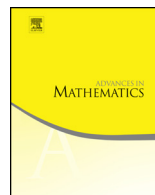




Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Positivity of Peterson Schubert calculus

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ARTICLE INFO

Article history:

Received 10 February 2022

Received in revised form 19 June 2024

Accepted 30 July 2024

Available online 21 August 2024

Communicated by L.K. Williams

MSC:

primary 14M15

secondary 14N15

Keywords:

Peterson varieties

Equivariant cohomology

Positivity

ABSTRACT

The Peterson variety is a subvariety of the flag manifold G/B equipped with an action of a one-dimensional torus, and a torus invariant paving by affine cells, called Peterson cells. We prove that the equivariant pull-backs of Schubert classes indexed by arbitrary Coxeter elements are dual (up to an intersection multiplicity) to the fundamental classes of Peterson cell closures. Dividing these classes by the intersection multiplicities yields a $\mathbb{Z}$ -basis for the equivariant cohomology of the Peterson variety. We prove several properties of this basis, including a Graham positivity property for its structure constants, and stability with respect to inclusion in a larger Peterson variety. We also find formulae for intersection multiplicities with Peterson classes. This explains geometrically, in arbitrary Lie type, recent positivity statements proved in type A by Goldin and Gorbunt.

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1. Introduction

Among the most important properties of the cohomology rings of complex flag manifolds is that they have a distinguished basis: the Schubert basis, $\{\sigma_v : v \in W\}$, indexed by the Weyl group W . The Schubert structure constants for the product in singular cohomology in this basis are *positive*, which means that all nonzero coefficients in the product expansion

$$\sigma_u \sigma_v = \sum_{w \in W} c_{uv}^w \sigma_w$$

are positive. This is a consequence of transversality properties, as the coefficients c_{uv}^w count the number of points in the intersection of three Schubert varieties translated in general (transversal) position. The study of positivity properties of these coefficients across a wide range of cohomology theories has spurred a large body of literature in algebraic geometry, representation theory, and combinatorics.

In the case of torus-equivariant cohomology, a conjecture by Peterson [43], proved by Graham [24], states that the Schubert structure constants for the torus equivariant cohomology rings of flag manifolds are polynomials in the simple roots with positive coefficients. Graham's proof relies on refined transversality techniques (see also [2]), which are expected to be useful in analyzing the equivariant cohomology ring of varieties related to flag manifolds.

Hessenberg varieties form a remarkable family of subvarieties of flag manifolds, and appear across multiple disciplines within mathematics; see [3] for a recent survey. In this paper we focus on a particular class of regular nilpotent Hessenberg varieties, namely Peterson varieties.

Peterson varieties may also be realized as a flat degeneration of certain regular semisimple Hessenberg varieties such as the permutohedral variety; see e.g., [34,35,1]. Peterson varieties share many properties with flag varieties, and provide a fertile ground for exploration. They initially appeared in the study of the quantum cohomology ring of (generalized) flag manifolds G/B in [37,43].

The purpose of this paper is to prove a positivity property of the equivariant cohomology ring of Peterson varieties, similar to that of flag varieties. This may be seen as a step towards investigating positivity and transversality properties of the (equivariant) cohomology ring of the larger family of Hessenberg varieties, little of which is known. We give next a more precise account of our results.

Let G be a complex, semisimple Lie group, let $B, B^- \subset G$ be opposite Borel subgroups, and let $T := B \cap B^-$ be the associated maximal torus. Let Δ be the set of positive simple roots corresponding to the choice of B , and let e be a principal nilpotent element contained in $\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space in $\mathfrak{g} := \text{Lie}(G)$ corresponding to the root α . Denote by W the Weyl group associated to (G, T) with length function $\ell : W \rightarrow \mathbb{N}$, and denote by $w_0 \in W$ the longest element. Let G^e be the centralizer of e in G . The Peterson variety

$$\mathbf{P} := \overline{G^e \cdot w_0 B} \hookrightarrow G/B, \quad (1)$$

is the closure of the G^e -orbit of $w_0 B$ inside the flag manifold G/B . It admits an action of a one-dimensional torus $S \subset T$ with finitely many fixed points. We denote by $\iota : \mathbf{P} \hookrightarrow G/B$ the inclusion.

In this manuscript we investigate $H_S^*(\mathbf{P}) := H_S^*(\mathbf{P}; \mathbb{Z})$, the integral S -equivariant cohomology ring of the Peterson variety. A presentation of $H_S^*(\mathbf{P}; \mathbb{Q})$ by generators and relations was given by Harada, Horiguchi and Masuda in [26].

Let $\sigma_{v_I} \in H_S^*(G/B)$ be a Schubert class indexed by some Coxeter element $v_I \in W$ for $I \subset \Delta$, let $p_I := \iota^*(\sigma_{v_I})$ be the pullback of σ_{v_I} along the inclusion $\iota : \mathbf{P} \rightarrow G/B$, and let $m(v_I)$ be the intersection multiplicity from Theorem 1.1 below. We show that $\left\{ \frac{p_I}{m(v_I)} \right\}_{I \subset \Delta}$ is a $H_S^*(pt)$ -basis of $H_S^*(\mathbf{P})$, and that this basis is dual to the equivariant Borel-Moore homology basis constituted by the fundamental classes of Peterson varieties (see Section 2). In particular, while the class p_I depends on the choice of v_I , the class $\frac{p_I}{m(v_I)}$ is independent of this choice.

In our main result we prove that the structure constants of multiplication are positive with respect to the basis $\left\{ \frac{p_I}{m(v_I)} \right\}$ in the sense of Graham [24]. This generalizes recent results in Lie type A by Goldin and Gorbunt [21], who found manifestly positive combinatorial formulae for the structure constants in question. Special cases of such formulae were found earlier by Harada and Tymoczko [29], and by Drellich [16].

We now present a more precise version of our results. For $I \subset \Delta$, let w_I be the maximal element of the Weyl group of I , and let $\mathbf{P}_I^\circ := \mathbf{P} \cap Bw_I B/B$. Tymoczko [50] and Bălibanu [7] proved that $\mathbf{P}_I^\circ \simeq \mathbb{C}^{|I|}$, and the cells $\mathbf{P}_I^\circ$ (called *Peterson cells*) form an affine paving of the Peterson variety. Consequently, the fundamental classes $[\mathbf{P}_I]_S \in H_{2|I|}^S(\mathbf{P})$, where $\mathbf{P}_I = \overline{\mathbf{P}_I^\circ}$, form a basis of $H_*^S(\mathbf{P})$ over $H_S^*(pt)$.

For $v \in W$, let $X^v := \overline{B^- v B/B}$ denote the (opposite) Schubert variety in G/B , and let $\sigma_v \in H_S^{2\ell(v)}(G/B)$ be the corresponding Poincaré dual class, satisfying the equality $\sigma_v \cap [G/B]_S = [X^v]_S$. Consider the pairing $\langle \cdot, \cdot \rangle : H_S^*(\mathbf{P}) \otimes H_*^S(\mathbf{P}) \rightarrow H_S^*(pt)$ of equivariant cohomology and equivariant homology defined by $\langle a, b \rangle = \int_{\mathbf{P}} a \cap b$; see §2. Our first result is the following (cf. Theorem 4.3 below):

Theorem 1.1 (Duality Theorem). *Let I, J be subsets of the set of simple roots Δ and let $v_I \in W$ be any Coxeter element for I . Then*

$$\langle \iota^* \sigma_{v_I}, [\mathbf{P}_J]_S \rangle = m(v_I) \delta_{I,J},$$

where $m(v_I) \in \mathbb{Z}_{>0}$ is the multiplicity of the (unique) intersection point of $X^{v_I} \cap \mathbf{P}_I$.

This follows because the varieties X^{v_I} and $\mathbf{P}_I$ intersect at a unique point, namely w_I , the longest element in the subgroup W_I determined by I . The remaining part of the proof exploits the poset structure of the affine paving by Peterson cells, along with the duality of Schubert classes in G/B .

A non-equivariant version of this theorem has appeared in the literature in type A in a recent preprint [4], and may be deduced in general Lie type from [32,31], where the intersection $X^{v_I} \cap \mathbf{P}(I)$ was analyzed. Besides working non-equivariantly, a key restriction in all these papers is that the Coxeter elements v_I are not arbitrary, but depend on a certain ordering of the simple roots. Our approach removes this restriction. In Section 7, we give algorithms to calculate the aforementioned multiplicities based on equivariant localization, and closed formulae for a particular basis; see also Theorem 1.3 below and the related discussion. A formula for $m(v_I)$ for any Coxeter element v_I was recently obtained in [25]; see Equation (19) below.

The duality theorem has several consequences. For each Coxeter element v_I for I , recall that $p_I := \iota^* \sigma_{v_I} \in H_S^{2|I|}(\mathbf{P})$. Then the classes $\left\{ \frac{p_I}{m(v_I)} \mid I \subseteq \Delta \right\}$ form a $H_S^*(pt)$ -basis of $H_S^*(\mathbf{P})$; see Corollary 4.4. By the duality theorem, the equivariant push forward $\iota_* : H_S^*(\mathbf{P}) \rightarrow H_S^*(G/B)$ is injective. Non-equivariantly, the injectivity was proved in [32].

The cocharacter h of T satisfying $\alpha(h) = 2$ for all $\alpha \in \Delta$ determines a one dimensional subtorus $S \subset T$, satisfying $\alpha|_S = \alpha'|_S$ for any $\alpha, \alpha' \in \Delta$; see Section 3.2. Consequently there is a well defined element $t \in H_S^*(pt)$ given by $t := \alpha|_S$ for $\alpha \in \Delta$.

Theorem 1.2 (Positivity). *Let I, J, K be subsets of Δ . The structure constants of multiplication, $c_{I,J}^K \in H_S^*(pt)$, given by*

$$p_I \cdot p_J = \sum_K c_{I,J}^K p_K \quad (2)$$

are polynomials in t with non-negative coefficients.

Theorem 1.2 (Theorem 5.3 below) generalizes several positivity statements to arbitrary Lie type, while providing a uniform proof in all cases. In the case where $|I| = 1$, i.e., p_I is a divisor class, a positive Monk-Chevalley formula for the structure constants $c_{I,J}^K$ was obtained by Harada and Tymoczko [29] in Lie type A, and in arbitrary Lie type by Drellich [16]. For general $c_{I,J}^K$, and in Lie type A, Goldin and Gorbutt [21] found a manifestly positive combinatorial formula for all equivariant coefficients $c_{I,J}^K$ in the expansion (2). A different combinatorial model computing these coefficients in non-equivariant cohomology was recently obtained in [4].

The proof of Theorem 1.2 relies on the duality theorem, and on positivity statements proved by Graham [24]. The structure constants of the multiplication $\sigma_u \cup \sigma_v \in H_S^*(G/B)$ are positive in the sense of Theorem 1.2. Hence it suffices to show that the coefficients $b_w^J \in H_S^*(pt)$ of the restricted classes $\iota^* \sigma_w = \sum_J b_w^J p_J$ ($w \in W$ arbitrary) satisfy the same positivity. By the duality theorem, the positivity of the coefficients b_w^J is equivalent to the positivity (in a suitable sense) of the coefficients c_I^v in the Schubert expansion

$$\iota_*[\mathbf{P}_I]_S = \sum_{v \in W} c_I^v [X_v]_S \in H_*^S(G/B), \quad (3)$$

where $[X_v]_S$ denotes the homology class of the Schubert variety $X_v := \overline{BvB/B}$ in G/B . In the non-equivariant case, this is clear. Indeed, by Kleiman transversality [33], the Schubert classes are a basis for the Chow group of G/B , and form a set of primitive generators for the cone of effective algebraic cycles. Since the Chow group is equal to $H_*(G/B)$ (cf. [20, Ex. 19.1.11]), the claim follows. Equivariantly, we deduce the positivity of c_I^v from a general positivity result of Graham [24] for expansions of fundamental classes of torus invariant varieties; cf. Theorem 5.2. A different geometric approach to positivity (in the non-equivariant setting) was pursued in [4]; see Remark 5.4 below.

Using equivariant localization, we obtain formulae for the multiplicities $m(v_I)$ in the duality theorem, and an effective algorithm to find the Schubert expansion from Equation (3); see Section 4.2. Our algorithms for the coefficients c_I^v , and for the multiplicities $m(v_I)$, rely on the restriction of the Schubert classes σ_{v_I} to the fixed points w_J in G/B (see Section 2). These *equivariant localizations* may be calculated using formulae developed by Andersen, Jantzen, and Soergel [5] and Billey [8].

Theorem 1.3.

- (a) Let I be a connected Dynkin diagram with the standard labelling, see [11], and set $v_I = s_1 s_2 \cdots s_n$. Then,

$$m(v_I) = \begin{cases} 1 & \text{if } I = A_n, \\ 2^{n-1} & \text{if } I = B_n, C_n, \\ 2^{n-2} & \text{if } I = D_n, \\ 72 = 2^3 \cdot 3^2 & \text{if } I = E_6, \end{cases} \quad m(v_I) = \begin{cases} 864 = 2^5 \cdot 3^3 & \text{if } I = E_7, \\ 51840 = 2^7 \cdot 3^4 \cdot 5 & \text{if } I = E_8, \\ 48 = 2^4 \cdot 3 & \text{if } I = F_4, \\ 6 = 2 \cdot 3 & \text{if } I = G_2. \end{cases}$$

- (b) Let $I_1, \dots, I_k$ be the connected components of a Dynkin diagram I , and let $v_1, \dots, v_k$ be any Coxeter elements for $I_1, \dots, I_k$ respectively. Then $v := v_1 \cdots v_k$ is a Coxeter element for I , and $m(v) = \prod m(v_j)$.

See Theorem 7.6 below. The factorization is related to the exponents of the Lie algebra of G , see §7 below, and also the recent paper [25]. Theorem 1.3 generalizes a result of Insko [31], who showed that when $I = A_n$, $m(s_1 \cdots s_n) = 1$. More generally, the theorem addresses [32, Question 1] by providing an explicit formula for these multiplicities; in particular, it disproves the conjecture by Insko and Tymoczko that the multiplicities are always 1 or 2 in classical Lie types. The proof of part (a) of Theorem 1.3 utilizes equivariant localization, while the proof of part (b) utilizes the stability property of Peterson classes explained below. Using parts (a) and (b) concurrently allows us to compute $m(v_I)$ for some Coxeter element v_I in each Dynkin diagram I , and hence allows us to construct the dual class $\frac{\iota^* \sigma_{v_I}}{m(v_I)}$ of any Peterson subvariety $\mathbf{P}_I \subset \mathbf{P}$.

Consider $I \subset \Delta$, a subset of the Dynkin diagram, and let G_I be a semisimple group with Dynkin diagram I . Let G_I/B_I and $\mathbf{P}(I)$ denote the flag variety and the Peterson

variety of G_I respectively, and let S_I be the one-dimensional subtorus defined analogously to S , and acting on $\mathbf{P}(I)$. There is a natural closed embedding $i : G_I/B_I \hookrightarrow G/B$, but unfortunately there may not be a morphism $S_I \rightarrow S$ which is compatible with this embedding. This leads to some technical subtleties explained in Section 6.2. The upshot is that there is an algebra isomorphism $H_S^*(pt; \mathbb{Q}) \simeq H_{S_I}^*(pt; \mathbb{Q})$, and induced maps $H_S^*(G/B; \mathbb{Q}) \xrightarrow{i^*} H_{S_I}^*(G_I/B_I; \mathbb{Q})$ and $H_*^{S_I}(G_I/B_I; \mathbb{Q}) \xrightarrow{i_*} H_*^S(G/B; \mathbb{Q})$. The stability theorem, proved in Proposition 6.5 and Theorem 6.6, is the following.

Theorem 1.4 (Stability). (a) $i(\mathbf{P}(I)) = \mathbf{P} \cap i(G_I/B_I) = \mathbf{P}_I$, as subsets of G/B .

(b) For $J \subset I$, we have $i_*([\mathbf{P}(J)]_{S_I}) = [\mathbf{P}_J]_S$, as classes in $H_*^S(\mathbf{P}; \mathbb{Q})$.

(c) Let $j : \mathbf{P}(I) \hookrightarrow \mathbf{P}$ be the restriction of $i : G_I/B_I \rightarrow G/B$. For $K \subset \Delta$, we have

$$j^*(p_K) = \begin{cases} p_K & \text{if } K \subset I, \\ 0 & \text{otherwise,} \end{cases}$$

as classes in $H_{S_I}^*(\mathbf{P}(I); \mathbb{Q})$.

Furthermore, in the non-equivariant case, the statements in (b) and (c) hold over $\mathbb{Z}$.

The proof of the stability theorem utilizes a common alternate description of the Peterson variety, namely

$$\mathbf{P} = \left\{ gB \in G/B \mid \text{Ad}(g^{-1})e \in \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha} \right\}, \quad (4)$$

where $\mathfrak{b} = \text{Lie}(B)$. In Appendix A, we take the opportunity to present a proof that the definitions (1) and (4) are equivalent, a matter of folklore implied by, and implicit in, Kostant's original work [37].

Acknowledgments: The authors thank an anonymous referee for careful reading and suggestions which helped improve the exposition of this paper. We thank Ana Bălibanu for explaining to us a proof of the irreducibility of Peterson variety, and Hiraku Abe and Tatsuya Horiguchi for explaining to us their results from [1,4,27]. LM would like to thank Sean Griffin for useful discussions and for pointing out the reference [42]. RG thanks Julianna Tymoczko for insightful conversations about Peterson varieties. The computer calculations in this paper were coded in SageMath [47]. We thank the authors of the Weyl Groups and Root Systems libraries in SageMath. RS would like to thank Camron Withrow for his help with the Sage code. During the preparation stages of this paper, LM enjoyed the hospitality and the wonderful environment of ICERM, as a participant to the special semester in ‘Combinatorial Algebraic Geometry’.

R. F. Goldin was supported in part by a National Science Foundation grant DMS-2152312. L. C. Mihalcea was supported in part by the Simons Collaboration Grant 581675 and by the National Science Foundation under grant DMS-2152294, and under grant DMS-1439786 while the author was in residence at the Institute for Computational

and Experimental Research in Mathematics in Providence, RI, during the Spring 2021 semester.

Conventions. We work over the field of complex numbers. By a variety we mean a reduced, irreducible scheme of finite type. Schemes defined as algebraic group orbits, and closures of group orbits, are always equipped with the induced reduced scheme structure; see, e.g., [46, tag 01IZ].

2. Equivariant (co)homology

Let X be a complex algebraic variety equipped with a left action of a torus T . We recall aspects of the T -equivariant homology and cohomology of X . We will use the Borel model of equivariant cohomology, and equivariant Borel-Moore homology, following the setup in Graham's paper [24]. We refer to [20, Ch 19], [19, Appendix B], [14, §2.6] for more details about cohomology and Borel-Moore homology.

Since we are working with algebraic varieties, our statements and proofs could have been written using the language of equivariant Chow groups [17]. For full results, this requires some additional properties of the operational Chow ring of *linear varieties* proved by Totaro [49]. Aware of this technicality, the reader may use the equivariant cycle map from [17] to freely swap between the Borel-Moore and Chow theories.

Fix an identification $T \cong (\mathbb{C}^*)^r$ and let $ET = (\mathbb{C}^\infty \setminus 0)^r$ be the universal T -bundle with classifying space $BT = (\mathbb{P}^\infty)^r$. The product $ET \times X$ has a right T -action given by $(e, y).t := (et, t^{-1}y)$. The action is free, and the orbit space $X_T := (ET \times X)/T$ is called the Borel mixing space of X . The universal T -bundle $ET \rightarrow BT$ admits finite dimensional approximations $ET_n \rightarrow BT_n$, where $ET_n = (\mathbb{C}^{n+1} \setminus 0)^r$ and $BT_n = (\mathbb{P}^n)^r$. These induce finite dimensional approximations of the Borel mixing space $X_{T,n} := (ET_n \times X)/T$, and inclusions $X_{T,n_1} \subset X_{T,n_2}$ for $n_1 < n_2$.

We define the equivariant cohomology ring by $H_T^*(X) := H^*(X_T)$; note that we have $H_T^i(X) = H^i(X_{T,n})$ for sufficiently large n . The equivariant Borel-Moore homology groups are defined via a limiting property,

$$H_i^T(X) := H_{i+2nr}^{BM}(X_{T,n}), \quad \text{for } n \gg 0$$

where the right hand side is the ordinary Borel-Moore homology. If $V \subset X$ is a closed T -stable subvariety of X of complex dimension d , its fundamental class $[V]_T$ is an element in $H_{2d}^T(X)$. The cap product gives the (total) equivariant homology $H_*^T(X) = \bigoplus_i H_i^T(X)$ a graded module structure over the equivariant cohomology ring $H_T^*(X)$.

If $X = pt$, then $H_T^*(pt) = H^*(BT)$ is naturally identified with the symmetric algebra $\text{Sym } \mathfrak{X}(T)$ of the character group $\mathfrak{X}(T) := \text{Hom}(T, \mathbb{C}^*)$ of T (written additively). For any map $S \rightarrow T$ of tori, we have a natural map of algebras $H_T^*(X) \rightarrow H_S^*(X)$, compatible with the algebra map $H_T^*(pt) \rightarrow H_S^*(pt)$ induced by $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$. Taking S to be the trivial subgroup in T , we obtain a ring homomorphism $H_T^*(X) \rightarrow H^*(X)$. (One can

show that this map is surjective for spaces with affine pavings in the sense of Lemma 2.1 below; we will not need this fact.)

The morphism $X_T \rightarrow BT$ that projects onto the first factor gives the equivariant cohomology $H_T^*(X)$ the structure of a graded algebra over $H_T^*(pt)$. In addition, the cap product $\cap$ endows the equivariant homology $H_*^T(X)$ with a graded module structure over $H_T^*(X)$. Equivalently, there is a compatibility of cap and cup products given by $(a \cup b) \cap c = a \cap (b \cap c)$, for $a, b \in H_T^*(X)$, $c \in H_*^T(X)$.

Each irreducible, T -stable, closed subvariety $Z \subset X$ of complex dimension k has a fundamental class $[Z]_T \in H_{2k}^T(X)$. If X is smooth and irreducible, then there exists a unique (Poincaré dual) class $\eta_Z \in H_T^{2(\dim X - k)}(X)$ such that $\eta_Z \cap [X]_T = [Z]_T$.

Any T -equivariant morphism of T -varieties $f : X \rightarrow Y$ induces a degree preserving pull-back morphism of $H_T^*(pt)$ -algebras $f^* : H_T^i(Y) \rightarrow H_T^i(X)$. For a point $x \in X$ fixed by the T action, the inclusion $\iota_x : \{x\} \rightarrow X$ induces a *localization* map $\iota_x^* : H_T^*(X) \rightarrow H_T^*(\{x\}) = H_T^*(pt)$.

If f is proper then there is a push-forward $f_* : H_i^T(X) \rightarrow H_i^T(Y)$, defined as follows. Let $Z \subset X$ be closed, irreducible and T -stable. Then $f_*[Z]_T = d_Z[f(Z)]_T$ if $\dim f(Z) = \dim Z$, where d_Z is the generic degree of the restriction $f : Z \rightarrow f(Z)$, and $f_*[Z]_T = 0$ if $\dim f(Z) < \dim Z$. The push-forward and pull-back are related by the usual projection formula $f_*(f^*(a) \cap c) = a \cap f_*(c)$.

An important particular case is when X is complete, thus $f : X \rightarrow pt$ is proper. For a homology class $c \in H_*^T(X)$, we denote by $\int_X c$ the class $f_*(c) \in H_*^T(pt)$.¹ Recall that the equivariant homology $H_*^T(pt)$ of a point is a free $H_T^*(pt)$ -module with basis $[pt]_T$. Therefore we identify $H_T^*(pt) = H_*^T(pt)$ via the map $a \mapsto a \cap [pt]_T$. Then we may define a pairing,

$$\langle \cdot, \cdot \rangle : H_T^*(X) \otimes_{H_T^*(pt)} H_*^T(X) \rightarrow H_T^*(pt); \quad \langle a, c \rangle := \int_X a \cap c. \quad (5)$$

We often abuse notation and for a *cohomology* class $a \in H_T^*(X)$ we write $\int_X a$ to mean $\int_X (a \cap [X]_T)$.

Following [20, Ex 1.9.1] (see also [24]) we say that a T -variety X admits a *T -stable affine paving* if it admits a filtration $X := X_n \supset X_{n-1} \supset \dots$ by closed T -stable subvarieties such that each $X_i \setminus X_{i-1}$ is a finite disjoint union of T -invariant varieties $U_{i,j}$ isomorphic to affine spaces $\mathbb{A}^i$. The following has been proved by Graham; see [24, Prop 2.1].

Lemma 2.1. *Assume X admits a T -stable affine paving, with cells $U_{i,j}$.*

(a) *The equivariant homology $H_*^T(X)$ is a free $H_T^*(pt)$ -module with basis $\{\overline{[U_{i,j}]_T}\}$.*

¹ We note that $f_*(c)$ agrees with “integration over the fiber” when X is smooth, justifying the notation.

- (b) If X is complete, the pairing from Equation (5) is perfect, and so we may identify $H_T^*(X) = \text{Hom}_{H_T^*(pt)}(H_*^T(X), H_T^*(pt))$.

3. Flag manifolds and Peterson varieties

In this section we recall some basic definitions about flag manifolds, Schubert varieties, and Peterson varieties. We mostly follow the setup in [50] and [7], from which we will need several important results.

3.1. Flag manifolds and Schubert varieties

Fix a complex semisimple Lie group G , a Borel subgroup $B \subset G$, $B^- \subset G$ an opposite Borel subgroup, and let $T := B \cap B^-$ be a maximal torus. Denote by Δ the system of simple positive roots associated to (G, B, T) and by $\Phi_\Delta^+ \subset \Phi_\Delta$ the set of positive roots included in the set of all roots. The Weyl group $W := N_G(T)/T$ is generated by simple reflections $s_i := s_{\alpha_i}$ where $\alpha_i \in \Delta$. Let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the length function and w_0 the longest element in W . Then $B^- = w_0 B w_0$.

Any subset $I \subset \Delta$ determines a Weyl subgroup $W_I := \langle s_i : \alpha_i \in I \rangle$ and a corresponding standard parabolic subgroup P_I . We denote by w_I the longest element of W_I . The flag manifold G/B is a smooth algebraic variety of complex dimension $\ell(w_0)$ with a transitive action of G given by left multiplication. The flag manifold has a stratification into finitely many B -orbits, respectively B^- -orbits, called the *Schubert cells*: $X_w^\circ := BwB/B \simeq \mathbb{C}^{\ell(w)}$ and $X^{w,\circ} := B^-wB/B \simeq \mathbb{C}^{\ell(w_0w)}$; we have

$$G/B = \bigsqcup_{w \in W} X_w^\circ = \bigsqcup_{w \in W} X^{w,\circ}. \quad (6)$$

The closures $X_w := \overline{X_w^\circ}$ and $X^w := \overline{X^{w,\circ}}$ are called *Schubert varieties* and *opposite Schubert varieties*, respectively. The *Bruhat order* is a partial order on W characterized by inclusions of Schubert varieties and opposite Schubert varieties. In particular, $X_v \subset X_w$ if and only if $v \leq w$, and $X^w \subset X^v$ if and only if $v \leq w$. Following Lemma 2.1, the homology classes $\{[X_v]_T \mid v \leq w\}$ form a basis of $H_*^T(X_w)$, while $\{[X^v]_T \mid w \leq v\}$ form a basis of $H_*^T(X^w)$.

The cohomology classes $\sigma_v \in H_T^*(X)$ Poincaré dual to the $[X^v]_T$, i.e. characterized by the equation $\sigma_v \cap [G/B]_T = [X^v]_T$, are called *Schubert classes*. Note that Lemma 2.1 also implies $\{\sigma_v \mid v \in W\}$ is a basis of $H_T^*(G/B)$ as a module over $H_T^*(pt)$. Under the pairing in Equation (5), the basis $\{\sigma_v \mid v \in W\}$ is dual to the basis $\{[X_v]_T \mid v \in W\}$, i.e., we have $\langle \sigma_v, [X_w]_T \rangle = \delta_{v,w}$.

3.2. The Peterson variety and Peterson cells

The Peterson variety appeared in the unpublished work of Peterson [43], in relation to the quantum cohomology of G/B ; we refer the reader to [37,45] for details.

We recall the definition of the Peterson variety. Let $\mathfrak{g} := \text{Lie}(G)$, $\mathfrak{h} := \text{Lie}(T)$, and consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{\Delta}} \mathfrak{g}_{\alpha}.$$

For each simple root $\alpha \in \Delta$, choose a root vector $e_{\alpha} \in \mathfrak{g}_{\alpha}$, and let

$$e := \sum_{\alpha \in \Delta} e_{\alpha}.$$

The element e is a regular nilpotent element in the Lie algebra $\mathfrak{b}$ of B ; see [36] or [15, Thm 4.1.6]. Denote by $G^e \subset G$ the stabilizer of e for the adjoint action of G on $\mathfrak{g}$. We have $G^e = (G^e)^{\circ} \times Z(G)$, where $(G^e)^{\circ}$ is the identity component of G^e , and $Z(G)$ the center of G . The identity component $(G^e)^{\circ}$ is a subgroup of the unipotent radical U of B , isomorphic to the affine variety $\mathbb{C}^n$, where $n := |\Delta|$ is the number of simple roots, i.e., the rank of G , cf. [36, Cor 5.3]. For instance, if $G = \text{SL}_n(\mathbb{C})$, then $(G^e)^{\circ}$ is the subgroup of upper triangular unipotent matrices with equal entries along each superdiagonal. The Peterson variety is defined by

$$\mathbf{P} := \overline{G^e \cdot w_0 B} \subset G/B. \quad (7)$$

This is an irreducible subvariety of G/B of dimension $|\Delta|$, singular in general.

For any $\omega \in \mathfrak{h}$ contained in the coroot lattice, the map $\varphi_{\omega} : \mathbb{C} \rightarrow \mathfrak{h}$ defined by $\varphi_{\omega}(z) = z\omega$ lifts to a cocharacter $\exp(\varphi_{\omega}) : \mathbb{C}^* \rightarrow T$. (Here the differential of $\exp(\varphi_{\omega})$ is equal to φ_{ω} . In complex differential geometry, the map $\exp(\varphi_{\omega})$ intertwines with the (non-algebraic) exponential maps $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ and $\exp : \mathfrak{h} \rightarrow T$; the cocharacter $\exp(\varphi_{\omega})$ is itself an algebraic map.) This identifies the coroot lattice of $\mathfrak{h}$ with a subset of the cocharacters of T . See, e.g., [23, Ch. 3, Prop. 1.15] (in the algebraic setting), or [18, p. 373-4] (in the manifold setting).

We take $h = \sum_{\alpha \in \Phi_{\Delta}^+} \alpha^{\vee}$ to be the sum of the positive coroots, and denote by $S \subset T$ the image of the cocharacter corresponding to h . Following [11, Ch 6, Prop 29], we have $\alpha(h) = 2$ for all $\alpha \in \Delta$, because h is equal to twice the sum of the fundamental coweights. In particular, it follows that $\alpha|S = \alpha'|S$ for any $\alpha, \alpha' \in \Delta$. We set $t := \alpha|S \in \mathfrak{X}(S) \subset H_S^*(pt)$.

Example 3.1. Consider $G = \text{SL}_n$, and let $T \subset G$ be the set of diagonal matrices:

$$T = \left\{ \left(\begin{array}{ccc} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_n \end{array} \right) \middle| z_1 \cdots z_n = 1 \right\}.$$

The α_i , $1 \leq i \leq n-1$, given by $\alpha_i(\text{diag}(z_1, \dots, z_n)) \mapsto z_i/z_{i+1}$, form a set of simple roots. The coroot h corresponds to the one-dimensional subtorus S given by

$$S = \left\{ \left(\begin{array}{cccc} z^{n-1} & 0 & 0 & 0 \\ 0 & z^{n-3} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & z^{-n+1} \end{array} \right) \middle| z \in \mathbb{C}^* \right\}.$$

The character t of S is the map given by $\text{diag}(z^{n-1}, z^{n-3}, \dots, z^{-n+1}) \mapsto z^2$.

Remark 3.2. The element t need not be a generator of the ring $H_S^*(pt)$. For example, if $G = SL_2$, we have

$$S = \left\{ \left(\begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right) \middle| z \in \mathbb{C}^* \right\}, \quad t : \left(\begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right) \mapsto z^2.$$

The character group $\mathfrak{X}(S)$ is generated by $t/2$, which is the map

$$\left(\begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right) \mapsto z.$$

However, we always have either $H_S^*(pt) = \mathbb{Z}[t]$, or $H_S^*(pt) = \mathbb{Z}[t/2]$.

Since $[h, e_\alpha] = 2e_\alpha$ for each simple root α , we have $[h, e] = 2e$, from which we observe that S normalizes G^e , cf. [37, Theorem 10], resulting in an action of the semidirect product $S \ltimes G^e$ on the Peterson variety.

The following was proved in classical types by Tymoczko [50, Thm 4.3] and generalized to all Lie types by Precup [44].

Proposition 3.3. *For $I \subset \Delta$, let w_I denote the longest element in the Weyl subgroup W_I .*

- (a) *The intersection $\mathbf{P} \cap Bw_I B/B$ is nonempty if and only if $w = w_I$ for some subset $I \subset \Delta$.*
- (b) *The set theoretic intersection $\mathbf{P}_I^\circ := \mathbf{P} \cap Bw_I B/B$ is an affine space of dimension $|I|$. In particular, its closure $\mathbf{P}_I$ is an irreducible subvariety of X_{w_I} .*

Some of the details proving part (b) are implicit in [7]. We take the opportunity to make these details explicit in Proposition A.1 below. We will refer to $\mathbf{P}_I^\circ$ as a *Peterson cell*; its closure $\mathbf{P}_I \subset X_{w_I}$ is an irreducible variety, and the Schubert cell decomposition of Schubert varieties yields an affine paving

$$\mathbf{P}_I = \bigsqcup_{J \subset I} \mathbf{P}_J^\circ.$$

Following [24, Prop 2.1(a)], the classes $\{[\mathbf{P}_I]_S \mid I \subset \Delta\}$ form a basis of $H_*^S(\mathbf{P})$. Observe that $S \subset T$ is a regular subtorus, thus the fixed point loci for S and T in G/B coincide, i.e., $(G/B)^T = (G/B)^S$; see e.g. [30, §24.2, §24.3]. It follows that

$$\mathbf{P}^S = (G/B)^S \cap \mathbf{P} = (G/B)^T \cap \mathbf{P} = \{w_I : I \subset \Delta\},$$

where we utilize the usual identification $(G/B)^T = W$.

For $I \subset \Delta$, an element $v \in W$ is called a *Coxeter element* for I if $v = s_{\alpha_1} \cdots s_{\alpha_k}$ for some enumeration $\alpha_1, \dots, \alpha_k$ of I . Recall the following result, cf. [32, Lemma 7]:

Proposition 3.4. *Let v_I be a Coxeter element for some subset $I \subset \Delta$. Then the intersection $X^{v_I} \cap \mathbf{P}_I$ is the single (possibly non-reduced) point w_I .*

Proof. The intersection $Y := X^{v_I} \cap \mathbf{P}_I$ is proper and S -stable. Any fixed point in $Y^S \subset \mathbf{P}_I^S$ is of the form w_J , for some $J \subset I$. On the other hand, since $w_J \in X^{v_I}$, we have $w_J \geq v_I$. Since v_I is a Coxeter element for W_I , $I \subset J$, and so $I = J$. Thus Y contains a unique S -fixed point; hence by [9, Prop 13.5], we have $Y = \{w_I\}$. $\square$

Corollary 3.5. *Let $\eta_I \in H_S^*(G/B)$ be the Poincaré dual of $[\mathbf{P}_I]_S \in H_*^S(G/B)$, v_I a Coxeter element for I , and τ_{w_I} the Poincaré dual of the point class $[w_I]_S$. Then*

$$\sigma_{v_I} \cup \eta_I = m(v_I) \tau_{w_I} \quad \text{and} \quad \int_{G/B} \sigma_{v_I} \cup \eta_I = m(v_I),$$

where $m(v_I)$ is the multiplicity of w_I in the intersection $X^{v_I} \cap \mathbf{P}_I$.

Proof. Observe from Lemma 2.1 that $H_S^*(G/B)$ is torsion-free, and hence the localization map $H_S^*(G/B) \rightarrow \bigoplus_{w \in W} H_S^*(w)$ is injective (over $\mathbb{Z}$); see [22, Cor 1.3.2, Thm 1.6.2] and [28, Thm 3.1]. By Proposition 3.4, the only potentially non-zero localization of $\sigma_{v_I} \cup \eta_I$ is at w_I , and therefore $\sigma_{v_I} \cup \eta_I = m(v_I) \tau_{w_I}$ for some integer $m(v_I)$. Under the specialization $H_S^*(G/B) \rightarrow H^*(G/B)$, the class τ_{w_I} maps to $1 \in H^*(G/B)$. It now follows from [19, Eq (31)] that $m(v_I)$ is the multiplicity of the intersection $X^{v_I} \cap \mathbf{P}_I$. $\square$

In Section 7, we provide a formula for $m(v_I)$ based on equivariant localization, and compute the value of $m(v_I)$ for certain Coxeter elements v_I .

4. Poincaré duality and consequences

Let G be a complex semisimple group, and $\iota : \mathbf{P} \hookrightarrow G/B$ the corresponding Peterson variety, as in Section 3. In Theorem 4.3, we construct a basis $\{p_I\}_{I \subset \Delta}$ of $H_S^*(\mathbf{P})$ dual (up to scaling) to the basis $\{[\mathbf{P}_I]_S\}_{I \subset \Delta}$ of $H_*^S(\mathbf{P})$. Theorem 4.3 relates the Schubert expansion of a Peterson class $[\mathbf{P}_I]_S$ to the expansion in the $\{p_I\}$ basis of the pull-backs $\iota^* \sigma_w$; the latter can be computed using equivariant localization and Gaussian elimination. We sketch an example in Section 4.2.

4.1. Peterson classes and duality

Lemma 4.1. *Let $I \subset \Delta$, and consider the expansion*

$$\iota_*[\mathbf{P}_I]_S = \sum_{v \in W} c_I^v [X_v]_S \in H_*^S(G/B).$$

Then $c_I^v = 0$ unless $v \leq w_I$.

Proof. By Lemma 2.1, the equivariant homology $H_*^S(X_{w_I})$ has a $H_*^S(pt)$ -basis given by the fundamental classes $[X_v]_S$, where $v \leq w_I$. Since $\mathbf{P}_I$ is a subvariety of X_{w_I} , we have $\iota_*[\mathbf{P}_I]_S = \sum_{v \leq w_I} c_I^v [X_v]_S$, for some $c_I^v \in H_*^S(pt)$. $\square$

Lemma 4.2. *Let $I \subset \Delta$, and consider the expansion*

$$\iota_*[\mathbf{P}_I]_S = \sum_{v \in W} c_I^v [X_v]_S \in H_*^S(G/B).$$

If v is a Coxeter element for $J \neq I$, then $c_I^v = 0$.

Proof. Suppose v is a Coxeter element for some subset $J \subset \Delta$ for which $c_I^v \neq 0$. Following Lemma 4.1, we have $v \leq w_I$, hence $J \subset I$. On the other hand, since the expansion is homogeneous, we have $|J| = \ell(v) \geq \dim \mathbf{P}_I = |I|$, and hence $J = I$. $\square$

Theorem 4.3 (Duality Theorem). *Let I, J be subsets of the set of simple roots Δ , and let v_I be a Coxeter element for I . We have*

$$\langle \iota^* \sigma_{v_I}, [\mathbf{P}_J]_S \rangle = m(v_I) \delta_{I,J},$$

where $m(v_I)$ is the multiplicity of the (unique) intersection point of $X^{v_I} \cap \mathbf{P}_I$. In particular, $m(v_I)$ is a positive integer.

Proof. Consider the Schubert expansion $\iota_*[\mathbf{P}_J]_S = \sum c_J^v [X_v]_S$. Then

$$\langle \iota^* \sigma_{v_I}, [\mathbf{P}_J]_S \rangle = \langle \sigma_{v_I}, \iota_*[\mathbf{P}_J]_S \rangle = c_J^{v_I},$$

since the set $\{\sigma_v\}_{v \in W}$ forms a dual basis to the fundamental classes $\{[X_v]_S\}_{v \in W}$. It follows from Lemma 4.2 that $c_J^{v_I} = 0$ for $I \neq J$. For $J = I$, Corollary 3.5 implies

$$c_I^{v_I} = \langle \sigma_{v_I}, \iota_*[\mathbf{P}_I]_S \rangle = \int_X \sigma_{v_I} \cup \eta_I = m(v_I) > 0.$$

Finally, $m(v_I) \in \mathbb{Z}_+$ because the pairing (5) has values in integral cohomology. $\square$

We record the following consequence of the Duality theorem.

Corollary 4.4. *For each $I \subset \Delta$, fix a Coxeter element v_I , and set $p_I := \iota^* \sigma_{v_I} \in H_S^*(\mathbf{P})$.*

- (a) *The classes $\left\{ \frac{p_I}{m(v_I)} \in H_S^*(\mathbf{P}) \mid I \subset \Delta \right\}$ form a $H_S^*(pt)$ -basis of $H_S^*(\mathbf{P})$.*
 (b) *The map $\iota_* : H_*^S(\mathbf{P}) \rightarrow H_*^S(G/B)$ is injective.*

Proof. By Theorem 4.3, the classes $\frac{p_I}{m(v_I)}$ are dual to the classes $[\mathbf{P}_I]_S$, and part (a) follows from Lemma 2.1. For part (b), observe that the pairing

$$\langle \sigma_{v_J}, \iota_* [\mathbf{P}_I]_S \rangle = m(v_I) \delta_{I,J},$$

along with the linear independence of the σ_{v_J} in $H_S^*(G/B)$, implies that the $\iota_* [\mathbf{P}_I]_S$ are linearly independent. It follows that the map $\iota_* : H_*^S(\mathbf{P}) \rightarrow H_*^S(G/B)$ is injective. $\square$

Remark 4.5. Part (a) of Corollary 4.4 was proved in various generalities, and for particular choices of Coxeter elements v_I , in [16,32,4]. The non-equivariant version of part (b) was proved in [32, Thm 2].

We also record the following immediate corollary, which will be utilized in the proof of the positivity statement Theorem 5.2.

Corollary 4.6. *For each $I \subset \Delta$, fix a Coxeter element v_I , and set $p_I := \iota^* \sigma_{v_I} \in H_S^*(\mathbf{P})$. Consider the expansions*

$$\iota^* \sigma_w = \sum_{J \subset \Delta} b_w^J p_J, \quad \iota_* [\mathbf{P}_I]_S = \sum_{u \in W} c_I^u [X_u]_S.$$

Then $c_I^u = m(v_I) b_u^I$ for all u , where $m(v_I) > 0$ is the coefficient from the Duality Theorem 4.3.

Proof. Using Theorem 4.3 and the equality $\langle \sigma_v, [X_u]_S \rangle_{G/B} = \delta_{u,v}$, we calculate,

$$c_I^u = \langle \sigma_u, \iota_* [\mathbf{P}_I]_S \rangle_{G/B} = \langle \iota^* \sigma_u, [\mathbf{P}_I]_S \rangle_{\mathbf{P}} = m(v_I) b_u^I.$$

Here the first equality follows from the definition of c_I^u , the second from the projection formula, and the third from Theorem 4.3 together with the definition of b_u^I . $\square$

4.2. Schubert expansion of the Peterson classes

In their study of certain regular Hessenberg varieties, Abe, Fujita and Zeng [1] found a beautiful closed formula for the non-equivariant Schubert expansions of the fundamental

classes of these varieties. For the Peterson varieties discussed here, their formula states that

$$\iota_*[\mathbf{P}] = \prod_{\alpha \in \Phi_{\Delta}^+ \setminus \Delta} c_1(G \times^B \mathbb{C}_{-\alpha}) \cap [G/B] \in H_*(G/B);$$

see Cor. 3.9 in [1]. However, since the line bundles $G \times^B \mathbb{C}_{-\alpha}$ are not globally generated, this formula involves cancellations. A manifestly positive formula was recently found by Nadeau and Tewari [42], and further investigated by Horiguchi [27], in relation to *mixed Eulerian numbers*. The origins of this approach lie in the realization of the Peterson variety as a flat degeneration of a smooth projective toric variety, called the (generalized) *permutohedral variety*; see [1,42]. The permutohedral variety is a regular semisimple Hessenberg variety; its cohomology ring has been classically studied e.g. by Klyachko [34,35]. In this section we present a different algorithm, which calculates the equivariant Schubert expansion of $\iota_*[\mathbf{P}]_S$. The algorithm is based on Corollary 4.6, and it depends on the multiplicities $m(v_I)$ for some choice of Coxeter elements v_I , $I \subset \Delta$. The values $m(v_I)$ for a particular such choice are computed in Theorem 7.6. It would be interesting to utilize this algorithm to extend the formulae from [1,42] to the equivariant setting; this will be left for elsewhere.

Proposition 4.7. *Fix Coxeter elements v_I for each subset $I \subset \Delta$, and consider the matrices,*

$$A_{u,I} = \iota_{w_I}^* \sigma_u, \quad C_{I,J} = \iota_{w_J}^* \sigma_{v_I}, \quad M_{I,J} = m(v_I) \delta_{I,J}.$$

Here A is a $|W| \times 2^{|\Delta|}$ matrix, and C and M are $2^{|\Delta|} \times 2^{|\Delta|}$ matrices. The fundamental classes $[\mathbf{P}_I]_S$ and $[X_u]_S$ are related by the matrix equation,

$$([\mathbf{P}_I]_S)_{I \subset \Delta} = (AC^{-1}M)^T ([X_u]_S)_{u \in W}. \quad (8)$$

Proof. Consider the commutative diagram,

$$\begin{array}{ccc} H_S^*(G/B) & \xrightarrow{\oplus \iota_u^*} & \bigoplus_{u \in W} H_S^*(u) \\ \downarrow \iota^* & & \downarrow \\ H_S^*(\mathbf{P}) & \xrightarrow{\oplus \iota_{w_I}^*} & \bigoplus_{I \subset \Delta} H_S^*(w_I). \end{array} \quad (9)$$

Let $\mathcal{Q}$ be the fraction field of the integral domain $H_S^*(pt)$, and let $R_{\mathcal{Q}} := R \otimes_{H_S^*(pt)} \mathcal{Q}$ for any $H_S^*(pt)$ -module R . The map $H_S^*(\mathbf{P}) \rightarrow \bigoplus_{I \subset \Delta} H_S^*(w_I)$ induces an isomorphism $H_S^*(\mathbf{P})_{\mathcal{Q}} \xrightarrow{\sim} \bigoplus_{I \subset \Delta} H_S^*(w_I)_{\mathcal{Q}}$; see [22, Cor 1.3.2, Thm 1.6.2, Thm 6.3]. Observe that $H_S^*(\mathbf{P})$

is torsion-free, and is naturally identified as a lattice in the $\mathcal{Q}$ -vector space $H_S^*(\mathbf{P})_{\mathcal{Q}}$. Let τ_I denote a generator of $H_S^*(w_I)$, and consider the column vectors,

$$\begin{aligned}\sigma &= (\iota^* \sigma_u)_{u \in W}, & \tau &= (\tau_I)_{I \subset \Delta}, \\ \mathbf{p} &= (p_I)_{I \subset \Delta}, & \mathbf{q} &= \left(\frac{p_I}{m(v_I)} \right)_{I \subset \Delta}.\end{aligned}$$

We have the following equalities in $H_S^*(\mathbf{P})_{\mathcal{Q}}$:

$$\mathbf{p} = M\mathbf{q}, \quad \sigma = A\tau, \quad \mathbf{p} = C\tau. \quad (10)$$

The matrix C is invertible since both $\{p_I\}_{I \subset \Delta}$ and $\{\tau_I\}_{I \subset \Delta}$ are bases for $H_S^*(\mathbf{P})_{\mathcal{Q}}$. We deduce that $\sigma = AC^{-1}M\mathbf{q}$. Equation (8) now follows from Corollary 4.6. $\square$

Remark 4.8. The coefficients $A_{w,I}$ and $C_{I,J}$ in Proposition 4.7 can be computed by composing the localization formula for the T -equivariant Schubert classes (cf. [5,8]) with the restriction map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$ defined by $\lambda \mapsto \lambda|_S$.

Remark 4.9. The invertibility of the matrix C in Proposition 4.7 can be directly deduced from the observation that $\iota_{w,J}^* \sigma_{v_I} \neq 0$ if and only if $I \subset J$, and hence C is upper triangular with respect to the partial order $I \leq J \iff I \subset J$.

Example 4.10. We use Proposition 4.7 to compute the Schubert expansion of $[\mathbf{P}]_S$ in the case $\Delta = B_2$, with $v_{\Delta} = s_1 s_2$. Set

$$p_{\phi} = \iota^* \sigma_{id}, \quad p_{\{1\}} = \iota^* \sigma_1, \quad p_{\{2\}} = \iota^* \sigma_2, \quad p_{\{1,2\}} = \iota^* \sigma_{12}.$$

Composing the localization formula for Schubert classes (cf. [5,8]) with the restriction map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$, we obtain the S -equivariant localizations of the Schubert classes:

$$\begin{pmatrix} \iota^* \sigma_{id} \\ \iota^* \sigma_1 \\ \iota^* \sigma_2 \\ \iota^* \sigma_{12} \\ \iota^* \sigma_{21} \\ \iota^* \sigma_{121} \\ \iota^* \sigma_{212} \\ \iota^* \sigma_{1212} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & t & 0 & 4t \\ 0 & 0 & t & 3t \\ 0 & 0 & 0 & 6t^2 \\ 0 & 0 & 0 & 6t^2 \\ 0 & 0 & 0 & 6t^3 \\ 0 & 0 & 0 & 6t^3 \\ 0 & 0 & 0 & 6t^4 \end{pmatrix} \begin{pmatrix} \tau_{\phi} \\ \tau_1 \\ \tau_2 \\ \tau_{12} \end{pmatrix}. \quad (11)$$

The 8×4 matrix in Equation (11) corresponds to the matrix A in Equation (10), and the matrix C is precisely its top 4×4 submatrix. The multiplicities $m(v_I)$ are computed in Theorem 7.6; we have $m(v_I) = 1$ for all $I \subsetneq B_2$ and $m(v_{\Delta}) = 2$, i.e.,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Applying Equation (8), we obtain

$$\begin{pmatrix} \iota_*[\mathbf{P}_\phi]_S \\ \iota_*[\mathbf{P}_{\{1\}}]_S \\ \iota_*[\mathbf{P}_{\{2\}}]_S \\ \iota_*[\mathbf{P}]_S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2t & 2t & 2t^2 \end{pmatrix} \begin{pmatrix} \iota_*[X_{id}]_S \\ \iota_*[X_1]_S \\ \iota_*[X_2]_S \\ \iota_*[X_{12}]_S \\ \iota_*[X_{21}]_S \\ \iota_*[X_{121}]_S \\ \iota_*[X_{212}]_S \\ \iota_*[X_{1212}]_S \end{pmatrix}.$$

In particular, we have $\iota_*[\mathbf{P}]_S = 2[X_{12}]_S + 2[X_{21}]_S + 2t[X_{121}]_S + 2t[X_{212}]_S + 2t^2[X_{1212}]_S$.

5. Positivity

We recall a theorem of Graham [24, Thm. 3.2], which plays a key role in the proof of our positivity results, Theorems 5.2 and 5.3.

Theorem 5.1. *Let B' be a connected solvable group with unipotent radical N' , and let $T' \subset B'$ be a maximal torus, so that $B' = T'N'$. Let $\alpha_1, \dots, \alpha_d$ be the weights of T' acting on $\text{Lie}(N')$. Let X be a scheme with a B' -action, and Y a T' -stable subvariety of X . Then there exist B' -stable subvarieties $D_1, \dots, D_k$ of X such that in the equivariant homology $H_*^{T'}(X)$,*

$$[Y]_{T'} = \sum f_i [D_i]_{T'},$$

where each $f_i \in H_{T'}^*(pt)$ is a linear combination of monomials in $\alpha_1, \dots, \alpha_d$ with non-negative integer coefficients.

Theorem 5.2. *Let I be a subset of Δ , let $\iota : \mathbf{P} \hookrightarrow G/B$ be the inclusion, and consider the Schubert expansion,*

$$\iota_*[\mathbf{P}_I]_S = \sum_{v \in W} c_I^v [X_v]_S.$$

Then $c_I^v \in H_S^*(pt)$ is a polynomial in t with non-negative coefficients.

Proof. We apply Graham's positivity theorem to the following situation: $Y = \mathbf{P}_I \subset X = G/B$, $T' = S$, and $B' = SU$, where U is the unipotent radical of B . We have $U \subset B' \subset B$, and since the U -orbits and B -orbits in G/B coincide, the B' -orbits in G/B are precisely the Schubert cells X_v° .

Observe that the restriction map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$ is given by $\alpha \mapsto ht(\alpha)t$ for $\alpha \in \Phi_{\Delta}^{+}$, where $ht(\alpha)$ is the height of α . It follows that the weights for the S -action on $Lie(U)$ are positive integer multiples of t . It follows from Theorem 5.1 that each $c_I^v \in H_S^*(pt)$ is a polynomial in t with non-negative coefficients. $\square$

Theorem 5.3. *Let $p_I := \iota^* \sigma_{v_I} \in H_S^*(\mathbf{P})$ for some Coxeter element v_I , and consider the multiplication in $H_S^*(\mathbf{P})$,*

$$p_I \cdot p_J = \sum_{K \subset \Delta} c_{I,J}^K p_K.$$

The structure constants $c_{I,J}^K \in H_S^(pt)$ are polynomials in t with non-negative coefficients.*

Proof. By Graham's equivariant positivity theorem [24, Prop 2.2, Thm 3.2], the structure constants $c_{u,v}^w$ in the expansion

$$\sigma_u \cdot \sigma_v = \sum c_{u,v}^w \sigma_w \in H_T^*(G/B)$$

are polynomials in the T -weights of $Lie(U)$ with non-negative coefficients. Then

$$p_I \cdot p_J = \iota^* \sigma_{v_I} \cdot \iota^* \sigma_{v_J} = \sum d_{u,v}^w \iota^* \sigma_w,$$

where $d_{u,v}^w$ is the image of $c_{u,v}^w$ under the restriction map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$; in particular, $d_{u,v}^w$ is a polynomial in t with non-negative coefficients. The result now follows from Theorem 5.2 and Corollary 4.6, since the classes $\iota^* \sigma_w$ expand into the classes p_K with coefficients having the same positivity property as the $d_{u,v}^w$. $\square$

Remark 5.4. In the recent preprint [21], Goldin and Gorbutt found a manifestly positive formula for the coefficients $c_{I,J}^K$, in Lie type A, and for a particular choice of the Coxeter elements v_I . While this paper was in preparation, a different combinatorial model, in the non-equivariant cohomology, appeared in the preprint [4] by Abe, Horiguchi, Kuwata and Zeng. They also provide a geometric proof of positivity (cf. Prop. 4.15 in [4]), which utilizes a ‘Giambelli formula’, writing the classes p_I in terms of products of pull-backs of the (effective) line bundles $GL_n \times^B \mathbb{C}_{-\omega_i}$ associated to the fundamental weights ω_i . This argument should extend to arbitrary Lie type if one utilizes instead the more general equivariant Giambelli formulae obtained by Drellich [16], specialized to ordinary cohomology.

6. Stability properties

In this section, we utilize a common alternate construction of the Peterson variety in order to prove a stability property of Peterson varieties. For each finite-type Dynkin

diagram Δ , we construct a variety $\mathbf{P}(\Delta)$ inside the flag manifold $\mathcal{Fl}(\Delta)$ which is isomorphic to the Peterson variety $\mathbf{P}$ corresponding to any group G whose Dynkin diagram is Δ . The equality $\mathbf{P}(\Delta) = \mathbf{P}$ is well-known to experts; in Appendix A, we present a proof following Kostant [37].

For $I \subset \Delta$, we show that there is a natural inclusion $\mathbf{P}(I) \hookrightarrow \mathbf{P}(\Delta)$ identifying $\mathbf{P}(I)$ with the Peterson cell closure $\mathbf{P}_I$. This implies that the fundamental classes $[\mathbf{P}_K]_S$ and the cohomology classes p_K are stable for the inclusion $\mathbf{P}(I) \hookrightarrow \mathbf{P}(\Delta)$, and that the Peterson Schubert varieties of [32] are simply Peterson varieties corresponding to smaller groups.

6.1. The flag manifold of a Dynkin diagram

Let Φ_Δ (resp. Φ_Δ^+ , W_Δ) denote the root system (resp. positive roots, Weyl group) corresponding to a finite-type Dynkin diagram Δ . Following [48], let $\mathfrak{g}_\Delta$ be the canonical complex semisimple Lie algebra associated to Δ . Recall that $\mathfrak{g}_\Delta$ comes with elements $\{e_\alpha, h_\alpha\}_{\alpha \in \Phi_\Delta}$, such that the h_α span a Cartan subalgebra $\mathfrak{h}_\Delta$ of $\mathfrak{g}_\Delta$, and the $(e_\alpha)_{\alpha \in \Phi_\Delta}$ form a Chevalley system for $(\mathfrak{g}_\Delta, \mathfrak{h}_\Delta)$; see [12, Ch 7, §2]. We denote by $\mathfrak{b}_\Delta$ (resp. $\mathfrak{b}_\Delta^-$) the Borel subalgebra of $\mathfrak{g}_\Delta$ spanned by $\mathfrak{h}_\Delta$ and the set $\{e_\alpha \mid \alpha \in \Phi_\Delta^+\}$ (resp. $\{e_\alpha \mid \alpha \in \Phi_\Delta^-\}$).

We fix a connected Lie group G with $\text{Lie}(G) = \mathfrak{g}_\Delta$. The adjoint action of G on $\mathfrak{g}_\Delta$ induces an action on the Grassmannian $\text{Gr}(\dim \mathfrak{b}_\Delta, \mathfrak{g}_\Delta)$. The orbit of $\mathfrak{b}_\Delta$ is closed, and it gives the *flag variety* $\mathcal{Fl}(\Delta)$; see [14, §3.1]. The Borel subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action giving the following description of the flag variety:

$$\mathcal{Fl}(\Delta) = \{\mathfrak{b} \subset \mathfrak{g}_\Delta \mid \mathfrak{b} \text{ a Borel subalgebra of } \mathfrak{g}_\Delta\}. \quad (12)$$

The stabilizer of $\mathfrak{b}_\Delta$ in G is the Borel subgroup $B \subset G$ satisfying $\mathfrak{b}_\Delta = \text{Lie}(B)$, hence we have the usual G -equivariant identification,

$$\varphi : G/B \xrightarrow{\sim} \mathcal{Fl}(\Delta). \quad (13)$$

For $I \subset \Delta$, the subalgebra of $\mathfrak{g}_\Delta$ spanned by $\{e_\alpha, h_\alpha\}_{\alpha \in \Phi_I}$ is precisely the Lie algebra $\mathfrak{g}_I$ associated to the Dynkin diagram I . We have $\mathfrak{h}_I = \mathfrak{h} \cap \mathfrak{g}_I$ and $\mathfrak{b}_I = \mathfrak{b}_\Delta \cap \mathfrak{g}_I$. Let T_I , B_I , and G_I be the connected subgroups of G corresponding to $\mathfrak{h}_I$, $\mathfrak{b}_I$ and $\mathfrak{g}_I$ respectively. The induced map $G_I/B_I \rightarrow G/B$ corresponds to an embedding $\mathcal{Fl}(I) \rightarrow \mathcal{Fl}(\Delta)$ via Equation (13). In Equation (14), we give a characterization of this embedding in terms of Equation (12).

Lemma 6.1. *If $\mathfrak{u} \subset \mathfrak{g}_\Delta$ is a $|\Phi_\Delta^+|$ -dimensional subalgebra containing only nilpotent elements, then its normalizer $N(\mathfrak{u}) = \{x \in \mathfrak{g} \mid \text{ad}(x)\mathfrak{u} \subset \mathfrak{u}\}$ is a Borel subalgebra of $\mathfrak{g}_\Delta$.*

Proof. Following [12, p. 162, Cor 2], every subalgebra $\mathfrak{u}$ containing only nilpotent elements is contained in some Borel subalgebra $\mathfrak{b}$, and further, $\mathfrak{u} \subset [\mathfrak{b}, \mathfrak{b}]$ [12, p. 91, Prop 5(b)]. Comparing dimensions, we deduce that $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$, and hence $N(\mathfrak{u}) = \mathfrak{b}$. $\square$

Let $\mathfrak{b}'_I$ be any Borel subalgebra of $\mathfrak{g}_I$. Observe that $\mathfrak{v}_I = \bigoplus_{\alpha \in \Phi_\Delta^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha$ is $\mathfrak{g}_I$ -stable, and hence it is an ideal in the $|\Phi_\Delta^+|$ -dimensional subalgebra $[\mathfrak{b}'_I, \mathfrak{b}'_I] \oplus \mathfrak{v}_I$. By [10, p. 71, Lemma 1], we see that $[\mathfrak{b}'_I, \mathfrak{b}'_I] \oplus \mathfrak{v}_I$ is a $|\Phi_\Delta^+|$ -dimensional subalgebra of $\mathfrak{g}_\Delta$ containing only nilpotent elements. Following Lemma 6.1, we see that $N([\mathfrak{b}'_I, \mathfrak{b}'_I] \oplus \mathfrak{v}_I)$ is a Borel subalgebra of $\mathfrak{g}_\Delta$. Hence we have an embedding,

$$i : \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta), \quad \mathfrak{b}'_I \mapsto N([\mathfrak{b}'_I, \mathfrak{b}'_I] \oplus \mathfrak{v}_I). \quad (14)$$

The embedding $i : \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta)$ is G_I -equivariant, and sends $\mathfrak{b}_I$ to $\mathfrak{b}_\Delta$. It follows that under the identifications $\mathcal{F}l(I) = G_I/B_I$ and $\mathcal{F}l(\Delta) = G/B$ of Equation (13), the map i is precisely the map $G_I/B_I \rightarrow G/B$ induced by the inclusion $G_I \hookrightarrow G$; observe that $B_I = B \cap G_I$ follows from, e.g., [9, §11.2, Corollary and Thm. 11.16].

We will say that a map of Lie groups $F : G_1 \rightarrow G_2$ *lifts* a Lie algebra map $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ if $\text{Lie}(G_i) = \mathfrak{g}_i$ for $i = 1, 2$, and f is the differential of F at the identity.

Remark 6.2. The inclusion $i : \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta)$ is f -equivariant for any map $f : G'_I \rightarrow G$ lifting the inclusion $\mathfrak{g}_I \hookrightarrow \mathfrak{g}_\Delta$.

Lemma 6.3. Fix $w \in W_I$, and let $\mathfrak{b}'_w = \mathfrak{h}_I \oplus \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_{w(\alpha)}$, and $\mathfrak{b}_w = \mathfrak{h}_\Delta \oplus \bigoplus_{\alpha \in \Phi_\Delta^+} \mathfrak{g}_{w(\alpha)}$. Consider the Schubert varieties

$$X_w^I = \overline{\text{Ad}(B_I)\mathfrak{b}'_w} \subset \mathcal{F}l(I) \quad \text{and} \quad X_w = \overline{\text{Ad}(B)\mathfrak{b}_w} \subset \mathcal{F}l(\Delta).$$

Then $i(\mathfrak{b}'_w) = \mathfrak{b}_w$ and $i(X_w^I) = X_w$. We view the X_w^I as B -varieties via this identification. Consider the Schubert classes $\sigma_w \in H_T^*(\mathcal{F}l(\Delta))$ and $\sigma_w^I \in H_T^*(\mathcal{F}l(I))$. We have

$$i_*[X_w^I]_T = [X_w]_T, \quad i^*\sigma_w = \sigma_w^I.$$

Proof. Since $w \in W_I$, we have $w(\Phi_\Delta^+ \setminus \Phi_I^+) = \Phi_\Delta^+ \setminus \Phi_I^+$, and hence

$$\bigoplus_{\alpha \in \Phi_\Delta^+} \mathfrak{g}_{w(\alpha)} = \mathfrak{v}_I \oplus \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_{w(\alpha)} = [\mathfrak{b}'_w, \mathfrak{b}'_w] \oplus \mathfrak{v}_I.$$

It follows that $i(\mathfrak{b}'_w) = \mathfrak{b}_w$. Next, since $B_I \subset B$, we have $i(X_w^I) \subset X_w$. Further, both varieties are irreducible of dimension $l(w)$, hence they are equal. Consequently, we have $i_*[X_w^I]_T = [X_w]_T$; since the Schubert classes σ_w (resp. σ_w^I) are dual to the fundamental classes $[X_w]_T$ (resp. $[X_w^I]_T$), we further obtain $i^*\sigma_w = \sigma_w^I$. $\square$

6.2. The Peterson variety

Given a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}_\Delta$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{b}$, let $\Phi_\mathfrak{h}$ denote the root system of $(\mathfrak{g}_\Delta, \mathfrak{h})$, and let $\Delta_\mathfrak{b} \subset \Phi_\mathfrak{h}$ be the set of simple roots for which $\mathfrak{b}$ is the Borel subalgebra corresponding to the positive roots. We define

$$\mathcal{H}(\mathfrak{b}) = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{b}}} \mathfrak{g}_{-\alpha, \mathfrak{h}} \quad (15)$$

where $\mathfrak{g}_{\alpha, \mathfrak{h}}$ is the root space corresponding to $\alpha \in \mathfrak{h}^*$.

Observe that the subspace $\mathcal{H}(\mathfrak{b})$ is independent of the choice of $\mathfrak{h}$. Indeed, any two Cartan subgroups $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{b}$ are conjugate via an inner automorphism of $\mathfrak{b}$ [12, Ch 7, §3, Prop 5]. Since $\mathcal{H}(\mathfrak{b})$ is stable under the adjoint action of $\mathfrak{b}$, the automorphism preserves $\mathcal{H}(\mathfrak{b})$. (Alternatively, $\mathcal{H}(\mathfrak{b}) = [\mathfrak{u}, \mathfrak{u}]^{\perp}$, where $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{b}$, and $\perp$ is taken with respect to the Killing form.)

Definition 6.4. Let $e := \sum_{\alpha \in \Delta} e_{\alpha}$. The *Peterson variety* $\mathbf{P}(\Delta)$ is defined by

$$\mathbf{P}(\Delta) := \{\mathfrak{b} \in \mathcal{F}l(\Delta) \mid e \in \mathcal{H}(\mathfrak{b})\}.$$

We recall that e is a regular nilpotent element of $\mathfrak{g}_{\Delta}$. Under the G -equivariant isomorphism $G/B \xrightarrow{\sim} \mathcal{F}l(\Delta)$ from Equation (13) we have

$$\begin{aligned} \mathbf{P}(\Delta) &= \{gB \in G/B \mid e \in \mathcal{H}(Ad(g)\mathfrak{b}_{\Delta})\} \\ &= \left\{ gB \in G/B \mid Ad(g^{-1})e \in H(\mathfrak{b}_{\Delta}) = \mathfrak{b}_{\Delta} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}e_{-\alpha} \right\}. \end{aligned} \quad (16)$$

Let G , G_I , and $\phi : G_I \rightarrow G$ be as in Section 6.1, and let $S_I \subset T_I$ be the one-dimensional torus corresponding to $h_I = \sum_{\alpha \in \Phi_I^+} \alpha^{\vee}$.

Proposition 6.5. Consider the map $i : \mathcal{F}l(I) \hookrightarrow \mathcal{F}l(\Delta)$ from Equation (14). Then $i(\mathbf{P}(I)) = \mathbf{P}_I$, as algebraic varieties. Furthermore, $\mathbf{P}_I$ is also equal to the set theoretic intersection $\mathbf{P}(\Delta) \cap \mathcal{F}l(I)$.

Proof. Let $e_I = \sum_{\alpha \in I} e_{\alpha}$ and $e_{\bar{I}} = \sum_{\alpha \in \Delta \setminus I} e_{\alpha}$, so that $e = e_I + e_{\bar{I}}$. Recall that

$$\mathbf{P}(I) = \{\mathfrak{b}'_I \in \mathcal{F}l(I) \mid e_I \in \mathcal{H}(\mathfrak{b}'_I)\}.$$

Consider $\mathfrak{b}'_I \in \mathcal{F}l(I)$, and set $i(\mathfrak{b}'_I) = \mathfrak{b}'$. We see from Equations (14) and (15) that

$$\mathcal{H}(\mathfrak{b}'_I) \oplus \mathfrak{v}_I \subset \mathcal{H}(\mathfrak{b}').$$

Suppose $\mathfrak{b}'_I \in \mathbf{P}(I)$. We have $e_{\bar{I}} \in \mathfrak{v}_I$, and hence

$$e_I \in \mathcal{H}(\mathfrak{b}'_I) \implies e = e_I + e_{\bar{I}} \in \mathcal{H}(\mathfrak{b}'_I) \oplus \mathfrak{v}_I \subset \mathcal{H}(\mathfrak{b}') \implies \mathfrak{b}' \in \mathbf{P}(\Delta).$$

We deduce that $i(\mathbf{P}(I)) \subset \mathbf{P}(\Delta)$.

Using the natural basis $\{e_{\alpha}, h_{\alpha}\}_{\alpha \in \Phi_{\Delta}}$ for $\mathfrak{g}_{\Delta}$, and its sub-basis of $\mathfrak{g}_I$, consider the $\mathfrak{g}_I$ -equivariant projection $\text{pr} : \mathfrak{g}_{\Delta} \rightarrow \mathfrak{g}_I$ defined by:

$$\mathrm{pr}(e_\alpha) = \begin{cases} e_\alpha & \text{if } \alpha \in \Phi_I, \\ 0 & \text{otherwise.} \end{cases} \quad \mathrm{pr}(h_\alpha) = \begin{cases} h_\alpha & \text{if } \alpha \in \Phi_I, \\ 0 & \text{otherwise.} \end{cases}$$

Now, suppose $\mathbf{b}' \in \mathbf{P}(\Delta)$. Then $e \in \mathcal{H}(\mathbf{b}')$, and hence

$$e_I = \mathrm{pr}(e) \in \mathrm{pr}(\mathcal{H}(\mathbf{b}')) = \mathcal{H}(\mathbf{b}'_I).$$

It follows that $\mathcal{F}l(I) \cap \mathbf{P}(\Delta) = \mathbf{P}(I)$. The equality $\mathbf{P}(I) = \mathbf{P}_I$ is a consequence of the observation that $\mathcal{F}l(I) = X_{w_I}$, and the irreducibility of $\mathbf{P}(I)$; see Lemma A.3. $\square$

We will denote by $j : \mathbf{P}(I) \rightarrow \mathbf{P}(\Delta)$ the inclusion induced by restricting i to $\mathbf{P}(I)$. In order to discuss stability for Peterson classes, we first need to construct algebra homomorphisms $H_S^*(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H_{S_I}^*(\mathcal{F}l(I); \mathbb{Q})$, compatible with restrictions to Peterson subvarieties. To this end, we replace S_I and S by a $\mathbb{C}^*$ ‘parameterizing’ (not necessarily injectively) these tori via the defining cocharacters $h_I : \mathbb{C}^* \rightarrow S_I$ and $h : \mathbb{C}^* \rightarrow S$. This $\mathbb{C}^*$ acts on $\mathcal{F}l(I)$, respectively on $\mathcal{F}l(\Delta)$, via its image $S_I \subset T_I$ and $S \subset T$. The embedding $\mathfrak{g}_I \rightarrow \mathfrak{g}$ is $\mathbb{C}^*$ -equivariant, and hence so is the embedding $\mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta)$ described in (14). These facts are summarized in the diagram below. The question marks signify that a map may not exist; see Remark 6.7 below.

$$\begin{array}{ccc} & \mathbb{C}^* & \\ h_I \swarrow & & \searrow h \\ S_I & \overset{??}{\dashrightarrow} & S \\ \downarrow & & \downarrow \\ \mathcal{F}l(I) & \hookrightarrow & \mathcal{F}l(\Delta) \end{array}$$

The cocharacter h induces an isomorphism $\mathrm{Lie}(\mathbb{C}^*) \rightarrow \mathrm{Lie}(S)$, and hence a ring isomorphism $H_S^*(pt; \mathbb{Q}) \rightarrow H_{\mathbb{C}^*}^*(pt; \mathbb{Q})$. (In general the corresponding map over integer coefficients, $H_S^*(pt; \mathbb{Z}) \rightarrow H_{\mathbb{C}^*}^*(pt; \mathbb{Z})$, may not be an isomorphism.) The identity map $\mathcal{F}l(\Delta) \rightarrow \mathcal{F}l(\Delta)$ is equivariant with respect to the cocharacter $h : \mathbb{C}^* \rightarrow S$, therefore by functoriality we have induced isomorphisms $H_S^*(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H_{\mathbb{C}^*}^*(\mathcal{F}l(\Delta); \mathbb{Q})$ and $H_{\mathbb{C}^*}^*(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H_S^*(\mathcal{F}l(\Delta); \mathbb{Q})$. Further, since $\mathbf{P}(\Delta)$ is S -stable, it inherits a $\mathbb{C}^*$ -action through h , giving isomorphisms

$$H_S^*(\mathbf{P}(\Delta); \mathbb{Q}) \xrightarrow{\sim} H_{\mathbb{C}^*}^*(\mathbf{P}(\Delta); \mathbb{Q}) \quad \text{and} \quad H_{\mathbb{C}^*}^*(\mathbf{P}(\Delta); \mathbb{Q}) \xrightarrow{\sim} H_S^*(\mathbf{P}(\Delta); \mathbb{Q}).$$

All these isomorphisms are natural with respect to the closed embedding $\mathbf{P}(\Delta) \subset \mathcal{F}l(\Delta)$. A similar discussion for the cocharacter h_I yields isomorphisms

$$H_{S_I}^*(\mathbf{P}(I); \mathbb{Q}) \xrightarrow{\sim} H_{\mathbb{C}^*}^*(\mathbf{P}(I); \mathbb{Q}) \quad \text{and} \quad H_{\mathbb{C}^*}^*(\mathbf{P}(I); \mathbb{Q}) \xrightarrow{\sim} H_{S_I}^*(\mathbf{P}(I); \mathbb{Q}),$$

natural with respect to $\mathbf{P}(I) \subset \mathcal{F}l(I)$. Consequently, the $\mathbb{C}^*$ -equivariant inclusion $j : \mathbf{P}(I) \rightarrow \mathbf{P}(\Delta)$ yields a pullback map,

$$H_S^*(\mathcal{F}l(\Delta); \mathbb{Q}) \rightarrow H_{S_I}^*(\mathcal{F}l(I); \mathbb{Q})$$

compatible with the algebra isomorphism $H_S^*(pt; \mathbb{Q}) \rightarrow H_{S_I}^*(pt; \mathbb{Q})$, and we obtain a commutative diagram,

$$\begin{array}{ccc} H_S^*(\mathbf{P}(\Delta); \mathbb{Q}) & \overset{j^*}{\dashrightarrow} & H_{S_I}^*(\mathbf{P}(I); \mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ H_{\mathbb{C}^*}^*(\mathbf{P}(\Delta); \mathbb{Q}) & \longrightarrow & H_{\mathbb{C}^*}^*(\mathbf{P}(I); \mathbb{Q}). \end{array} \quad (19)$$

In a similar fashion, we also obtain a pushforward $j_* : H_{S_I}^*(\mathbf{P}(I); \mathbb{Q}) \rightarrow H_S^*(\mathbf{P}(\Delta); \mathbb{Q})$.

The following is the main result of this section.

Theorem 6.6. *Consider the map $i : \mathcal{F}l(I) \hookrightarrow \mathcal{F}l(\Delta)$ from Equation (14).*

- (a) *For $J \subset I$, we have $i_*[\mathbf{P}_J]_{S_I} = [\mathbf{P}_J]_S$ in $H_S^*(\mathcal{F}l(\Delta); \mathbb{Q})$.*
- (b) *Let $j^* : H_S^*(\mathbf{P}(\Delta); \mathbb{Q}) \rightarrow H_{S_I}^*(\mathbf{P}(I); \mathbb{Q})$ denote the pullback induced from the inclusion $\mathbf{P}(I) \hookrightarrow \mathbf{P}(\Delta)$. For $K \subset \Delta$, we have*

$$j^* p_K = \begin{cases} p_K & \text{if } K \subset I, \\ 0 & \text{otherwise.} \end{cases}$$

In the non-equivariant case, the equalities in (a) and (b) hold with integral coefficients.

Proof. For $J \subset I \subset \Delta$, the inclusions $\mathcal{F}l(J) \xrightarrow{i'} \mathcal{F}l(I) \xrightarrow{i} \mathcal{F}l(\Delta)$ are $\mathbb{C}^*$ -equivariant for the action given by the cocharacters h_J, h_I and h , respectively. By Proposition 6.5, we have $i'(\mathbf{P}(J)) = \mathbf{P}_J \subset \mathbf{P}(I)$ and $i(i'(\mathbf{P}(J))) = \mathbf{P}_J \subset \mathbf{P}(\Delta)$, and consequently $[\mathbf{P}_J]_{\mathbb{C}^*} = i_*(i'_*([\mathbf{P}(J)]_{\mathbb{C}^*})) = i_*([\mathbf{P}_J]_{\mathbb{C}^*})$ in $H_{\mathbb{C}^*}^*(\mathcal{F}l(\Delta))$. Then part (a) follows because the $\mathbb{C}^*$ -equivariance may be replaced by the S_I , respectively S -equivariance, as explained above. Part (b) follows from Lemma 6.3 and the commutativity of the diagram,

$$\begin{array}{ccc} \mathbf{P}(I) & \xrightarrow{j} & \mathbf{P}(\Delta) \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{F}l(I) & \xrightarrow{i} & \mathcal{F}l(\Delta) \end{array}$$

utilizing again that all maps are $\mathbb{C}^*$ -equivariant.

In the non-equivariant case, all (co)homology morphisms are defined over $\mathbb{Z}$, and the classes $[\mathbf{P}_I]$ and p_I are integral, by their definition. This finishes the proof. $\square$

Remark 6.7. The reader may wonder whether an algebra map $H_S^*(\mathcal{F}l(\Delta)) \rightarrow H_{S_I}^*(\mathcal{F}l(I))$ may be directly constructed from the inclusion $i : \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta)$, equivariant with respect to a map $\varphi_I : S_I \rightarrow S$. The requirement that i is φ_I -equivariant implies that

the differential $d\varphi_I : \text{Lie}(S_I) \rightarrow \text{Lie}(S)$ must send $h_I \mapsto h$. (Note that this is *not* the restriction of the natural map $\text{Lie}(T_I) \hookrightarrow \text{Lie}(T)$.) The existence of a lift $S_I \rightarrow S$ of this Lie algebra map cannot be guaranteed. For instance, consider the inclusion $G_I := \text{SL}_3 \subset G := \text{SL}_4$ given by the natural embedding of Dynkin diagrams $A_2 \subset A_3$. The tori S_I and S are the images of cocharacters

$$h_I(z) = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} \quad \text{and} \quad h(z) = \begin{pmatrix} z^3 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-3} \end{pmatrix},$$

respectively. In this case there is no group homomorphism $\varphi_I : S_I \rightarrow S$ satisfying $\varphi_I(h_I(z)) = h(z)$.

Corollary 6.8. *Let $I \subset \Delta$ and assume that the map $\text{Lie}(S_I) \rightarrow \text{Lie}(S)$ sending $h_I \mapsto h$ lifts to a map $\varphi_I : S_I \rightarrow S$. Then the push-forward and pull-back maps*

$$j^* : H_S^*(\mathbf{P}(\Delta)) \rightarrow H_{S_I}^*(\mathbf{P}(I)) \quad \text{and} \quad j_* : H_*^{S_I}(\mathbf{P}(I)) \rightarrow H_*^S(\mathbf{P}(\Delta))$$

may be defined with $\mathbb{Z}$ coefficients. In particular, the statements in Theorem 6.6 also hold over $\mathbb{Z}$.

Proof. The claim follows because the G_I -equivariant map $i : \mathcal{F}l(I) \rightarrow \mathcal{F}l(\Delta)$ from (14) restricts to the φ_I -equivariant map $j : \mathbf{P}(I) \rightarrow \mathbf{P}(\Delta)$. Then j_* and j^* may be defined over $\mathbb{Z}$. $\square$

Remark 6.9. The results of this section can be extended to the case of reductive groups G and a one-dimensional torus $S \subset T$ satisfying $\alpha|_S = \beta|_S$ for all simple roots α, β . For G semisimple, there is a unique $S \subset T$ satisfying this condition. For an arbitrary reductive group G , this may not determine S uniquely.

It is common in the literature on type A Peterson varieties to use the group $G = \text{GL}_n$ and the one-dimensional torus $S = \text{diag}(z^n, z^{n-1}, \dots, z)$. In this case, we have an identification between the one dimensional subtori of GL_n and GL_{n+1} given by $\text{diag}(z^n, z^{n-1}, \dots, z) \mapsto \text{diag}(z^{n+1}, z^n, \dots, z)$. Then the diagram in (17), and hence the statements in Theorem 6.6, hold over $\mathbb{Z}$.

7. Intersection multiplicities

Different choices of Coxeter elements v_I lead to different bases $\{p_I = \iota^* \sigma_{v_I}\}$ for $H_S^*(\mathbf{P}; \mathbb{Q})$. By Theorem 4.3, the transition matrix between two such bases $\{p_I\}$ and $\{p'_I\}$ is diagonal, with entries given by ratios

$$\frac{m(v_I)}{m(v'_I)} = \frac{\langle p_I, [\mathbf{P}_I]_S \rangle}{\langle p'_I, [\mathbf{P}_I]_S \rangle}.$$

Table 1

The exponents of Dynkin Diagrams; see [13, Ch. 10].

Diagram	Exponents	Diagram	Exponents
A_n	$1, 2, \dots, n$	F_4	$1, 5, 7, 11$
B_n, C_n	$1, 3, \dots, 2n - 1$	E_6	$1, 4, 5, 7, 8, 11$
D_n	$1, 3, \dots, 2n - 3, n - 1$	E_7	$1, 5, 7, 9, 11, 13, 17$
G_2	$1, 5$	E_8	$1, 7, 11, 13, 17, 19, 23, 29$

It is natural to ask whether there are choices for the Coxeter elements v_I for which $m(v_I) = 1$, and more generally, to ask for formulae for the $m(v_I)$. In Proposition 7.3, we give a formula for $m(v_I)$ in terms of the localization of the Schubert variety X^{v_I} at the point w_I , and in Theorem 7.6, we use this formula to compute $m(v_I)$ for certain Coxeter elements v_I . Theorem 7.6 settles Question 1 of [32] for all classical types. As a further application of Proposition 7.3(b), we show in Example 7.4 that not all choices of v_I lead to $m(v_I) = 1$ in type A, and in Example 7.5 that for $I \in \{B_2, C_2\}$, there is no Coxeter element v_I for which $m(v_I) = 1$.

7.1. The exponents of a Dynkin diagram

Let Δ be a Dynkin diagram with n nodes. The exponents $m_1, \dots, m_n$ of Δ are fundamental invariants, appearing in many contexts. We will utilize the following two characterizations found in [13, Ch. 10]; see also [36]:

- (1) Let $\mathfrak{g}$ be the Lie algebra with Dynkin diagram Δ , and let $\{e, f, h\}$ be an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}$, such that e is a regular nilpotent element in $\mathfrak{g}$; see [40, 15]. The $\mathfrak{sl}_2$ -decomposition of $\mathfrak{g}$ is precisely $\oplus V(2m_i)$, where $V(k)$ denotes the irreducible finite dimensional $\mathfrak{sl}_2$ -representation with highest weight k .
- (2) Let a_i be the number of roots of height i in Φ_Δ^+ . Then $(a_1, \dots, a_k)$ is a partition, and the conjugate partition is precisely $(m_1, \dots, m_n)$.

Throughout this section, we will denote by $m_1, \dots, m_n$, the exponents of Δ .

Lemma 7.1. *The weights for the S -action on $\text{Lie}(G^e)$ are precisely $m_1 t, \dots, m_n t$.*

Proof. Recall that $S \subset T$ corresponds to the cocharacter h satisfying $\alpha(h) = 2$ for all $\alpha \in \Delta$, and that $[h, e] = 2e$. Identifying $\mathfrak{X}(S)$ as a lattice in $\text{Lie}(S)^*$, we view t as an element of $\text{Lie}(S)^*$. Let $\varpi \in \text{Lie}(S)^*$ be the fundamental weight dual to h , i.e., given by $\varpi(h) = 1$. Comparing the weights of the h -action and S -action on e , we deduce that $t = 2\varpi$.

Consider now an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}$, with e (resp. h) as the nilpositive (resp. neutral) element. Since e is a principal nilpotent element of $\mathfrak{g}$, the decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$ -representation is given by $\mathfrak{g} = \oplus V(2m_i \varpi) = \oplus V(m_i t)$. Now, simply observe that

$$\text{Lie}(G^e) = \{x \in \text{Lie}(U) \mid [e, x] = 0\} = \ker(ad(e))$$

is spanned by the highest weight vectors in $\mathfrak{g}$, whose weights are precisely $m_1 t, \dots, m_n t$. $\square$

Lemma 7.2. *The S -equivariant Euler class of the tangent space $T_{w_I} \mathcal{F}l(I)$ is $(\prod m_i!) t^N$, where $N = \dim \mathcal{F}l(I)$.*

Proof. Observe that the map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$ is given by $\alpha \mapsto t$, for all $\alpha \in \Delta$. Consequently, the T -weight space $\mathfrak{g}_\alpha$, for $\alpha \in \Phi_I^+$, is an S -weight space of weight $ht(\alpha)t$. The tangent space at w_I admits a T -decomposition,

$$T_{w_I}(G/B) = \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha;$$

hence the S -equivariant Euler class of $T_{w_I}(G/B)$ is $t^{a_1}(2t)^{a_2} \dots (kt)^{a_k}$, where a_i is the number of roots of height i in Φ_I^+ . Following Section 7.1, the partition $(a_1, \dots, a_k)$ is conjugate to $(m_1, \dots, m_n)$; consequently, the S -equivariant Euler class of $T_{w_I}(G/B)$ is precisely $m_1! m_2! \dots m_n! t^N$. $\square$

We are now ready to calculate the multiplicities $m(v_I)$ using the map in cohomology obtained by restricting to the fixed point set.

Proposition 7.3. *Let $\iota_w^* : H_S^*(G/B) \rightarrow H_S^*(w)$ be the map induced by the inclusion $wB/B \hookrightarrow G/B$. Define $b \in \mathbb{Z}$ by $\iota_{w_I}^* \sigma_{v_I} = bt^n$.*

- (a) We have $m(v_I) = \frac{b}{m_1 \dots m_n}$.
 (b) Suppose X^{v_I} is smooth at w_I . Let $\{\beta_1, \dots, \beta_n\} = \{\alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I\}$. Then

$$m(v_I) = \frac{ht(\beta_1) \dots ht(\beta_n)}{m_1 \dots m_n}.$$

Proof. Recall from Corollary 3.5 that

$$\sigma_{v_I} \cup \eta_I = m(v_I) \tau_{w_I}, \quad (18)$$

where η_I and τ_{w_I} are Poincaré dual to $[\mathbf{P}_I]_S$ and $[w_I]_S$, respectively, in $H_S^*(G/B)$.

We restrict both sides to w_I under the map $\iota_{w_I}^* : H_S^*(G/B) \rightarrow H_S^*(w_I)$. By Theorem 6.6, we may assume $\Delta = I$, so that the tangent space $T_{w_I} \mathbf{P}_I = \text{Lie}(G^e)$ has S -weights $m_1 t, \dots, m_n t$ as described in Lemma 7.1. Following Proposition A.1 and Lemma 7.2, we see that the S -equivariant Euler class at w_I of the normal bundle of $\mathbf{P}$ is $\frac{m_1! m_2! \dots m_n! t^N}{m_1 m_2 \dots m_n t^n}$. Applying $\iota_{w_I}^*$ to both sides of Equation (18) yields

$$\frac{m_1! m_2! \dots m_n! t^N}{m_1 m_2 \dots m_n t^n} \iota_{w_I}^* \sigma_{v_I} = m(v_I) \iota_{w_I}^* \tau_{w_I}.$$

Using Lemma 7.2, we have $\iota_{w_I}^* \tau_{w_I} = m_1! m_2! \cdots m_n! t^N$, and part (a) follows.

For part (b), since X^{v_I} is smooth at w_I , the normal space of X^{v_I} at w_I is spanned by

$$\{\mathfrak{g}_\alpha \mid \alpha \in \Phi_I^+, s_\alpha w_I \not\leq v_I\} = \{\mathfrak{g}_\alpha \mid \alpha \in \Phi_I^+, s_\alpha \not\leq v_I w_I\};$$

see [38, Cor 12.1.10]. Part (b) now follows from (a), along with the observation that the map $\mathfrak{X}(T) \rightarrow \mathfrak{X}(S)$ is given by $\beta \mapsto ht(\beta)t$. $\square$

Example 7.4. Let $I = A_3$, and $v_I = s_1 s_3 s_2$. Then $m(v_I) = 2$.

Example 7.5. For $I \in \{B_2, C_2\}$, we have $m(v_I) = 2$ for every Coxeter element v_I .

In [32, Question 1], Insko and Tymoczko conjecture that $m(v_I) = 1$ for certain Coxeter elements, when I is contained in some sub-diagram of type A , and that $m(v_I) = 2$ otherwise. As an application of Proposition 7.3, we compute $m(v_I)$ for one Coxeter element in each Dynkin diagram; this formula proves their conjecture in type A , and disproves it in other cases.

Theorem 7.6.

(a) Let I be a connected Dynkin diagram with the standard labelling (see [11]), and set $v_I = s_1 s_2 \cdots s_n$. Then,

$$m(v_I) = \begin{cases} 1 & \text{if } I = A_n, \\ 2^{n-1} & \text{if } I = B_n, C_n, \\ 2^{n-2} & \text{if } I = D_n, \\ 72 = 2^3 \cdot 3^2 & \text{if } I = E_6, \end{cases} \quad m(v_I) = \begin{cases} 864 = 2^5 \cdot 3^3 & \text{if } I = E_7, \\ 51840 = 2^7 \cdot 3^4 \cdot 5 & \text{if } I = E_8, \\ 48 = 2^4 \cdot 3 & \text{if } I = F_4, \\ 6 = 2 \cdot 3 & \text{if } I = G_2. \end{cases}$$

(b) Let $I_1, \dots, I_k$ be the connected components of a Dynkin diagram I , and let $v_1, \dots, v_k$ be Coxeter elements for $I_1, \dots, I_k$ respectively. Then $v := v_1 \cdots v_k$ is a Coxeter element for I , and $m(v) = \prod_{j=1}^k m(v_j)$.

Proof of Theorem 7.6. If I is a diagram of classical type, the variety X^{v_I} is smooth at w_I , cf. [32, Thm 3]. Consequently, we can use Proposition 7.3(b) to compute $m(v_I)$. We show the details of the calculations in Appendix B. For the exceptional cases, a computer calculation suffices: we use the localization formula (cf. [5,8]) to compute $\iota_{w_I}^* \sigma_{v_I}$, and apply Proposition 7.3(a).

Following Theorem 6.6, we may assume $\Delta = I$. The integer $m(v)$ is the multiplicity of the intersection of X^v with $\mathbf{P}$. We have

$$\mathcal{F}l(I) = \prod \mathcal{F}l(I_j), \quad X^v = \prod X^{v_j}, \quad \mathbf{P}(I) = \prod \mathbf{P}(I_j),$$

and hence the multiplicity $m(v)$ is the product of the multiplicities $m(v_j)$. $\square$

Remark 7.7. We conjecture for all Coxeter elements v_I a type-independent formula for the intersection multiplicity, namely:

$$m(v_I) = \frac{|\mathcal{R}(v_I)| |W_I|}{|I|! \det(C_I)} = \mathcal{R}(v_I) \prod_{\alpha \in I} a_\alpha. \quad (19)$$

Here $\mathcal{R}(v_I)$ is the set of reduced expressions for v_I , W_I is the Weyl group of I , C_I is the Cartan matrix of the Dynkin diagram I , and the integers a_α are the coefficients of the highest root $\theta_I = \sum_{\alpha \in I} a_\alpha \alpha$ of I . The second equality follows from [11, p. 297]. For the Coxeter elements in Theorem 7.6, we have verified this formula with type-by-type calculations. For a type-independent proof, see [25].

Remark 7.8. The formula $m(v_I) = 1$ for $I = A_n$ was first obtained by Insko in [31], who proved that the scheme-theoretic intersection $X^{v_I} \cap \mathbf{P}_I$ is reduced.

Corollary 7.9. *Suppose I is contained in some sub-diagram J of type A , and let v be the Coxeter element of I obtained by multiplying the simple reflections in increasing order (for the standard type A labelling of nodes in J). Then $m(v) = 1$.*

Proof. Observe that each connected component $I_j \subset I$ is of type A . Let v_j be the Coxeter element of I_j obtained by multiplying the simple reflections in increasing order (for the standard type A labelling of nodes in I_j), so that $v = \prod v_j$. Following Theorem 7.6, we have $m(v_j) = 1$, and $m(v) = \prod m(v_j) = 1$. $\square$

Appendix A. Two definitions of the Peterson variety

In this section we recall the affine paving of the Peterson variety (Appendix A.1), and we show in Proposition A.4 that the two definitions of the Peterson variety,

$$\begin{aligned} \mathbf{P} &:= \overline{G^e \cdot w_0 B} \quad , \\ \mathbf{P}(\Delta) &:= \left\{ gB \in G/B \left| \operatorname{Ad}(g^{-1})e \in \operatorname{Lie}(B) \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}e_{-\alpha} \right. \right\}, \end{aligned} \quad (20)$$

agree. These results are well-known to experts, but either some statements are only implicitly present in the literature, or we present slightly different proofs. A key point is the irreducibility of $\mathbf{P}(\Delta)$, which we prove utilizing results of Kostant [37]. We also present in Remark A.5 an alternate proof following [6, 44, 1], as explained to us by Bălibanu. Their arguments extend to the wider setting of regular Hessenberg varieties.

A.1. Paving by affines

For $I \subset \Delta$, let $\mathbf{P}(I)^\circ$ be the *Peterson cell*, $\mathbf{P}(I)^\circ = \mathbf{P}(I) \setminus \bigcup_{J \subsetneq I} \mathbf{P}(J)$. Let U_I be the unipotent Lie group corresponding to the Dynkin diagram I , and let A_I be the centralizer of e_I in U_I ; see [48].

The following proposition was proved in various cases by Tymoczko [50, Theorem 4.3] and Bălibanu [7, Section 6]. Following the exposition in [7], we recall the main steps in the proof.

Proposition A.1.

- (a) The group A_I acts transitively and faithfully on $\mathbf{P}(I)^\circ$.
- (b) $\mathbf{P}(\Delta) = \bigsqcup_{I \subset \Delta} \mathbf{P}(I)^\circ$ is a paving by affines.
- (c) The intersection $\mathbf{P}(\Delta) \cap X_w^\circ$ is nonempty if and only if $w = w_I$ for some subset $I \subset \Delta$.

Proof. Following [7, Prop 6.3], we have $\mathbf{P}(I)^\circ = A_I w_I B / B$, i.e., A_I acts transitively on $\mathbf{P}(I)^\circ$. Further, U_I acts faithfully on the Schubert cell $X_{w_I}^\circ$, and hence the action of $A_I \subset U_I$ is faithful at the point w_I . Next, observe that $\mathbf{P}(I)^\circ$ is a principal space for A_I , hence is an affine space. Finally, the observation $\mathbf{P}(I)^\circ \subset X_{w_I}^\circ$, along with part (b) implies that $\mathbf{P}(\Delta) \cap X_w^\circ$ is empty unless $w = w_I$ for some I . $\square$

A.2. Equivalence of two definitions of the Peterson variety

Lemma A.2. ([37]) *The variety $\mathbf{P}(\Delta)$ is locally irreducible at the point $1B$.*

Proof. Let U^- be the unipotent radical of the opposite Borel subgroup B^- , and let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}$. Consider the map $\eta : U^- \rightarrow \mathfrak{g}$ given by $u \mapsto \text{Ad}(u^{-1})e$. Following [37, Thm. 17], the map η induces an isomorphism,

$$\eta : U^- \xrightarrow{\sim} (e + \mathfrak{b}^-) \cap \mathcal{N},$$

where $\mathfrak{b}^- = \text{Lie}(B^-)$. Recall that the U^- -orbit of $1B$ is an open set (namely the opposite Schubert cell) in G/B . Hence $Z := \mathbf{P}(\Delta) \cap U^-B/B$ is an open neighborhood of $1B$ in $\mathbf{P}(\Delta)$, and it suffices to show that Z is irreducible. Let $\mathfrak{f} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}$, and let $\mathcal{N}^{reg}$ denote the set of regular nilpotent elements in $\mathfrak{g}$. Since the intersection $U^- \cap B$ is trivial, it follows that $Z = \eta^{-1}(\mathfrak{b} \oplus \mathfrak{f})$. Then

$$\begin{aligned} Z &= \eta^{-1}(\mathfrak{b} \oplus \mathfrak{f}) \cong (\mathfrak{b} \oplus \mathfrak{f}) \cap (e + \mathfrak{b}^-) \cap \mathcal{N} \\ &= (e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N} = (e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N}^{reg}, \end{aligned}$$

where the last equality is from [37, §3.2]. The result now follows from the irreducibility of $(e + \mathfrak{h} + \mathfrak{f}) \cap \mathcal{N}^{reg}$, cf. [37, Thm. 6]. $\square$

Lemma A.3. *The variety $\mathbf{P}(\Delta)$ is irreducible.*

Proof. Let Y be an irreducible component of $\mathbf{P}(\Delta)$. Recall from Appendix A.1 that the (connected) group $A_\Delta = \text{Stab}_U(e)$ acts on $\mathbf{P}(\Delta)$, hence it acts on Y . Since Y is a closed (hence projective) subvariety of G/B and since A_Δ is solvable, Y admits an A_Δ -fixed point by [9, Thm.10.4]. This point must necessarily be $1B$, as this is the unique A_Δ -fixed point in G/B . In other words, every irreducible component of $\mathbf{P}(\Delta)$ contains $1B$; the irreducibility of $\mathbf{P}(\Delta)$ now follows from the local irreducibility of $\mathbf{P}(\Delta)$ at $1B$; see Lemma A.2. $\square$

Proposition A.4. *The two definitions of the Peterson variety in Equation (20) agree, i.e., $\mathbf{P} = \mathbf{P}(\Delta)$.*

Proof. Observe that $G^e = A_\Delta \times Z(G)$, where $Z(G)$ is the center of G , cf. [37, p. 9]. Since $Z(G) \subset B$, we have an equality $A_\Delta \cdot w_0 B = G^e \cdot w_0 B \subset \mathbf{P} \cap \mathbf{P}(\Delta)$. It follows from Proposition A.1 that $G^e \cdot w_0 B$ is an open subset of $\mathbf{P}(\Delta)$, and since $\mathbf{P}(\Delta)$ is irreducible by Lemma A.3, $\mathbf{P} = \mathbf{P}(\Delta)$. $\square$

Remark A.5. We recall an alternate proof of the irreducibility of the variety $\mathbf{P}(\Delta)$, following Precup [44, Cor 14] and [6, Lemma 7.1], as explained to us by Bălibanu.

Let $H = \mathfrak{b} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha})$, and consider the variety $\mathcal{Z} = G \times^B H$, equipped with the map $\mathcal{Z} \rightarrow \mathfrak{g}$ given by $(g, x) \mapsto \text{Ad}(g)x$. For x a regular semisimple element of $\mathfrak{g}$, the fiber $\mathcal{Z}_x$ has dimension $|\Delta|$; see [44, Cor 3]. Since regular semisimple elements are dense in $\mathfrak{g}$, it follows from [41, Ch. 1, §8, Thms. 2, 3] that each irreducible component of the fiber $\mathcal{Z}_e = \mathbf{P}(\Delta)$ has dimension greater than or equal to $|\Delta|$, and hence $\mathbf{P}(\Delta)$ is pure-dimensional. Following [20, §1.5], the fundamental classes of the irreducible components of $\mathbf{P}(\Delta)$ freely generate the top Chow group of $\mathbf{P}(\Delta)$. Since there is a unique top-dimensional cell in the affine paving of Proposition A.1(b), it follows that $\mathbf{P}(\Delta)$ has a unique irreducible component.

Appendix B. Intersection multiplicities for classical diagrams

We present here the details of our calculation in Theorem 7.6 of the intersection multiplicities $m(v_I)$ for classical Dynkin diagrams.

B.1. Type A

Let V be a vector space with orthonormal basis $\epsilon_1, \dots, \epsilon_n$. The vectors $\{\epsilon_i - \epsilon_j\}$ form a root system with Dynkin diagram A_{n-1} . A choice of simple roots is $\alpha_i = \epsilon_i -$

ϵ_{i+1} for $1 \leq i < n$, and the Weyl group is naturally identified with the symmetric group on $\{\epsilon_1, \dots, \epsilon_n\}$. We calculate $v_I w_I = [1, n, \dots, 2]$, and $\{\alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I\} = \{\epsilon_1 - \epsilon_i \mid 2 \leq i \leq n\}$. Now, $ht(\epsilon_1 - \epsilon_i) = i - 1$. Consequently Proposition 7.3 and Table 1 yield $m(v_I) = 1$.

B.2. Type B

Let V be a vector space with orthonormal basis $\epsilon_1, \dots, \epsilon_n$. The vectors $\{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm\epsilon_i\}$ form a root system with Dynkin diagram B_n . A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = \epsilon_n$.

Let S_{2n} be the symmetric group on the letters $\{1, \dots, n, \bar{n}, \dots, \bar{1}\}$, and let $r_{ij} \in S_{2n}$ be the transposition switching the letters i and j . The Weyl group W can be viewed as the subgroup of S_{2n} generated by the reflections,

$$s_{\epsilon_i - \epsilon_j} = r_{ij} r_{i\bar{j}}, \quad s_{\epsilon_i + \epsilon_j} = r_{i\bar{j}} r_{ij}, \quad s_{\epsilon_i} = r_{i\bar{i}}, \quad 1 \leq i < j \leq n.$$

Given $v, w \in W$, if $v \leq w$ in the Bruhat order on W , then $v \leq w$ in the Bruhat order on S_{2n} ; see [39, §6.1.1]. We calculate

$$v_I w_I = [\bar{2}, \dots, \bar{n}, 1, \bar{1}, n, \dots, 2]$$

and $\{\alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I\} = \{\epsilon_1 + \epsilon_i \mid 2 \leq i \leq n\} \cup \{\epsilon_1\}$. Now, $ht(\epsilon_1) = n$, and $ht(\epsilon_1 + \epsilon_i) = 2n + 1 - i$. Following Proposition 7.3 and Table 1, we have

$$m(v_I) = \frac{n(n+1) \cdots (2n-1)}{(1)(3) \cdots (2n-1)} = 2^{n-1}.$$

B.3. Type C

Let V be a vector space with orthonormal basis $\epsilon_1, \dots, \epsilon_n$. The set of vectors $\{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\}$ forms a root system with Dynkin diagram C_n . A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = 2\epsilon_n$. The Weyl group of C_n is isomorphic the Weyl group of B_n . We calculate $\{\alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I\} = \{\epsilon_1 + \epsilon_i \mid 2 \leq i \leq n\} \cup \{2\epsilon_1\}$. Now, $ht(2\epsilon_1) = 2n - 1$, and $ht(\epsilon_1 + \epsilon_i) = 2n - i$, for $2 \leq i \leq n$. Following Proposition 7.3 and Table 1, we have

$$m(v_I) = \frac{n(n+1) \cdots (2n-1)}{(1)(3) \cdots (2n-1)} = 2^{n-1}.$$

B.4. Type D

Let V be a vector space with orthonormal basis $\epsilon_1, \dots, \epsilon_n$. The set of vectors $\{\pm\epsilon_i \pm \epsilon_j\}$ forms a root system with Dynkin diagram D_n . A choice of simple roots is $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < n$, and $\alpha_n = \epsilon_{n-1} + \epsilon_{n+1}$.

Let S_{2n} be the symmetric group on the letters $\{1, \dots, n, \bar{n}, \dots, \bar{1}\}$, and let $r_{ij} \in S_{2n}$ be the transposition switching the letters i and j . The Weyl group W can be viewed as the subgroup of S_{2n} generated by the reflections,

$$s_{\epsilon_i - \epsilon_j} = r_{ij}, \quad s_{\epsilon_i + \epsilon_j} = r_{i\bar{j}} r_{\bar{i}, j}, \quad 1 \leq i < j \leq n.$$

Given $v, w \in W$, if $v \leq w$ in the Bruhat order on W , then $v \leq w$ in the Bruhat order on S_{2n} ; see [39, §7.1.1]. A simple computation yields

$$v_I w_I = \begin{cases} [\bar{2}, \dots, \overline{n-1}, 1, n] & \text{if } n \text{ is even,} \\ [\bar{2}, \dots, \overline{n-1}, 1, \bar{n}] & \text{if } n \text{ is odd.} \end{cases}$$

Observe that $\{\alpha \in \Phi_I^+ \mid s_\alpha \not\leq v_I w_I\} = \{\epsilon_1 + \epsilon_i \mid 2 \leq i \leq n\} \cup \{\epsilon_1 - \epsilon_n\}$. Now $ht(\epsilon_1 - \epsilon_n) = n - 1$, and $ht(\epsilon_1 + \epsilon_i) = 2n - 1 - i$, for $2 \leq i \leq n$. Consequently, we deduce from Proposition 7.3 and Table 1 that

$$m(v_I) = \frac{(n-1)(n-1)n \cdots (2n-3)}{(1)(3) \cdots (2n-3)(n-1)} = \frac{(n-1) \cdots (2n-3)}{(1)(3) \cdots (2n-3)} = 2^{n-2}.$$

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