FIFTH GRADE STUDENTS' MULTIPLICATIVE REASONING WHEN ENGAGING WITH PARTIAL PRODUCTS

Sanampreet Gill
Kent State University
sgill13@kent.edu

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Temitope Egbedeyi
Kent State University
tegbedey@kent.edu

Despite the importance of multiplicative reasoning (MR) in mathematics, there is limited empirical study of the role of MR in multi-digit multiplication tasks. This study examined children's multiplicative reasoning within their use of partial products. This mixed methods study of 16 fifth-grade students revealed that half of participants did not attend to partial products, and those that did tended to operate with the second or third multiplicative concept.

Introduction

Although there is significant scholarship on multi-digit addition and subtraction, research on multi-digit multiplication has, historically, been limited in scope and magnitude (Fuson, 2003; Harrison, 2013; Hickendorff et al., 2019). Prior work has typically either focused on the use of symbolic strategies and associated reasoning (Hickendorff et al., 2019), or the relationship between students visual and symbolic approaches to multi-digit multiplication (Izsàk, 2005; Larsson et al., 2017). In particular, the role of partial products in multiplication is often noted, but analysis of how children reason about them is largely absent from the literature (Fuson, 2003; Iszák, 2005). This study emerged as a contribution to the existing gap to gain more insight into students' multiplicative reasoning and conceptual knowledge, with the purpose being to examine how children's multiplicative reasoning relates to how they solve multi-digit multiplication tasks.

Theoretical Framework & Literature Review

The current paper uses Scheme Theory to examine children's multiplicative reasoning, with particular focus on the multiplicative concepts as described by Hackenberg (2010). Prior to multiplicative reasoning, students demonstrate pre-multiplicative reasoning by using a standard algorithm or counting by 1s to solve the multiplication task. For instance, a child may use a standard algorithm to solve 4x19. However, when "carrying the 3," they may not be able to explain that it represents 30. If pressed to represent 4x30 or 4x3 visually, they will be unable to do so. At the first multiplicative concept (MC1), a child is anticipated to use any strategy like skip-counting, repeated addition, and doubling to solve the given task. with a limited ability to visualize part of the partial products. For example, a child may solve 4x19 successfully by doubling 19 (or 19+19) and then doubling 38 to get 76, but when "carrying the 3," they may not

be able to explain that it represents 30. If pressed to represent 4x30 visually, they may do it with some difficulty (but will easily be able to do so for 4x3).

At the second multiplicative concept (MC 2), a child is anticipated to solve and represent four partial products with an area model (arrays rods or base 10 blocks) but explain it conceptually. For example, a child may be able to solve 19x24 successfully with a standard algorithm but unable to represent all the partial products on a dot paper (i.e., visually shows 10x20 and 9x4, but not the other two partial products). When a child can solve and represent all the partial products with an area model, dot paper, or other manipulative, such a child is considered to be at the third multiplicative concept; during this stage, the child is anticipated to solve and interpret four or more partial products/quotients. For instance, a child solves 19x24 successfully with a standard algorithm and represents it visually on dot paper, area model, or base 10 blocks (i.e., visually shows 10x20, 10x4, 9x20, and 9x4).

As noted earlier, examination of children's reasoning with multi-digit multiplication has been limited in scope and magnitude (Fuson, 2003; Harrison, 2013; Hickendorff et al., 2019). A clear indication from much of the research on students' symbolic approaches to multiplication suggests that good recall of multiplication facts and accuracy in answers is an indicator of success (Javornik & Lipovec, 2020; Lin & Kubina, 2005). However, many students are unable to meaningfully connect their symbolic algorithmic work with a drawn representation (Hurst & Hurrell, 2016; Larsson et al., 2017). Indeed, Larsson et al. (2017) found that, when pressed for explanation, such students relied on repeated addition instead of any multiplicative reasoning. Examining how children used bundles of sticks when modeling multi-digit multiplication, Hurst and Hurrell (2016) found that although all students "seemed to have a robust recall of multiplication facts…this facility to recall multiplication facts was not an indicator that the students had a conceptual understanding of multiplication" (p. 37).

Key in understanding children's skill and reasoning with multi-digit multiplication is their developed understanding of partial products (Hickendorff et al., 2019). Most scholars examining students' conceptual understanding of partial products have advocated introducing it with arrays or some other form of area model to allow for visualizing the decomposition of numbers into their base-ten components (Izsàk, 2005; Young-Loveridge & Mills, 2009). In examining this scholarship, it appears that students may initially learn to solve problems with two partial products (i.e., 15×24 = 15×20 + 15×4) before learning to do so with four or more partial products

(i.e., 15×24 = 10×20 + 10×4 + 5×20 + 5×4). Despite this assumption in the literature, there is little empirical study of how this aligns with different levels of multiplicative reasoning. Such is the central focus in the current paper: to examine the relationship between children's multiplicative reasoning and their strategies in using partial products.

Methods

Sample & Procedure

Participants included a convenience sample of 16 fifth-grade students. Participants completed the Multiplicative Reasoning Assessment (MRA). The MRA is a 21-item assessment of children's multiplicative concepts. Validity evidence includes psychometric data for internal structure and test content, and cognitive interviews for response processes (Kosko, 2019). Raw dichotomous responses on the MRA were transformed via a Rasch model into a continuous scale. Four students, representing each of the multiplicative reasoning levels assessed by the MRA (pre-multiplicative, MC1, MC2, & MC3) were purposefully assigned within one of each of four conditions. Each condition provided a specific visual: base-ten blocks, array rods, or dot array paper as a visual for explaining their multiplication (see Figure 1). Visualizations were modeled with the task 2×12, which was considered accessible to students not yet able to skip-count given the connection to doubling. Next, participants were asked to solve four multiplication tasks, with follow-up questions for assessing their reasoning, use of the visual, and any written algorithms used: 4×19, 15×24, 8×33, and 13×27.

During each interview, participants wore a Pupil Core eye-tracking headset, which recorded their pupil-gaze and front view. Eye tracking is an exploratory method used to analyze students' behavior through gaze patterns, which allows researchers to capture real-time data on how students engage with tasks. It can distinguish gaze patterns, pupil dilation, and fixations (Duchowski, 2007). In general, numerous but shorter duration fixations tend to indicate lower working memory load and higher ability whereas fewer but longer duration fixations indicate more demands on working memory (Rayner, 2006). However, where such fixations matters as much as their duration and number. In the present study, we examined pupil-gaze qualitative in concert with other video-based interview data. Given the exploratory nature of our analysis, this seemed appropriate.

A data-transformation variant of convergent mixed methods design was used in the current study (Creswell & Plano Clark, 2018). We collected interview data of students with eye-tracking

glasses and examined their spoken explanations, hand gestures, eye-gaze, use of manipulatives, and written mathematics. This multi-modal data allowed for a richer analysis of the meaning conveyed by students' actions. Following qualitative analysis, emergent themes were transformed into ordinal data and used in a correlational analysis with MRA scores.

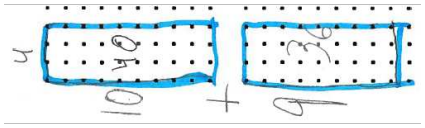
Figure 1

Visualizations Used in Interviews

Base Ten Blocks

Array Rods

Dot Array Paper



Array Rods +

Analysis & Results

Qualitative Analysis & Findings

We engaged in three rounds of qualitative analysis across all three authors, focusing on how students conveyed multiplicative meaning. Students' written work and eye-tracking videos were used to examine gestures, eye-gaze, use of manipulatives, verbal descriptions and written mathematics. After an initial round of individual open coding, authors met and examined emergent themes for overlap. A second round of analysis confirmed three emergent themes for analysis: students did not attend to partial products; students attended to no more than two partial products; students attended to more than two partial products. We used these themes to examine students' work with particular tasks in the next round of analysis, as some students did not consistently convey their use of partial products. In the paragraphs below, we briefly summarize these themes with three cases with particular attention to students' solving 15×24.

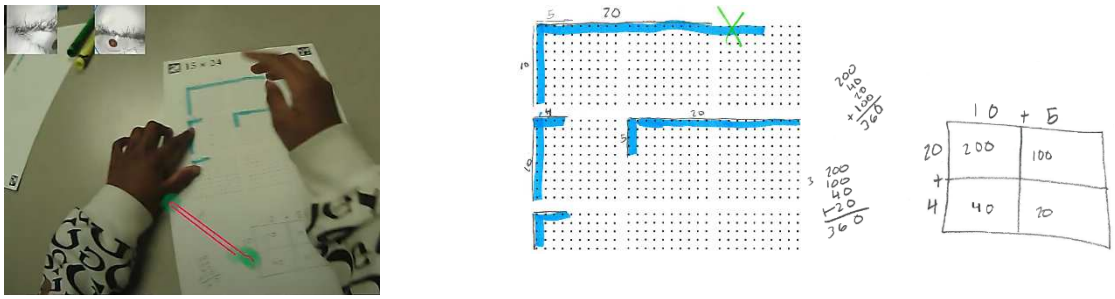
Did Not Attend to Partial Products

Half of participants were unable to attend to partial products. Many demonstrated either this by using additive reasoning or doubling approaches, but such evidence often did not become apparent until they were asked to explain or visually represent their reasoning. Among these individuals, Amanda was of particular interest because she could easily solve any problem, so long as it was represented symbolically. Amanda was able to quickly able to use the box method

to find a solution to 15×24 before adding each multiplicand. Yet, when asked to demonstrate her reasoning with arrays, Amanda began counting ten dots and five dots to draw two lines in the top-left corner of the array sheet. Following a brief exchange with the interviewer, she redrew it as 10 and 20, before mimicking the arrangement for each box in her box model. While checking to see if her visuals matched her numbers in the written algorithm, Amanda would put one finger from each hand to point to each number on the exterior (either of the array or the box), as her eye-gaze glanced down to the box model for reference (see Figure 2). In each instance, it appears that Amanda focused on the multiplicands but never the relationship between them to produce the product. Later in the interview, when tasked with solving 13×27, Amanda did something similar. However, the interviewer pressed her to explain what 3 times 20 meant with the array, asking "how do you know there are 60 dots?" Like prior tasks, Amanda's eye-gaze focused on the multiplicands but never the interior of the rectangular array that *could* be inferred from her work. She then stated that "3 plus 20 is 23, but if I did it by times, I know that 3 times 2 is 6. Then it's 60 because you add a zero." When asked how she knew there were 21 dots for 3×7, Amanda said "I just know my numbers…my third-grade teacher had these songs that would help us remember…" This interaction confirmed that despite fluency with rotely memorized multiplication facts, Amanda did not understand how to represent or explain multiplication with either single or multi-digit whole numbers.

Figure 2

Amanda's Written Work and Eye-tracking for 15×24



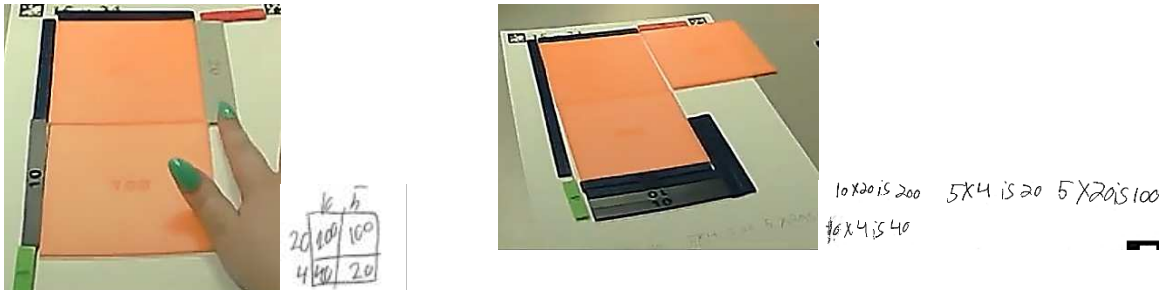
Attend to No More than Two Partial Products

A quarter of participants demonstrated an ability to explain or visually represent their reasoning with no more than two partial products. For example, Kelsey was able to easily model problems like 4×19 and 8×33 with the provided array rods and then explain how the visual represented the problem. When asked to do so with 15×24, Kelsey carefully placed a 10 and 5

rod for 15 at the top, and two 10s and a 4 rod on the left for 24. She then verbally talked through multiplying 10 times 20, looked for and found two 100s before stating “okay and then 5 times 4 is 20” (see Figure 3). After confirming she had finished and was satisfied with her response, the interviewer asked, “have you done this in a different way – a written way?” Kelsey said she had used a box to do so and then was provided with a pencil to show her written approach (see Figure 3). After finishing the written algorithm, Kelsey proceeded to immediately add a 100 block and a 40 block to her visual, but the arrangement did not create a rectangle that coincided with her model. This contrasted her approach when working with tasks requiring only two partial products. Other interview participants demonstrating this theme similarly skipped partial products to multiply, or did not separate a multiplier into factors at all (i.e., 10×24 + 5×24). Thus, regardless of visual used, Kelsey’s approach is representative of actions demonstrated with this theme.

Figure 3

Kelsey (left) and Justin’s (right) Visualizations and Approach to Solving 15×24



Attend to More Than Two Partial Products

A quarter of our participants demonstrated the ability to engage with more than two partial products, indicating a higher level of multiplicative reasoning. Justin was able to visually represent his reasoning on simpler tasks (i.e., 4×19 & 8×33). When prompted to solve 15×24, Justin initially placed two 10s and a 4 on the top and a ten and five on the left but later adjusted so that it would fit on the sheet of paper (see Figure 3). Next, Justin laid out two 100 blocks and began to use 10s to fill in the remaining space. Throughout, Justin’s eye-gaze scanned back-and-forth between the perimeter and area of the array, suggesting a possible coordination between the rods representing area and those representing its dimensions. Halfway through his construction of the area, Justin paused and tapped his pencil while his eye-gaze indicated looking off in the distance. Justin removed five 10s he had placed at the top and replaced them with a 100 (See

Figure 3). We infer from this that Justin anticipated iterating the other five 10s (i.e., 50) and unitized the ten 10s as a 100. Despite this not canonically representing an area model (i.e., the 100 does not line up perfectly with its multiplicands), this was considered as an act of anticipating three levels of units (skip counting the 10s, grouping five 10s as 50, and regrouping/anticipating that two sets of five 10s is one 100s). When prompted to explain his model, Justin wrote "10×20 is 200; 5×4 is 20; 5×20 is 100; 10×4 is 40" and was able to coordinate his written mathematics with the visual representation.

Quantitative Analysis & Results

Following quantitative analysis, the three themes that emerged were quantitititized into an ordinal variable (0 = students did not attend to partial products; 1= students attended to no more than two partial products; 2 =students attended to more than two partial products.). Recall that some participants displayed different levels of partial product use in different tasks, so we created composite scores (averages) for these individuals. Quantitization allowed us to examine the correlation between students' MRA scores and their use of partial products, as observed in the interviews. We used the Spearman rank coefficient given our sample size (n = 16). Additionally, the Spearman rho coefficient is ideal for examining ordinal and continuous data (Siegel & Castellan, 1988). Results indicate a statistically significant and strong correlation (ρ = .82, $p < .001$). Next, we visually examined a scatterplot of the correlated data and observed that students demonstrating MC2 or MC3, as assessed with the MRA, appeared to be those who were consistently attending to partial products during interviews.

Discussion

The purpose of this exploratory study was to examine how children's multiplicative reasoning is related to solving multi-digit multiplication tasks. Results and findings indicated that children operating at the first multiplicative concept or with pre-multiplicative reasoning could not meaningfully use, visually represent, or explain partial products. A primary implication is that students need to be able to anticipate two levels of units or more (MC2 & MC3) to meaningfully understand partial products. These findings highlight a need for instructional approaches to foster students' ability to anticipate and visualize partial products. For example, encouraging both visual and concrete representations is essential for students to gain a conceptual understanding of partial products. Additionally, teachers can provide scaffolding to support students' transitioning from working with two partial products to four partial products.

This study also contributes to understanding how eye tracking technology can provide insights into children's cognitive processes engaged with mathematical tasks. Further study is needed, but this study fills a gap identified by many who have examined multi-digit multiplication (Izsàk, 2005; Hickendorff et al., 2019).

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