THE q-SCHUR CATEGORY AND POLYNOMIAL TILTING MODULES FOR QUANTUM GL_n

JONATHAN BRUNDAN

Abstract. The *q-Schur category* is a $\mathbb{Z}[q,q^{-1}]$-linear monoidal category closely related to the q-Schur algebra. We explain how to construct it from coordinate algebras of quantum GL_n for all $n \geq 0$. Then we use Donkin's work on Ringel duality for q-Schur algebras to make precise the relationship between the q-Schur category and a $\mathbb{Z}[q,q^{-1}]$-form for the $U_q\mathfrak{gl}_n$-web category of Cautis, Kamnitzer and Morrison. We construct explicit integral bases for morphism spaces in the latter category, and extend the Cautis-Kamnitzer-Morrison theorem to polynomial representations of quantum GL_n at a root of unity over a field of any characteristic.

1. Introduction

In this article, we revisit some algebra from the 1990s using the diagrammatic technique of string calculus for strict monoidal categories which has become ubiquitous in this area since then. The initial goal is to give a self-contained construction of a strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal category, the *q-Schur category*, together with three important bases for its morphism spaces. The path algebra of this category is Morita equivalent to the direct sum of the q-Schur algebras $S_q(n,n)$ of Dipper and James [DJ89] for all $n \geq 0$. In that context, all three bases were studied in detail already 30 years ago, and this part of the article is mainly expository. Indeed, there are already many generalizations in the literature—cyclotomic [DJM98], affine [Gre99, MS19, MS21], and 2-categorical [Wil11, MSV13, Web17], to name but a few.

Once the general framework is in place, we use the q-Schur category to define a $\mathbb{Z}[q,q^{-1}]$-form for the positive half of the $U_q\mathfrak{gl}_n$-web category of Cautis, Kamnitzer and Morrison [CKM14], complete with bases for its morphism spaces as free $\mathbb{Z}[q,q^{-1}]$-modules. Integral bases in the latter category have previously been constructed in the unpublished paper of Elias [Eli15], and their existence also follows theoretically from [AST18], but the relationship to the known bases for the q-Schur algebra is not apparent from that work. We also explain how the canonical basis fits into this picture, something which is not mentioned at all in [Eli15].

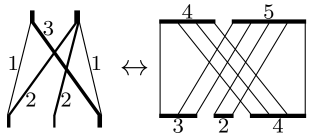
Our starting point is the definition of a strict $\mathbb{Z}$-linear monoidal category called the *Schur category*, denoted simply by **Schur**, from [BEAEO20, Def. 4.2]. The object set of **Schur** is the set Λ_s of all *strict compositions*, that is, sequences $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of positive integers for $\ell \geq 0$, with tensor product of objects defined by concatenation. For strict compositions λ and μ, the morphism space $\mathrm{Hom}_{\mathbf{Schur}}(\mu, \lambda)$ is zero unless $r := \sum_i \lambda_i = \sum_i \mu_i$, in which case this morphism space is a free $\mathbb{Z}$-module with a distinguished *standard basis* parametrized by the set $(S_\lambda \backslash S_r / S_\mu)_{\min}$ of minimal length representatives for the double cosets of the parabolic subgroups S_λ and S_μ in the symmetric group S_r. Vertical composition making **Schur** into a $\mathbb{Z}$-linear category is defined by *Schur's product rule* as in the classical Schur algebra (see [Gre07, 2.3b]), and the horizontal composition making it into a monoidal category is induced by the natural embeddings $S_a \times S_b \hookrightarrow S_{a+b}$.

As usual with strict monoidal categories, it is convenient to represent morphisms in **Schur** by certain string diagrams; the vertical composition $f \circ g$ of morphisms f and g is obtained

2020 *Mathematics Subject Classification.* 17B10, 18D10, 81R10.

Research supported in part by NSF grants DMS-2101703 and DMS-2348840.

by stacking the string diagram for f on top of the one for g, and their horizontal composition $f \star g$ is obtained by stacking f to the left of g. We represent the standard basis elements for $\mathrm{Hom}_{\mathbf{Schur}}(\mu, \lambda)$ by $\lambda \times \mu$ *double coset diagrams*[1], such as the diagram on the left:



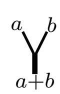
$$\leftrightarrow (2\,5\,8\,4\,7\,3\,6) \in (S_{(4,5)} \backslash S_9 / S_{(3,2,4)})_{\min} \leftrightarrow A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{bmatrix}.$$

In this double coset diagram, there are strings of various thicknesses indicated by the numerical labels. Thick strings at the bottom split into thinner strings, which are allowed to cross each other forming a *reduced* diagram for a permutation in the middle of the picture, before merging back into thick strings at the top. Subsequently, we will index $S_\lambda \backslash S_r / S_\mu$-double cosets also by the set $\mathrm{Mat}(\lambda, \mu)$ consisting of matrices of non-negative integers whose row and column sums are the entries of the compositions λ and μ, respectively. The ij-entry $a_{i,j}$ of the matrix A records the thickness of the string that connects the ith thick string at the top to the jth thick string at the bottom of the corresponding double coset diagram.

The q-analog of the Schur category is a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category denoted q-**Schur** whose specialization at $q = 1$ recovers **Schur**. In our approach, q-**Schur** is defined from the outset to be the $\mathbb{Z}[q, q^{-1}]$-linear category with the same objects as **Schur**, tensor product of objects being by concatenation as before. Its morphism spaces are defined so that $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ is the free $\mathbb{Z}[q, q^{-1}]$-module with a *standard basis* $\{\xi_A \,|\, A \in \mathrm{Mat}(\lambda, \mu)\}$, which we represent graphically by almost the same double coset diagrams as above, except that we replace each singular crossing  with a positive crossing . Then we need rules for computing vertical and horizontal compositions of standard basis vectors. Horizontal composition is defined by horizontally stacking diagrams just as in **Schur**. Vertical composition is defined by the q-analog of Schur's product rule; see (4.8) and (4.9). Although there is no simple closed formula for this in general, it can be computed algorithmically using relations in Manin's quantized coordinate algebra $\mathcal{O}_q(n)$ of $n \times n$ matrices from [Man88].

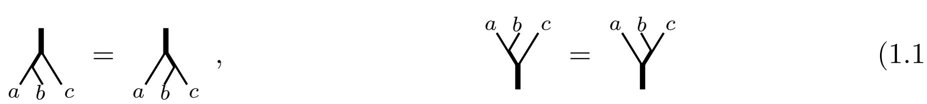
Theorem 1. *As a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category, q-**Schur** is generated by the objects (r) for $r > 0$ and morphisms called* merges *and* splits *represented by the diagrams*

$$: (a) \star (b) \to (a+b),$$

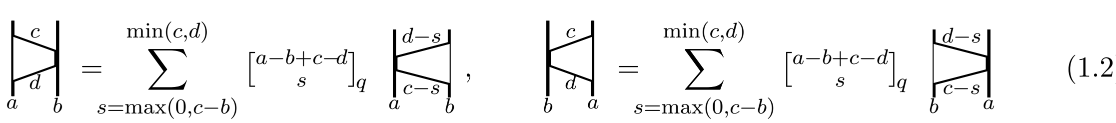


$$: (a+b) \to (a) \star (b)$$

for $a, b > 0$, subject to the associativity *and* coassociativity relations



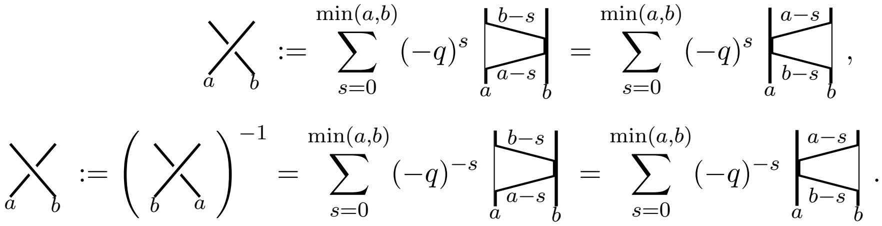
for $a, b, c > 0$, and one of the equivalent square-switch relations



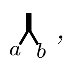
for $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$. Here, $\begin{bmatrix} n \\ s \end{bmatrix}_q$ is the q-binomial coefficient (3.2)*, and splits/merges with a string of thickness zero should be interpreted as identities.*

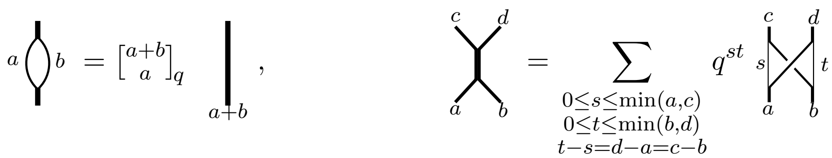
[1] Called "chicken foot diagrams" in [BEAEO20].

There is a braiding making q-**Schur** into a braided monoidal category. This is defined on generating objects by the *positive crossings*, with inverses denoted by *negative crossings*:



The following gives another presentation for q-**Schur** with even simpler relations, but with the positive crossings as a third family of generating morphisms; setting $q = 1$ in this recovers the presentation for **Schur** derived in [BEAEO20].

Theorem 2. *The monoidal category q-**Schur** is generated by the objects (r) for $r > 0$ and the morphisms*  ,  *and*  *for $a, b > 0$, subject only to the relations (1.1) together with*

 (1.3)

for $a, b, c, d > 0$ with $a + b = c + d$.

The presentations for q-**Schur** in Theorems 1 and 2 are not new, e.g., the relations can be found in [LT21] (with a different choice of normalization for the positive crossings coming from quantum SL_n rather than quantum GL_n). We give complete proofs here, rather than attempting to adapt related results already in the literature such as [Dot03]. Our general approach to the definition of q-**Schur**, equipping each of its morphism spaces with a standard basis over $\mathbb{Z}[q, q^{-1}]$ from the outset with structure constants which can computed algorithmically, facilitates calculations which seem quite awkward otherwise; e.g., see Corollary 6.2 for a formula for the composition of two positive crossings. The ability to compute products effectively is also exploited in the proof of the straightening formula in Lemma 7.4.

This straightening formula is the key ingredient in the proof of Theorem 3, which constructs a second basis for morphism spaces in q-**Schur**. We formulate this in terms of the path algebra

$$H := \bigoplus_{\lambda,\mu\in\Lambda_s} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda) \tag{1.4}$$

viewed as a locally unital algebra with distinguished idempotents $\{1_\lambda \mid \lambda \in \Lambda_s\}$ arising from the identity endomorphisms of the objects of q-**Schur**. Multiplication in H is induced by composition. Let Λ^+ be the subset of Λ_s consisting of all *partitions*, that is, ordered sequences $\kappa = (\kappa_1 \geq \cdots \geq \kappa_\ell)$ of positive integers for $\ell \geq 0$. For $\lambda \in \Lambda_s$ and $\kappa \in \Lambda^+$, we denote the usual set of all semistandard tableaux of shape κ and content λ by $\mathrm{Std}(\lambda, \kappa)$. For $P \in \mathrm{Std}(\lambda, \kappa)$, let $A(P) \in \mathrm{Mat}(\lambda, \kappa)$ be the matrix whose ij-entry records the number of times i appears on row j of P. For the definition of "symmetrically-based quasi-hereditary algebra" used in the statement of the theorem, see Definition 7.1. The triangular bases in this definition are *cellular bases* in the sense of [GL96]. However, the axioms are simpler than the ones for a cellular algebra; they are also more restrictive since it follows automatically that the underlying algebra is a split quasi-hereditary algebra with duality in the sense of [CPS90].

Theorem 3. *The locally unital algebra $H = \bigoplus_{\lambda,\mu\in\Lambda_s} 1_\lambda H 1_\mu$ is a symmetrically-based quasi-hereditary algebra with weight poset Λ^+ ordered by the dominance ordering $\leq$, anti-involution*

$\mathtt{T} : H \to H, \xi_A \mapsto \xi_{A^{\mathtt{T}}}$, *and triangular basis consisting of the* codeterminants $\xi_{A(P)}\xi_{A(Q)^{\mathtt{T}}}$ *for* $(P, Q) \in \bigcup_{\lambda,\mu\in\Lambda_s,\kappa\in\Lambda^+} \mathrm{Std}(\lambda,\kappa) \times \mathrm{Std}(\mu,\kappa)$.

There is a third remarkable basis in this subject, the canonical basis, which appeared originally in the context of q-Schur algebras in [BLM90] and was studied in detail by Du [Du92a, Du92b, Du95] from the perspective of Hecke algebras; see also [DDPW08, Ch. 9]. To define it, take $\lambda, \mu \in \Lambda_s$ such that $r := \sum_i \lambda_i = \sum_i \mu_i$, and $A \in \mathrm{Mat}(\lambda, \mu)$. Writing $d_A^+ \in (S_\lambda \backslash S_r / S_\mu)_{\max}$ for the maximal length double coset representative indexed by A, let

$$\theta_A := \sum_{B\in\mathrm{Mat}(\lambda,\mu)} q^{\ell(d_A^+)-\ell(d_B^+)} P_{d_A^+,d_B^+}(q^{-2})\xi_B,$$

where $P_{x,y}(t) \in \mathbb{Z}[t]$ is the Kazhdan-Lusztig polynomial for $x, y \in S_r$. Then the *canonical basis* for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ is $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$. The canonical basis can also be defined in terms of the *bar involution* $- : q\text{-}\mathbf{Schur} \to q\text{-}\mathbf{Schur}$, the anti-linear strict monoidal functor which fixes objects and the generating merge and split morphisms: θ_A is the unique morphism in $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ such that $\overline{\theta}_A = \theta_A$ and $\theta_A \equiv \xi_A \pmod{\sum_{B\in\mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]\xi_B}$. Note also that the bar involution interchanges positive and negative crossings.

The canonical basis makes the path algebra H into a "standardly full-based algebra" using the language of [DR98], with the same weight poset and cell ideals as the ones arising from the codeterminant basis in Theorem 3. This follows from the results in [DR98, §5.3], which imply that the canonical basis is cellular, hence, equivalent to a triangular basis; see Remark 7.6 for a precise statement. Theorem 3 could be deduced as a consequence of this like in [DR98, §5.5]. It could also be deduced from R. M. Green's construction [Gre96] of the q-analog of J. A. Green's codeterminant basis for the Schur algebra. The short self-contained proof of Theorem 3 given here is similar to the one in [Gre96] (and in [Woo93] when $q = 1$), but incorporates simplifications made possible by working in the less constrained setting of the q-Schur category. Analogous bases for cyclotomic q-Schur algebras of all levels (not merely level one) have been constructed in [DJM98, Th. 6.6] by a different method.

At least one of these new bases (codeterminant or canonical) is needed in order to understand a certain truncation $q\text{-}\mathbf{Schur}_n$ of the q-Schur category. By definition, this is the quotient of q-**Schur** by the two-sided tensor ideal $\mathbf{I}_n$ generated by the identity endomorphisms $1_{(r)}$ for $r > n$. The presentation for $q\text{-}\mathbf{Schur}_n$ arising from Theorem 1 makes it clear that it is a version of Cautis-Kamnitzer-Morrison's $U_q\mathfrak{gl}_n$-web category, or rather, its positive half involving only upward-pointing strings. The ideal $\mathbf{I}_n$ is compatible with the basis from Theorem 3. Consequently, the path algebra of $q\text{-}\mathbf{Schur}_n$ is also a symmetrically-based quasi-hereditary algebra with triangular basis given by the images of the codeterminants $\xi_{A(P)}\xi_{A(Q)^{\mathtt{T}}}$ for all pairs (P, Q) of semistandard tableaux whose shape κ satisfies $\kappa_1 \leq n$. This basis is of a similar nature to the integral bases for morphism spaces in this category constructed in [Eli15]. The canonical basis also induces a cellular basis for the path algebra of $q\text{-}\mathbf{Schur}_n$.

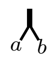
Let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q, q^{-1}]$-algebra in some way, and consider the specialization $q\text{-}\mathbf{Schur}_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}_n$. Also let U_n be Lusztig's $\mathbb{Z}[q, q^{-1}]$-form for the quantized enveloping algebra $U_q\mathfrak{gl}_n$ with Chevalley generators $E_i, F_i\,(1 \leq i \leq n-1)$ and $D_i^{\pm1}\,(1 \leq i \leq n)$. Let $U_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} U_n$. We view it as a Hopf algebra with comultiplication Δ satisfying

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes D_i^{-1}D_{i+1}, \quad \Delta(F_i) = F_i \otimes 1 + D_iD_{i+1}^{-1} \otimes F_i, \quad \Delta(D_i) = D_i \otimes D_i.$$

The natural $U_n(\Bbbk)$-module V is the vector space with basis $v_1, \ldots, v_n$ such that $E_iv_j = \delta_{i+1,j}v_i$, $F_iv_j = \delta_{i,j}v_{i+1}$, $D_iv_j = q^{\delta_{i,j}}v_j$. Its rth quantum exterior power $\bigwedge^r V$ is a certain quotient of the rth tensor power $V^{\otimes r}$ with a basis given by the monomials $v_{i_1} \wedge \cdots \wedge v_{i_r}$ that are images of the pure tensors $v_{i_1} \otimes \cdots \otimes v_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$. Let $q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$, the category of *polynomial tilting modules*, be the full additive Karoubian monoidal subcategory of $U_n(\Bbbk)$-mod generated by the exterior powers $\bigwedge^r V$ for all $r \geq 0$. This is a braided (but not rigid) monoidal category with braiding $c : - \otimes - \stackrel{\sim}{\Rightarrow} - \otimes^{\mathrm{rev}} -$ defined so that

$$c_{V,V} : V \otimes V \to V \otimes V, \qquad v_i \otimes v_j \mapsto \begin{cases} v_j \otimes v_i & \text{if } i < j, \\ q^{-1} v_j \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i - (q - q^{-1}) v_i \otimes v_j & \text{if } i > j. \end{cases} \tag{1.5}$$

If $\Bbbk$ is of characteristic 0 and the image of q in $\Bbbk$ is not a root of unity, q-$\mathbf{Tilt}_n^+(\Bbbk)$ is a semisimple Abelian category, and the following theorem can be deduced from [CKM14].

Theorem 4. *There is a $\Bbbk$-linear monoidal functor $\Sigma_n : q\text{-}\mathbf{Schur}_n(\Bbbk) \to q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$ taking the generating object (r) to $\bigwedge^r V$, the merge*  *to the natural surjection $\bigwedge^a V \otimes \bigwedge^b V \twoheadrightarrow \bigwedge^{a+b} V$, and the split*  *to the inclusion $\bigwedge^{a+b} V \hookrightarrow \bigwedge^a V \otimes \bigwedge^b V$ defined by*

$$v_{i_1} \wedge \cdots \wedge v_{i_{a+b}} \mapsto q^{-ab} \sum_{w \in (S_{a+b}/S_a \times S_b)_{\min}} (-q)^{\ell(w)} v_{i_{w(1)}} \wedge \cdots \wedge v_{i_{w(a)}} \otimes v_{i_{w(a+1)}} \wedge \cdots \wedge v_{i_{w(a+b)}}$$

for $1 \leq i_1 < \cdots < i_{a+b} \leq n$. This functor is full and faithful, and it induces a monoidal equivalence between the additive Karoubi envelope of q-$\mathbf{Schur}_n(\Bbbk)$ and q-$\mathbf{Tilt}_n^+(\Bbbk)$.

The monoidal functor Σ_n of Theorem 4 is not a braided monoidal functor—it takes the positive crossing  to $(-1)^{ab} c^{-1}_{\bigwedge^b V, \bigwedge^a V}$ rather than to $c_{\bigwedge^a V, \bigwedge^b V}$. This twist, which may at first seem inconvenient, is reasonable since the proof involves some Ringel duality—the generating object (r) of the q-Schur category corresponds more naturally to the rth quantum symmetric power of the natural module rather than its exterior power.

There is one more important explanation to be made: subsequently, the notation q-**Schur** will be used to denote a slightly larger version of the q-Schur category than appears in this introduction, with objects that are indexed by *all* compositions, not just strict ones. In other words, we adjoin an additional generating object (0) which is isomorphic but not equal to the strict identity object $\mathbb{1}$. We prefer to use the same notation for both versions—it should be clear from context whether we are working with or without strings of thickness zero. The natural inclusion of the q-Schur category as defined in the introduction into the one with 0-strings is a monoidal equivalence, making it easy to go back and forth between the two versions. One advantage of q-Schur category *with* 0-strings is that there is a surjective algebra homomorphism from U_n to the path algebra of the full subcategory whose objects are compositions with exactly n parts. Actually, it is more convenient to work with Lusztig's modified form $\dot{U}_n$ here; see (8.1). Using this connection, the approach to q-Schur algebras taken in [Dot03], exploiting Lusztig's refined Peter-Weyl theorem for $\dot{U}_n$ [Lus10, Sec. 29.3], could be adapted to give yet another approach to the results here.

Acknowledgements. The first draft of this article was written in February 2020. I would like to thank Elijah Bodish, one of the few mathematicians I saw at the University of Oregon for many months after that, for his interest in this material. This paper is dedicated to Gary Seitz who had a profound influence on my life—indeed, without Gary's encouragement and mentorship, I would never have found my way to Oregon in the first place.

2. Double coset combinatorics

A *composition* $\lambda \vDash r$ is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of non-negative integers summing to r. We write $\ell(\lambda)$ for the total number ℓ of parts, which is allowed to be zero, and $|\lambda|$ for the sum of the parts. We emphasize that we treat compositions of different lengths as being different, e.g., $() \neq (0) \neq (0,0)$. A *partition* $\lambda \vdash r$ is a composition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vDash r$ whose parts satisfy $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. For partitions, we allow to write λ_r even if $r > \ell(\lambda)$, in which case $\lambda_r = 0$. We denote the sets of all compositions and all partitions by Λ and Λ^+, respectively. Let $\leq$ be the usual dominance ordering on Λ^+.

We denote the transposition $(i\ i{+}1)$ in the symmetric group S_r by s_i, $\ell : S_r \to \mathbb{N}$ is the length function, and $w_r \in S_r$ is the longest element. Elements of S_r act on the *left* on the set $\{1, \dots, r\}$. There is also a *right* action of S_r on $\mathbb{Z}^r$ by place permutation: for $\boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$ and $w \in S_r$, the r-tuple $\boldsymbol{i} \cdot w$ has jth entry $i_{w(j)}$. For $\lambda = (\lambda_1, \dots, \lambda_\ell) \vDash r$, the set

$$\mathrm{I}_\lambda := \big\{\boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r \bigm| \#\{k = 1, \dots, r \mid i_k = i\} = \lambda_i \text{ for all } i \in \{1, \dots, \ell(\lambda)\}\big\} \tag{2.1}$$

is a single orbit under this action. Also let $\boldsymbol{i}^\lambda = (i_1^\lambda, \dots, i_r^\lambda)$ denote the unique element of I_λ whose entries are in weakly increasing order. Its stabilizer in S_r is the parabolic subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$.

For $\lambda, \mu \vDash r$, the symmetric group S_r acts diagonally on the right on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$. The orbits are parametrized by the set $\mathrm{Mat}(\lambda, \mu)$ of all $\ell(\lambda) \times \ell(\mu)$ matrices with non-negative integer entries such that the entries in the ith row sum to λ_i and the entries in the jth column sum to μ_j for all $i \in \{1, \dots, \ell(\lambda)\}$ and $j \in \{1, \dots, \ell(\mu)\}$. For $A = (a_{i,j}) \in \mathrm{Mat}(\lambda, \mu)$, the corresponding S_r-orbit on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$ is

$$\Pi_A := \left\{ (\boldsymbol{i}, \boldsymbol{j}) \in \mathrm{I}_\lambda \times \mathrm{I}_\mu \;\middle|\; \begin{array}{l} \#\{k = 1, \dots, r \mid (i_k, j_k) = (i, j)\} = a_{i,j} \\ \text{for all } i \in \{1, \dots, \ell(\lambda)\}, j \in \{1, \dots, \ell(\mu)\} \end{array} \right\}. \tag{2.2}$$

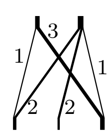
The set $\mathrm{Mat}(\lambda, \mu)$ is actually just one of many different sets used in the literature to parametrize the orbits of S_r on $\mathrm{I}_\lambda \times \mathrm{I}_\mu$. Another is by the set $\mathrm{Row}(\lambda, \mu)$ of *row tableaux* of shape μ and content λ, that is, left justified arrays with μ_1 boxes in row 1 (the top row), μ_2 boxes in row 2, and so on, with boxes filled with integers so that entries are weakly increasing in order from left to right along each row, and there are a total of λ_1 entries equal to 1, λ_2 equal to 2, and so on. We use the explicit bijection

$$A : \mathrm{Row}(\lambda, \mu) \to \mathrm{Mat}(\lambda, \mu) \tag{2.3}$$

taking $P \in \mathrm{Row}(\lambda, \mu)$ to the matrix $A(P) \in \mathrm{Mat}(\lambda, \mu)$ whose ij-entry records the number of times i appears on row j of P. The inverse bijection maps $A \in \mathrm{Mat}(\lambda, \mu)$ to the row tableau $P \in \mathrm{Row}(\lambda, \mu)$ whose jth row is equal to $1^{a_{1,j}}\ 2^{a_{2,j}}\ \cdots\ \ell^{a_{\ell(\lambda),j}}$.

A third way to parametrize orbits is by the double coset diagrams introduced already in the introduction. We gave already there an example in which $\lambda = (4, 5)$, $\mu = (3, 2, 4)$, for which the matrix $A \in \mathrm{Mat}(\lambda, \mu)$, the corresponding double coset diagram, and the corresponding row tableau $P \in \mathrm{Row}(\lambda, \mu)$ are

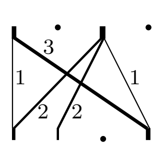
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \end{bmatrix} \quad \leftrightarrow$$



$$\leftrightarrow \quad P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 2 \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \tag{2.4}$$

Unlike in the introduction, we are now allowing compositions with parts equal to 0, so double coset diagrams can also have strings labelled by 0. In fact, it is harmless to omit these zero thickness strings from the diagram entirely, but one should mark their endpoints. Here is an example with $\lambda = (4, 0, 5, 0)$ and $\mu = (3, 2, 0, 4)$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \leftrightarrow$$



$$\leftrightarrow \quad P = \begin{array}{l} \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline 3 & 3 \\ \hline \end{array} \\ \\ \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline \end{array} \end{array} \tag{2.5}$$

Two other sets in bijection with $\mathrm{Mat}(\lambda,\mu)$ are the sets $(S_\lambda\backslash S_r/S_\mu)_{\min}$ and $(S_\lambda\backslash S_r/S_\mu)_{\max}$ of minimal length and maximal length double coset representatives. For $A \in \mathrm{Mat}(\lambda,\mu)$, we denote the corresponding elements of $(S_\lambda\backslash S_r/S_\mu)_{\min}$ and $(S_\lambda\backslash S_r/S_\mu)_{\max}$ by d_A and d_A^+, respectively.

Lemma 2.1. *Given $\lambda,\mu \vDash r$ and $A \in \mathrm{Mat}(\lambda,\mu)$, let $\lambda^- \vDash r$ (resp., $\mu^+ \vDash r$) be obtained by reading the entries of A in order along rows starting with the top row (resp., in order down columns starting with the leftmost column). We have that*

$$\big(S_\lambda d_A\big) \cap \big(d_A S_\mu\big) = d_A S_{\mu^+} = S_{\lambda^-}\ d_A.$$

Every element $w \in S_\lambda d_A S_\mu$ can be written uniquely as $x d_A\, y$ for $x \in (S_\lambda/S_{\lambda^-})_{\min}$, $y \in S_\mu$ (resp., $x d_A\, y$ for $x \in S_\lambda$, $y \in (S_{\mu^+}\backslash S_\mu)_{\min}$), and we have that $\ell(x d_A\, y) = \ell(x) + \ell(d_A) + \ell(y)$.

Proof. This follows from [DJ89, Lemma 1.6]. □

The double coset diagram gives a convenient visual way to translate $A \in \mathrm{Mat}(\lambda,\mu)$ into the minimal length double coset representatives d_A. Alternatively, to obtain d_A, let $(i_1,\dots,i_r) \in I_\lambda$ be the sequence $\zeta(P)$ obtained by reading the entries of the corresponding row tableau P from left to right along rows, starting with the top row. Then replace the λ_1 entries equal to 1 in this sequence by $1,\dots,\lambda_1$ in increasing order, the λ_2 entries equal to 2 by $\lambda_1+1,\dots,\lambda_1+\lambda_2$ in increasing order, and so on. The result is d_A written in one-line notation. To compute d_A^+, we instead start from the sequence $\bar{\zeta}(P)$ obtained by reading entries of P from right to left along rows, starting with the top row. Then we replace the entries 1 by $\lambda_1,\dots,1$ in decreasing order, the entries 2 by $\lambda_1+\lambda_2,\dots,\lambda_1+1$ in decreasing order, and so on. In the example (2.4), $\zeta(P) = (1,2,2,2,2,1,1,1,2)$ so $d_A = (1,5,6,7,8,2,3,4,9)$, and $\bar{\zeta}(P) = (2,2,1,2,2,2,1,1,1)$ so $d_A^+ = (9,8,4,7,6,5,3,2,1)$.

Let $\le$ be the *Bruhat ordering* on the symmetric group (so the identity element is *minimal*). This restricts to partial orders on the sets $(S_\lambda\backslash S_r/S_\mu)_{\min}$ and $(S_\lambda\backslash S_r/S_\mu)_{\max}$, such that

$$d_A \le d_B \qquad \Leftrightarrow \qquad d_A^+ \le d_B^+ \tag{2.6}$$

if d_A and d_B are minimal length double coset representatives and d_A^+ and d_B^+ are the corresponding maximal ones (this coincidence is proved in [HS05]). Using the bijections between these sets, we transport the Bruhat order to partial orders on $\mathrm{Row}(\lambda,\mu)$ and $\mathrm{Mat}(\lambda,\mu)$. The resulting partial order on $\mathrm{Mat}(\lambda,\mu)$ is given explicitly in terms of matrices by

$$A \le B \Leftrightarrow \left(\sum_{i=1}^{s}\sum_{j=1}^{t} a_{i,j} \ge \sum_{i=1}^{s}\sum_{j=1}^{t} b_{i,j} \text{ for all } s \in \{1,\dots,\ell(\lambda)\}, t \in \{1,\dots,\ell(\mu)\}\right). \tag{2.7}$$

One finds this elementary combinatorial observation in many places in the literature, e.g., see [BLM90] which also explains the geometric origin of this ordering.

3. The quantized coordinate algebra

The ring $\mathbb{Z}[q,q^{-1}]$ has a bar involution $-$ which sends q to q^{-1}. We will use the term "anti-linear map" for a $\mathbb{Z}$-module homomorphism between $\mathbb{Z}[q,q^{-1}]$-modules which intertwines q and q^{-1} in this way. For $\mathbb{Z}[q,q^{-1}]$-modules, $V \otimes W$ means tensor product over $\mathbb{Z}[q,q^{-1}]$ and V^* denotes $\mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(V,\mathbb{Z}[q,q^{-1}])$. We will need the quantum integer

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \tag{3.1}$$

and quantum binomial coefficient

$$\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q[n-1]_q\cdots[n-s+1]_q}{[s]_q[s-1]_q\cdots[1]_q}, \tag{3.2}$$

which we interpret as zero in case $s < 0$. These satisfy the Pascal-type recurrence relation:

$$\begin{bmatrix} n \\ s \end{bmatrix}_q = q^s \begin{bmatrix} n-1 \\ s \end{bmatrix}_q + q^{s-n} \begin{bmatrix} n-1 \\ s-1 \end{bmatrix}_q = q^{-s} \begin{bmatrix} n-1 \\ s \end{bmatrix}_q + q^{n-s} \begin{bmatrix} n-1 \\ s-1 \end{bmatrix}_q. \tag{3.3}$$

The following play the role of the binomial theorem for positive and negative exponents:

$$\prod_{s=1}^{n} (1 + q^{2s-n-1}x) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q x^s, \qquad \prod_{s=1}^{n} \frac{1}{(1 + q^{2s-n-1}x)} = \sum_{s=0}^{n} \begin{bmatrix} -n \\ s \end{bmatrix}_q x^i. \tag{3.4}$$

Here are some more identities that will be needed later.

Lemma 3.1. *For $n \geq 0$, we have that* $\sum_{s=0}^{n} (-1)^s q^{s(n-1)} \begin{bmatrix} n \\ s \end{bmatrix}_q = \delta_{n,0}$.

Proof. Set $x = -q^{-n-1}$ in the first identity from (3.4). □

Lemma 3.2. *For $m, n \in \mathbb{Z}$ and $s \geq 0$, we have that* $\sum_{a+b=s} q^{mb-na} \begin{bmatrix} m \\ a \end{bmatrix}_q \begin{bmatrix} n \\ b \end{bmatrix}_q = \begin{bmatrix} m+n \\ s \end{bmatrix}_q$.

Proof. This is proved by a standard argument using (3.4). See also [Fie23, Prop. 4.1(5)] (where this is called the Chu-Vandermonde convolution formula). □

Lemma 3.3. *For $m \in \mathbb{Z}$ and $s \geq 0$, we have that* $\sum_{a+b=s} (-q)^{-b} \begin{bmatrix} m+a \\ a \end{bmatrix}_q \begin{bmatrix} m \\ b \end{bmatrix}_q = q^{ms}$.

Proof. This is the q-analog of [BEAEO20, Lem. A.1]. See [BK22, Lem. 3.1(3)] for its proof. □

Let $\mathcal{O}_q(n)$ be Manin's quantized coordinate algebra of $n \times n$ matrices [Man88], which is the $\mathbb{Z}[q, q^{-1}]$-algebra on generators $\{x_{i,j} \mid 1 \leq i, j \leq n\}$ subject to the relations

$$x_{i,j} x_{k,l} = \begin{cases} x_{k,l} x_{i,j} & \text{if } i < k \text{ and } j > l, \\ x_{k,l} x_{i,j} - (q - q^{-1}) x_{i,l} x_{k,j} & \text{if } i > k \text{ and } j > l, \\ q^{-1} x_{k,l} x_{i,j} & \text{if } i = k, j > l, \\ q x_{k,l} x_{i,j} & \text{if } i < k, j = l. \end{cases} \tag{3.5}$$

We view $\mathcal{O}_q(n)$ as a bialgebra with comultiplication $\Delta : \mathcal{O}_q(n) \to \mathcal{O}_q(n) \otimes \mathcal{O}_q(n)$ and counit $\varepsilon : \mathcal{O}_q(n) \to \mathbb{Z}[q, q^{-1}]$ defined by

$$\Delta(x_{i,k}) = \sum_{j=1}^{n} x_{i,j} \otimes x_{j,k}, \qquad \varepsilon(x_{i,j}) = \delta_{i,j}. \tag{3.6}$$

Lemma 3.4. *In $\mathcal{O}_q(2)$, we have for $a, b \geq 0$ that*

$$x_{2,2}^a x_{1,1}^b = \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2} (q^{-1} - q)^s [s]_q^! \begin{bmatrix} a \\ s \end{bmatrix}_q \begin{bmatrix} b \\ s \end{bmatrix}_q x_{2,1}^s x_{1,1}^{b-s} x_{2,2}^{a-s} x_{1,2}^s.$$

Proof. Use induction on a to check that $x_{2,2}^a x_{1,1} = x_{1,1} x_{2,2}^a - (q - q^{-1})[a] x_{2,1} x_{2,2}^{a-1} x_{1,2}$. This treats the case $b = 1$. Then proceed by induction on b using (3.3). □

Lemma 3.5. *In $\mathcal{O}_q(2)$, we have for $a \geq 0$ and $i, j \in \{1, 2\}$ that*

$$\Delta(x_{i,j}^a) = \sum_{s=0}^{a} \begin{bmatrix} a \\ s \end{bmatrix}_q x_{i,1}^s x_{i,2}^{a-s} \otimes x_{2,j}^{a-s} x_{1,j}^s.$$

Proof. Exercise. □

The character group of the n-dimensional torus consisting of diagonal matrices in GL_n is naturally identified with the Abelian group $\mathbb{Z}^n$, with standard coordinates $\varepsilon_1,\dots,\varepsilon_n$. There is a scalar product on $\mathbb{Z}^n$ such that $\varepsilon_i\cdot\varepsilon_j=\delta_{i,j}$. We also have the *dominance order* on $\mathbb{Z}^n$ defined by $\lambda\leq\mu$ if the difference $\mu-\lambda$ is a sum of simple roots $\alpha_i:=\varepsilon_i-\varepsilon_{i+1}$ for $i=1,\dots,n-1$.

The algebra $\mathcal{O}_q(n)$ admits two different gradings. It is $\mathbb{Z}$-graded with $x_{i,j}$ in degree one, and it is bigraded by the character group $\mathbb{Z}^n$ with $x_{i,j}$ of bidegree $(\varepsilon_i,\varepsilon_j)$:

$$\mathcal{O}_q(n)=\bigoplus_{r\geq 0}\mathcal{O}_q(n,r)=\bigoplus_{\lambda,\mu\in\mathbb{Z}^n}\mathcal{O}_q[\lambda,\mu]. \tag{3.7}$$

These two gradings are compatible with each other:

$$\mathcal{O}_q(n,r)=\bigoplus_{\lambda,\mu\in\Lambda(n,r)}\mathcal{O}_q[\lambda,\mu], \tag{3.8}$$

where $\Lambda(n,r):=\{\lambda\vDash r\mid\ell(\lambda)=n\}$ is the set of all $\lambda\in\mathbb{Z}^n$ such that $\lambda_1,\dots,\lambda_n\geq 0$ and $\lambda_1+\cdots+\lambda_n=r$. It is also important to observe that $\mathcal{O}_q(n,r)$ is a subcoalgebra of $\mathcal{O}_q(n)$.

Let

$$\mathrm{I}(n,r):=\left\{\boldsymbol{i}=(i_1,\dots,i_r)\in\mathbb{Z}^r\;\middle|\;1\leq i_1,\dots,i_r\leq n\right\}=\bigcup_{\lambda\in\Lambda(n,r)}\mathrm{I}_\lambda. \tag{3.9}$$

For $\boldsymbol{i},\boldsymbol{j}\in\mathrm{I}(n,r)$, we use the shorthand $x_{\boldsymbol{i},\boldsymbol{j}}:=x_{i_1,j_1}\cdots x_{i_r,j_r}$. Then $\mathcal{O}_q(n,r)$ is free as a $\mathbb{Z}[q,q^{-1}]$-module with the following basis, which we call the *normally-ordered monomial basis*:

$$\left\{x_{\boldsymbol{i},\boldsymbol{j}}\;\middle|\;\boldsymbol{i},\boldsymbol{j}\in\mathrm{I}(n,r),j_1\leq\cdots\leq j_r\text{ and }i_s\geq i_{s+1}\text{ when }j_s=j_{s+1}\right\}. \tag{3.10}$$

There are several different proofs of this, e.g., in [Bru06, §6] it is derived from another realization of $\mathcal{O}_q(n)$ as a braided tensor product of quantum symmetric algebras; normally-ordered here corresponds to the "terminal double indexes" in [Bru06]. Another relevant basis is

$$\left\{x_{\boldsymbol{i},\boldsymbol{j}}\;\middle|\;\boldsymbol{i},\boldsymbol{j}\in\mathrm{I}(n,r),i_1\geq\cdots\geq i_r\text{ and }j_s\leq j_{s+1}\text{ when }i_s=i_{s+1}\right\}. \tag{3.11}$$

This is the monomial basis in [Bru06] indexed by "initial double indexes".

Following [Bru06, Theorem 16], the *bar involution* on $\mathcal{O}_q(n)$ is the anti-linear map

$$-:\mathcal{O}_q(n)\to\mathcal{O}_q(n) \tag{3.12}$$

which fixes all of the generators $x_{i,j}$ and satisfies

$$\overline{xy}=q^{\lambda\cdot\mu-\lambda'\cdot\mu'}\,\overline{y}\,\overline{x} \tag{3.13}$$

for x of bidegree (λ,λ') and y of bidegree (μ,μ'). It is indeed an involution. Moreover:

Lemma 3.6. *The bar involution is an anti-linear coalgebra automorphism.*

Proof. Let $\overline{\Delta}$ denote the composition $-\otimes-\circ\Delta$. We must show that $\overline{\Delta}(x)=\Delta(\overline{x})$ for any $x\in\mathcal{O}_q(n)$. This follows by induction on degree. □

For $\lambda,\mu\in\Lambda(n,r)$, recall the set $\mathrm{Mat}(\lambda,\mu)$ of matrices with these row and column sums from §2, which parametrizes the orbits Π_A of S_r on $I_\lambda\times I_\mu$. For $A\in\mathrm{Mat}(\lambda,\mu)$, let

$$x_A:=x_{\boldsymbol{i},\boldsymbol{j}}\text{ for }(\boldsymbol{i},\boldsymbol{j})\in\Pi_A\text{ such that }j_1\leq\cdots\leq j_d\text{ and }i_k\geq i_{k+1}\text{ when }j_k=j_{k+1}. \tag{3.14}$$

In other words, if A corresponds to $P\in\mathrm{Row}(\lambda,\mu)$ under (2.3) then $\boldsymbol{i}=\swarrow\!\!(P)$ and $\boldsymbol{j}=\boldsymbol{i}^\mu$; the notation $\swarrow\!\!(P)$ means the sequence obtained by reading the entries of P in the order suggested by the arrow. Hence $\boldsymbol{i}=\boldsymbol{i}^\lambda\cdot d(A)w_0$ where w_0 is the longest element of S_r. The set $\{x_A\mid\lambda,\mu\in\Lambda(n,r),A\in\mathrm{Mat}(\lambda,\mu)\}$ is the normally-ordered monomial basis of $\mathcal{O}_q(n,r)$ from (3.10), we have merely parametrized it in a more convenient way. By [Bru06, Theorem 16] again, the image of the normally-ordered monomial x_A under the bar involution is

$$\overline{x}_A:=x_{\boldsymbol{i},\boldsymbol{j}}\text{ for }(\boldsymbol{i},\boldsymbol{j})\in\Pi_A\text{ such that }i_1\geq\cdots\geq i_r\text{ and }j_k\leq j_{k+1}\text{ when }i_k=i_{k+1}. \tag{3.15}$$

In other words, if A^{T} corresponds to $Q \in \mathrm{Row}(\mu,\lambda)$ under (2.3) then $\boldsymbol{i} = \boldsymbol{i}^\lambda \cdot w_r$ and $\boldsymbol{j} = \trianglelefteq(Q)$. The set $\{\overline{x}_A \mid \lambda,\mu \in \Lambda(n,r), A \in \mathrm{Mat}(\lambda,\mu)\}$ is the basis for $\mathcal{O}_q(n,r)$ from (3.11).

Recall the partial order (2.7) on $\mathrm{Mat}(\lambda,\mu)$. The bar involution acts on the normally-ordered monomial basis in a unitriangular fashion:

$$\overline{x}_A = x_A + \big(\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } x_B\text{'s for } B > A\big)\,.$$

This may be seen explicitly by using the relations (3.5) to rewrite (3.15) in terms of normally-ordered monomials. So one can apply Lusztig's Lemma to define another basis for $\mathcal{O}_q[\lambda,\mu]$, the *dual canonical basis* $\{b_A \mid A \in \mathrm{Mat}(\lambda,\mu)\}$. The dual canonical basis element b_A is the unique bar-invariant vector in $\mathcal{O}_q[\lambda,\mu]$ such that $b_A \equiv x_A \pmod{\sum_{B\in\mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]x_B}$. The dual canonical basis is discussed further in [Bru06] (and many other places). In particular, the polynomials $p_{A,B}(q) \in \mathbb{Z}[q]$ defined from

$$x_B = \sum_{A\in\mathrm{Mat}(\lambda,\mu)} p_{A,B}(q) b_A \tag{3.16}$$

are (renormalized) Kazhdan-Lusztig polynomials: writing $P_{x,y}(t) \in \mathbb{Z}[t]$ for the usual Kazhdan-Lusztig polynomial associated to $x,y \in S_r$, we have that

$$p_{A,B}(q) = q^{\ell(d_A^+)-\ell(d_B^+)} P_{d_A^+,d_B^+}(q^{-2}). \tag{3.17}$$

This is explained in [Bru06, Rem. 10]. We have that $p_{A,B}(q) = 0$ unless $A \geq B$, $p_{A,A}(q) = 1$, and $p_{A,B}(q) \in q\mathbb{N}[q]$ if $A > B$. The last assertion, which follows from positivity of Kazhdan-Lusztig polynomials, will not be needed here.

Lemma 3.7. *Suppose we are given $A', B' \in \mathrm{Mat}(\lambda',\mu')$ for $\lambda',\mu' \in \Lambda(n,r)$ and $1 \leq i,j \leq n$ such that $\lambda'_i = \mu'_j = 0$. Let A and B be the matrices obtained from A' and B' by removing the ith row and jth column. Then $p_{A,B}(q) = p_{A',B'}(q)$.*

Proof. This is clear from the nature of the defining relations (3.5) for $\mathcal{O}_q(n)$: they only depend on the relative positions of the indices in the total order on the set $\{1,\dots,n\}$, not on the actual values. □

Example 3.8. For $\lambda,\mu \in \Lambda(2,r)$ and $A \in \mathrm{Mat}(\lambda,\mu)$, we have that

$$b_A = x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}-\min(a_{1,1},a_{2,2})} \left(x_{1,1}x_{2,2} - qx_{2,1}x_{1,2}\right)^{\min(a_{1,1},a_{2,2})} x_{2,2}^{a_{2,2}-\min(a_{1,1},a_{2,2})} x_{1,2}^{a_{1,2}}.$$

This follows from a special case of [Bru06, Theorem 20], which gives a closed formula for the dual canonical basis element b_A for all $A \in \mathrm{Mat}(\lambda,\mu)$ providing either λ or μ has at most two non-zero parts. Expanding the binomial gives

$$b_A = x_A - q^M \begin{bmatrix} m\\ 1\end{bmatrix}_q x_{A+B} + q^{2(M-1)} \begin{bmatrix} m\\ 2\end{bmatrix}_q x_{A+2B} - \cdots + (-1)^m q^{m(M+1-m)} \begin{bmatrix} m\\ m\end{bmatrix}_q x_{A+mB}$$

where $m := \min(a_{1,1},a_{2,2})$, $M := \max(a_{1,1},a_{2,2})$ and $B := \left[\begin{smallmatrix} -1 & 1\\ 1 & -1\end{smallmatrix}\right]$.

Lemma 3.9. *There is a surjective bialgebra homomorphism*

$$\mathtt{Y}^* : \mathcal{O}_q(m+n) \twoheadrightarrow \mathcal{O}_q(m)\otimes\mathcal{O}_q(n), \qquad x_{i,j} \mapsto \begin{cases} x_{i,j}\otimes 1 & \text{if } 1 \leq i,j \leq m,\\ 1 \otimes x_{i-m,j-m} & \text{if } m+1 \leq i,j \leq m+n,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, this intertwines the bar involution on $\mathcal{O}_q(m+n)$ with the bar involution $-\otimes-$ on $\mathcal{O}_q(m)\otimes\mathcal{O}_q(n)$.

Proof. The existence of this algebra homomorphism follows from the relations. Then one checks that it is a coalgebra homomorphism too. Finally, for the statement about the bar involution, note for an $m\times m$ matrix A and an $n \times n$ matrix B that $\mathtt{Y}^*$ sends $x_{\mathrm{diag}(A,B)}$ to $x_A \otimes x_B$ and $\overline{x}_{\mathrm{diag}(A,B)}$ to $\overline{x}_A \otimes \overline{x}_B$. □

There is also an anti-linear algebra anti-automorphism

$$\overline{\mathtt{T}}^* : \mathcal{O}_q(n) \to \mathcal{O}_q(n), \qquad x_{i,j} \mapsto x_{j,i}. \tag{3.18}$$

This is a coalgebra anti-automorphism, i.e., $\overline{\mathtt{T}}^* \otimes \overline{\mathtt{T}}^* \circ \Delta = \mathtt{P} \circ \Delta \circ \overline{\mathtt{T}}^*$ where $\mathtt{P}$ is the tensor flip. Comparing (3.14) and (3.15), we see that $\overline{\mathtt{T}}^*(x_A) = \overline{x}_{A^\mathtt{T}}$ where $A^\mathtt{T}$ is the transpose matrix. Since $\overline{\mathtt{T}}^*$ is an involution, it follows that it commutes with the bar involution. Let

$$\mathtt{T}^* := - \circ \overline{\mathtt{T}}^* = \overline{\mathtt{T}}^* \circ - : \mathcal{O}_q(n) \to \mathcal{O}_q(n). \tag{3.19}$$

This is a linear coalgebra anti-automorphism (but *not* an algebra anti-automorphism) which commutes with the bar involution and sends x_A to $x_{A^\mathtt{T}}$. It follows that

$$\mathtt{T}^*(b_A) = b_{A^\mathtt{T}}. \tag{3.20}$$

The dual canonical basis element b_A indexed by $A = I_n$, which is minimal in the Bruhat order, is the *quantum determinant*

$$\det_q := \sum_{w \in S_n} (-q)^{\ell(w)} x_{w(1),1} \cdots x_{w(n),n}. \tag{3.21}$$

This is central in $\mathcal{O}_q(n)$. It is also a group-like element, i.e., $\Delta(\det_q) = \det_q \otimes \det_q$ and $\varepsilon(\det_q) = 1$. The coordinate algebra of the *quantum general linear group q-GL_n* is the Ore localization of $\mathcal{O}_q(n)$ at the quantum determinant. The bialgebra structure on $\mathcal{O}_q(n)$ extends to make this into a Hopf algebra. We will not work explicitly with this Hopf algebra here, but its existence underpins all subsequent language and notation.

By a *polynomial representation of q-GL_n* we mean a right $\mathcal{O}_q(n)$-comodule. We use the notation $\mathrm{Hom}_{q\text{-}GL_n}(-,-)$ to denote morphisms in the category of polynomial representations. Since $\mathcal{O}_q(n)$ is a bialgebra, this is a monoidal category. For example, we have the *natural representation* of q-GL_n, which is the free $\mathbb{Z}[q,q^{-1}]$-module V with basis $v_1, \dots, v_n$ and comodule structure map $\eta : V \to V \otimes \mathcal{O}_q(n,1)$ defined from

$$\eta(v_j) = \sum_{i=1}^n v_i \otimes x_{i,j}. \tag{3.22}$$

It is a polynomial representation of degree 1, hence, its rth tensor power $V^{\otimes r}$ is a polynomial representation of degree r, meaning that it is a right $\mathcal{O}_q(n,r)$-comodule.

The category of polynomial representations of q-GL_n is also braided, with braiding c that is uniquely determined by requiring that $c_{V,V} \in \mathrm{End}_{q\text{-}GL_n}(V \otimes V)$ is the $\mathbb{Z}[q,q^{-1}]$-linear map defined by (1.5). We have that $(c_{V,V} + q)(c_{V,V} - q^{-1}) = 0$, hence, $c_{V,V}$ has eigenvalues $-q$ and q^{-1}. After localizing at $[2] = q + q^{-1}$, the tensor square $V \otimes V$ decomposes as the direct sum of the corresponding eigenspaces. The q^{-1}-eigenspace is spanned by

$$\{v_j \otimes v_i + q v_i \otimes v_j \mid 1 \le i < j \le n\} \cup \{v_k \otimes v_k \mid 1 \le k \le n\}. \tag{3.23}$$

The *quantum exterior algebra*

$$\textstyle\bigwedge(V) = \bigoplus_{r \ge 0} \bigwedge^r V \tag{3.24}$$

is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the quadratic tensors from (3.23). This is studied in [PW91] (see also [Bru06, §5]), where it is proved that $\bigwedge^r V$ is free as a $\mathbb{Z}[q,q^{-1}]$-module with basis

$$\big\{v_I := v_{i_1} \wedge \cdots \wedge v_{i_r} \bigm| I = \{i_1 < \cdots < i_r\} \subseteq \{1,\dots,n\}\big\}.$$

The comodule structure map η for $\bigwedge^r V$ satisfies $\eta(v_J) = \sum_I v_I \otimes x_{I,J}$ where

$$x_{I,J} := \sum_{w \in S_r} (-q)^{\ell(w)} x_{i_{w(1)},j_1} \cdots x_{i_{w(r)},j_r} \tag{3.25}$$

for $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$. These so-called *quantum minors* include the quantum determinant (3.21) as a special case.

4. The q-Schur algebra

We continue to work over $\mathbb{Z}[q, q^{-1}]$ like in the previous section. The *q-Schur algebra* is the $\mathbb{Z}[q, q^{-1}]$-linear dual

$$S_q(n, r) := \mathcal{O}_q(n, r)^* = \bigoplus_{\lambda,\mu \in \Lambda(n,r)} \mathcal{O}_q[\lambda, \mu]^*. \tag{4.1}$$

It is an algebra with multiplication $S_q(n, r) \otimes S_q(n, r) \to S_q(n, r)$ defined by the dual map to the restriction $\mathcal{O}_q(n, r) \to \mathcal{O}_q(n, r) \otimes \mathcal{O}_q(n, r)$ of the comultiplication on $\mathcal{O}_q(n)$. For this, we are identifying $f \otimes g \in S_q(n, r) \otimes S_q(n, r)$ with an element of $(\mathcal{O}_q(n, r) \otimes \mathcal{O}_q(n, r))^*$ so that $\langle f \otimes g, x \otimes y\rangle := \langle f, x\rangle\langle g, y\rangle$ for $f, g \in S_q(n, r), x, y \in \mathcal{O}_q(n, r)$.

The unit element $1 \in S_q(n, r)$ is the restriction of the counit ε to $\mathcal{O}_q(n, r)$. For $\lambda \in \Lambda(n, r)$, let 1_λ be the function which is equal to ε on $\mathcal{O}_q[\lambda, \lambda]$ and is zero on all other summands $\mathcal{O}_q[\lambda, \mu]$ in the decomposition (3.8). This defines mutually orthogonal idempotents $\{1_\lambda \mid \lambda \in \Lambda(n, r)\}$ in $S_q(n, r)$ whose sum is the identity. Moreover, $1_\lambda S_q(n, r)1_\mu = \mathcal{O}_q[\lambda, \mu]^*$.

The dual map to the bar involution on $\mathcal{O}_q(n, r)$ defines a bar involution on $S_q(n, r)$ which we denote with the same notation, so $\langle \overline{f}, x\rangle = \overline{\langle f, \overline{x}\rangle}$ for $f \in S_q(n, r), x \in \mathcal{O}_q(n, r)$. Lemma 3.6 implies that $- : S_q(n, r) \to S_q(n, r)$ is an anti-linear algebra automorphism. The dual of the restriction $\mathcal{O}_q(m + n, r) \to \bigoplus_{a+b=r} \mathcal{O}_q(m, a) \otimes \mathcal{O}_q(n, b)$ of the homomorphism $\mathtt{Y}^*$ from Lemma 3.9 defines an injective algebra homomorphism

$$\mathtt{Y}_r : \bigoplus_{a+b=r} S_q(m, a) \otimes S_q(n, b) \hookrightarrow S_q(m + n, r), \qquad \xi_A \otimes \xi_B \mapsto \xi_{\mathrm{diag}(A,B)}. \tag{4.2}$$

This intertwines the bar involutions $- \otimes -$ on each $S_q(m, a) \otimes S_q(n, b)$ with the bar involution on $S_q(m + n, r)$. The dual of (3.19) gives us a transposition involution $\mathtt{T} : S_q(n, r) \to S_q(n, r)$. This is a linear algebra anti-automorphism.

The dual bases to $\{x_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$ and $\{b_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$ give bases for $1_\lambda S_q(n, r)1_\mu$ denoted $\{\xi_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$, the *standard basis*, and $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$, the *canonical basis*. The canonical basis element $\theta_A \in 1_\lambda S_q(n, r)1_\mu$ is the unique bar-invariant element such that $\theta_A \equiv \xi_A \pmod{\sum_{B\in \mathrm{Mat}(\lambda,\mu)} q\mathbb{Z}[q]\xi_B}$. In fact, we have that $\theta_A = \xi_A+$(a $q\mathbb{N}[q]$-linear combination of ξ_B for $B < A$), because by (3.16) we have that

$$\theta_A = \sum_{B\in \mathrm{Mat}(\lambda,\mu)} p_{A,B}(q)\xi_B, \tag{4.3}$$

where $p_{A,B}(q)$ is the Kazhdan-Lusztig polynomial from (3.17). There is also a geometric construction of the canonical basis via intersection cohomology. This is explained in [BLM90, §1.4], where the standard basis element ξ_A is denoted $[A]$ and θ_A is denoted $\{A\}$ (up to some renormalization).

The counit ε is zero on all of the normally-ordered monomials in $\mathcal{O}_q[\lambda, \lambda]$ except for $x_{1,1}^{\lambda_1} \cdots x_{n,n}^{\lambda_n}$, proving the first equality in

$$1_\lambda = \xi_{\mathrm{diag}(\lambda_1,\ldots,\lambda_n)} = \theta_{\mathrm{diag}(\lambda_1,\ldots,\lambda_n)}. \tag{4.4}$$

The second equality follows because $\overline{\xi}_A = \xi_A +$ (a $\mathbb{Z}[q, q^{-1}]$-linear combination of ξ_B's for $B < A$) and $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is minimal in the Bruhat ordering, so $\xi_{\mathrm{diag}(\lambda_1,\ldots,\lambda_n)}$ is bar invariant. More generally, since the homomorphism $\mathtt{Y}_r$ is bar equivariant, we have that

$$\mathtt{Y}_r(\theta_A \otimes \theta_B) = \theta_{\mathrm{diag}(A,B)}. \tag{4.5}$$

Also, by (3.20), we have that

$$\mathtt{T}(\xi_A) = \xi_{A^\intercal}, \qquad\qquad \mathtt{T}(\theta_A) = \theta_{A^\intercal}. \tag{4.6}$$

Example 4.1. For $A \in \mathrm{Mat}(\lambda,\mu)$ with $\lambda,\mu \in \Lambda(2,r)$ we have that

$$\theta_A = \sum_{s=0}^{\min(a_{1,2},a_{2,1})} q^{s(s+\max(a_{1,1},a_{2,2}))} \begin{bmatrix} s+\min(a_{1,1},a_{2,2}) \\ s \end{bmatrix}_q \xi_{A-sB} \tag{4.7}$$

where $B := \left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. This follows by inverting the transition matrix from Example 3.8.

For $n \times n$ matrices A, B, C with non-negative integer entries, define

$$Z(A,B,C) := \langle \xi_A \otimes \xi_B, \Delta(x_C)\rangle \in \mathbb{Z}[q,q^{-1}]. \tag{4.8}$$

These are the *structure constants* for multiplication in the standard basis of the q-Schur algebra: we have that

$$\xi_A \circ \xi_B := \sum_C Z(A,B,C)\xi_C. \tag{4.9}$$

This formula can be viewed as a q-analog of Schur's product rule. For a completely different approach to the definition of these structure constants (counting points over a finite field), see [BLM90, §1.1]. The structure constants have following stabilization property, which will be relevant in the next section.

Lemma 4.2. *Suppose we are given $A' \in \mathrm{Mat}(\lambda',\mu'), B' \in \mathrm{Mat}(\mu',\nu')$ and $C' \in \mathrm{Mat}(\lambda',\nu')$ for $\lambda',\mu',\nu' \in \Lambda(n,r)$ and $1 \leq i,j,k \leq n$ such that $\lambda'_i = \mu'_j = \nu'_k = 0$. Let A,B,C be the matrices obtained by removing the ith row and jth column of A', the jth row and kth column of B', and the ith row and kth column of C', respectively. Then we have that $Z(A,B,C) = Z(A',B',C')$.*

Proof. This follows for the same reason as Lemma 3.7. □

Let H_r be the Hecke algebra of the symmetric group, that is, the $\mathbb{Z}[q,q^{-1}]$-algebra on generators $\tau_1,\dots,\tau_{r-1}$ subject to the relations

$$(\tau_i + q)(\tau_i - q^{-1}) = 0, \qquad \tau_i\tau_j = \tau_j\tau_i \text{ if } |i-j|>1, \qquad \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}. \tag{4.10}$$

For $w \in S_r$, we have the corresponding element $\tau_w \in H_r$ defined from a reduced expression for w, and the elements $\{\tau_w \mid w \in S_r\}$ give a basis for H_w as a free $\mathbb{Z}[q,q^{-1}]$-module. Recall also that the Hecke algebra has its own anti-linear bar involution $- : H_w \to H_w, \tau_w \mapsto \tau_{w^{-1}}^{-1}$.

Lemma 4.3. *Suppose that $r \leq n$ and let $\omega := (1^r\ 0^{n-r}) \in \Lambda(n,r)$. There is an algebra isomorphism $H_r \xrightarrow{\sim} 1_\omega S_q(n,r)1_\omega$ sending τ_w to the standard basis element ξ_A for the matrix $A \in \mathrm{Mat}(\omega,\omega)$ such that $a_{w(i),i} = 1$ for $i = 1,\dots,r$ and all other entries are zero. This map intertwines the bar involutions on H_r and $S_q(n,r)$.*

Proof. Check that the relation

$$\tau_w\tau_i = \begin{cases} \tau_{ws_i} & \text{if } w(i) < w(i+1) \\ \tau_{ws_i} - (q-q^{-1})\tau_w & \text{if } w(i) > w(i+1) \end{cases}$$

holds in $S_q(n,r)$ by explicitly calculating the corresponding structure constants. This is well known so we omit the details. □

Let V be the natural representation of q-GL_n. In addition to our definition of $S_q(n,r)$ by dualizing $\mathcal{O}_q(n,r)$, and the approach in [BLM90] where the q-Schur algebra arises as the endomorphism algebra of a permutation representation of the finite general linear group, the q-Schur algebra can be realized as an endomorphism algebra for an action of the Hecke algebra H_r on the tensor space $V^{\otimes r}$. To explain this, note that $V^{\otimes r}$ has basis $v_{\boldsymbol{i}} := v_{i_1} \otimes \cdots \otimes v_{i_r}$ for

$\boldsymbol{i} \in \mathrm{I}(n,r)$. There is a *right* action of H_r on $V^{\otimes r}$ such that τ_i acts as the braiding $1^{\otimes(i-1)} \otimes c_{V,V} \otimes 1^{r-i-1}$ from (1.5). Since $V^{\otimes r}$ is a polynomial representation of degree r, it is a left $S_q(n,r)$-module. The action of H_r commutes with the action of $S_q(n,r)$. Hence, there is a well-defined algebra homomorphism

$$S_q(n,r) \to \mathrm{End}_{H_r}\left(V^{\otimes r}\right). \tag{4.11}$$

This homomorphism is actually an algebra *isomorphism*. There are several ways to see this, e.g., it can be deduced from [DJ86]. In fact, in [DJ86], the authors work with a different realization of the right H_r-module $V^{\otimes r}$ as a direct sum of permutation modules. In this form, one obtains a basis for the endomorphism algebra on the right hand side of (4.11) quite easily from the Mackey theorem, and then just need to check that this basis is also the image of the standard basis for $S_q(n,r)$ under the homomorphism (4.11). Since this is quite important for us, we go through some details in the next paragraph.

For $\lambda \in \Lambda(n,r)$, let H_λ be the parabolic subalgebra of H_r associated to S_λ. Let X_λ be the the free $\mathbb{Z}[q,q^{-1}]$-module of rank one with basis m_λ viewed as a right H_λ-module so that $m_\lambda \tau_i = q^{-1} m_\lambda$ for each $\tau_i \in H_\lambda$. The (right) *permutation module* is the induced module $M(\lambda) := X_\lambda \otimes_{H_\lambda} H_r$. There is a unique H_r-module homomorphism

$$f_\lambda : M(\lambda) \to 1_\lambda V^{\otimes r}, \qquad m_\lambda \otimes 1 \mapsto v_{\boldsymbol{i}^\lambda}. \tag{4.12}$$

This is actually an *isomorphism* because the vectors $\left\{m_\lambda \otimes \tau_w \,\middle|\, w \in (S_\lambda \backslash S_r)_{\min}\right\}$ give a basis for $M(\lambda)$, and f_λ maps them to the basis $\{v_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathrm{I}_\lambda\}$ for $1_\lambda V^{\otimes r}$. Summing over all $\lambda \in \Lambda(n,r)$, this gives us an H_r-module isomorphism

$$f : \bigoplus_{\lambda \in \Lambda(n,r)} M(\lambda) \xrightarrow{\sim} V^{\otimes r}. \tag{4.13}$$

The following lemma explains how to transport the natural action of $S_q(n,r)$ on $V^{\otimes r}$ through f to obtain an action on this direct sum of permutation modules.

Lemma 4.4. *Suppose that $\lambda, \mu \in \Lambda(n,r)$ and $A \in \mathrm{Mat}(\lambda,\mu)$. The diagram*

$$\begin{array}{ccc} 1_\mu V^{\otimes r} & \xrightarrow{\ \xi_A\ } & 1_\lambda V^{\otimes r} \\ {\scriptstyle f_\mu}\big\uparrow & & \big\uparrow{\scriptstyle f_\lambda} \\ M(\mu) & \longrightarrow & M(\lambda) \end{array}$$

commutes, where the top map is defined by acting on the left with ξ_A, and the bottom map is the H_r-module homomorphism sending

$$m_\mu \otimes 1 \mapsto \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} q^{\ell(w_0)-\ell(w)} m_\lambda \otimes \tau_{d_A} \tau_w, \tag{4.14}$$

where $\mu^+ \vDash r$ is as in Lemma 2.1 and w_0 is the longest element of $(S_{\mu^+} \backslash S_\mu)_{\min}$.

Proof. The comodule structure map η of $V^{\otimes r}$ satisfies $\eta(v_{\boldsymbol{j}}) = \sum_{\boldsymbol{i} \in \mathrm{I}(n,r)} v_{\boldsymbol{i}} \otimes x_{\boldsymbol{i},\boldsymbol{j}}$. Hence, for $\boldsymbol{j} \in \mathrm{I}_\mu$, we have that

$$\xi_A v_{\boldsymbol{j}} = \sum_{\boldsymbol{i} \in \mathrm{I}_\lambda} \langle \xi_A, x_{\boldsymbol{i},\boldsymbol{j}} \rangle v_{\boldsymbol{i}}. \tag{4.15}$$

By the definition (3.14), we have that $x_A = x_{\boldsymbol{i}^\lambda \cdot d_A w_0, \boldsymbol{i}^\mu}$. The S_μ-orbit of $\boldsymbol{i}^\lambda \cdot d_A w_0$ is $\{\boldsymbol{i}^\lambda \cdot d_A w | w \in (S_{\mu^+} \backslash S_\mu)_{\min}\}$. Also for $w \in (S_{\mu^+} \backslash S_\mu)_{\min}$ we have that $x_{\boldsymbol{i}^\lambda \cdot d_A w, \boldsymbol{i}^\mu} = q^{\ell(w_0)-\ell(w)} x_{\boldsymbol{i}^\lambda \cdot d_A w_0, \boldsymbol{i}^\mu}$ as one needs to use the last relation in (3.5) a total of $\ell(w_0) - \ell(w)$ times. Putting this together shows that

$$\xi_A v_{\boldsymbol{i}^\mu} = \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} q^{\ell(w_0)-\ell(w)} v_{\boldsymbol{i}^\lambda \cdot d_A w}.$$

The lemma now follows since f_λ sends $m_\lambda \otimes 1$ to $v_{\boldsymbol{i}^\lambda}$, f_μ sends $m_\mu \otimes 1$ to $v_{\boldsymbol{i}^\mu}$, and $v_{\boldsymbol{i}^\lambda \cdot d_A w} = v_{\boldsymbol{i}^\lambda} \tau_{d_A} \tau_w$ as $i_1^\lambda \leq \cdots \leq i_r^\lambda$. □

Let m be another natural number. For $\lambda \in \Lambda(m,r)$, let $Y(\lambda)$ be the free $\mathbb{Z}[q,q^{-1}]$-module of rank one with basis n_λ viewed as a left H_λ-module so that $\tau_i n_\lambda = -q n_\lambda$ for each $\tau_i \in H_\lambda$. The (left) signed permutation module is the induced module $N(\lambda) := H_r \otimes_{H_\lambda} Y(\lambda)$.

Lemma 4.5. *There is an algebra isomorphism* $S_q(m,r) \xrightarrow{\sim} \operatorname{End}_{H_r}\left(\bigoplus_{\lambda \in \Lambda(m,r)} N(\lambda)\right)$ *sending* $\xi_A \in 1_\lambda S_q(m,r) 1_\mu$ *to the unique* H_r*-module homomorphism such that*

$$1 \otimes n_\mu \mapsto \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w^{-1} \tau_{d_A}^{-1} \otimes n_\lambda$$

where μ^+ *is as in Lemma 2.1 and* w_0 *is the longest element of* $(S_{\mu^+} \backslash S_\mu)_{\min}$*, and* $1 \otimes n_\nu \mapsto 0$ *for* $\nu \neq \mu$.

Proof. We start from the algebra isomorphism (4.11). Using (4.13) and Lemma 4.4, and replacing n by m, this gives us an algebra isomorphism $S_q(m,r) \xrightarrow{\sim} \operatorname{End}_{H_r}\left(\bigoplus_{\lambda \in \Lambda(m,r)} M(\lambda)\right)$ such that $\xi_A \in 1_\lambda S_q(m,r) 1_\mu$ acts on $m_\mu \otimes 1 \in M(\mu)$ according to (4.14), and it acts as zero on all other summands. Then we use the algebra anti-automorphism $H_r \to H_r, \tau_x \mapsto (-1)^{\ell(x)} \tau_x^{-1}$. The pull-back of the right H_r-module $M(\lambda)$ along this map is isomorphic the left H_r-module $N(\lambda)$, there being a unique isomorphism such that $m_\lambda \otimes \tau_x \mapsto (-1)^{\ell(x)} \tau_x^{-1} \otimes n_\lambda$ for all $x \in S_r$. We deduce that $\operatorname{End}_{H_r}\left(\bigoplus_{\lambda \in \Lambda(m,r)} M(\lambda)\right) \cong \operatorname{End}_{H_r}\left(\bigoplus_{\lambda \in \Lambda(m,r)} N(\lambda)\right)$. It just remains to note that the action of $\xi_A \in 1_\lambda S(m,r) 1_\mu$ on $\bigoplus_{\lambda \in \Lambda(m,r)} M(\lambda)$ translates into the action on $\bigoplus_{\lambda \in \Lambda(m,r)} N(\lambda)$ described explicitly in the statement of the lemma. □

The goal now is to replace H_r and the signed permutation modules $N(\lambda)$ in Lemma 4.5 with the quantum general linear group q-GL_n and its polynomial representations

$$\textstyle\bigwedge^\lambda V := \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_{\ell(\lambda)}} V. \tag{4.16}$$

Lemma 4.6. *Take* $\lambda, \mu \in \Lambda(m,r)$ *and* $A \in \operatorname{Mat}(\lambda,\mu)$. *There is a unique* q-GL_n*-module homomorphism* $\phi_A : \bigwedge^\mu V \to \bigwedge^\lambda V$ *such that the diagram*

$$\begin{array}{ccc} V^{\otimes r} & \longrightarrow & V^{\otimes r} \\ \big\downarrow & & \big\downarrow \\ \bigwedge^\mu V & \xrightarrow{\phi_A} & \bigwedge^\lambda V \end{array}$$

commutes, where the vertical maps are the quotient maps and the top map is right multiplication by $\sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w^{-1} \tau_{d_A}^{-1}$ *where* μ^+ *is defined as in Lemma 2.1 and* w_0 *is the longest element of* $(S_{\mu^+} \backslash S_\mu)_{\min}$.

Proof. By the definition of quantum exterior powers, the kernel of the projection $V^{\otimes r} \twoheadrightarrow \bigwedge^\mu V$ is generated by the kernels of the endomorphisms $\tau_j - q^{-1} = \tau_j^{-1} - q$ for all j with $s_j \in S_\mu$. So we need to show for such a j and $v \in V^{\otimes r}$ with $v\tau_j^{-1} = qv$ that the vector

$$v' := \sum_{w \in (S_{\mu^+} \backslash S_\mu)_{\min}} (-1)^{\ell(w)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} v \tau_w^{-1} \tau_{d_A}^{-1}$$

is in the sum of the kernels of the maps $\tau_i^{-1} - q$ for all i with $s_i \in S_\lambda$. We have that

$$(S_{\mu^+} \backslash S_\mu)_{\min} = X \sqcup Xs_j \sqcup Y$$

such that $\ell(xs_j) = \ell(x) + 1$ for all $x \in X$, and $ys_jy^{-1} \in S_{\mu^+}$ for all $y \in Y$. This follows from [DJ89, Lemma 1.1]. For $x \in X$, we have that

$$(-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(x)} v\tau_x^{-1}\tau_{d_A}^{-1} + (-1)^{\ell(xs_j)+\ell(d_A)} q^{\ell(w_0)-\ell(xs_j)} v\tau_j^{-1}\tau_x^{-1}\tau_{d_A}^{-1} = 0$$

as $v\tau_j^{-1} = qv$. This implies that

$$v' = \sum_{y \in Y} (-1)^{\ell(y)+\ell(d_A)} q^{\ell(w_0)-\ell(y)} v\tau_y^{-1}\tau_{d_A}^{-1}.$$

It remains to show for $y \in Y$ that $v\tau_y^{-1}\tau_{d_A}^{-1}$ is in the kernel of $\tau_i^{-1} - q$ for some i with $s_i \in S_\lambda$. We have that $d_Ays_j = td_Ay$ for $t := d_A(ys_jy^{-1})d_A^{-1}$. Since $ys_jy^{-1} \in S_{\mu^+}$, we deduce using Lemma 2.1 that $t \in S_\lambda$ (in fact, it is in $S_{\lambda^-} \leq S_\lambda$ in the notation from the lemma), and that $\ell(td_Ay) = \ell(t)+\ell(d_A)+\ell(y)$. Since $\ell(d_Ays_j) \leq \ell(d_A)+\ell(y)+1$, we deduce that $\ell(t) = 1$. Hence, $t = s_i$ for some i such that $s_i \in S_\lambda$. Moreover $v\tau_y^{-1}\tau_{d_A}^{-1}\tau_i^{-1} = v\tau_j^{-1}\tau_y^{-1}\tau_{d_A}^{-1} = qv\tau_y^{-1}\tau_{d_A}^{-1}$. □

The following theorem is the quantum analog of [Don93, Proposition 3.11]. See also [Don98, 4.2(19)] for a closely related result already in the quantum setting.

Theorem 4.7. *Fix $m, r \in \mathbb{N}$. For any $n \geq 0$, there is a surjective algebra homomorphism*

$$g_n : S_q(m,r) \twoheadrightarrow \operatorname{End}_{q\text{-}GL_n}\left(\bigoplus_{\lambda \in \Lambda(m,r)} \bigwedge^\lambda V\right) \tag{4.17}$$

sending $\xi_A \in 1_\lambda S_q(m,r)1_\mu$ to the endomorphism that is equal to the homomorphism ϕ_A from Lemma 4.6 on the summand $\bigwedge^\mu V$, and is zero on all other summands. Moreover, g_n is an isomorphism if $n \geq r$.

Proof. Using the base change functor $\Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} -$, it suffices to prove the analogous statement when $\mathbb{Z}[q,q^{-1}]$ is replaced by a field $\Bbbk$ and q is any non-zero element. In the remainder of the proof, we assume we are working over a field in this way, writing $q\text{-}GL_n(\Bbbk)$ for the quantum general linear group over $\Bbbk$, whose coordinate algebra is $\Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{O}_q(n)$. The category of polynomial representations of $q\text{-}GL_n(\Bbbk)$ is a highest weight category satisfying standard homological properties. This is justified e.g. in [PW91] or [Don98][2]. In the next paragraph, we treat the case that $n \geq r$. Then the existence and surjectivity of g_n for $n < r$ follows from the existence and surjectivity of g_N for $N \geq r$ by an argument involving truncation to the subgroup $q\text{-}GL_n < q\text{-}GL_N$ using [Don98, 4.2(11)] (this requires the standard homological properties).

So now assume that $n \geq r$ and that we are working over a field. We must show that g_n is a well-defined algebra isomorphism. To see this, we use the *Schur functor*, that is, the idempotent truncation functor $\pi : S_q(n,r)\text{-mod} \to H_r\text{-mod}$ defined by the idempotent 1_ω, notation as in Lemma 4.3. This sends an $S_q(n,r)$-module to its ω-weight space viewed as an H_r-module via the isomorphism from that lemma. We have that $\pi\left(\bigwedge^\lambda V\right) \cong N(\lambda)$, there being a unique such isomorphism sending the canonical image of $v_1 \otimes \cdots \otimes v_r$ in $\bigwedge^\lambda V$ to $1 \otimes n_\lambda$. Moreover, the Schur functor induces an isomorphism

$$\operatorname{Hom}_{S_q(n,r)}(\bigwedge^\mu V, \bigwedge^\lambda V) \xrightarrow{\sim} \operatorname{Hom}_{H_r}(N(\mu), N(\lambda)).$$

[2]It can also be deduced by using the results of §7 to show that $S_q(n,r)$ is a split quasi-hereditary algebra.

This follows by general principles (e.g., see [JS92, Th. 2.12]) because the head of $\bigwedge^\mu V$ and the socle of $\bigwedge^\lambda V$ are p-restricted, i.e., they only involve irreducible modules L which are not annihilated by π. Indeed, these modules are both submodules and quotient modules of the tensor space $V^{\otimes r}$, which has p-restricted head and socle because $V^{\otimes r} \cong S_q(n,r)1_\omega$ by the isomorphisms (4.11) and (4.12), hence,

$$\mathrm{Hom}_{S_q(n,r)}(L, V^{\otimes r}) \cong \mathrm{Hom}_{S_q(n,r)}(V^{\otimes r}, L) \cong \mathrm{Hom}_{S_q(n,r)}(S_q(n,r)1_\omega, L) \cong 1_\omega L$$

for any self-dual module L. Consequently, π induces an algebra isomorphism

$$\mathrm{End}_{q\text{-}GL_n}\left(\bigoplus_{\lambda\in\Lambda(m,r)} \bigwedge\nolimits^\lambda V\right) \cong \mathrm{End}_{H_r}\left(\bigoplus_{\lambda\in\Lambda(m,r)} N(\lambda)\right).$$

Composing this with the isomorphism from Lemma 4.5 gives the desired isomorphism g_n.

It just remains to identify the endomorphism $g_n(\xi_A)$ with ϕ_A. For this, it suffices to check for $\xi_A \in 1_\lambda S_q(m,r)1_\mu$ that the maps $g_n(\xi_A)$ and ϕ_A are equal on the canonical image of $v_1\otimes\cdots\otimes v_r$ in $\bigwedge^\mu V$. By the definition from Lemma 4.6, ϕ_A sends this vector to the canonical image of

$$\sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} (v_1\otimes\cdots\otimes v_r)\tau_w^{-1}\tau_{d_A}^{-1}$$

in $\bigwedge^\lambda V$. On the other hand, $g_n(\xi_A)$ takes this vector to the image of

$$\sum_{w\in(S_{\mu^+}\backslash S_\mu)_{\min}} (-1)^{\ell(x)+\ell(d_A)} q^{\ell(w_0)-\ell(w)} \tau_w\tau_{d_A}^{-1}(v_1\otimes\cdots\otimes v_r)$$

where the left action of H_r on $1_\omega V^{\otimes r}$ comes from the left action of $S_q(n,r)$ via the isomorphism of Lemma 4.3. Now observe for any $x \in S_r$ that $\tau_x(v_1\otimes\cdots\otimes v_r) = (v_1\otimes\cdots\otimes v_r)\tau_x$ as, by the definitions, both vectors are equal to $v_{x(1)}\otimes\cdots\otimes v_{x(r)}$. □

5. The q-Schur category

It is easy to adapt (4.8) to define $Z(A,B,C) \in \mathbb{Z}[q,q^{-1}]$ for $A \in \mathrm{Mat}(\lambda,\mu)$, $B \in \mathrm{Mat}(\mu,\nu)$, $C \in \mathrm{Mat}(\lambda,\nu)$ and compositions $\lambda,\mu,\nu \vDash r$ that are not necessarily of the same length. To do so, we pick any $n \geq \ell(\lambda),\ell(\mu),\ell(\nu)$ and let λ',μ' and ν' be compositions of length n obtained from λ,μ and ν by adding some extra entries equal to zero. Let $A' \in \mathrm{Mat}(\lambda',\mu')$, $B' \in \mathrm{Mat}(\mu',\nu')$ and $C' \in \mathrm{Mat}(\lambda',\nu')$ be the matrices obtained by inserting corresponding rows and columns of zeros into A, B and C; see (2.4) and (2.5) for an example. Then we define $Z(A,B,C)$ to be the structure constant $Z(A',B',C')$ for the q-Schur algebra $S_q(n,r)$ exactly as defined earlier. The stability from Lemma 4.2 implies that this is well-defined independent of all choices.

The following theorem defines the *q-Schur category with 0-strings*. The version without 0-strings discussed in the introduction is the full subcategory with object set $\Lambda_s \subset \Lambda$.

Theorem 5.1. *There is a $\mathbb{Z}[q,q^{-1}]$-linear strict monoidal category q-**Schur** with*

- *objects that are all compositions $\lambda \in \Lambda$;*
- *for $\lambda \vDash r$ and $\mu \vDash s$, the morphism space $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda)$ is $\{0\}$ unless $r = s$, and it is the free $\mathbb{Z}[q,q^{-1}]$-module with basis $\{\xi_A \mid A \in \mathrm{Mat}(\lambda,\mu)\}$ if $r = s$;*
- *tensor product of objects is defined by concatenation of compositions;*
- *tensor product of morphisms (horizontal composition) is defined by $\xi_A \star \xi_B := \xi_{\mathrm{diag}(A,B)}$;*
- *vertical composition of morphisms is defined as in (4.9).*

The strict identity object $\mathbb{1}$ is the composition of length zero, and the identity endomorphism 1_λ of an object $\lambda \in \Lambda$ is $\xi_{\mathrm{diag}(\lambda_1,\ldots,\lambda_{\ell(\lambda)})}$.

Proof. Most of the axioms of strict monoidal category are straightforward to check. The fact that vertical composition is associative is a consequence of associativity of multiplication in the q-Schur algebra. To check the interchange law, we must show that

$$(\xi_A \star 1_\sigma) \circ (1_\mu \star \xi_B) = (1_\lambda \star \xi_B) \circ (\xi_A \star 1_\rho)$$

for $\lambda, \mu \vDash a, \sigma, \rho \vDash b$ and $A \in \mathrm{Mat}(\lambda, \mu), B \in \mathrm{Mat}(\sigma, \rho)$, that is,

$$\xi_{\mathrm{diag}(A,\sigma_1,\dots,\sigma_{\ell(\sigma)})} \circ \xi_{\mathrm{diag}(\mu_1,\dots,\mu_{\ell(\mu)},B)} = \xi_{\mathrm{diag}(\lambda_1,\dots,\lambda_{\ell(\lambda)},B)} \circ \xi_{\mathrm{diag}(A,\rho_1,\dots,\rho_{\ell(\rho)})}.$$

Using the stability from Lemma 4.2, we may assume that $\ell(\lambda) = \ell(\mu) = m$ and $\ell(\sigma) = \ell(\rho) = n$. We have that $(\xi_A \otimes 1_\sigma)(1_\mu \otimes \xi_B) = (1_\lambda \otimes \xi_B)(A \otimes 1_\rho)$ in the algebra $S_q(m, a) \otimes S_q(n, b)$. Now apply the algebra homomorphism Υ_{a+b} from (4.2). □

Remark 5.2. It is clear from the definition that the path algebra of the full subcategory of q-**Schur** generated by objects in $\Lambda(n, r)$ may be identified with the q-Schur algebra, that is,

$$S_q(n, r) = \bigoplus_{\lambda,\mu \in \Lambda(n,r)} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda). \tag{5.1}$$

By (4.11) and Lemma 4.4, we have that $1_\lambda S_q(n, r) 1_\mu \cong \mathrm{Hom}_{H_r}(M(\mu), M(\lambda))$ for $\lambda, \mu \in \Lambda(n, r)$. It follows that the full subcategory of q-**Schur** generated by objects in $\Lambda(r) := \bigcup_{n \geq 0} \Lambda(n, r)$ is isomorphic to the category q-**Schur**(r) with object set $\Lambda(r)$ and morphism spaces

$$\mathrm{Hom}_{q\text{-}\mathbf{Schur}(r)}(\mu, \lambda) := \mathrm{Hom}_{H_r}(M(\mu), M(\lambda)), \tag{5.2}$$

with the natural composition law. The categories q-**Schur**(r) for all r can then be assembled to obtain an alternative approach to the definition of q-**Schur**, with tensor product arising from the bifunctors q-**Schur**$(r) \times q$-**Schur**$(s) \to q$-**Schur**$(r + s)$ induced by the natural embeddings $H_r \otimes H_s \hookrightarrow H_{r+s}$. We have emphasized the based approach in Theorem 5.1 since it allows composition of standard basis elements to be computed effectively using the coalgebra structure on $\mathcal{O}_q(n)$. This will be used several times later on.

We have defined the morphism space $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ so that it comes equipped with the standard basis $\{\xi_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$. We can also transfer the canonical basis from the q-Schur algebra to q-**Schur**, as follows. Take any $\lambda, \mu \vDash r$ and $A, B \in \mathrm{Mat}(\lambda, \mu)$. There is a corresponding Kazhdan-Lusztig polynomial $p_{A,B}(q) \in \mathbb{Z}[q]$. To define this, we again pick any $n \geq \ell(\lambda), \ell(\mu)$, add extra zeros to λ and μ to make them into compositions of the same length n, and add corresponding rows and columns of zeros to A and B to obtain $A', B' \in \mathrm{Mat}(\lambda', \mu')$. Then we let $p_{A,B}(q) := p_{A',B'}(q)$, where the latter polynomial comes from (4.3). This is well defined independent of the choices thanks to Lemma 3.7. It is also clear from the proof of that lemma that the slightly more general polynomials $p_{A,B}(q)$ still satisfy (3.17). Let

$$\theta_A := \sum_{B \in \mathrm{Mat}(\lambda,\mu)} p_{A,B}(q) \xi_B, \tag{5.3}$$

thereby defining the canonical basis $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$ for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$.

Lemma 5.3. *There is an anti-linear strict monoidal functor* $-: q$-**Schur** $\to q$-**Schur** *which is the identity on objects and, on the morphism space* $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$*, is the unique anti-linear map which fixes the canonical basis* $\{\theta_A \mid A \in \mathrm{Mat}(\lambda, \mu)\}$*.*

Proof. Since the bar involution for q-Schur algebras is an anti-linear algebra automorphism, this prescription gives a well-defined anti-linear functor. To see that it is strict monoidal, it suffices to observe that $\theta_A \star \theta_B = \theta_{\mathrm{diag}(A,B)}$. This follows from (4.5). □

Similarly, we upgrade the involution $\mathtt{T}$ on $S_q(n,r)$ to a strict linear monoidal functor

$$\mathtt{T} : q\text{-}\mathbf{Schur} \to (q\text{-}\mathbf{Schur})^{\mathrm{op}} \tag{5.4}$$

which is the identity on objects, commutes with the bar involution, and sends $\xi_A \mapsto \xi_{A^{\mathtt{T}}}$, $\theta_A \mapsto \theta_{A^{\mathtt{T}}}$. This follows by (4.6).

Theorem 5.4. *There is a full $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor Σ_n from q-**Schur** to the category of polynomial representations of q-GL_n sending the object $\lambda \vDash d$ to the polynomial representation $\bigwedge^\lambda V$ of degree r from* (4.16)*, and the morphism ξ_A for $\lambda, \mu \vDash r$ and $A \in \mathrm{Mat}(\lambda,\mu)$ to the homomorphism $\phi_A : \bigwedge^\mu V \to \bigwedge^\lambda V$ from Lemma 4.6.*

Proof. To see that Σ_n is a functor, we must show that $\Sigma_n(\xi_A \circ \xi_B) = \Sigma_n(\xi_A) \circ \Sigma_n(\xi_B)$ for $A \in \mathrm{Mat}(\lambda,\mu)$, $B \in \mathrm{Mat}(\mu,\nu)$, $\lambda,\mu,\nu \vDash r$ and $r \geq 0$. In view of the definition of vertical composition in q-**Schur**, this follows because we have that $\phi_A \circ \phi_B = \sum_{C \in \mathrm{Mat}(\lambda,\nu)} Z(A,B,C)\phi_C$ by Theorem 4.7, taking $m \geq \ell(\lambda), \ell(\mu), \ell(\nu)$. The same theorem also shows that Σ_n is full. Finally, to see that Σ_n is a monoidal functor, we need to check that $\phi_A \otimes \phi_B = \phi_{\mathrm{diag}(A,B)}$. This is clear from the explicit description of these maps given by Lemma 4.6. □

Remark 5.5. (1) Using the final statement from Theorem 4.7, the proof of Theorem 5.4 also shows that the functor Σ_n defines an *isomorphism* $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda) \xrightarrow{\sim} \mathrm{Hom}_{q\text{-}GL_n}(\bigwedge^\mu V, \bigwedge^\lambda V)$ providing $n \geq |\lambda|, |\mu|$. So one could say that Σ_n is *asymptotically faithful* as $n \to \infty$. In Corollary 8.4 below, we will give an explicit description of the kernel of Σ_n, that is, the tensor ideal of q-**Schur** consisting of the morphisms that it annihilates.
(2) Let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q,q^{-1}]$-algebra in some way, and consider the specialization $q\text{-}\mathbf{Schur}(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}$. The functor Σ_n in Theorem 5.4 induces a $\Bbbk$-linear monoidal functor from q-**Schur**$(\Bbbk)$ to the category of polynomial representations of q-$GL_n(\Bbbk)$. By the proofs of Theorems 4.7 and 5.4, this induced functor is also full.

By *merges*, *splits*, and *positive crossings*, we mean the morphisms $\xi_{[a\ b]}$, $\xi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$, and $\xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ for $a, b \geq 0$. The images $\phi_{[a\ b]}$, $\phi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$, and $\phi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ of these special morphisms under the functor Σ_n from Theorem 5.4 are the natural projection

$$\textstyle\bigwedge^a V \otimes \bigwedge^b V \twoheadrightarrow \bigwedge^{a+b} V, \qquad v \otimes w \mapsto v \wedge w, \tag{5.5}$$

the inclusion

$$\begin{gathered}\textstyle\bigwedge^{a+b} V \hookrightarrow \bigwedge^a V \otimes \bigwedge^b V, \\ v_{i_1} \wedge \cdots \wedge v_{i_{a+b}} \mapsto q^{-ab} \sum_{w \in (S_{a+b}/S_a \times S_b)_{\min}} (-q)^{\ell(w)} v_{i_{w(1)}} \wedge \cdots \wedge v_{i_{w(a)}} \otimes v_{i_{w(a+1)}} \wedge \cdots \wedge v_{i_{w(a+b)}},\end{gathered} \tag{5.6}$$

and the isomorphism

$$(-1)^{ab} c^{-1}_{\bigwedge^b V, \bigwedge^a V} : \textstyle\bigwedge^a V \otimes \bigwedge^b V \xrightarrow{\sim} \bigwedge^b V \otimes \bigwedge^a V \tag{5.7}$$

where $c_{\bigwedge^b V, \bigwedge^a V} : \bigwedge^b V \otimes \bigwedge^a V \to \bigwedge^a V \otimes \bigwedge^b V$ is the braiding on the monoidal category of polynomial representations of q-GL_n. This follows from the explicit description of ϕ_A in Lemma 4.6. We refer to the morphisms $\overline{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ for $a, b \geq 0$ as *negative crossings*. The following lemma implies that the image of $\overline{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ under the functor Σ_n is the isomorphism

$$(-1)^{ab} c_{\bigwedge^a V, \bigwedge^b V} : \textstyle\bigwedge^a V \otimes \bigwedge^b V \xrightarrow{\sim} \bigwedge^b V \otimes \bigwedge^a V. \tag{5.8}$$

Lemma 5.6. *For $a, b \geq 0$, we have that $\overline{\xi}_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = \xi^{-1}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]}$. Also the merge and split morphisms $\xi_{[a\ b]}$ and $\xi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$ are invariant under the bar involution, hence, they coincide with the canonical basis elements $\theta_{[a\ b]}$ and $\theta_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}$.*

Proof. The part about merge and split morphisms is trivial as these matrices are not comparable to any other in the Bruhat ordering. For first part, we show that $\overline{\xi}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = 1_{(a,b)}$. This identity (with a and b switched) together with the image of this identity under the bar involution implies the result. Take any $A \in \mathrm{Mat}((a,b),(a,b))$ and consider the coefficient of ξ_A when the product $\overline{\xi}_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} \in S_q(2, a+b)$ is expanded in terms of the standard basis. Since multiplication in $S_q(2, a+b)$ is dual to comultiplication in $\mathcal{O}_q(2, a+b)$, this coefficient is equal to the $x_{2,1}^b x_{1,2}^a \otimes x_{2,1}^a x_{1,2}^b$-coefficient of $\Delta\left(x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}} x_{2,2}^{a_{2,2}} x_{1,2}^{a_{1,2}}\right)$ when expanded in terms of the basis $\{\overline{x}_B \otimes x_C \mid B, C \in \mathrm{Mat}((a,b),(a,b),\}$. By Lemma 3.5,

$$\begin{aligned}\Delta\left(x_{2,1}^{a_{2,1}} x_{1,1}^{a_{1,1}} x_{2,2}^{a_{2,2}} x_{1,2}^{a_{1,2}}\right) = \sum_{a'_{2,1}=0}^{a_{2,1}} \sum_{a'_{1,1}=0}^{a_{1,1}} \sum_{a'_{2,2}=0}^{a_{2,2}} \sum_{a'_{1,2}=0}^{a_{1,2}} \begin{bmatrix} a_{2,1} \\ a'_{2,1} \end{bmatrix}_q \begin{bmatrix} a_{1,1} \\ a'_{1,1} \end{bmatrix}_q \begin{bmatrix} a_{2,2} \\ a'_{2,2} \end{bmatrix}_q \begin{bmatrix} a_{1,2} \\ a'_{1,2} \end{bmatrix}_q \times \\ x_{2,1}^{a'_{2,1}} x_{2,2}^{a_{2,1}-a'_{2,1}} x_{1,1}^{a'_{1,1}} x_{1,2}^{a_{1,1}-a'_{1,1}} x_{2,1}^{a'_{2,2}} x_{2,2}^{a_{2,2}-a'_{2,2}} x_{1,1}^{a'_{1,2}} x_{1,2}^{a_{1,2}-a'_{1,2}} \otimes \\ x_{2,1}^{a_{2,1}-a'_{2,1}} x_{1,1}^{a'_{2,1}} x_{2,1}^{a_{1,1}-a'_{1,1}} x_{1,1}^{a'_{1,1}} x_{2,2}^{a_{2,2}-a'_{2,2}} x_{1,2}^{a'_{2,2}} x_{2,2}^{a_{1,2}-a'_{1,2}} x_{1,2}^{a'_{1,2}}.\end{aligned}$$

To get $x_{2,1}^a x_{1,2}^b$ on straightening using (3.5) into the normal order in the second tensor position, we must have that $a'_{2,1} = a'_{1,1} = 0$, $a'_{1,2} = a_{1,2}$ and $a'_{2,2} = a_{2,2}$. This term is

$$x_{2,2}^{a_{2,1}} x_{1,2}^{a_{1,1}} x_{2,1}^{a_{2,2}} x_{1,1}^{a_{1,2}} \otimes x_{2,1}^b x_{1,2}^a = q^{-a_{2,1}a_{2,2}-a_{1,1}a_{1,2}} x_{2,1}^{a_{2,2}} x_{2,2}^{a_{2,1}} x_{1,1}^{a_{1,2}} x_{1,2}^{a_{1,1}} \otimes x_{2,1}^b x_{1,2}^a. \tag{5.9}$$

Because we are using the ordering from (3.11) for monomials in the first tensor (rather than the normal ordering), we only get a non-zero coefficient when $A = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, when the coefficient is 1. This shows the product is $1_{(a,b)}$. □

Remark 5.7. By a similar argument to the proof of Lemma 5.6, one can also prove the following "quadratic relation":

$$\xi_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]} = \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2-s(a+b-2s)} (q^{-1}-q)^s [s]_q^! \, \xi_{\left[\begin{smallmatrix} a-s & s \\ s & b-s \end{smallmatrix}\right]}.$$

Indeed, from (5.9), the coefficient of $\xi_{\left[\begin{smallmatrix} a-s & s \\ s & b-s \end{smallmatrix}\right]}$ in $\xi_{\left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ is $q^{-s(a-s)-s(b-s)}$ times the coefficient of $x_{2,1}^b x_{1,2}^a$ when $x_{2,1}^{b-s} x_{2,2}^s x_{1,1}^s x_{1,2}^{a-s}$ is expanded in terms of the normally-ordered monomial basis. The latter coefficient is $q^{-s(s-1)/2}(q^{-1}-q)^s [s]_q^!$ by Lemma 3.4.

More generally, a *merge of n strings* is a morphism of the form ξ_A for a $1 \times n$ matrix A, an *split of n strings* is a morphism of the form ξ_A for an $n \times 1$ matrix A, and a *positive permutation of n strings* is a morphism of the form ξ_A for an $n \times n$ matrix A such that in each row and column there is at most one non-zero entry. Positive permutations of n strings can be parametrized instead by $w \in S_n$ and a composition μ of length n, setting

$$\tau_{w;\mu} := \xi_A \quad \text{where } A \in \mathrm{Mat}(\mu \cdot w^{-1}, \mu) \text{ has } a_{w(1),1} = \mu_1, \ldots, a_{w(n),n} = \mu_n. \tag{5.10}$$

If $\mu \in \Lambda(n,r)$ then the same formula defines an element of $1_{\mu \cdot w^{-1}} S_q(n,r) 1_\mu$; for example, for $w \in S_r \leq S_n$, the image of $\tau_w \in H_r$ under the isomorphism of Lemma 4.3 is $\tau_{w;\omega}$. For

$1 \leq i < n$, we have that

$$\tau_{s_i;\mu} = 1_{(\mu_1,\dots,\mu_{i-1})} \star \xi_{\left[\begin{smallmatrix} 0 & \mu_{i+1} \\ \mu_i & 0 \end{smallmatrix}\right]} \star 1_{(\mu_{i+2},\dots,\mu_n)}. \tag{5.11}$$

So $\tau_{s_i;\mu}$, which we call a *simple permutation of n strings*, is a positive crossing tensored on the left and right with the appropriate identity morphisms. The following lemma implies that any positive permutation of n strings can be obtained by composing simple permutations.

Lemma 5.8. *Suppose that $\mu \in \Lambda(n,r)$ and $w \in S_n$ is a permutation such that $w(i) < w(i+1)$ for some $1 \leq i < n$. Then $\tau_{ws_i;\mu} = \tau_{w;\mu\cdot s_i} \circ \tau_{s_i;\mu}$.*

Proof. It suffices to prove the analogous statement in the q-Schur algebra $S_q(n,r)$. There is a left action of S_n on $\mathrm{I}(n,r)$ by its action on entries. This commutes with the right action of S_r. We claim that the left action of $S_q(n,r)$ on $V^{\otimes r}$ satisfies $\tau_{w;\mu} v_{\boldsymbol{i}^\mu} = v_{w\cdot\boldsymbol{i}^\mu}$. To see this, the normally-ordered monomial in $\mathcal{O}_q(n,r)$ that is dual to the standard basis vector $\tau_{w;\mu}$ is $x_{w\cdot\boldsymbol{i}^\mu,\boldsymbol{i}^\mu}$. Moreover, $w\cdot\boldsymbol{i}^\mu$ is the only $\boldsymbol{i} \in \mathrm{I}(n,r)$ such that $x_{w\cdot\boldsymbol{i}^\mu,\boldsymbol{i}^\mu}$ appears in the normally-ordered monomial basis expansion of $x_{\boldsymbol{i},\boldsymbol{i}^\mu}$. So the claim follows from (4.15).

To prove the lemma, it suffices to show that $\tau_{w;\mu\cdot s_i}\tau_{s_i;\mu}$ and $\tau_{ws_i;\mu}$ act in the same way on $v_{\boldsymbol{i}^\mu}$. The latter gives $v_{ws_i\cdot\boldsymbol{i}^\mu}$ by the claim.Also $\tau_{s_i;\mu} v_{\boldsymbol{i}^\mu} = v_{s_i\cdot\boldsymbol{i}^\mu}$. So we are reduced to checking that $\tau_{w;\mu\cdot s_i} v_{s_i\cdot\boldsymbol{i}^\mu} = v_{ws_i\cdot\boldsymbol{i}^\mu}$. Let $d \in (S_{\mu\cdot s_i}\backslash S_r)_{\min}$ be the unique Grassmann permutation such that $\boldsymbol{i}^{\mu\cdot s_i}\cdot d = s_i\cdot\boldsymbol{i}^\mu$. The action of H_r on $V^{\otimes r}$ was defined using (1.5), from which we see that $v_{\boldsymbol{i}^{\mu\cdot s_i}}\tau_d = v_{\boldsymbol{i}^{\mu\cdot s_i}\cdot d}$. Similarly, because $w(i) < w(i+1)$, we get that $v_{w\cdot\boldsymbol{i}^{\mu\cdot s_i})}\tau_d = v_{(w\cdot\boldsymbol{i}^{\mu\cdot s_i})\cdot d}$. So

$$\begin{aligned}\tau_{w;\mu\cdot s_i} v_{s_i\cdot\boldsymbol{i}^\mu} &= \tau_{w;\mu\cdot s_i} v_{\boldsymbol{i}^{\mu\cdot s_i}\cdot d} = \tau_{w;\mu\cdot s_i} v_{\boldsymbol{i}^{\mu\cdot s_i}}\tau_d = v_{w\cdot\boldsymbol{i}^{\mu\cdot s_i}}\tau_d \\ &= v_{(w\cdot\boldsymbol{i}^{\mu\cdot s_i})\cdot d} = v_{w\cdot(\boldsymbol{i}^{\mu\cdot s_i}\cdot d)} = v_{w\cdot(s_i\cdot\boldsymbol{i}^\mu)} = v_{ws_i\cdot\boldsymbol{i}^\mu}.\end{aligned}$$

□

A special case of the next lemma implies that

$$\xi_{[\,a_1+\cdots+a_s\ b_1+\cdots+b_t\,]} \circ \left(\xi_{[\,a_1\ \cdots\ a_s\,]} \star \xi_{[\,b_1\ \cdots\ b_t\,]}\right) = \xi_{[\,a_1\ \cdots\ a_s\ b_1\ \cdots\ b_t\,]}, \tag{5.12}$$

$$\left(\xi_{[\,a_1\ \cdots\ a_s\,]^{\mathtt{T}}} \star \xi_{[\,b_1\ \cdots\ b_t\,]^{\mathtt{T}}}\right) \circ \xi_{[\,a_1+\cdots+a_s\ b_1+\cdots+b_t\,]^{\mathtt{T}}} = \xi_{[\,a_1\ \cdots\ a_s\ b_1\ \cdots\ b_t\,]^{\mathtt{T}}}, \tag{5.13}$$

for $a_1,\dots,a_s,b_1,\dots,b_t \geq 0$. Hence, all merges/splits of n strings can be expressed as compositions of tensor products of merges/splits of 2 strings and appropriate identity morphisms.

Lemma 5.9. *Suppose that $\lambda,\mu \vDash r$, $A \in \mathrm{Mat}(\lambda,\mu)$ and $1 \leq i \leq \ell(\lambda), 1 \leq j \leq \ell(\mu)$.*

(a) *Let B be obtained from A by replacing its ith row by two rows of length $\ell(\mu)$, the first of which has entries $a_{i,1},\dots,a_{i,j},0,\dots,0$ with sum λ_i', and the second has entries $0,\dots,0,a_{i,j+1},\dots,a_{i,\ell(\mu)}$ with sum λ_i'' (so $\lambda_i'+\lambda_i''=\lambda_i$). Then we have that*

$$\xi_A = \left(1_{(\lambda_1,\dots,\lambda_{i-1})} \star \xi_{[\,\lambda_i'\ \lambda_i''\,]} \star 1_{(\lambda_{i+1},\dots,\lambda_{\ell(\lambda)})}\right) \circ \xi_B.$$

(b) *Let B be obtained from A by replacing its jth column by two columns of length $\ell(\lambda)$, the first of which has entries $a_{1,j},\dots,a_{i,j},0,\dots,0$ with sum μ_j', and the second has entries $0,\dots,0,a_{i+1,j},\dots,a_{\ell(\lambda),j}$ with sum μ_j'' (so $\mu_j'+\mu_j''=\mu_j$). Then we have that*

$$\xi_A = \xi_B \circ \left(1_{(\mu_1,\dots,\mu_{j-1})} \star \xi_{\left[\begin{smallmatrix}\mu_j' \\ \mu_j''\end{smallmatrix}\right]} \star 1_{(\mu_{j+1},\dots,\mu_{\ell(\mu)})}\right).$$

Proof. We just prove (b). Then (a) follows on applying $\mathtt{T}$. By the way that composition in q-**Schur** is defined, the statement we are trying to prove reduces to the following claim about multiplication in $S_q(n,r)$:

Suppose that we are given $\lambda,\mu \in \Lambda(n,r)$, $A \in \mathrm{Mat}(\lambda,\mu)$ and $1 \leq i \leq n, 1 \leq j \leq n-1$ with $\mu_{j+1} = 0$. Let $B \in \mathrm{Mat}(\lambda,\mu')$ be obtained from A by replacing the jth and $(j+1)$th columns *with* $[a_{i,1}\ \cdots\ a_{i,j}\ 0\ \cdots\ 0]^{\mathrm{T}}$ *and* $[0\ \cdots\ 0\ a_{i+1,j}\ \cdots\ a_{n,j}]^{\mathrm{T}}$*, respectively, and* $C \in \mathrm{Mat}(\mu', \mu)$ *be* $\mathrm{diag}\Big(\mu_1, \dots, \mu_{j-1}, \begin{bmatrix} \mu'_j & 0 \\ \mu'_{j+1} & 0 \end{bmatrix}, \mu_{j+2}, \dots, \mu_n\Big)$*. Then we have that* $\xi_A = \xi_B \xi_C$ *in* $S_q(n,r)$*.*

To see this, it suffices to show for $D \in \mathrm{Mat}(\lambda, \mu)$ that g_D, the $x_B \otimes x_C$-coefficient of $\Delta(x_D)$ when expanded in terms of normally-ordered monomials, is equal to $\delta_{A,D}$. We have that

$$x_B = x_{\boldsymbol{i}_1,(1^{\mu_1})} \cdots x_{\boldsymbol{i}_{j-1},((j-1)^{\mu_{j-1}})} \big(x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}\big) x_{\boldsymbol{i}_{j+2},((j+2)^{\mu_{j+2}})} \cdots x_{\boldsymbol{i}_n,(n^{\mu_n})},$$
$$x_C = x_{1,1}^{\mu_1} \cdots x_{j-1,j-1}^{\mu_{j-1}} \big(x_{j+1,j}^{\mu'_{j+1}} x_{j,j}^{\mu'_j}\big) x_{j+2,j+2}^{\mu_{j+2}} \cdots x_{n,n}^{\mu_n},$$
$$x_D = x_{\boldsymbol{j}_1,(1^{\mu_1})} \cdots x_{\boldsymbol{j}_{j-1},((j-1)^{\mu_{j-1}})} \big(x_{n,j}^{d_{n,j}} \cdots x_{1,j}^{d_{1,j}}\big) x_{\boldsymbol{j}_{j+2},((j+2)^{\mu_{j+2}})} \cdots x_{\boldsymbol{j}_n,(n^{\mu_n})}$$

where $\boldsymbol{i}_k := (n^{a_{n,k}}\ \cdots\ 1^{a_{1,k}})$ and $\boldsymbol{j}_k = (n^{d_{n,k}}\ \cdots\ 1^{d_{1,k}})$. It is easy to see that the $x_{\boldsymbol{i}_k,(k^{\mu_k})} \otimes x_{k,k}^{\mu_k}$-coefficient of $\Delta\big(x_{\boldsymbol{j}_k,(k^{\mu_k})}\big)$ is 0 unless $\boldsymbol{j}_k = \boldsymbol{i}_k$, when it is 1. This implies that $g_D = 0$ unless $\boldsymbol{j}_k = \boldsymbol{i}_k$ for each $k = 1, \dots, j-1, j+2, \dots, n$ in which case, by weight considerations, we have that $d_{1,j} = a_{1,j}, \dots, d_{n,j} = a_{n,j}$, hence, $D = A$. Thus, we are reduced to showing that $g_A = 1$.

Our argument shows that g_A is the coefficient of $x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}} \otimes x_{j+1,j}^{\mu'_{j+1}} x_{j,j}^{\mu'_j}$ in the normally-ordered expansion of

$$\Delta\big(x_{n,j}^{a_{n,j}} \cdots x_{1,j}^{a_{1,j}}\big) = \sum_{\boldsymbol{k} \in \mathrm{I}(n,r)} x_{(n^{a_{n,j}}\ \dots\ 1^{a_{1,j}}),\boldsymbol{k}} \otimes x_{\boldsymbol{k},(j^{\mu_j})}.$$

To complete the proof, we claim for $\boldsymbol{k} \in \mathrm{I}(n,r)$ that $x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}$ appears with non-zero coefficient in the normally-ordered expansion of $x_{(n^{a_{n,j}}\ \dots\ 1^{a_{1,j}}),\boldsymbol{k}}$ if and only if $\boldsymbol{k} = ((j+1)^{\mu'_{j+1}}\ j^{\mu'_j})$, in which case the coefficient is 1. Certainly, $\boldsymbol{k}$ must be a permutation of $((j+1)^{\mu'_{j+1}}\ j^{\mu'_j})$. For any such $\boldsymbol{k}$ and any $\boldsymbol{h}$ that is a permutation of $(n^{a_{n,j}}\ \cdots\ 1^{a_{1,j}})$, we define the *height* of the monomial $x_{\boldsymbol{h},\boldsymbol{k}}$ to be $\sum_s h_s$ where the sum is over all $1 \leq s \leq \mu_j$ such that $k_s = j$. The monomial $x_{i,j}^{a_{i,j}} \cdots x_{1,j}^{a_{1,j}} x_{n,j+1}^{a_{n,j}} \cdots x_{i+1,j+1}^{a_{i+1,j}}$ is of height $\sum_{k=1}^{i} k a_{k,j}$. Also $x_{(n^{a_{n,j}}\ \dots\ 1^{a_{1,j}}),\boldsymbol{k}}$ is of the same height if $\boldsymbol{k} = ((j+1)^{\mu'_{j+1}}\ j^{\mu'_j})$, and otherwise its height is strictly bigger. In order to straighten $x_{(n^{a_{n,j}}\ \dots\ 1^{a_{1,j}}),\boldsymbol{k}}$, we need to use the commutation relations $x_{p,j+1}x_{p,j} = q^{-1}x_{p,j}x_{p,j+1}$ and $x_{p,j+1}x_{q,j} = x_{q,j}x_{p,j+1} - (q-q^{-1})x_{p,j}x_{q,j+1}$ for $p > q$. Monomials arising from the "error term" $x_{p,j}x_{q,j+1}$ are of strictly greater height, so do not contribute to the coefficient, and the others have the same height. The claim follows. ∎

Now take any $A \in \mathrm{Mat}(\lambda, \mu)$ and define λ^-, μ^+ as in Lemma 2.1. Note that $n := \ell(\lambda^-) = \ell(\mu^+) = \ell(\lambda)\ell(\mu)$. We can convert A into a matrix $A^\circ \in \mathrm{Mat}(\lambda^-, \mu^+)$ with at most one non-zero entry in each row and column by applying a sequence of the operations $A \mapsto B$ described in Lemma 5.9(a)–(b). For example:

$$\begin{bmatrix} 1&0&3\\2&2&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0\\0&0&3\\2&2&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0\\0&0&0\\0&0&3\\2&2&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0\\0&0&0\\0&0&3\\2&0&0\\0&2&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0\\0&0&0\\0&0&3\\2&0&0\\0&2&0\\0&0&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&3\\0&2&0&0\\0&0&2&0\\0&0&0&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&3\\0&2&0&0&0\\0&0&0&2&0\\0&0&0&0&1 \end{bmatrix} \mapsto \begin{bmatrix} 1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&3&0\\0&2&0&0&0&0\\0&0&0&2&0&0\\0&0&0&0&0&1 \end{bmatrix}.$$

The $n \times n$ matrix A° obtained in this way is uniquely determined. It corresponds to the permutation of n strings arising from the middle part of the double coset diagram of A. Lemma 5.9 plus (5.12) and (5.13) gives us an explicit algorithm to express the standard basis element ξ_A as a composition

$$\xi_A = \xi_{A^-} \circ \xi_{A^\circ} \circ \xi_{A^+} \tag{5.14}$$

where ξ_{A^-} is a tensor product of $\ell(\lambda)$ merges of $\ell(\mu)$ strings and ξ_{A^+} is a tensor product of $\ell(\mu)$ split of $\ell(\lambda)$ strings. The double coset diagrams of $A^- \in \mathrm{Mat}(\lambda, \lambda^-)$ and $A^+ \in \mathrm{Mat}(\mu^+, \mu)$ are given explicitly by the top part or the bottom part of the diagram of A, respectively.

Lemma 5.10. *For* $a, b \geq 0$*, we have that* $\xi_{[\,a\ b\,]} \circ \xi_{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q \xi_{[\,a+b\,]}$*.*

Proof. The $x_{1,1}^a x_{1,2}^b \otimes x_{2,1}^b x_{1,1}^a$-coefficient of $\Delta(x_{1,1}^{a+b})$ is $\begin{bmatrix} a+b \\ a \end{bmatrix}_q$ by Lemma 3.5. $\square$

Lemma 5.11. *For $a, b, c, d \geq 0$ with $a + b = c + d$, we have that*

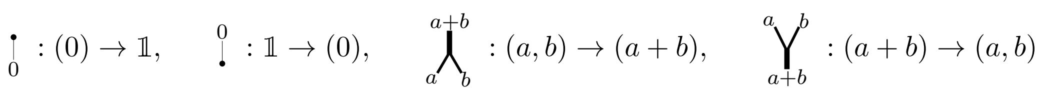
$$\theta_{\left[\begin{smallmatrix} 0 & c \\ a & d-a \end{smallmatrix}\right]} = \xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} a & b \end{smallmatrix}\right]} = \sum_{s=0}^{\min(a,c)} q^{s(s+d-a)} \xi_{\left[\begin{smallmatrix} s & c-s \\ a-s & s+d-a \end{smallmatrix}\right]} \qquad \text{if } a \leq d \text{ and } b \geq c,$$

$$\theta_{\left[\begin{smallmatrix} c-b & b \\ d & 0 \end{smallmatrix}\right]} = \xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} a & b \end{smallmatrix}\right]} = \sum_{t=0}^{\min(b,d)} q^{t(t+c-b)} \xi_{\left[\begin{smallmatrix} t+c-b & b-t \\ d-t & t \end{smallmatrix}\right]} \qquad \text{if } a \geq d \text{ and } b \leq c.$$

Proof. We just prove this when $a \leq d$, the other case is similar. Since the merge $\xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]}$ and the split $\xi_{\left[\begin{smallmatrix} a & b \end{smallmatrix}\right]}$ are bar invariant, and the canonical basis element θ_A is the unique bar invariant element equal to ξ_A plus a $q\mathbb{Z}[q]$-linear combination of other ξ_B, the first equality follows from the second one. To prove the second equality, we must show that the $\xi_{\left[\begin{smallmatrix} s & c-s \\ a-s & s+d-a \end{smallmatrix}\right]}$-coefficient of $\xi_{\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]} \circ \xi_{\left[\begin{smallmatrix} a & c+d-a \end{smallmatrix}\right]}$ is $q^{s(s+d-a)}$. This is the $x_{2,1}^d x_{1,1}^c \otimes x_{1,1}^a x_{1,2}^{c+d-a}$-coefficient in $\Delta(x_{2,1}^{c-s} x_{1,1}^s x_{2,2}^{s+d-a} x_{1,2}^{c-s})$, which may be computed by the same argument as was used in the proof of Lemma 5.6. $\square$

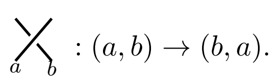
6. Presentations

We start now to represent morphisms in q-**Schur** by string diagrams. Let $\mathbb{1}$ be the strict identity object, that is, the composition $()$ of length zero. For $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \Lambda$, the identity endomorphism 1_λ in q-**Schur** will be represented by a sequence of strings labelled from left to right by $\lambda_1, \dots, \lambda_\ell$, which we think of as indicating the *thicknesses* of the strings. We are including strings of zero thickness. For $a, b \geq 0$, we use the string diagrams

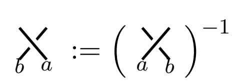


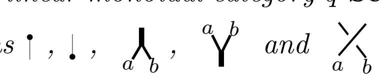
(6.1)

to denote the standard basis vectors ξ_A where A is the 0×1 matrix, the 1×0 matrix, the matrix $\left[\begin{smallmatrix} a & b \end{smallmatrix}\right]$ or the matrix $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, respectively. Henceforth, in string diagrams for morphisms in q-**Schur**, we will omit thickness labels on strings when they are implicitly determined by the other labels. We represent the positive crossing $\xi_{\left[\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}\right]}$ by the string diagram



(6.2)

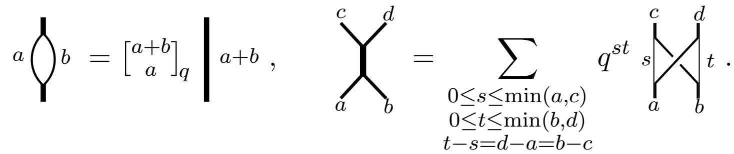
This morphism is invertible by Lemma 5.6, so it makes sense to define .

Theorem 6.1. *The $\mathbb{Z}[q, q^{-1}]$-linear monoidal category q-**Schur** is generated by the objects (r) for $r \geq 0$ and the morphisms*  *for $a, b \geq 0$, subject only to the following relations for $a, b, c, d \geq 0$ with $a + b = c + d$:*

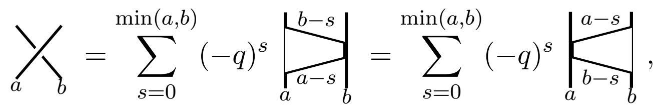
(6.3)

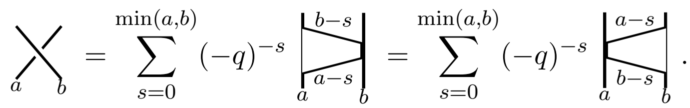
(6.4)


(6.5)


(6.6)

Positive and negative crossings can be written in terms of other generating morphisms since we have that


(6.7)


(6.8)

Moreover, the following hold:

(a) There is a unique braiding $c : -\star- \stackrel{\sim}{\Rightarrow} -\star^{\mathrm{rev}}-$ *making* q-**Schur** *into a braided monoidal category such that* $c_{(a),(b)} =$  .

(b) For any $A \in \mathrm{Mat}(\lambda, \mu)$*, the standard basis element* ξ_A *is represented as a string diagram by the double coset diagram for* A *with all crossings drawn as positive crossings.*

(c) The anti-linear involution $-$: q-**Schur** $\to$ q-**Schur** *is defined on string diagrams by interchanging positive and negative crossings.*

(d) The linear isomorphism $\mathtt{T} : q$-**Schur** $\to q$-**Schur**$^{\mathrm{op}}$ *maps a string diagram to its rotation through 180° around a horizontal axis.*

Before we prove this, some comments. The relations (6.3) imply that $(0) \cong \mathbb{1}$. The relations (6.4) mean that splits and merges with a string of thickness zero can be expressed in terms of the other generating morphisms, hence, can be eliminated from any string diagram. Using (6.3), (6.4) and the definition of negative crossings, the second relation in (6.6) implies that

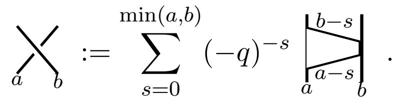

(6.9)

This means that crossings involving a string of thickness zero can also be expressed in terms of other morphisms, so these can be eliminated from string diagrams too. Then all remaining strings of thickness zero can be contracted to dots on the top and bottom boundaries. In this way, any string diagram is equivalent to one without strings of thickness zero. The relation (6.5) means that we can introduce further diagrams as shorthands more general splits and merges of n strings. For example, splits and merges of 3 strings are


(6.10)

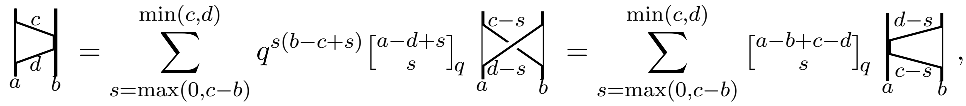
By (5.12) and (5.13), these are the standard basis vectors $\xi_{[\,a\ b\ c\,]}$ and $\xi_{\left[\begin{smallmatrix} a\\ b\\ c\end{smallmatrix}\right]}$, respectively.

Proof of Theorem 6.1. Let q-**Schur**$'$ be the strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal category defined by the generators and relations in the statement of the theorem. We also *define* the negative crossings in q-**Schur**$'$ by setting

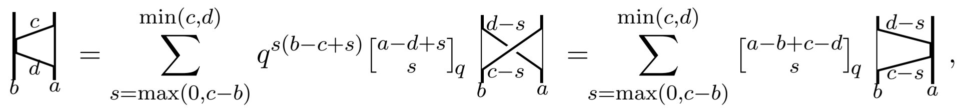


(6.11)

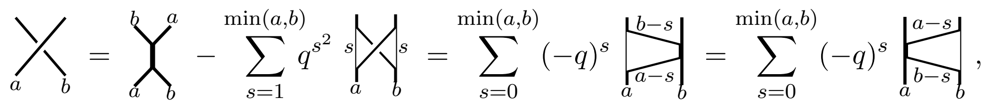
At this point, some calculations are needed to deduce the following additional relations from the defining relations in q-**Schur**$'$ (for all $a, b, c, d \geq 0$ that make sense):



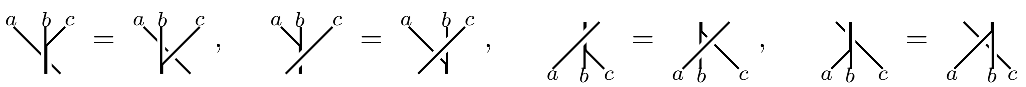
(6.12)



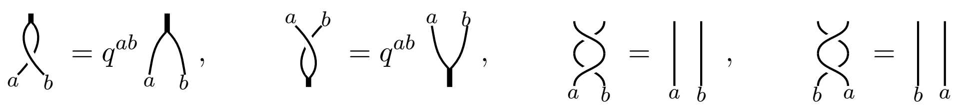
(6.13)



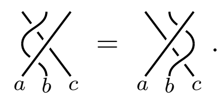
(6.14)



(6.15)

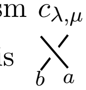


(6.16)

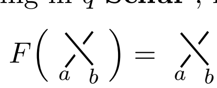


(6.17)

The derivations of these relations are similar to those in the appendix of [BEAEO20] (which treats the $q = 1$ case); see the appendix to the version of this article available on the `arxiv`.

Now we prove (a) but for the presented category q-**Schur**$'$ rather than q-**Schur** itself; then (a) for q-**Schur** follows at the end when we have established that q-**Schur**$' \cong q$-**Schur**. We need natural isomorphisms $c_{\lambda,\mu} : \lambda \star \mu \xrightarrow{\sim} \mu \star \lambda$ for all compositions λ, μ. Given that $c_{(a),(b)}$ is the positive crossing, there is no choice for the definition of more general $c_{\lambda,\mu}$ in order for the hexagon axioms for a braided monoidal category to hold: it must be defined by composing positive crossings according to a reduced expression for the Grassmann permutation taking $1, \dots, \ell(\lambda)$ to $\ell(\mu)+1, \dots, \ell(\mu)+\ell(\lambda)$ and $\ell(\lambda)+1, \dots, \ell(\lambda)+\ell(\mu)$ to $1, \dots, \ell(\mu)$. As any two reduced expressions for a Grassmann permutation are equivalent by commuting braid relations, the resulting morphism is well defined by the interchange law. The morphism $c_{\lambda,\mu}$ is an isomorphism since the positive crossing  is invertible; its two-sided inverse is  according to the last two relations in (6.16). Naturality follows from (6.15).

Next, we show that there a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal functor $F : q\text{-}\mathbf{Schur}' \to q\text{-}\mathbf{Schur}$ taking $(r) \mapsto (r)$ and the generating morphisms of q-**Schur**$'$ to the morphisms in q-**Schur** represented by the same diagrams. To prove this, we just need to check relations: the relations (6.3) and (6.4) are trivial to check in q-**Schur**, (6.5) follows from (5.12) and (5.13), and (6.6) follows from Lemmas 5.10 and 5.11. By definition, the functor F takes the positive crossing in q-**Schur**$'$ to the positive crossing in q-**Schur**, so the identity (6.7) in q-**Schur** follows by applying F to (6.14). We have observed already that the negative crossing in q-**Schur**$'$ is the two-sided inverse of the positive crossing in q-**Schur**$'$, hence,



since the negative crossing in q-**Schur** is also the inverse of the positive crossing by the original definition. To prove that (6.8) holds in q-**Schur**, the first equality follows by applying F to (6.11). The second equality follows by applying the bar involution to the second equality of (6.7), remembering that this fixes splits and merges in q-**Schur** thanks to Lemma 5.6.

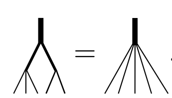
For any $A \in \mathrm{Mat}(\lambda, \mu)$, let ξ'_A be the morphism in q-**Schur**$'$ obtained by taking the (reduced) double coset diagram for A, replacing all crossings by positive crossings, and interpreting the result as a morphism by composing generators as the diagram suggests. The resulting morphism is well defined independent of the choices made when doing this. For the split of $\ell(\mu)$ strings at the bottom and the merge of $\ell(\lambda)$ strings at the top, this depends on (6.5) as explained in the comments after the statement of the theorem. For the permutation of $\ell(\lambda)\ell(\mu)$ strings in the middle, one needs to draw the diagram according to a choice of a reduced expression, but the resulting morphism is independent of this by (6.17). We are ready to prove (b) by showing that $F(\xi'_A) = \xi_A$. This follows from the factorization of ξ_A explained in (5.14), together with Lemma 5.8 and (5.12) and (5.13), since these results show ξ_A can be obtained from merges, splits and positive crossings in exactly the same way as ξ'_A is obtained from the correspnding generating morphisms for q-**Schur**$'$.

Now we can prove that F is an isomorphism. It is clear that it defines a bijection between the object sets of q-**Schur**$'$ and q-**Schur** (both are identified with Λ). Since the morphisms $\xi_A (A \in \mathrm{Mat}(\lambda, \mu))$ form a basis for $\mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu, \lambda)$ by the definition of q-**Schur**, we deduce using the previous paragraph that F is full. It just remains to show that it is faithful, which we do by proving that the morphisms ξ'_A $(A \in \mathrm{Mat}(\lambda, \mu))$ span $\mathrm{Hom}_{q\text{-}\mathbf{Schur}'}(\mu, \lambda)$ as a $\mathbb{Z}[q, q^{-1}]$-module. This follows from our next claim, since the merge and split morphisms f described in the claim for all λ, λ' generate q-**Schur**$'$ as a $\mathbb{Z}[q, q^{-1}]$-linear category by (6.7).

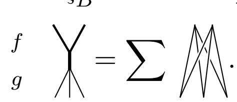
Claim. *For any $\lambda, \lambda', \mu \in \Lambda$, $A \in \mathrm{Mat}(\lambda, \mu)$ and $f : \lambda \to \lambda'$ that consists of a merge or split of 2 strings tensored on the left and/or right by some identity morphisms, the composition $f \circ \xi'_A$ is a $\mathbb{Z}[q, q^{-1}]$-linear combination of the morphisms ξ'_A $(A \in \mathrm{Mat}(\lambda, \mu))$.*

To prove the claim, there are two cases:

- Suppose first that f has a merge of two strings connecting to the ith and $(i+1)$th thick strings at the top of ξ'_A. The double coset diagram of A has a merge of r strings at its ith vertex and merge of s strings at its $(i+1)$th vertex. We use (6.5) to convert $f \circ \xi'_A$ into a diagram which has a merge of $(r+s)$ strings at its ith vertex. For example:


(6.18)

  The permutation arising in the middle section of the resulting diagram is not necessarily reduced, but it can be converted to a scalar multiple of some ξ'_B using the relations (6.5), (6.6), (6.16) and (6.17).
- Now suppose that f has a split connecting to the ith vertex at the top of the double coset diagram of A. Say this vertex in the double coset diagram is part of an n-fold merge. Using (6.5), (6.6) and (6.15), we rewrite the composition of the split in f and this merge in ξ'_A as a sum of other ξ'_B. For example:

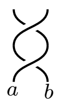


(6.19)

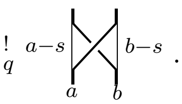
  Then compose these diagrams with the remainder of the diagram, using (6.15) then (6.5) again to commute the splits at the bottom of this part of the resulting diagrams downwards past the positive crossings in ξ'_A.

All that is left is to prove (c) and (d). Part (c) follows because the bar involution on q-**Schur** fixes merges and splits and interchanges positive and negative crossings by Lemma 5.6; it obviusly fixes the other two generating morphisms $\uparrow$ and $\downarrow$. Part (d) follows using (b) because T takes ξ_A to $\xi_{A^{\mathtt{T}}}$. □

Corollary 6.2. *In q-**Schur**, we have that*



$$= \sum_{s=0}^{\min(a,b)} q^{-s(s-1)/2+r(a+b-2s)}(q^{-1}-q)^s[s]^!_q$$



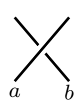
Proof. This is a translation of Remark 5.7 into the graphical description of q-**Schur** provided by the theorem. □

We also have the following theorem, which gives an alternative presentation for q-**Schur** with fewer generators and relations.

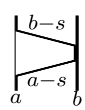
Theorem 6.3. *The strict $\mathbb{Z}[q,q^{-1}]$-monoidal category q-**Schur** is generated by the objects (r) for $r \geq 0$ and the morphisms $\uparrow$, $\downarrow$, merge and split subject only to the relations (6.3) and (6.4) for $a, b \geq 0$, (6.5) for $a, b, c > 0$, and one of the two square-switch relations from (1.2) for all $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$.*

Proof. Let q-**Schur**$'$ be the strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal category defined by the new presentation in the statement of the theorem, assuming for clarity that the *second* relation in (1.2) is the chosen one. All of the relations of q-**Schur**$'$ hold in q-**Schur** thanks to (6.13). So there is a strict $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor $F : q\text{-}\mathbf{Schur}' \rightarrow q\text{-}\mathbf{Schur}$ taking $(r) \mapsto (r)$ and the generating morphisms for q-**Schur**$'$ to the morphisms represented by the same diagrams in q-**Schur**. In the next paragraph, we show that F is an isomorphism, proving the theorem for this choice of square-switch. The proof of the theorem if one instead chooses the first square-switch relation from (1.2), i.e., the one that is known to hold in q-**Schur** by (6.12), is very similar—one simply needs to rotate all calculations in a vertical axis.

To prove that F is an isomorphism, we use the presentation from Theorem 6.1 to construct a two-sided inverse $G : q\text{-}\mathbf{Schur} \rightarrow q\text{-}\mathbf{Schur}'$. This is defined on objects so that $(r) \mapsto (r)$ and, on generating morphisms, it maps the positive crossing to



$$:= \sum_{s=0}^{\min(a,b)} (-q)^s$$



$$\in \mathrm{Hom}_{q\text{-}\mathbf{Schur}'}((a) \star (b) \rightarrow (b) \star (a)) \tag{6.20}$$

and the other generating morphisms for q-**Schur** to the morphisms represented by the same diagrams in q-**Schur**$'$. That G is indeed a two-sided inverse of F follows using (6.7). It remains to show that G is well defined, which is another relations check. The relations (6.3) and (6.4) hold in q-**Schur**$'$ by its definition. If one or more of a, b, c is zero, the relations (6.5) follow easily from (6.3) and (6.4), so the relations (6.5) also hold in q-**Schur**$'$ for all $a, b, c \geq 0$. The first relation from (6.6) follows from the chosen square-switch relation taking $b = 0$ and $c = d$. It remains to show that the second relation from (6.6) holds in q-**Schur**$'$ using only (6.3) to (6.5) and square-switch. This is explained in the appendix to the `arxiv` version of this paper; see (a) of the corollary there. □

7. A straightening formula for codeterminants

Definition 7.1. Let $\mathcal{O}$ be a commutative Noetherian ring and $K = \bigoplus_{\lambda,\mu\in\Lambda} 1_\lambda K 1_\mu$ be a locally unital $\mathcal{O}$-algebra with (mutually orthogonal) distinguished idempotents 1_λ ($\lambda \in \Lambda$) for some index set Λ. We say that K is a *based quasi-hereditary algebra* with *weight poset* Λ^+ if we are given a subset $\Lambda^+ \subseteq \Lambda$, an upper finite partial order $\leq$ on Λ^+, and finite sets $X(\lambda,\kappa) \subset 1_\lambda K 1_\kappa$ and $Y(\kappa,\lambda) \subset 1_\kappa K 1_\lambda$ for $\lambda \in \Lambda, \kappa \in \Lambda^+$, such that the following axioms hold:

- The products xy for $(x,y) \in \bigcup_{\lambda,\mu\in\Lambda}\bigcup_{\kappa\in\Lambda^+} X(\lambda,\kappa)\times Y(\kappa,\mu)$ give a basis for K as a free $\mathcal{O}$-module. We refer to this as the *triangular basis*.
- For $\lambda,\mu \in \Lambda^+$, we have that $X(\lambda,\mu) \neq \varnothing \Rightarrow \lambda \leq \mu$, $Y(\lambda,\mu) \neq \varnothing \Rightarrow \lambda \geq \mu$, and $X(\lambda,\lambda) = Y(\lambda,\lambda) = \{1_\lambda\}$.

We say that it is a *symmetrically-based quasi-hereditary algebra* if in addition there is an algebra anti-involution $\mathtt{T} : K \to K$ such that $Y(\kappa,\lambda) = \mathtt{T}(X(\lambda,\kappa))$ for all $\lambda \in \Lambda$ and $\kappa \in \Lambda^+$ (in this case, there is no need to specify $Y(\kappa,\lambda)$ in the first place).

Remark 7.2. When $\mathcal{O}$ is a field, Definition 7.1 is [BS24, Def. 5.1]. When the set Λ is finite, it is a simplified version of the definition of based quasi-hereditary algebra given in [KM20]. In that case, as explained in detail in [KM20], K is also a standardly full-based algebra in the sense of [DR98], and a split quasi-hereditary algebra in the sense of [CPS90]. In the symmetrically-based case, K is a cellular algebra in the sense of [GL96], and when K is the path algebra of an $\mathcal{O}$-linear category $\mathbf{C}$ with object set Λ, Definition 7.1 is equivalent to $\mathbf{C}$ being a strictly object-adapted cellular category in the sense of [EL16, Def. 2.1] (the opposite partial order is used there). The far-reaching consequences for the representation theory of K are well known, and are discussed in these references.

For the remainder of the section, K is the path algebra

$$K := \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda) \tag{7.1}$$

of the q-Schur category with 0-strings. This is a locally unital $\mathbb{Z}[q,q^{-1}]$-algebra with the distinguished system $\{1_\lambda \,|\, \lambda \in \Lambda\}$ of mutually orthogonal idempotents coming from the identity endomorphisms of the objects of q-**Schur**. Recall the set $\mathrm{Row}(\lambda,\mu)$ of *row tableaux* of shape μ and content λ from §2, and the bijection $A : \mathrm{Row}(\lambda,\mu) \xrightarrow{\sim} \mathrm{Mat}(\lambda,\mu)$ from (2.3). We start now to index the standard and canonical bases by the sets $\mathrm{Row}(\lambda,\mu)$ instead of $\mathrm{Mat}(\lambda,\mu)$, introducing the shorthands

$$\varphi_P := \xi_{A(P)}, \qquad\qquad \beta_P := \theta_{A(P)} \tag{7.2}$$

for $P \in \mathrm{Row}(\lambda,\mu)$. For a partition κ, let $\mathrm{Std}(\lambda,\kappa)$ be the usual set of *semistandard tableau of shape* κ *and content* λ, that is, is the subset of $\mathrm{Row}(\lambda,\kappa)$ consisting of the row tableaux of shape κ and content λ whose entries are also strictly increasing down columns.

Lemma 7.3. *For $\lambda,\mu \vDash r$, the $\mathbb{Z}[q,q^{-1}]$-module $1_\lambda K 1_\mu$ is spanned by the products $\varphi_P \mathtt{T}(\varphi_Q)$ for $P \in \mathrm{Row}(\lambda,\kappa), Q \in \mathrm{Row}(\mu,\kappa)$, where κ is the dominant conjugate of μ.*

Proof. The dominant conjugate κ of μ is the unique partition whose parts are a permutation of the non-zero parts of μ. Using a morphism of the form $\tau_{w;\mu}$ from (5.10), we deduce $\mu \cong \kappa$ in q-**Schur**. Consequently, any element of $1_\lambda K 1_\mu = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\lambda)$ is a morphism which factors through κ. Since the morphisms φ_P for $P \in \mathrm{Row}(\lambda,\kappa)$ give the standard basis for $1_\lambda K 1_\kappa = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\kappa,\lambda)$ and the morphisms $\mathtt{T}(\varphi_Q)$ for $Q \in \mathrm{Row}(\mu,\kappa)$ give the standard basis for $1_\kappa K 1_\mu = \mathrm{Hom}_{q\text{-}\mathbf{Schur}}(\mu,\kappa)$, we deduce that the products $\varphi_P\mathtt{T}(\varphi_Q)$ span $1_\lambda K 1_\mu$. □

Now we come to the main combinatorial lemma. To formulate it, we use certain lexicographic total orders on tableaux and partitions. On partitions, $\geq_{\mathrm{lex}}$ is just the usual lexicographical ordering; it is a refinement of the dominance ordering on partitions into a total order. To define the required ordering $\leq_{\mathrm{lex}}$ on tableaux of the same shape, given any tableau T, we let $\swarrow\!(T)$ be the sequence obtained by reading its entries in order from right to left along rows, starting with the top row. Then we declare that $S \leq_{\mathrm{lex}} T$ if and only if $\swarrow\!(S) \leq_{\mathrm{lex}} \swarrow\!(T)$ in the lexicographic ordering on sequences.

Lemma 7.4. *For $\lambda \vDash r$, $\kappa \vdash r$ and $P \in \mathrm{Row}(\lambda,\kappa)$ which is* not *semistandard, φ_P can be written as a $\mathbb{Z}[q,q^{-1}]$-linear combination of the following elements:*

- *φ_S for $S \in \mathrm{Row}(\lambda,\kappa)$ with $S <_{\mathrm{lex}} P$;*
- *$\varphi_{P'}\mathtt{T}(\varphi_{Q'})$ for $P' \in \mathrm{Row}(\lambda,\kappa')$ and $Q' \in \mathrm{Row}(\kappa,\kappa')$ of shape $\kappa' \vdash r$ with $\kappa' >_{\mathrm{lex}} \kappa$.*

Proof. Take P as in the statement. Since P is not semistandard, we may choose $a \geq 1$ and $0 \leq m < n \leq \kappa_{a+1}$ so that the entries of P in rows a and $(a+1)$ look like

$$\begin{array}{ccccccccccccccc} i_1 & \leq & \cdots & \leq & i_m & < & i_{m+1} & \leq & \cdots & \leq & i_n & \leq & i_{n+1} & \leq & \cdots & \leq & i_{\kappa_a} \\ &&&&&& \vee\!\!| &&&&&&&&&& \\ j_1 & \leq & \cdots & \leq & j_m & \leq & j_{m+1} & = & \cdots & = & j_n & < & j_{n+1} & \leq & \cdots & \leq & j_{\kappa_{a+1}}. \end{array}$$

Let U be the row tableau which is identical to P everywhere except in rows a and $(a+1)$, which are replaced by *three* (possibly empty) rows as in the diagram:

$$\begin{array}{lllllllll} i_1 & \leq & \cdots & \leq & i_m &&&& \\ j_1 & \leq & \cdots & \leq & j_n & \leq & i_{m+1} & \leq & \cdots \leq i_{\kappa_a} \\ j_{n+1} & \leq & \cdots & \leq & j_{\kappa_{a+1}}. &&&& \end{array}$$

Let μ be the shape of the tableau U. Let V be the row tableau of shape κ and content μ with all entries on row b equal to b for $b < a$, entries $a^m\ (a+1)^{\kappa_a - m}$ on row a, entries $(a+1)^n\ (a+2)^{\kappa_{a+1}-n}$ on row $a+1$, and all entries on row b equal to $b+1$ for $b > a+1$. Expanding in terms of the standard basis, we have that

$$\varphi_U\varphi_V = \sum_{S \in \mathrm{Row}(\lambda,\kappa)} g_S \varphi_S \tag{7.3}$$

for coefficients $g_S \in \mathbb{Z}[q,q^{-1}]$. We claim that $g_S = 0$ unless $S \leq_{\mathrm{lex}} P$ and that $g_P = 1$. This suffices to prove the lemma. Indeed, assuming the claim, we rearrange (7.3) to obtain

$$\varphi_P = \varphi_U\varphi_V - \sum_{S <_{\mathrm{lex}} P} g_S\varphi_S.$$

The second term on the right hand side is already of the desired form. To understand the first term, note that the first $(a-1)$ rows of U are of lengths $\kappa_1,\dots,\kappa_{a-1}$, and it also has a row of length $\kappa_a + n - m > \kappa_a$. Consequently, the dominant conjugate of the shape μ of U is greater than κ in the ordering $>_{\mathrm{lex}}$. So, by Lemma 7.3, the first term can be rewritten as a sum $\varphi_{P'}\mathtt{T}(\varphi_{Q'})$ for row tableaux P', Q' of dominant shape $\kappa' >_{\mathrm{lex}} \kappa$. This is also of the desired form.

It just remains to prove the claim. Take $S \in \mathrm{Row}(\lambda,\kappa)$. Recalling that $x_S = x_{\leftrightarrows(S),\boldsymbol{i}^\kappa}$ and $x_U = x_{\leftrightarrows(U),\boldsymbol{i}^\mu}$, the definition of multiplication in K gives that g_S is the $x_{\leftrightarrows(U),\boldsymbol{i}^\mu} \otimes x_{\leftrightarrows(V),\boldsymbol{i}^\kappa}$-coefficient of

$$\Delta(x_{\leftrightarrows(S),\boldsymbol{i}^\kappa}) = \sum_{\boldsymbol{k}\in \mathrm{I}_\mu} x_{\leftrightarrows(S),\boldsymbol{k}} \otimes x_{\boldsymbol{k},\boldsymbol{i}^\kappa}$$

when expanded in terms of the normally-ordered monomial basis. To straighten $x_{\boldsymbol{k},\boldsymbol{i}^\kappa}$ into normal order, we only need the fourth relation from (3.5), and see that this coefficient is non-zero if and only if $\boldsymbol{k} = \leftrightarrows(R)$ for a tableau R of shape κ (not necessarily a row tableau) that is obtained from V by shuffling entries within rows a and $(a+1)$. Moreover, the coefficient is 1 in the case that $R = V$. To complete the proof, we show for such a tableau R that the $x_{\leftrightarrows(U),\boldsymbol{i}^\mu}$-coefficient of $x_{\leftrightarrows(S),\leftrightarrows(R)}$ is zero unless $S \leq_{\mathrm{lex}} P$, it is 1 if $S = P$ and $R = V$, and it is zero if $S = P$ and $R \neq V$. Suppose the entries in rows a and $(a+1)$ of S are

$$\begin{array}{ccccc} i'_1 & \leq & \cdots & \leq & i'_{\kappa_a} \\ j'_1 & \leq & \cdots & \leq & j'_{\kappa_{a+1}}. \end{array}$$

In order to convert the monomial $x_{\trianglerighteq(S),\trianglerighteq(R)}$ into normal order, we must apply the relations to commute products of the form $x_{i'_c,a+1}x_{i'_b,a}$ for $1 \leq b < c \leq \kappa_a$ or $x_{j'_c,a+2}x_{j'_b,a+1}$ for $1 \leq b < c \leq \kappa_{a+1}$. This can be done using the second and third relations from (3.5). We deduce that

$$x_{\trianglerighteq(S),\trianglerighteq(R)} = \sum_{\substack{v \in (S_{\kappa_r}/S_m \times S_{\kappa_a - m})_{\min} \\ w \in (S_{\kappa_{a+1}}/S_n \times S_{\kappa_{a+1}-n})_{\min}}} g_{v,w} x_{\trianglerighteq(T_{v,w}),\boldsymbol{i}^\mu}$$

for some scalars $g_{v,w} \in \mathbb{Z}[q,q^{-1}]$ with $g_{1,1} = \delta_{R,V}$, where $T_{v,w}$ is the tableau of shape μ obtained from S by replacing its rows a and $(a+1)$ by three rows according to the diagram:

$$\begin{array}{lllllll} i'_{v(1)} & \leq & \cdots & \leq & i'_{v(m)} & & \\ j'_{w(1)} & \leq & \cdots & \leq & j'_{w(n)} & i'_{v(m+1)} \leq \cdots \leq i'_{v(\kappa_a)} \\ j'_{w(n+1)} & \leq & \cdots & \leq & j'_{w(\kappa_{a+1})}. & \end{array}$$

In particular, if $S = P$ then $T_{1,1} = U$. Using the fourth relation, the $x_{\trianglerighteq(U),\boldsymbol{i}^\mu}$-coefficient of $x_{\trianglerighteq(T_{v,w}),\boldsymbol{i}^\mu}$ is non-zero if and only if $T_{v,w} \sim_{\text{row}} U$, i.e., they have the same entries in each row counted with multiplicity, and the coefficient is 1 if $T_{v,w} = U$. Now it remains to check that

- $T_{v,w} \sim_{\text{row}} U \Rightarrow S \leq_{\text{lex}} P$;
- $T_{v,w} \sim_{\text{row}} U$ and $S = P \Rightarrow (v,w) = (1,1)$.

To see this, suppose that $T_{v,w} \sim_{\text{row}} U$. All rows of S are clearly equal to the corresponding rows of P except perhaps for rows a and $(a+1)$. Also the sequences $i'_{v(1)} \leq \cdots \leq i'_{v(m)}$ and $j'_{w(n+1)} \leq \cdots \leq j'_{w(\kappa_{a+1})}$ are equal to $i_1 \leq \cdots \leq i_m$ and $j_{n+1} \leq \cdots \leq j_{\kappa_{a+1}}$, respectively. So the ath row of S is obtained by taking all of the entries in the ath row of U together with $\kappa_a - m$ entries from row $(a+1)$, and row $(a+1)$ of S is obtained by taking all of the remaining entries from row $(a+1)$ of U plus all of the entries in row $(a+2)$. It follows that $S \leq_{\text{lex}} P$. Moreover, if $S = P$, then we must have that $v = w = 1$ due to the assumptions that $i_m < i_{m+1}$ and $j_n < j_{n+1}$. □

Theorem 7.5. *The path algebra $K = \bigoplus_{\lambda,\mu\in\Lambda} 1_\lambda K 1_\mu$ of q-***Schur** *is a symmetrically-based quasi-hereditary algebra. The required data from Definition 7.1 is as follows:*

- *The weight poset is the set $\Lambda^+ \subset \Lambda$ of partitions ordered by the dominance ordering.*
- *The anti-involution $\mathtt{T} : K \to K$ is the transposition map arising from* (5.4)*.*
- *$X(\lambda,\kappa) = \{\varphi_P \mid P \in \mathrm{Std}(\lambda,\kappa)\}$.*

In particular, for $\lambda, \mu \vDash r$, the codeterminants

$$\left\{\varphi_P \mathtt{T}(\varphi_Q) \;\middle|\; \kappa \vdash r, P \in \mathrm{Std}(\lambda,\kappa), Q \in \mathrm{Std}(\mu,\kappa)\right\} \tag{7.4}$$

give a basis for $1_\lambda K 1_\mu$ as a free $\mathbb{Z}[q,q^{-1}]$-module.

Proof. The second axiom follows because there is a unique semistandard tableau of shape and content κ, and there only exist semistandard tableaux of shape κ' and content κ if $\kappa \leq \kappa'$. It just remains to check for $\lambda, \mu \vDash r$ that the set (7.4), is a basis for $1_\lambda K 1_\mu$ as a free $\mathbb{Z}[q,q^{-1}]$-module. By the original definition, $1_\lambda K 1_\mu$ is a free $\mathbb{Z}[q,q^{-1}]$-module with basis labelled by $\mathrm{Mat}(\lambda,\mu)$. It is well known that $|\mathrm{Mat}(\lambda,\mu)| = \sum_{\kappa\vdash r} |\mathrm{Std}(\lambda,\kappa) \times \mathrm{Std}(\mu,\kappa)|$, e.g., this follows from the Robinson-Schensted-Knuth-type correspondence in (7.5) below. So the set (7.4) is of size $\leq \operatorname{rank} 1_\lambda K 1_\mu$. It remains to show that the set (7.4) spans $1_\lambda K 1_\mu$ as a $\mathbb{Z}[q,q^{-1}]$-module.

By Lemma 7.3, the elements $\varphi_P \mathtt{T}(\varphi_Q)$ for $P \in \mathrm{Row}(\lambda,\kappa), Q \in \mathrm{Row}(\mu,\kappa)$ and $\kappa \vdash r$ span $1_\lambda K 1_\mu$. To complete the proof, we show by induction on the lexicographic orderings that any such $\varphi_P \mathtt{T}(\varphi_Q)$ can be written as a $\mathbb{Z}[q,q^{-1}]$-linear combination of $\varphi_{P'} \mathtt{T}(\varphi_{Q'})$ such that either $P' \in \mathrm{Std}(\lambda,\kappa), Q' \in \mathrm{Std}(\mu,\kappa)$ with $P' \leq_{\text{lex}} P, Q' \leq_{\text{lex}} Q$, or $P' \in \mathrm{Std}(\lambda,\kappa'), Q' \in \mathrm{Std}(\mu,\kappa')$ for $\kappa' >_{\text{lex}} \kappa$. Applying $\mathtt{T}$ if necessary, we may assume that P is not semistandard. Applying Lemma 7.4, we see that $\varphi_P \mathtt{T}(\varphi_Q)$ is a linear combination of elements $\varphi_S \mathtt{T}(\varphi_Q)$ for $S \in \mathrm{Row}(\lambda,\kappa)$

with $S <_{\text{lex}} P$, and $\varphi_{P'}\mathtt{T}(\varphi_{Q'})\mathtt{T}(\varphi_Q) = \varphi_{P'}\mathtt{T}(\varphi_Q\varphi_{Q'})$ with P' of shape $\kappa' >_{\text{lex}} \kappa$. Both types of elements can then be expanded into the required form by induction; for the second type, one first expands $\varphi_Q\varphi_{Q'}$ as a sum of terms φ_R for $R \in \text{Row}(\mu, \kappa')$, then applies $\mathtt{T}$ to obtain a linear combination of $\varphi_{P'}\mathtt{T}(\varphi_R)$'s, before invoking the induction hypothesis. □

Remark 7.6. Let us explain how the canonical basis fits into this picture. In [DR98, §5.3], one finds a Robinson-Schensted-Knuth-type correspondence giving a bijection

$$\text{Mat}(\lambda, \mu) \xrightarrow{\sim} \bigcup_{\kappa \in \Lambda^+} \text{Std}(\lambda, \kappa) \times \text{Std}(\mu, \kappa), \qquad A \mapsto (P(A), Q(A)), \tag{7.5}$$

which we explain more fully shortly. Also let $\kappa(A)$ be the common shape of the tableaux $P(A)$ and $Q(A)$ and recall (7.2). Then [DR98, Th. 5.3.3] can be reformulated as follows:

Theorem. *The path algebra K of q-**Schur** has another triangular basis*

$$\left\{\beta_P \mathtt{T}(\beta_Q) \;\middle|\; (P, Q) \in \textstyle\bigcup_{\lambda,\mu \in \Lambda, \kappa \in \Lambda^+} \text{Std}(\lambda, \kappa) \times \text{Std}(\mu, \kappa)\right\} \tag{7.6}$$

making it a symmetrically-based quasi-hereditary algebra with $X(\lambda, \kappa) = \{\beta_P \mid P \in \text{Std}(\lambda, \kappa)\}$ and all other data as in Theorem 7.5. Moreover, for $A \in \text{Mat}(\lambda, \mu)$ we have that

$$\theta_A \equiv \beta_{P(A)}\mathtt{T}(\beta_{Q(A)}) \pmod{\textstyle\sum_{B \in \text{Mat}(\lambda,\mu) \text{ with } \kappa(B) > \kappa(A)} \mathbb{Z}[q, q^{-1}]\theta_B}. \tag{7.7}$$

So the canonical basis is a cellular basis which is equivalent to the triangular basis (7.6)*, i.e., it defines the same two-sided cell ideals and induces the same basis in each two-sided cell.*

To define the map (7.5) explicitly, take $A \in \text{Mat}(\lambda, \mu)$ corresponding to $R \in \text{Row}(\lambda, \mu)$ under the bijection (2.3). Let $\boldsymbol{i} = (i_1, \dots, i_r) \in I_\lambda$ be the sequence $\nwarrow(R)$. Then we use *column insertion*[3] to insert $i_1, \dots, i_r$ in order into the empty tableau, to end up with a semistandard tableau $P(A) \in \text{Std}(\lambda, \kappa)$ for some $\kappa \vdash r$. We also obtain another semistandard tableau $Q(A) \in \text{Std}(\mu, \kappa)$, namely, the *recording tableau* defined so that the entry of the box that gets added at the rth step of the algorithm is i^μ_r. This concise description of the map (7.5) is equivalent to the more complicated description in [DR98, §5.3]. It takes some combinatorial work (omitted here) to establish the equivalence. For example, suppose that $A = \left[\begin{smallmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{smallmatrix}\right]$, $\lambda = (3, 1)$ and $\mu = (1, 1, 2)$. Then $\boldsymbol{i} = (1, 2, 1, 1)$ and $\boldsymbol{i}^\mu = (1, 2, 3, 3)$. Column insertion of the sequence $\boldsymbol{i}$ gives $\varnothing \xrightarrow{1} \begin{array}{|c|}\hline 1\\\hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}$. So we get that

$$P(A) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}, \qquad Q(A) = \begin{array}{|c|c|c|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}, \qquad \kappa(A) = (3, 1).$$

8. Tilting modules

For $n \geq 0$, let $\mathbf{I}_n$ be the two-sided tensor ideal of q-**Schur** generated by the identity morphisms $1_{(r)}$ for all $r > n$, then set $q\text{-}\mathbf{Schur}_n := q\text{-}\mathbf{Schur}/\mathbf{I}_n$. This is a strict $\mathbb{Z}[q, q^{-1}]$-linear monoidal category.

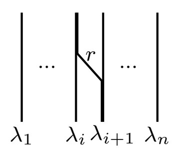
Theorem 8.1. *The path algebra K_n of q-$\mathbf{Schur}_n$ is a symmetrically-based quasi-hereditary algebra, with one possible triangular basis arising from the images of the codeterminants from* (7.4) *for all $\kappa \in \Lambda^+$ satisfying $\kappa_1 \leq n$, and another one given by the images of the canonical basis products from* (7.6) *for the same κ. Also the images of the canonical basis elements θ_A for $A \in \bigcup_{\lambda,\mu \in \Lambda} \text{Mat}(\lambda, \mu)$ such that $\kappa(A)_1 \leq n$ give a cellular basis for K_n.*

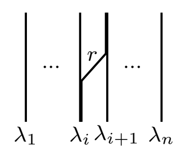
[3] We mean the following algorithm to insert i into a semistandard tableau: start with the first column; if i is bigger than all entries in the column then we add i to the bottom of that column and stop; otherwise, we find the smallest entry j in the column that is greater than or equal to i, replace that entry by i, then repeat to insert j into the next column to the right.

Proof. The two-sided tensor ideal $\mathbf{I}_n$ is equal to the ordinary two-sided ideal of q-**Schur** generated by the morphisms 1_κ for all partitions $\kappa \in \Lambda^+$ with $\kappa_1 > n$. This follows because every object $\lambda \in \Lambda$ which has some part $r > n$ is isomorphic to such a partition κ. Hence, $\mathbf{I}_n$ corresponds to the two-sided ideal $I_n \lhd K$ of the path algebra K of q-**Schur** generated by the idempotents 1_κ for all $\kappa \in \Lambda^+$ with $\kappa_1 > n$, and $K_n = K/I_n$. The set $\{\kappa \in \Lambda^+ \mid \kappa_1 > n\}$ is an upper set in the poset Λ^+, hence, I_n is a *cell ideal* in the based quasi-hereditary algebra K. Consequently, by [BS24, Cor. 5.6], the quotient algebra K_n is also a symmetrically-based quasi-hereditary algebra with bases as described in the statement of the theorem. □

Now let $\Bbbk$ be a field viewed as a $\mathbb{Z}[q, q^{-1}]$-algebra in some way, and consider the $\Bbbk$-linear monoidal categories $q\text{-}\mathbf{Schur}(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}$ and $q\text{-}\mathbf{Schur}_n(\Bbbk) := \Bbbk \otimes_{\mathbb{Z}[q,q^{-1}]} q\text{-}\mathbf{Schur}_n$. From the bases as free $\mathbb{Z}[q, q^{-1}]$-modules discussed in the proof of Theorem 8.1, it follows that $q\text{-}\mathbf{Schur}_n(\Bbbk)$ may identified with the quotient of $q\text{-}\mathbf{Schur}(\Bbbk)$ by the two-sided tensor ideal $\mathbf{I}_n(\Bbbk)$ generated by the morphisms $1_{(r)}$ for $r > n$.

Let $q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$ be the monoidal category of polynomial tilting modules for $q\text{-}GL_n(\Bbbk)$, that is, the full additive Karoubian monoidal subcategory of the category of polynomial representations of $q\text{-}GL_n(\Bbbk)$ generated by the exterior powers $\bigwedge^r V$ for $1 \leq r \leq n$. Here, to avoid too much more notation, we are re-using $\bigwedge^r V$ to denote the specializations of the $\mathbb{Z}[q, q^{-1}]$-modules from before. Note also that we defined the braided monoidal category $q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$ in the introduction in a different way in terms modules over the algebra $U_n(\Bbbk)$, but the two definitions are equivalent. This identification requires the specific choice of comultiplication Δ described in the introduction in order for the induced homomorphism $\dot{U}_n \to K$ to map

$E_i^{(r)} 1_\lambda \mapsto$ 

, $\qquad F_i^{(r)} 1_\lambda \mapsto$ 

(8.1)

for $1 \leq i < n, r \geq 0$ and $\lambda \in \mathbb{N}^n$ with $\lambda_{i+1} \geq r$ or $\lambda_i \geq r$, respectively (they map to zero for all other λ). To see that the defining relations of $\dot{U}_n$ hold in K, most of them are easy, indeed, this is the origin of the square-switch relation. The Serre relation is deduced from the other relations in [CKM14, Lem. 2.2.1].

Remark 8.2. When $0 \leq a - d \leq b - c$, the expressions in (6.12) and (6.13) are the canonical basis elements $\theta_{\left[\begin{smallmatrix} a-d & c \\ d & b-c \end{smallmatrix}\right]}$ and $\theta_{\left[\begin{smallmatrix} b-c & d \\ c & a-d \end{smallmatrix}\right]}$ from Example 4.1. They are also the images under the homomorphism (8.1) of the canonical basis elements $E^{(c)} F^{(d)} 1_{(a,b)}$ and $F^{(c)} E^{(d)} 1_{(b,a)}$ of $\dot{U}_2$.

The monoidal functor Σ_n from Theorem 5.4 extends to define a $\Bbbk$-linear monoidal functor $q\text{-}\mathbf{Schur}(\Bbbk) \to q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$. Since $\bigwedge^r V = \{0\}$ for $r > n$, this factors through the quotient $q\text{-}\mathbf{Schur}_n(\Bbbk)$ to induce a $\Bbbk$-linear monoidal functor $\bar{\Sigma}_n : q\text{-}\mathbf{Schur}_n(\Bbbk) \to q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$.

Theorem 8.3. *For any field $\Bbbk$, the functor $\bar{\Sigma}_n : q\text{-}\mathbf{Schur}(\Bbbk) \to q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$ induces a $\Bbbk$-linear monoidal equivalence between the additive Karoubi envelope of $q\text{-}\mathbf{Schur}_n(\Bbbk)$ and $q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$.*

Proof. We saw already in Remark 5.5(2) that $\bar{\Sigma}_n$ is full. It is dense by the definition of $q\text{-}\mathbf{Tilt}_n^+(\Bbbk)$. It just remains to show that it is faithful. Thus, we must show that the surjective $\Bbbk$-linear map $\operatorname{Hom}_{q\text{-}\mathbf{Schur}_n(\Bbbk)}(\mu, \lambda) \twoheadrightarrow \operatorname{Hom}_{q\text{-}GL_n(\Bbbk)}(\bigwedge^\mu V, \bigwedge^\lambda V)$ induced by the functor is also injective for any $\lambda, \mu \vDash r$. By Theorem 8.1, we know that the morphism space on the left is of dimension $\sum_{\kappa \vdash r} |\operatorname{Std}(\lambda, \kappa) \times \operatorname{Std}(\mu, \kappa)|$. This is also the dimension of $\operatorname{Hom}_{q\text{-}GL_n(\Bbbk)}(\bigwedge^\mu V, \bigwedge^\lambda V)$. Indeed, in the highest weight category of polynomial representations of $q\text{-}GL_n(\Bbbk)$, the tilting module $\bigwedge^\mu V$ has a filtration with sections that are standard modules $\Delta(\kappa')$ for partions κ with $\kappa_1 \leq n$, and $\bigwedge^\lambda V$ has a filtration with sections that are costandard modules $\nabla(\kappa')$ for the same

κ. By the Littlewood-Richardson rule, the multiplicities $(\bigwedge^\mu V : \Delta(\kappa'))$ and $(\bigwedge^\mu V : \nabla(\kappa'))$ are $|\operatorname{Row}(\mu,\kappa)|$ and $|\operatorname{Row}(\lambda,\kappa)|$. Since $\dim \operatorname{Ext}^i_{q\text{-}GL_n(\Bbbk)}(\Delta(\lambda),\nabla(\mu)) = \delta_{\lambda,\mu}\delta_{i,0}$, this is enough to prove that $\operatorname{Hom}_{q\text{-}GL_n(\Bbbk)}(\bigwedge^\mu V, \bigwedge^\lambda V)$ has the same dimension as $\operatorname{Hom}_{q\text{-}\mathbf{Schur}_n(\Bbbk)}(\mu,\lambda)$. □

Corollary 8.4. *The kernel of the full monoidal functor Σ_n from Theorem 5.4 is equal to $\mathbf{I}_n$.*

Proof. Let $\mathbf{J}_n$ be the kernel of Σ_n. Since $\bigwedge^r V = \{0\}$ for $r > n$, it is clear that $\mathbf{I}_n \subseteq \mathbf{J}_n$. Hence, Σ_n factors through the quotient to induce a full $\mathbb{Z}[q,q^{-1}]$-linear monoidal functor from q-$\mathbf{Schur}_n$ to the category of polynomial representations of q-GL_n. To prove that $\mathbf{J}_n = \mathbf{I}_n$, thereby proving the corollary, it remains to show that this induced functor is also faithful. This follows because it remains an isomorphism on base change to $\mathbb{Q}(q)$ by a special case of Theorem 8.3. □

Proofs of results in the introduction. Recall that in the introduction we were discussing the q-Schur category without 0-strings. This is the full subcategory of the q-Schur category with 0-strings generated by the objects Λ_s. The path algebra H of the category without 0-strings from (1.4) is the idempotent truncation $H = \bigoplus_{\lambda,\mu\in\Lambda_s} 1_\lambda K 1_\mu$ of the path algebra K of the category with 0-strings from (7.1). The set Λ^+ indexing the special idempotents is a subset of $\Lambda_s \subset \Lambda$. In view of this, Theorem 3 follows immediately from Theorem 7.5. Every object of the q-Schur category with 0-strings is isomorphic to an object of the q-Schur category without 0-strings. So the two path algebras K and H are Morita equivalent, and the restriction of the equivalence from Theorem 8.3 remains an equivalence. Theorem 4 follows. Finally, we explain how to establish the presentations in Theorems 1 and 2. These are similar to the ones in Theorem 6.3 and Theorem 6.1, respectively, but we have omitted the relations involving the generators ꜛ and ꜜ. Instead, the relations (1.2) and (1.3) need to be interpreted in a different way when strings labelled by 0 are present—simply omit those strings so that the splits and merges become identity morphisms. That these relations hold follows from the ones in Theorems 6.1 and 6.3 by contracting strings of thickness zero. To complete the proof of Theorem 2, one needs to show that we have a full set of relations. This follows by a straightening argument which is the same as the one used in the proof of Theorem 6.1. Then Theorem 1 follows from Theorem 2 by the same argument that was used to deduce Theorem 6.3 from Theorem 6.1.

References

[AST18] H. H. Andersen, C. Stroppel, and D. Tubbenhauer. Cellular structures using U_q-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018.

[BEAEO20] J. Brundan, I. Entova-Aizenbud, P. Etingof, and V. Ostrik. Semisimplification of the category of tilting modules for GL_n. *Adv. Math.*, 375:107331, 37, 2020.

[BK22] J. Brundan and A. Kleshchev. Odd Grassmannian bimodules and derived equivalences for spin symmetric groups. arXiv:2203.14149, 2022.

[BLM90] A. A. Beilinson, G. Lusztig, and R. D. MacPherson. A geometric setting for the quantum deformation of GL_n. *Duke Math. J.*, 61(2):655–677, 1990.

[Bru06] J. Brundan. Dual canonical bases and Kazhdan-Lusztig polynomials. *J. Algebra*, 306(1):17–46, 2006.

[BS24] J. Brundan and C. Stroppel. Semi-infinite highest weight categories. *Mem. Amer. Math. Soc.*, 293(1459):vii+152, 2024.

[CKM14] S. Cautis, J. Kamnitzer, and S. Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014.

[CPS90] E. Cline, B. Parshall, and L. Scott. Integral and graded quasi-hereditary algebras. I. *J. Algebra*, 131(1):126–160, 1990.

[DDPW08] B. Deng, J. Du, B. Parshall, and J. P. Wang. *Finite Dimensional Algebras and Quantum Groups*, volume 150 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2008.

[DJ86] R. Dipper and G. D. James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc. (3)*, 52(1):20–52, 1986.

[DJ89] R. Dipper and G. D. James. The q-Schur algebra. *Proc. London Math. Soc. (3)*, 59(1):23–50, 1989.

[DJM98] R. Dipper, G. D. James, and A. Mathas. Cyclotomic q-Schur algebras. *Math. Z.*, 229(3):385–416, 1998.

[Don93] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[Don98] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[Dot03] S. Doty. Presenting generalized q-Schur algebras. *Represent. Theory*, 7:196–213, 2003.

[DR98] J. Du and H. Rui. Based algebras and standard bases for quasi-hereditary algebras. *Trans. Amer. Math. Soc.*, 350(8):3207–3235, 1998.

[Du92a] J. Du. Canonical bases for irreducible representations of quantum GL_n. *Bull. London Math. Soc.*, 24(4):325–334, 1992.

[Du92b] J. Du. Kazhdan-Lusztig bases and isomorphism theorems for q-Schur algebras. In *Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989)*, volume 139 of *Contemp. Math.*, pages 121–140. Amer. Math. Soc., 1992.

[Du95] J. Du. Canonical bases for irreducible representations of quantum GL_n. II. *J. London Math. Soc. (2)*, 51(3):461–470, 1995.

[EL16] B. Elias and A. Lauda. Trace decategorification of the Hecke category. *J. Algebra*, 449:615–634, 2016.

[Eli15] B. Elias. Light ladders and clasp conjectures. arXiv:1510.06840, 2015.

[Fie23] P. Fiebig. Representations and binomial coefficients. arXiv:2310.08090, 2023.

[GL96] J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[Gre96] R. M. Green. A straightening formula for quantized codeterminants. *Comm. Algebra*, 24(9):2887–2913, 1996.

[Gre99] R. M. Green. The affine q-Schur algebra. *J. Algebra*, 215(2):379–411, 1999.

[Gre07] J. A. Green. *Polynomial Representations of* GL_n, volume 830 of *Lecture Notes in Mathematics*. Springer, Berlin, augmented edition, 2007.

[HS05] C. Hohlweg and M. Skandera. A note on Bruhat order and double coset representatives. arXiv:math/0511611, 2005.

[JS92] J. C. Jantzen and G. M. Seitz. On the representation theory of the symmetric groups. *Proc. London Math. Soc. (3)*, 65(3):475–504, 1992.

[KM20] A. Kleshchev and R. Muth. Based quasi-hereditary algebras. *J. Algebra*, 558:504–522, 2020.

[LT21] G. Latifi and D. Tubbenhauer. Minimal presentations of $\mathfrak{gl}_n$-web categories. arXiv:2112.12688, 2021.

[Lus10] G. Lusztig. *Introduction to Quantum Groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[Man88] Y. I. Manin. Quantum Groups and Non-commutative Geometry. Université de Montréal, 1988.

[MS19] V. Miemietz and C. Stroppel. Affine quiver Schur algebras and p-adic GL_n. *Selecta Math. (N.S.)*, 25(2):Paper No. 32, 66, 2019.

[MS21] R. Maksimau and C. Stroppel. Higher level affine Schur and Hecke algebras. *J. Pure Appl. Algebra*, 225(8):Paper No. 106442, 44, 2021.

[MSV13] M. Mackaay, M. Stošić, and P. Vaz. A diagrammatic categorification of the q-Schur algebra. *Quantum Topol.*, 4(1):1–75, 2013.

[PW91] B. Parshall and J. P. Wang. Quantum linear groups. *Mem. Amer. Math. Soc.*, 89(439):vi+157, 1991.

[Web17] B. Webster. Comparison of canonical bases for Schur and universal enveloping algebras. *Transform. Groups*, 22(3):869–883, 2017.

[Wil11] G. Williamson. Singular Soergel bimodules. *Int. Math. Res. Not. IMRN*, 20:4555–4632, 2011.

[Woo93] D. Woodcock. Straightening codeterminants. *J. Pure Appl. Algebra*, 88(1-3):317–320, 1993.

Department of Mathematics, University of Oregon, Eugene, OR, USA
URL: pages.uoregon.edu/brundan, *ORCiD*: orcid.org/0009-0009-2793-216X
Email address: brundan@uoregon.edu