subset K_{\vec{j}}^{(i)}(G)$ and clearly

$$\theta(z_{\vec{j},1}^{(i)}) = v^{2^{1+\Sigma\vec{j}}}, \;\; \theta(z_{\vec{j},2}^{(i)}) = (vw^{-1})^{2^{\Sigma\vec{j}}}, \;\; \theta(z_{\vec{j},3}^{(i)}) = v^{2^{\Sigma\vec{j}}} w^{-2^{\Sigma\vec{j}}}.$$

This finishes the proof of the second statement. $\quad\square$

10. A new generic construction of units in integral group rings

In this section we introduce a new generic construction of units in $\mathbb{Z}G$ which in fact are elements of $\mathrm{SL}_1(\mathbb{Z}G)$ (see (25)). These elements originated as the pullback of the products of the H-units in twisted group rings from Definition 6.9 along the transgression morphisms of Section 3.2. However, they can also be defined directly as will be shown in Section 10.2. Furthermore, as explained in Section 10.1, those units of Definition 6.9 can be constructed in general finite dimensional semisimple F-algebras, with F a number field. Finally, in Section 10.3, we will give an infinite family of groups where the newly constructed units contain the bicyclic units as subgroup of infinite index. In particular, these elements are indeed a new step towards the problem of describing generators of $\mathcal{U}(\mathbb{Z}G)$, up to commensurability, generically in $\mathbb{Z}G$. Interestingly, these units will also yield the first generic construction of free groups of rank larger than 2, see Theorem 10.6.

Notation. Recall that $\tilde{H} = \sum_{h \in H} h$ for any finite subgroup H in an algebra A and $\widehat{H} = \frac{1}{|H|} \tilde{H}$. If A is semisimple we will denote by $\mathrm{PCI}(A)$ the set of primitive central idempotents of A.

10.1. Restricted construction of units for orders in general semisimple algebras

Let A be a finite dimensional semisimple F-algebra, with F a number field, and let $\mathcal{O}$ be a $\mathbb{Z}$-order in A. Inspired by Definition 6.9 and Remark 6.10 we define the following.

Definition 10.1. Let $x, t \in \mathcal{U}(\mathcal{O})$ be torsion units such that $[t, x] \in N_{\mathcal{U}(\mathcal{O})}(\langle t \rangle)$ is of finite order and $\widetilde{[t, x]} = 0$. Then the elements

$$u_{x^{\pm 1}, \widetilde{t}} := 1 + x^{\pm 1}\widetilde{t} \text{ and } u_{\widetilde{t}, x^{\pm 1}} := 1 + \widetilde{t}x^{\pm 1}$$

will be called *primitive H-units*.

As the name suggests, the elements in Definition 10.1 are indeed invertible elements in $\mathcal{U}(\mathcal{O})$, as proven below. The reason for adding, compared to Definition 6.9, primitive in the name will be clarified later on in Remark 10.10.

Theorem 10.2. *Let $x, t \in \mathcal{U}(\mathcal{O})$ be as in Definition 10.1. Then $u_{x^{\pm 1}, \widetilde{t}}$ and $u_{\widetilde{t}, x^{\pm 1}}$ are unipotent units in $\mathcal{U}(\mathcal{O})$. In particular they are of infinite order.*

We will prove that $\left(x^{\pm}\widetilde{t}\right)^2 = 0$, hence the inverse of $u_{x^{\pm 1}, \widetilde{t}}$ is $1 - x^{\pm 1}\widetilde{t}$. Similarly for $u_{\widetilde{t}, x^{\pm 1}}$.

Example 10.3. The archetypical example is a tuple (x, t) of torsion units such that $[t, x] \in F^*$ is a root of unity. In that case $[t, x]$ is both central and $\widetilde{[t, x]} = 0$. If $A = R^{\alpha}[G]$ is some twisted group ring and $x, t \in G$, then the linear independence of the basis elements yields that $u_{x^{\pm 1}, \widetilde{t}}$ and $u_{\widetilde{t}, x^{\pm 1}}$ are non-trivial (i.e. $\neq 1$) if $\tilde{x} \neq 0$ in A. However, in general, there is no transparent characterisation of being non-trivial.

In order to prove Theorem 10.2 we need the following lemma which was shared to us by Ángel del Río and whose proof depends on useful identities from [9].

Lemma 10.4. *Let $G = \langle g, a \rangle$ be a finite meta-cyclic group with $\langle g \rangle$ a normal subgroup such that $o(g) = o(ga)$. Then $o(a) \mid o(g)$.*

Proof. Denote $m = o(g)$, $n = [G : \langle g \rangle] = \frac{|G|}{m}$ and $s = [G : \langle a \rangle] = [\langle g \rangle : \langle g \rangle \cap \langle a \rangle]$. Then s divides m and $o(a) = \frac{|G|}{s} = \frac{mn}{s}$.

For a prime p denote by $v_p(k)$ the p-adic valuation of $k \in \mathbb{Z}_{\geq 0}$. Suppose that $o(a) = \frac{nm}{s}$ does not divide $m = o(g)$, or equivalently n does not divide s, i.e. $v_p(n) > v_p(s)$ for some prime p.

Let i be the smallest non-negative integer such that $a^{-1}ga = g^i$ and $1 \leq t \leq m$ such that $a^n = g^t$. Note that $g = a^{-n}ga^n = g^{i^n}$ and so $i^n \equiv 1 \mod m$. Next, by using formula [9, eq. (2.2)] (notice that in [9] the role of a and g are interchanged) we have $(ag)^n = a^n g^{S(i|n)} = g^{t+S(i|n)}$ with $S(i|n) := \sum_{j=0}^{n-1} i^j$. Then, by the assumption, $m = o(g) = o(g^{-1}(ga)g) = o(ag) = n\frac{m}{\gcd(m, t+S(i|n))}$ and hence $n = \gcd(m, t + S(i|n))$. In particular n divides both $t + S(i|n)$ and m. Also, $\frac{nm}{s} = o(a) = n.o(g^t) = \frac{nm}{\gcd(m,t)}$. Therefore $s = \gcd(m, t)$. All together we deduce that

$$v_p(s) = \min\{v_p(t), v_p(m)\} < v_p(n) = \min\{v_p(m), v_p(t + S(i|n))\}.$$

This entails that $v_p(t) = v_p(s) < v_p(n)$. Consequently also $p \mid m$ and $v_p(t) < v_p(t + S(i|n))$. In particular $v_p(t) = v_p(t + S(i|n) - t) = v_p(S(i|n))$, hence $v_p(S(i|n)) < v_p(n)$.

Now let q be the multiplicative order of i modulo p, i.e. the smallest integer q with $p \mid i^q - 1$. Since $i^n \equiv 1 \mod m$ and p divides m, it follows that q divides n. Moreover, q divides $p - 1$. Hence $v_p(n) = v_p(\frac{n}{q}) \geq 1$. From [9, Lemma 8.2] it follows that in case p is odd or $p = 2$ and $i^q \equiv 1 \mod 4$, we have $v_p(S(i^q|\frac{n}{q})) = v_p(\frac{n}{q})$. Otherwise $p = 2, v_p(i^q + 1) \geq 2$ and $v_2(S(i^q|\frac{n}{q})) = v_2(\frac{n}{q}) + v_2(i^q + 1) - 1 > v_2(\frac{n}{q})$. So all together, $v_p(S(i^q|\frac{n}{q})) \geq v_p(\frac{n}{q})$.

Moreover, by [9, Lemma 8.1], $S(i|n) = S(i|q)S(i^q|\frac{n}{q})$. Thus, $v_p(\frac{n}{q}) = v_p(n) > v_p(S(i|n)) = v_p(S(i|q)) + v_p(S(i^q|\frac{n}{q})) \geq v_p(\frac{n}{q})$, which is clearly a contradiction. The contradiction came from the assumption that $v_p(n) > v_p(s)$ for some prime p. Hence we conclude that $v_p(n) \leq v_p(s)$ for all primes p, that is, n divides s and hence $o(a) \mid o(g)$. $\quad\square$

We can now proceed to the proof of the theorem.

Proof of Theorem 10.2. Assume $x, t \in \mathcal{U}(\mathcal{O})$ are torsion units such that $[t, x] \in N_{\mathcal{U}(\mathcal{O})}(\langle t \rangle)$ is of finite order and $\widetilde{[t, x]} = 0$. For simplicity of notation, denote $[t, x] = a$. Since $x^{-1}tx = ta$ we have that $o(t) = o(ta)$. Thus $G = \langle a, t \rangle$ satisfies the conditions of Lemma 10.4 and hence $o(a) \mid o(t)$.

Next, recall the notation $S(s \mid n) := \sum_{j=0}^{n-1} s^j$ for any numbers $s, n \in \mathbb{N}$. Since $a \in N_{\mathcal{U}(\mathcal{O})}(\langle t \rangle)$ there is some $i \neq 0$ such that $ta = at^i$. In other words, $tx = xat^i$ and $x^{-1}t = at^ix^{-1}$. Via induction one directly obtains that

$$x^{-1}t^k = a^k t^{iS(i|k)}x^{-1} \text{ and } t^kx = xa^k t^{iS(i|k)}. \tag{44}$$

These formulas to swap t and x enable us to compute $\widetilde{t}x^{\pm 1}\widetilde{t}$. We claim that:

$$\widetilde{t}x^{-1}\widetilde{t} = \frac{o(t)}{o(a)}\widetilde{a}\,\widetilde{t}\,x^{-1} \text{ and } \widetilde{t}x\widetilde{t} = \frac{o(t)}{o(a)}x\,\widetilde{t}\,\widetilde{a}.$$

In order to prove this claim, denote $\frac{o(t)}{o(a)} = m \in \mathbb{N}$ and rewrite

$$\widetilde{t} = \sum_{\ell=0}^{m-1} t^{\ell \cdot o(a)} \left(\sum_{j=0}^{o(a)-1} t^j \right).$$

Then, using the rule in (44) and that $a\widetilde{t} = \widetilde{t}a$, because a normalizes $\langle t \rangle$, we obtain that

$$\widetilde{t}x\widetilde{t} = x \left(\sum_{\ell=0}^{m-1} a^{l\, o(a)} t^{iS(i|l\, o(a))} \left(\sum_{j=0}^{o(a)-1} a^j t^{iS(i|j)} \right) \right) \widetilde{t}$$

$$= x \left(\sum_{\ell=0}^{m-1} t^{iS(i|l\, o(a))} \left(\sum_{j=0}^{o(a)-1} a^j \widetilde{t} \right) \right)$$

$$= x\widetilde{t} \left(\sum_{\ell=0}^{m-1} \widetilde{a} \right).$$

In the second and last equality we have used that $\widetilde{t}t^j = \widetilde{t}$ for every j. The expression for $\widetilde{t}x^{-1}\widetilde{t}$ is computed analogously. Now, since by assumption $\widetilde{a} = 0$, we obtain that $\left(x^{\pm 1}\widetilde{t}\right)^2 = 0 = (\widetilde{t}x^{\pm 1})^2$. Consequently, $u^n_{x^{\pm 1},\widetilde{t}} = 1 + nx\widetilde{t}$ for all $n \in \mathbb{Z}$. In particular it is invertible with inverse $1 - x^{\pm 1}\widetilde{t}$ and has infinite order. Similarly for $u_{\widetilde{t},x^{\pm 1}}$. $\quad\square$

10.2. General construction and properties of H-units

In this section we start with constructing a class of units in $\mathrm{SL}_1(\mathbb{Z}G)$ from any triple (g, h, Q) satisfying the properties mentioned in the definition below. A first new feature of these, that we will obtain in Theorem 10.6, is that they produce free groups of large rank. Furthermore, as shown later in Section 10.3, the group consisting of these units can contain the bicyclic units as subgroup of infinite index and hence also up to commensurability they are new.

Definition 10.5. Let G be a finite group and (g, h, Q) a triple satisfying

- Q is a normal subgroup in $\langle g, h, Q \rangle$,
- $[g, h]Q \in \mathcal{Z}(\langle g, h, Q \rangle / Q)$ and $o(hQ) = 2$.

Denote $\widetilde{(\pm g)}_Q := \sum_{i=0}^{o(gQ)-1} (\pm g)^i$ and for any tuple $(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4_{\geq 0}$ define the element

$$v_{(x_1,x_2,y_1,y_2)} := 1 + \frac{1}{2|Q|}\widetilde{Q}(1 - [g,h])\Big(h\left[x_1\,\widetilde{(-g)}_Q + x_2\,\widetilde{g}_Q\right] + y_1\,\widetilde{(-g)}_Q + y_2\,\widetilde{g}_Q\Big). \quad (45)$$

A quadruple $(x_1, x_2, y_1, y_2) \in \mathbb{Z}^4_{\geq 0}$ will be called *admissible for* (g, h, Q) if

$$y_1 + y_2 + o(gQ)\,y_1 y_2 = o(gQ)\,x_1 x_2 \text{ and } x_2 \pm x_1 \equiv 0 \equiv y_2 \pm y_1 \mod 2|Q|. \quad (46)$$

The elements $v_{(x_1,x_2,y_1,y_2)}$ for admissible (x_1, x_2, y_1, y_2) will be called *an H-unit*.

The first condition to be admissible will exactly correspond to being invertible (more precisely, to belong to $\mathrm{SL}_1(\mathbb{Z}G)$) and the second condition yields that the element is in $\mathbb{Z}G$.

For a fixed triple (g, h, Q) as in Definition 10.5, the set of elements of the form (45) for admissible quadruples (46) *will be denoted* $\mathcal{H}(g, h, Q)$ and

$$\mathcal{H}(G) = \langle \mathcal{H}(g, h, Q) \mid (g, h, Q) \text{ as in Definition 10.5} \rangle \quad (47)$$

is the group generated by all *H*-units.

Among others, the next result says that $\mathcal{H}(g, h, Q)$ is not only a set, but even a subgroup of the group of reduced norm 1 elements.

Theorem 10.6. *Let (g, h, Q) be a triple as in Definition 10.5. Then,*

(1) $\mathcal{H}(g, h, Q)$ *is a finitely generated subgroup of* $\mathrm{SL}_1(\mathbb{Z}G)$ *and*
   $v_{(x_1, x_2, y_1, y_2)}^{-1} = v_{(-x_1, -x_2, y_2, y_1)}$,
(2) $\mathcal{H}(g, h, Q) \neq 1$ *if and only if* $[g, h] \notin \langle g \rangle Q$.

Moreover, for $\mathcal{H}(g, h, Q) \neq 1$,

(3) *if*[38] $o(gQ)|Q| = 2$, *then* $\mathcal{H}(g, h, Q) \cong F_3 \times C_2$, *and*
(4) *if* $o(gQ)|Q| > 2$, *then* $\mathcal{H}(g, h, Q) \cong F_n$ *with* $n = 1 + \frac{(o(gQ)|Q|)^3}{6} \prod_p (1 - \frac{1}{p^2})$, *where the product runs over the prime divisors* p *of* $o(gQ)|Q|$.

As $\mathcal{H}(g, h, Q)$ is finitely generated, one only needs to construct $v_{(x_1, x_2, y_1, y_2)}$ for a finite number of admissible (x_1, x_2, y_1, y_2). We have not tried to give a precise upperbound, but in principle this could be done. In fact the generators should somehow correspond to the 'minimal solutions' of the equations in (46).

For any $a, c \in G$, with $c \notin N_G(\langle a \rangle)$, consider the corresponding bicyclic unit $b(a, c) = 1 + (1 - a)c\tilde{a}$. It was proven by Marciniak and Sehgal [43] that $\langle b(a, c), b(a, c)^* \rangle \cong F_2$, where $b(a, c)^*$ is the image of $b(a, c)$ under the canonical involution of $\mathbb{Q}G$. For nilpotent groups G a similar statement holds for their torsion variant [19,44]: the set consisting of the unipotent units belonging to $\mathrm{Bic}(G)$ and that form a free product with $b(a, c)$ a is profinitely dense in $\mathrm{Bic}(G)$ [21]. However, Theorem 10.6 yields the first generic construction of large free groups (different from taking artificially a copy of F_n inside F_2). See [16] for a survey on constructing free subgroups of $\mathcal{U}(\mathbb{Z}G)$.

Proof of Theorem 10.6. To start, note that $(-x_1, -x_2, y_2, y_1)$ is admissible if (x_1, x_2, y_1, y_2) is. Moreover, it directly follows from the $x_2 \pm x_1 \equiv 0 \equiv y_2 \pm y_1 \mod 2|Q|$ congruences that both elements in the statement are in $\mathbb{Z}G$. To prove the statements of the result, we will "locate" precisely the H-units inside $\mathbb{Q}[\langle g, h, Q \rangle]$.

First we consider when $\mathcal{H}(g, h, Q)$ is trivial, i.e. when all elements $v_{(x_1, x_2, y_1, y_2)}$ equal 1. Or equivalently, all $\frac{1}{2|Q|} \widetilde{Q}(1 - [g, h]) \left(h \left[x_1 \widetilde{(-g)}_Q + x_2 \widetilde{g}_Q \right] + y_1 \widetilde{(-g)}_Q + y_2 \widetilde{g}_Q \right) = 0$. Since $\mathbb{Z}\langle g, h, Q \rangle \frac{\widetilde{Q}}{|Q|} \cong \mathbb{Z}(\langle g, h, Q \rangle / Q)$, it easily follows that we may assume that $Q = 1$; and thus $o(h) = 2$ and $[g, h] \in \mathcal{Z}(\langle g, h \rangle)$. Hence, we need to verify when all $(1 - [g, h]) \left(h \left[x_1 \widetilde{(-g)}_Q + x_2 \widetilde{g}_Q \right] + y_1 \widetilde{(-g)}_Q + y_2 \widetilde{g}_Q \right) = 0$. Of course, if $[g, h] = 1$ then the latter always holds. Hence, for the remaining of the proof of part (2) we may also assume that $1 \neq [g, h]$. Now, as $o(h) = 2$ and $1 \neq [g, h] \in \mathcal{Z}(\langle g, h \rangle)$ one has that $o([g, h]) = 2$. Furthermore, $o(g)$ is even and g^2 is central in $\langle g, h \rangle$.

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[38] Since we assume $\mathcal{H}(g, h, Q)$ to be non-trivial, by part (2), $o(gQ)|Q| = 2$ if and only if $Q = 1$ and $o(g) = 2$.

Now suppose that $[g, h] \in \langle g \rangle$. Write $[g, h] = g^k$ and thus $h^{-1}gh = g^{k+1}$. As $o(h^{-1}gh) = o(g)$ it follows that $k+1$ is odd and thus $[g, h] \in \langle g^2 \rangle$ is central. Therefore the required triviality follows from $(1-[g, h])(\widetilde{g}_Q) = 0 = (1-[g, h])\widetilde{(-g)}_Q$. So we have shown that if $[g, h] \in \langle g \rangle$ then $\mathcal{H}(g, h, Q) = 1$. Conversely, assume $\mathcal{H}(g, h, Q) = 1$, in particular $v(1, -1, 1, -1) = 1$ and, because $o(g)$ is even, we thus get $(1-[g, h])(h+1)(\widetilde{(-g)}_Q + \widetilde{g}_Q) = (1-[g, h])(h+1)\widetilde{(g^2)}_Q = 0$. In other words, $(h+1)\widetilde{g^2_Q} = [g, h](h+1)\widetilde{g^2_Q}$. Clearly, a support argument in the group ring $\mathbb{Z}\langle g, h \rangle$ yields that $[g, h] \in \langle g^2 \rangle$ or $[g, h] \in h\langle g^2 \rangle$. However, the latter is impossible as elements in $h\langle g^2 \rangle$ are not central in $\langle g, h \rangle$. Hence, we have shown that if $\mathcal{H}(g, h, Q) = 1$ then $[g, h] \in \langle g^2 \rangle$, and thus part (2) of the result follows.

From now on we may assume that $\mathcal{H}(g, h, Q) \neq 1$. In particular, according to the above $[g, h] \notin \langle g \rangle Q$. Furthermore, as mentioned above, we thus also have that $o([g, h]Q) = 2$.

Consider the central idempotent $e := \langle Q, \widehat{-[g, h]} \rangle$ in $\mathbb{Q}[\langle g, h, Q \rangle]$ and the associated decomposition $\mathbb{Q}[\langle g, h, Q \rangle] = \mathbb{Q}[\langle g, h, Q \rangle](1 - e) \oplus \mathbb{Q}[\langle g, h, Q \rangle]e$. Notice that $v_{(x_1, x_2, y_1, y_2)}(1 - e) = 1 - e$, i.e. the projection on the first part is trivial. Thus we need to prove the desired statements (1), (3) and (4) within the second component. For this, put $\overline{g} = gQ$ and $\overline{h} = hQ$. As mentioned above, $o(\overline{g})$ is even and note that $\mathbb{Q}[\langle g, h, Q \rangle]e \cong \mathbb{Q}^\alpha[\langle \overline{g} \rangle \times \langle \overline{h} \rangle]$ with $[\alpha] \in H^2(\langle \overline{g} \rangle \times \langle \overline{h} \rangle, \mathbb{Z}^*)$ determined by $u_{\overline{g}}^{o(\overline{g})} = 1 = u_{\overline{h}}^{o(\overline{h})}$ and $[u_{\overline{g}}, u_{\overline{h}}] = -1$. Under that isomorphism $v_{(x_1, x_2, y_1, y_2)}$ corresponds to the element

$$1 + u_{\overline{h}}[x_1 \widetilde{(-u_{\overline{g}})} + x_2 \widetilde{u_{\overline{g}}}] + y_1 \widetilde{(-u_{\overline{g}})} + y_2 \widetilde{u_{\overline{g}}}.$$

A direct verification yields that $\widetilde{(-u_{\overline{g}})}.\widetilde{u_{\overline{g}}} = 0$ and $\pm \widetilde{u_{\overline{g}}}^2 = o(\overline{g})\pm\widetilde{u_{\overline{g}}}$. Also, $[x_1 \widetilde{(-u_{\overline{g}})} + x_2 \widetilde{u_{\overline{g}}}] u_{\overline{h}} = u_{\overline{h}}[x_2 \widetilde{(-u_{\overline{g}})} + x_1 \widetilde{u_{\overline{g}}}]$. Using all this, one could straightforwardly compute the image of the product $v_{(x_1, x_2, y_1, y_2)} v_{(-x_1, -x_2, y_2, y_1)}$ and deduce that it is equal to 1 if and only if $y_1 + y_2 + o(\overline{g}) y_1 y_2 = o(\overline{g}) x_1 x_2$. However, by decomposing $\mathbb{Q}^\alpha[\langle \overline{g} \rangle \times \langle \overline{h} \rangle]$ further one can also give the following conceptual explanation, which moreover also will yield the remainder of the result.

Remark that $\overline{g}^2$ is central and hence, by Theorem 3.2,

$$\mathbb{Q}^\alpha[\langle \overline{g} \rangle \times \langle \overline{h} \rangle] \cong \bigoplus_{\chi \in \mathrm{Lin}(\langle \overline{g}^2 \rangle, \mathbb{Q})} \mathbb{Q}(\chi)^{\mathrm{Tra}_\alpha(\chi)}[C_2 \times C_2] \cong \bigoplus_{d \mid o(\overline{g}^2)} \mathbb{Q}(\zeta_d)^{\alpha_d}[C_2 \times C_2].$$

Explicitly, denoting $C_2 \times C_2 = \langle a, b \rangle$, $[\alpha_d] \in H^2(\langle a, b \rangle, \langle \pm \zeta_d \rangle)$ is determined by $u_a^2 = \zeta_d, u_b^2 = 1$ and $u_a u_b = -u_b u_a$. The projections onto the direct summands are given by the transgression maps from Proposition 3.3, hence $u_{\overline{g}} \mapsto u_a, u_{\overline{h}} \mapsto u_b$ and $u_{\overline{g}^2} \mapsto \zeta_d u_1$. Next, notice that $\widetilde{(\pm u_{\overline{g}})} = (1 \pm u_{\overline{g}}) . (\widetilde{u_{\overline{g}}^2})$. Due to this, $\widetilde{(\pm u_{\overline{g}})}$ maps to 0 in every direct summand except for the one indexed by the trivial representation of $\langle \overline{g}^2 \rangle$. In other words, there is only a single component of $\mathbb{Q}[\langle g, h, Q \rangle]$ where $v_{(x_1, x_2, y_1, y_2)}$ has a nontrivial projection and this component is isomorphic to $\mathbb{Q}^{\alpha_1}[\langle a, b \rangle]$ with $u_a^2 = 1 = u_b^2$ and $[u_a, u_b] = -1$. In that component the projection of $v_{(x_1, x_2, y_1, y_2)}$ is

$$1 + o(\overline{g}^2)\big(u_b[x_1(1 - u_a) + x_2(1 + u_a)] + y_1(1 - u_a) + y_2(1 + u_a)\big).$$

Restricting to $\mathbb{Z}[\langle g, h, Q \rangle]$, we can now compose with the isomorphism ϕ from Section 8.1, defined in (36). This composition is defined by

$$u_a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u_b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_{ab} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By doing so we have obtained a ring morphism $\Phi : \mathbb{Z}[\langle g, h, Q \rangle] \to \widetilde{D}$ which on $\mathcal{H}(g, h, Q)$ is defined as

$$\Phi : v_{(x_1, x_2, y_1, y_2)} \mapsto \begin{pmatrix} 1 + 2\,o(\overline{g}^2)\,y_2 & 2o(\overline{g}^2)\,x_1 \\ 2o(\overline{g}^2)\,x_2 & 1 + 2\,o(\overline{g}^2)\,y_1 \end{pmatrix} = \begin{pmatrix} 1 + o(\overline{g})\,y_2 & o(\overline{g})\,x_1 \\ o(\overline{g})\,x_2 & 1 + o(\overline{g})\,y_1 \end{pmatrix}.$$
(48)

Moreover, the elements of $\mathcal{H}(g, h, Q)$ being trivial on all the other components, Φ is injective on $\mathcal{H}(g, h, Q)$. In particular, see (24), $nr(v_{(x_1, x_2, y_1, y_2)}) = 1$ if $\det(\Phi(v_{(x_1, x_2, y_1, y_2)})) = 1$. The latter holds as $\det(\Phi(v_{(x_1, x_2, y_1, y_2)})) = 1 + o(\overline{g})(y_1 + y_2) + o(\overline{g})^2 y_1 y_2 - o(\overline{g})^2 x_1 x_2 = 1$, by the definition of an admissible quadruple. In fact, since Φ is injective on $\mathcal{H}(g, h, Q)$, the first condition of being admissible is equivalent to $\det(\Phi(v_{(x_1, x_2, y_1, y_2)})) = 1$. Now, we also see that the inverse is

$$\Phi(v_{(x_1, x_2, y_1, y_2)})^{-1} = \begin{pmatrix} 1 + o(\overline{g})\,y_1 & -o(\overline{g})\,x_1 \\ -o(\overline{g})\,x_2 & 1 + o(\overline{g})\,y_2 \end{pmatrix}.$$

Hence indeed $v_{(x_1, x_2, y_1, y_2)}^{-1} = v_{(-x_1, -x_2, y_2, y_1)}$. In conclusion, $\mathcal{H}(g, h, Q)$ is a subgroup of $\mathrm{SL}_1(\mathbb{Z}G)$.

All the above in fact yields more. Namely that

$$\Phi : \mathcal{H}(g, h, Q) \hookrightarrow \Gamma(o(\overline{g})),$$

where $\Gamma(n)$ is the principal congruence subgroup of level n of $\mathrm{SL}_2(\mathbb{Z})$. However, Φ is not onto due to the congruences $x_2 \pm x_1 \equiv 0 \equiv y_2 \pm y_1 \mod 2|Q|$.

Claim. $x_2 \pm x_1 \equiv 0 \equiv y_2 \pm y_1 \mod 2|Q|$ *if and only if* $x_i = |Q|l_i, y_i = |Q|t_i$ *for* $l_i, t_i \in \mathbb{N}$ *such that* $l_1 \equiv l_2 \mod 2$ *and* $t_1 \equiv t_2 \mod 2$.

Indeed, if $x_2 \pm x_1 \equiv 0 \mod 2|Q|$, then $2x_1 \equiv 0$ and $2x_2 \equiv 0 \mod 2|Q|$. Hence $x_2 \equiv x_1 \equiv 0 \mod |Q|$ and consequently $x_i = |Q|l_i$, for some $l_i \in \mathbb{N}$. Now, as $2|Q|$ divides $x_2 \pm x_1$, one must have that $2 \mid l_1 \pm l_2$, as desired. Conversely, x_i of that form clearly satisfy $x_2 \pm x_1 \mod 2|Q|$. The proof for the y_i is exactly the same, hence the claim follows.

Now denote

$$V_m = \left\{ \begin{pmatrix} 1 + m\,l_2 & m\,t_1 \\ m\,t_2 & 1 + m\,l_1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid l_1 \equiv l_2 \text{ and } t_1 \equiv t_2 \mod 2 \right\}. \quad (49)$$

Notice that the groups $U_i \cong \phi(U_i)$ from the proof of Proposition 8.3 are equal to V_{2^i}. By the claim above

$$\Phi(\mathcal{H}(g,h,Q)) = V_{o(\overline{g})|Q|}. \tag{50}$$

Since, $\Gamma(2m) \le V_m \le \Gamma(m)$ for every $m \ge 1$, we have obtained that $\Phi(\mathcal{H}(g,h,Q))$ is a finite index subgroup in $SL_2(\mathbb{Z})$. In particular it is finitely generated, finishing the proof of (1).

Finally, to obtain (3) and (4) we recall the well-known fact that $SL_2(\mathbb{Z}) \cong C_4 \star_{\langle -1 \rangle} C_6$ with $C_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ and $C_6 = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \rangle$. Hence, by Kurosh's theorem, $\Gamma(m)$ is free if and only if it is torsion-free. Moreover, all periodic subgroups are conjugated to a subgroup of C_4 or C_6. As $\Gamma(m)$ is normal in $SL_2(\mathbb{Z})$, to verify when $\Gamma(m)$ is free it is enough to verify explicitly which powers of the two matrices are in $\Gamma(m)$. By doing so we see that $\Gamma(m)$ is torsion-free if $m \ne 2$ and -1 is the only torsion element in $\Gamma(2)$. As a consequence, the same conclusion is valid for V_m instead of $\Gamma(m)$. Also recall, e.g. see the proof of Proposition 8.3, that $V_2 = U_1 \cong F_3 \times \langle -1 \rangle$. All this applied to $\Phi(\mathcal{H}(g,h,Q)) = V_{o(\overline{g})|Q|}$ yields that $\mathcal{H}(g,h,Q)$ is a finitely generated free group, except if $|Q| = 1$ and $o(\overline{g}) = 2$. In this case $\mathcal{H}(g,h,Q) \cong F_3 \times C_2$.

When $m := o(\overline{g})|Q| > 2$ the rank of the free group V_m can be computed. In order to do so, recall that $o(\overline{g})$ is even. In particular $\mathcal{H}(g,h,Q) \cong V_m$ with m even. Next, with the same method as in the proof of Proposition 8.3, it can be shown that $[\Gamma(m) : V_m] = 2$ when m is even. Also, because $V_m \le F_2 \le \Gamma(2) = F_2 \times C_2$, we need to compute $[F_2 : V_m] = [\Gamma(2) : V_m]/2$. This can readily be done using (39):

$$[\Gamma(2) : V_m] = [\Gamma(2) : \Gamma(m)]\,[\Gamma(m) : V_m] = 2\frac{[SL_2(\mathbb{Z}) : \Gamma(m)]}{[SL_2(\mathbb{Z}) : \Gamma(2)]} = 2\frac{m^3}{6}\prod_{p|m}(1 - \frac{1}{p^2}).$$

Finally, by Nielsen-Schreier's formula, we obtain that $V_m \cong F_n$ with $n = 1 + \frac{m^3}{6}\prod_{p|m}(1 - \frac{1}{p^2})$. $\quad \square$

In order to make further use of it in the text, we explicitly state the following fact that has been noticed in the previous proof.

Remark 10.7. Let (g,h,Q) be a triple as in Definition 10.5. Note that $[g,h] \notin Q$ yields that $o([g,h]Q) = 2$. Consequently, $\frac{1}{2}\widehat{Q}(1 - [g,h]) = \widehat{Q}\,(\widehat{-[g,h]}) = \langle Q, \widehat{-[g,h]} \rangle$.

Interestingly, a further inspection of the proof of Theorem 10.6 yields the following matrix description of $\mathcal{H}(g,h,Q)$.

Theorem 10.8. *Let (g, h, Q) be a triple such that $\mathcal{H}(g, h, Q) \neq 1$ and let $H = \langle g, h, Q \rangle$. Then there exists a unique primitive central idempotent e of $\mathbb{Q}H$ such that[39]*

$$\mathcal{H}(g, h, Q) = \mathrm{SL}_1(\mathbb{Z}H) \cap (1 - e + \mathbb{Q}He).$$

Moreover, $\mathcal{H}(g, h, Q) = 1 - e + V_m$ for $m = o(gQ)|Q|$ and V_m defined in (49). Also, considering any maximal order $\mathcal{O}$ in $\mathbb{Q}He$ and denoting by $\Gamma(m)$ the principal congruence subgroup of level $m = o(gQ)|Q|$ in $\mathrm{SL}_1(\mathcal{O})$, one has that

$$\Gamma(2m) \leq V_m \leq \Gamma(m) \ and \ [\Gamma(m) : V_m] = 2.$$

In particular, $\mathcal{H}(g, h, Q)$ is a finite index normal subgroup of $1 - e + \mathcal{U}(\mathbb{Z}He)$.

Proof. Note that the proof of Theorem 10.6 in fact works locally, in the sense that it is carried out in $\mathbb{Q}[\langle g, h, Q \rangle]$ rather than $\mathbb{Q}[G]$. Furthermore, the morphism Φ defined in (48) in fact coincides with the projection onto its simple component $\mathbb{Q}^{\alpha_1}[\langle a, b \rangle]$ (notations as in the proof). More precisely, this projection factors through $\mathbb{Q}^{\alpha}[\langle \overline{g} \rangle \times \langle \overline{h} \rangle]$. Projecting thereon is given by first multiplying with the central idempotent e and then the subsequent projection onto $\mathbb{Q}^{\alpha_1}[\langle a, b \rangle]$ is given by multiplying with e_T (we use the subscript T because the component arises from the trivial character). The element $e_T e$ is a primitive central idempotent of $\mathbb{Q}[\langle g, h, Q \rangle]$ and Φ is a concrete realization of the composition of multiplying with $e_T e$ followed by applying ϕ (defined in (36)). In particular Φ is injective on $\mathbb{Q}H \cap (1 - e_T e + \mathbb{Q}He_T e)$.

Next, as $g e_T e = a$ and $h e_T e = b$, one has that $o(g e_T e) = 2 = o(h e_T e)$. Using this a direct verification yields that any element in $\mathbb{Q}H \cap (1 - e_T e + \mathbb{Q}He_T e)$ can be rewritten in the form (45) for some $x_1, x_2, y_1, y_2 \in \mathbb{Q}$. However, as explained in the proof of Theorem 10.6, via the injectivity of Φ, such an element is in $\mathrm{SL}_1(\mathbb{Z}G)$ if and only if (x_1, x_2, y_1, y_2) is admissible for (g, h, Q). Hence $\mathcal{H}(g, h, Q) = \mathrm{SL}_1(\mathbb{Z}H) \cap (1 - e_T e + \mathbb{Q}He_T e)$ and thus $e_T e$ is the desired primitive central idempotent of $\mathbb{Q}H$. The moreover part of the statement of the result was explicitly obtained in the proof of Theorem 10.6, see (50). The final finite index assertion holds because $\mathbb{Q}He_T e \cong M_2(\mathbb{Q})$ hence the centre of the unit group of an order therein is finite and therefore $\mathrm{SL}_1(\mathbb{Z}Ge)$ is of finite index in $\mathcal{U}(\mathbb{Z}He)$.

It remains to prove that $\mathcal{H}(g, h, Q)$ is normal in $1 - e + \mathcal{U}(\mathbb{Z}He)$ or in other words that V_m is normal in $\mathcal{U}(\mathbb{Z}He) \cong \mathcal{U}(\mathbb{Z}^{\alpha_1}[\langle a, b \rangle])$. A presentation of the latter group was obtained in Proposition 8.2. The image under ϕ of the generators are the following matrices:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

---

[39] Confusing at first, $1 - e + \mathbb{Q}He$ however consists simply of the elements of $\mathbb{Q}H$ projecting to the identity in all simple components of $\mathbb{Q}H$ except the simple component $(\mathbb{Q}H)e$ where any element of $\mathbb{Q}He$ is taken.

A direct verification yields that these matrices normalize V_m for every m, finishing the proof. $\square$

Note also that, unlike primitive H-units, the elements $v_{(x_1,x_2,y_1,y_2)}$ are not necessarily unipotent. For instance, using the map Φ in (48) we see that

$$-1 + v_{(x_1,x_2,y_1,y_2)} \text{ is nilpotent } \Leftrightarrow \begin{cases} y_1 = -y_2 \\ y_1 y_2 = x_1 x_2 \end{cases} \tag{51}$$

Finally, it would be interesting to investigate how different H-units interact.

Question 10.9. Let (g_i, h_i, Q_i), with $i = 1, 2$, be two different triples as in Definition 10.5. What is the structure of the group $\langle \mathcal{H}(g_1, h_1, Q_1), \mathcal{H}(g_2, h_2, Q_2) \rangle$?

A first interesting contribution to Question 10.9 would be to determine when it is the direct product of $\mathcal{H}(g_1, h_1, Q_1)$ and $\mathcal{H}(g_2, h_2, Q_2)$.

Remark 10.10. The proof of Theorem 10.6 has shown that locally, i.e. in $\mathbb{Q}[\langle g, h, Q \rangle]$, $v_{(x_1,x_2,y_1,y_2)}$ injects via Φ to an element of $\mathrm{SL}_1(\mathbb{Z}^\alpha[C_2 \times C_2])$. By Proposition 8.2, $\Phi(v_{(x_1,x_2,y_1,y_2)})$ is a product of elements as in Definition 10.1. In particular the elements in the latter are somehow the smallest, hence the name primitive. Besides, the map Φ is directly related with decomposing $\langle g, h, Q \rangle$ as a non-split extension of $\langle Q, [g, h], g^2 \rangle$ by $C_2 \times C_2$. Thus the non-triviality of that second cohomology group was explicitly necessary for the existence of H-units. This clarifies our choice of the name for those units.

10.3. H-units extend bicyclic units with infinite index

Recall that a simple quotient of $\mathbb{Q}[G]$ is called an *exceptional component of type II* if it is of the form[40] $\mathrm{M}_2(\mathbb{Q}(\sqrt{-d}))$ or $\mathrm{M}_2\left(\left(\frac{-a,-b}{\mathbb{Q}}\right)\right)$ with a and b strict positive integers and $d \in \mathbb{N}$ (i.e. $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ is a totally definite quaternion algebra) and it is called an *exceptional component of type I* if it is a division algebra which is not a totally definite quaternion algebra. The terminology 'exceptional' refers to the fact that in their absence the bicyclic units are of finite index in $\mathrm{SL}_1(\mathbb{Z}G)$, e.g. see Corollary 6.7. Therefore, as $\mathrm{SL}_1(\mathcal{O})$ for $\mathcal{O}$ an order in a division algebra has no unipotent units, the focus of current research is on the type II exceptional components

As demonstrated by [2, Appendix A] the most recurrent[41] component of that type is $\mathrm{M}_2(\mathbb{Q})$ and this is a simple quotient of $\mathbb{Q}G$ if and only if G surjects onto D_8 or S_3. It

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[40] The division algebras appearing in these matrix algebras are exactly those having an order with finite unit group [2, Theorem 2.10.].

[41] $\mathbb{Q}G$ has an exceptional 2×2 component exactly when it maps onto one of the 52 groups in [13]. Now the table in [2, Appendix A] says that only 16 of them have no simple component of the type $\mathrm{M}_2(\mathbb{Q})$. Among nilpotent groups there are only 5 such groups.

follows that, when $3 \nmid |G|$ and $\mathrm{M}_2(\mathbb{Q})$ is a simple component of $\mathbb{Q}G$, one has a triple (g, h, Q) satisfying the following:

$$g, h \in N_G(Q) \text{ and } \langle g, h, Q \rangle / Q \cong D_8. \tag{52}$$

In fact the stronger properties $Q \vartriangleleft G$ and $G/Q = \langle gQ, hQ \rangle = \langle ghQ, hQ \rangle \cong D_8$ are satisfied. We will call a triple satisfying (52) a D_8-*triple*.

Convention. For a D_8-triple (g, h, Q) we will always assume that $o(gQ) = o(hQ) = 2$ and $D_8 \cong C_4 \rtimes C_2 = \langle ghQ \rangle \rtimes \langle hQ \rangle$.

Using H-units built on D_8-triples we can describe generically a subgroup of finite index for the following class of groups.

Theorem 10.11. *Let G be a 2-group such that the only exceptional components of $\mathbb{Q}G$ are of the form $\mathrm{M}_2(\mathbb{Q})$, then $\langle \mathrm{Bic}(G), \mathcal{H}(G) \rangle$ is of finite index in $\mathrm{SL}_1(\mathbb{Z}G)$. Consequently, $\langle \mathcal{B}(G), \mathcal{H}(G) \rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$.*

In the above $\mathcal{B}(G)$ denotes the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by the bicyclic and Bass units (see [26, Section 1.2.] for definitions).

Proof. Denote by $\mathcal{E}_{exc}$ the set of primitive central idempotents e such that $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$. For $e \in \mathcal{E}_{exc}$ consider the associated projection $\pi_e : G \to Ge$ (which is the restriction of the natural projection from $\mathcal{U}(\mathbb{Q}G)$ to $\mathcal{U}(\mathbb{Q}G)e$). Since G is a 2-group one has that $\pi_e(G) = Ge \cong D_8$. Consider $Q = \ker(\pi_e)$ and take $g, h \in G$ such that $Ge \cong \langle \pi_e(g) \rangle \rtimes \langle \pi_e(h) \rangle$, with $o(gQ) = 4, o(hQ) = 2$ and $[gQ, hQ] = (gQ)^2$. In this way we obtain a D_8-triple (gh, h, Q) such that $1 \neq \mathcal{H}(gh, h, Q)$ (by Theorem 10.6) and $G = \langle gh, h, Q \rangle$. This allows to apply Theorem 10.8 to conclude that $H_e := \mathcal{H}(gh, h, Q) \leq \mathrm{SL}_1(\mathbb{Z}G)$ is of finite index in $1 - e + \mathcal{U}(\mathbb{Z}Ge)$.

Now consider $\mathcal{E} := \mathrm{PCI}(\mathbb{Q}G) \setminus \mathcal{E}_{exc}$ and take $e' \in \mathcal{E}$, i.e. $\mathbb{Q}Ge' \ncong \mathrm{M}_2(\mathbb{Q})$. By assumption, if $\mathbb{Q}Ge'$ is a division algebra it needs to be a totally definite quaternion algebra, and hence $\mathrm{SL}_1(\mathbb{Z}Ge')$ is finite [37]. It now remains to consider the case that $\mathbb{Q}Ge'$ is not a division algebra, say $\mathrm{M}_n(D)$, and let $\mathcal{O}$ be an order in the division algebra D. In this case classical arguments can be used. Namely, for such a component Corollary 6.7 gives a $y \in \mathbb{Z}$ and subgroup $1 - e' + E_n(y\mathcal{O}) \leq \mathrm{Bic}(G)$. Because $\mathbb{Q}Ge'$ is non-exceptional, the solution to the subgroup congruence problem in higher rank (e.g. see [26, Theorem 11.2.3]) yields that $E_{e'} := E_n(y\mathcal{O})$ is of finite index in $\mathrm{GL}_n(\mathcal{O})$. All together we obtained a subgroup $H = \prod_e H_e \times \prod_{e'} E_{e'}$ of $\mathrm{SL}_1(\mathbb{Z}G)$ which is of finite index in $\prod_{e \in \mathrm{PCI}(\mathbb{Q}G))} \mathrm{SL}_1(\mathbb{Z}Ge)$, hence also in $\mathrm{SL}_1(\mathbb{Z}G)$. The second part now classically follows by a Bass-Milnor Theorem [26, Theorem 11.1.2] which says that the Bass units map to a subgroup of finite index in the Whitehead group $K_1(\mathbb{Z}G) := \mathrm{GL}(\mathbb{Z}G)^{ab}$ of $\mathbb{Z}G$ and hence [26, Prop. 9.5.11] jointly with H it is of finite index in $\mathcal{U}(\mathbb{Z}G)$. In particular also $\langle \mathcal{B}(G), \mathcal{H}(G) \rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$. $\square$

Remark 10.12. The first part of the proof of Theorem 10.11 shows that when G does not map onto S_3 one can extend the Jespers-Leal theorem [29] by including also the exceptional component $M_2(\mathbb{Q})$. Concretely, let $e \in \mathrm{PCI}(\mathbb{Q}G)$ such that $\mathbb{Q}Ge \cong M_2(\mathbb{Q})$ and $3 \nmid |Ge|$. Then $\mathrm{SL}_1(\mathbb{Z}G)$ contains a subgroup W_e, consisting of H-units, that is of finite index in $1 - e + \mathcal{U}(\mathbb{Z}Ge) \subset 1 - e + \mathbb{Q}Ge$. Moreover W_e contains a principal congruence subgroup of level $2o(gQ)|Q| = 2 \cdot 2 \cdot \frac{|G|}{8} = \frac{|G|}{2}$ of $\mathrm{SL}_1(\mathbb{Z}Ge)$. More precisely, $W_e \cong V_{|G|/4}$.

The condition $3 \nmid |Ge|$ stems from the fact that the proof of Theorem 10.6 requires Proposition 8.2. On turn the latter originates from a splitting of D_8 and needs a precise understanding on subgroups of small index in $\mathrm{SL}_2(\mathbb{Z})$. Using [48], the necessary tools seem to exist to extend the results to S_3. In particular we expect that the above and Theorem 10.11 extends to non 2-groups.

By a result of Jespers and del Río [25] natural examples for finite groups G as in Theorem 10.11 can be found when $\mathcal{U}(\mathbb{Z}G)$ is virtually a direct product of free products of abelian groups. Such groups have been classified and the 2-groups of this type are isomorphic to $K \times C_2^n$ with K one of the following types:

$\mathcal{G}_1$: $\langle x, y \mid x^4 = y^4 = 1$ and $y^2, x^2 = [x,y]$ central $\rangle$,
$\mathcal{G}_2$: $\langle x, y_1, \ldots, y_n \mid x^4 = y_i^2 = [y_i, y_j] = 1$ and $x^2, [x, y_i]$ central $\rangle$,
$\mathcal{G}_3$: $\langle x, y_1, \ldots, y_n \mid x^4 = y_i^4 = y_i^2[x, y_i] = [y_i, y_j] = 1$ and x^2, y_i^2 central $\rangle$,
$\mathcal{G}_4$: $\langle x, y_1, \ldots, y_n \mid x^2 = y_i^2 = [y_i, y_j] = [[x, y_i], y_j] = [x, y_i]^2 = 1 \rangle$,
$\mathcal{G}_5$: $\langle x, y_1, \ldots, y_n \mid x^2 = y_i^4 = y_i^2[x, y_i] = [y_i, y_j] = [[x, y_i], x] = 1 \rangle$,
$\mathcal{G}_6$: $\langle x, y_1, \ldots, y_n \mid x^4 = y_i^4 = x^2y_1^2 = y_i^2[x, y_i] = [y_i, y_j] = [y_i^2, x] = 1 \rangle$,
$\mathcal{G}_7$: $\langle x, y_1, \ldots, y_n \mid x^4 = x^2y_i^4 = y_i^{-2}[x, y_i] = [y_i, y_j] = 1 \rangle$.

Remark 10.13. In [25] the relation $y_i^2[x, y_i] = 1$ is written for the groups in class $\mathcal{G}_7$ (which are the groups of 'type (g)' in [25]). Inspection of the proof however shows that it must be $y_i^{-2}[x, y_i] = 1$.

The group in $\mathcal{G}_1$ is simply $C_4 \rtimes C_4$ with C_4 acting by inversion. It is known [26, Corollary 12.7.2] that $\mathrm{Bic}(C_4 \rtimes C_4)$ is of infinite index in $\mathrm{SL}_1(\mathbb{Z}G)$.

For groups G as above, [25, Theorem 1.3] tells that the non-division algebra components of $\mathbb{Q}G$ are isomorphic to $M_2(\mathbb{Q})$. Moreover, $\mathbb{Q}G$ does not have exceptional components of type I. Thus we can use Theorem 10.11 to obtain that $\mathcal{H}(G)$ is a so-called congruence subgroup of $\mathrm{SL}_1(\mathbb{Z}G)$. More importantly, as a byproduct we obtain a more precise version of [12, Theorem 1.1] and a classification-free proof.

Corollary 10.14. Let $G = K \times C_2^n$, with K a group in $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_7$, and let q denote the number of simple components of the type $M_2(\mathbb{Q})$ in $\mathbb{Q}G$. Then, there exist D_8-triples (g_i, h_i, Q_i), $1 \leq i \leq q$, such that $\langle \mathcal{H}(g_i, h_i, Q_i) \rangle \cong \prod_{i=1}^q \mathcal{H}(g_i, h_i, Q_i)$ is a finite index normal subgroup of $\mathrm{SL}_1(\mathbb{Z}G)$. In particular, the H-units $\mathcal{H}(G)$ are of finite index despite

that $\mathrm{Bic}(G)$ *can be of infinite index. Moreover, if* $G \not\cong D_8$, $\langle \mathcal{H}(g_i, h_i, Q_i) \rangle \cong F_n^q$ *with* $n = 1 + \frac{|G|^3}{2.4^4}$.

In fact $\langle \mathcal{H}(g_i, h_i, Q_i) \rangle$ is the largest finite index subgroup in $\mathrm{SL}_1(\mathbb{Z}G)$ which is the direct product of free groups. This follows from Theorem 10.8, saying that $\mathcal{H}(g_i, h_i, Q_i) = \mathrm{SL}_1(\mathbb{Z}G) \cap 1 - e + \mathbb{Q}Ge$ for an associated $e \in \mathrm{PCI}(\mathbb{Q}G)$, and the indecomposability of $SL_1(\mathcal{O})$ for $\mathcal{O}$ any order in some $\mathrm{M}_n(D)$ ($n \neq 1$) [38]. That $\prod_{e \in \mathrm{PCI}(\mathbb{Q}G)} \mathrm{SL}_1(\mathbb{Z}G) \cap 1 - e + \mathbb{Q}Ge$ is the largest direct product of free groups was already obtained by del Río and Ruiz in [12, Theorem 1.1]. However our proof is uniform, i.e. we do not use the classification for G, and yield more explicit generators.

Proof of Corollary 10.14. Let $e \in \mathrm{PCI}(\mathbb{Q}G)$. By [25, Theorem 1.3], either $\mathbb{Q}Ge$ is a totally definite quaternion algebra or it is isomorphic to $\mathrm{M}_2(\mathbb{Q})$. In the former case $\mathrm{SL}_1(\mathbb{Z}Ge)$ is finite, so we only need to consider the case $\mathbb{Q}Ge \cong \mathrm{M}_2(\mathbb{Q})$. As pointed out in Remark 10.12, the proof of Theorem 10.11 gives a D_8-triple (g_e, h_e, Q_e) such that $\mathcal{H}(g_e, h_e, Q_e)$ is of finite index in $1 - e + \mathcal{U}(\mathbb{Z}Ge)$. In particular, picking one such triple for every e now yields a subgroup of $\mathrm{SL}_1(\mathbb{Z}G)$ which is even of finite index in the overgroup $\prod_e \mathrm{SL}_1(\mathbb{Z}Ge)$ such that $\langle \mathcal{H}(g_i, h_i, Q_i) \rangle \cong \prod_{i=1}^q \mathcal{H}(g_i, h_i, Q_i)$. Following Theorem 10.8, $\mathcal{H}(g_e, h_e, Q_e)$ is isomorphic to a certain group V_m which is normal in $\mathcal{U}(\mathbb{Z}Ge)$. This implies that $\prod_{i=1}^q \mathcal{H}(g_i, h_i, Q_i)$ is normal in $\prod_e \mathcal{U}(\mathbb{Z}Ge)$ and in particular in the subgroup $\mathcal{U}(\mathbb{Z}G)$. Now, as mentioned earlier for example in the group $\mathcal{G}_1$ the bicyclic units are of infinite index. This finishes the first part.

That $\mathcal{H}(g_i, h_i, Q_i) \cong F_n$ with n as in the statement is a combination of Theorem 10.6 and Remark 10.12. More precisely, the latter says that $\mathcal{H}(g_i, h_i, Q_i) \cong V_m$ with $m = o(gQ)|Q| = \frac{|G|}{4}$. Now would $q \neq 0$ and $o(gQ)|Q| = 2$, then $Q = 1$ and $o(g) = 2$. This however entails that $G \cong D_8$, which was excluded. Therefore, Theorem 10.6 yields that $\mathcal{H}(g_i, h_i, Q_i)$ is a free group and as m is a 2-power the product in Theorem 10.6 only runs over the prime divisor 2, yielding the desired formula. $\quad\square$

To finish this section, we exhibit a first surprising application of H-units which indicate a first of many new possible paths of research.

Example of a normal complement via H-units

Consider

$$\Gamma = C_4 \rtimes C_4 := \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1} \rangle,$$

a group in the class $\mathcal{G}_1$. Recall that $\mathbb{Q}[\Gamma] \cong \mathbb{Q}[C_2 \times C_2] \oplus 2\mathbb{Q}(i) \oplus \left(\frac{-1,-1}{\mathbb{Q}} \right) \oplus \mathrm{M}_2(\mathbb{Q})$. In particular we can apply all the results above. Concretely, notice that $(ab, b, \langle b^2 \rangle)$ is a D_8-triple fulfilling the non-triviality condition of Theorem 10.6 with $o(\overline{ab})|\langle b^2 \rangle| = 4$. As there is only one matrix component and $\mathcal{Z}(\mathcal{U}(\mathbb{Z}\Gamma))$ is finite, Theorem 10.6 now yields that

$\mathcal{H}(ab, b, \langle b^2 \rangle) \cong F_{1 + \frac{4^3 \cdot 3}{6 \cdot 4}} = F_9$ finite index normal subgroup in $\mathcal{U}(\mathbb{Z}\Gamma)$.

Furthermore, by Theorem 10.8, $\mathcal{H}(ab, b, \langle b^2 \rangle)) = \mathrm{SL}_1(\mathbb{Z}\Gamma) \cap (1 - e + \mathbb{Q}\Gamma e)$ with $\mathbb{Q}\Gamma e \cong \mathrm{M}_2(\mathbb{Q})$ and $\mathcal{H}(ab, b, \langle b^2 \rangle)) = 1 - e + V_4$ is a normal subgroup. Furthermore,

$$\mathrm{SL}_1(\mathbb{Z}\Gamma) = \langle \mathcal{H}(ab, b, \langle b^2 \rangle)), G \cap \mathrm{SL}_1(\mathbb{Z}\Gamma) \rangle.$$

The latter can be seen from the facts: (i) $\mathrm{SL}_1(\mathbb{Z}\Gamma) \leq 1 \times \mathrm{SL}_1\left(\frac{-1,-1}{\mathbb{Z}}\right) = Q_8 \times \mathrm{SL}_2(\mathbb{Z})$ and (ii) $\Gamma \cap \mathrm{SL}_1(\mathbb{Z}\Gamma) = \langle a^2 \rangle$ (all other $g \in \Gamma$ have a non-trivial projection in one of the commutative components) and with a^2 mapping to -1 in $\mathrm{SL}_1\left(\frac{-1,-1}{\mathbb{Z}}\right)$. With a bit more of work[42] one can prove that $\mathcal{U}(\mathbb{Z}\Gamma) = \langle \mathcal{H}(ab, b, \langle b^2 \rangle), \pm\Gamma \rangle$. All together we recover [28, Theorem 5.1.] (also see [4, Example 5.5]):

$$\mathcal{U}(\mathbb{Z}[\Gamma]) \cong F_9 \rtimes \pm\Gamma$$

where $F_9 = \mathcal{H}(ab, b, \langle b^2 \rangle)$. This description of the free group F_9 yields a *new set of generators.*

A full presentation can also be obtained. A presentation was given in [4, Example 5.5], using [28, Theorem 5.1.], but our methods will yield a considerably more symmetric presentation. Indeed, as $F_9 = \mathcal{H}(ab, b, \langle b^2 \rangle) = 1 - e + V_4$, it is enough to compute the action of $\pi_e(\Gamma) = \langle \pi_e(ab), \pi_e(b) \rangle \cong D_8$ on V_4, all seen as subgroups of $\mathrm{GL}_2(\mathbb{Z})$. To do so, recall that Proposition 8.2 and its proof yields a matrix representation of all elements and also record the required actions. More precisely, $V_2 = \langle w^2, v^2, w^{-1}v, -1 \rangle$ with $w = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Also, $\pi_e(ab) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\pi_e(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Via a direct application of the Reidemeister-Schreier method one obtains that

$$V_4 = \langle v^4, w^4, w^2 v^{-2}, x, v^2 x v^{-2} \mid x \in S \rangle \tag{53}$$

with $S = \{(w^{-1}v)^2, (wv^{-1})^2, (w^{-1}v)(wv^{-1})^{-1}\}$. Using the action in Proposition 8.2 (or the matrices above) one readily verifies that

$$y^{\pi_e(ab)} = y^{-1} \text{ for } y \in \{v^4, w^4, (w^{-1}v)(wv^{-1})^{-1}\}$$

and

$$z^{\pi_e(b)} = z^{-1} \text{ for } z \in \{w^2 v^{-2}, (w^{-1}v)^2, (wv^{-1})^2\}.$$

Furthermore, $(w^2 v^{-2})^{\pi_e(ab)} = w^{-4}(w^2 v^{-2})v^4$ and

---

[42] For example, via the methods as in the proof of Claim 4 in the proof of Theorem 10.16 one can prove that $\mathcal{U}(\mathbb{Z}\Gamma) = \langle \pm\Gamma, \mathrm{SL}_1(\mathbb{Z}G) \rangle$. The latter fact can alternatively be deduced from [28, Theorem 5.1].

$$\left((w^{-1}v)(wv^{-1})^{-1}\right)^{\pi_e(b)} = (w^{-1}v)^2(w^{-1}v)(wv^{-1})^{-1}(wv^{-1})^2.$$

The remaining actions are computed similarly, yielding all together the following presentation:

$$V(\mathbb{Z}\Gamma) = \langle s, t, u, x_1, x_2, x_3, y_1, y_2, y_3 \rangle \rtimes \langle a, b \rangle \cong F_9 \rtimes \Gamma$$

where the action is the following:

$$
\begin{array}{lll}
s^{ab} = s^{-1} & x_1^{ab} = x_2 & y_1^{ab} = s^{-1}y_2 s \\
t^{ab} = t^{-1} & x_2^{ab} = x_1 & y_2^{ab} = s^{-1}y_1 s \\
u^{ab} = t^{-1}us & x_3^{ab} = x_3^{-1} & y_3^{ab} = s^{-1}y_3 s \\
s^b = t & x_1^b = x_1^{-1} & y_1^b = uy_1^{-1}u^{-1} \\
t^b = s & x_2^b = x_2^{-1} & y_2^b = uy_2^{-1}u^{-1} \\
u^b = u^{-1} & x_3^b = x_1^{-1}x_3 x_2 & y_3^b = uy_1^{-1}y_3 y_2 u^{-1}
\end{array}
$$

The previous example naturally raises the following question:

Question 10.15. When is $\langle \mathrm{Bic}(G), \mathcal{H}(G) \rangle$ a normal complement for the trivial units? Also, when is $\mathrm{SL}_1(\mathbb{Z}G)$ a complement?

10.4. Applications to the abelianisation conjectures

Recall that $\mathcal{U}(\mathbb{Z}G)$ is a finitely generated group and hence $\mathcal{U}(\mathbb{Z}G)^{ab}$ is the direct product of a finitely generated free abelian group $\mathbb{Z}^n$ and a finite abelian group. One calls n the rank of the abelian group $\mathcal{U}(\mathbb{Z}G)^{ab}$. Also recall that the centre $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ of $\mathcal{U}(\mathbb{Z}G)$ is finitely generated, hence its rank is finite. Finally, recall that $\mathcal{U}(\mathbb{Z}G) = \pm V(\mathbb{Z}G)$ with $V(\mathbb{Z}G)$ the *group of invertible elements with augmentation one*.

We finish the article with another application of H-units. We do this by giving an answer, for the class of groups considered above, to the following recent questions on the abelianisation of the unit group of $\mathbb{Z}G$:

(R1) Is the rank of the abelian groups $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ and $\mathcal{U}(\mathbb{Z}G)^{ab}$ equal? In particular if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite, is $\mathcal{U}(\mathbb{Z}G)^{ab}$ also finite? (see [2, Question 7.8 and Proposition 7.9], the labelling of the questions is taken from [4].)

(P) If $V(\mathbb{Z}G)^{ab}$ contains an element of prime order p, then so does G^{ab}? (see [4, page 2])

(E1) Is $\exp V(\mathbb{Z}G)^{ab} = \exp G^{ab}$? (see [4, page 2])

The labelling of the questions is taken from [4]. In question (E1) by $\exp \Gamma$ of a group Γ we mean the least common multiple of the elements of finite order in Γ. Note that question (E2) implies question (P).

In general, as proven in [2, Proposition 6.1],

$$\operatorname{rank}\mathcal{U}(\mathbb{Z}\mathrm{G})^{\mathrm{ab}} \geq \operatorname{rank}\mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})).$$

In case $\mathbb{Q}G$ does not have exceptional components the above is an equality by [31, Theorem 6.3], and hence question (R1) has a positive answer in that case. However in [4, Theorem D] one group, where $\mathbb{Q}G$ has an exceptional component of the form $\mathrm{M}_2(\mathbb{Q}(i))$, was found where the rank of the abelianisation is non-zero but the centre of the unit group is finite (thus the rank of $\mathcal{U}(\mathbb{Z}G)^{ab}$ is larger than expected). Surprisingly, crucially using $\mathcal{H}(G)$, for each group in $\mathcal{G}_i \times C_2^n$, for $1 \leq i \leq 7$, we obtain an expression for the exponent and rank of the abelianisation.

Recall that for $G \in \mathcal{G}_i$ and $e \in \mathrm{PCI}(\mathbb{Q}G)$, one has that $\mathbb{Q}Ge$ either is a division algebra that is not exceptional of type I or it is a simple component of type $M_2(\mathbb{Q})$ (and thus of exceptional type II). The set of primitive central idempotents e of the latter type we will denote (as in the proof of Theorem 10.11) as $\mathcal{E}_{exc}$. Also recall that for $e \in \mathcal{E}_{exc}$ the group G can be written as an extension as follows:

$$1 \to N \longrightarrow G \xrightarrow{\pi_e} Ge \cong D_8 = \langle a : a^4 = 1\rangle \rtimes \langle b : b^2 = 1\rangle \to 1.$$

Consider the following property:

$$(\star)\, \forall e \in \mathcal{E}_{exc}, \exists\, g, h \in G : o(\pi_e(g)) = o(ab) \text{ and } o(\pi_e(h)) = o(b). \tag{54}$$

Note that this property is satisfied if G is a split extension of D_8. Under the additional property $(\star)$ we will give a positive answer to (R1) and (P), despite that such groups may have arbitrarily many exceptional components.

Theorem 10.16. *Let $G = K \times C_2^n$ with K a group in $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_7$ and π the natural map of $\mathcal{U}(\mathbb{Z}G)$ onto $\mathcal{U}(\mathbb{Z}G)^{ab}$. Then*

$$\operatorname{rank}\mathcal{U}(\mathbb{Z}\mathrm{G})^{\mathrm{ab}} = \operatorname{rank}\mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})) + \operatorname{rank}\pi(\langle\mathcal{H}(\mathrm{G})_{\mathrm{un}}\rangle),$$

where $\mathcal{H}(G)_{un} = \{x \in \mathcal{H}(G) \mid x \text{ is unipotent }\}$. Furthermore,

$$\exp(V(\mathbb{Z}G)^{ab}) = \operatorname{lcm}\left(\exp(G^{ab}), \exp\left(V(\mathbb{Z}G)/\mathcal{Z}(V(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle\mathrm{Bic}(G), \pm G\rangle)\right)^{ab}\right).$$

Moreover, if G satisfies $(\star)$, then $cl_{\mathcal{U}(\mathbb{Z}G)}(\langle\mathrm{Bic}(G), \pm G\rangle)$, i.e. the normal closure of the bicyclic and trivial units, together with the centre $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$ and (R1) and (P) have a positive answer.

Recall that for the group $\mathcal{G}_1$ the group $\langle\mathrm{Bic}(G), \pm G\rangle$ is of infinite index, which we expect to be a rather general phenomena for the classes of groups considered. Thus it is somehow surprising that under property $(\star)$ their normal closure is of finite index.

Remark 10.17. (1) The combination of [20, Lemma 4.4] with Theorem 10.8 yields that $\langle \mathcal{H}(G)_{un} \rangle$ is of infinite index in $\mathrm{SL}_1(\mathbb{Z}G)$ whenever $|G| \geq 32$ and $G = K \times C_2^n$ with $K \in \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_7$. Thus for this class of groups the rank-formula reduces the problem of determining the abelianisation to a significantly smaller group.

(2) The proof of Theorem 10.16 will furthermore yield that

$$\mathrm{rank}\, \mathcal{U}(\mathbb{Z}\mathrm{G})^{\mathrm{ab}} = \mathrm{rank}\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})) + \mathrm{rank}\, \left(\mathcal{U}(\mathbb{Z}\mathrm{G})/\mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G}))\, \mathrm{cl}_{\mathcal{U}(\mathbb{Z}\mathrm{G})}(\langle \mathrm{Bic}(\mathrm{G}), \pm\mathrm{G} \rangle) \right)^{\mathrm{ab}}.$$

Clearly $D_8 \in \mathcal{G}_4$ and $G = D_8 \times C_2^n$ is a cut group (i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite) and G satisfies $(\star)$, thus Theorem 10.16 yields that

$$\mathrm{rank}\, (\mathcal{U}(\mathbb{Z}[\mathrm{D}_8 \times \mathrm{C}_2^n])^{\mathrm{ab}}) = 0$$

for all n, answering Question 7.6 in case $G = D_8$.

Proof of the torsion-free part of Theorem 10.16. Recall that taking abelianisation is right exact. More precisely, for any group Γ and normal subgroup N ons has the short exact sequence

$$1 \to \pi_\Gamma(N) \to \Gamma^{ab} \to (\Gamma/N)^{ab} \to 1 \tag{55}$$

where $\pi_\Gamma : \Gamma \to \Gamma^{ab}$ is the canonical projection and thus $\pi_\Gamma(N) \cong N/N \cap [\Gamma, \Gamma]$.

In particular

$$\mathrm{rank}\, (\Gamma^{\mathrm{ab}}) = \mathrm{rank}\, (\pi_\Gamma(\mathrm{N})) + \mathrm{rank}\, ((\Gamma/\mathrm{N})^{\mathrm{ab}}). \tag{56}$$

Furthermore, it follows from the proof of [26, Proposition 5.5.1] that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap \mathrm{SL}_1(\mathbb{Z}G)$ is finite. Since $[\mathcal{U}(\mathbb{Z}G), \mathcal{U}(\mathbb{Z}G)] \leq \mathrm{SL}_1(\mathbb{Z}G)$ and thus also $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \cap [\mathcal{U}(\mathbb{Z}G), \mathcal{U}(\mathbb{Z}G)]$ is finite, this implies that

$$\mathrm{rank}\, \pi(\langle \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})), \mathrm{N} \rangle) = \mathrm{rank}\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})) + \mathrm{rank}\, \pi(\mathrm{N}) \tag{57}$$

for every normal subgroup N in $\mathcal{U}(\mathbb{Z}G)$ such that $\pi(N) \cap \pi(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)))$ is finite. The latter condition is for example satisfied for a subgroup of $\langle \mathrm{SL}_1(\mathbb{Z}G), \pm G \rangle$. Moreover, if $N \leq \mathrm{SL}_1(\mathbb{Z}G)$, then

$$\mathrm{rank}\, (\mathrm{SL}_1(\mathbb{Z}\mathrm{G})/\mathrm{N})^{\mathrm{ab}} = \mathrm{rank}\, (\mathrm{SL}_1(\mathbb{Z}\mathrm{G})\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G})/\mathrm{N}\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G}))^{\mathrm{ab}} \tag{58}$$

and thus, because $\langle \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)), \mathrm{SL}_1(\mathbb{Z}G) \rangle$ is of finite index in $\mathcal{U}(\mathbb{Z}G)$ (see [26, Proposition 5.5.1]),

$$\mathrm{rank}\, (\mathrm{SL}_1(\mathbb{Z}\mathrm{G})/\mathrm{N})^{\mathrm{ab}} \geq \mathrm{rank}\, (\mathcal{U}(\mathbb{Z}\mathrm{G})/\mathrm{N}\, \mathcal{Z}(\mathcal{U}(\mathbb{Z}\mathrm{G}))^{\mathrm{ab}}. \tag{59}$$

Consider $\mathcal{E}_{exc} := \{e \in \text{PCI}(\mathbb{Q}G) \mid \mathbb{Q}Ge \cong \text{M}_2(\mathbb{Q})\}$ and let $e_1, \ldots, e_q$ be its distinct elements. For the rest of the proof consider the subgroup $\langle \pm G, \text{Bic}(G), \prod_{i=1}^{q} \mathcal{H}(g_i, h_i, Q_i) \rangle$ delivered by Corollary 10.14, where by Theorem 10.8,

$$\mathcal{H}(g_i, h_i, Q_i) = \text{SL}_1(\mathbb{Z}G) \cap (1 - e_i + \mathbb{Q}Ge_i).$$

As explained in the proof of Theorem 10.8, for each i we have a morphism as in (48), which we now denote by Φ_i, is the composition of projecting to $\mathbb{Q}Ge_i$ with the isomorphism ϕ defined in (36) (Section 8.1). Moreover, by Remark 10.7, one has that $e_i = \langle Q_i, \widehat{-[g_i, h_i]} \rangle = \widehat{Q_i} \frac{(1-[g_i,h_i])}{2}$. By Theorem 10.8, $\mathcal{H}(g_i, h_i, Q_i) = 1 - e_i + V_{o(g_iQ_i)|Q_i|}$. Note that $o(g_iQ_i)|Q_i| = 2|Q_i| = |G|/4$ does not depend on i, and thus $|Q_i| = 2^{m-1}$ for some positive integer m.

First we choose $N = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic}(G), \pm G \rangle)$. It follows from[43] [4, Proposition 3.1] that any bicyclic unit of $\mathbb{Z}G$ is a product of commutators of elements of $\langle \text{Bic}(G), \pm G \rangle$. Hence, $\pi(cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic}(G), \pm G \rangle))$ is finite. Consequently, because of (56), and (57) we obtain that:

$$\text{rank}\,\mathcal{U}(\mathbb{Z}G)^{\text{ab}} = \text{rank}\left(\pi(\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic(G)} \rangle))\right)$$
$$+ \text{rank}\left(\mathcal{U}(\mathbb{Z}G)/\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic(G)}, \pm G \rangle)\right)^{\text{ab}}$$
$$= \text{rank}\,\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) + \text{rank}\left(\mathcal{U}(\mathbb{Z}G)/\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic(G)}, \pm G \rangle)\right)^{\text{ab}}.$$
$$(60)$$

Thus, if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \text{Bic}(G), \pm G \rangle)$ is of finite index, then (R1) has a positive answer.

Now consider the bicyclic unit $b_{\widetilde{h_i}, g_i} := 1 + \widetilde{h_i}g_i(1 - h_i^{-1})$. Using that $o(h_iQ) = 2$ one sees that

$$e_i b_{\widetilde{h_i}, g_i} = e_i \left(1 + \frac{o(h_i)}{o(h_iQ)}(1 + h_i)g_i(1 - h_i)\right) = e_i\left(1 + o(h_i)(1 + h_i)g_i\right)$$

$$= e_i v_{(\frac{-o(h_i)}{2}, \frac{o(h_i)}{2}, \frac{-o(h_i)}{2}, \frac{o(h_i)}{2})}.$$

Therefore, using (48), composing with ϕ gives that

$$\Phi_i(b_{\widetilde{h_i}, g_i}) = \begin{pmatrix} 1 + o(h) & -o(h) \\ o(h) & 1 - o(h) \end{pmatrix}.$$

Next consider the bicyclic units $b_{h_i, \widetilde{g_i}} := 1 + (1 - g_i)h_i\widetilde{g_i}$ and $b_{\widetilde{g_i}, h_i} := 1 + \widetilde{g_i}h_i(1 - g_i)$. Analogously as with the preceding unit one verifies that

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[43] In [4] the result is only proven for bicyclic units of the form $1 + (1 - h)g\widetilde{h}$, but the same proof works for those of the form $1 + \widetilde{h}g(1 - h)$.

$$e_i b_{h_i, \tilde{g}_i} = e_i \left(1 + \frac{o(g_i)}{o(g_i Q_i)} (1 - g_i) h_i (1 + g_i) \right) = e_i v_{(0, o(g_i), 0, 0)}$$
$$e_i b_{\tilde{g}_i, h_i} = e_i \left(1 + \frac{o(g_i)}{o(g_i Q_i)} (1 + g_i) h_i (1 - g_i) \right) = e_i v_{(o(g_i), 0, 0, 0)}.$$

Thus after composing with ϕ we obtain the matrices

$$\Phi_i(b_{h_i, \tilde{g}_i}) = \begin{pmatrix} 1 & 2o(g_i) \\ 0 & 1 \end{pmatrix} \text{ and } \Phi_i(b_{\tilde{g}_i, h_i}) = \begin{pmatrix} 1 & 0 \\ 2o(g_i) & 1 \end{pmatrix}.$$

Now suppose that G has property $(\star)$. In particular, $o(h_i) = 2 = o(g_i)$ and so the matrices obtained above are simply those already encountered in Proposition 8.2. Namely:

$$(\Phi_i(b_{\widetilde{h_i}, g_i}), \Phi_i(b_{h_i, \tilde{g}_i}), \Phi_i(b_{\tilde{g}_i, h_i})) = (vw^{-1}, w^2, v^2).$$

Therefore, by (37) we obtain that

$$\Phi_i(\pm \langle b_{h_i, \tilde{g}_i}, b_{\tilde{g}_i, h_i}, b_{\widetilde{h_i}, g_i} \rangle) \cong V_2$$
$$= \{ \begin{pmatrix} 1 + 2\,l_2 & 2\,t_1 \\ 2\,t_2 & 1 + 2\,l_1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid l_1 \equiv l_2 \text{ and } t_1 \equiv t_2 \mod 2 \}.$$

In particular, because of the description given in (50), we get that $\mathcal{H}(g_i, h_i, Q_i) \cong \Phi_i(\mathcal{H}(g_i, h_i, Q_i)) \lhd \Phi_i(\pm \langle b_{h_i, \tilde{g}_i}, b_{\tilde{g}_i, h_i}, b_{\widetilde{h_i}, g_i} \rangle)$. Denote

$$H := cl_{\mathcal{U}(\mathbb{Z}G)} \left(\langle b_{h_i, \tilde{g}_i}, b_{\tilde{g}_i, h_i}, b_{\widetilde{h_i}, g_i}, g_i, h_i \mid 1 \le i \le q \rangle \right).$$

Next, note that

$$[H, \mathcal{H}(g_i, h_i, Q_i)] = [1 - e_i + He_i, \mathcal{H}(g_i, h_i, Q_i)] \cong [\Phi_i(H), \Phi_i(\mathcal{H}(g_i, h_i, Q_i))],$$

with $\Phi_i(g_i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\Phi_i(h_i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We will prove that H is of finite index in $\prod_{i=1}^{q} \mathcal{U}(\mathbb{Z}Ge_i)$. Since the latter contains $\mathcal{U}(\mathbb{Z}G)$ and $H \le cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), \pm G \rangle)$, this would finish the proof of the last statement of the Theorem.

Recall that $\mathcal{H}(g_i, h_i, Q_i) = 1 - e_i + V_{|G|/4}$. Hence, by the above descriptions,

$$[H, \prod_{i=1}^{q} \mathcal{H}(g_i, h_i, Q_i)] = \prod_{i=1}^{q} 1 - e_i + [\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_{|G|/4}]. \qquad (61)$$

We claim that $V_{|G|/4} / [\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_{|G|/4}]$ is finite. To prove this we need following general group theoretical inequality:

Claim 1. *Let $N_2 \le N_1$ be normal finite index subgroups of some group Γ. Denote $[N_1 / [\Gamma, N_2] : N_2 / [\Gamma, N_2]] = n$ which divides $[N_1 : N_2]$. If $N_1 / [\Gamma, N_1]$ is finite, then*

$N_2/[\Gamma, N_2]$ *is finite. Furthermore,* $|[\Gamma, \Gamma]/[\Gamma, N_2]|$ *divides* $([\Gamma : N_1]n)^{[s/2]+1}$ *where s is the product of all p^{e_p} with e_p the maximum exponent of p dividing $n[\Gamma : N_1]$.*

Indeed, to prove the claim and for notation simplicity we denote $M := [\Gamma, N_2]$. Note that N_2/M is central in Γ/M and thus the latter is central-by-finite. Hence, by a well-known theorem of Schur, $[\Gamma/M, \Gamma/M]$ is finite. Moreover, by [59, Theorem 1], $|[\Gamma/M, \Gamma/M]| \mid ([\Gamma/M : \mathcal{Z}(G)/M])^{[t/2]+1}$ with t the product of all p^{l_p} with l_p the maximum exponent of p dividing $[\Gamma/M : \mathcal{Z}(\Gamma)/M]$. As $[\Gamma/M : \mathcal{Z}(\Gamma)/M] \mid [\Gamma/M : N_1/M].[N_1/M : N_2/M]$ it also divides the multiple mentioned in the statement of the claim. Now, as $[\Gamma, N_1]/M \leq [\Gamma, \Gamma]/M$ is finite, and by assumption also $N_1/[\Gamma, N_1]$, we obtain that N_1/M is finite. In particular N_2/M is finite as desired.

Using this we can now prove that:

Claim 2. $V_{|G|/4}/[\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_{|G|/4}]$ *is finite.*

Applying Claim 1 to $\Gamma = \langle V_2, \Phi(g_i), \Phi(h_i) \rangle$, $N_2 = V_{|G|/4}$ and $N_1 = V_2$ we see that it is enough to prove that

$$V_2/[\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_2] \text{ is finite.}$$

The latter can be seen via an explicit set of generators. Concretely, following (37), $V_2 = \langle w^2, v^2, w^{-1}v, -1 \rangle$ with $w = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. It is readily computed that

$$v^{\Phi_i(h_i)} = w, v^{\Phi_i(g_i)} = v^{-1} \text{ and } w^{\Phi_i(g_i)} = w^{-1}. \tag{62}$$

From this one verifies that the square of each generator is a single commutator. Therefore, $V_2/[\langle \Phi_i(g_i), \Phi_i(h_i) \rangle, V_2]$ is an elementary abelian 2-group, finishing the proof of Claim 2.

Thus all together we have proven that when G satisfies property $(\star)$, then $\prod_{i=1}^q [H, \mathcal{H}(g_i, h_i, Q_i)]$ is of finite index in $\prod_{i=1}^q \mathcal{U}(\mathbb{Z}Ge_i)$. But by the normality of H in $\mathcal{U}(\mathbb{Z}G)$ we have that $[H, \mathcal{H}(g_i, h_i, Q_i)] \leq H$ for each i and hence $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) H$ indeed is of finite index in $\mathcal{U}(\mathbb{Z}G)$. As mentioned earlier, by (60) this implies that (R1) holds for G if it satisfies $(\star)$.

Now consider again a general $G \in \mathcal{G}_j \times C_2^n$. We will now apply (56) to $\Gamma = \mathcal{U}(\mathbb{Z}G)$ and $N = \langle \mathcal{Z}(\mathcal{U}(\mathbb{Z}G)), \mathcal{H}(G)_{un} \rangle$. Using (57), we see that the first part of the statement follows if $(\mathcal{U}(\mathbb{Z}G)/\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)).\langle \mathcal{H}(G)_{un} \rangle)^{ab}$ is finite.

For this recall that each $\mathcal{H}(g_i, h_i, Q_i)$ contains the element $1 - e_i + \begin{pmatrix} 1 & 2^m \\ 0 & 1 \end{pmatrix}$ (for simplicity, we abuse notation by writing matrices in e_i-part). As $V_{o(g_i Q_i).|Q_i|}$ is normal in $\mathcal{U}(\mathbb{Z}Ge_i)$, the H-units $\mathcal{H}(g_i, h_i, Q_i)$ contain the group $A_i := 1 - e_i + cl_{\mathcal{U}(\mathbb{Z}Ge_i)}(\langle \begin{pmatrix} 1 & 2^m \\ 0 & 1 \end{pmatrix} \rangle)$. Notice that $A_i \leq \langle \mathcal{H}(G)_{un} \rangle$. Now as abelianisation is right exact and by (58) it is enough

to prove that $(\mathrm{SL}_1(\mathbb{Z}G)/\prod_i A_i)^{ab}$ is finite. To prove the latter it is sufficient to show that $\prod_{i=1}^q (\mathrm{SL}_1(\mathbb{Z}Ge_i)/A_i)^{ab}$ is finite. This is directly verified using the presentation in Proposition 8.2 as also $\begin{pmatrix} 1 & 0 \\ 2^m & 1 \end{pmatrix} = 1 - e_i + h^{-1}e_i \begin{pmatrix} 1 & 2^m \\ 0 & 1 \end{pmatrix} he_i \in A_i$. This finishes the proof of the torsion-free part of the Theorem. $\quad\square$

Next,

Proof of the torsion part of Theorem 10.16. For the torsion statement we consider the short exact sequence (55) for $\Gamma = V(\mathbb{Z}G)$ and $N = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G \rangle)$. Hereby the following is crucial:

Claim 3. *For $\Gamma = V(\mathbb{Z}G)$ and $N = \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))\, cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G \rangle)$ holds:*

$$\exp(\Gamma^{ab}) = lcm(\exp(\pi(N)), \exp(\Gamma/N)^{ab}). \tag{63}$$

When $\pi(N)$ is finite Claim 3 is clear, as the claim even holds more generally for any short exact sequence of finitely generated abelian groups with finite kernel. In particular, recalling that $\pi(cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G \rangle))$ is finite, we have that Claim 3 holds when $G \notin \mathcal{G}_7 \times C_2^n$ as for such groups G we have that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite and hence also so is $\pi(N)$.

To handle the family $\mathcal{G}_7 \times C_2^n$ we need to do some more work. First recall that by [30, Theorem 6] and [25, beginning of Section 5] the only simple components of $\mathbb{Q}[G]$ are of the form $\mathbb{Q}, \mathbb{Q}(i), \left(\frac{-1,-1}{\mathbb{Q}} \right), \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{2})} \right)$, or $\mathrm{M}_2(\mathbb{Q})$. Therefore by [30, Lemmas 2 and 3] and [25, Lemma 5.3] G is a subgroup of $C_2^{n_1} \times C_4^{n_2} \times Q_8^{n_3} \times D_8^{n_4} \times Q_{16}^{n_5}$ for some $n_1, \ldots, n_5 \in \mathbb{N}$ with $n_5 = 0$ if $K \notin \mathcal{G}_7$. Recall that both Q_8 and Q_{16} have a unique subgroup of order 2 which moreover is central. Thus the second part of Claim 4, see below, holds for every subgroup of $C_2^{n_1} \times C_4^{n_2} \times Q_8^{n_3} \times D_8^{n_4} \times Q_{16}^{n_5}$. Also note that G has a normal subgroup, say A, so that both groups A and G/A are abelian and also $\exp(G/A)$ divides 4. A result of Cliff, Sehgal and Weiss (see [54, Theorem 31.1]) yields that $V_{tf}(\mathbb{Z}G) := V(\mathbb{Z}G) \cap (1 + \ker(\omega_A)\ker(\omega_G))$ is a torsion-free normal subgroup of $V(\mathbb{Z}G)$ and $V(\mathbb{Z}G) = V_{tf}(\mathbb{Z}G)\, G \cong V_{tf}(\mathbb{Z}G) \rtimes G$.

Claim 4. *if $x \in V_{tf}(\mathbb{Z}G)$ is such that $x^m \in \mathcal{Z}(V(\mathbb{Z}G))$ for some $m \in \mathbb{Z}_{>0}$, then $x \in \mathcal{Z}(V(\mathbb{Z}G))$. Furthermore, $g^{o(g)/2} \in \mathcal{Z}(G)$ for every $g \in G$ with $o(g) > 2$.*

Take $e \in \mathrm{PCI}(\mathbb{Q}G)$. Note that the Cliff-Sehgal-Weiss result also holds for the quotient groups Ge of G. Thus also $V(\mathbb{Z}[Ge]) = V_{tf}(\mathbb{Z}[Ge]) \rtimes Ge$ has such a decomposition. Furthermore, in view of the explicit description of $V_{tf}(\mathbb{Z}G)$, the decompositions are compatible in the following sense: the natural epimorphism $\pi_e : G \to Ge$ extends to an epimorphism $\phi_e : \mathbb{Z}G \to \mathbb{Z}[Ge]$ and its restriction yields a morphism $\phi_e : V(\mathbb{Z}G) \to V(\mathbb{Z}[Ge])$. It follows from the description of the basis of the kernel of a relative augmentation map that $\phi_e(V_{tf}(\mathbb{Z}G)) \subseteq V_{tf}(\mathbb{Z}[Ge])$ and $\phi_e(\mathcal{Z}(V(\mathbb{Z}[G]))) \subseteq \mathcal{Z}(V(\mathbb{Z}[Ge]))$. This compatibility combined with the assumption yields that

$$\phi_e(x)^m \in V_{tf}(\mathbb{Z}[Ge]) \cap \mathcal{Z}(V(\mathbb{Z}[Ge])). \tag{64}$$

Now if Ge is abelian or isomorphic to Q_8 or D_8 then $\mathcal{Z}(V(\mathbb{Z}[Ge]))$ is finite. Thus $\phi_e(x)^m = e$ by (64) and, since $\phi_e(x)$ belongs to the torsion-free group $V_{tf}(\mathbb{Z}[Ge])$, we thus even get $\phi_e(x) = e$ (in particular it is central). Next consider the case that $Ge \cong Q_{16}$ the quaternion group of order 16. From the description of the unit group of $\mathbb{Z}Q_{16}$ obtained in [32, Theorem 4] it follows that $V_{tf}(\mathbb{Z}[Ge]) = V_{tf}(\mathbb{Z}Q_{16})$ is the direct product of an infinite cyclic group, which moreover is $V_{tf}(\mathbb{Z}Q_{16}) \cap \mathcal{Z}(V(\mathbb{Z}[Ge]))$, and a non-abelian free group. Thus (64) can only happen if $\phi_e(x) \in \mathcal{Z}(V(\mathbb{Z}[Ge]))$.

So, we have shown that $\phi_e(x)$ is central for each $e \in \mathrm{PCI}(\mathbb{Q}G)$. Therefore xe, the projection of x in the simple component $\mathbb{Q}Ge$, is central for each e. Hence, x itself is central and the claim follows.

Proof of Claim 3. By Claim 4, and because $G \subseteq N$, we can choose a transversal $\mathcal{T}$ of N in Γ such that $\mathcal{T} \subseteq V_{tf}(\mathbb{Z}G)$ and the only elements in $\mathcal{T}$ with some power central are the central elements. Now, for $x \in \Gamma^{ab}$ write $x = t\,y\,z$, with $t \in \pi(\mathcal{T}), z \in \pi(\mathcal{Z}(V(\mathbb{Z}G)))$ and $y \in \pi(\langle \mathrm{Bic}(G), G\rangle)$. If x is periodic, then so are t, y, z. To see this, recall that $\pi(cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G\rangle))$ is finite. Thus for some positive integer n we have t^n is a central unit. However, by the choice of the transversal, this implies that also t is periodic. Consequently, the remaining component z also needs to be periodic. With this Claim 3 now follows directly.

The central and bicyclic units contribute as predicted by conjecture (E1):

Claim 5. $\exp(\pi\left(\mathcal{Z}(V(\mathbb{Z}G))\, cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G\rangle)\right)) = \exp(G^{ab})$.

First we note that $\exp(G^{ab}) \mid \exp(\pi(N))$ with $N = \mathcal{Z}(V(\mathbb{Z}G))\, cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G\rangle)$. To see this, consider the relative augmentation $\omega_{G'} : \mathbb{Z}[G] \to \mathbb{Z}[G/G']$. As $\mathcal{U}(\mathbb{Z}[G/G'])$ is abelian, we have an induced morphism $\widetilde{\omega_{G'}} : V(\mathbb{Z}G)^{ab} \to V(\mathbb{Z}[G/G'])$. Since $G' \subseteq [V(\mathbb{Z}G), V(\mathbb{Z}G)]$ and $\widetilde{\omega_{G'}}(g[V(\mathbb{Z}G), V(\mathbb{Z}G)]) = gG'$ for $g \in G$, we have that any periodic element of G/G' is a $\widetilde{\omega_{G'}}|_{\pi(N)}$-image of a periodic element of $\pi(N)$. Hence it follows that $\exp(G/G') \mid \exp(\pi(N))$.

Thus in order to prove Claim 5, it is enough to show that $o(\pi(\alpha))$ divides $\exp(G^{ab})$ for every element α of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)), \mathrm{Bic}(G)$ and G. For the latter this trivially is true. Now consider a central unit $\alpha \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$. By [26, Proposition 5.5.1] one has that $\mathcal{Z}(V(\mathbb{Z}G)) \cap [V(\mathbb{Z}G), V(\mathbb{Z}G)] \leq \mathcal{Z}(V(\mathbb{Z}G)) \cap \mathrm{SL}_1(\mathbb{Z}G)$ is finite. Thus if $o(\pi(\alpha))$ is finite, then α itself is a periodic unit. Consequently, by a result of Berman and Higman [52, Corollary 7.1.3] (or see Theorem 5.5), $\alpha \in G$ and in particular $o(\pi(\alpha)) \mid \exp(G^{ab})$.

Next consider $\alpha = 1 + (1-x)y\widetilde{x} \in \mathrm{Bic}(G)$ whose inverse is $\alpha^{-1} = 1 - (1-x)y\widetilde{x}$. We will prove that

$$o(\pi(\alpha)) \mid 2. \tag{65}$$

If $o(x) = 2$, then (65) was obtained in [4, Proposition 3.1]. So suppose that $o(x) > 2$ and thus $o(x) = 4$ or 8. However, $o(x) = 8$ only occurs for the class $\mathcal{G}_7$ and in that case x generates a normal subgroup [25, Lemma 5.7]. In particular if $o(x) = 8$, then $\alpha = 1$. In conclusion, we may suppose that $o(x) = 4$.

It was noticed in the proof of [4, Proposition 3.1] that $[\alpha^{-1}, x^k] = 1 + (1-x)(1-x^{-k})y\widetilde{x}$ for any non-negative integer k. Consequently, for $\mathcal{I}$ a subset of $\{1, \ldots, o(x)\}$ one has that

$$\prod_{k \in \mathcal{I}} [\alpha^{-1}, x^k] = 1 + \sum_{k \in \mathcal{I}} (1-x)(1-x^{-k})y\widetilde{x}$$
$$= 1 + |\mathcal{I}|(1-x)y\widetilde{x} - (1-x)(\sum_{k \in \mathcal{I}} x^{-k})y\widetilde{x}.$$

Now take $\mathcal{I} = \{o(x)-1, o(x)\}$ and using that $\widetilde{x} = (1+x)(1+x^2)$ we see that

$$(1-x)(\sum_{k \in \mathcal{I}} x^{-k})y\widetilde{x} = (1-x)(1+x)y(1+x)(1+x^2) = (1-x^2)(1+x^2)y(1+x) = 0,$$

where we used that x^2 is central (by Claim 4). Altogether we have proven that $\alpha^2 = 1 + 2(1-x)y\widetilde{x} \in [V(\mathbb{Z}G), V(\mathbb{Z}G)]$, yielding (65).

The statement (65) also holds for the units $1 + \widetilde{x}y(1-x)$ and follows from an analogue proof. This finishes the proof of Claim 5.

Claim 2 together with (63) yields the upper bound for $\exp(V(\mathbb{Z}G)^{ab})$ we were looking for.

Now suppose that G satisfies $(\star)$, then we already have proven that the group generated by $\mathcal{Z}(V(\mathbb{Z}G))$ and $cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), G \rangle)$ is of finite index in $V(\mathbb{Z}G)$. More precisely, recall that $\mathcal{H}(g_i, h_i, Q_i) = 1 - e_i + V_{|G|/4}$ and also the identification from (61). With this we can reformulate Claim 2 saying that $\prod_i \mathcal{H}(g_i, h_i, Q_i)/[cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), g_i, h_i \rangle), \mathcal{H}(g_i, h_i, Q_i)]$ is finite. To control the exponent of the latter we will pass over to an overgroup:

$$\left| \frac{\mathcal{H}(g_i, h_i, Q_i)}{[cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), g_i, h_i \rangle), \mathcal{H}(g_i, h_i, Q_i)]} \right| \text{ divides } \left| \frac{\langle V_2, \Phi(g_i), \Phi(h_i) \rangle}{[\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_{|G|/4}]} \right|.$$

Next note that the proof of Claim 2 entails that $(\langle V_2, \Phi(g_i), \Phi(h_i) \rangle)^{ab}$ is a finite elementary abelian 2-group. Therefore using the explicit bound from[44] Claim 1 in the setting of Claim 2 yields that

$$\exp \left(\langle V_2, \Phi(g_i), \Phi(h_i) \rangle / [\langle V_2, \Phi(g_i), \Phi(h_i) \rangle, V_{|G|/4}] \right)$$
$$\text{divides } 2 \left([\langle V_2, \Phi(g_i), \Phi(h_i) \rangle : V_{|G|/4}] \right)^{[t/2]+1}$$

where t is the product of all p^{t_p} with t_p the maximum exponent of p dividing $[\langle V_2, \Phi(g_i), \Phi(h_i) \rangle : V_{|G|/4}]$. Now, recalling that $V_{2^i} \cong U_i$ by (38), Proposition 8.3

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[44] More precisely we replace $[\Gamma : N_1]n$ by the multiple $[\Gamma : N_1].[N_1 : N_2] = [\Gamma : N_2]$.

says that $[V_2 : V_{|G|/4}]$ is a power of two. Furthermore, Proposition 8.2 yields that $[\langle V_2, \Phi(g_i), \Phi(h_i) \rangle : V_2]$ is a 2-power. Summarized we have proven that

$$\exp \left(\prod_i \frac{\mathcal{H}(g_i, h_i, Q_i)}{[cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), g_i, h_i \rangle), \mathcal{H}(g_i, h_i, Q_i)]} \right) \text{ is a power of 2.}$$

Combining this with the fact that $[\mathcal{U}(\mathbb{Z}Ge_i) : V_{|G|/4}]$ is 2-power and

$$\exp \left(\frac{V(\mathbb{Z}G)}{\mathcal{Z}(V(\mathbb{Z}G)) \, cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), \pm G \rangle)} \right) \text{ divides}$$

$$\exp \left(\prod_i \frac{\mathcal{U}(\mathbb{Z}Ge_i)}{[cl_{V(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), g_i, h_i \rangle), \mathcal{H}(g_i, h_i, Q_i)]} \right)$$

we obtain that 2 is the only prime divisor of the left hand side quotient group. Therefore, as $\exp(G^{ab}) \mid \exp(G) = 4$ and thanks to the value obtained for $\exp V(\mathbb{Z}G)^{ab}$ conjecture (P) holds, finishing the proof. $\quad\square$

The statement that $cl_{\mathcal{U}(\mathbb{Z}G)}(\langle \mathrm{Bic}(G), \pm G \rangle)$ is of finite index when G satisfies $(\star)$ does not mention H-units. However we would like to emphasize that the proof needed them and hence the result is truly a combined use of bicyclic and H-units. In upcoming work by the first author a systematic study of the abelianisation of H-units and their role on the rank of $\mathcal{U}(\mathbb{Z}G)^{ab}$ will be done.

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