



Humanizing Proof-based Mathematics Instruction Through Experiences Reading Rich Proofs and Mathematician Stories

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Abstract The Reading and Appreciating Mathematical Proofs (RAMP) project seeks to provide novel resources for teaching undergraduate introduction to proof courses centered around reading activities. These reading activities include (1) reading rich proofs to learn new mathematics through proofs as well as to learn how to read proofs for understanding and (2) reading mathematician stories to humanize proving and to legitimize challenge and struggle. One of the guiding analogies of the project is thinking about learning proof-based mathematics like learning a genre of literature. We want students to read

We have adopted the mathematical practice of alphabetizing the names of the authors for this paper.

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interesting proofs so they can appreciate what is exciting about the genre and how they can engage with it. Proofs were selected by eight professors in mathematics who as curriculum co-authors collected intriguing mathematical results and added stories of their experience becoming mathematicians. As mathematicians of colour and/or women mathematicians, these co-authors speak to the challenges they faced in their mathematical history, how they overcame these challenges, and the key role mentors and community have played in that process. These novel opportunities to learn to read and read to learn in the proof-based context hold promise for supporting student learning in new ways. In this commentary, we share how we have sought to humanize proof-based mathematics both in the reading materials and in our classroom implementation thereof.

Résumé Le projet «Reading and Appreciating Mathematical Proofs (RAMP)» (Lire et reconnaître la valeur des preuves mathématiques) vise à générer de nouvelles ressources pour l'enseignement des cours d'introduction à la preuve, centrées sur des activités de lecture. Celles-ci comprennent 1) la lecture de preuves étoffées pour apprendre de nouvelles mathématiques par le biais de preuves ainsi que pour apprendre à lire les preuves pour mieux comprendre la matière et 2) la lecture d'histoires racontées par des mathématiciens pour humaniser la vérification des preuves et pour légitimer les défis et les difficultés. L'une des analogies qui orientent le projet consiste à considérer l'apprentissage des mathématiques fondées sur les preuves comme celui d'un genre littéraire. Nous voulons que les étudiants lisent des preuves pertinentes afin qu'ils puissent apprécier ce les rend attrayantes et voir comment ils peuvent s'y intéresser. En tant que coauteurs du programme, huit professeurs de mathématiques ont choisi les preuve et ont recueilli des résultats mathématiques intrigants et ajouté des récits sur leur expérience en tant que mathématiciens. Ces coauteurs, des mathématiciens de couleur ou des femmes mathématiciennes (ou les deux) racontent les défis auxquels ils ont été confrontés au cours de leur cheminement en mathématiques, la manière avec laquelle ils ont surmonté ces défis et le rôle clé joué par les mentors et la communauté dans ce processus. Ces nouvelles occasions d'apprendre à lire et de lire pour apprendre dans un contexte fondé sur la preuve sont prometteuses afin de soutenir l'apprentissage des élèves de façon novatrice. Dans ce commentaire, nous partageons comment nous avons cherché à humaniser les mathématiques fondées sur la preuve, à la fois dans les documents de lecture et dans leur mise en œuvre en classe.

Keywords Humanizing mathematics · Reading to learn · Mathematical proof · Diverse representation

Introduction

What if we taught an introduction to proofs course like a great literature course? If we conceptualize learning mathematical proof like learning a genre of literature, then we might prioritize students reading compelling examples from the genre. Reading such rich proof texts might help students answer, “what is there to be excited about in proof?” Furthermore, we might employ the helpful

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adage: learning to read and reading to learn. (When) Do we teach students to read proof effectively? These questions and ideas provided some of the seeds of our grant project¹ in which we are developing curricular tools for teaching introduction to proof courses with major reading components. We refer to this as the Reading and Appreciating Mathematical Proofs (RAMP) project.

As we explored the possibilities of developing tools for teaching such a course, we saw much potential for trying to humanize proof-based mathematics. A key component of the grant project, and the specific goal of this commentary, has been exploring what humanizing proof-based mathematics means and how it can be carried out productively and appropriately. We imagined engaging students with the human aspects of proofs and exposing them to the diverse perspectives of mathematicians who created these proofs. This entails two kinds of readings: (1) Rich Proofs that are challenging, yet accessible to novice, undergraduate readers; and (2) Author Stories that present mathematician reflections on their experiences in the mathematical sciences (c.f., Garcia et al., 2023; Harris et al., 2021; Henrich et al., 2010). In analogy to literature courses reading great stories, we also wanted Rich Proofs to prove something interesting, in contrast with the many proofs that justify claims students find obvious. The grant team, comprised of mathematics education researchers, partnered with eight co-author mathematicians — women mathematicians and mathematicians of color — to prepare the Rich Proofs and to share their experiences in Author Stories. This has opened a collaboration among our author team who have a broad range of experiences working to foster inclusive inquiry in proof-based classrooms as well as supporting diversity, access, and mentorship in the mathematical sciences more broadly. We hope in this report to continue our process of learning with and from one another and to offer some of our insights to a broader community.

We have piloted the RAMP materials in a few classrooms and will continue to implement, refine, and develop instructional support materials. Future research reports will share findings from classroom implementations regarding student learning and experiences. Though midway through our project, this commentary marks an opportunity for us to reflect together about what we have been learning about the needs and the opportunities for humanizing proof-based mathematics. In the next section, we review some relevant literature regarding why proofs need to be humanized in ways that give students access and opportunity to participate. We also identify literature that has informed and inspired us about how to pursue these goals. After that, we briefly describe the process by which we developed this paper, share some samples of Rich Proofs and Author Stories, and a brief account of how we are implementing these in the classroom. This will provide the context for the remainder of the paper in which we share our collective reflections on what it means to humanize proof-based mathematics and how we have been learning more about it through the RAMP project. These sections include some attention to the challenges we face or anticipate in trying to use these materials and support others in doing so.

Literature Review

For many students, undergraduate proof-based courses create a challenging transition after the primarily computational mathematics prior (Gueudet 2008; Selden, 2012; Stylianides et al., 2017; Tall, 2008). This transition occasions enculturation into the communication and argumentation of theoretical research mathematics. While there are ways to teach such courses building on student understandings and arguments, many proof courses simply impose mathematical norms and values to which students must adapt. Undergraduate proof-based mathematics courses then have several features that have the potential to promote an exclusionary and dehumanized experience for students

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stemming from both the aims of enculturation and the proof object itself (Weber & Melhuish, 2022). For instance, Balacheff (1988) highlighted how proof language seeks to decontextualize, depersonalize, and detemporalize.

How are students invited to engage in the practices of proving in ways that build upon their own experiences, understandings, and epistemology? Brown (2014) has cautioned that the acceptant of proof as the argumentation form can easily stem from “deference to authority, and, at worst, obedience” where the only way to legitimately participate is to emulate practice from a “larger community” (p. 327) that may be quite distant from students themselves (see also, Brown, *in press*). In fact, numerous studies have pointed to the prevalence of stereotypes of mathematicians that include both the racial and gender factors (e.g. white, male), and also particular personality traits (e.g. socially isolated and awkward, brilliant and obsessed; Di Martino et al., 2023; Moreau et al., 2009; Piatek-Jimenez, 2008). These images can serve to exclude students, and those who do not recognize themselves in the mathematician community may not continue in the field (e.g. Piatek-Jimenez, 2008). Furthermore, the version of disciplinary mathematics portrayed often hides the humanity in the activity where proofs are presented as static objects often perceived as “a formal necessity required by the teacher” (Alibert & Thomas, 1991, p. 216) rather than a key part of discovery process. The proof production process is often carefully hidden since it is not done in the formal representation system and involves a rather nonlinear process of moving between formal and informal, experimenting, and hitting false starts (Melhuish et al., 2022b).

This artificial dehumanization is then readily apparent in the proof product itself. As Davis and Hersh (1981) commented, the formal proof is “to conceal any sign that the author or the reader is a human being” (p. 36). Discussing the norms and values of proof, Dawkins and Weber (2017) elaborated that “mathematical proof is written without reference to author or reader’s agency” and that the “proof is an autonomous object, not a description of a problem-solving process”. The proof is an outcome of mathematical activity, rather than the mathematical activity itself. To a large degree, it is stripped of the author’s voice and follows many conventions and norms that are largely invariant in the genre (e.g. Lew & Mejía-Ramos, 2019; Selden & Selden, 2013). That is, introduction to proof serves to not only transform students’ ways of arguing into those of mathematicians, but also to shift communication to match a highly formalized style apparently devoid of human agency. As argued by Hottinger (2016), such mathematics emphasizes “logic, neutrality, a lack of emotional connection, and a separation between the knower and the object of knowledge” (p. 15). Hottinger (2016) and other scholars (e.g. Battey & Leyva, 2016) have suggested these traits reflect stereotypical white and patriarchal spaces. That is, not only is the product devoid of human voice, but also the features that are retained are likely to be more inviting to specific cultural backgrounds of some students.

We suggest these features call for focused action to learn how we might humanize courses, such as introduction to proof, while maintaining their apprenticeship aims. In this way, we still provide students with the rules of the game, and also open space such that more students feel included, and that mathematics is humanized. There are many ways that humanizing has been defined in the literature about mathematics education. As Langer-Osuna and Nasir (2016) explained, much of this work focuses on humanizing those who have been othered, such as African Americans, whose humanity has been obscured by dominant discourse and structural elements of the USA. Tan et al. (2022) explained that “humanizing mathematics education seeks to firm and cultivate complex, intersectional, and gifted cultural beings through inclusivity, equity, and social justice” (p. 875), and involves teachers not just attending to mathematical content, but “seeing [their] students as whole human beings” (Kalinec-Craig & del Rosario Zavala, 2019). As argued by Yeh and Otis (2019) in commitment to “more humanizing pedagogy”, education can be “a site of social reproduction”, but also as “a potential site for transformation”. Humanizing can serve to change views on who can do math and what it means to do math in ways that actively include students.

When we consider the undergraduate proof level, we suggest this work involves not just seeing students as whole human beings, but also seeing mathematicians as whole human beings rather than some group of people separate from most of the students in the courses. As advocated for by Jett et al. (2015), culturally responsive pedagogy can include having students learn about research mathematicians and “mathematical [contributions] from culturally diverse groups in general and black people in particular” (p. 287). Recent research has demonstrated that culturally sensitive curricula — for instance, those including Diversity Represented and Positive Depictions dimensions — are significant predictors of positive interest in higher education (Quinlan et al., 2024). There is a need to counteract stereotypes and dominant narratives of who does mathematics in courses where emulating mathematicians is emphasized. Further, we suggest the need to disrupt the image of mathematicians as socially inept and uniquely brilliant to better serve students who do not see cultural characteristics of themselves. As noted in Cervia’s (2019) study of female scientists, we need to be sure to share stories not just of people from disenfranchised backgrounds, but also those that disrupt other elements of what the popular discourse around a subject area entail. Finally, we note the need to see mathematics as a fundamental human activity. A major component of humanizing mathematics, as advocated for by mathematicians, involves not only “study[ing] mathematics in itself, but as an *activity*, developed by humans in a variety of different settings” (Pais, 2018, pp. 235–236). Doing and conducting mathematics is not a flawless enterprise that the finalized proof product might represent, but rather filled with challenges, failures, and breakthroughs.

RAMP Project Materials and Our Process of Reflection

Pursuing the novel vision for teaching an introduction to proof course focused on reading proofs required us to make several strategic choices and develop new tools for instruction. In this section, we outline some of these aspects of the project to provide context for the reflections on humanizing mathematics that we share in subsequent sections. We end this section by presenting the method by which we engaged in a process of collective reflection on what we have been learning.

Choices Regarding the Materials

The RAMP project is not developing a full curriculum for teaching introduction to proof, but rather a set of curricular materials that can be implemented in most any course taught using a textbook or instructor notes. Having reviewed commonly used textbooks (Dawkins et al., 2022), we recognize the common set of topics covered in such courses (see also Schumacher et al., 2015) and the existence of a range of adequate tools for teaching much of this content². Many such courses cover sets, logic, quantification, proof techniques (including contradiction and induction), relations, functions, and cardinality.

We have not tried to reinvent the wheel, but rather to create opportunities to engage students in the two kinds of reading described in the introduction. The RAMP materials contain 27 Rich Proofs and 12 Author Stories, which is far more than we would expect any course could cover in a semester. The proofs cover a range of topics including analysis, algebra, number theory, combinatorics, graph theory, and topology. Our intention is that the Rich Proofs be paired with topics (e.g. functions, relations, cardinality) to exemplify how those topics are used in more complex settings. The various Rich Proofs are thus modular and can be placed into a course wherever they fit according to instructor goals. We expect some of the more challenging readings would serve nicely as honours projects to be completed in parallel to the course.

² We specifically appreciate Hammack (2013) for providing a good text for free.

Because we expect students to read the Rich Proofs without much prior introduction in class, they all contain front matter introducing definitions, notation, motivation, and prior theorems. Unlike many proof courses that try to prove any theorems they use, we instead chose to cite prior theorems so students can jump straight to the proof of interest. Some of the front matter or epilogues to the proofs also contain personal explanations about when the author first encountered this proof, why the proof was chosen, or shout-outs to collaborators and mentors connected to the proof. This is an instance of how Rich Proof authors added personal voices to the text in preparing these readings for students. To enhance this when teaching the course, we have had students read the Author Stories at home from the same author who prepared their Rich Proof reading from class.

We have implemented all the Rich Proof readings as classroom discussion activities so we can support students in learning how to read proofs for understanding. While this builds on some of our prior work orchestrating classroom activities around proof (Melhuish et al., 2022a), this has also required ongoing design work to implement well. We, like most mathematics instructors, have little experience leading a class in group reading activities. We needed to develop resources for implementation, specifically group-worthy questions to guide the reading process. We call questions group-worthy when they are challenging enough to foster conversation (rather than being immediately answered by one student) and yet accessible to their shared work. As a team, we workshopped and lab tested comprehension questions (using Mejía-Ramos et al.'s, 2012, framework) to help students learn what they need to investigate to make sense of proofs. We have assigned these questions as understanding checks before students come to class (having read the text at home), as discussion questions in class, or as later homework questions to extend their learning. Figure 1 displays excerpts from two Author Stories. Figure 2 shows sections of Rich Proof texts along with associated comprehension questions.

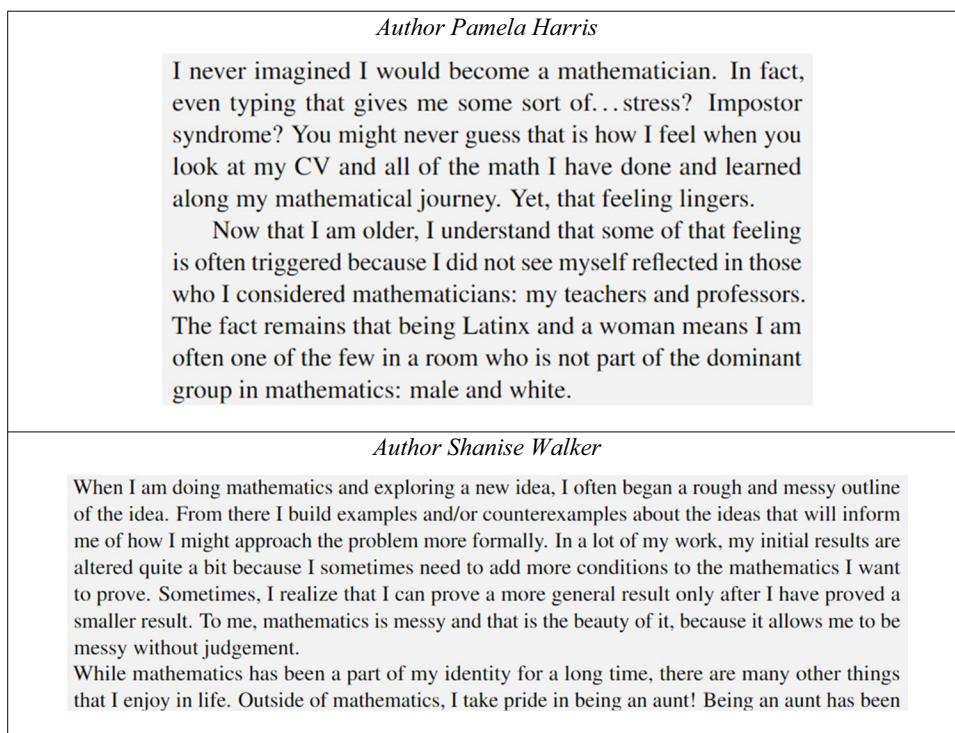


Fig. 1 Excerpts from author stories

A Brief Description of Implementation

This section will help the reader imagine how these Rich Proofs and Author Stories were used in our initial implementations. Our introduction to proof classes range from 15 to 40 students depending upon the institution and semester. We implement group work throughout the semester, including days on which we read Rich Proofs in class. The desks in our classrooms are movable so that students can form clusters of three or four and the instructor can move about the room. Depending upon the length of the text, we sometimes had students read the background information before coming to class. We generally projected the proof for the class to see as well as giving students printed copies they could write on. We segmented each proof into sections to pace student progress through the proof. We usually give students in each group roles or give group members different work they are responsible for to foster interaction and support collaboration (Melhuish et al., 2022a). For each section, we gave students the comprehension questions described in the previous section. During small group discussion, the instructor moved around the room to hear student ideas and identify what topics needed to be discussed as a whole class before moving forward with the proof reading. Moving back and forth between small group and whole class discussion, we progress through the proof text. Figure 3 presents the course schedule from a recent pilot of the RAMP materials. In that course, students read eight Rich Proofs and Author Stories, with each Rich Proof reading occupying one 80-min class session (half of a week of class).

We have integrated the readings with assessment in a couple of ways. First, some homework assignments ask students to adapt the proof techniques from the Rich Proofs to similar statements or to answer further questions about challenging parts of a proof read in class. Second, our exams sometimes included proof comprehension questions, though usually on shorter proofs. For instance, we may present students with the sequence of claims in a proof and ask them to identify the warrants for those claims. We have also provided proofs without a theorem statement and asked students to identify which statement the proof proves.

Choices Regarding Authorship

At the outset of this project, the grant PI team recognized dangers in drawing upon the work and stories of women mathematicians and mathematicians of colour. It is easy for our female and minoritized colleagues to be expected to do extra work in academia without getting proper credit or compensation. Even worse, others may usurp the credit due to these scholars. To help avoid these pitfalls, we have worked to maintain the grant project as a partnership in which authorship is shared among mathematicians and mathematics educators. This is both to honor the contributions of all as well as to share academia's "coin of the realm", which is authorship in publication.

This paper represents another expression of this commitment to partnership. All the RAMP authors were invited to contribute to this reflection, as well as the graduate students who have been working on the grant. Due to time commitments, not all the authors of the RAMP materials were available to participate. All of those who chose to participate wrote reflections guided by the following prompts:

- What do you think it means to humanize proof-based mathematics?
- Has being involved with the RAMP project influenced your ideas on humanizing proof-based mathematics? If so, how?
- What experiences (in your teaching practice or as part of the project) helped you understand the need to humanize proof-based mathematics?
- What experiences (in your teaching practice or as part of the project) have given you images of what it could look like to humanize proof-based mathematics for undergraduate students?

Author Rebecca Garcia

Theorem text with lemmas:

10.2.2 Proof that the Closed Formula Works

To make notation for the proof a bit easier, let's set $\phi = \frac{1+\sqrt{5}}{2}$ and its conjugate by $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. The lemma below contains some useful facts about ϕ and $\bar{\phi}$ that we will use later. You can prove them directly from the definitions.

Lemma 10.2.1 Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. Then the following statements hold:

1. $\phi + \bar{\phi} = 1$. Alternatively, $\phi = 1 - \bar{\phi}$ and $\bar{\phi} = 1 - \phi$.
2. $\phi^{-1} = -\bar{\phi}$. Alternatively, $\bar{\phi} = -\phi^{-1}$.
3. $\bar{\phi}^{-1} = -\phi$. Alternatively, $\phi = -\bar{\phi}^{-1}$.

Theorem 10.2.2 For $n = 0, 1, 2, 3, \dots$, the n^{th} Fibonacci number F_n is determined by the following formula:

$$F_n := \frac{1}{\sqrt{5}} [\phi^n - \bar{\phi}^n].$$

Discussion questions:

Now read up to (and including) Theorem 10.2.2

4. How would you find the 100th term using the closed formula?
5. Approximately what numbers are ϕ and $\bar{\phi}$?
6. Pick one of the lemmas. Can you argue that it is true?

Author Aris Winger

Proof text:

16.2.2 The Third Real Analysis Statement

Theorem 16.2.1 Every bounded sequence a_n has a cluster point, a point around which there are infinitely many elements of a_n .

Proof. The Proof is in cases. Let S be the set of elements of the sequence. So $S = \{a_n : n \geq 1\}$. Either the set S is finite or infinite.

If S is finite, then that means infinitely many elements of the sequence are the same number or multiple numbers. Any one of those numbers is then an accumulation point and the case is over.

If S is infinite, then we begin with an argument my mathematical hero James Donaldson taught me that I will never forget.

Since a_n is bounded, S is bounded. So there are numbers L and M with all the elements of S in the interval $[L, M]$. So the set $[L, M]$ has infinitely many elements of S . Split the interval down the

middle. Now the interval has infinitely many elements on at least one side of the midpoint of S . Pick such a side. That's an interval $[L_1, M_1]$. This new set has infinitely many members of S . But we can split this set into two equal sub-intervals, with at least one side which has infinitely many elements of S . Pick it. Let's call that set $[L_2, M_2]$. We can continue in this fashion with sets $[L_n, M_n]$ each of which is a subset of the previous one and whose length is half the length of the previous one as well. You might note that these sets are collapsing!

By Theorem 15.2.1, all of these sets have just one element in common. We can call it z . This is the element we are looking for. We think z is a cluster point of a_n , an accumulation point of S . Here is the argument. Take any interval with z in the middle. Then it eventually contains one of these collapsing sets. Each of these collapsing sets has infinitely elements of a_n . Therefore, every interval around z has infinitely many elements of S . This means z has elements of S crowding around it. ■

Discussion questions:

6. Why does one of the two half intervals always have to have infinitely many elements of S ?
7. Explain why these intervals being constructed must meet the three conditions for a sequence of intervals being collapsing.

Now read the final section of the proof.

8. How does the proof identify the number z ?
9. Why must z be a cluster point?
10. Summarize the whole proof in your own words.

Fig. 2 Rich proof text excerpts and comprehension questions

- What are ongoing tensions you see in trying to teach proof-based mathematics as envisioned in the RAMP materials?
- How do you hope the RAMP materials might benefit students and instructors in introduction to proof courses?

A few authors then compiled these reflections, organized them by themes, and prepared a draft of this paper. All contributing authors on this manuscript then had a chance to review the draft and provide feedback before submission. In this way, we have attempted to produce a shared product that reflects our different perspectives and learning journeys.

What Does It Mean to Humanize Proof-Based Mathematics Courses?

We organize this main section of the paper around a few key points of our shared reflection. First, we consider what precisely gets humanized, recognizing at least four very appropriate answers: humanizing proof texts, humanizing the classroom, humanizing the process of reading and writing proofs, and humanizing mathematicians. Second, we consider the challenges learned in humanizing mathematics using the RAMP materials as a tool. Some of these challenges reflect issues we have already attempted to address while others relate to future work we need to do in design, implementation, or dissemination.

Humanizing Proof Texts as the Product of Human Reasoning and Choice

One of the first ways we recognize that RAMP activities can help humanize mathematics is by connecting proofs to the people who wrote them (in this case, maybe not the original prover, but the author who wrote the given text). We already mentioned how some of the Rich Proof texts connect the proofs to personal stories. Even when those touches are not present, the Author Stories invite students to think of proofs as coming from a prover. Though we did not anticipate this, having a range of proof authors allows students to see differences among how mathematicians write. This may offer students some amount of insight into what stays the same across proof texts (norms and conventions) and what is more open to author choice. While proofs are, compared to many other genres, a technical text with lots of symbols and lots of internal structure, that does not mean that they do not leave room for author voice and personal style. Inviting students to read a range of authors in a short time may portray this better than most other classroom resources we have seen.

Humanizing the Proof-Based Classroom

Many of us have been employing inquiry-oriented instruction (Andrews-Larson et al., 2021; Kuster et al., 2018; Laursen & Rasmussen, 2019; Melhuish et al., 2022a; Rasmussen & Kwon, 2007; Schoenfeld & Kilpatrick, 2013) in our classes for some time, which potentially opens the classroom to more student contributions. Even within that body of experience, the RAMP curriculum has opened deeper ways to invite and build upon student experiences and stories. This happens in several ways. First, many of the Author Stories speak to challenges that the mathematicians faced at various stages of their academic journeys. These include challenges to learn complex content, experiences of imposter syndrome, and wrestling with belonging, as well as being minoritized and marginalized by professors or peers. As students reflect on these readings, we have seen how students get new opportunities to share their own stories of struggle. Especially at the crucial transition from computational courses to proof-based courses, many students experience new challenges to their sense of competence in mathematics. Reading about the struggles of research mathematicians helps students recognize that their current struggles are

Day	Date	Chapter	Topic
1	Aug 22		Introduction to the course/ Relations between sets
2	Aug 24		Relations between sets
3	Aug 29		When statements are true and false
4	Aug 31		When statements are true and false
5	Sept 5		What proofs prove (numbers)
6	Sept 7		What proofs prove (numbers)
7	Sept 12		What proofs prove (geometry)
8	Sept 14		What proofs prove (comparing)
9	Sept 19		What proofs prove (mystery)
10	Sept 21		Multiply quantified statements
11	Sept 26		Multiply quantified statements – “So Many Primes” DW2
12	Sept 28		Proof by contradiction
13	Oct 3		Proof by contradiction – “Increasing bounded sequences” AW
14	Oct 5		Review
15	Oct 10		Midterm exam
16	Oct 12	10	Induction
17	Oct 17	10	Induction – “Closed formula for Fibonacci numbers” RG
18	Oct 19	11	Relations
19	Oct 24	11	Relations – “Towers of Hanoi” ADL
20	Oct 26	11	Relations
21	Oct 31	11	Relations – “Fields and Cryptography” KM
22	Nov 2	12	Functions
23	Nov 7	12	Functions – “Permutations with no peaks” PH
24	Nov 9	12	Functions
25	Nov 14	14	Cardinality
26	Nov 16	14	Cardinality – “The set of rational numbers is countable” PP
27	Nov 21	14	Cardinality
28	Nov 28	14	Cardinality – “Coloring graphs” SW
29	Nov 30		Comprehensive review
	Dec 5		FINAL EXAM (11 AM - 1:30 PM)

Fig. 3 Example course schedule with rich proofs integrated with related topics

not indicative of their overall capability to learn. Furthermore, some of our students from historically marginalized communities have expressed gratitude to read about mathematicians whose backgrounds reflect their own. Bringing Author Stories to the course helps students bring their own stories to the course as well.

A second way that the RAMP materials can humanize the classroom is by opening more space for personal story in general. Inspired by this project, some of us have felt more compelled to share our stories with our students. As we explore our own stories as mathematicians, we recognize broader resources to draw on for teaching our classes. Reflexively, the third way RAMP opens the classroom is by trying to draw upon a broader range of student competencies. One of the motivations for the RAMP project was to focus on reading as a learning goal. Literacy instruction has long distinguished reading, writing, speaking, and listening as distinct, but related parts of linguistic competence. Proof instruction is far behind in terms of conceptualizing what it means to productively engage and teach students to do all of these. One great benefit we see in our focus on collective reading is that we may tap into a host of existing student competencies that may not be rewarded in much standard instruction (see David & Zazkis, 2020, regarding what is commonly covered in Introduction to Proof courses around the country). Most proof-oriented courses primarily reward proof writing. Many inquiry-oriented courses put a high premium on speaking. As we seek to broaden the ways that students can contribute to the classroom, we hope we will open more opportunities for students to excel in different ways and will provide more diverse opportunities to learn.

Humanizing the Proof Reading and Writing Processes

Inquiry-oriented instruction often tries to help students learn mathematical processes (such as proving or defining) by engaging in the process collectively. Proofs have long been noted as challenging because they generally hide the thought process by which they were produced, conveying a clean and logical flow that is unreflective of the false-starts and revisions that occurred in their creation (e.g., Selden & Selden, 2013; Thurston, 1994). We similarly note that reading a proof for comprehension is a non-linear process that can be messy and effortful (Inglis & Alcock, 2012). Bringing reading into the classroom activity can help students humanize the process of reading proofs. These processes require practice and learning. As we noted above, most proof-based courses make no specific provision to teach students how to read, which means that reading to understand remains part of the implicit curriculum for such courses (Selden & Selden, 2003). Reading mathematical proofs involves very specific skills such as making sense of new notation and reading equations as conveying conceptual relationships, which computational courses require much less (e.g. Dawkins & Zazkis, 2021). Group reading in the RAMP classroom makes notational choices a part of collective reflection. Students can discuss why notation was chosen and work to develop a shared understanding of the dense text.

Indeed, reflecting on author choice is a key affordance of RAMP activities. RAMP works to bring the author behind the text into the conversation, which provides opportunities for students to recognize they are engaged in a human interchange; they are not mere recipients of information from a faceless authority. Inquiry-oriented instruction (Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007) has long been identified by a mutual responsibility among all members of a classroom to inquire into one another's thinking. We add to this a reflexive sense of responsibility between reader and author to accomplish effective communication. We hope that by reading together, students can recognize when an author has not done an effective job communicating. The responsibility to communicate is shared between the two parties. This is another way that the challenges students face can be recast not as a lack of ability on the part of the student, but (at times) as indicating how well the author has done their job to aid the reader.

Finally, as suggested by the title of the project, we hope that RAMP activities will support students in appreciating the power of mathematical proofs. We mentioned in the introduction that we want students to learn to read and read to learn. However, many introduction to proof courses contain a high volume of proofs of claims that students find obvious. This is an understandable choice given that the goal of the course is for students to learn to prove. However, RAMP opens the opportunity for students to also read proofs that teach them new and fascinating ideas, which is a big part of why mathematicians love proving. Would we teach a literature course with all banal stories? We hope that the Rich Proofs of RAMP help students see proof as a rich genre that can be beautiful and enriching, even while it may be challenging.

Humanizing Mathematicians, So Students Might Imagine Their Own Mathematical Futures

RAMP provides stories of modern mathematicians who represent a broad array of backgrounds and life experiences. We hope that among the stories available, students can find authors with whom they can identify and connect, either in parts of their personal stories or in the challenges faced and overcome. The author stories in the RAMP curriculum are valuable to combat the common association of mathematics with old, white men, which is often fostered by textbook references to the originators of the content being learned. However, the disenfranchising effect of proofs does not require any reference to European mathematicians. Proving provides a challenge to a wide array of the undergraduates we teach, inviting the sense that it is an elite practice for only elite learners. So long as proof-based mathematics is viewed as the territory of the few, most-talented mathematics thinkers, students are likely to interpret that they are not part of that community — as evidenced by their struggles — and they are not welcomed into

that community — as evidenced by the lack of support they feel in learning proving. RAMP represents members of the mathematical community instead seeking to invite undergraduate learners in using their position of perceived legitimacy. The mathematical house does not belong to anyone, but those of us who have spaces in the house we are comfortable in have a responsibility to welcome novices in a supportive environment.

Student reflections and feedback during our pilot semester mirrored many of the themes in the previous paragraph, which we expected since these were prime motivators behind the project. A recurrent theme that we did not expect was consistent student reflection on their own plans for graduate study. Many of the Author Stories speak to mathematicians' pathways to graduate school and experiences along that journey. Far more than in any previous semester, the pilot instructors found students asking for advice and input about their own plans for future study. The RAMP readings helped students reflect not only on their current experiences as mathematics learners, but also upon their current position in a long process of mathematical growth and learning. It seems that reading about and connecting with mathematicians helped our students conceptualize themselves as mathematicians in the making.

What Are RAMP's Challenges for Humanizing Proof-Based Courses?

We are a group of practitioners committed, for a range of reasons, to humanizing our proof-based courses. However, we all recognize that we are learners along a path of continual learning about how to effectively realize these goals. As such, we are thankful for this opportunity to learn together and imagine new pathways forward. We hope we are developing actionable resources and replicable visions for how to humanize introduction to proof courses. RAMP activities provide instructors with rich opportunities to approach their students from an asset lens by building on their strengths and opportunities to grow in a supportive community. However, alongside the potential we perceive, we recognize a set of challenges to using this set of resources and making them useful for other instructors.

One of the central challenges we recognize in the design and use of the materials regards how to engage students with the Author Stories. One reason is that students often perceive these activities as not directly tied to the mathematics they are learning. Likely, many instructors who assign Author Stories in their courses will need to “sell” the activity to their students to motivate its intent and value. In our pilots of this and similar activities, we further recognize that only some of our students really “buy in” with many others reading and reflecting in a perfunctory manner. While this disappoints us on one level, we note two things. One is that we value the benefit to those students who take it seriously enough to make us persist in the assignment. Second, we know students in our courses do not “buy in” to many other things we do, so this is not a particularly unique aspect of Author Stories. Another problem with implementing Author Stories well relates to the prompts we assign to guide student reflection. We have used a few different response prompts so far, but repeated use seems to lead to diminishing returns. Like our comprehension questions for reading Rich Proofs, we likely need to develop a battery of prompts and questions to keep the reflections feeling fresh if students read very many Author Stories.

Content coverage is another challenge in teaching with RAMP materials as well as encouraging others to do the same. This is a perennial concern with any kind of inquiry instruction or active learning (Johnson et al., 2013, 2018; Yoshinobu & Jones, 2012). One reason that we picked introduction to proof as an appropriate space to implement these reading activities is that this is a relatively new course to the curriculum (Schumacher et al., 2015) that may be less bound by the tradition of content as compared to algebra or analysis. We have seen introduction to proof courses that follow the standard approach we described in the introduction, those that merge the course with content such as discrete mathematics or number theory, approaches that are slower introductions to analysis or algebra, and finally courses that only focus on problem solving and independent proof writing. We judge that this diversity reveals a lack of clear obligation to any one set of content. Further, as we talk to instructors around the country, we get the sense that most departments recognize the need for such

a course and yet very few feel that their approach is adequately meeting the need. We thus think that our novel approach does not necessarily have to somehow show its adequacy as compared to the status quo.

The real issue is how to compare the value of teaching specific mathematical topics and examples to providing experiences such as reading complex proofs and connecting to mathematician stories. The choice we have made is to shift the focus of an introduction to proof course from coverage of pre-requisite skills for later courses toward a very different set of goals. Do students appreciate proofs as a rich intellectual achievement and as a compelling epistemological activity? Do students think that proofs can contain beauty and insight? Do students recognize proving as a human activity in which they can engage? Have students been exposed to the range of mathematical domains expressing a breadth of questions and techniques? Do students learn how to read for understanding and how to teach themselves by interacting with their textbook? We think the RAMP curriculum can cover a lot of ground that is often left untouched by conventional approaches. We judge this is worth the cost of covering traditional topics less thoroughly.

Finally, we acknowledge there are other real challenges in instructors navigating how to use the RAMP materials in their own course and context. They must make many choices about which proofs to read, how to order them, how to connect them to student opportunities to prove, and so forth. While options can be helpful, they can also be overwhelming while planning a course. We are working to develop more support materials to make the tools user-friendly and to offer a few reasonable reading paths for instructors who do not want to make it all up from scratch.

Conclusions

The RAMP project has worked to re-envision the introduction to proof course as a literature course in which students learn to read and read to learn. As a key component of this, we want to humanize proof-based mathematics. This has many facets, as we explored in the previous sections. We want to humanize proof texts and the mathematicians who write them. We want to humanize classroom interactions as well as the processes of proof reading and writing. In all cases, we want students to recognize these as human activities in which they can participate. Students should feel invited in, allowed to struggle, and supported to overcome challenges. At the very least, by bringing challenging activities into the public space of the classroom, students can know they are not alone in their struggles. They are not only joined by their classmates, but by the very mathematicians who prepared the texts they are reading.

As mathematics instructors committed to humanizing our own classrooms, we recognize the challenges inherent to imagining new instructional opportunities and finding adequate tools to use in the classroom. We are hopeful that we have produced a set of useful tools and actionable images of how to teach that we can offer to the mathematical community. While we have recognized several challenges inherent to pursuing this form of instruction, we hope that we have cast a vision for the potential benefits that will be compelling to others.

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Declarations

Competing Interest The authors declare no competing interests.

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