PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 11, Pages 187–199 (June 24, 2024)
https://doi.org/10.1090/bproc/218

A NOTE ON SURFACES IN $\mathbb{CP}^2$ AND $\mathbb{CP}^2\#\mathbb{CP}^2$

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(Communicated by Shelly Harvey)

Abstract. In this brief note, we investigate the $\mathbb{CP}^2$-genus of knots, i.e., the least genus of a smooth, compact, orientable surface in $\mathbb{CP}^2 \smallsetminus \mathring{B}^4$ bounded by a knot in S^3. We show that this quantity is unbounded, unlike its topological counterpart. We also investigate the $\mathbb{CP}^2$-genus of torus knots. We apply these results to improve the minimal genus bound for some homology classes in $\mathbb{CP}^2\#\mathbb{CP}^2$.

1. Introduction

The genus function G_X of a smooth 4-manifold X, as a function $H_2(X;\mathbb{Z}) \to \mathbb{Z}_{\geq 0}$, is defined for $\alpha \in H_2(X;\mathbb{Z})$ as

$$G_X(\alpha) = \min\{g(F) \mid i\colon F \to X, i_*([F]) = \alpha\}, \tag{1.1}$$

where the minimum is taken over all smooth embeddings i of smooth, closed, oriented surfaces F. A triumph of modern gauge theory consists of Kronheimer and Mrowka's determination of $G_{\mathbb{CP}^2}$, i.e., the solution to the minimal genus problem in $\mathbb{CP}^2$, also called the Thom conjecture [KM94]. They showed that if $h \in H_2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ is a generator and $d \neq 0$ is an integer, then

$$G_{\mathbb{CP}^2}(d \cdot h) = \frac{(|d|-1)(|d|-2)}{2},$$

and $G_{\mathbb{CP}^2}(0) = 0$. In this paper we will consider two generalisations of this: in one direction we will study a relative version of the genus bound, for surfaces with boundary, embedded in a punctured $\mathbb{CP}^2$ (Section 1.1); in another direction, we will examine the classical minimal smooth genus problem in the connected sum of *two* copies of $\mathbb{CP}^2$ (Section 1.3).

1.1. **Relative minimal genus problem in $\mathbb{CP}^2$.**

The $\mathbb{CP}^2$-*genus* of a knot $K \subseteq S^3$, denoted by $g_{\mathbb{CP}^2}(K)$, is the least genus of a smooth, compact, connected, orientable, properly embedded surface $F \subseteq \mathbb{CP}^2 \smallsetminus \mathring{B}^4 =: (\mathbb{CP}^2)^\times$ bounded by $K \subseteq \partial(\mathbb{CP}^2)^\times \cong S^3$. Similarly, there is the *topological* $\mathbb{CP}^2$-*genus*, denoted by

Received by the editors March 6, 2023, and, in revised form, December 11, 2023.
2020 *Mathematics Subject Classification.* Primary 57R40, 57K10, 57N70.
This project was funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 893282. The fourth author was partially supported by the *Élvonal (Frontier) grant* KKP144148 of the NKFIH. This material was partly based upon work supported by the National Science Foundation under Grant No. DMS-1928930, while the fourth author was in residence at the Simons Laufer Mathematical Sciences Institute (previously known as MSRI) Berkeley, California, during the Fall 2022 semester.

$g^{\mathrm{top}}_{\mathbb{CP}^2}(K)$, the least genus of a surface F as above which is only locally flat and embedded. A knot K is said to be slice in $\mathbb{CP}^2$ if $g_{\mathbb{CP}^2}(K) = 0$, and topologically slice in $\mathbb{CP}^2$ if $g^{\mathrm{top}}_{\mathbb{CP}^2}(K) = 0$.

It was shown in [KPRT22, Corollary 1.12 (2)] that $g^{\mathrm{top}}_{\mathbb{CP}^2}(K) \leq 1$ for every knot K. By contrast, we show that $g_{\mathbb{CP}^2}$ is unbounded.

Theorem 1.1. *Let $n \geq 0$. There exists a topologically slice knot K with $g_{\mathbb{CP}^2}(K) \geq n$.*

This result answers [Nou09, Question 2.1], which asked for a knot which is topologically slice in $\mathbb{CP}^2$ but not smoothly slice. The largest value of $g_{\mathbb{CP}^2}$ previously in the literature was $g_{\mathbb{CP}^2}(T_{2,17}) = 7$ from [Nou09, Theorem 1.2]. Our examples can be taken to be connected sums of the untwisted, negative clasped Whitehead double of the left-handed trefoil knot. Since these are topologically slice in B^4, they have trivial $g^{\mathrm{top}}_{\mathbb{CP}^2}$.

On the constructive side, we also give a method to find knots with trivial $g^{\mathrm{top}}_{\mathbb{CP}^2}$ that are not necessarily topologically slice in B^4.

Theorem 1.2. *Let $K \subseteq S^3$ be a knot. If* $\mathrm{Arf}(K) = 0$ *then* $g^{\mathrm{top}}_{\mathbb{CP}^2}(K) = 0$.

Note that the above result does not give a complete characterization of knots which are topologically slice in $\mathbb{CP}^2$, since for example the right-handed trefoil T satisfies $g_{\mathbb{CP}^2}(T) = g^{\mathrm{top}}_{\mathbb{CP}^2}(T) = 0$, as an unknotting number one knot, but has $\mathrm{Arf}(T) = 1$. The proof of Theorem 1.2 uses a result of Freedman and Quinn on when an immersed disk with an algebraically dual sphere is homotopic to a locally flat embedding [FQ90, Theorem 10.5 (1)] (see Theorem 2.6).

The $\mathbb{CP}^2$-genus was previously studied by [Yas91, Yas92, Nou09, Pic20]. Specifically, Yasuhara [Yas91, Yas92] studied the sliceness of certain torus knots in $\mathbb{CP}^2$. Pichelmeyer [Pic20] studied the $\mathbb{CP}^2$-genus for low crossing knots and alternating knots. Ait Nouh showed in [Nou09] that for $3 \leq q \leq 17$,

$$g_{\mathbb{CP}^2}(T_{2,q}) = g_4(T_{2,q}) - 1 = \frac{q-3}{2},$$

and asked whether the equality

$$g_{\mathbb{CP}^2}(T_{p,q}) = g_4(T_{p,q}) - 1 = \frac{(p-1)(q-1)}{2} - 1$$

holds for all $p, q > 0$. Theorem 2.8 shows that this is not the case, and yields Corollary 1.3.

Corollary 1.3. *For every $n \geq 1$,*

$$g_{\mathbb{CP}^2}(T_{n,n-1}) \leq \begin{cases} g_4(T_{n,n-1}) - \frac{n-2}{2} & \text{if } n \equiv 0 \mod 2, \\ g_4(T_{n,n-1}) - \frac{n-1}{2} & \text{if } n \equiv 1 \mod 2. \end{cases}$$

Indeed the above inequality holds for the least genus of null-homologous surfaces bounded by $T_{n,n-1}$, as we indicate in the proof.

1.2. **Relative minimal genus problems in other 4-manifolds.** Given a closed, smooth 4-manifold M, let $M^\times$ denote the punctured manifold $M \smallsetminus \mathring{B}^4$. The *$M$-genus* of a knot $K \subseteq S^3$, denoted by $g_M(K)$, generalizes the definition of Equation (1.1) and is defined as

$$g_M(K) = \min\{g(\Sigma) \mid i\colon \Sigma \to M^\times, i(\partial\Sigma) = K\},$$

where the minimum is taken over all smooth, compact, orientable, properly embedded surfaces $\Sigma \subseteq M^\times$ bounded by $K \subseteq \partial M^\times \cong S^3$. A knot K is said to be *slice* in M if $g_M(K) = 0$. One may also consider the *topological M-genus*, denoted by $g_M^{\text{top}}(K)$, the least genus of a surface F as above which is only locally flat and embedded. Note that $g_M(K) \leq g_{S^4}(K)$ and $g_M^{\text{top}}(K) \leq g_{S^4}^{\text{top}}(K)$ for all M.

The smooth and topological S^4-genera correspond to the usual smooth and topological slice genera of knots and have been studied extensively. In particular, there exist infinitely many knots with trivial topological S^4-genus and nontrivial smooth S^4-genus [Gom86, End95]. Any such knot can be used to produce a nonstandard smooth structure on $\mathbb{R}^4$ [Gom85] and in general slicing knots in S^4 is connected to major open questions in 4-manifold topology, such as the smooth Poincaré conjecture and the topological surgery conjecture [FGMW10, CF84] (see also [KOPR21]). Slicing knots in more general 4-manifolds has also been fruitful, e.g. in revealing structure within the knot concordance group [COT03, COT04, CT07, CHL09, CHL11], and in distinguishing between smooth concordance classes of topologically slice knots [CHH13, CH15, CK21]. In [MMP20] it was shown that the set of knots which bound smooth, null-homologous disks in a 4-manifold M can distinguish between smooth structures on M, that is, there are examples of homeomorphic smooth 4-manifolds M_1, M_2 and a knot $K \subset S^3$ which bounds a smooth, null-homologous disk in $M_1^\times$, but does not bound such a disk in $M_2^\times$. It is an open question whether the set of slice knots can distinguish between smooth structures on a manifold.

It was shown in [KPRT22, Corollary 1.12] using Freedman's disk embedding theorem [Fre82, FQ90] that $g_M^{\text{top}}(K) = 0$ for every closed, simply connected 4-manifold M other than S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $*\mathbb{CP}^2$, and $*\overline{\mathbb{CP}^2}$. Here $\overline{M}$ denotes the 4-manifold M with its orientation reversed, and $*\mathbb{CP}^2$ is the topological 4-manifold homotopy equivalent, but not homeomorphic, to $\mathbb{CP}^2$, constructed by Freedman in [Fre82]. As mentioned before, $g_{\mathbb{CP}^2}^{\text{top}}(K) \leq 1$ for every knot K. Moreover, $g_{*\mathbb{CP}^2}^{\text{top}}(K) \leq 1$ for every knot K as well. Further, $g_{\overline{\mathbb{CP}^2}}^{\text{top}}(K) = g_{\mathbb{CP}^2}^{\text{top}}(-K)$ for every knot K, where $-K$ is the mirror image. Therefore, the topological problem for closed, simply connected 4-manifolds with positive second Betti number is better understood. By contrast, many open questions remain in the smooth setting. It is a straightforward consequence of the Norman trick that all knots in S^3 are slice in both $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2 \cong \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ [Nor69, Corollary 3 and Remark], [Suz69, Theorem 1] (see also [Ohy94, Liv21]). By Theorem 1.1 the $\mathbb{CP}^2$-genus can be arbitrarily large. It is open whether there is a knot that is not slice in the $K3$-surface – if such a knot exists, its unknotting number must be more than 21 [MM22].

1.3. **Minimal genus problem in $\mathbb{CP}^2\#\mathbb{CP}^2$.** Above we provided a possible extension of the Thom conjecture from $\mathbb{CP}^2$ to $(\mathbb{CP}^2)^\times$, by replacing closed surfaces with surfaces having boundary knotted in $S^3 = \partial(\mathbb{CP}^2)^\times$. Another extension of Kronheimer-Mrowka's seminal result on $G_{\mathbb{CP}^2}$ would be the determination of the corresponding function for other closed 4-manifolds, for example $G_{\mathbb{CP}^2\#\mathbb{CP}^2}$ of $\mathbb{CP}^2\#\mathbb{CP}^2$. (The cases of $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $S^2 \times S^2$ were resolved by Ruberman in [Rub96].) Gauge-theoretic methods are less effective for this 4-manifold – the lower bounds, resting on variants of Furuta's $\frac{10}{8}$-theorem, from [Bry97] provide sharp results only in a handful of cases (see [Bry97, Corollary 1.7]), while constructions are rare.

Let $(n,d) = nh_1 + dh_2 \in H_2(\mathbb{CP}^2\#\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ denote a second homology class in $\mathbb{CP}^2\#\mathbb{CP}^2$, with the convention that h_i is a generator in the i^{th} summand and $n,d \in \mathbb{Z}$. A naïve approach to construct a surface representing the class (n,d) is to take the connected sum of the genus-minimizing surfaces in the two $\mathbb{CP}^2$-components (representing the desired homology class), giving a surface of genus $G_{\mathbb{CP}^2}(n\cdot h) + G_{\mathbb{CP}^2}(d\cdot h)$. This straightforward upper bound has been improved by 1 in [Nou14] for an infinite family of homology classes; below we implement roughly the same idea to show that the difference between the value of $G_{\mathbb{CP}^2\#\mathbb{CP}^2}$ can arbitrarily differ from the above naïve bound. For all $k \geq 0$, let $\widetilde{G}_{\mathbb{CP}^2}(k) := (k-1)(k-2)/2$. This function agrees with $G_{\mathbb{CP}^2}(k\cdot h)$ for all k except 0, where it is off by 1.

Theorem 1.4. *Suppose that $(n,d) \in H_2(\mathbb{CP}^2\#\mathbb{CP}^2;\mathbb{Z})$ is a homology class with $n > d \geq 0$.*

- *For $n \equiv d \mod 2$ we have*
$$\left(\widetilde{G}_{\mathbb{CP}^2}(n) + \widetilde{G}_{\mathbb{CP}^2}(d)\right) - G_{\mathbb{CP}^2\#\mathbb{CP}^2}(n,d) \geq \frac{n-3d}{2};$$
- *for $n \not\equiv d \mod 2$ we have*
$$\left(\widetilde{G}_{\mathbb{CP}^2}(n) + \widetilde{G}_{\mathbb{CP}^2}(d)\right) - G_{\mathbb{CP}^2\#\mathbb{CP}^2}(n,d) \geq \frac{n-3d+1}{2}.$$

Note that this bound is significant only when the right hand side is positive; in this case it shows that for suitably chosen (n,d) the difference between
$$G_{\mathbb{CP}^2\#\mathbb{CP}^2}(n,d)$$
and the naïve guess $G_{\mathbb{CP}^2}(n) + G_{\mathbb{CP}^2}(d)$ can be arbitrarily large. An independent proof of Theorem 1.4 will appear in a forthcoming paper of Stefan Mihajlović.

2. Genera of knots

Given a closed 4-manifold M, we identify $H_2(M^\times, \partial(M^\times);\mathbb{Z}) \cong H_2(M^\times;\mathbb{Z}) \cong H_2(M,\mathbb{Z})$. We will use the following results from [Vir75, Gil81, OS03].

Lemma 2.1. *Let $K \subseteq S^3$ be a knot which bounds a smooth, compact, connected, orientable, properly embedded surface Σ of genus g in $(\mathbb{CP}^2)^\times$ such that $[\Sigma] = d[\mathbb{CP}^1]$ in $H_2((\mathbb{CP}^2)^\times;\mathbb{Z})$.*

1. [OS03, Theorem 1.1] *For the τ-invariant from Heegaard Floer homology, we have*
$$g \geq -\tau(K) + \frac{|d|(1-|d|)}{2}.$$
2. [Gil81, Vir75] *If d is even, then*
$$2g+1 \geq \left|\frac{d^2}{2} - 1 - \sigma(K)\right|.$$
3. [Gil81, Vir75] *If d is divisible by an odd prime p, then*
$$2g+1 \geq \left|\frac{p^2-1}{2p^2}d^2 - 1 - \sigma_p(K)\right|,$$
where $\sigma_p(K) := \sigma_K\left(e^{\pi i \frac{p-1}{p}}\right)$ and σ_K denotes the Levine-Tristram signature function.

First we construct knots with arbitrarily large $\mathbb{CP}^2$-genus, which is the main ingredient in the proof of Theorem 1.1.

Proposition 2.2. *Fix $g_0 \geq 0$ and $c_0 > \frac{3}{2}\sqrt{2g_0+2} > 1$. Let K be a knot with vanishing Levine-Tristram signature function and such that the τ-invariant from Heegaard Floer homology satisfies*

$$-\tau(K) \geq g_0 - \frac{c_0(1-c_0)}{2}.$$

Then $g_{\mathbb{CP}^2}(K) \geq g_0$.

Proof. Let Σ be a genus g surface in $(\mathbb{CP}^2)^\times$ with $\partial\Sigma = K$ and such that $[\Sigma] = d[\mathbb{CP}^1]$ in second homology. By Lemma 2.1(1), we have that

$$g \geq -\tau(K) + \frac{|d|(1-|d|)}{2} \geq g_0 - \frac{c_0(1-c_0)}{2} + \frac{|d|(1-|d|)}{2}.$$

Then either $g \geq g_0$, in which case we are done, or $g < g_0$ and so $|d| > c_0 > 1$.

So assume that $|d| > c_0$ and let p be a prime factor of $|d|$. If $p = 2$, then by Lemma 2.1(2),

$$2g+1 \geq \frac{d^2}{2} - 1 > \frac{c_0^2}{2} - 1 > \frac{9}{4}(2g_0+2) - 1 > 2g_0 + 1.$$

It follows that $g \geq g_0$, as desired.

If p is odd, then $\frac{1}{2} - \frac{1}{2p^2} \geq \frac{1}{2} - \frac{1}{2(3)^2} = \frac{4}{9}$ and so by Lemma 2.1(3)

$$2g+1 \geq \frac{p^2-1}{2p^2}d^2 - 1 > \frac{p^2-1}{2p^2}c_0^2 - 1 > \left(\frac{1}{2} - \frac{1}{2p^2}\right)\frac{9}{4}(2g_0+2) - 1 \geq 2g_0 + 1.$$

It follows that $g \geq g_0$, as desired. □

Proof of Theorem 1.1. It suffices to find topologically slice knots satisfying the hypothesis of Proposition 2.2. There are many such knots, e.g. one could take connected sums of a knot with trivial Alexander polynomial and negative τ-invariant. As a concrete example, let K be the untwisted, negative clasped Whitehead double of the left-handed trefoil knot. By [Hed07, Theorem 1.5], we know that $\tau(K) = -1$, and since $\Delta_K(t) = 1$ it is topologically slice [FQ90, Section 11.7], hence its Levine-Tristram signatures all vanish. Then the knots $\#^\ell K$, for $\ell \gg 0$, satisfy the hypothesis of Proposition 2.2 with $g_0 = n$. □

Remark 2.3. If one only desires a knot K with $g^{\text{top}}_{\mathbb{CP}^2}(K) = 0$ and $g_{\mathbb{CP}^2}(K) \geq n$ for a given n, then by Theorem 1.2 and Proposition 2.2 any knot K with trivial Arf invariant, vanishing Levine-Tristram signature function, and satisfying $-\tau(K) \geq n - \frac{c_0(1-c_0)}{2}$ for $c_0 > \frac{3}{2}\sqrt{2n+2}$, will work.

Remark 2.4. In [Pic20, Conjecture 1], Pichelmeyer conjectured that two knots with equal ordinary signature and Arf invariant must have equal $g_{\mathbb{CP}^2}$. The knots from Theorem 1.1 with $g^{\text{top}}_{\mathbb{CP}^2}(K_n) = 0$ and $g_{\mathbb{CP}^2}(K_n) \geq n$ $(n \to \infty)$ contradict the conjecture: since they are topologically slice in B^4, they have equal (indeed trivial) Arf invariant and Levine-Tristram signature function, and one can choose an infinite subfamily realizing distinct $\mathbb{CP}^2$-genera.

Next we will prove Theorem 1.2 from Section 1. For this we will need the following formulation of the Arf invariant of a knot. Recall that the *self-intersection* *number* of a properly, generically immersed disk $f\colon (D^2, \partial D^2) \looparrowright (M, \partial M)$ restricting to an embedding on the boundary, where M is a simply connected 4-manifold, is the signed count of double points of f. Here and in the rest of the paper, the symbol $\looparrowright$ denotes a *generic immersion* of a compact surface, i.e., an immersion such that the singular set is a closed, discrete subset of M consisting only of transverse double points, each of whose preimages lies in the interior of the surface. We will only use this notion where the target manifold is smooth, but we note that there is an analogous notion in the topological setting [FQ90, Chapter 1], [PR21, Section 11.1]. We will also use (possibly immersed) Whitney disks in the argument. For more details about Whitney disks, see, for example, [PR21].

Proposition 2.5 ([Mat78],[FK78],[CST14, Lemma 10])**.** *Let $K \subseteq S^3$ be a knot bounding a generically immersed disk Δ in B^4 with trivial self-intersection number. This implies that there is a collection $\{W_i\}$ of framed, generically immersed Whitney disks pairing up the self-intersections of Δ and intersecting Δ in transverse double points in the interiors $\{\mathring{W}_i\}$.*

For any such collection $\{W_i\}$, we have the equality

$$\operatorname{Arf}(K) = \sum_i \Delta \cdot \mathring{W}_i \mod 2.$$

We will also need the following result of Freedman-Quinn, giving a sufficient condition under which an immersion of a disk is homotopic to an embedding.[1] Since the statement in [FQ90] does not match the one below exactly, we point out that Theorem 2.6 is also an immediate corollary of [KPRT22, Theorem 1.2], using [KPRT22, Lemma 5.5].

Theorem 2.6 ([FQ90, Theorem 10.5 (1)])**.** *Let M be a simply connected 4-manifold. Let $f\colon (D^2, \partial D^2) \looparrowright (M, \partial M)$ be a proper, generic immersion with trivial self-intersection number, restricting to an embedding on the boundary, and admitting an algebraically dual sphere, i.e., there is a sphere $C\colon S^2 \looparrowright M$ with $f \cdot C \equiv 1 \mod 2$. Assume further that there exists a collection $\{W_i\}$ of framed, generically immersed Whitney disks pairing up the self-intersections of f and intersecting the image of f in transverse double points in the interiors $\{\mathring{W}_i\}$ so that $\sum_i f \cdot \mathring{W}_i \equiv 0 \mod 2$.*

Then f is homotopic to a locally flat embedding relative to the boundary.

Note that the above result differs from that proven by Freedman in [Fre82] in that the algebraically dual sphere C is not required to have trivial normal bundle, necessitating the additional condition on the Whitney disks.

Proof of Theorem 1.2. Let Δ be a generically immersed disk bounded by K in B^4, obtained e.g. as the trace of a null-homotopy. By adding local self-intersections if necessary, we can assume that the signed count of double points of Δ equals zero, i.e., that Δ has vanishing self-intersection number. Then these double points can be paired by a collection of framed, generically immersed Whitney disks $\{W_i\}$, whose

[1] For the convenience of the reader we have stated [FQ90, Theorem 10.5 (1)] in the special case where the ambient manifold is simply connected. The more general formulation in [FQ90] had an error, which was detected and corrected by Stong in [Sto94]. The error is related to elements of order two in the fundamental group of the ambient manifold and is not relevant here. For more details, see [Sto94, KPRT22].

interiors intersect Δ in isolated points (see, for example, [PR21, Proposition 11.10]). By Proposition 2.5, we know that $0 = \mathrm{Arf}(K) = \sum_i \Delta \cdot \mathring{W}_i \mod 2$.

Perform a connected sum of B^4 with $\mathbb{CP}^2$, to obtain $(\mathbb{CP}^2)^\times$. By choosing the location of the connected sum carefully, we ensure that Δ and $\{W_i\}$ also lie in $(\mathbb{CP}^2)^\times$, and are in particular disjoint from $\mathbb{CP}^1 \subseteq (\mathbb{CP}^2)^\times$. Tube Δ into $\mathbb{CP}^1$, and call the result Δ'. Since $\mathbb{CP}^1$ is embedded, the collection $\{W_i\}$ pairs up all the double points of Δ'. A push-off C of $\mathbb{CP}^1$ provides an algebraically dual sphere for Δ'. Then by Theorem 2.6, the disk Δ' is homotopic to a locally flat embedding, which is the desired slice disk. □

Remark 2.7. The disk that we have constructed in the proof of Theorem 1.2 is a generator of $H_2((\mathbb{CP}^2)^\times; \mathbb{Z})$. By [KPRT22, Theorem 1.6], one sees that a knot K bounds a locally flat, embedded disk in $(\mathbb{CP}^2)^\times$ representing a generator of $H_2((\mathbb{CP}^2)^\times; \mathbb{Z})$ if and only if $\mathrm{Arf}(K) = 0$ (see also [KPRT22, Proof of Corollary 1.12]).

We finish this section by considering the $\mathbb{CP}^2$-genera of certain torus knots.

Theorem 2.8. *Let $n > d \geq 0$.*

- *If $n \equiv d \mod 2$, the torus knot $T_{n,n-1}$ bounds a surface $\Sigma \subseteq (\mathbb{CP}^2)^\times$ with degree d and genus*

$$g(\Sigma) = \left(\frac{n+d-2}{2}\right)^2 + \left(\frac{n-d-2}{2}\right)^2.$$

- *If $n \not\equiv d \mod 2$, the torus knot $T_{n,n-1}$ bounds a surface $\Sigma \subseteq (\mathbb{CP}^2)^\times$ with degree d and genus*

$$g(\Sigma) = \left(\frac{n+d-1}{2}\right)\cdot\left(\frac{n+d-3}{2}\right) + \left(\frac{n-d-1}{2}\right)\cdot\left(\frac{n-d-3}{2}\right).$$

The proof of Theorem 2.8 is based on a certain torus knot twisting technique, which has already appeared in the literature (see e.g. [McC21, Figure 7] for a recent appearance).

Proof. For the case $n \equiv d \mod 2$, set $a = \frac{n+d}{2}$ and $b = \frac{n-d}{2}$. Figure 2.1 shows a concordance from $K_0 := T_{a,2a-1}\#T_{b,2b-1}$ to $K_1 := T_{n,n-1}$ in $\mathbb{CP}^2 \setminus (B^4 \sqcup B^4)$, with degree $d = a - b$.

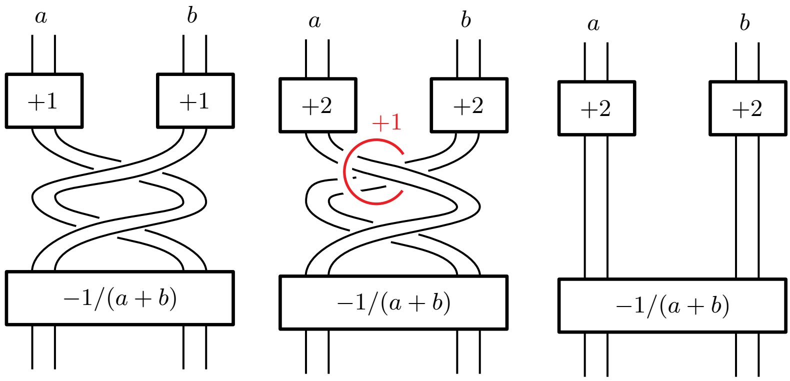

FIGURE 2.1. The figure on the left shows the torus knot $T_{a+b,a+b-1}$. A box with an integer denotes that many full twists (right-handed for positive, left-handed for negative integers). The box containing $-\frac{1}{a+b}$ indicates a negative fractional twist, i.e., the rightmost strand passes across to the extreme left, above all the other strands. The second figure shows how to change $a \cdot b$ crossings with a blow-up, at the expense of adding a full twist on each of the two bundles of parallel strands. The third figure shows a simplified diagram of the knot in the second figure, after one forgets the (+1)-framed 2-handle. From the last figure it is clear that the knot in question is $T_{a,2a-1} \# T_{b,2b-1}$.

We can cap off this concordance with a minimum genus surface for K_0 in B^4, thus obtaining the desired surface Σ, with genus

$$g(\Sigma) = g_4(T_{a,2a-1}) + g_4(T_{b,2b-1}) = (a-1)^2 + (b-1)^2.$$

For the case $n \not\equiv d \mod 2$, set $a = \frac{n+d-1}{2}$ and $b = \frac{n-d-1}{2}$. Instead of the concordance in Figure 2.1, we now use the one illustrated in Figure 2.2. As before, we cap off with a minimum genus surface for K_0 in B^4, thus obtaining the desired surface Σ, which now has genus

$$g(\Sigma) = g_4(T_{a,2a+1}) + g_4(T_{b,2b+1}) = a \cdot (a-1) + b \cdot (b-1). \qquad \square$$

Proof of Corollary 1.3. The corollary follows by applying Theorem 2.8 with $d = 0$.

More explicitly, if $n \equiv 0 \mod 2$, Theorem 2.8 provides a smooth surface Σ in $(\mathbb{CP}^2)^{\times}$ of genus

$$g(\Sigma) = \frac{(n-2)(n-2)}{2},$$

which is an upper bound to $g_{\mathbb{CP}^2}(T_{n,n-1})$. Thus, the difference

$$g_4(T_{n,n-1}) - g_{\mathbb{CP}^2}(T_{n,n-1})$$

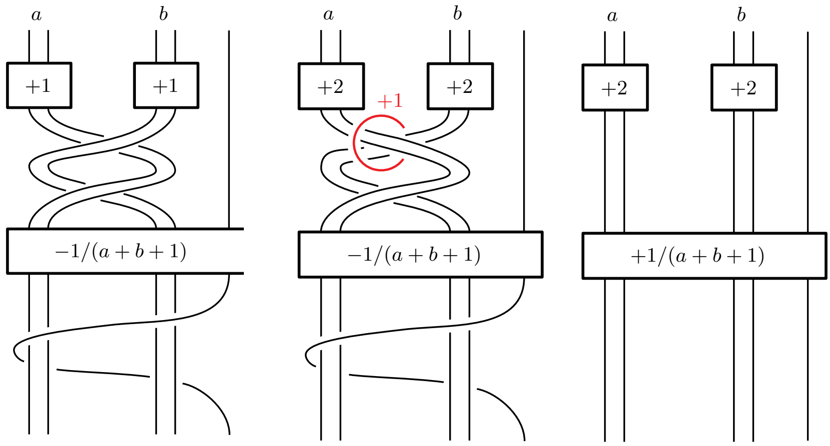

FIGURE 2.2. The figure on the left shows the torus knot $T_{a+b+1,a+b}$. The second figure shows how to change $a \cdot b$ crossing with a blow-up, at the expense of adding a full twist on each of the two bundles of parallel strands. After simplifying the diagram (third figure), the knot is identified with $T_{a,2a+1}\#T_{b,2b+1}\#T_{1,0} = T_{a,2a+1}\#T_{b,2b+1}$. (Note the sign change in the fractional twist in the third figure.)

is at least

$$\begin{aligned} g_4(T_{n,n-1}) - g(\Sigma) &= \frac{(n-1)(n-2)}{2} - \frac{(n-2)(n-2)}{2} \\ &= \frac{n-2}{2} \cdot ((n-1)-(n-2)) \\ &= \frac{n-2}{2}. \end{aligned}$$

The case $n \equiv 1 \mod 2$ is very similar: in this case Theorem 2.8 provides a smooth surface Σ in $(\mathbb{CP}^2)^\times$ of genus

$$g(\Sigma) = \frac{(n-1)(n-3)}{2},$$

and therefore the difference $g_4(T_{n,n-1}) - g_{\mathbb{CP}^2}(T_{n,n-1})$ is at least

$$\begin{aligned} g_4(T_{n,n-1}) - g(\Sigma) &= \frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-3)}{2} \\ &= \frac{n-1}{2} \cdot ((n-2)-(n-3)) \\ &= \frac{n-1}{2}. \end{aligned}$$

□

3. THE GENUS FUNCTION IN $\mathbb{CP}^2\#\mathbb{CP}^2$

Using Theorem 2.8, it is quite straightforward to complete the proof of Theorem 1.4. The strategy is the same as for the proof of [Nou14, Theorem 1.2].

Proof of Theorem 1.4. We first note that the knot $-T_{n,n-1}$, the mirror of the $(n, n-1)$ torus knot, is slice in $\mathbb{CP}^2$, i.e., $g_{\mathbb{CP}^2}(-T_{n,n-1}) = 0$, and that in addition the slice disk can be chosen to represent n-times the generator in relative homology. To see this, start from a description of the unknot as the closure of the braid $\sigma_1\sigma_2\cdots\sigma_{n-1}$, as in Figure 3.1, which (being unknotted) bounds a disk in S^3, and hence in B^4. If we attach a $(+1)$-framed 2-handle to B^4 along the red unknot U, which encircles all the strands of the braid once, then we obtain a Kirby diagrammatic description of $(\mathbb{CP}^2)^\times$, together with a knot K sitting in the new boundary $S^3 = \partial\left((\mathbb{CP}^2)^\times\right)$; note that by construction K bounds a disk in $(\mathbb{CP}^2)^\times$, in homology class given by $\mathrm{lk}(K, U) = n$. Upon doing a Rolfsen twist, K is revealed to be the knot $-T_{n,n-1}$.

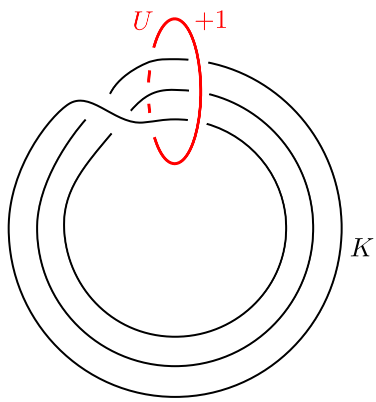

FIGURE 3.1. Attaching a $(+1)$-framed 2-handle to B^4 along an unknot gives the standard handle decomposition for $(\mathbb{CP}^2)^\times$. In the resulting surgery description of S^3, the knot K depicted above appears unknotted, though blowing down the $(+1)$-surgery curve to get the empty surgery diagram for S^3 reveals that $K = -T_{n,n-1}$. The image of the standard slice disk for the unknot in B^4 in $(\mathbb{CP}^2)^\times$ is the desired slice disk for K, and represents n-times the generator of relative second homology. The case $n = 3$ is shown.

Taking the boundary connected sum of $((\mathbb{CP}^2)^\times, T_{n,n-1})$ and $((\mathbb{CP}^2)^\times, -T_{n,n-1})$ together with the surface described in the proof of Theorem 2.8 in the first and the previously described slice disk in the second summand (and taking the connected sum at points of the knots) we get a surface in $(\mathbb{CP}^2\#\mathbb{CP}^2)^\times$ with genus given in Theorem 2.8. In addition, the boundary of this surface is the connected sum $T_{n,n-1}\# -T_{n,n-1}$, which is a slice knot in B^4 (as is any knot of the form $K\# -K$). This shows that the above surface with boundary can be capped off by a disk to provide a closed surface in $\mathbb{CP}^2\#\mathbb{CP}^2$ with the same genus, representing the homology class (n, d). The proof of the theorem now follows after some simple arithmetic. $\square$

Acknowledgments

We are grateful to Stefan Mihajlović for pointing out an off-by-one error in the $d = 0$ case of Theorem 1.4 in a previous version of the paper, and to the anonymous referee for their comments.

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