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Structure constants in equivariant oriented cohomology of flag varieties

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Abstract
We introduce generalized Demazure operators for the equivariant oriented cohomology of the flag variety, which have specializations to various Demazure operators and Demazure–Lusztig operators in both equivariant cohomology and equivariant K-theory. In the context of the geometric basis of the equivariant oriented cohomology given by certain Bott–Samelson classes, we use these operators to obtain formulas for the structure constants arising in different bases. Specializing to divided difference operators and Demazure operators in singular cohomology and K-theory, we recover the formulas for structure constants of Schubert classes obtained in Goldin and Knutson (Pure Appl Math Q 17(4):1345–1385, 2021). Two specific specializations result in formulas for the the structure constants for cohomological and K-theoretic stable bases as well; as a corollary we reproduce a formula for the structure constants of the Segre–Schwartz–MacPherson basis previously obtained by Su (Math Zeitschrift 298:193–213, 2021). Our methods involve the study of the formal affine Demazure algebra, providing a purely algebraic proof of these results.

Keywords Equivariant oriented cohomology · Schubert classes · Structure constants

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1 Introduction

Flag varieties G/B are among the most studied varieties in topology and algebraic geometry. They have a cellular decomposition by Schubert cells, whose closures are called Schu-

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bert varieties. Schubert varieties are invariant under a torus action and, consequently, their torus-equivariant cohomology is spanned as a module by the Schubert classes. Throughout this paper, "cohomology" (as opposed to "generalized cohomology") refers to Borel-Moore cohomology.

Other classes associated to Schubert varieties in the equivariant cohomology H_T^* and equivariant K-theory K_T of the flag variety G/B include Chern–Schwartz–MacPherson (CSM) classes and Motivic Chern (mC) classes, studied in [1, 2, 17–20, 22]. These classes coincide with the corresponding *stable bases* of Maulik-Okounkov [16] for H_T^* and K_T, of the Springer resolutions. Due to this fact, we always refer to the CSM classes as the cohomological stable basis, and to the mC classes as the K-theoretic stable basis. These classes behave like Schubert classes in their corresponding theories. Roughly speaking, Schubert classes in $H_T^*(G/B)$ and $K_T(G/B)$ are constructed by *Demazure operators* (also called *divided difference operators*), and elements of the stable bases are constructed by *Demazure–Lusztig operators.* All these operators generate various Hecke-type algebras.

Structure constants of Schubert classes are central objects in Schubert calculus, appearing in important questions of representation theory and combinatorics. In [11], the first author and Knutson obtain formulas for the structure constants in $H_T^*(G/B)$ and $K_T(G/B)$ using geometric properties of Bott–Samelson resolutions of Schubert varieties. They pull-back the Schubert classes to the equivariant cohomology (or equivariant K-theory) of Bott–Samelson variety, apply the cup product in this variety, then push-forward back to G/B. In [21], Su generalized this method to the so-called Segre–Schwartz–MacPherson (SSM) classes, a variant form of CSM classes.

We are interested in generalized cohomology theories, called oriented cohomology theories, defined by Levine and Morel [15]. These cohomologies are contravariant functors defined on the category of smooth projective varieties over a field k of characteristic 0 to the category of commutative rings, such that for proper maps, there is a push-forward map on cohomology groups. Examples include Chow rings (cohomology), K-theory and algebraic cobordism. Chern classes are defined for each oriented cohomology theory $\mathbb{h}$, and there is an associated formal group law F defined over $R = \mathbb{h}(\mathrm{pt})$. The machinery works equivariantly as well, resulting in a cohomology theory $\mathbb{h}_T$ with an associated formal group law F defined over $R = \mathbb{h}_T(pt)$.

For flag varieties, generalizing work of Kostant and Kumar [13, 14] on equivariant cohomology and equivariant K-theory of flag varieties, the ring $\mathbb{h}_T(G/B)$ has a nice algebraic model, constructed in Hoffmann et al. in [12], and studied in [5–7] by Calmès, Zainoulline, and the second author. One can define the (formal) Demazure operators X_α associated to each simple root α. These operators generate a non-commutative algebra, called the formal affine Demazure algebra $\mathbf{D}_F$. It is a free left $\mathbb{h}_T(\mathrm{pt})$-module with basis $\{X_{I_w} \mid w \in W\}$, where X_{I_w} is, roughly speaking, a product of the operators X_α, with I_w indicating a reduced word expression for w.

The algebra $\mathbf{D}_F$ is also a co-commutative co-algebra, where the coproduct comes from the twisted Leibniz rule of the operator X_α. Taking the $\mathbb{h}_T(\mathrm{pt})$-dual, one obtains a commutative ring $\mathbf{D}_F^*$, a free $\mathbb{h}_T(\mathrm{pt})$-module isomorphic to $\mathbb{h}_T(G/B)$, together with a dual basis $\{X_{I_w}^* \mid w \in W\}$. Indeed, for equivariant Chow group/cohomology/K-theory, $X_{I_w}^*$ coincides, up to various normalizations, to the Schubert class associated with w. Then $H_T^*(G/B)$ and $K_T(G/B)$ are achieved with the same module basis, and a restricted coefficient ring: a polynomial ring for $H_T^*(G/B)$ and Laurent polynomial ring for $K_T(G/B)$.

We notice that the product structure on $\mathbf{D}_F^*$ is obtained by dualizing the coproduct structure of $\mathbf{D}_F$. It follows that the structure constants of the basis $X_{I_w}^*$ may be deduced from the twisted Leibniz rule of the product $X_{\beta_1} X_{\beta_2} \cdots X_{\beta_k}$ for a reduced word $s_{\beta_1} \cdots s_{\beta_k}$ of $w \in W$. This is the main idea of the proof of Theorem 3.7, which implies the main result, Theorem 4.1. Specializing $\mathbb{h}_T$ to equivariant cohomology and equivariant K-theory, we recover the formulas of the first author and Knutson in [11].

In the case of $H_T^*(G/B)$ and $K_T(G/B)$, replacing the Demazure operators X_α by the Demazure–Lusztig operators T_α and τ_α^-, one obtains the stable bases for $H_T^*(G/B)$ and $K_T(G/B)$, respectively. Both the cohomology stable basis and the K-theory stable basis can be described in an analogous fashion to the story for Schubert classes. That is, the Demazure–Lusztig operators generate a degenerate affine Hecke algebra (for equivariant cohomology) and an affine Hecke algebra (for equivariant K-theory). The dual elements to products of these operators are essentially the cohomological/K-theoretic stable bases, so their respective twisted Leibniz rules result in a formula for the structure constants of stable bases. For instance, for cohomology, we recover the formula of Su [21] (see Remark 6.6).

To work with the Demazure operators X_α and Demazure–Lusztig operators T_α at the same time, we define a general operator Z_α (see Sect. 3) in a ring containing $\mathbf{D}_F$, which can be specialized to X_α and T_α. Our main results are Theorems 4.1 and 6.3, which state a formula for structure constants of the basis determined by Z_α and apply it to the cohomological stable basis.

The paper is organized as follows: In Sect. 2 we recall necessary notation introduced by the second author in [5–7]. We recall the definition of a Demazure element, the formal affine Demazure algebra, its dual, and relation with $\mathbb{h}_T(G/B)$. In Sect. 3 we prove the twisted Leibniz rule for the operator Z_α, which is used to derive the structure constants of the basis $Z_{I_w}^*$ in Sect. 4. In Sect. 5, we specialize our result to Demazure operators in cohomology and K-theory, and recover the formulas in [11]. In Sect. 6 we specialize our result to Demazure–Lusztig operators in cohomology, which, as a by-product, recovers the formula due to Su in [21]. In Sect. 7 we consider Demazure–Lusztig operators in K-theory and obtain a formula for the structure constants of the K-theoretic stable basis. In Sect. 8, for equivariant oriented cohomology, we generalize some results of Kostant-Kumar ([13, Proposition 4.32], [14, Lemma 2.25]) by relating our formula for structure constants with a restriction formula of Schubert classes.

2 Preliminary

We follow notation used in [5–7]. Let $\Sigma \hookrightarrow \Lambda^\vee, \alpha \mapsto \alpha^\vee$ be a semi-simple root datum of rank n. That is, Σ is the finite set of roots, Λ is the lattice and $\Lambda^\vee$ is its dual. Let $\{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots, Σ^+ and Σ^- be the set of positive and negative roots, respectively.

Let W be the Weyl group generated by the associated simple reflections $s_i := s_{\alpha_i}$. Denote by $\leq$ the Bruhat order, and let $\ell(v)$ be the length of an element $v \in W$. Note that W acts on Λ since it preserves the root system. For each sequence $I = (i_1, \dots, i_k)$ with $i_j \in [n]$, denote the product $s_{i_1} \cdots s_{i_k} \in W$ by $\prod I$, in which we keep track both of the concatenated sequence of simple reflections and the resulting element of W. If $\prod I$ is a reduced word expression for the resulting Weyl group element, we say that I is a *reduced sequence*. Following [11, Sect. 1], define the *Demazure product*

$$\widetilde{\prod} I = s_{i_1} \cdots s_{i_k}$$

subject to the braid relations and $s_i^2 = s_i$ for all i. Observe that $\prod I = \widetilde{\prod} I$ when I is a reduced sequence. When I is a reduced sequence for w, we may denote it by I_w and abuse notation by calling it a reduced word for w. Finally, let I^{rev} denote the sequence obtained by reversing the sequence I.

Let F be a formal group law over the coefficient ring R. Examples of formal group laws include the additive formal group law $F_a = x + y$ and the multiplicative formal group law $F_m = x + y - xy$. Suppose the root datum together with the formal group law satisfy the regularity condition of [5, Lemma 2.7]. This guarantees that all the properties that we use from [5–7] hold. Indeed, the regularity condition guarantees that the elements $x_\alpha, \alpha \in \Lambda$ defined in S below are non-zero-divisors. In particular, the Demazure operators X_α for simple roots α are well defined.

Let G be a split semi-simple linear algebraic group with maximal torus T and a Borel subgroup B. Let the associated root datum of G be $\Sigma \hookrightarrow \Lambda^\vee$, so Λ is the group of characters of T.

Let $\mathbb{h}$ be an oriented cohomology theory of Levine and Morel. Roughly speaking, it is a contravariant functor from the category of smooth projective varieties to the category of commutative rings such that there is a push-forward map for any proper map. The Chern classes of vector bundles are defined. Associated to $\mathbb{h}$, there is a formal group law is F defined over $R = \mathbb{h}(\text{pt})$. That is, the first Chern class of line bundles over a smooth projective variety X satisfies

$$c_1^{\mathbb{h}}(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1^{\mathbb{h}}(\mathcal{L}_1), c_1^{\mathbb{h}}(\mathcal{L}_2)).$$

For example, F_a (resp. F_m) is associated to the Chow group (or cohomology) (resp. K-theory). Both can be extended to the torus equivariant setting. We assume the equivariant oriented cohomology theory $\mathbb{h}_T$ is Chern-complete over the point for T, that is, the ring $\mathbb{h}_T(\text{pt})$ is separated and complete with respect to the topology induced by the γ-filtration [5, Definition 2.2]. In particular, this includes the completed equivariant Chow ring, the completed equivariant K-theory and equivariant algebraic cobordism.

Let S be the formal group algebra defined in [4]:

$$S = R[[\Lambda]]_F := R[[x_\lambda | \lambda \in \Lambda]]/J_F, \tag{1}$$

where J_F is the closure of the ideal generated by x_0 and $x_{\lambda+\mu} - F(x_\lambda, x_\mu)$, for all $\lambda, \mu \in \Lambda$. Indeed, if $\{t_1, \ldots, t_n\}$ is a basis of Λ, then S is (non-canonically) isomorphic to $R[[x_{t_1}, \ldots, x_{t_n}]]$. According to [5, Sect. 3], $S \cong \mathbb{h}_T(\text{pt})$ with x_λ corresponding to $c_1^{\mathbb{h}}(\mathcal{L}_\lambda)$ where $\mathcal{L}_\lambda$ is the line bundle associated to $\lambda \in \Lambda$. Since $x_{-\lambda}$ is the formal inverse of x_λ, i.e. $F(x_\lambda, x_{-\lambda}) = 0$ in S, we may write

$$x_{-\lambda} = -x_\lambda + \text{higher degree terms} \in S.$$

Define $Q := S[\frac{1}{x_\alpha} | \alpha \in \Sigma]$. We will frequently need the special element of Q given by $\kappa_\lambda := \frac{1}{x_\lambda} + \frac{1}{x_{-\lambda}}$. Note that κ_λ actually belongs to S. Note also that the action of W on Λ induces an action of W on S.

Example 2.1 Two cases of the formal product appear widely in the literature [4, Sect. 2].

1. If $F = F_a$ with $R = \mathbb{Z}$, then $\mathbb{h}$ is the cohomology/Chow groups, and $S \cong \text{Sym}_{\mathbb{Z}}(\Lambda)^\wedge$ (with $x_\lambda \mapsto \lambda$) is the completion of the polynomial ring at the augmentation ideal. In this case $x_{-\lambda} = -x_\lambda$ and $\kappa_\lambda = 0$.
2. If $F = F_m$ with $R = \mathbb{Z}$, then $\mathbb{h}$ is K-theory, and $S \cong \mathbb{Z}[\Lambda]^\wedge$ (with $x_\lambda \mapsto 1 - e^{-\lambda}$) is the completion of the Laurent polynomial ring at the augmentation ideal. In this case $x_{-\lambda} = \frac{x_\lambda}{x_\lambda - 1}$, and $\kappa_\lambda = 1$.

To obtain equivariant cohomology $H_T^*(X)$ and equivariant K-theory $K_T(X)$, we restrict the coefficient ring to $S^a = Sym[\Lambda]$ and $S^m = \mathbb{Z}[\Lambda]$, respectively.

2.1 The operator algebras Q_W and D_F

This paper is concerned with various divided difference operators acting on $\mathbb{h}_T(G/B)$, the equivariant oriented cohomology of G/B. To create an algebraic framework for these operators, following [6, 7] we localize S at $\{x_\alpha\}$ to create an algebra out of this localization and the Weyl group, as follows.

Let S be the ring described in (1), and let $Q := S[\frac{1}{x_\alpha}|\alpha \in \Sigma]$. Define $Q_W := Q \rtimes R[W]$, as a left Q-module with basis $\{\delta_w\}$, $w \in W$.

We shall see that Q_W acts on its dual space Q_W^*, which is identified with $Q \otimes_S \mathbb{h}_T(G/B)$, the cohomlogy of G/B with inverted Chern classes.

We impose a product on Q_W by

$$(p\delta_w)(p'\delta_{w'}) = pw(p')\delta_{ww'}, \quad \text{for all } p, p' \in Q, \text{ and } w, w' \in W,$$

using the natural W action on Q induced from that on Λ and extending linearly. Note that Q identified with $Q\delta_e$ is a subring of Q_W under this product, where $e \in W$ denotes the identity element of W. We routinely abuse notation and write δ_α for δ_{s_α}, and use $1 = \delta_e$ to denote the identity element of Q_W. The ring Q_W acts on Q by

$$p\delta_w \cdot p' = pw(p'), \quad \text{for all } p, p' \in Q.$$

The action of Q_W on Q induces a coproduct structure on Q_W as follows. Let $\eta = \sum_{w\in W} q_w\delta_w \in Q_W$. Then

$$\eta \cdot (pq) = \sum_w q_w w(pq) = \sum_w q_w w(p)w(q) = \sum_w q_w(\delta_w \cdot p)(\delta_w \cdot q).$$

This action factors through the coproduct $\Delta : Q_W \to Q_W \otimes_Q Q_W$

$$\Delta(\eta) = \sum_w q_w \Delta(\delta_w) = \sum_w q_w \delta_w \otimes \delta_w. \tag{2}$$

In other words, the coproduct structure on Q_W is induced from the Q_W-action on Q.

For any simple root α we define the Demazure element X_α and the push-pull element Y_α in Q_W:

$$X_\alpha = \frac{1}{x_\alpha}(1 - \delta_\alpha) \qquad \text{and} \qquad Y_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_\alpha.$$

We observe the relationship $Y_\alpha = \kappa_\alpha - X_\alpha$. In particular, if $F = F_a$ (resp. $F = F_m$), then $Y_\alpha = -X_\alpha$ (resp. $Y_\alpha = 1 - X_\alpha$).

The way Q_W acts on Q implies that X_α acts in a fashion similar to the Demazure operator defined in [8] (and there denoted D_α). In particular, $X_\alpha \cdot S \subset S$ and, for any $r \in R$, $X_\alpha \cdot r = 0$ and $\delta_\alpha \cdot r = s_\alpha(r) = r$.

Let $\mathbf{D}_F$ be the R-subalgebra of Q_W

$$\mathbf{D}_F = \langle S, X_{\alpha_1}, \ldots, X_{\alpha_n} \rangle$$

generated by S and the elements $X_\alpha \in Q_W$ for simple roots α, and call it the *formal affine Demazure algebra*. It is also generated by S and $\{Y_\alpha : \alpha \text{ simple}\}$. As a left S module, $\mathbf{D}_F$ is also free with basis $\{X_{I_w}\}_{w\in W}$, or with basis $\{Y_{I_w}\}_{w\in W}$; see [6, Proposition 7.7].

Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced word decomposition and $I_w = (i_1, \ldots, i_k)$ the corresponding sequence of reflections. Define

$$X_{I_w} = X_{\alpha_{i_1}} \cdots X_{\alpha_{i_k}} \qquad \text{and} \qquad Y_{I_w} = Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_k}}.$$

In particular, $X_{(i)} = X_{\alpha_i}$ and $Y_{(i)} = Y_{\alpha_i}$, though we eliminate parentheses when there is no confusion. We write $X_e := 1 \in Q_W$ to indicate X_I when I is the empty sequence.

The Demazure and push-pull elements have the following properties:

Lemma 2.2 [24, Proposition 3.2] *Let α and β be simple roots. The following identities hold in Q_W:*

1. $X_\alpha^2 = \kappa_\alpha X_\alpha, \quad Y_\alpha^2 = \kappa_\alpha Y_\alpha$.
2. $X_\alpha p = s_\alpha(p) X_\alpha + X_\alpha \cdot p, \quad p \in Q$.
3. *If* $(s_\alpha s_\beta)^2 = e$, *then* $X_\alpha X_\beta = X_\beta X_\alpha$.
4. *If* $(s_\alpha s_\beta)^3 = e$, *then* $X_\beta X_\alpha X_\beta - X_\alpha X_\beta X_\alpha = \kappa_{\alpha\beta} X_\alpha - \kappa_{\beta\alpha} X_\beta$, *where*
$$\kappa_{\alpha\beta} = \frac{1}{x_{\alpha+\beta} x_\beta} - \frac{1}{x_{\alpha+\beta} x_{-\alpha}} - \frac{1}{x_\alpha x_\beta}.$$
Furthermore, $\kappa_{\alpha\beta} \in S$ *by* [12, Lemma 6.7].
5. *Suppose* $s_\alpha s_\beta$ *has order* m *with* $m = 4$ *or* 6, *and* I_w *is a choice of reduced word for* $w \in W$. *Then*
$$\underbrace{X_\alpha X_\beta X_\alpha \cdots}_{m} - \underbrace{X_\beta X_\alpha X_\beta \cdots}_{m} = \sum_{v \in W} c_{I_v} X_{I_v},$$
where $c_{I_v} = 0$ *if* $v \not\leq \underbrace{s_\alpha s_\beta s_\alpha \cdots}_{m}$. *Moreover,* $c_{I_v} = 0$ *if* $\ell(v) = m - 1$ *or* $v = e$.

Lemma 2.2 (4)–(5) imply that the operators X_α (and similarly Y_α) do not satisfy braid relations for general F. For $F = F_a$ or $F = F_m$, they do; in these cases, the coefficients $\kappa_{\alpha\beta}$ and c_{I_v} all vanish. In general, X_{I_w} and Y_{I_w} depend on the choice of I_w due to this failure of braid relations.

For the purposes of this paper, we fix a reduced sequence I_w of w for each $w \in W$. While the specific coefficients and calculations regarding X_{I_w} and Y_{I_w} depend on this choice, statements regarding bases and ring phenomena do not.

By construction, $\{\delta_v : v \in W\}$ form a basis of Q_W as a module over Q. In [6], and extended in [7], the second author proves that $\{X_{I_v} : v \in W\}$ and $\{Y_{I_v} : v \in W\}$ also form bases of Q_W as a module over Q, and that the change of basis matrix from $\{X_{I_v}\}$ (or from $\{Y_{I_v}\}$) to $\{\delta_v\}$ consists of elements of S. In particular, $\{\delta_v\}$ are elements of $\mathbf{D}_F$. The lower-triangularity of the change of bases matrices is expressed in the following lemma.

Lemma 2.3 [7, Lemma 3.2, Lemma 3.3] *For each $v \in W$, choose a reduced decomposition of v and let I_v be its corresponding sequence. There exist elements $a^X_{I_w,v} \in Q$ for $v \in W$, and $b^X_{w,I_v} \in S$ such that*
$$X_{I_w} = \sum_{v \leq w} a^X_{I_w,v}\, \delta_v, \ \ and \ \ \delta_w = \sum_{v \leq w} b^X_{w,I_v} X_{I_v}.$$
Similarly, there exist $a^Y_{I_w,v} \in Q$ and $b^Y_{w,I_v} \in S$ such that
$$Y_{I_w} = \sum_{v \leq w} a^Y_{I_w,v}\, \delta_v, \ \ and \ \ \delta_w = \sum_{v \leq w} b^Y_{w,I_v} Y_{I_v}.$$

Notice that nonzero coefficients b^X_{w,I_v} are elements of S with $v \leq w$.

Example 2.4 Consider the root datum A_2, with

$$W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, w_0\},$$

where w_0 is the longest element and s_i is the reflection corresponding to α_i for $i = 1, 2$. We fix the reduced sequence $I_{w_0} = (1, 2, 1)$ for w_0. For simplicity, let $\alpha_{13} = \alpha_1 + \alpha_2$. By direct computation,

$$\begin{aligned}
\delta_e &= X_e & \delta_{s_1 s_2} &= 1 - x_1 X_{(1)} - x_{\alpha_{13}} X_{(2)} + x_{\alpha_1} x_{\alpha_{13}} X_{(1,2)},\\
\delta_1 &= 1 - x_{\alpha_1} X_{(1)} & \delta_{s_2 s_1} &= 1 - x_{\alpha_2} X_{(2)} - x_{\alpha_{13}} X_{(1)} + x_2 x_{\alpha_{13}} X_{(2,1)},\\
\delta_2 &= 1 - x_{\alpha_2} X_{(2)} & \delta_{w_0} &= 1 - x_{\alpha_{13}} X_{(2)} - (x_{\alpha_1} + x_{\alpha_2} - \kappa_{\alpha_1} x_{\alpha_1} x_{\alpha_2}) X_{(1)}\\
& & &\quad + x_{\alpha_1} x_{\alpha_{13}} X_{(1,2)} + x_{\alpha_2} x_{\alpha_{13}} X_{(2,1)} - x_{\alpha_1} x_{\alpha_2} x_{\alpha_{13}} X_{I_{w_0}}.
\end{aligned}$$

2.2 The dual operator algebras

The dual Q-module

$$Q_W^* = \mathrm{Hom}_Q(Q_W, Q) \cong \mathrm{Hom}(W, Q),$$

contains a natural basis $\{f_w\}_{w \in W}$ dual to $\{\delta_w\}_{w \in W}$, defined by

$$\langle f_w, \delta_v \rangle = \begin{cases} 1 & \text{if } w = v; \\ 0 & \text{otherwise.} \end{cases}$$

One may think of Q_W^* as the T-equivariant oriented cohomology of W with the trivial T action, tensored with Q. In particular,

$$Q_W^* = Q \otimes_S \mathbb{h}_T(W) = Q \otimes_S \mathbb{h}_T(G/B).$$

The module Q_W^* forms a ring with product $f_w f_v = 1$ if an only if $w = v$, and 0 otherwise, extended linearly to all elements of Q_W^*, and unity $\mathbf{1} = \sum_{w \in W} f_w$. This product structure is equivalent to the one induced from the coproduct structure (see Sect. 4 below).

The ring Q_W acts on Q_W^* by

$$\langle z \cdot f, z' \rangle = \langle f, z' z \rangle, \quad \text{for all } z, z' \in Q_W,\ f \in Q_W^*.$$

In the bases $\{\delta_w\}$ of Q_W and $\{f_w\}$ of Q_W^*, the action has explicit formulation

$$p\delta_w \cdot (q f_v) = q v w^{-1}(p) f_{vw^{-1}}, \text{ for all } p, q \in Q. \tag{3}$$

Denote

$$\mathrm{pt}_w = \left(\prod_{\alpha < 0} x_\alpha \right) \cdot f_w = w \left(\prod_{\alpha < 0} x_\alpha \right) f_w \in Q_W^*.$$

Let $\mathbf{D}_F^* := \mathrm{Hom}_S(\mathbf{D}_F, S) \subset Q_W^*$ be the dual S-module to $\mathbf{D}_F$. It is proved in [7, Lemma 10.3] that $\mathrm{pt}_w \in \mathbf{D}_F^*$. Let

$$\begin{aligned}
\zeta_{I_w}^X &= X_{I_w^{\mathrm{rev}}} \cdot \mathrm{pt}_e, \text{ and}\\
\zeta_{I_w}^Y &= Y_{I_w^{\mathrm{rev}}} \cdot \mathrm{pt}_e .
\end{aligned}$$

Then $\{\zeta_{I_w}^X\}$ forms a basis of D_F^* over S, as does $\{\zeta_{I_w}^Y\}$.

Finally, let $\{X_{I_w}^*\}$, (respectively $\{Y_{I_w}^*\}$) be the bases dual to $\{X_{I_w}\}$ (resp. $\{Y_{I_w}\}$) in $\mathbf{D}_F^*$, which are also Q-basis of Q_W^*.

The classes $X^*_{I_v}$ for each $v \in W$ are determined by duality. Under the dual pairing,

$$\langle X^*_{I_v}, \delta_w\rangle = \left\langle X^*_{I_v}, \sum_{u\in W} b^X_{w,I_u} X_{I_u}\right\rangle = b^X_{w,I_v}.$$

Set $X^*_{I_v} = \sum_{u\in W} m_{I_v,u} f_u$, which implies

$$\langle X^*_{I_v}, \delta_w\rangle = \left\langle \sum_{u\in W} m_{I_v,u} f_u, \delta_w\right\rangle = m_{I_v,w},$$

and thus $X^*_{I_v} = \sum_{w\in W} b^X_{w,I_v} f_w$.

Example 2.5 Consider the root datum A_2, with $W = \{e, s_1, s_2, s_1s_2, s_2s_1, w_0\}$. Fix the reduced sequence $w_0 = s_1s_2s_1$. The calculations from Example 2.4 imply

$$\begin{aligned}
X^*_e &= \mathbf{1} = \sum_{w\in W} f_w, & X^*_{(1,2)} &= x_{\alpha_1}x_{\alpha_{13}}(f_{s_1s_2} + f_{w_0})\\
X^*_{(1)} &= -x_{\alpha_1}(f_{s_1} + f_{s_1s_2}) - x_{\alpha_{13}} f_{s_2s_1} - y f_{w_0}, & X^*_{(2,1)} &= x_{\alpha_2}x_{\alpha_{13}}(f_{s_2s_1} + f_{w_0})\\
X^*_{(2)} &= -x_{\alpha_2}(f_{s_2} + f_{s_2s_1}) - x_{\alpha_{13}}(f_{s_1s_2} + f_{w_0}), & X^*_{I_{w_0}} &= -x_{\alpha_1}x_{\alpha_2}x_{\alpha_{13}} f_{w_0},
\end{aligned}$$

where $y = x_{\alpha_1} + x_{\alpha_2} - \kappa_{\alpha_1}x_{\alpha_1}x_{\alpha_2}$. In case $F = F_a$ or F_m, we have $y = x_{\alpha_{13}}$.

The following proposition explains the relationship between the algebraic construction above and equivariant oriented cohomology of G/B.

For each reduced sequence I_w, let $\mathcal{X}_{I_w} \to G/B$ denote the Bott–Samelson resolution. The push-forward in $\mathtt{h}_T$ of the fundamental class along this resolution is called the *Bott–Samelson class of* I_w, which we denote by η_{I_w}. Define a map

$$\Phi : \mathbf{D}^*_F \longrightarrow \mathtt{h}_T(G/B)$$

given by $\Phi(\zeta^Y_{I_w}) = \eta_{I_w}$ and $\Phi(\mathbf{1}) = [G/B]$, the fundamental class of G/B, and extended as a module over S.

Proposition 2.6 *The isomorphism Φ satisfies the following properties:*

1. [5, Theorem 8.2, Lemma 8.8] *The map Φ is a functorial isomorphism.*
2. [7, Theorem 14.7] *The basis $\{\Phi(X^*_{I_w}) : w \in W\}$ (resp. $\{\Phi(Y^*_{I_w})) : w \in W\}$) is dual to $\Phi(\zeta^X_{I_w})$ (resp. $\Phi(\zeta^Y_{I_w})$) via the nondegenerate dual pairing on $\mathtt{h}_T(G/B)$ given by multiplying and pushing forward to a point.*
3. [5, Corollary 6.4] *Let $i_w : wB \hookrightarrow G/B$ be the inclusion of the T-fixed point corresponding to $w \in W$, and $(i_w)_* : \mathtt{h}_T(wB) \to \mathtt{h}_T(G/B)$ be the pushforward map. Then $\Phi(\mathrm{pt}_w) = (i_w)_*(1)$.*
4. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{D}^*_F \arrow[r, hook] \arrow[d, "\cong"] & Q^*_W \arrow[d, "\cong"] \\
\mathtt{h}_T(G/B) \arrow[r, hook, "\bigoplus_{w\in W} i^*_w"] & Q \otimes_S \mathtt{h}_T(W),
\end{tikzcd}$$

where the top horizontal map is the embedding of the S-module into the Q-module Q^*_W.

By specializing the formal group law to F_a or F_m, respectively, and restricting S to $R[\Lambda]/J_F$, we obtain a map $\Phi^H : \mathbf{D}_F^* \to H_T^*(G/B)$ or $\Phi^K : \mathbf{D}_F^* \to K_T(G/B)$ to the equivariant cohomology or equivariant K-theory. The map remains an isomorphism over the corresponding module. From now on we will not distinguish between $\mathbf{D}_F^*$ and $\mathbb{h}_T(G/B)$.

Example 2.7 Let $X(w) = \overline{BwB/B}$ be the Schubert variety and $Y(w) = \overline{B^-wB/B}$ be the opposite Schubert variety. For $H_T^*(G/B)$ (with $F = F_a$) or $K_T(G/B)$ (with $F = F_m$), we write w for I_w since X_{I_w} and Y_{I_w} are independent of the reduced sequence.

1. [11, Sect. 1.2] For $H_T^*(G/B)$, $\zeta_w^Y = [X(w)]$, and $\zeta_w^X = (-1)^{\ell(w)}[X(w)]$, where each homology class is identified with its dual cohomology class. Then $Y_w^* = [Y(w)]$ and similarly $X_w^* = (-1)^{\ell(w)}[Y(w)]$.
2. [3, Sect. 3] For $K_T(G/B)$, $\zeta_w^Y = [\mathcal{O}_{X(w)}]$ is the class of the structure sheaf of $X(w)$, $Y_w^* = [\mathcal{O}_{Y(w)}(-\partial Y(w))]$, $\zeta_w^X = (-1)^{\ell(w)}[\mathcal{O}_{X(w)}(-\partial X(w))]$, and $X_w^* = (-1)^{\ell(w)}[\mathcal{O}_{Y(w)}]$.

3 Generalized Demazure operators and the generalized Leibniz rule

In this section, we generalize the operators X_{I_v} and Y_{I_v} on $\mathbb{h}_T(G/B)$ to a more general class of elements of Q_W, and prove the generalized Leibniz rule for $\mathbf{D}_F$ acting on Q. We use this result to compute the coproduct structure in Q_W, and then the product structure in Q_W^*.

Let $\{a_\alpha, b_\alpha \in Q : \alpha \in \Sigma\}$ be a set of elements with the property that, for all $w \in W$,

$$w(a_\alpha) = a_{w(\alpha)}, \qquad w(b_\alpha) = b_{w(\alpha)}, \text{ and } b_\alpha \text{ are all invertible in } Q.$$

For any simple root α, define operators $Z_\alpha \in Q_W$ by

$$Z_\alpha = a_\alpha + b_\alpha \delta_\alpha.$$

Clearly X_α and Y_α result from Z_α as special cases of a_α and b_α. For any sequence $I = (i_1, \ldots, i_k)$, define $Z_I \in Q_W$ by

$$Z_I = Z_{\alpha_{i_1}} Z_{\alpha_{i_2}} \cdots Z_{\alpha_{i_k}}.$$

We call Z_I *generalized Demazure operators.*

As before, we choose a reduced word expression I_v for each $v \in W$.

Lemma 3.1 *The set of generalized Demazure operators $\{Z_{I_v}\}$ forms a basis of Q_W as a module over Q.*

Proof This follows from the fact that $b_\alpha \in Q$ is invertible for all simple roots α (hence, for all roots α). ◻

Remark 3.2 Note that $Z_\alpha \in \mathbf{D}_F$ if and only it satisfies the residue condition [23, Definition 3.7]. If this is satisfied, then $Z_{I_v} \in \mathbf{D}$ and equivalently, $Z_{I_v}^* \in \mathbf{D}_F^*$. Moreover, Z_{I_v} forms a basis of $\mathbf{D}_F$ if and only if $\frac{1}{b_\alpha} \in S$ for all α. For example, this holds for X_α, Y_α, but fails for T_α considered in Sects. 6 and 7. This is precisely why the stable basis is only a basis after localization.

Lemma 3.3 *For any sequence J, define coefficients $c_{J,I_w} \in Q$ by*

$$Z_J = \sum_{w \in W} c_{J,I_w} Z_{I_w}, \tag{4}$$

Then $c_{J,I_w} = 0$ unless $w \leq \widetilde{\prod} J$.

Proof Clearly $Z_\alpha = a_\alpha + b_\alpha \delta_\alpha$ has support on $\{w : w \le s_\alpha\}$. An immediate observation of the product in Q_W shows inductively that Z_J may be expressed as a Q-linear combination of δ_v for $v \le \widetilde{\prod} J$.

For any $v \in W$ and reduced sequence $I_v = (i_1, \ldots, i_k)$, let $\gamma_j = \alpha_{i_j}$ for $j = 1, \ldots, k$. The coefficient of δ_v in Z_{I_v} is

$$b_{\gamma_1} s_{\gamma_1}(b_{\gamma_2}) s_{\gamma_1} s_{\gamma_2}(b_{\gamma_3}) \ldots s_{\gamma_1} \ldots s_{\gamma_{k-1}}(b_{\gamma_k}).$$

In particular, since b_{γ_j} is invertible, so is $w(b_{\gamma_j})$ for any Weyl group element w, and thus the coefficient of δ_v in Z_{I_v} is nonzero.

Let $A = \{w \in W : c_{J,I_w} \neq 0 \text{ and } w \nleq \widetilde{\prod} J\}$, and assume A is nonempty. Pick $v \in A$ to be a maximal element of A in the Bruhat order. By support considerations, the only terms contributing to the coefficient of δ_v in (4) is $c_{J,I_v} Z_{I_v}$. Since the coefficient of δ_v in Z_{I_v} is a unit, we conclude $c_{J,I_v} = 0$, contrary to assumption. □

The structure constants c_{J,I_w} reflect geometric properties in some special cases (see Sect. 5). When $Z_\alpha = X_\alpha$ for all α or $Z_\alpha = Y_\alpha$ for all α, and $F = F_a$, the coefficients in the sum (4) vanish unless J is a reduced word for w, in which case $c_{J,I_w} = 1$; this reflects the property that the pushforward map in homology sends the orientation class $[BS_J]$ to the Schubert variety $X(w)$ when J is a reduced word for w. When $Z_\alpha = X_\alpha$ for all α or $Z_\alpha = Y_\alpha$ for all α, and $F = F_m$, coefficients vanish except when the Demazure product of J is w, which occurs exactly once and results in $c_{J,I_w} = 1$. In this case, the K-theoretic pushforward of $[\mathcal{O}_{BS_J}]$ is the structure sheaf of $X(w)$ when $w = \widetilde{\prod} J$. More generally, Z_J is an (equivariant) operator whose dual has support only on those fixed points in the Schubert variety $X(w)$, where $w = \widetilde{\prod} J$.

We have the following lemma describing the action of Z_α on a product.

Lemma 3.4 *For a simple root α, and $p, q \in Q$, we have*

$$Z_\alpha \cdot (pq) = \frac{a_\alpha(a_\alpha + b_\alpha)}{b_\alpha} pq - \frac{a_\alpha}{b_\alpha}[(Z_\alpha \cdot p)q + p(Z_\alpha \cdot q)] + \frac{1}{b_\alpha}(Z_\alpha \cdot p)(Z_\alpha \cdot q).$$

Proof One just has to plug in $Z_\alpha = a_\alpha + b_\alpha \delta_{s_\alpha}$ and use the definition of the action $\delta_{s_\alpha} \cdot p = s_\alpha(p)$. A comparison of both sides yields the identity. □

The coefficients occurring in Lemma 3.4 may be generalized to the case of the action of Z_I on a product pq.

Definition 3.5 For each simple root α, let $Z_\alpha = a_\alpha + b_\alpha \delta_\alpha$ with $a_\alpha, b_\alpha \in Q$ and b_α invertible. Let $I = (i_1, \ldots, i_k)$ be a sequence of indices of simple roots, with $\gamma_j := \alpha_{i_j}$ corresponding to the jth entry of I. For $E, F \subset \{1, \ldots k\}$, define the **Leibniz coefficients** $\mathbf{C}^I_{E,F} \in Q$ by

$$\mathbf{C}^I_{E,F} = (B^Z_1 B^Z_2 \cdots B^Z_k) \cdot 1, \tag{5}$$

where the operators $B^Z_j \in Q_W$ are given by

$$B^Z_j = \begin{cases} \frac{1}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \in E \cap F, \\ -\frac{a_{\gamma_j}}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ a_{\gamma_j} + \frac{a^2_{\gamma_j}}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \notin E \cup F. \end{cases} \tag{6}$$

Example 3.6 Let $\gamma_j = \alpha_{i_j}$ indicate the jth root listed in the sequence I. If $Z = X$, then using the specific choice of coefficients for the Demazure operator yields

$$B_j^X = \begin{cases} -x_{\gamma_j}\delta_{\gamma_j}, & \text{if } j \in E \cap F, \\ \delta_{\gamma_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ X_{\gamma_j}, & \text{if } j \notin E \cup F. \end{cases}$$

Similarly, if $Z = Y$ indicate the push-pull operators,

$$B_j^Y = \begin{cases} x_{\gamma_j}\delta_{\gamma_j}, & \text{if } j \in E \cap F, \\ \frac{x_{\gamma_j}}{x_{-\gamma_j}}\delta_{\gamma_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ \frac{1}{x_{-\gamma_j}} + \frac{x_{\gamma_j}}{(x_{-\gamma_j})^2}\delta_{\gamma_j}, & \text{if } j \notin E \cup F. \end{cases}$$

Now we prove the main technical result of this paper, generalizing [6, Lemma 4.8].

Theorem 3.7 (Generalized Leibniz Rule) *Let Z_I be a generalized Demazure operator for $I = (i_1, \ldots, i_k)$, and let $\gamma_j = \alpha_{i_j}$ denote the jth simple root in the list. Then for any $p, q \in Q$,*

$$Z_I \cdot (pq) = \sum_{E,F\subset[k]} \mathbf{C}^I_{E,F}(Z_E \cdot p)(Z_F \cdot q),$$

where $\mathbf{C}^I_{E,F}$ are the Leibniz coefficients defined in (5)

Proof For any simple root α, observe the following two identities:

$$a_\alpha(1-\delta_\alpha) + \frac{a_\alpha(a_\alpha + b_\alpha)}{b_\alpha}\delta_\alpha = a_\alpha + \frac{a_\alpha^2}{b_\alpha}\delta_\alpha = \frac{a_\alpha}{b_\alpha}Z_\alpha\delta_\alpha, \tag{7}$$

$$Z_\alpha \cdot (pq) = a_\alpha(p - s_\alpha(p))q + s_\alpha(p)(Z_\alpha \cdot q). \tag{8}$$

We prove the theorem by induction on k. If $k = 1$, the theorem holds by Lemma 3.4.

Now assume it holds for all I with $\ell(I) < k$, and let $I = (i_1, \ldots, i_k)$. Let $J = (i_2, \ldots, i_k)$ and let $\alpha = \alpha_{i_1}$. We have

$$\begin{aligned} Z_I \cdot (pq) &= (Z_\alpha Z_J)\cdot(pq) = Z_\alpha \cdot (Z_J \cdot (pq)) \\ &= Z_\alpha \cdot \left[\sum_{E,F\subset\{2,\ldots,k\}} \mathbf{C}^J_{E,F}(Z_E\cdot p)(Z_F\cdot q)\right] \\ &= \sum_{E,F\subset\{2,\ldots,k\}} a_\alpha\left[\mathbf{C}^J_{E,F} - s_\alpha(\mathbf{C}^J_{E,F})\right](Z_E\cdot p)(Z_F\cdot q) \\ &\quad + \sum_{E,F\subset\{2,\ldots,k\}} s_\alpha(\mathbf{C}^J_{E,F})Z_\alpha\cdot[(Z_E\cdot p)(Z_F\cdot q)] \text{ by Equation (8)} \\ &= \sum_{E,F\subset\{2,\ldots,k\}} a_\alpha\left[\mathbf{C}^J_{E,F} - s_\alpha(\mathbf{C}^J_{E,F})\right](Z_E\cdot p)(Z_F\cdot q) \\ &\quad + \sum_{E,F\subset\{2,\ldots,k\}} s_\alpha(\mathbf{C}^J_{E,F})\frac{a_\alpha(a_\alpha+b_\alpha)}{b_\alpha}(Z_E\cdot p)(Z_F\cdot q) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{E,F\subset\{2,\ldots,k\}} s_\alpha(\mathbf{C}^J_{E,F})\frac{a_\alpha}{b_\alpha}\left[(Z_\alpha Z_E\cdot p)\right](Z_F\cdot q)+(Z_E\cdot p)(Z_\alpha Z_F\cdot q)]\\
&+ \sum_{E,F\subset\{2,\ldots,k\}} s_\alpha(\mathbf{C}^J_{E,F})\frac{1}{b_\alpha}(Z_\alpha Z_E\cdot p)(Z_\alpha Z_F\cdot q) \text{ by Lemma 3.4}\\
= &\sum_{E,F\subset\{2,\ldots,k\}} \left[\left(a_\alpha+\frac{a_\alpha^2}{b_\alpha}\delta_\alpha\right)\cdot \mathbf{C}^J_{E,F}\right](Z_E\cdot p)(Z_F\cdot q)\\
&- \sum_{E,F\subset\{2,\ldots,k\}} \left(\frac{a_\alpha}{b_\alpha}\delta_\alpha\cdot \mathbf{C}^J_{E,F}\right)[(Z_\alpha Z_E\cdot p)(Z_F\cdot q)+(Z_E\cdot p)(Z_\alpha Z_F\cdot q)]\\
&+ \sum_{E,F\subset\{2,\ldots,k\}} \left(\frac{1}{b_\alpha}\delta_\alpha\cdot \mathbf{C}^J_{E,F}\right)(Z_\alpha Z_E\cdot p)(Z_\alpha Z_F\cdot q) \text{ by Equation (7).}
\end{aligned}
$$

Comparing the coefficients with $B_1^Z\cdot(\mathbf{C}^J_{E,F})$ from (6), we see that they coincide. The proof then follows by induction. □

The following corollary follows immediately. We see in Sect. 8 that the Leibniz coefficients $\mathbf{C}^I_{[k],E}$ arise as factors in summands of specific structure constants in Schubert calculus, justifying the name. Here $[k]=\{1,2,\ldots,k\}$.

Corollary 3.8 (Generalized Billey's Formula) *Let $I=(i_1,\ldots,i_k)$ be a sequence of indices of simple roots, and denote $m_j=s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(a_{i_j})$ and $n_j=s_{i_1}s_{i_2}\cdots s_{i_{j-1}}(b_{i_j})$. For $E\subset[k]$, we have*

$$\mathbf{C}^I_{[k],E}=\mathbf{C}^I_{E,[k]}=(-1)^{k-|E|}\prod_{j\in[k]\setminus E} m_j\prod_{j\in[k]} n_j^{-1}.$$

As a consequence of Theorem 3.7, [6, Proposition 9.5] and the coproduct defined in Eq. (2), we obtain the following theorem.

Theorem 3.9 *Let $Z_\alpha=a_\alpha+b_\alpha\delta_\alpha\in Q_W$ with b_α invertible, then for any $I=(i_1,\ldots,i_k)$, we have*

$$\triangle(Z_I)=\sum_{E,F\subset[k]}\mathbf{C}^I_{E,F}Z_E\otimes Z_F,$$

where $\mathbf{C}^I_{E,F}$ are defined in Definition 3.5.

We specialize Theorem 3.7 to the elements X_I and Y_I. For any index j, the operators B_j^X and B_j^Y preserve S under the action of Q_W on Q, and thus $B_j^X, B_j^Y\in\mathbf{D}_F$ (see [6, Remark 7.8]). The first statement in the next corollary is the result [6, Proposition 9.5].

Corollary 3.10 *For the Demazure elements X_α, and $I=(i_1,\ldots,i_k)$, we have*

$$X_I\cdot(pq)=\sum_{E,F\subset[k]}\mathbf{A}^I_{E,F}(X_E\cdot p)(X_F\cdot q),$$

where $\mathbf{A}^I_{E,F}=(B_1^XB_2^X\cdots B_k^X)\cdot 1$ with $B_j^X\in\mathbf{D}_F$ defined in Example 3.6. Similarly, for the push-pull elements Y_α and $I=(i_1,\ldots,i_k)$, we have

$$Y_I\cdot(pq)=\sum_{E,F\subset[k]}\mathbf{B}^I_{E,F}(Y_E\cdot p)(Y_F\cdot q),$$

where $\mathbf{B}^I_{E,F}=(B_1^YB_2^Y\cdots B_k^Y)\cdot 1$, and $B_j^Y\in\mathbf{D}_F$ is defined in Example 3.6.

4 The structure constants of equivariant oriented cohomology of flag varieties

In this section we prove the main result, i.e., the formulas of structure constants of $Z^*_{I_w}$ in $\mathbb{h}_T(G/B)$, with resulting formulas for the structure constants of $X^*_{I_w}$ and of $Y^*_{I_w}$.

Let $\{Z^*_{I_w}\}$ be the basis of Q^*_W (as a module over Q) dual to the basis $\{Z_{I_w}\}$ of Q_W introduced in Sect. 3.

Theorem 4.1 *For any $u, v \in W$, the product $Z^*_{I_u}\ Z^*_{I_v}$ is given by*

$$Z^*_{I_u}\ Z^*_{I_v} = \sum_{w \geq u,v} \mathbb{c}^{I_w}_{I_u,I_v} Z^*_{I_w},$$

where

$$\mathbb{c}^{I_w}_{I_u,I_v} = \sum_{E,F \subset [\ell(w)]} \mathbf{C}^{I_w}_{E,F} c_{E,I_u} c_{F,I_v} \in Q,$$

$\mathbf{C}^{I_w}_{E,F} \in Q$ are the Leibniz coefficients given in Definition 3.5. As before, the Q elements c_{E,I_u} and c_{F,I_v} are defined as constants appearing in the expansion

$$Z_J = \sum_{w \in W} c_{J,I_w} Z_{I_w}. \tag{9}$$

Example 4.2 Consider the A_3-case. Consider $I_u = (2, 3, 1, 2, 1)$, $I_v = (1, 2, 3, 2, 1)$, then $\mathbb{c}^{I_w}_{I_u,I_v} = 0$ unless $w = w_0$ is the longest element. Fix $I_{w_0} = (1, 2, 3, 1, 2, 1)$, in which case we have

$$\begin{aligned}\mathbf{C}^{I_{w_0}}_{\{2,3,4,5,6\},\{1,2,3,5,6\}} &= B^Z_1 B^Z_2 B^Z_3 B^Z_4 B^Z_5 B^Z_6 \cdot 1 \\ &= \frac{a_{\alpha_1} a_{\alpha_2}}{b_{\alpha_1} b_{\alpha_2} b_{\alpha_1+\alpha_2} b_{\alpha_2+\alpha_3} b_{\alpha_1+\alpha_2+\alpha_3}},\end{aligned}$$

and $c_{\{2,3,4,5,6\},I_u} = c_{\{1,2,3,5,6\},I_v} = 1$. Therefore,

$$Z^*_{I_u} \cdot Z^*_{I_v} = \frac{a_{\alpha_1} a_{\alpha_2}}{b_{\alpha_1} b_{\alpha_2} b_{\alpha_1+\alpha_2} b_{\alpha_2+\alpha_3} b_{\alpha_1+\alpha_2+\alpha_3}} Z^*_{I_{w_0}}.$$

Proof of Theorem 4.1 The coproduct structure Δ on Q_W (Eq. (2)) naturally induces a product on Q^*_W. For all $f, g \in Q^*_W$ and $\sum_{w \in W} q_w \delta_w \in Q_W$,

$$\begin{aligned}\left\langle f\ g, \sum_{w \in W} q_w \delta_w \right\rangle &= \langle f \otimes g, \Delta(\sum_{w \in W} q_w \delta_w)\rangle \\ &= \langle f \otimes g, \sum_{w \in W} q_w \delta_w \otimes \delta_w\rangle \\ &= \sum_{w \in W} q_w \langle f, \delta_w\rangle\langle g, \delta_w\rangle.\end{aligned}$$

Note that this product corresponds to the product on Q_W^* introduced at the beginning of Sect. 2.2 since

$$\langle f_u\, f_v, \sum_{w\in W} q_w\delta_w\rangle = \begin{cases} \langle f_u, \sum q_w\delta_w\rangle = \sum_w q_w\langle f_u, \delta_w\rangle, & \text{if } u = v; \\ 0, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} q_u, & \text{if } u = v; \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 3.9 we have

$$\begin{aligned} \triangle(Z_{I_w}) &= \sum_{E,F\subset[\ell(w)]} \mathbf{C}^{I_w}_{E,F} Z_E \otimes Z_F \\ &= \sum_{E,F\subset[\ell(w)]} \mathbf{C}^{I_w}_{E,F} \left[\left(\sum_{u\in W} c_{E,I_u} Z_{I_u}\right) \otimes \left(\sum_{v\in W} c_{F,I_v} Z_{I_v}\right)\right] \\ &= \sum_{u,v\in W} \left[\sum_{E,F\subset[\ell(w)]} \mathbf{C}^{I_w}_{E,F} c_{E,I_u} c_{F,I_v}\right] Z_{I_u} \otimes Z_{I_v} \\ &= \sum_{u,v\in W} \mathbb{c}^{I_w}_{I_u,I_v} Z_{I_u} \otimes Z_{I_v}. \end{aligned}$$

Finally we obtain the coefficient by calculating the pairing:

$$\langle Z^*_{I_u}\, Z^*_{I_v}, Z_{I_w}\rangle = \langle Z^*_{I_u} \otimes Z^*_{I_v}, \Delta(Z_{I_w})\rangle = \mathbb{c}^{I_w}_{I_u,I_v}.$$

Let $I_w|_E$ be the subsequence obtained from restricting I_w to E. Since $w = \tilde{\prod} I_w \geq \tilde{\prod}(I_w|_E)$ for any $E \subset [\ell(w)]$, by Lemma 3.3, $\mathbb{c}^{I_w}_{I_u,I_v} = 0$ unless $u \leq w$ and $v \leq w$. $\square$

The coproduct structure on the left Q-module Q_W restricts to a coproduct structure on the left S-module $\mathbf{D}_F$ [6, Theorem 9.2]. Consequently, the embedding $\mathbf{D}_F^* \subset Q_W^*$ is an embedding of subrings. So the structure constants of the S-bases $\{X^*_{I_w}\}$ and $\{Y^*_{I_w}\}$ in $\mathbf{D}^*_F$ are precisely those of the Q-bases $\{X^*_{I_w}\}$ and $\{Y^*_{I_w}\}$ in Q^*_W.

Specializing Theorem 4.1 to the X-operators, we have

$$X^*_{I_u}\, X^*_{I_v} = \sum_{w\geq v, w\geq u} \mathbb{a}^{I_w}_{I_u,I_v} X^*_{I_w},$$

with

$$\mathbb{a}^{I_w}_{I_u,I_v} = \sum_{E,F\subset[\ell(w)]} \mathbf{A}^{I_w}_{E,F} c_{I_w|_E,I_u} c_{I_w|_F,I_v}, \tag{10}$$

where c_{I,I_v} are the coefficients that occur in the expansion $X_I = \sum_v c_{I,I_v} X_{I_v}$. It follows from [6, Theorem 9.2 and Proposition 7.7] that $\mathbf{A}^{I_w}_{E,F} \in S$, that $c_{I,I_w} \in S$, so $\mathbb{a}^{I_w}_{I_u,I_v} \in S$. Similarly, specializing to the Y-operators, the structure constants for $Y^*_{I_w}$ are denoted by $\mathbb{b}^{I_w}_{I_v,I_u}$ and can be expressed as

$$\mathbb{b}^{I_w}_{I_u,I_v} = \sum_{E,F\subset[\ell(w)]} \mathbf{B}^{I_w}_{E,F} c_{I_w|_E,I_u} c_{I_w|_F,I_v},$$

where now the coefficients c_{I,I_v} are those appearing in the expansion of Y_I. As before, $\mathbf{B}^{I_w}_{E,F} \in S$ and $c_{I,I_w} \in S$, so $\mathbb{b}^{I_w}_{I_u,I_v} \in S$. In Sect. 5 we show that these coefficients simplify in the case that $F = F_a$ or $F = F_m$, resulting in Theorem 1 from [11]. It is worth noting that the formula (10) can be used to prove the Leray-Hirsch Theorem for flag varieties (see [9]).

Example 4.3 Assume the root datum is of type A_1, then $W = \{e, s_1\}$. We calculate the basis change explicitly:

$$X_e^* = f_e + f_{s_1}, \quad X_{(1)}^* = -x_{\alpha_1} f_{s_1},$$

and then we may obtain the products directly:

$$X_e^* X_e^* = X_e^*, \qquad X_e^* X_{(1)}^* = X_{(1)}^*, \qquad X_{(1)}^* X_{(1)}^* = -x_{\alpha_1} X_{(1)}^*,$$

and note that it agrees with Theorem 4.1 with $Z = X$.

Example 4.4 Consider the root datum A_2, with $W = \{e, s_1, s_2, s_1s_2, s_2s_1, w_0\}$. For the longest element w_0, we fix the reduced sequence $I_{w_0} = s_1s_2s_1$.

We use the calculation in Example 2.5, and the product structure on Q_W^* to obtain the multiplication table for $\{X_{I_v}\}$. Recall that $f_u f_v = 1$ if $u = v$ and 0 otherwise, and that $X_e^* = f_e + f_{s_1} + f_{s_2} + f_{s_1s_2} + f_{s_2s_1} + f_{w_0}$. If $X_w = \sum_u a_u f_u$, we have

$$X_w^* X_e^* = \left(\sum_u a_u f_u\right)\left(\sum_v f_v\right) = \sum_u a_u f_u = X_w^*$$

for all $w \in W$. Similarly,

$$\begin{aligned} X_{I_{w_0}}^* \; X_{(1,2)}^* &= (-x_{\alpha_1} x_{\alpha_2} x_{\alpha_{13}} f_{w_0}) \left(x_{\alpha_1} x_{\alpha_{13}} (f_{s_1s_2} + f_{w_0})\right) \\ &= -x_{\alpha_1}^2 x_{\alpha_2} x_{\alpha_{13}}^2 f_{w_0} \\ &= x_{\alpha_1} x_{\alpha_{13}} X_{I_{w_0}}^*. \end{aligned}$$

The other products are as follows:

$$\begin{array}{lllllll}
X_{I_{w_0}}^* \; X_{I_{w_0}}^* & = & -x_1 x_2 x_{\alpha_{13}} X_{I_{w_0}}^* & & X_{I_{w_0}}^* \; X_{(2,1)}^* & = & x_2 x_{\alpha_{13}} X_{I_{w_0}}^* \\
X_{(1)}^* \; X_{(2)}^* & = & X_{(1,2)}^* + X_{(2,1)}^* + \kappa_1 X_{I_{w_0}}^* & & X_{I_{w_0}}^* \; X_{(2)}^* & = & -x_{\alpha_{13}} X_{I_{w_0}}^* \\
X_{(2)}^* \; X_{(2)}^* & = & \frac{x_{\alpha_{13}} - x_2}{x_1} X_{(1,2)}^* - x_2 X_{(2)}^*, & & X_{(2,1)}^* \; X_{(2,1)}^* & = & x_2 x_{\alpha_{13}} X_{(2,1)}^* \\
X_{(2,1)}^* \; X_{(2)}^* & = & \frac{x_{\alpha_{13}} - x_2}{x_1} X_{I_{w_0}}^* - x_2 X_{(2,1)}^*, & & X_{I_{w_0}}^* \; X_{(1)}^* & = & -y X_{I_{w_0}}^* \\
X_{(1)}^* \; X_{(1)}^* & = & -x_1 X_{(1)}^* + \frac{x_{\alpha_{13}} - x_1}{x_2} X_{(2,1)}^* & & X_{(1,2)}^* \; X_{(1,2)}^* & = & x_1 x_{\alpha_{13}} X_{(1,2)}^* \\
 & & + \frac{x_1 y + x_{\alpha_{13}}^2 - x_1 x_{\alpha_{13}} - y^2}{x_1 x_2 x_{\alpha_{13}}} X_{I_{w_0}}^*. & & & &
\end{array}$$

Here y was defined in Example 2.5.

One can check that the above coefficients $\mathbb{c}_{I_u, I_v}^{I_w}$ agree with the formula (10). Note that when computing $\mathbb{c}_{1,1}^{I_{w_0}}$, one needs to compute the following coefficients:

$$\mathbf{A}_{\{3\},\{3\}}^{I_{w_0}}, \; \mathbf{A}_{\{1,3\},\{3\}}^{I_{w_0}}, \; \mathbf{A}_{\{3\},\{1,3\}}^{I_{w_0}}, \; \mathbf{A}_{\{1,3\},\{1,3\}}^{I_{w_0}}.$$

As an application, we consider the case of a partial flag variety. Let K be a subset of $[n]$. Let P_K be the standard parabolic subgroup, $W_K < W$ the corresponding subgroup, and $W^K \subset W$ be the set of minimal length representatives of W/W_K. We say a set of reduced sequences I_w is K-compatible if for each $w = uv$, $u \in W^K$, $v \in W_K$, we have $I_w = I_u \cup I_v$, i.e., I_w is the concatenation of I_u with I_v.

Theorem 4.5 *Suppose the set $\{I_w\}$ is K-compatible. Then for any $v, u \in W^K$, we have*

$$X^*_{I_u} X^*_{I_v} = \sum_{w \in W^K, w \geq v, w \geq u} \mathbb{a}^{I_w}_{I_u, I_v} X^*_{I_w}.$$

Proof It follows from [7, Corollary 8.4] that $X^*_{I_u}, u \in W^K$ is a basis of $(Q^*_W)^{W_K}$. Moreover, from Lemma 4.3 of loc.it., we know $\delta_w \cdot (ff') = (\delta_w \cdot f)(\delta_w \cdot f')$. Therefore, $X^*_{I_u} X^*_{I_v} \in (Q^*_W)^{W_K}$, so is a linear combination of $X^*_{I_w}$, $w \in W^K$. □

Geometrically, under the assumption of this theorem, it follows from [7, Corollary 8.4] that $\{X^*_{I_w}\}_{w \in W^K}$ is a basis of $(\mathbf{D}^*_F)^{W_K} \cong \mathbb{h}_T(G/P_K)$. So the product $X^*_{I_u} X^*_{I_v}$, $u, v \in W^K$ is a linear combination of $X^*_{I_w}$, $w \in W^K$.

Corollary 4.6 *Let $F = F_a$ or F_m, and suppose $u \in W$ satisfies that $u \in W^K$ for some K and u is the longest element in W^K. Then for any $v \in W^K$, $\mathbb{a}^w_{u,v} = 0$ for any $w \in W$, unless $w = u$.*

Proof In these cases, the braid relations are satisfied, so the structure constants do not depend on the choice of reduced sequences. In other words, fixing u and K, we can assume we have chosen K-compatible reduced sequences. Then Theorem 4.5 applies, which implies that for any $v \in W^K$, $w \in W$, we have $\mathbb{a}^{I_w}_{u,v} = 0$ unless $w \in W^K$ and $w \geq u$. Since u is maximal in W^K, so $w = u$. □

5 Structure constants in cohomology and K-theory

We restrict our attention to $H^*_T(G/B)$ and $K_T(G/B)$ to recover formulas in [11] of structure constants of Schubert classes for cohomology ($F = F_a$) and K-theory ($F = F_m$). We first simplify the coefficients c^X_{I,I_w} and c^Y_{I,I_w} in these two cases. Recall that, when the formal group law is $F = F_a$ or $F = F_m$, the braid relations are satisfied for $Z_\alpha = X_\alpha$ and $Z_\alpha = Y_\alpha$. We consider the equivariant oriented cohomology together with either the additive or multiplicative formal group law, and restrict the coefficient ring to S^a or S^m.

Lemma 5.1 *Let J be a word in the Weyl group. As in Lemma 3.3, define coefficients c_{J,I_w} by*

$$Z_J = \sum_{w \in W} c_{J,I_w} Z_{I_w}.$$

1. *Let $F = F_a$. If $Z_\alpha = X_\alpha$ or $Z_\alpha = Y_\alpha$, then*

$$c_{J,I_w} = \begin{cases} 1, & \textit{if } J \textit{ is a reduced word for } w; \\ 0, & \textit{else}. \end{cases}$$

2. *Let $F = F_m$. If $Z_\alpha = X_\alpha$ or $Z_\alpha = Y_\alpha$, then*

$$c_{J,I_w} = \begin{cases} 1, & \textit{if } w = \widetilde{\prod} J; \\ 0, & \textit{else}. \end{cases}$$

Proof When $F = F_a$ or $F = F_m$, it is well-known that the braid relations are satisfied. We write $c_{J,w}$ for the coefficient c_{J,I_w}. When $F = F_a$, $Z_\alpha^2 = 0$, so if J is not reduced, $Z_J = 0$. If J is reduced and $\prod J = w$, then $Z_J = Z_w$, so $c_{J,w} = 1$ and $c_{J,v} = 0$ for $v \neq w$.

When $F = F_m$, we have $Z_\alpha^2 = Z_\alpha$ and thus $Z_J = Z_w$ where $w := \widetilde{\prod} J$. It follows that $c_{J,w} = 1$ and $c_{J,v} = 0$ for $v \neq w$. $\square$

Example 5.2 For $H^*(G/B)$ and $F = F_a$, as described in Example 2.7 and Proposition 2.7, the element ζ_w^X in $\mathbf{D}_F^*$ corresponds under a natural isomorphism

$$\mathbf{D}_{F_a}^* \longrightarrow \mathbb{h}_T(G/B)$$

to the equivariant cohomology class Poincaré dual to $[X(w)]$, where $[X(w)]$ is the homology class of the Schubert variety. Furthermore, the first Chern classes of the corresponding line bundles are $x_\alpha = \alpha$ for all simple roots α.

For each $w \in W$, fix a reduced sequence I_w. From the specialization of Theorem 4.1, we have defining relations

$$Y_u^* \, Y_v^* = \sum_{w \geq u, w \geq v} \mathbb{b}_{v,u}^{I_w} Y_w^*$$

for $\mathbb{b}_{v,u}^{I_w}$. Then

$$\begin{aligned}
\mathbb{b}_{u,v}^{I_w} &= \sum_{E,F \subset [\ell(w)]} \mathbf{B}_{E,F}^{I_w} c_{E,I_u}^Y c_{F,I_v}^Y \text{ by Theorem 4.1,} \\
&= \sum_{\substack{E,F \text{ reduced} \\ \text{for } u,v}} \mathbf{B}_{E,F}^{I_w}, \text{ by Lemma 5.1(1)}
\end{aligned}$$

where the second sum is over E, F whose corresponding products of reflections are reduced and equal to u, v respectively. Recall that

$$\mathbf{B}_{E,F}^{I_w} = (B_1^Y B_2^Y \cdots B_{\ell(w)}^Y) \cdot 1,$$

with

$$B_j^Y = \begin{cases} x_{\beta_j} \delta_{\beta_j}, & \text{if } j \in E \cap F, \\ \delta_{\beta_j}, & \text{if } j \in E \text{ or } F, \text{ but not both,} \\ Y_{\beta_j}, & \text{if } j \notin E \cup F. \end{cases}$$

with $\beta_j = \alpha_{i_j}$.

The coefficients $\mathbb{b}_{u,v}^{I_w}$ coincide with the structure constants c_{uv}^w in [11, Theorem 1]. Note that in this case, $Y_\alpha = -X_\alpha$, so $\zeta_w^Y = (-1)^{\ell(w)} \zeta_w^X$, and thus $X_w^* = (-1)^{\ell(w)} Y_w^*$. Therefore,

$$\mathbb{a}_{u,v}^{I_w} = (-1)^{\ell(w)+\ell(u)+\ell(v)} \mathbb{b}_{u,v}^{I_w}.$$

Example 5.3 For $K_T(G/B)$ (and $F = F_m$), we have $x_\alpha = 1 - e^{-\alpha}$. The action of X_α (resp. $Y_{-\alpha}$) on $K_T(pt)$ corresponds to the action of the ordinary (resp. isobaric) Demazure operator in [11].

Fixing a reduced sequence I_w for each w, we have

$$X_u^* \, X_v^* = \sum_{w \geq u, w \geq v} \mathbb{a}_{u,v}^{I_w} X_w^* = \sum_{w \geq v, w \geq u} \sum_{E,F} \mathbf{A}_{E,F}^{I_w} X_w^*,$$

where by Lemma 5.1(2), the second sum is over all $E, F \subset [\ell(w)]$ such that $\widetilde{\prod} E = u$ and $\widetilde{\prod} F = v$. Here, we have

$$B_j^X = \begin{cases} -(1-e^{-\beta_j})\delta_{\beta_j}, & \text{if } j \in E \cap F, \\ \delta_{\beta_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ X_{\beta_j}, & \text{if } j \notin E \cup F, \end{cases}$$

where $\beta_j = \alpha_{i_j}$.

The classes $\{\xi_w : w \in W\}$ in [11] are defined as the dual basis to $[\mathcal{O}_{X(w)}(-\partial X(w))]$ under the pairing obtained by taking the equivariant cap product and pushing forward to a point. Each ξ_w coincides with the Poincaré dual class to $[\mathcal{O}_{Y(w)}]$. In Example 2.7 we note that $X_w^* = (-1)^{\ell(w)}[\mathcal{O}_{Y(w)}]$, and thus $\xi_w = (-1)^{\ell(w)} X_w^*$. Therefore,

$$\xi_u\, \xi_v = (-1)^{\ell(u)+\ell(v)+\ell(w)} \sum_{w \geq u,v} \mathbb{a}_{u,v}^{I_w} \xi_w.$$

It follows that the coefficients $(-1)^{\ell(u)+\ell(v)+\ell(w)} \mathbb{a}_{u,v}^{I_w}$ coincide with a_{uv}^w in [11], as is clear from the formula.

With the observation that the classes $\{\mathring{\xi}_w : w \in W\}$ defined in [11] satisfy $\mathring{\xi}_w = Y_w^*$, a similar argument implies that $\mathbb{b}_{u,v}^w$ coincide with the structure constants $\mathring{a}_{u,v}^w$ defined in [11].

Example 5.4 Let $F = F_a$. Consider the A_2 case. If $I_w = s_1s_2s_1$, $u = s_1$, $v = s_1s_2$, then

$$\mathbb{b}_{s_1,s_1s_2}^{s_1s_2s_1} = \mathbf{B}_{\{1\},\{1,2\}}^{s_1s_2s_1} + \mathbf{B}_{\{3\},\{1,2\}}^{s_1s_2s_1} = \begin{pmatrix} \alpha_1\delta_1 & \delta_2 & Y_1 \\ +\delta_1 & \delta_2 & \delta_1 \end{pmatrix} \cdot 1 = 0 + 1 = 1.$$

Similarly,

$$\mathbb{b}_{s_1,s_2s_1}^{s_1s_2s_1} = \mathbf{B}_{\{1\},\{2,3\}}^{s_1s_2s_1} + \mathbf{B}_{\{3\},\{2,3\}}^{s_1s_2s_1} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_1 \\ +Y_1 & \delta_2 & \alpha_1\delta_1 \end{pmatrix} \cdot 1 = 1 - 1 = 0.$$

For the A_3 case, one can also compute

$$\begin{aligned} \mathbb{b}_{s_2s_3s_2,s_1s_2s_1}^{s_1s_2s_3s_1s_2} &= \mathbf{B}_{\{2,3,5\},\{1,2,4\}}^{s_1s_2s_3s_1s_2} + \mathbf{B}_{\{2,3,5\},\{2,4,5\}}^{s_1s_2s_3s_1s_2} \\ &= \begin{pmatrix} \delta_1 & \alpha_2\delta_2 & \delta_3 & \delta_1 & \delta_2 \\ +Y_1 & \alpha_2\delta_2 & \delta_3 & \delta_1 & \alpha_2\delta_2 \end{pmatrix} \cdot 1 \\ &= (\alpha_1+\alpha_2)+\alpha_3. \end{aligned}$$

Example 5.5 Let $F = F_m$. Consider the A_3 case, with $I_w = s_1s_2s_3s_1s_2$, $u = s_2s_3s_2$, $v = s_1s_2s_1$. We have

$$\begin{aligned} \mathbb{a}_{s_2s_3s_2,s_1s_2s_1}^{s_1s_2s_3s_1s_2} &= \mathbf{A}_{\{2,3,5\},\{1,2,4\}}^{s_1s_2s_3s_1s_2} + \mathbf{A}_{\{2,3,5\},\{2,4,5\}}^{s_1s_2s_3s_1s_2} + \mathbf{A}_{\{2,3,5\},\{1,2,4,5\}}^{s_1s_2s_3s_1s_2} \\ &= \begin{pmatrix} \delta_1 & -x_2\delta_2 & \delta_3 & \delta_1 & \delta_2 \\ +X_1 & -x_2\delta_2 & \delta_3 & \delta_1 & -x_2\delta_2 \\ +\delta_1 & -x_2\delta_2 & \delta_3 & \delta_1 & -x_2\delta_2 \end{pmatrix} \cdot 1 \\ &= -x_{1+2} + \frac{x_{1+2+3}x_2 - x_{2+3}x_{1+2}}{x_1} + x_{2+3}x_{1+2} \\ &= x_{\alpha_2} - x_{\alpha_1+2\alpha_2+\alpha_3}. \end{aligned}$$

6 Structure constants of cohomological stable bases

In this section, we let $F = F_a$ and $R = R^a = \mathbb{Z}[h]$. We recall the definition of the cohomological stable basis of Maulik-Okounkov, and generalize Su's formula of structure constants for Segre–Schwartz–MacPherson classes (Theorem 6.3). We use the twisted group algebra language for cohomology, whose K-theory version was given in [22]. As the framework and proofs are very similar to earlier sections, we will only review essential properties. Some of the notation introduced below is restricted to this section only.

Let $R^a = \mathbb{Z}[h]$, $S^a = \mathrm{Sym}_{R^a}(\Lambda)$ and $Q^a = \mathrm{Frac}(S^a)$. Define

$$Q^a_W = Q^a \rtimes_{R^a} R^a[W]$$

with Q^a-basis δ_w, $w \in W$. For simplicity we introduce the following notation:

$$\widehat{\alpha} = h - \alpha, \quad \alpha_{w_0} = \prod_{\alpha>0} \alpha, \quad \widehat{\alpha}_{w_0} = \prod_{\alpha>0} (h - \alpha).$$

Finally, for any simple root α, define an operator associated to this root by

$$T_\alpha = -h\frac{1}{\alpha}(1 - \delta_\alpha) - \delta_\alpha = -\frac{h}{\alpha} + \frac{\widehat{\alpha}}{\alpha}\delta_\alpha \in Q^a_W.$$

By direct computation, the set $\{T_\alpha\}_{\alpha\in\{\alpha_1,\ldots,\alpha_n\}}$ satisfies the braid relations, and $T_\alpha^2 = 1$. Indeed, the algebra generated by $\{T_\alpha\}$ is called the degenerate (or graded) Hecke algebra. Note that T_α is a special case of Z_α, occurring over $R = R^a$.

For any sequence $I = (i_1, \ldots, i_\ell)$ (not necessarily reduced), we define the *Demazure–Lusztig operator*

$$T_I = T_{\alpha_{i_1}} \ldots T_{\alpha_{i_\ell}}$$

in cohomology to be the product of the operators indicated in the list I. It follows from the relations that, if I and I' are two sequences with $w := \prod I = \prod I'$, then $T_I = T_{I'}$, and we denote it T_w. The set $\{T_w |\ w \in W\}$ is a basis of Q^a_W.

Let $(Q^a_W)^*$ be the Q^a-dual of Q^a_W, and let $\{T^*_w\} \subseteq (Q^a_W)^*$ be the dual basis. Denote the basis of $(Q^a_W)^*$ dual to $\{\delta_w \in Q^a_W\}$ by $\{f_w\}$, as in Sect. 2. The identity of the ring $(Q^a_W)^*$ is denoted by $\mathbf{1} = \sum_{w\in W} f_w$. The ring Q^a_W acts on $(Q^a_W)^*$ via the $\cdot$-action, given as before by

$$\langle z \cdot q^*, z' \rangle = \langle q^*, z'z \rangle \quad \text{for } z, z' \in Q^a_W, q^* \in (Q^a_W)^*.$$

It induces a W-action on $(Q^a_W)^*$ via the embedding $W \subset Q^a_W$. Let $((Q^a_W)^*)^W$ denote the Weyl-invariant subgroup of $(Q^a_W)^*$.

In this section only, denote by $\widehat{Y} \in Q^a_W$ the element

$$\sum_{w\in W} \delta_w \frac{1}{\alpha_{w_0}\widehat{\alpha}_{w_0}} = \sum_{w\in W} \delta_w \frac{1}{\prod_{\alpha>0} \alpha(h - \alpha)}.$$

The map $\widehat{Y} \cdot \_ : (Q^a_W)^* \to ((Q^a_W)^*)^W = Q^a\mathbf{1}$ is the algebraic analogue of the composition of the map

$$Q^a \otimes_{S^a} H^*_{T\times\mathbb{C}^*}(T^*G/B) \cong Q^a \otimes_{S^a} H^*_{T\times\mathbb{C}^*}(G/B) \to Q^a \otimes_{S^a} H^*_{T\times\mathbb{C}^*}(\mathrm{pt}),$$

where the last map is the equivariant pushforward of cohomology class on G/B to a point on the second term. The proofs in [7, Lemma 7.1] and [22, Lemma 5.1] easily extend to show that, for any $f, g \in (Q^a_W)^*$,

$$\widehat{Y} \cdot ((T_\alpha \cdot f) \cdot g) = \widehat{Y} \cdot (f \cdot (T_\alpha \cdot g)).$$

Definition 6.1 We define two bases of $(Q_W^a)^*$ as a module over Q^a. Let

$$\begin{aligned}\mathrm{stab}_w^+ &= T_{w^{-1}} \cdot (\alpha_{w_0} f_e), \text{ and}\\ \mathrm{stab}_w^- &= (-1)^{\ell(w_0)} T_{w^{-1}w_0} \cdot (\alpha_{w_0} f_{w_0}).\end{aligned}$$

Then $\{\mathrm{stab}_w^+ : w \in W\}$ and $\{\mathrm{stab}_w^- : w \in W\}$ each form a basis for $(Q_W^a)^*$ as a module over Q^a. We call these bases the **cohomological stable bases.** See [20] for more details.

It is immediate from the definition that stab_w^+ has support on $\{f_v : v \le w\}$ and stab_w^- has support on $\{f_v : v \ge w\}$.

The following lemma is the analogue of Theorem 5.7 and Lemma 5.6 in [22]. The first identity was due to Maulik-Okounkov originally.

Lemma 6.2 *We have*

$$\widehat{Y} \cdot \left[\mathrm{stab}_v^+ \cdot \mathrm{stab}_u^-\right] = (-1)^{\ell(w_0)} \delta_{v,u} \mathbf{1}, \quad \widehat{Y} \cdot \left[\mathrm{stab}_v^+ \cdot \widehat{\alpha}_{w_0} T_u^*\right] = \delta_{v,u} \mathbf{1}.$$

Define structure constants $\mathbb{c}_{u,v}^w \in Q^a$ by the equation

$$\mathrm{stab}_u^- \cdot \mathrm{stab}_v^- = \sum_{w \in W} \mathbb{c}_{u,v}^w \,\mathrm{stab}_w^- .$$

We now present the main result about the stable basis $\{\mathrm{stab}_w^-\}$.

Theorem 6.3 *The classes* stab_w^- *and the coefficients* $\mathbb{c}_{u,v}^w$ *satisfy the following properties:*

1. *We have* $\mathrm{stab}_w^- = (-1)^{\ell(w_0)} \widehat{\alpha}_{w_0} T_w^*$.
2. *For each* $w \in W$*, fix a reduced sequence* I_w*. Then*

$$\mathbb{c}_{u,v}^w = \sum_{\substack{E, F \subset [\ell(w)] \\ \prod(I_w|_E) = u, \prod(I_w|_F) = v}} \widehat{\alpha}_{w_0}^2 \mathbf{t}_{E,F}^{I_w},$$

where $\mathbf{t}_{E,F}^{I_w} = (B_1^T B_2^T \cdots B_k^T) \cdot 1$ *with*

$$B_j^T = \begin{cases} \frac{\alpha_{i_j}}{\widehat{\alpha}_{i_j}} \delta_{i_j}, & \text{if } j \in E \cap F, \\ \frac{h}{\widehat{\alpha}_{i_j}} \delta_{i_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ -\frac{h}{\alpha_{i_j}} + \frac{h^2}{\alpha_{i_j}\widehat{\alpha}_{i_j}} \delta_{i_j}, & \text{if } j \notin E \cup F. \end{cases}$$

Proof (1). This follows from Lemma 6.2 above.

(2). For each $w \in W$, we fix a reduced decomposition. We have

$$\mathrm{stab}_u^- \cdot \mathrm{stab}_v^- = (-1)^{\ell(w_0)} \widehat{\alpha}_{w_0} T_u^* \cdot (-1)^{\ell(w_0)} \widehat{\alpha}_{w_0} T_v^* = \widehat{\alpha}_{w_0}^2 T_u^* \cdot T_v^*.$$

Therefore, it suffices to consider the structure constants for T_u^*. But the elements T_u are an instantiation of Z_{I_u} with the coefficient ring R^a, with $a_{\alpha_{i_j}} = -h/\alpha_{i_j}$ and $b_{\alpha_{i_j}} = \widehat{\alpha}_{i_j}/\alpha_{i_j}$. Thus Theorem 4.1 indicates how to multiply the corresponding dual elements, resulting in B_j^T defined as above. □

When $h = -1$, the Demazure Lusztig operator T_α specializes to the operator considered by Su in [21], allowing us to recover his formula for the structure constants from the SSM classes from Theorem 6.3.

Example 6.4 Consider the A_2-case. If $I_w = s_1s_2s_1$, $u = v = s_1$, then

$$\begin{aligned}
\mathbb{t}^{s_1s_2s_1}_{s_1,s_1} &= \widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_{13}(\mathbf{t}^{121}_{\{1\},\{3\}} + \mathbf{t}^{121}_{\{3\},\{1\}} + \mathbf{t}^{121}_{\{1\},\{1\}} + \mathbf{t}^{121}_{\{3\},\{3\}}) \\
&= \widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_{13}\begin{pmatrix} \frac{h}{\widehat{\alpha}_1}\delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_1}\delta_1 \\ \frac{h}{\widehat{\alpha}_1}\delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_1}\delta_1 \\ \frac{\alpha_1}{\widehat{\alpha}_1}\delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2\widehat{\alpha}_2}\delta_2 & -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1\widehat{\alpha}_1}\delta_1 \\ -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1\widehat{\alpha}_1}\delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2\widehat{\alpha}_2}\delta_2 & \frac{\alpha_1}{\widehat{\alpha}_1}\delta_1 \end{pmatrix} \cdot 1 \\
&= h^2(h + \alpha_1).
\end{aligned}$$

If $I_w = s_1s_2s_1$, $u = s_1$, $v = s_1s_2$, then

$$\begin{aligned}
\mathbb{t}^{s_1s_2s_1}_{s_1,s_1s_2} &= \widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_{13}(\mathbf{t}^{121}_{\{1\},\{1,2\}} + \mathbf{t}^{121}_{\{3\},\{1,2\}}) \\
&= \widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_{13}\begin{pmatrix} \frac{\alpha_1}{\widehat{\alpha}_1}\delta_1 & \frac{h}{\widehat{\alpha}_2}\delta_2 & -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1\widehat{\alpha}_1}\delta_1 \\ \frac{h}{\widehat{\alpha}_1}\delta_1 & \frac{h}{\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_1}\delta_1 \end{pmatrix} \cdot 1 = h^2(h + \alpha_1).
\end{aligned}$$

Similarly, for $v' = s_2s_1$, we have

$$\begin{aligned}
\mathbb{t}^{s_1s_2s_1}_{s_1,s_2s_1} &= \widehat{\alpha}^2_{w_0}(\mathbf{t}^{s_1s_2s_1}_{\{1\},\{2,3\}} + \mathbf{t}^{s_1s_2s_1}_{\{3\},\{2,3\}}) \\
&= \widehat{\alpha}_1\widehat{\alpha}_2\widehat{\alpha}_{13}\begin{pmatrix} \frac{h}{\widehat{\alpha}_1}\delta_1 & \frac{h}{\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_1}\delta_1 \\ -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1\widehat{\alpha}_1}\delta_1 & \frac{h}{\widehat{\alpha}_2}\delta_2 & \frac{\alpha_1}{\widehat{\alpha}_1}\delta_1 \end{pmatrix} \cdot 1 \\
&= h^3 - h^2\widehat{\alpha}_{13} = h^2(\alpha_1 + \alpha_2).
\end{aligned}$$

Example 6.5 Consider the A_3 case. For $I_w = s_1s_2s_3s_1s_2$, $u = s_2s_3s_2$, $v = s_1s_2s_1$, with $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$ for $1 \le i < j \le 4$, we have

$$\begin{aligned}
\mathbb{t}^{s_1s_2s_3s_1s_2}_{s_2s_3s_2,s_1s_2s_1} &= \widehat{\alpha}_{w_0}(\mathbf{t}^{s_1s_2s_3s_1s_2}_{\{2,3,5\},\{1,2,4\}} + \mathbf{t}^{s_1s_2s_3s_1s_2}_{\{2,3,5\},\{2,4,5\}}) \\
&= \widehat{\alpha}_{w_0}\begin{pmatrix} \frac{h}{\widehat{\alpha}_1}\delta_1 & \frac{\alpha_2}{\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_3}\delta_3 & \frac{h}{\widehat{\alpha}_1}\delta_1 & \frac{h}{\widehat{\alpha}_2}\delta_2 \\ -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1\widehat{\alpha}_1}\delta_1 & \frac{\alpha_2}{\widehat{\alpha}_2}\delta_2 & \frac{h}{\widehat{\alpha}_3}\delta_3 & \frac{h}{\widehat{\alpha}_1}\delta_1 & \frac{\alpha_2}{\widehat{\alpha}_2}\delta_2 \end{pmatrix} \cdot 1 \\
&= h^4\widehat{\alpha}_3(\alpha_1 + \alpha_2) + h^3\widehat{\alpha}_3(h\alpha_3 + \alpha_1\alpha_2 + \alpha_2^2 + \alpha_2\alpha_3) \\
&= h^3\widehat{\alpha}_3(h + \alpha_2)(\alpha_1 + \alpha_2 + \alpha_3). \\
\mathbb{t}^{12312}_{232,1} &= \widehat{\alpha}_{w_0}(\mathbf{t}^{12312}_{\{2,3,5\},\{1\}} + \mathbf{t}^{12312}_{\{2,3,5\},\{4\}}) \\
&= h^5(\widehat{\alpha}_2 + 2\widehat{\alpha}_3). \\
\mathbb{t}^{12312}_{232,2} &= \widehat{\alpha}_{w_2}(\mathbf{t}^{12312}_{\{2,3,5\},\{2\}} + \mathbf{t}^{12312}_{\{2,3,5\},\{5\}}) \\
&= h^4(3h^2 + h\alpha_2 + (\alpha_{24})\widehat{\alpha}_{14}).
\end{aligned}$$

Remark 6.6 In [21, Theorem 1.1], the authors find a formula for the structure constants of $\sigma^*_w \in (Q^a_W)^*$, where

$$\sigma_i = \frac{1 + \alpha_i}{\alpha_i}\delta_i - \frac{1}{\alpha_i} \in Q^a_W.$$

This is equal to our $-T_\alpha$ with $\hbar = -1$.

7 Structure constants for K-theoretic stable bases

In this section, we give a formula of the structure constants of the K-theory stable basis. Similar to our strategy in Sect. 6, we use the twisted group algebra method. This method was introduced by Su, Zhao and the second author in [22]; we only recall the definitions below. Here we use $F = F_m$ and $R = R^m = \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Let $S^m = R^m[\Lambda]$. We use the following notation in this section:

$$x_{\pm\alpha} = 1 - e^{\mp\alpha}, \quad \hat{x}_\alpha = 1 - qe^{-\alpha}, \quad \hat{x}_w = \prod_{\alpha>0, w^{-1}\alpha<0} \hat{x}_\alpha, \quad q_w = q^{\ell(w)}.$$

Let $Q^m = \mathrm{Frac}(S^m)$ and apply the twisted group algebra construction to obtain the module

$$Q^m_W = Q^m \rtimes_{R^m} R^m[W].$$

Define the operator τ_α^- by

$$\tau_\alpha^- = \frac{q-1}{1-e^\alpha} + \frac{1-qe^{-\alpha}}{1-e^\alpha}\delta_\alpha \in Q^m_W.$$

Observe that τ_α^- is a special case of Z_α when $Q = Q^m$.

A simple calculation shows that $(\tau_\alpha^-)^2 = (q-1)\tau_\alpha^- + q$, and that $\{\tau_\alpha\}$ satisfies the braid relations. It follows that the K-theoretic Demazure–Lusztig operator τ_w^-, given by the product

$$\tau_w^- = \tau_{\alpha_{i_1}}^- \tau_{\alpha_{i_2}}^- \cdots \tau_{\alpha_{i_\ell}}^-,$$

is independent of choice of reduced word $s_{i_1}s_{i_2}\cdots s_{i_\ell}$ for w. The set $\{\tau_w^-, w \in W\}$ is a Q^m-basis of Q^m_W.

For each not-necessarily reduced sequence $I = (i_1, \ldots, i_\ell)$, let τ_I^- be the concatenation $\tau_{\alpha_{i_1}}^- \cdots \tau_{\alpha_{i_\ell}}^-$, and define the structure constants $c^{\tau^-}_{I,w} \in R^m$ by the equations

$$\tau_I^- = \sum_{w\in W} c^{\tau^-}_{I,w} \tau_w^-. \tag{11}$$

Lemma 7.1 *The coefficients $c^{\tau^-}_{I,w} \in R^m$ in (11) satisfy the following:*

1. *For all $w \in W$ and sequences I, $c^{\tau^-}_{I,w} = 0$ unless $w \le \widetilde{\prod} I$.*
2. *If I is reduced, then*

$$c^{\tau^-}_{I,w} = \begin{cases} 0 & \text{if } w \neq \prod I \\ 1 & \text{if } w = \prod I. \end{cases}$$

Proof Statement (1) follows from the quadratic relation $(\tau_\alpha^-)^2 = (q-1)\tau_\alpha^- + q$.

Statement (2) follows from the braid relations satisfied by the τ_α^-. □

The analogous statement to Theorem 3.7 is the following proposition.

Proposition 7.2 (K-Stable Leibniz Rule) *If $I = (i_1, \ldots, i_k)$, we have*

$$\tau_I^- \cdot (pq) = \sum_{E,F\subset[k]} P^I_{E,F}(\tau_{I|_E} \cdot p)(\tau_{I|_F} \cdot q), \quad p, q \in Q.$$

where $P^I_{E,F} = (B_1^{\tau^-} B_2^{\tau^-} \cdots B_k^{\tau^-}) \cdot 1$ *with* $B_j^{\tau^-} \in Q_W^m$ *defined by*

$$B_j^{\tau^-} = \begin{cases} \frac{1-e^{\alpha_{i_j}}}{1-qe^{-\alpha_{i_j}}}\delta_{i_j}, & \text{if } j \in E \cap F, \\ \frac{1-q}{1-qe^{-\alpha_{i_j}}}\delta_{i_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\ \frac{q-1}{1-qe^{-\alpha_{i_j}}}\tau^-_{\alpha_{i_j}}\delta_{i_j}, & \text{if } j \notin E \cup F. \end{cases}$$

Similar to Sect. 6, we take the dual $(Q_W^m)^*$, and Q_W^m acts on $(Q_W^m)^*$ via the $\cdot$-action. Indeed, we have

$$(Q_W^m)^* \cong Q^m \otimes_{S^m} K_{\mathbb{C}^*\times T}(G/B) \cong Q^m \otimes_{S^m} K_{\mathbb{C}^*\times T}(T^*G/B).$$

Definition 7.3 [22, Definition 5.3, Theorem 5.4] The K-theoretic stable basis elements are defined by

$$\mathrm{stab}_w^- = q_{w_0} q_w^{-1/2} (\tau^-_{w_0 w})^{-1} \cdot \big(\prod_{\alpha>0}(1-e^\alpha) f_{w_0}\big) \in (Q_W^m)^*.$$

Moreover, by [22, Theorem 5.4, Theorem 6.5], we have

$$\mathrm{stab}_w^- = q_w^{1/2} \hat{x}_{w_0} (\tau_w^-)^*.$$

The following theorem gives a formula for the structure constants of the K-theory stable basis:

Theorem 7.4 *Let* $\{\mathrm{stab}_w^- \mid w \in W\}$ *denote the K-theory stable basis of* $(Q_W^m)^*$. *Define coefficients* $p^w_{u,v} \in Q^m$ *by the equation*

$$\mathrm{stab}_u^- \cdot \mathrm{stab}_v^- = \sum_{w \ge u, w \ge v} p^w_{u,v} \, \mathrm{stab}_w^- .$$

Then

$$p^w_{u,v} = q^{\frac{1}{2}(\ell(u)+\ell(v)-\ell(w))} \hat{x}_{w_0} \sum P^{I_w}_{E,F} c^{\tau^-}_{I_w|_E,u} c^{\tau^-}_{I_w|_F,v},$$

where the sum is over all $E, F \subset [\ell(w)]$ *such that* $\widetilde{\prod}(I_w|_E) \ge u$ *and* $\widetilde{\prod}(I_w|_F) \ge v$, *and coefficients* $c^\tau_{I_w|_F,v}$ *are given in Lemma 7.1*

Proof The proof follows a similar argument as that of Theorem 6.3. □

Remark 7.5 Due to the quadratic relation $(\tau_\alpha^-)^2 = (q-1)\tau_\alpha^- + q$, it is difficult to express the sum in terms of formulas in Sects. 5 and 6. Indeed, this is also the reason why it is difficult to express the restriction formula of stab_w^- in [22] in terms of an AJS-Billey-Graham-Willems type formula.

8 The restriction formula

In this section we relate the structure constants of $Z^*_{I_w}$ with its restriction coefficients. This generalizes such relations in cohomology and K-theory due to Kostant and Kumar in [13, Proposition 4.32] and [14, Lemma 2.25].

Let Z_α be given in Definition 3.5. Following Lemma 2.3, we obtain coefficients $b^Z_{u,I_w} \in Q$ using the defining relations

$$\delta_u = \sum_{w\in W} b^Z_{u,I_w} Z_{I_w},$$

Then $Z^*_{I_v} = \sum_u b^Z_{u,I_v} f_u$, i.e., $Z^*_{I_v}(\delta_u) = b^Z_{u,I_v}$. We call b^Z_{u,I_v} the restriction coefficients of $Z^*_{I_v}$.

Theorem 8.1 *For any $w \in W$, define the matrix $\mathfrak{p}^Z_w$ with $\mathfrak{p}^Z_w(u,v) = \mathbb{c}^{I_v}_{I_w,I_u}$, the matrix $\mathfrak{b}^Z$ with $\mathfrak{b}^Z(u,v) = b^Z_{v,I_u}$, and the matrix $\mathfrak{b}^Z_w$ with $\mathfrak{b}^Z_w(u,v) = \delta_{u,v} b^Z_{u,I_w}$. Then*

$$\mathfrak{p}^Z_w = \mathfrak{b}^Z \cdot \mathfrak{b}^Z_w \cdot (\mathfrak{b}^Z)^{-1}.$$

Proof We have

$$\begin{aligned}
(\mathfrak{p}^Z_w \cdot \mathfrak{b}^Z)(u,v) &= \sum_{z\in W} \mathfrak{p}^Z_w(u,z)\mathfrak{b}^Z(z,v) = \sum_{z\in W} \mathbb{c}^{I_z}_{I_w,I_u} b^Z_{v,I_z} \\
&= \sum_{z\in W} \mathbb{c}^{I_z}_{I_w,I_u} Z^*_{I_z}(\delta_v) = (Z^*_{I_u}\cdot Z^*_{I_w})(\delta_v) \\
&= Z^*_{I_u}(\delta_v)\cdot Z^*_{I_w}(\delta_v) = b^Z_{v,I_u} b^Z_{v,I_w} \\
&= \sum_{z\in W} b^Z_{z,I_u}\delta_{z,v} b^Z_{z,I_w} = \sum_{z\in W} \mathfrak{b}^Z(u,z)\mathfrak{b}^Z_w(z,v) \\
&= (\mathfrak{b}^Z\cdot\mathfrak{b}^Z_w)(u,v).
\end{aligned}$$

□

Corollary 8.2 *For any $v, w \in W$, we have*

$$\mathbb{c}^{I_v}_{I_w,I_v} = b^Z_{v,I_w}.$$

In particular, $\mathbb{c}^{I_v}_{I_w,I_v}$ does not depend on the choice of I_v.

Proof Denote $Z_{I_w} = \sum_{v\le w} a^Z_{I_w,v}\delta_v$. Then the matrix $\mathfrak{a}^Z$ with $\mathfrak{a}^Z(u,v) = a^Z_{I_v,u}$ is the inverse of $\mathfrak{b}^Z$. Theorem 8.1 implies that

$$\begin{aligned}
\mathbb{c}^{I_v}_{I_w,I_v} &= \mathfrak{p}^Z_w(v,v) \\
&= \sum_{z_1,z_2\in W} \mathfrak{b}^Z(v,z_1)\mathfrak{b}^Z_w(z_1,z_2)\mathfrak{a}^Z(z_2,v) \\
&= \sum_{z_1\ge v, z_2\le v} b^Z_{z_1,I_v}\delta_{z_1,z_2} b^Z_{z_1,I_w} a^Z_{I_v,z_2} \\
&= \sum_{v\le z_1\le v} b^Z_{z_1,I_v} b^Z_{z_1,I_w} a^Z_{I_v,z_1} \\
&= b^Z_{v,v} b^Z_{v,I_w} a^Z_{v,v} = b^Z_{v,I_w} = Z^*_{I_w}(\delta_v).
\end{aligned}$$

□

Remark 8.3 Corollary 8.2 can also be proved with the following argument: since $Z^*_{I_w}(\delta_u) = b^Z_{u,I_w} = 0$ unless $u \geq w$,

$$(Z^*_{I_u} \cdot Z^*_{I_v})(\delta_u) = \left(\sum_{w \geq u, w \geq v} \mathbb{c}^{I_w}_{I_u,I_v} Z^*_{I_w}\right)(\delta_u)$$
$$= \mathbb{c}^{I_u}_{I_u,I_v} Z^*_{I_u}(\delta_u) = \mathbb{c}^{I_u}_{I_u,I_v} b^Z_{u,I_u}.$$

On the other hand,

$$(Z^*_{I_u} \cdot Z^*_{I_v})(\delta_u) = Z^*_{I_u}(\delta_u) Z^*_{I_v}(\delta_u) = b^Z_{u,I_u} b^Z_{u,I_v}.$$

Therefore, $\mathbb{c}^{I_u}_{I_u,I_v} = b^Z_{u,I_v}$.

Remark 8.4 As mentioned in [11], specializing Corollary 8.2 and Examples 5.2 and 5.3 to cohomology or K-theory, and Z_α to the X_α and Y_α-operators, one recovers the AJS/Billey formula and Graham-Willems formula of restriction coefficients of Schubert classes, which are obtained by using root polynomials.

Example 8.5 Consider the A_2-case with $w = s_1$, $v = s_1 s_2 s_1$. We compute $b^X_{v,w} = X^*_{I_w}(\delta_v)$. For $\mathbf{A}^{I_v}_{[3],E}$, we only need to consider the following three:

$$\mathbf{A}^{I_v}_{[3],\{1\}} = -x_1, \quad \mathbf{A}^{I_v}_{[3],\{3\}} = -x_2, \quad \mathbf{A}^{I_v}_{[3],\{1,3\}} = x_1 x_2.$$

On the other hand, $c^X_{I_v|_E,I_w} = 1$ when $E = \{1\}, \{3\}$, and $X_1 X_1 = \kappa_1 X_1$. So $c^X_{I_v|_{\{1,3\}},s_1} = \kappa_1$. Therefore,

$$b^X_{w,I_v} = -x_1 - x_2 + \kappa_1 x_1 x_2.$$

In particular, if $F = F_a$, then $b^X_{w,I_v} = -x_1 - x_2$, and if $F = F_m$, then $b^X_{w,I_v} = -x_1 - x_2 + x_1 x_2 = -x_{1+2}$, with $x_{1+2} = x_{\alpha_1+\alpha_2}$.

Example 8.6 Let $w = s_1 s_2$, $v = s_1 s_2 s_3 s_1 s_2$. Let us compute $b^X_{v,I_w} = X^*_{I_w}(\delta_v)$. We write $X_{ijk\cdots}$ for $X_i X_j X_k \cdots$, $x_{\pm i \pm j} = x_{\pm\alpha_i \pm\alpha_j}$ and $\kappa_{\pm i,\pm j} = \kappa_{\pm\alpha_i,\pm\alpha_j}$. To compute $\mathbf{A}^{I_v}_{[6],E}$, we only need to consider

$$\mathbf{A}^{I_v}_{[6],\{1,2\}} = x_1 x_{1+2}, \quad \mathbf{A}^{I_v}_{[6],\{1,5\}} = x_1 x_{2+3},$$
$$\mathbf{A}^{I_v}_{[6],\{4,5\}} = x_2 x_{2+3}, \quad \mathbf{A}^{I_v}_{[6],\{1,2,5\}} = -x_1 x_{1+2} x_{2+3},$$
$$\mathbf{A}^{I_v}_{[6],\{1,4,5\}} = -x_1 x_2 x_{2+3}, \quad \mathbf{A}^{I_v}_{[6],\{1,2,4,5}\} = x_1 x_{1+2} x_2 x_{2+3}.$$

On the other hand, $c^X_{I_v|_E,I_w} = 1$ when $E = \{1,2\}, \{1,5\}, \{4,5\}$. Concerning $X_{I_w|_{\{1,2,5\}}} = X_{122}$, since

$$X_1 X_2 X_2 = X_1 \kappa_2 X_2 = s_1(\kappa_2) X_{12} + \Delta_1(\kappa_2) X_2 = \kappa_{1+2} X_{12} + \Delta_1(\kappa_2) X_2,$$

so

$$c^X_{I_v|_{\{1,2,5\}},I_w} = \kappa_{1+2}.$$

For $X_{I_w|_{\{1,4,5\}}} = X_{112}$, from $X_1 X_1 X_2 = \kappa_1 X_1 X_2$, we get

$$c^X_{I_v|_{\{1,4,5\}},I_w} = \kappa_1.$$

Lastly, for $X_{I_w|_{\{1,2,4,5\}}} = X_{1122}$, from Lemma 2.2 we know

$$\begin{aligned}X_{1212} &= X_1(X_{121} + \kappa_{12}X_1 - \kappa_{21}X_2)\\ &= \kappa_1 X_{121} + X_1\kappa_{12}X_1 - X_1\kappa_{21}X_2\\ &= \kappa_1 X_{121} + s_1(\kappa_{12})X_1^2 + \Delta_1(\kappa_{12})X_1 - s_1(\kappa_{21})X_{12} - \Delta_1(\kappa_{21})X_2\\ &= \kappa_1 X_{121} + \kappa_{-1,1+2}\kappa_1 X_1 + \Delta_1(\kappa_1)X_1 - \kappa_{1+2,-1}X_{12} - \Delta_1(\kappa_{21})X_2,\end{aligned}$$

so

$$c^X_{I_v|_{\{1,2,4,5\}}, I_w} = s_1(\kappa_{21}) = -\kappa_{1+2,-1}.$$

Therefore,

$$\begin{aligned}&b^X_{s_1s_2, s_1s_2s_3s_1s_2}\\ &\quad= \mathbf{A}^{I_v}_{[6],\{1,2\}} + \mathbf{A}^{I_v}_{[6],\{1,5\}} + \mathbf{A}^{I_v}_{[6],\{4,5\}}\\ &\qquad+ \mathbf{A}^{I_v}_{[6],\{1,2,5\}}\kappa_{1+2} + \mathbf{A}^{I_v}_{[6],\{1,4,5\}}\kappa_1 + \mathbf{A}^{I_v}_{[6],\{1,2,4,5\}}(-\kappa_{1+2,-1})\\ &\quad= x_1x_{1+2} + x_1x_{2+3} + x_2x_{2+3} - x_1x_{1+2}x_{2+3}\kappa_{1+2} - x_1x_2x_{2+3}\kappa_1 - x_1x_{1+2}x_{2+3}\kappa_{1+2,-1}\\ &\quad= x_1x_{1+2} + x_1x_{2+3} + x_2x_{2+3} - x_{2+3}(x_1 + x_2 + \frac{x_1}{x_{-1}}x_{1+2}).\end{aligned}$$

In particular, if $F = F_a$, then

$$b^X_{s_1s_2, s_1s_2s_3s_1s_2} = \alpha_1(\alpha_1 + \alpha_2) + \alpha_1(\alpha_2 + \alpha_3) + \alpha_2(\alpha_2 + \alpha_3).$$

If $F = F_m$, then

$$b^X_{s_1s_2, s_1s_2s_3s_1s_2} = x_1x_{1+2} + x_1x_{2+3} + x_2x_{2+3} - x_1x_{2+3}(x_{1+2} + x_2).$$

These agree with the result computed by using root polynomials.

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