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Rational homotopy type and nilpotency of mapping spaces between projective spaces

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Abstract. The rational homotopy type of a mapping space is a way to describe the structure of the space using the algebra of its homotopy groups and the differential graded algebra of its cochains. An L_∞-model is a graded Lie algebra with a family of higher-order brackets satisfying the generalized Jacobi identity and antisymmetry. It can be used to study the rational homotopy type of a space. The nilpotency index of an L_∞-model is useful in understanding a space's algebraic structure. In this paper, we compute the rational homotopy type of the component of some mapping spaces between projective spaces and determine the nilpotency index of corresponding L_∞-models.

Анотація. Раціональний гомотопічний тип простору відображень — це спосіб опису структури цього простору за допомогою алгебри його гомотопічних груп та диференціальної градуйованої алгебри його коланцюгів. В свою чергу, L_∞-модель — це градуйована алгебра Лі, оснащена сім'єю дужок вищого порядку, яка задовольняє узагальнену тотожність Якобі та антисиметрію. Її можна використовувати як інструмент для вивчення раціонального гомотопічного типу простору, а індекс нільпотентності L_∞-моделі також допомагає зрозуміти алгебраїчну структуру даного простору. В представленій роботі обчислено раціональний гомотопічний тип компонент лінійної зв'язності деяких просторів відображень між кватерніонними проективними просторами та визначено індекс нільпотентності відповідних L_∞-моделей.

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1. Introduction

Given a pair of topological spaces X and Y, the space of all continuous maps from X to Y is denoted by $\mathrm{map}(X, Y)$ and equipped with the compact open topology. In general, the space $\mathrm{map}(X, Y)$ is disconnected. For any map $f\colon X \to Y$, let $\mathrm{map}(X, Y; f) \subseteq \mathrm{map}(X, Y)$ be the path component that contains f. The goal is to classify, up to homotopy, the path component that contains f.

The study of the rational homotopy type of mapping spaces was initiated by Thom (in [11]), by considering an Eilenberg-MacLane as a codomain. Haefliger (in [7]) gave the first description of a Sullivan model for mapping spaces. From the Haefliger model, Buijs et al. (in [2]) applied the notion of L_∞-algebra introduced by Lada in [8] to describe an L_∞-model of $\mathrm{map}(X; Y; f)$.

The rational homotopy type of mapping spaces into spheres and complex projective spaces was determined by Møller and Raussen (see [9]). Gatsinzi in [6] gives another proof of Møller and Raussen's result for the rational homotopy type of the mapping space of the component of the natural inclusion between complex projective spaces, $i\colon \mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+k}$, and showed that it has the rational homotopy type of a product of odd dimensional spheres and a complex projective space.

In this paper, we consider projective spaces $\mathbb{KP}^n$ over $\mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, and $\mathbb{OP}^n$ for $n = 1, 2$. We give a different proof of Møller and Raussen result in [9] for the quaternionic case. We also compute the rational homotopy type of the component of the inclusion between real and quaternionic projective spaces using L_∞-models of mapping spaces. For octonionic projective spaces, we only consider the inclusion $\mathbb{OP}^1 \hookrightarrow \mathbb{OP}^2$. In particular, we show that the rational homotopy type of the component of the natural inclusion between quaternionic and octonionic projective spaces is a product of a projective space and odd dimensional spheres. Moreover, we compute the nilpotency index of the component of mapping spaces containing the inclusion $\mathbb{KP}^n \hookrightarrow \mathbb{KP}^{n+k}$, $k \geqslant 1$, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ as well as the inclusion $i_{1,1}\colon \mathbb{OP}^1 \hookrightarrow \mathbb{OP}^2$.

The organization of this paper is as follows. In Section 2, we recall Sullivan models of simply connected spaces and define L_∞-models of function spaces; in Section 3 we determine the rational homotopy type of mapping spaces between projective spaces and in the fourth section the nilpotency index of mapping spaces between projective spaces will be computed.

2. Sullivan minimal models and L_∞-models of function spaces

In this section, some of the basic definitions and terms that will be used in our discussions are given. The main references of the concepts are [2] and [3].

Definition 2.1. A differential graded algebra (*dga*) is a graded algebra $A = \oplus_{n\geqslant 0} A^n$ together with a differential $d\colon A^n \to A^{n+1}$ of degree $+1$ such that $d \circ d = 0$ and $d(ab) = (da)b + (-1)^{|a|}a(db)$ for $a, b \in A$.

A *dga* satisfying $ab = (-1)^{|a||b|}ba$, for all $a, b \in A$ is called a commutative differential graded algebra (*cdga*).

Definition 2.2. If a chain of quasi-isomorphisms of commutative cochain algebras connects two *cdga*'s (A, d) and (C, d), then they are said to have the same homotopy type, i.e.

$$(A, d) \xrightarrow{\cong} (B(0), d) \xleftarrow{\cong} \cdots \xrightarrow{\cong} (B(k), d) \xleftarrow{\cong} (C, d).$$

A *Sullivan* algebra $(\wedge V, d)$ is a free *cdga* spanned by a positively graded vector space with an increasing sequence of graded subspaces:

$$V(0) \subset V(1) \subset V(2) \subset \cdots ,$$

such that $d = 0$ on $V(0)$ and $d\colon V(k) \to \wedge V(k-1)$, for $k \geqslant 1$. A Sullivan model for a simply connected space X is a quasi-isomorphism

$$(\wedge V, d) \xrightarrow{\cong} \mathcal{A}_{PL(X)},$$

where $\mathcal{A}_{PL(X)}$ is the *cdga* of piecewise linear forms on X (see [10]), and it is called minimal if $\operatorname{Im} d \subset \wedge^{\geqslant 2} V$. If $(\wedge V, d)$ is a Sullivan model of X, then there is an isomorphism of algebras

$$H^*(\wedge V, d) \cong H^*(X; \mathbb{Q}).$$

Moreover, if $(\wedge V, d)$ is minimal and V is of finite type, then there is an isomorphism of vector spaces $V^n \cong \operatorname{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbb{Q})$.

A *cdga* model of X is a *cdga* (A, d) with the same rational homotopy type as $\mathcal{A}_{PL(X)}$. If $f\colon X \to Y$ is a map between two simply connected spaces of finite type, then there is a *cdga* map $\phi\colon (\wedge V, d) \to (A, d)$, called model of f, where $(\wedge V, d)$ and (A, d) are respective *cdga*'s models of Y and X (see [6]).

Consider the minimal Sullivan model $(\wedge V, d)$ of a simply connected space X. We say that X is a *formal space* if there is a quasi-isomorphism

$$(\wedge V, d) \xrightarrow{\simeq} (H^*(X, \mathbb{Q}), 0)$$

between its Sullivan model and its rational cohomology; where projective spaces and spheres are some examples of formal spaces.

The notion of L_∞-algebra was first introduced by Lada (in [8]) and we recall its definition below.

Definition 2.3. An L_∞*-algebra* is a graded vector space $L = \oplus_{n\geqslant 1} L_n$ with a family of linear maps, $\ell_k \colon L^{\otimes k} \to L$, of degree $k-2$, for $k \geqslant 1$, satisfying:

(i) graded skew symmetry: $\ell_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \operatorname{sgn}(\sigma)\epsilon_\sigma \ell_k(x_1, \dots, x_k)$, for $\sigma \in S_k$,

(ii) generalized Jacobi identity:

$$\sum_{i+j=n+1} \sum_{\sigma\in S(i,n-i)} \operatorname{sgn}(\sigma)\epsilon(\sigma)\ell_j(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0,$$

where $\sigma \in S_n$ is an $(i, n-i)$ shuffle and $\epsilon(\sigma)$ is the Koszul sign of σ, which is defined by the relation $x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$ and depends on degree of x_i's.

In the definition, if $\ell_1 = 0$, then the L_∞-algebra L is called *minimal*. The definition above coincides with that of a differential graded Lie algebra if $\ell_k = 0$, for $k \geqslant 3$, where ℓ_1 is the differential and ℓ_2 is the Lie bracket. An L_∞-algebra L of finite type is said to be an L_∞*-model* of X if

$$C^\infty(L) = (\wedge(sL)^\#, d)$$

is a Sullivan model of X (see [2]).

Definition 2.4. Given a morphism $\phi \colon (A, d) \to (B, d)$ of two *cdga*'s, a ϕ-derivation of degree k is a linear map $\theta \colon A^* \to B^{*-k}$ such that

$$\theta(xy) = \theta(x)\phi(y) + (-1)^{k|x|}\phi(x)\theta(y).$$

Let us denote the graded vector space of all ϕ-derivations of degree k by $\operatorname{Der}_k(A, B; \phi)$. Then, there is a differential

$$\delta \colon \operatorname{Der}_k(A, B; \phi) \to \operatorname{Der}_{k-1}(A, B; \phi)$$

of chain complexes given by $\delta\theta = d\theta - (-1)^{|\theta|}\theta d$ and

$$\operatorname{Der}(A, B; \phi) = \bigoplus_{k\in\mathbb{Z}} \operatorname{Der}_k(A, B; \phi)$$

is a chain complex. In particular, if $A = B$, then $\operatorname{Der}_k(A, A; \operatorname{Id}_A)$ is the usual chain complex of derivations on A. If $A = \wedge V$, then there is an isomorphism

$$\operatorname{Der}(\wedge V, B; \phi) \xrightarrow{\cong} \operatorname{Hom}(V, B)$$

via an identification map $\theta \mapsto \theta|_V$. In particular, if $A = \wedge V$, where $\{v_1, \dots, v_t, \dots\}$ is a basis of V, then we denote by (v_i, a) the unique ϕ-derivation θ such that:

$$\theta(v_j) = \begin{cases} a, & \text{for } i = j, a \in B; \\ 0, & \text{for } i \neq j. \end{cases}$$

Define the subspace of $\operatorname{Der}(\wedge V, B; \phi)$ of positive derivations, denoted by $\widetilde{\operatorname{Der}}(\wedge V, A; \phi)$, as:

$$\widetilde{\operatorname{Der}}_i(\wedge V, A; \phi) = \begin{cases} \operatorname{Der}_i(\wedge V, B; \phi), & \text{for } i > 1, \\ Z \operatorname{Der}_1(\wedge V, B; \phi) = \{\theta \in \operatorname{Der}_1(\wedge V, B; \phi) : \delta\theta = 0\}. \end{cases}$$

Let $\varphi_1, \dots, \varphi_i \in \widetilde{\operatorname{Der}}(\wedge V, B; \phi)$ be ϕ-derivations of degrees $q_1, \dots, q_i$ respectively. Then, their bracket operation of length i is defined by

$$[\varphi_1, \dots, \varphi_i](v) = \\ = (-1)^{q_1+\cdots+q_i-1} \sum \Big(\sum_{j_1,\dots,j_i} \epsilon \phi(v_1 \cdots \hat{v}_{j_1} \cdots \hat{v}_{j_i} \cdots v_k) \varphi_1(v_{j_1}) \cdots \varphi_i(v_{j_i}) \Big),$$

where $dv = \sum v_1 \cdots v_k$ and ϵ is the Koszul sign. We may desuspend the bracket operation to define a set of linear maps $\{\ell_j\}_{j \geqslant 1}$ on $s^{-1}\widetilde{\operatorname{Der}}(\wedge V, B; \phi)$, each with degree $j - 2$.

For $j = 1, \ell_1(s^{-1}\varphi) = -s^{-1}\delta\varphi$ and for $j \geqslant 2$,

$$\ell_j(s^{-1}\varphi_1, \dots, s^{-1}\varphi_j) = (-1)^{\epsilon_j} s^{-1}[\varphi_1, \dots, \varphi_j], \text{ where } \epsilon_j = \sum_{i=1}^{j-1} (j-i)|\varphi_i|.$$

This endows $s^{-1}\widetilde{\operatorname{Der}}(\wedge V, B; \phi)$ with an L_∞ structure, which is a model of $\operatorname{map}(X, Y; f)$ [2, Lemma 3.3].

3. Rational homotopy type of mapping spaces between projective spaces

Gatsinzi (in [6]) showed that the rational homotopy type of the component of the natural inclusion between complex projective spaces is a product of a complex projective space and odd dimensional spheres. Here, we consider the component of the inclusion $i\colon \mathbb{KP}^n \hookrightarrow \mathbb{KP}^{n+k}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ and of the inclusion $\mathbb{O}P^1 \hookrightarrow \mathbb{O}P^2$. First let us consider the real projective space $\mathbb{RP}^n$ and we have the following result.

Theorem 3.1. *Consider the inclusion* $i\colon \mathbb{RP}^m \hookrightarrow \mathbb{RP}^{m'}$ *for* $m < m'$. *Then, the rational homotopy type of* $\operatorname{map}(\mathbb{R}^m, \mathbb{RP}^{m'}; i)$ *is:*

(i) *a product of odd dimensional spheres if* m *is even and* m' *is odd;*

(ii) *a product of an odd and an even dimensional sphere, and an Eilenberg-MacLane space if both* m *and* m' *are even;*

(iii) *a product of odd dimensional spheres if if both* m *and* m' *are odd;*

(iv) *the total space of a non trivial fibration over a product of 2 spheres by an Einleberg-MacLane space if* m *is odd and* m' *even.*

Proof. There is a covering projection $\mathbb{Z}_2 \to \mathbb{S}^n \to \mathbb{RP}^n$ and hence $\mathbb{RP}^n$ has the rational homotopy type of $\mathbb{S}^n$, for $n > 1$. Therefore, $\operatorname{map}(\mathbb{RP}^m, \mathbb{RP}^{m'}; i)$ and $\operatorname{map}(\mathbb{S}^m, \mathbb{S}^{m'}; i)$ have the same rational homotopy type. Moreover, the inclusions $i\colon \mathbb{S}^m \hookrightarrow \mathbb{S}^{m'}$ are homotopy trivial as $\pi_i(\mathbb{S}^{m'}) = 0$ for $i < m'$.

Here, we mimic the Gatsinzi's (presented in [6]) construction to get an L_∞-model of the component of the inclusion $i\colon \mathbb{S}^m \hookrightarrow \mathbb{S}^{m'}$.

(1) Let $m = 2n$ and $m' = 2n + 2k - 1$, for $k \geqslant 1$. Following the construction given in [6], a model of the inclusion i is given by

$$\phi\colon (\wedge y_{2n+2k-1}, 0) \to \big(\wedge(x_{2n})/(x_{2n}^2), 0\big), \text{ where } \phi(y_{2n+2k-1}) = 0.$$

Then,

$$\widetilde{\operatorname{Der}}\big((\wedge y_{2n+2k-1}, 0), (\wedge(x_{2n})/(x_{2n}^2), 0); \phi\big) = \langle \alpha_{2n+2k-1}, \alpha_{2k-1}\rangle,$$

where $\alpha_{2n+2k-1} = (y_{2n+2k-1}, 1)$ and $\alpha_{2k-1} = (y_{2n+2k-1}, x_{2n})$. For degree reason, every bracket is trivial. Thus $\operatorname{map}(\mathbb{S}^{2n}, \mathbb{S}^{2n+2k-1}; i)$ has the rational homotopy of $\mathbb{S}^{2k-1} \times \mathbb{S}^{2n+2k-1}$.

(2) Let $m = 2n$ and $m' = 2n + 2k$ for $k \geqslant 1$. Consider the inclusion $i\colon \mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2k}$. Then, a model of i is given by

$$\phi\colon \big(\wedge(y_{2(n+k)}, y_{4(n+k)-1}), d\big) \to \big(\wedge(x_{2n})/(x_{2n}^2), 0\big),$$

where

$$dy_{2(n+k)} = 0, \qquad dy_{4(n+k)-1} = y_{2(n+k)}^2,$$

and both $\phi(y_{2(n+k)})$ and $\phi(y_{4(n+k)-1})$ are equal to zero.

Note that

$$\widetilde{\operatorname{Der}}\,\big((\wedge(y_{2(n+k)}, y_{4(n+k)-1}), d), (\wedge(x_{2n})/(x_{2n}^2), 0); \phi\big)$$

is spanned by the set

$$\big\{\alpha_{2(n+k)},\ \alpha_{2k},\ \alpha_{4(n+k)-1},\ \alpha_{2n+4k-1}\big\},$$

where

$$\begin{aligned} \alpha_{2(n+k)} &= (y_{2(n+k)}, 1)\\ \alpha_{2k} &= (y_{2(n+k)}, x_{2n})\\ \alpha_{4(n+k)-1} &= (y_{4(n+k)-1}, 1)\\ \alpha_{2n+4k-1} &= (y_{4(n+k)-1}, x_{2n}) \end{aligned}$$

Here, the only nonzero bracket is $[\alpha_{2(n+k)}, \alpha_{2(n+k)}] = -2\alpha_{4(n+k)-1}$. The L_∞-model (L, ℓ_i) of $\mathrm{map}(\mathbb{S}^{2n}, \mathbb{S}^{2n+2k}; i)$ is spanned by the set

$$\{s^{-1}\alpha_{2(n+k)},\ s^{-1}\alpha_{2k},\ s^{-1}\alpha_{4(n+k)-1},\ s^{-1}\alpha_{2n+4k-1}\}$$

and thus, a Sullivan model of $\mathrm{map}(\mathbb{S}^{2n}, \mathbb{S}^{2n+2k}; i)$ is:

$$C^\infty(L) = \big(\wedge(z_{2(n+k)}, z_{4(n+k)-1}), d\big) \otimes \big(\wedge(z_{2k}, z_{2n+4k-1}), 0\big),$$

where $dz_{2(n+k)} = 0, dz_{4(n+k)-1} = z^2_{2(n+k)}$. Therefore, $\mathrm{map}(\mathbb{S}^{2n}, \mathbb{S}^{2n+2k}; i)$ has the rational homotopy type of

$$\mathbb{S}^{2n+2k} \times \mathbb{S}^{2n+4k-1} \times K(\mathbb{Z}, 2k).$$

(3) Let $m = 2n+1$ and $m' = 2n+2k+1$ for $k \geqslant 1$. Following similar work as above, we can obtain that

$$\widetilde{\mathrm{Der}}\big((\wedge(y_{2n+2k+1}), 0), (\wedge(x_{2n+1}), 0); \phi\big)$$

is spanned by the set $\{\alpha_{2n+2k+1},\ \alpha_{2k}\}$, where $\alpha_{2n+2k+1} = (y_{2n+2k+1}, 1)$, $\alpha_{2k} = (y_{2n+2k+1}, x_{2n+1})$ and all brackets are trivial. Thus, a Sullivan model of $\mathrm{map}(\mathbb{S}^{2n+1}, \mathbb{S}^{2n+2k+1}; i)$ is $(\wedge(z_{2n+2k+1}, z_{2k}), 0)$. Therefore,

$$\mathrm{map}(\mathbb{S}^{2n+1}, \mathbb{S}^{2n+2k+1}; i) \cong_{\mathbb{Q}} \mathbb{S}^{2n+2k+1} \times K(\mathbb{Z}, 2k).$$

(4) Let $m = 2n+1$ and $m' = 2n+2k$ for $k \geqslant 1$. A Sullivan model of i is given by

$$\phi\colon (\wedge(y_{2(n+k)}, y_{4(n+k)-1}), d) \to (\wedge(x_{2n+1}), 0),$$

where $dy_{2(n+k)} = 0, dy_{4(n+k)-1} = y^2_{2(n+k)}$ and ϕ is the zero map. The vector space

$$\widetilde{\mathrm{Der}}\big((\wedge(y_{2n+2k}, y_{4(n+k)-1}), d), (\wedge(x_{2n+1}), 0); \phi\big)$$

is spanned by

$$\{\alpha_{2(n+k)},\ \alpha_{4(n+k)-1},\ \alpha_{2k-1},\ \alpha_{2n+4k-2}\},$$

where

$$\begin{aligned}
\alpha_{2(n+k)} &= (y_{2(n+k)}, 1)\\
\alpha_{4(n+k)-1} &= (y_{4(n+k)-1}, 1)\\
\alpha_{2k-1} &= (y_{2(n+k)}, x_{2n+1})\\
\alpha_{2n+4k-2} &= (y_{4(n+k)-1}, x_{2n+1})
\end{aligned}$$

and the only non zero brackets are $[\alpha_{2(n+k)}, \alpha_{2(n+k)}] = 2\alpha_{4(n+k)-1}$ and $[\alpha_{2(n+k)}, \alpha_{2k-1}] = 2\alpha_{2n+4k-2}$.

The Sullivan model of $\mathrm{map}(\mathbb{S}^{2n+1}, \mathbb{S}^{2n+2k+1}; i)$ is given by

$$\big(\wedge(z_{2(n+k)}, z_{4(n+k)-1}, z_{2k-1}, z_{2n+4k-2}), d\big),$$

where

$$dz_{2(n+k)} = dz_{2k-1} = 0, \qquad dz_{4(n+k)-1} = z_{2(n+k)}^2$$

and

$$dz_{2n+4k-2} = z_{2n+2k}z_{2k-1}.$$

This is a model of the total space E of a fibration

$$K(\mathbb{Z}, 2n+4k-2) \to E \to \mathbb{S}^{2k-1} \times \mathbb{S}^{2n+2k}.$$

It is classified by a map $f\colon \mathbb{S}^{2k-1} \times \mathbb{S}^{2n+2k} \to \mathbb{S}^{2n+4k-1}$ with a Sullivan model

$$\psi\colon \big(\wedge x_{2n+4k-1}, 0\big) \to \big(\wedge x_{2k-1}, 0\big) \otimes \big(\wedge(x_{2n+2k})/(x_{2n+2k}^2), 0\big),$$

where $\psi(x_{2n+4k-1}) = x_{2n+2k}x_{2k-1}$, which is not trivial. □

We now proceed to determine the rational homotopy type of mapping spaces of maps between quaternionic projective spaces.

As $\mathbb{HP}^n$ is a homogeneous space, one can apply [4, Proposition 15.18] to derive its minimal Sullivan model, which is given by $(\wedge(y_4, y_{4n+3}), d)$, where $dy_4 = 0$, and $dy_{4n+3} = y_4^{n+1}$. Moreover, there is a quasi-isomorphism

$$\psi\colon \big(\wedge(y_4, y_{4n+3}), d\big) \xrightarrow{\cong} \big(\wedge(y_4)/(y_4^{n+1}), 0\big).$$

Hence, $\mathbb{HP}^n$ is formal.

The inclusion $i_{n,k}\colon \mathbb{HP}^n \to \mathbb{HP}^{n+k}$, $k \geqslant 0$ has a Sullivan model

$$\phi\colon \big(\wedge(x_4, x_{4(n+k)+3}), d\big) \to \big(\wedge(y_4, y_{4n+3}), d\big),$$

where $\phi(x_4) = y_4$ and $\phi(x_{4(n+k)+3}) = \alpha$ such that $d\alpha = y_4^{n+1}$. Then, the composition $\tilde{\phi} = \psi \circ \phi$ is also a model of $i_{n,k}$.

Theorem 3.2. *Denote the constant map and the identity map on* $\mathbb{HP}^n$ *by* c *and* Id *respectively. Then:*

(i) *the rational homotopy type of the mapping space* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n; c)$ *is a product of odd dimensional spheres and* $\mathbb{HP}^n$*,*

(ii) *the rational homotopy type of* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n; \mathrm{Id})$ *is a product of odd dimensional spheres.*

Proof. (i) A model of c is given by

$$\tilde{\phi}\colon \big(\wedge(x_4, x_{4n+3}), d\big) \to \big(\wedge(y_4)/(y_4^{n+1}), 0\big),$$

where $\tilde{\phi}(x_4) = \tilde{\phi}(x_{4n+3}) = 0$. The vector space of $\tilde{\phi}$-derivations is spanned by

$$\theta_4 = (x_4, 1) \qquad \text{and} \qquad \alpha_{4t-1} = (x_{4n+3}, y_4^{n-t+1}),$$

for $t = 1, \ldots, n+1$.

Following the definition of $\tilde{\phi}$, it is obvious that $\delta\theta_4 = \delta\alpha_{4t-1} = 0$ for all t. The only non-zero bracket is

$$\underbrace{[\theta_4, \ldots, \theta_4]}_{n+1} = -(n+1)!\alpha_{4n+3}.$$

An L_∞-model (L, ℓ_i) of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n; c)$ is spanned by

$$\{s^{-1}\theta_4, s^{-1}\alpha_{4t-1},\ t = 1, \ldots, n+1\},$$

where

$$\ell_{n+1}(s^{-1}\theta_4, \ldots, s^{-1}\theta_4) = -(n+1)!s^{-1}\alpha_{4n+3}$$

and all other brackets are zero. Thus,

$$C^\infty(L) \cong \big(\wedge(z_4, z_{4n+3}), d\big) \otimes \big(\wedge(z_3, z_7, \ldots, z_{4n-1}), 0\big),$$

where $dz_4 = 0$ and $dz_{4n+3} = z_4^{n+1}$.

Therefore, the rational homotopy type of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n; c)$ is of a product of odd dimensional spheres and an n-dimensional quaternionic projective space: that is, $\mathbb{HP}^n \times \mathbb{S}^3 \times \mathbb{S}^7 \times \cdots \times \mathbb{S}^{4n+3}$.

(ii) A model of the identity map is given by

$$\tilde{\phi}\colon \big(\wedge(x_4, x_{4n+3}), d\big) \to \big(\wedge(x_4)/(x_4^{n+1}), 0\big),$$

where $\tilde{\phi}(x_4) = x_4$ and $\tilde{\phi}(x_{4n+3}) = 0$.

The vector space of $\tilde{\phi}$-derivations is spanned by

$$\{\theta_4 = (x_4, 1),\ \alpha_{4j-1} = (x_{4n+3}, x_4^{n-j+1}),\ \text{ for } j = 1, \ldots, n+1\}.$$

A straightforward computation gives us nonzero brackets

$$\delta\theta_4 = -(n+1)\alpha_3 \qquad \text{and} \qquad \underbrace{[\theta_4, \ldots, \theta_4]}_{j} = -(n+1)!\alpha_{4j-1},$$

for $2 \leqslant j \leqslant n+1$. An L_∞-model (L, ℓ_j) of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n; \mathrm{Id})$ is spanned by

$$\{s^{-1}\theta_4,\ s^{-1}\alpha_3,\ s^{-1}\alpha_7,\ \ldots,\ s^{-1}\alpha_{4n+3}\}.$$

Moreover,

$$\ell_j(s^{-1}\theta_4, \ldots, s^{-1}\theta_4) = s^{-1}\underbrace{[\theta_4, \ldots, \theta_4]}_{j} = -(n+1)!s^{-1}\alpha_{4j-1},$$

for $j = 1, \ldots, n+1$, and all other brackets are zero. Thus, a Sullivan model for $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^n;\, \mathrm{Id})$ is given by

$$C^\infty(L) = \big(\wedge(z_4, z_3, z_7, \ldots, z_{4n+3}), d\big),$$

where $dz_4 = 0$ and $dz_3 = z_4$, $dz_7 = z_4^2$, $\ldots$, $dz_{4n+3} = z_4^{n+1}$. Moreover, the ideal generated by $\{z_3, z_4\}$ is acyclic, hence

$$\big(\wedge(z_4, z_3, z_7, \ldots, z_{4n+3}), d\big) \cong \big(\wedge(z_3, z_4), d\big) \otimes \big(\wedge(z_7, \ldots, z_{4n+3}), 0\big),$$

where $dz_3 = z_4$, which is quasi-isomorphism to $(\wedge(z_7, \dots, z_{4n+3}), 0)$.

Therefore, the component of the identity has the rational homotopy type of $\mathbb{S}^7 \times \mathbb{S}^{11} \times \cdots \times \mathbb{S}^{4n+3}$. □

Theorem 3.3. *Consider the natural inclusion $i_{n,k}\colon \mathbb{HP}^n \hookrightarrow \mathbb{HP}^{n+k}$, for $k \geqslant 1$. The mapping space* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k}; i_{n,k})$ *has the rational homotopy type of a product of a k-dimensional quaternionic projective space and odd dimensional spheres.*

Proof. A model of the inclusion $i_{n,k}$ is given by

$$\tilde{\phi}\colon \big(\wedge(x_4, x_{4(n+k)+3}), d\big) \to \big(\wedge(y_4)/(y_4^{n+1}), 0\big),$$

where $\tilde{\phi}(x_4) = y_4$, and $\tilde{\phi}(x_{4(n+k)+3}) = 0$.

The vector space of $\tilde{\phi}$-derivations is spanned by

$$\{\theta_4,\ \alpha_{4k+3},\ \alpha_{4k+7},\ \alpha_{4(n+k)+3}\}$$

where

$$\begin{aligned}\theta_4 &= (x_4, 1),\\ \alpha_{4k+3} &= (x_{4(n+k)+3}, y_4^n),\\ \alpha_{4k+7} &= (x_{4(n+k)+3}, y_4^{n-1}),\\ &\vdots\\ \alpha_{4(n+k)+3} &= (x_{4(n+k)+3}, 1).\end{aligned}$$

A direct computation shows that the only nonzero brackets are

$$\underbrace{[\theta_4, \dots, \theta_4]}_{k+j} = c_j \alpha_{4(k+j)-1} \quad \text{for} \quad j = 1, 2, \dots, n+1,$$

where $c_j = (n+k+1)(n+k)\cdots(n+k-j)$.

Hence, an L_∞-model (L, ℓ_j) of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k}; i_{n,k})$ is spanned by

$$\{s^{-1}\theta_4, s^{-1}\alpha_{4k+3}, s^{-1}\alpha_{4k+7}, \dots, s^{-1}\alpha_{4(n+k)+3}\},$$

where $\ell_{k+j}(s^{-1}\theta_4, \dots, s^{-1}\theta_4) = c_j s^{-1}\alpha_{4(k+j)-1}$, for $j = 1, \dots, n+1$.

Thus, its Sullivan model is given by

$$C^\infty(L) = \big(\wedge(z_4, z_{4k+3}, z_{4k+7}, \dots, z_{4(n+k)+3}), d\big),$$

where $dz_4 = 0, dz_{4k+4j-1} = b_j z_4^{k+j}$ and $b_j = -c_j$, for $j = 1, 2, \dots, n+1$. However, we may assume that all $b_j = 1$ by making a suitable change of variables.

A subsequent change of variables

$$u_{4k+7} = z_{4k+7} - z_4 z_{4k+3,},$$

$$u_{4k+11} = z_{4k+11} - z_4^2 z_{4k+3},$$
$$u_{4k+15} = z_{4k+15} - z_4^3 z_{4k+3},$$
$$\vdots$$
$$u_{4k+4n+3} = z_{4(n+k)+3} - z_4^n z_{4k+3},$$

yields an isomorphic model

$$\big(\wedge(z_4, z_{4k+3}), d\big) \otimes \big(\wedge(u_{4k+7}, \ldots, u_{4(n+k)+3}), 0\big),$$

where $dz_4 = 0$ and $dz_{4k+3} = z_4^{k+1}$. Therefore, the rational homotopy type of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k}; i_{n,k})$ is $\mathbb{HP}^k \times \mathbb{S}^{4k+7} \times \cdots \times \mathbb{S}^{4n+4k+3}$. □

One could also perform similar computations for the natural inclusion

$$i_{1,1} \colon \mathbb{S}^8 \cong \mathbb{OP}^1 \hookrightarrow \mathbb{OP}^2 = \mathbb{S}^8 \cup_\sigma e^{16}$$

for the Hopf map $\sigma \colon \mathbb{S}^{15} \to \mathbb{S}^8$, between Cayley projective spaces. A Sullivan model of $i_{1,1}$ is given by

$$\phi \colon \big(\wedge(x_8, x_{23}), d\big) \to \big(\wedge(y_8)/(y_8^2), 0\big),$$

where $dx_8 = 0$, $dx_{23} = x_8^3$, $\phi(x_8) = y_8$ and $\phi(x_{23}) = 0$.

The vector space

$$\widetilde{\mathrm{Der}}\,\big((\wedge(x_8, x_{23}), d), (\wedge(y_8)/(y_8^2), 0; \phi)\big)$$

is spanned by $\{\beta_8, \alpha_{15}, \alpha_{23}\}$, where

$$\beta_8 = (x_8, 1), \qquad \alpha_{23} = (x_{23}, 1), \qquad \alpha_{15} = (x_{23}, y_8).$$

Here the only nonzero brackets are

$$\ell_2(\beta_8, \beta_8) = \alpha_{15} \qquad \text{and} \qquad \ell_3(\beta_8, \beta_8, \beta_8) = \alpha_{23}.$$

So, a Sullivan model of $\mathrm{map}(\mathbb{S}^8, \mathbb{OP}^2; i_{1,1})$ is given by

$$(\wedge(z_8, z_{15}, z_{23}), d),$$

where $dz_8 = 0, dz_{15} = z_8^2$ and $dz_{23} = z_8^3$.

By a change of variable $u_{23} = z_{23} - z_8 z_{15}$, one obtains an isomorphic model

$$\big(\wedge(z_8, z_{15}), d\big) \otimes \big(\wedge u_{23}, 0\big),$$

where $dz_8 = 0, dz_{15} = z_8^2$.

Thus, $\mathrm{map}(\mathbb{S}^8, \mathbb{OP}^2; i_{1,1})$ has the rational homotopy type of $\mathbb{OP}^1 \times \mathbb{S}^{23}$.

4. Nilpotency of mapping spaces between projective spaces

The nilpotency index of a ring R, denoted by $\mathrm{nil}R$, is the least positive integer n such that $R^n = 0$. Let L be an L_∞-algebra. Consider the lower central descending series $F^1L = L \supseteq F^2L \supseteq \cdots$, where F^iL is spanned by all possible bracket expressions one can form using at least i elements from L, that is,

$$F^iL = \sum_{i_1+i_2+\cdots+i_k \geqslant i} [F^{i_1}L, \dots, F^{i_k}L],$$

(see [1, 2]).

Definition 4.1. An L_∞-algebra L is said to be *nilpotent* if there exists a positive integer i such that $F^iL = 0$. If L is a nilpotent L_∞-algebra, the *nilpotency index* of L, denoted by $\mathrm{nil}L$, is the positive integer i_0 such that $F^iL = 0$ for $i > i_0$ and $F^{i_0} \neq 0$.

If X is a simply connected CW-complex of finite type with minimal L_∞-model L, then the rational nilpotency index of X, denoted by $\mathrm{nil}_{\mathbb{Q}}(X)$, is defined as $\mathrm{nil}L$ [2, Definition 4.1]. It is shown in [2, Corrollary 4.3] that if $c\colon X \to Y$ is the constant map and X is a formal finite CW-complex, then

$$\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(X, Y; c)) \leqslant \mathrm{nil}_{\mathbb{Q}}(Y).$$

In this section, we show that a similar result holds for the inclusion

$$i_{n,k}\colon \mathbb{HP}^n \hookrightarrow \mathbb{HP}^{n+k} \quad \text{for } k \geqslant 1.$$

Proposition 4.2. *The mapping space* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k})$, *for* $k \geqslant 1$, *is nilpotent with nilpotency index* $n + k + 1$.

Proof. It follows from the L_∞-model L of $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k})$,

$$\ell_{k+n+1} \neq 0 \ \text{ and } \ \ell_j = 0 \ \text{ for } \ j > n + k + 1. \qquad \square$$

Theorem 4.3. *The rational nilpotency index of* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k})$ *is equal to the nilpotency indexes of* $H^*(\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k}), \mathbb{Q})$.

Proof. The cohomology algebra $H^*(\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k}; i_{n,k}), \mathbb{Q})$ is isomorphic to

$$\wedge(z_4)/(z_4^{k+1}) \otimes (\otimes_{i=1}^{n}(\wedge z_{4(k+1+i)-1})).$$

Hence,

$$H^*(\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k}; i_{n,k}), \mathbb{Q})$$

has the nilpotency index of $k + 1 + n$. $\square$

Corollary 4.4. *The rational nilpotency indexes of* $\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k})$ *and* $\mathbb{HP}^{n+k}$ *are equal.*

Proof. Note that, $\mathbb{HP}^n$ has minimal L_∞-model $L = \langle y_3, y_{4n+2}\rangle$, where the only nonzero bracket is $\ell_{n+1}(y_3, \ldots, y_3) = y_{4n+2}$.

Therefore, $\mathrm{nil}_{\mathbb{Q}}(\mathbb{HP}^n) = n + 1$, and so

$$\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(\mathbb{HP}^n, \mathbb{HP}^{n+k};\ i_{n,k})) = \mathrm{nil}_{\mathbb{Q}}(\mathbb{HP}^{n+k}) = n + k + 1. \qquad \square$$

In the proof of Proposition 10 in [5], similar computations for the component of the inclusion $i_{n,k}\colon \mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+k}$ shows that

$$\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(\mathbb{CP}^n, \mathbb{CP}^{n+k}; i_{n,k})) = \mathrm{nil}_{\mathbb{Q}}(\mathbb{CP}^{n+k}) = n + k + 1.$$

As $\mathbb{S}^n$ is coformal, $\pi_*(\Omega\mathbb{S}^n) \otimes \mathbb{Q} = (\mathbb{L}(x_{n-1}), 0)$ is a minimal L_∞-model for $\mathbb{S}^n$, when endowed with the Samelson product.

Therefore

$$\mathrm{nil}_{\mathbb{Q}}(\mathbb{S}^n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

From the computation of the L_∞-model of $\mathrm{map}(\mathbb{RP}^m, \mathbb{RP}^{m'}; i)$ in the proof of Theorem 3.2, one can deduce the following result.

Corollary 4.5. *The rational nilpotency index,*

$$\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(\mathbb{S}^m, \mathbb{S}^{m'}; i)) = \mathrm{nil}_{\mathbb{Q}}(\mathbb{S}^{m'}).$$

In a similar way, from the computation of the L_∞-model of $\mathrm{map}(\mathbb{S}^8, \mathbb{OP}^2; i)$ in Section 3, we conclude that:

Corollary 4.6. *The rational nilpotency index* $\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(\mathbb{S}^8, \mathbb{OP}^2); i_{1,1})$ *and* $\mathrm{nil}_{\mathbb{Q}}(\mathbb{OP}^2)$ *are equal.*

Then, we finally summarize all the above cases in the following Theorem.

Theorem 4.7. *The rational nilpotency indexes*

$$\mathrm{nil}_{\mathbb{Q}}(\mathrm{map}(\mathbb{KP}^n, \mathbb{KP}^{n+k}; i_{n,k})) \qquad \textit{and} \qquad \mathrm{nil}_{\mathbb{Q}}(\mathbb{KP}^{n+k})$$

are equal for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *or* $\mathbb{O}$.

We conclude our work with the following example.

Example 4.8. Consider $i_{3,2}\colon \mathbb{HP}^3 \to \mathbb{HP}^5$. The minimal Sullivan model of the mapping space $\mathrm{map}(\mathbb{HP}^3, \mathbb{HP}^5; i_{3,2})$ is

$$(\wedge(z_4, z_{11}), d) \otimes (\wedge(z_{15}, z_{19}, z_{23}), 0)$$

where $dz_4 = 0, d_{11} = z_4^3$ and its cohomology is

$$A = \wedge(z_4)/(z_4^3) \otimes \wedge(z_{15}, z_{19}, z_{23})$$

for which $\mathrm{nil}(A^{\geqslant 1}) = 6$. Indeed the highest non-zero cohomology class is $[z_4^2 \cdot z_{15} \cdot z_{19} \cdot z_{23}]$ and $(A^+)^6 = 0$.

5. Conclusion

In this work, we computed the rational homotopy type of mapping spaces between projective spaces. In the case of an inclusion $i_{n,k} \colon \mathbb{HP}^n \hookrightarrow \mathbb{HP}^{n+k}$, for $k \geqslant 1$, we showed that the mapping space has the rational homotopy type of a product of a k-dimensional projective space and odd dimensional spheres. We also showed that the rational nilpotency indexes of $\mathrm{map}(\mathbb{KP}^n, \mathbb{KP}^{n+k}; i_{n,k})$ and $\mathbb{KP}^{n+k}$ are equal.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] Alexander Berglund. Rational homotopy theory of mapping spaces via Lie theory for L_∞-algebras. *Homology Homotopy Appl.*, 17(2):343–369, 2015. doi:10.4310/HHA.2015.v17.n2.a16.

[2] Urtzi Buijs, Yves Félix, and Aniceto Murillo. L_∞ rational homotopy of mapping spaces. *Rev. Mat. Complut.*, 26(2):573–588, 2013. doi:10.1007/s13163-012-0105-z.

[3] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001. doi:10.1007/978-1-4613-0105-9.

[4] Yves Félix, John Oprea, and Daniel Tanré. *Algebraic models in geometry*, volume 17 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2008.

[5] J.-B. Gatsinzi. On the genus of elliptic fibrations. *Proc. Amer. Math. Soc.*, 132(2):597–606, 2004. doi:10.1090/S0002-9939-03-07203-4.

[6] J.-B. Gatsinzi. Rational homotopy type of mapping spaces between complex projective spaces and their evaluation subgroups. *Commun. Korean Math. Soc.*, 37(1):259–267, 2022. doi:10.4134/CKMS.c200431.

[7] André Haefliger. Rational homotopy of the space of sections of a nilpotent bundle. *Trans. Amer. Math. Soc.*, 273(2):609–620, 1982. doi:10.2307/1999931.

[8] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995. doi:10.1080/00927879508825335.

[9] Jesper Michael Møller and Martin Raussen. Rational homotopy of spaces of maps into spheres and complex projective spaces. *Trans. Amer. Math. Soc.*, 292(2):721–732, 1985. doi:10.2307/2000242.

[10] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977. URL: http://www.numdam.org/item?id=PMIHES_1977__47__269_0.

[11] René Thom. L'homologie des espaces fonctionnels. In *Colloque de topologie algébrique, Louvain, 1956*, pages 29–39. Georges Thone, Liège; Masson & Cie, Paris, 1957.

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