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Numerical analysis of a 1/2-equation model of turbulence

Wei-Wei Han [a], Rui Fang [b], William Layton [b,*]

[a] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China
[b] Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260, USA

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ABSTRACT

The recent 1/2-equation model of turbulence is a simplification of the standard Kolmogorov–Prandtl 1-equation URANS model. In tests, the 1/2-equation model produced comparable velocity statistics to a full 1-equation model with lower computational complexity. There is little progress in the numerical analysis of URANS models due to the difficulties in treating the coupling between equations and the nonlinearities in highest-order terms. The numerical analysis herein on the 1/2-equation model has independent interest and is also a first numerical analysis step to address the couplings and nonlinearities in a full 1-equation model. This report develops a complete numerical analysis of the 1/2-equation model. Stability, convergence, and error estimates are proven for a semi-discrete and fully discrete approximation. Finally, numerical tests are conducted to validate the predictions of the convergence theory.

1. Introduction

Fluids transport and mix heat, chemical species, and contaminants. Numerical analysis, supporting accurate simulation of fluid velocities, of laminar flows of incompressible, viscous fluids is increasingly understood. For flows at higher Reynolds numbers, simulations based on URANS (Unsteady Reynolds Averaged Navier–Stokes) models are an essential, if not wholly understood or completely reliable, tool for prediction, design, and control. Almost all fundamental issues are unresolved in the numerical analysis of URANS models.

In turbulence modeling, appending an ordinary differential equation (ODE) to determine eddy viscosity parameters in the momentum equation is considered a 1/2-equation model, [1]. An added ODE can be used to increase accuracy without increasing complexity by allowing a parameter to vary coherently. It can also be used to decrease complexity by replacing an appended partial differential equation (PDE) with an ODE representing the aggregate behavior of its solution. A closed ODE in time for the space average of $k(x,t)$, the turbulent kinetic energy, has been obtained in [2]. It was found there that the resulting 1/2-equation model produced, at lower complexity, velocity statistics close to the full 1-equation model. The question then arises as to the accuracy and reliability of numerical simulations of the 1/2-equation model. This report resolves that question, giving a complete stability, convergence, and error analysis for the 1/2-equation URANS model

$$v_t - \nabla\cdot([2\nu + \mu(y)k(t)\tau]\,\nabla^s v) + v\cdot\nabla v + \nabla q = f(x) \text{ and } \nabla\cdot v = 0, \tag{1.1}$$

$$\frac{d}{dt}k(t) + \frac{\sqrt{2}}{2}\tau^{-1}k(t) = \frac{1}{|\Omega|}\int_\Omega \mu(y)k(t)\tau|\nabla^s v(x,t)|^2 dx, \tag{1.2}$$

where $\nabla^s v$ = the symmetric part of the gradient, τ = model time-scale, y = wall normal distance to no-slip boundaries, L = diameter (Ω), $k(t)$ = the space average of the 1-equation approximates turbulent kinetic energy and $\mu(y) = 0.55\,(y/L)^2$.

The challenge in the model's numerical analysis is dealing with the non-monotone nonlinearity in the highest derivative, eddy viscosity terms and with the cubic nonlinearity in the right-hand side (RHS) of the $k(t)$ -equation. These are all features shared by the full 1-equation model. While the numerical analysis of the full 1-equation model remains intractable, the analysis herein may give new ideas for analyzing the model terms beyond the laminar case of the Navier–Stokes equations (NSE). Traditional methods for non-monotone, higher than quadratic nonlinearities are limited to small time or small data. Turbulence develops over long times and for large data. Thus, methods for the numerical analysis of the NSE developed for laminar flows or for small time or small data, are inadequate. The only previous work on numerical analysis of URANS models occurs in important papers [3–6], based on other model simplifications than herein.

* Corresponding author.
E-mail addresses: hanweiwei@stu.xjtu.edu.cn (W.-W. Han), ruf10@pitt.edu (R. Fang), wjl@pitt.edu (W. Layton).

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Error analysis of a model requires estimation of the deviation of the model solution from its discrete approximation. It thus builds on how model uniqueness proofs estimate the deviation of model solutions. In Section 3, Theorem 3.1, we prove the uniqueness of strong solutions to the model (1.1)–(1.2). The main new issue in the proof is dealing with the various nonlinearities and coupling. Section 4 proves stability, convergence, and error estimates for a spatially discrete, continuous time approximation. This proof builds on the analysis of uniqueness in Section 3. Section 5 presents a fully discrete numerical analysis. Section 6 shows the numerical results. Since the model's accuracy was studied in [2], the tests in Section 6 focus on verifying the numerical accuracy predicted in the error analysis in Section 4 and Section 5.

For the 1/2 equation model, impose $v(x,t)=0$ on the domain boundary and the initial condition $v(x,0)=v_0(x)$. The $k(t)$-equation is typically initialized after some start-up period. Thus, we impose the initial condition $k\left(t^{\star}\right)=k_0>0$ for $k(t)$ at some $t^{\star}$ (and consider $k(t)=0$ at earlier times). We impose the usual condition that the model pressure has a mean zero over the flow domain.

1.1. Related work

URANS models approximate time averages

$$v(x,t)\simeq \overline{u}(x,t):=\frac{1}{\tau}\int_{t-\tau}^{t}u(x,t')dt' \tag{1.3}$$

of solutions of the NSE. The most common URANS model begins with an eddy viscosity closure [7,8] of the time-averaged NSE with eddy viscosity given by the Kolmogorov–Prandtl formula $\nu_T=\mu\sqrt{k}l$ with Kolmogorov's choice, $l=\sqrt{2k}\tau$, τ := a time scale (see also [9–11] for recent developments). The variable $k(x,t)$ satisfies an accepted equation modeling the turbulent kinetic energy evolution and $l(x,t)$, the turbulent length scale, has many different specifications of increasing complexity. We make Kolmogorov's choice, $l=\sqrt{2k}\tau$. For the full 1-equation model [12–15], simulations require solving the two coupled nonlinear PDEs

$$v_t-\nabla\cdot([2\nu+\mu k\tau]\nabla^s v)+v\cdot\nabla v+\nabla q=f(x),\ \nabla\cdot v=0,$$
$$k_t-\nabla\cdot(\mu k\tau\nabla k)+v\cdot\nabla k+\frac{\sqrt{2}}{2}\tau^{-1}k=\mu k\tau|\nabla^s v|^2.$$

The model studied herein (1.1)–(1.2) is obtained by space averaging the above $k-$equation. Specifically, defining the space averaging $k(t)=\frac{1}{|\Omega|}\int_\Omega k(x,t)dx$ and replacing $k(x,t)$ in the 1-equation model by $k(t)$ in the eddy viscosity terms converts the full 1-equation model to 1/2-equation model (1.1)–(1.2).

The main challenge herein arises from the nonlinearity in the eddy viscosity term and the RHS of (1.2). To our knowledge, the only previous large data numerical analysis of fluids' models with similar non-monotone, nonlinear eddy viscosity was in [3–5]. Their work studied the models under various simplifying assumptions (different from the space averaging used to simplify herein). The model uniqueness proof in Section 3 uses a regularity assumption to treat the NSE convection term. This necessity reflects the fact that the uniqueness of time averages (1.3) of solutions of the NSE is as little understood as solution uniqueness. It allows our analysis to focus on the coupling and higher degree of nonlinearity introduced by the turbulence model.

Finite time averaging and ensemble averaging are the two most common approaches to URANS modeling. The k-Eq. (1.2) studied herein was derived in [2] by space averaging the standard k-equation derived independently by Prandtl [16] and Kolmogorov [17] (for more details see [12–15,18,19]). It makes use of the following turbulence length scale $l=\sqrt{2k}\tau$, proposed by Kolmogorov [17] and mentioned as an option by Prandtl [16]. The idea of 1/2-equation modeling is from Johnson and King [20], see also Wilcox [1], Section 3.7. The idea is to take a calibration parameter that must be pre-specified and allow it to be determined by local flow conditions through solving an ODE (considered as a '1/2 equation' in turbulence modeling). In the pioneering paper, Johnson and King [20] posed the ODE in the streamwise variable x. In the derivation in [2] and herein, the ODE is formulated in time.

2. Preliminaries

The common Sobolev spaces and Lebesgue spaces on Ω will be denoted by $W^{k,p}(\Omega)$ and $L^p(\Omega)$ respectively [21], equipped with the norms $\|\cdot\|_{k,p}$ and $\|\cdot\|_{L^p}$. As for $p=2$, we adopt $W^k(\Omega)$ equipped with the norm $\|\cdot\|_k$ to replace $W^{k,p}(\Omega)$ for short. With respect to the $L^2(\Omega)$ space, $(\cdot,\cdot)$ and $\|\cdot\|$ will indicate the inner product and norm. Next, we introduce the space $L^p(0,T;X)$ with the following norm

$$\|\cdot\|_{L^p(X)}:=\left(\int_0^T\|\cdot\|_X^p dt\right)^{1/p}\quad \text{if}\quad 1\le p<\infty,$$
$$\|\cdot\|_{L^p(X)}:=\operatorname{ess}\sup_{t\in[0,T]}\|\cdot\|_X\quad\text{if}\quad p=\infty.$$

We define discrete time notations as follows. The time step is denoted by $\Delta t>0$ and $t_n=n\Delta t, n=0,1,\ldots,N=\frac{T}{\Delta t}$. Given a Banach space X, we define the following norms:

$$|\,\|v\|\,|_{l^p(X)}:=\left(\Delta t\sum_{n=0}^{N}\|v^n\|_X^p\right)^{1/p}\quad\text{and}\quad |\,\|v\|\,|_{l^\infty(X)}:=\max_{0\le n\le N}\|v^n\|_X.$$

Define the following spaces W and Q for velocity and pressure, respectively.

$$W:=H_0^1(\Omega)^d=\{u\in H^1(\Omega)^d : u|_{\partial\Omega}=0\},$$
$$Q:=L_0^2(\Omega)=\left\{\varphi\in L^2(\Omega): \int_\Omega\varphi dx=0\right\},$$
$$V:=\{u\in W : (\nabla\cdot u,p)=0,\forall p\in Q\}.$$

We will employ the standard skew-symmetric trilinear form:

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v,w)+\frac{1}{2}((\nabla\cdot u)v,w) \\ &= \frac{1}{2}((u\cdot\nabla)v,w)-\frac{1}{2}((u\cdot\nabla)w,v) \quad \forall u,v,w\in W. \end{aligned} \tag{2.1}$$

For the trilinear form, we have the following bounds [22,23]:

$$b(u,v,w)\le \begin{cases} C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla w\|\|\nabla v\|, \\ C\|\nabla u\|\|\nabla w\|\|\nabla v\|, \\ C\|\nabla u\|\|w\|_2\|v\|. \end{cases} \tag{2.2}$$

The spatial discretization will use the classical finite element method. Suppose Ω_h be a regular mesh of Ω and $\bar{\Omega}=\cup_{M\in\Omega_h}M$. The finite element velocity and pressure spaces W_h and Q_h are:

$$\begin{aligned} W_h &= \{u_h\in W\cap C^0(\Omega)^d : u_h|_M\in P_k(M)^d, \forall M\in\Omega_h\}, \\ Q_h &= \{p_h\in Q\cap C^0(\Omega) : p_h|_M\in P_{k-1}(M), \forall M\in\Omega_h\}, \end{aligned}$$

where $P_k(M)$ denotes the kth order polynomial space on M and $k\ge 2$. We assume a quasi-uniform mesh. There hold [24,25] the following properties for (W_h,Q_h):

$$\begin{aligned} &\inf_{u_h\in W_h}\{\|u-u_h\|+h\|u-u_h\|_1\}\le Ch^{k+1}|u|_{k+1} \quad \forall u\in W\cap H^{k+1}(\Omega)^d, \\ &\inf_{p_h\in Q_h}\|q-p_h\|\le Ch^k|q|_k \quad \forall q\in H^k(\Omega)\cap Q, \end{aligned} \tag{2.3}$$

in which $h=$: maximum triangle diameter in Ω_h. Furthermore, we suppose that W_h and Q_h satisfy the discrete inf–sup condition:

$$\inf_{p_h\in Q_h}\sup_{u_h\in W_h}\frac{(p_h,\nabla\cdot u_h)}{\|p_h\|\|u_h\|_1}\ge\beta_0>0, \tag{2.4}$$

where β_0 is a constant. The discrete divergence-free space is:

$$V_h=\{u_h\in W_h : (\nabla\cdot u_h,p_h)=0, \forall p_h\in Q_h\}. \tag{2.5}$$

Note that the Taylor–Hood element satisfies all the above conditions with $k=2$.

The following discrete Gronwall's inequality from [26] will be used.

Lemma 2.1. *Suppose that G, Δt, and d_n, e_n, a_n, b_n (for integer $n\ge 0$) be nonnegative numbers such that*

$$e_N+\Delta t\sum_{n=0}^{N}d_n\le\Delta t\sum_{n=0}^{N-1}b_ne_n+\Delta t\sum_{n=0}^{N}a_n+G,$$

for $\forall N\ge 1$ and $\forall\Delta t>0$, then

$$e_N+\Delta t\sum_{n=0}^{N}d_n\le\exp\left(\Delta t\sum_{n=0}^{N-1}b_n\right)\left(\Delta t\sum_{n=0}^{N}a_n+G\right).$$

3. Model uniqueness

In this section, we prove the uniqueness of strong solutions of the 1/2-equation model (1.1)–(1.2). A proof of model uniqueness gives insight into critical terms in the model numerical analysis of Section 4 and Section 5. In particular, we assume that the 1/2-equation model has a solution satisfying the classical condition of Ladyzhenskaya

$$\int_0^T\|\nabla v\|^4dt<\infty, \tag{3.1}$$

which is sufficient to prove uniqueness to the NSE, [23], as well as (1.1)–(1.2), Theorem 3.1 below. Next, we use the following lemma from [2].

Lemma 3.1. *With respect to the 1/2-equation model (1.1)–(1.2). Recall that $k(t^\star)>0$. Then $k(t)>0$ for all $t>t^\star$. For strong solutions, there holds the following energy equality*

$$\frac{d}{dt}\left[\frac{1}{|\Omega|}\int_\Omega\frac{1}{2}|v(x,t)|^2dx+k(t)\right]+\frac{1}{|\Omega|}\int_\Omega\nu|\nabla^s v(x,t)|^2dx+\frac{\sqrt{2}}{2}\tau^{-1}k(t)=\frac{1}{|\Omega|}\int_\Omega f\cdot v(x,t)\,dx.$$

The following uniform in T bounds on energy and dissipation rates hold:

$$\begin{aligned} &\frac{1}{|\Omega|}\int_\Omega\frac{1}{2}|v(x,t)|^2dx\le C, \quad \frac{1}{T}\int_0^T\left\{\frac{1}{|\Omega|}\int_\Omega\nu|\nabla^s v(x,t)|^2dx+\frac{\sqrt{2}}{2}\tau^{-1}k(t)\right\}dt\le C, \\ &\frac{1}{T}\int_0^T\left\{\frac{1}{|\Omega|}\int_\Omega\left(\nu+\nu_T\right)|\nabla^s v(x,t)|^2dx\right\}dt\le C, \quad \frac{1}{|\Omega|}\int_\Omega\frac{1}{2}|v(x,t)|^2dx+k(T)\le C, \end{aligned} \tag{3.2}$$

where the constant $C<\infty$ depends on $f, v_0(x), k(t^*), \nu, T$.

We now present the main result of this section.

Theorem 3.1. *Assume that (3.1) holds for a solution of (1.1)–(1.2). Then the solution is unique.*

Proof. Let (v_1, k_1) and (v_2, k_2) be two different solutions of (1.1)–(1.2) with the same data. Set $\phi = v_1 - v_2$ and $e(t) = k_1(t) - k_2(t)$. By subtraction, there holds

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \nu\|\nabla\phi\|^2 + \int_\Omega \phi\cdot\nabla v_1\cdot\phi dx + \mu\tau\int_\Omega \left(k_1\nabla^s v_1 - k_2\nabla^s v_2\right) : \nabla^s\phi dx = 0, \\ \frac{d}{dt}e(t) + \frac{\sqrt{2}}{2}\tau^{-1}e(t) = \varepsilon_1(t) - \varepsilon_2(t), \end{cases} \tag{3.3}$$

where $\varepsilon_i(t) = \frac{1}{2|\Omega|}\int_\Omega \mu\tau k_i|\nabla v_i|^2 dx$ since $2\|\nabla^s v\|^2 = \|\nabla v\|^2$ from $\nabla\cdot v = 0$.

Let the two key model terms be denoted by

$$A := \mu\tau\int_\Omega \left(k_1\nabla^s v_1 - k_2\nabla^s v_2\right) : \nabla^s\phi dx, \quad B := \varepsilon_1(t) - \varepsilon_2(t).$$

By adding and subtracting $\int_\Omega k_1\nabla v_2 : \nabla\phi dx$, we have

$$\begin{aligned} A &= \mu\tau\int_\Omega \left(k_1\nabla^s v_1 - k_1\nabla^s v_2 + k_1\nabla^s v_2 - k_2\nabla^s v_2\right) : \nabla^s\phi dx \\ &= \mu\tau\int_\Omega k_1\|\nabla^s\phi\|^2 dx + \mu\tau\left(k_1 - k_2\right)\int_\Omega \nabla^s v_2 : \nabla^s\phi dx \\ &\geq \mu\tau\frac{1}{2}\int_\Omega k_1\|\nabla\phi\|^2 dx - \frac{\nu}{2}\|\nabla^s\phi\|^2 - \frac{\mu^2\tau^2}{2\nu}\|\nabla^s v_2\|^2\left(k_1 - k_2\right)^2(t) \\ &= \mu\tau\frac{1}{2}\int_\Omega k_1\|\nabla\phi\|^2 dx - \frac{\nu}{4}\|\nabla\phi\|^2 - \frac{\mu^2\tau^2}{4\nu}\|\nabla v_2\|^2(k_1 - k_2)^2(t). \end{aligned}$$

Substituting the above inequality into Eq. (3.3), it yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \frac{3\nu}{4}\|\nabla\phi\|^2 + \int_\Omega \phi\cdot\nabla v_1\cdot\phi dx + \frac{1}{2}\mu\tau\int_\Omega k_1|\nabla\phi|^2 dx \\ &\quad\leq \frac{\mu^2\tau^2}{4\nu}\|\nabla v_2\|^2\left(k_1 - k_2\right)^2(t). \end{aligned} \tag{3.4}$$

Note that the term on the RHS of (3.4) belongs to $L^1(0,T)$ since $k(t) \in L^\infty(0,T)$ and $\|\nabla v\|^2 \in L^1(0,T)$. We now need an equation for $e(t)^2 = \left(k_1 - k_2\right)^2(t)$. We have

$$\frac{1}{2}\frac{d}{dt}e^2(t) + \frac{\sqrt{2}}{2}\tau^{-1}e^2(t) = \left(\varepsilon_1(t) - \varepsilon_2(t)\right)\cdot e(t). \tag{3.5}$$

Adding (3.4) to (3.5) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\phi\|^2 + e^2(t)\right) + \frac{3\nu}{4}\|\nabla\phi\|^2 + \frac{1}{2}\mu\tau\int_\Omega k_1|\nabla\phi|^2 dx + \frac{\sqrt{2}}{2}\tau^{-1}e^2(t) + \int_\Omega \phi\cdot\nabla v_1\cdot\phi dx \\ &\qquad\leq \frac{\mu^2\tau^2}{4\nu}\|\nabla v_2\|^2\left(k_1 - k_2\right)^2(t) + \left(\varepsilon_1(t) - \varepsilon_2(t)\right)\cdot e(t). \end{aligned} \tag{3.6}$$

We now deal with the last term on the RHS of (3.6) in what follows.

$$\begin{aligned} &\left(\varepsilon_1(t) - \varepsilon_2(t)\right)\cdot e(t) = \frac{1}{2|\Omega|}\mu\tau e(t)\int_\Omega \left(k_1\nabla v_1 : \nabla v_1 - k_1\nabla v_1 : \nabla v_2\right) dx \\ &+ \frac{1}{2|\Omega|}\mu\tau e(t)\int_\Omega \left(k_1\nabla v_1 : \nabla v_2 - k_2\nabla v_1 : \nabla v_2 + k_2\nabla v_1 : \nabla v_2 - k_2\nabla v_2 : \nabla v_2\right) dx \\ &= \frac{1}{2|\Omega|}\mu\tau e(t)\int_\Omega \left(k_1\nabla v_1 : \nabla\phi + \left(k_1 - k_2\right)\nabla v_1 : \nabla v_2 + k_2\nabla v_2 : \nabla\phi\right) dx. \end{aligned}$$

Then

$$\begin{aligned} &\left(\varepsilon_1(t) - \varepsilon_2(t)\right)\cdot e(t) \leq \frac{1}{2|\Omega|}e(t)\sqrt{\mu\tau\int_\Omega k_1|\nabla\phi|^2 dx}\sqrt{\mu\tau\int_\Omega k_1|\nabla v_1|^2 dx} \\ &+ \frac{1}{2|\Omega|}\mu\tau e^2(t)\sqrt{\int_\Omega |\nabla v_1|^2 dx}\sqrt{\int_\Omega |\nabla v_2|^2 dx} + \frac{1}{2|\Omega|}\mu\tau e(t)\int_\Omega \left[\left(k_2 - k_1\right) + k_1\right]\nabla v_2 : \nabla\phi dx \\ &\leq \frac{1}{2}\frac{\mu\tau}{2}\int_\Omega k_1|\nabla\phi|^2 dx + \frac{1}{2|\Omega|^2}\left(\frac{\mu\tau}{2}\int_\Omega k_1|\nabla v_1|^2 dx\right)e^2(t) + \frac{1}{2|\Omega|}\mu\tau e^2(t)\|\nabla v_1\|\|\nabla v_2\| \\ &+ \frac{1}{2|\Omega|^2}\left(\mu\tau\|\nabla\phi\|\|\nabla v_2\|\right)e^2(t) + \frac{1}{2}\frac{\mu\tau}{4}\int_\Omega k_1|\nabla\phi|^2 dx + \frac{1}{2|\Omega|^2}\left(\mu\tau\int_\Omega k_1|\nabla v_2|^2 dx\right)e^2(t). \end{aligned} \tag{3.7}$$

Let

$$\begin{aligned} a(t) = &\frac{1}{2|\Omega|}\mu\tau\left(\frac{1}{2|\Omega|}\int_\Omega k_1|\nabla v_1|^2 dx + \|\nabla v_1\|\|\nabla v_2\| + \frac{1}{|\Omega|}\|\nabla\phi\|\|\nabla v_2\| + \frac{1}{|\Omega|}\int_\Omega k_1|\nabla v_2|^2 dx\right) \\ &+ \frac{\mu^2\tau^2}{4\nu}\|\nabla v_2\|^2. \end{aligned}$$

Note that $a(t) \in L^1(0,T)$ by Lemma 3.1.

There remains the standard NSE term which is bounded in a standard way as in [22,23]:

$$\int_\Omega \phi\cdot\nabla v_1\cdot\phi dx \leq C\|\phi\|^{1/2}\|\nabla\phi\|^{3/2}\|\nabla v_1\| \leq \frac{\nu}{4}\|\nabla\phi\|^2 + C\|\nabla v_1\|^4\|\phi\|^2. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), it yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\phi\|^2+e^2(t)\right)+\frac{\nu}{2}\|\nabla\phi\|^2+\frac{\mu\tau}{8}\int_\Omega k_1|\nabla\phi|^2dx+\frac{\sqrt{2}}{2}\tau^{-1}e^2(t) \le \left(C\|\nabla v_1\|^4+a(t)\right)\left(\|\phi\|^2+e^2(t)\right). \tag{3.9}$$

From (3.1) we have $\left(C\|\nabla v_1\|^4+a(t)\right)\in L^1(0,T)$. Uniqueness follows from Gronwall's inequality. □

4. The semi-discrete approximation

The semi-discretization scheme is as follows. Suppose $v_h(x,0)$ is the given approximation of initial condition $v_0(x)$. Find $v_h:[0,T]\to W_h$ and $q_h:[0,T]\to Q_h$ for $\forall w_h\in W_h,\forall p_h\in Q_h$ satisfying

$$\begin{aligned}\left(\frac{\partial v_h}{\partial t},w_h\right)+\left(2\nu+\mu\tau k_h(t)\right)\left(\nabla^s v_h,\nabla^s w_h\right)+b\left(v_h,v_h,w_h\right)-\left(\nabla\cdot w_h,q_h\right)&=\left(f,w_h\right),\\ \left(\nabla\cdot v_h,p_h\right)&=0.\end{aligned} \tag{4.1}$$

As for the k-equation, now we have

$$\frac{dk_h(t)}{dt}+\frac{\sqrt{2}}{2}\tau^{-1}k_h(t)=\frac{1}{2|\Omega|}\int_\Omega \mu\tau k_h(t)|\nabla v_h|^2dx. \tag{4.2}$$

Next, let $e:=v-v_h=\eta-\phi_h$, where $\eta:=v-\tilde{V}$ and $\phi_h:=v_h-\tilde{V}$, $\tilde{V}\in\left\{w_h\in W_h|\left(\nabla\cdot w_h,p_h\right)=0,\forall p_h\in Q_h\right\}$. We begin by presenting the stability of the semi-discrete approximation as follows.

Theorem 4.1. *Under the same assumption as in Lemma 3.1, then the energy equality holds*

$$\frac{d}{dt}\left\{\frac{1}{2|\Omega|}\|v_h(x,t)\|^2+k_h(t)\right\}+\frac{\nu}{2|\Omega|}\|\nabla v_h(x,t)\|^2+\frac{\sqrt{2}}{2}\tau^{-1}k_h(t) =\frac{1}{|\Omega|}\left(f(x,t),v_h(x,t)\right). \tag{4.3}$$

Furthermore, there holds the following uniform in T bounds on the energy and dissipation rate:

$$\begin{aligned}\frac{1}{2}\|v_h(x,T)\|^2&\le C<\infty,\\ \frac{1}{T}\int_0^T\left(\nu+\frac{1}{2}\mu\tau k_h(t)\right)\|\nabla v_h(x,t)\|^2dt&\le C<\infty,\\ \frac{1}{2}\|v_h(x,T)\|^2+|\Omega|k_h(T)&\le C<\infty,\\ \frac{1}{T}\int_0^T\left(\frac{\nu}{2}\|\nabla v_h(x,t)\|^2+\frac{\sqrt{2}}{2}\tau^{-1}|\Omega|k_h(t)\right)dt&\le C<\infty.\end{aligned} \tag{4.4}$$

Proof. Taking $w_h=v_h$ in (4.1) and using the skew-symmetric property of the trilinear term lead to:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|v_h(x,t)\|^2+\left(\nu+\frac{1}{2}\mu\tau k_h(t)\right)\|\nabla v_h(x,t)\|^2&=\left(f(x,t),v_h(x,t)\right)\\ &\le\frac{\nu}{2}\|\nabla v_h(x,t)\|^2+\frac{1}{2\nu}\|f(x,t)\|_{-1}^2.\end{aligned} \tag{4.5}$$

Note that $k_h(t)$ is always nonnegative. Then a differential inequality implies that

$$\begin{aligned}\frac{1}{2}\|v_h(x,T)\|^2&\le C<\infty,\\ \frac{1}{T}\int_0^T\left(\nu+\frac{1}{2}\mu\tau k_h(t)\right)\|\nabla v_h(x,t)\|^2dt&\le C<\infty.\end{aligned} \tag{4.6}$$

Furthermore, taking $w_h=v_h$ in (4.1) and multiplying by $|\Omega|$ on both sides of (4.2), then adding the two equations, it will yield:

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|v_h(x,t)\|^2+|\Omega|k_h(t)\right\}+\nu\|\nabla v_h(x,t)\|^2+\frac{\sqrt{2}}{2}\tau^{-1}|\Omega|k_h(t)\\ &=\left(f(x,t),v_h(x,t)\right)\le\frac{\nu}{2}\|\nabla v_h(x,t)\|^2+\frac{1}{2\nu}\|f(x,t)\|_{-1}^2.\end{aligned} \tag{4.7}$$

Once again, a standard differential inequality leads to

$$\begin{aligned}\frac{1}{2}\|v_h(x,T)\|^2+|\Omega|k_h(T)&\le C<\infty,\\ \frac{1}{T}\int_0^T\left(\frac{\nu}{2}\|\nabla v_h(x,t)\|^2+\frac{\sqrt{2}}{2}\tau^{-1}|\Omega|k_h(t)\right)dt&\le C<\infty.\end{aligned} \tag{4.8}$$

□

To prove an error estimate, the next theorem requires a strong solution, (3.1) and $v\in L^4(0,T;L^2(\Omega)), v_t\in L^2(0,T;H^{-1}(\Omega))$. To extract the convergence rate from the interpolation errors on the RHS requires further smoothness as follows:

$$\begin{aligned}&v\in L^\infty(0,T;W\cap H^{k+1})\cap L^4\left(0,T;H^{k+1}\right);\\ &v_t\in L^2(0,T;H^{k+1});\quad q\in L^2(0,T;Q\cap H^k);\quad k(t)\in L^\infty(0,T).\end{aligned} \tag{4.9}$$

Theorem 4.2. *Assume $(v, q, k(t))$ are solutions of the 1/2-equation model satisfying the regularity (4.9). Then the following error estimate holds*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|v - v_h\|^2 + \sup_{t\in[0,T]} \left(k(t) - k_h(t)\right)^2 + \nu \int_0^T \|\nabla v - \nabla v_h\|^2 dt \\
&\le C \left(\|v_0 - v_h(0)\|^2 + \left(k(0) - k_h(0)\right)^2 \right) \\
&+ C \inf_{w_h\in V_h} \left(\|\nabla(v - w_h)\|^2_{L^4(0,T,L^2)} + \sup_{t\in[0,T]} \|v - w_h\|^2 \right) \\
&+ C \inf_{w_h\in V_h, p_h\in Y_h} \int_0^T \left(\|q - p_h\|^2 + \|(v - w_h)_t\|^2_{-1} + \|\nabla v - \nabla w_h\|^2 \right) dt,
\end{aligned} \tag{4.10}$$

in which C is a positive constant depending on $v_0, \nu, f, T, \int_0^T \|\nabla v\|^4 dt$.

Proof. A variational formulation of strong solution of the 1/2-equation model is

$$\left(v_t, w_h\right) + (2\nu + \mu\tau k(t)) \left(\nabla^s v, \nabla^s w_h\right) + b\left(v, v, w_h\right) - \left(\nabla\cdot w_h, q\right) = \left(f, w_h\right) \quad \forall w_h \in W_h.$$

Subtracting (4.1) from the above and choosing $w_h \in \left\{w_h \in W_h | \left(\nabla\cdot w_h, p_h\right) = 0, \forall p_h \in Q_h\right\}$ yield

$$\begin{aligned}
&\left(\frac{\partial\phi_h}{\partial t}, w_h\right) + 2\nu\left(\nabla^s\phi_h, \nabla^s w_h\right) = \left(\eta_t, w_h\right) + \nu\left(\nabla\eta, \nabla w_h\right) + \mu\tau k(t)\left(\nabla^s v, \nabla^s w_h\right) \\
&- \mu\tau k_h(t)\left(\nabla^s v_h, \nabla^s w_h\right) + b\left(v, v, w_h\right) - b\left(v_h, v_h, w_h\right) - \left(\nabla\cdot w_h, q\right).
\end{aligned}$$

After arranging the above equation and taking $w_h = \phi_h$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \nu\|\nabla\phi_h\|^2 + \frac{1}{2}\mu\tau k(t)\|\nabla\phi_h\|^2 &= \left(\eta_t, \phi_h\right) + \nu\left(\nabla\eta, \nabla\phi_h\right) \\
&- \left(\nabla\cdot\phi_h, q\right) + b\left(\eta, v, \phi_h\right) - b\left(\phi_h, v, \phi_h\right) + b\left(v_h, \eta, \phi_h\right) \\
&+ \mu\tau k(t)\left(\nabla^s\eta, \nabla^s\phi_h\right) + \mu\tau\left(k(t) - k_h(t)\right)\left(\nabla^s v_h, \nabla^s\phi_h\right) = \sum_{i=1}^{8} T_i,
\end{aligned} \tag{4.11}$$

where we subtract and add the term $\mu\tau k(t)\left(\nabla^s v_h, \nabla^s\phi_h\right)$.

Next, we will bound each term of the RHS of (4.11). With the help of the Cauchy Schwarz and Young's inequality, we have

$$T_1 = \left(\eta_t, \phi_h\right) \le \frac{\nu}{14}\|\nabla\phi_h\|^2 + \frac{C}{\nu}\|\eta_t\|^2_{-1}, \tag{4.12}$$

$$T_2 = \nu\left(\nabla\eta, \nabla\phi_h\right) \le \frac{\nu}{14}\|\nabla\phi_h\|^2 + C\nu\|\nabla\eta\|^2, \tag{4.13}$$

$$T_3 = \left(\nabla\cdot\phi_h, q\right) = \left(\nabla\cdot\phi_h, q - p_h\right) \le \frac{\nu}{14}\|\nabla\phi_h\|^2 + C\frac{1}{\nu}\|q - p_h\|^2 \quad \forall p_h \in Q_h. \tag{4.14}$$

As for those trilinear terms, we obtain

$$\begin{aligned}
T_4 = b\left(\eta, v, \phi_h\right) &\le C\|\nabla\eta\|\|\nabla v\|\|\nabla\phi_h\| \\
&\le \frac{\nu}{14}\|\nabla\phi_h\|^2 + \frac{C}{\nu}\|\nabla\eta\|^2\|\nabla v\|^2,
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
T_5 = b\left(\phi_h, v, \phi_h\right) &\le C\|\phi_h\|^{1/2}\|\nabla\phi_h\|^{3/2}\|\nabla v\| \\
&\le \frac{\nu}{14}\|\nabla\phi_h\|^2 + C(\nu)\|\nabla v\|^4\|\phi_h\|^2,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
T_6 = b\left(v_h, \eta, \phi_h\right) &\le C\|v_h\|^{1/2}\|\nabla v_h\|^{1/2}\|\nabla\eta\|\|\nabla\phi_h\| \\
&\le \frac{\nu}{14}\|\nabla\phi_h\|^2 + C\frac{1}{\nu}\|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2.
\end{aligned} \tag{4.17}$$

We bound those nonlinear eddy viscosity terms as follows:

$$\begin{aligned}
&T_7 = \mu\tau k(t)\left(\nabla^s\eta, \nabla^s\phi_h\right) \le \frac{\mu\tau}{8}k(t)\|\nabla\phi_h\|^2 + \frac{1}{2}\mu\tau k(t)\|\nabla\eta\|^2, \\
&T_8 = \mu\tau\left(k(t) - k_h(t)\right)\left(\nabla^s v_h, \nabla^s\phi_h\right) \le \frac{\nu}{14}\|\nabla\phi_h\|^2 + \frac{C\mu^2\tau^2}{\nu}\|\nabla v_h\|^2\left(k(t) - k_h(t)\right)^2.
\end{aligned} \tag{4.18}$$

Substituting (4.12)–(4.18) into (4.11), then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \frac{\nu}{2}\|\nabla\phi_h\|^2 + \frac{3\mu\tau}{8}k(t)\|\nabla\phi_h\|^2 \\
&\quad\le C(\nu)\|\nabla v\|^4\|\phi_h\|^2 + \frac{C}{\nu}\|\eta_t\|^2_{-1} + C(\nu + \mu\tau k(t))\|\nabla\eta\|^2 \\
&\quad+ \frac{C}{\nu}\|q - p_h\|^2 + \frac{C}{\nu}\|\nabla v\|^2\|\nabla\eta\|^2 + \frac{C}{\nu}\|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2 \\
&\quad+ \frac{C\mu^2\tau^2}{\nu}\|\nabla v_h\|^2\left(k(t) - k_h(t)\right)^2.
\end{aligned} \tag{4.19}$$

To deal with the last term in the above equation, we need to introduce the k-equation. Subtracting (4.2) from (1.2) and multiplying by $e_k(t) = \left(k(t) - k_h(t)\right)$, it will obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}e_k(t)^2 + \frac{\sqrt{2}}{2}\tau^{-1}e_k(t)^2 &= \frac{1}{2|\Omega|}\mu\tau e_k(t)\left[(k(t)\nabla v, \nabla v) - \left(k_h(t)\nabla v_h, \nabla v_h\right)\right] \\ &= \frac{1}{2|\Omega|}\mu\tau e_k(t)\left[\left(k(t)\nabla v, \nabla v - \nabla v_h\right) + \left(e(t)\nabla v, \nabla v_h\right) + \left(\nabla v - \nabla v_h, k_h(t)\nabla v_h\right)\right] \\ &= \sum_{i=9}^{11} T_i, \end{aligned} \tag{4.20}$$

where we add and subtract the term $\left(k(t)\nabla v, \nabla v_h\right)$ and $\left(k_h(t)\nabla v, \nabla v_h\right)$.

We will bound the three terms T_i, where $i = 9, 10$ and 11 as follows. As for the first term T_9, we have

$$\begin{aligned} T_9 &= \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(k(t)\nabla v, \nabla v - \nabla v_h\right) = \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(k(t)\nabla v, \nabla\eta - \nabla\phi_h\right) \\ &= \frac{1}{2|\Omega|}\mu\tau e_k(t)(k(t)\nabla v, \nabla\eta) - \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(k(t)\nabla v, \nabla\phi_h\right) \\ &\le \frac{\sqrt{2}}{4}\tau^{-1}e_k(t)^2 + \frac{C}{|\Omega|^2}\tau^3\mu^2 k(t)^2\|\nabla v\|^2\|\nabla\eta\|^2 + \frac{1}{2|\Omega|}\mu\tau k(t)e_k(t)\|\nabla v\|\|\nabla\phi_h\| \\ &\le \frac{\sqrt{2}}{4}\tau^{-1}e_k(t)^2 + \frac{C}{|\Omega|^2}\tau^3\mu^2 k(t)^2\|\nabla v\|^2\|\nabla\eta\|^2 + \frac{\mu\tau k(t)}{8}\|\nabla\phi_h\|^2 + \frac{C}{|\Omega|^2}\mu\tau k(t)\|\nabla v\|^2 e_k(t)^2. \end{aligned} \tag{4.21}$$

Similarly, we have

$$T_{10} = \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(e_k(t)\nabla v, \nabla v_h\right) \le \frac{1}{2|\Omega|}\mu\tau\|\nabla v\|\|\nabla v_h\|e_k(t)^2. \tag{4.22}$$

and

$$\begin{aligned} T_{11} &= \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(\nabla v - \nabla v_h, k_h(t)\nabla v_h\right) = \frac{1}{2|\Omega|}\mu\tau e_k(t)\left(\nabla v - \nabla v_h, \left(k(t) - e_k(t)\right)\nabla v_h\right) \\ &= \frac{1}{2|\Omega|}\mu\tau e_k(t)\left[k(t)\left(\nabla\eta, \nabla v_h\right) - k(t)\left(\nabla\phi_h, \nabla v_h\right) - e_k(t)\left(\nabla\eta, \nabla v_h\right) + e_k(t)\left(\nabla\phi_h, \nabla v_h\right)\right] \\ &\le \frac{1}{2|\Omega|}\mu\tau\|\nabla v_h\|^2 e_k(t)^2 + \frac{1}{2|\Omega|}\mu\tau k(t)^2\|\nabla\eta\|^2 + \frac{\mu\tau k(t)}{8}\|\nabla\phi_h\|^2 \\ &+ \frac{C}{|\Omega|^2}\mu\tau k(t)\|\nabla v_h\|^2 e_k(t)^2 + \frac{1}{2|\Omega|}\mu\tau\left(\|\nabla v\| + \|\nabla v_h\|\right)\|\nabla v_h\|e_k(t)^2. \end{aligned} \tag{4.23}$$

Substituting (4.21)–(4.23) into (4.20) and adding (4.19), then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi_h\|^2 + \frac{1}{2}\frac{d}{dt}e_k(t)^2 + \frac{\nu}{2}\|\nabla\phi_h\|^2 + \frac{\mu\tau}{8}k(t)\|\nabla\phi_h\|^2 + \frac{\sqrt{2}}{4}\tau^{-1}e_k(t)^2 \\ &\le C(\nu)\|\nabla v\|^4\|\phi_h\|^2 + \left(\frac{C\mu^2\tau^2}{\nu}\|\nabla v_h\|^2 + C(|\Omega|)\mu\tau(1 + k(t))\left(\|\nabla v\|^2 + \|\nabla v_h\|^2\right)\right)e_k(t)^2 \\ &+ \frac{C}{\nu}\|\eta_t\|_{-1}^2 + C(\nu + \mu\tau k(t))\|\nabla\eta\|^2 + \frac{C}{\nu}\|q - p_h\|^2 + \frac{C}{\nu}\|\nabla v\|^2\|\nabla\eta\|^2 \\ &+ \frac{C}{\nu}\|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2 + \frac{C}{|\Omega|^2}\tau^3\mu^2 k(t)^2\|\nabla v\|^2\|\nabla\eta\|^2 + \frac{1}{2|\Omega|}\mu\tau k(t)^2\|\nabla\eta\|^2. \end{aligned} \tag{4.24}$$

Let $b(t) = \max\left\{C(\nu)\|\nabla v\|^4, \left(\frac{C\mu^2\tau^2}{\nu}\|\nabla v_h\|^2 + C(|\Omega|)\mu\tau(1 + k(t))\left(\|\nabla v\|^2 + \|\nabla v_h\|^2\right)\right)\right\}$. Since $\|\nabla v\| \in L^4(0, T)$, we know $b(t) \in L^1(0, T)$, then

$$B(t) := \int_0^t b\left(t'\right)dt' < \infty.$$

Multiplying by the integrating factor $e^{-B(t)}$ gives

$$\begin{aligned} &\frac{d}{dt}\left[e^{-B(t)}\left(\|\phi_h\|^2 + e_k(t)^2\right)\right] + e^{-B(t)}\nu\|\nabla\phi_h\|^2 \\ &\le Ce^{-B(t)}\left\{\|\eta_t\|_{-1}^2 + \|q - p_h\|^2 + \left(\nu + \mu\tau k(t) + \|\nabla v\|^2\right)\|\nabla\eta\|^2\right\} \\ &+ Ce^{-B(t)}\left\{\|v_h\|\|\nabla v_h\| + \tau^3\mu^2 k(t)^2\|\nabla v\|^2 + \mu\tau k(t)^2\right\}\|\nabla\eta\|^2. \end{aligned} \tag{4.25}$$

Integrating over the time interval $[0, T]$ and multiplying by $e^{B(t)}$, it yields

$$\begin{aligned} &\|\phi_h(T)\|^2 + e_k(T)^2 + \nu\int_0^T\|\nabla\phi_h\|^2 dt \le C(T, \nu)\|\phi_h(0)\|^2 \\ &+ C(T, \nu, |\Omega|)\int_0^T\left\{\|\eta_t\|_{-1}^2 + \|q - p_h\|^2 + \|\nabla v\|^2\|\nabla\eta\|^2 + \|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2 + \|\nabla\eta\|^2\right\}dt. \end{aligned} \tag{4.26}$$

In particular, when dealing with those terms of the RHS of (4.26), we need to pay attention to the terms containing $\nabla\eta$. For example,

$$\begin{aligned} \int_0^T\|\nabla v\|^2\|\nabla\eta\|^2 dt &\le \left(\int_0^T\|\nabla v\|^4 dt\right)^{1/2}\left(\int_0^T\|\nabla\eta\|^4 dt\right)^{1/2} \\ &\le \|\nabla v\|_{L^4(0,T;L^2)}\|\nabla\eta\|_{L^4(0,T;L^2)}^2. \end{aligned} \tag{4.27}$$

From Theorem 4.1, we have $\|v_h\|$ uniformly bounded in time by data. Thus we can bound the term $\int_0^T \|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2 dt$ as follows:

$$\begin{aligned}\int_0^T \|v_h\|\|\nabla v_h\|\|\nabla\eta\|^2 dt &\le C\int_0^T \|\nabla v_h\|\|\nabla\eta\|^2 dt\\ &\le C\|\nabla v_h\|_{L^2(0,T;L^2)}\|\nabla\eta\|^2_{L^4(0,T;L^2)}.\end{aligned}\tag{4.28}$$

Finally, combining the triangle inequality with (4.26)–(4.28) gives the final result. □

5. The fully discrete approximation

We analyze the following full discretization scheme which employs the backward Euler (BE) method to discrete the time interval. We chose the simple BE time discretization for the analysis so we can focus the analysis on the new terms in the model. Given v_h^n, k_h^n, finding $v_h^{n+1}\in W_h, q_h^{n+1}\in Q_h, k_h^{n+1}$ satisfies for $\forall w_h\in W_h, \forall p_h\in Q_h$

$$\begin{cases}\left(\dfrac{v_h^{n+1}-v_h^n}{\Delta t}, w_h\right) + \left(2\nu+\mu\tau k_h^n\right)\left(\nabla^s v_h^{n+1},\nabla^s w_h\right) + b\left(v_h^n, v_h^{n+1}, w_h\right) - \left(\nabla\cdot w_h, q_h^{n+1}\right)\\ \qquad = \left(f^{n+1}, w_h\right),\\ \left(\nabla\cdot v_h^{n+1}, p_h\right) = 0,\\ \dfrac{k_h^{n+1}-k_h^n}{\Delta t} + \dfrac{\sqrt{2}}{2}\tau^{-1}k_h^{n+1} = \dfrac{1}{2|\Omega|}\displaystyle\int_\Omega \mu\tau k_h^n|\nabla v_h^{n+1}|^2 dx.\end{cases}\tag{5.1}$$

The main challenge of the numerical analysis is dealing with the non-monotone nonlinearity in the eddy viscosity and the RHS of the k-equation. To streamline the analysis, we assume that the k-equation (a linear constant coefficient ODE) is solved exactly. However, it still contains errors since its RHS depends on the highly nonlinear energy dissipation rate and the approximation velocity. Further, $k_h(t)$ is evaluated at t_n not t_{n+1} in ν_T. Based on this assumption, we can obtain

$$\begin{aligned}&\left(\frac{v_h^{n+1}-v_h^n}{\Delta t}, w_h\right) + \left(2\nu+\mu\tau k_h(t_n)\right)\left(\nabla^s v_h^{n+1},\nabla^s w_h\right) + b\left(v_h^n, v_h^{n+1}, w_h\right)\\ &\qquad - \left(\nabla\cdot w_h, q_h^{n+1}\right) = \left(f^{n+1}, w_h\right).\end{aligned}\tag{5.2}$$

Taking $t=t_{n+1}$ in (1.1) yields

$$\begin{aligned}&\left(\frac{v^{n+1}-v^n}{\Delta t}, w_h\right) + \left(2\nu+\mu\tau k(t_{n+1})\right)\left(\nabla^s v^{n+1},\nabla^s w_h\right) + b\left(v^{n+1}, v^{n+1}, w_h\right)\\ &\qquad - \left(\nabla\cdot w_h, q^{n+1}\right) = \left(f^{n+1}, w_h\right) + \left(R_v^{n+1}, w_h\right),\end{aligned}\tag{5.3}$$

where $R_v^{n+1} = -\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\left(t-t_{n+1}\right)v_{tt}dt$.

Before presenting the error estimates, we first show that the numerical scheme (5.2) is unconditionally stable.

Theorem 5.1. *The scheme (5.1) is unconditionally stable:*

$$\begin{aligned}&\|v_h^N\|^2 + 2|\Omega|k_h^N + \sum_{n=0}^{N-1}\|v_h^{n+1}-v_h^n\|^2 + \Delta t\sum_{n=0}^{N-1}\left(\nu\|\nabla v_h^{n+1}\|^2 + \sqrt{2}\tau^{-1}|\Omega|k_h^{n+1}\right)\\ &\qquad \le \|v_h^0\|^2 + 2|\Omega|k_h^0 + \sum_{n=0}^{N-1}C\Delta t\|f^{n+1}\|_{-1}^2.\end{aligned}\tag{5.4}$$

Proof. First, taking $w_h = 2\Delta t v_h^{n+1}$ in the first equation of (5.1), it yields

$$\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|v_h^{n+1}-v_h^n\|^2 + 2\left(\nu+\frac{1}{2}\mu\tau k_h^n\right)\Delta t\|\nabla v_h^{n+1}\|^2 = 2\Delta t\left(f^{n+1}, v_h^{n+1}\right).\tag{5.5}$$

Then multiply both sides of the third equation of (5.1) by $2|\Omega|\Delta t$, we will have

$$2|\Omega|\left(k_h^{n+1}-k_h^n\right) + \sqrt{2}\tau^{-1}|\Omega|\Delta t k_h^{n+1} = \Delta t\mu\tau k_h^n\|\nabla v_h^{n+1}\|^2.\tag{5.6}$$

Adding (5.5) and (5.6) and using the Young's inequality give

$$\begin{aligned}&\|v_h^{n+1}\|^2 - \|v_h^n\|^2 + \|v_h^{n+1}-v_h^n\|^2 + 2|\Omega|\left(k_h^{n+1}-k_h^n\right) + \sqrt{2}\tau^{-1}|\Omega|\Delta t k_h^{n+1}\\ &\qquad + \nu\Delta t\|\nabla v_h^{n+1}\|^2 \le C\Delta t\|f^{n+1}\|_{-1}^2.\end{aligned}\tag{5.7}$$

Summing up from $n=0$ to $N-1$, we get the final result. □

We assume the following regularity:

$$\begin{aligned}v &\in L^\infty(0,T;W\cap H^{k+1})\cap L^4\left(0,T;H^{k+1}\right)\cap l^4(0,T;H^{k+1});\\ v_t &\in L^4(0,T;H^1)\cap L^2(0,T;H^{k+1});\quad v_{tt}\in L^4(0,T;H^1)\\ q &\in L^2(0,T;Q\cap H^k)\cap l^2(0,T;H^k);\quad k(t)\in L^\infty(0,T)\cap W^{1,4}(0,T).\end{aligned}\tag{5.8}$$

Before performing the derivation, we introduce some notations:

$$\begin{aligned}e^{n+1} &:= v^{n+1} - v_h^{n+1} = \eta^{n+1} - \phi_h^{n+1},\ \text{where}\\ \eta^{n+1} &:= \left(v^{n+1}-U_h\right), \phi_h^{n+1} := \left(v_h^{n+1}-U_h\right),\quad \text{and}\quad U_h\in V_h.\end{aligned}$$

Theorem 5.2. *Suppose that the solutions of 1/2-equation model are sufficiently smooth and satisfy the regularity requirements (5.8). Then there exists a positive constant $C(v,q,k,\nu,T)$ such that*

$$\begin{aligned}
&\|v^N - v_h^N\|^2 + \nu\Delta t \sum_{n=0}^{N-1} \|\nabla\left(v^{n+1} - v_h^{n+1}\right)\|^2 \\
&\le C\exp\left(\Delta t\sum_{n=0}^{N-1}\|v^{n+1}\|_2^2\right)\left(\inf_{U_h\in V_h} |\,\|\nabla\left(v-U_h\right)\|\,|_{l^4(L^2)}^2\left(|\,\|\nabla v_h\|\,|_{l^2(L^2)}^2 + |\,\|\nabla v\|\,|_{l^4(L^2)}^2\right)\right.\\
&+ \inf_{U_h\in V_h}\left(|\|\nabla\left(v-U_h\right)\||_{l^2(L^2)}^2 + \|\left(v-U_h\right)_t\|_{L^2(H^{-1})}^2\right)\\
&+ C|\,\|v_h\|\,|_{l^2(L^2)}^2 \inf_{U_h\in V_h, p_h\in Y_h}\left(\|q-p_h\|_{L^2(L^2)}^2 + \|\left(v-U_h\right)_t\|_{L^2(H^{-1})}^2 + \|\nabla v - \nabla U_h\|_{L^2(L^2)}^2\right)\\
&+ C|\,\|v_h\|\,|_{l^2(L^2)}^2 \inf_{U_h\in V_h}\left(\|\nabla\left(v-U_h\right)\|_{L^4(0,T,L^2)}^2 + \sup_{t\in[0,T]}\|v-U_h\|^2\right)\\
&+ C|\,\|v_h\|\,|_{l^2(L^2)}^2\left(\|v_0 - v_h(0)\|^2 + \left(k(0)-k_h(0)\right)^2\right) + \inf_{p_h\in Q_h}|\|q-p_h\||_{l^2(L^2)}^2\\
&\left.+C\Delta t^2\left(|\,\|\nabla v\|\,|_{l^4(L^2)}^4 + \|k_t\|_{L^4(0,T)}^4 + \|\nabla v_t\|_{L^4(L^2)}^4 + \|v_{tt}\|_{L^2(H^{-1})}^2\right)\right).
\end{aligned} \tag{5.9}$$

Proof. The analysis requires bounding several terms common to the NSE and several new terms, the analytical contribution here. The eddy viscosity terms are treated in (5.15) to (5.17). The terms corresponding to the energy dissipation rate errors (the RHS of the k-equation) are in (5.17). Subtracting (5.2) from (5.3) and taking $w_h \in V_h$, it yields

$$\begin{aligned}
&\left(\frac{e^{n+1}-e^n}{\Delta t}, w_h\right) + 2\nu\left(\nabla^s e, \nabla^s w_h\right) + \mu\tau k\left(t_{n+1}\right)\left(\nabla^s v^{n+1}, \nabla^s w_h\right) - \mu\tau k_h\left(t_n\right)\left(\nabla^s v_h^{n+1}, \nabla^s w_h\right)\\
&\qquad + b\left(v^{n+1}, v^{n+1}, w_h\right) - b\left(v_h^n, v_h^{n+1}, w_h\right) - \left(\nabla\cdot w_h, q^{n+1}\right) = \left(R_v^{n+1}, w_h\right).
\end{aligned}$$

By adding and subtracting $\mu\tau k\left(t_n\right)\left(\nabla v^{n+1}, \nabla w_h\right)$, $b\left(v^n, v^{n+1}, w_h\right)$, $b\left(v_h^n, v^{n+1}, w_h\right)$, and taking $w_h = \phi_h^{n+1}$ in the above equation, we have

$$\begin{aligned}
&\left(\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}, \phi_h^{n+1}\right) + \nu\left(\nabla\phi_h^{n+1}, \nabla\phi_h^{n+1}\right) = \left(R_v^{n+1}, \phi_h^{n+1}\right) + \left(\frac{\eta^{n+1}-\eta^n}{\Delta t}, \phi_h^{n+1}\right)\\
&+ 2\nu\left(\nabla^s\eta^{n+1}, \nabla^s\phi_h^{n+1}\right) - \left(\nabla\cdot\phi_h^{n+1}, q^{n+1}\right) + \mu\tau\left(k\left(t_{n+1}\right) - k\left(t_n\right)\right)\left(\nabla^s v^{n+1}, \nabla^s\phi_h^{n+1}\right)\\
&+ \mu\tau k\left(t_n\right)\left(\nabla^s\left(v^{n+1} - v_h^{n+1}\right), \nabla^s\phi_h^{n+1}\right) + \mu\tau\left(k\left(t_n\right) - k_h\left(t_n\right)\right)\left(\nabla^s v_h^{n+1}, \nabla^s\phi_h^{n+1}\right)\\
&+ b\left(v^{n+1} - v^n, v^{n+1}, \phi_h^{n+1}\right) + b\left(\eta^n, v^{n+1}, \phi_h^{n+1}\right)\\
&- b\left(\phi_h^n, v^{n+1}, \phi_h^{n+1}\right) + b\left(v_h^n, \eta^{n+1}, \phi_h^{n+1}\right).
\end{aligned} \tag{5.10}$$

Next, we will bound each term of the RHS of (5.10). As for the first three terms, we bound these terms as follows:

$$\left(R_v^{n+1}, \phi_h^{n+1}\right) \le \|R_v^{n+1}\|_{-1}\|\nabla\phi_h^{n+1}\| \le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\|R_v^{n+1}\|_{-1}^2, \tag{5.11}$$

$$\begin{aligned}
\left(\frac{\eta^{n+1}-\eta^n}{\Delta t}, \phi_h^{n+1}\right) &\le \|\nabla\phi_h^{n+1}\|\|\frac{\eta^{n+1}-\eta^n}{\Delta t}\|_{-1}\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}}\|\eta_t\|_{-1}^2 dt,
\end{aligned} \tag{5.12}$$

$$2\nu\left(\nabla^s\eta^{n+1}, \nabla^s\phi_h^{n+1}\right) \le 2\nu\|\nabla^s\phi_h^{n+1}\|\|\nabla^s\eta^{n+1}\| \le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\|\nabla\eta^{n+1}\|^2, \tag{5.13}$$

$$\begin{aligned}
\left(\nabla\cdot\phi_h^{n+1}, q^{n+1}\right) &= \left(\nabla\cdot\phi_h^{n+1}, q^{n+1} - p_h\right) \le C\|\nabla\phi_h^{n+1}\|\|q^{n+1} - p_h\|\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\|q^{n+1} - p_h\|^2.
\end{aligned} \tag{5.14}$$

Furthermore, we will deal with the three terms originating from the nonlinear eddy viscosity terms, which makes it new. We have

$$\begin{aligned}
&\mu\tau\left(k\left(t_{n+1}\right) - k\left(t_n\right)\right)\left(\nabla^s v^{n+1}, \nabla^s\phi_h^{n+1}\right)\\
&\le \frac{1}{2}\mu\tau\left(k\left(t_{n+1}\right) - k\left(t_n\right)\right)\|\nabla v^{n+1}\|\|\nabla\phi_h^{n+1}\|\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\|\nabla v^{n+1}\|^2\left(k\left(t_{n+1}\right) - k\left(t_n\right)\right)^2\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\Delta t\|\nabla v^{n+1}\|^2\int_{t_n}^{t_{n+1}}\left(k_t\right)^2 dt\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\Delta t^2\|\nabla v^{n+1}\|^4 + C\Delta t\int_{t_n}^{t_{n+1}}\left(k_t\right)^4 dt,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&\mu\tau k\left(t_n\right)\left(\nabla^s\left(v^{n+1} - v_h^{n+1}\right), \nabla^s\phi_h^{n+1}\right)\\
&= \mu\tau k\left(t_n\right)\left(\nabla^s\eta^{n+1}, \nabla^s\phi_h^{n+1}\right) - \mu\tau k\left(t_n\right)\left(\nabla^s\phi_h^{n+1}, \nabla^s\phi_h^{n+1}\right)\\
&\le \frac{\mu\tau k\left(t_n\right)}{4}\|\nabla\phi_h^{n+1}\|^2 + C\mu\tau k\left(t_n\right)\|\nabla\eta^{n+1}\|^2 - \frac{1}{2}\mu\tau k\left(t_n\right)\|\nabla\phi_h^{n+1}\|^2,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&\mu\tau\left(k\left(t_n\right) - k_h\left(t_n\right)\right)\left(\nabla^s v_h^{n+1}, \nabla^s\phi_h^{n+1}\right)\\
&\le \epsilon\nu\|\nabla\phi_h^{n+1}\|^2 + C\mu^2\tau^2\|\nabla v_h^{n+1}\|^2\left(k\left(t_n\right) - k_h\left(t_n\right)\right)^2.
\end{aligned} \tag{5.17}$$

As for those trilinear terms, we can obtain

$$\begin{aligned} b\left(v^{n+1}-v^{n}, v^{n+1}, \phi_{h}^{n+1}\right) &\leq C\|\nabla\left(v^{n+1}-v^{n}\right)\|\|\nabla v^{n+1}\|\|\nabla \phi_{h}^{n+1}\| \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\|\nabla\left(v^{n+1}-v^{n}\right)\|^{2}\|\nabla v^{n+1}\|^{2} \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\Delta t^{2}\|\nabla v^{n+1}\|^{2}\int_{t_{n}}^{t_{n+1}}\|\nabla v_{t}\|^{2}\,dt \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\Delta t^{2}\|\nabla v^{n+1}\|^{4}+C\Delta t\int_{t_{n}}^{t_{n+1}}\|\nabla v_{t}\|^{4}\,dt, \end{aligned} \tag{5.18}$$

$$\begin{aligned} b\left(\eta^{n}, v^{n+1}, \phi_{h}^{n+1}\right) &\leq C\|\nabla \eta^{n}\|\|\nabla v^{n+1}\|\|\nabla \phi_{h}^{n+1}\| \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\|\nabla \eta^{n}\|^{2}\|\nabla v^{n+1}\|^{2}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} b\left(\phi_{h}^{n}, v^{n+1}, \phi_{h}^{n+1}\right) &\leq C\|\phi_{h}^{n}\|\|v^{n+1}\|_{2}\|\nabla \phi_{h}^{n+1}\| \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\|v^{n+1}\|_{2}^{2}\|\phi_{h}^{n}\|^{2}, \end{aligned} \tag{5.20}$$

$$\begin{aligned} b\left(v_{h}^{n}, \eta^{n+1}, \phi_{h}^{n+1}\right) &\leq C\|v_{h}^{n}\|^{1/2}\|\nabla v_{h}^{n}\|^{1/2}\|\nabla \eta^{n+1}\|\|\nabla \phi_{h}^{n+1}\| \\ &\leq \epsilon\nu\|\nabla \phi_{h}^{n+1}\|^{2}+C\|v_{h}^{n}\|\|\nabla v_{h}^{n}\|\|\nabla \eta^{n+1}\|^{2}. \end{aligned} \tag{5.21}$$

Substituting (5.11)–(5.21) into (5.10) and taking $\epsilon=\frac{1}{20}$, then it yields

$$\begin{aligned} &\frac{1}{2\Delta t}\|\phi_{h}^{n+1}\|^{2}-\frac{1}{2\Delta t}\|\phi_{h}^{n}\|^{2}+\frac{1}{2\Delta t}\|\phi_{h}^{n+1}-\phi_{h}^{n}\|^{2}+\frac{\nu}{2}\|\nabla \phi_{h}^{n+1}\|^{2}+\frac{\mu\tau k\left(t_{n}\right)}{2}\|\nabla \phi_{h}^{n+1}\|^{2} \\ &\leq C\|v^{n+1}\|_{2}^{2}\|\phi_{h}^{n}\|^{2}+C\mu^{2}\tau^{2}\|\nabla v_{h}^{n+1}\|^{2}\left(k\left(t_{n}\right)-k_{h}\left(t_{n}\right)\right)^{2} \\ &+C\|R_{v}^{n+1}\|_{-1}^{2}+C\|\nabla \eta^{n+1}\|^{2}+C\|q^{n+1}-p_{h}\|^{2}+C\mu\tau k\left(t_{n}\right)\|\nabla \eta^{n+1}\|^{2} \\ &+C\Delta t^{2}\|\nabla v^{n+1}\|^{4}+C\Delta t\left(\int_{t_{n}}^{t_{n+1}}\left(k_{t}\right)^{4}+\|\nabla v_{t}\|^{4} dt\right) \\ &+C\|\nabla \eta^{n}\|^{2}\nabla v^{n+1}\|^{2}+C\|v_{h}^{n}\|\|\nabla v_{h}^{n}\|\|\nabla \eta^{n+1}\|^{2}+\frac{C}{\Delta t}\int_{t^{n}}^{t^{n+1}}\|\eta_{t}\|_{-1}^{2}dt. \end{aligned} \tag{5.22}$$

Sum from $n=0$ to $N-1$ and multiply $2\Delta t$, we have

$$\begin{aligned} &\|\phi_{h}^{N}\|^{2}-\|\phi_{h}^{0}\|^{2}+\sum_{n=0}^{N-1}\|\phi_{h}^{n+1}-\phi_{h}^{n}\|^{2}+\Delta t\sum_{n=0}^{N-1}\left(\nu\|\nabla \phi_{h}^{n+1}\|^{2}+\mu\tau k\left(t_{n}\right)\|\nabla \phi_{h}^{n+1}\|^{2}\right) \\ &\leq C\Delta t\sum_{n=0}^{N-1}\|v^{n+1}\|_{2}^{2}\|\phi_{h}^{n}\|^{2}+C\Delta t\sum_{n=0}^{N-1}\Big\{\|q^{n+1}-p_{h}\|^{2}+\|\nabla \eta^{n+1}\|^{2} \\ &+\mu\tau k\left(t_{n}\right)\|\nabla \eta^{n+1}\|^{2}+\|\nabla \eta^{n}\|^{2}\|\nabla v^{n+1}\|^{2} \\ &+\|v_{h}^{n}\|\|\nabla v_{h}^{n}\|\|\nabla \eta^{n+1}\|^{2}+\|\nabla v_{h}^{n+1}\|^{2}\left(k\left(t_{n}\right)-k_{h}\left(t_{n}\right)\right)^{2}\Big\} \\ &+C\Delta t^{2}\left(|\,\|\nabla v\|\,|_{l^{4}(L^{2})}^{4}+\|k_{t}\|_{L^{4}(0,T)}^{4}+\|\nabla v_{t}\|_{L^{4}(L^{2})}^{4}+\|v_{tt}\|_{L^{2}(H^{-1})}^{2}\right)+\|\eta_{t}\|_{L^{2}(H^{-1})}^{2}. \end{aligned} \tag{5.23}$$

Further, by using the discrete Gronwall's inequality, then it follows

$$\begin{aligned} &\|\phi_{h}^{N}\|^{2}+\sum_{n=0}^{N-1}\|\phi_{h}^{n+1}-\phi_{h}^{n}\|^{2}+\Delta t\sum_{n=0}^{N-1}\left(\nu\|\nabla \phi_{h}^{n+1}\|^{2}+\mu\tau k\left(t_{n}\right)\|\nabla \phi_{h}^{n+1}\|^{2}\right) \\ &\leq C\exp\left(\Delta t\sum_{n=0}^{N-1}\|v^{n+1}\|_{2}^{2}\right)\Big\{\left(|\|\nabla\eta\||_{l^{2}(L^{2})}^{2}+|\,\|\nabla\eta\|\,|_{l^{4}(L^{2})}^{2}\left(|\,\|\nabla v_{h}\|\,|_{l^{2}(L^{2})}^{2}+|\,\|\nabla v\|\,|_{l^{4}(L^{2})}^{2}\right)\right) \\ &+C|\,\|v_{h}\|\,|_{l^{2}(L^{2})}^{2}\max_{0\leq n\leq N}\left(k\left(t_{n}\right)-k_{h}\left(t_{n}\right)\right)^{2}+|\|q-p_{h}\||_{l^{2}(L^{2})}^{2} \\ &+C\Delta t^{2}\left(|\,\|\nabla v\|\,|_{l^{4}(L^{2})}^{4}+\|k_{t}\|_{L^{4}(0,T)}^{4}+\|\nabla v_{t}\|_{L^{4}(L^{2})}^{4}+\|v_{tt}\|_{L^{2}(H^{-1})}^{2}\right)+\|\eta_{t}\|_{L^{2}(H^{-1})}^{2}+\|\phi_{h}^{0}\|^{2}\Big\}. \end{aligned} \tag{5.24}$$

By using triangle inequality and Theorem 4.2, it can obtain the final result. □

6. Numerical tests

To test the convergence rates of the numerical schemes, we use a test problem from [27], which describes the fluid flow between offset circles. Due to not knowing the analytical solution, we will use the numerical results on the finer mesh as the reference solution to compute the convergence rates. Herein, we define the following computational domain:

$$\Omega=\{(x_1,x_2): x_1^2+x_2^2\leq r_1^2\cap(x_1-c_1)^2+(x_2-c_2)^2\geq r_2^2\},$$

in which $c=(c_1,c_2)=(\frac{1}{2},0), r_1=1, r_2=0.1$. In addition, the counterclockwise force drives the flow $f(x_1,x_2,t)=(4x_1\min(t,1)(1-x_1^2-x_2^2),-4x_2\min(t,1)(1-x_1^2-x_2^2))$. The no-slip boundary conditions will be enforced on both the inner and outer circles. Herein we set $\tau=0.1$, $\mu=0.55$, $\nu=10^{-4}$, $L=1$, $U=1$ and $\mathrm{Re}=\frac{UL}{\nu}$. We simulate the NSE before turning on the 1/2-equation model at $t^*=1$.

Initial and boundary conditions. The following initialization strategy from [11] is used:

$$k(x,1)=\frac{1}{2\tau^2}l^2(x),$$

Table 1
Errors and convergence rates in time.

dt	$\max_{t_n} \|u - u_h\|_{2,0}$	Rate	$\int_0^T \|\nabla u - \nabla u_h\|_{2,0}^2$	Rate
$8e-3$	0.01187	–	23.53	–
$6e-3$	0.00897	0.97	18.50	0.92
$4e-3$	0.00578	1.08	11.95	1.03
$2e-3$	0.00213	1.43	3.43	1.40

Table 2
Errors and convergence rates for velocity in space.

h	$\max_{t_n} \|u_h - u_{3/4 \cdot h}\|_{2,0}$	Rate	$\int_0^T \|\nabla u_h - \nabla u_{3/4 \cdot h}\|_{2,0}^2$	Rate
1/60	0.045145	–	2725.76	–
$(\frac{3}{4})^1 \cdot 1/60$	0.028904	1.55	1922.15	0.61
$(\frac{3}{4})^2 \cdot 1/60$	0.011953	3.07	648.01	1.89
$(\frac{3}{4})^3 \cdot 1/60$	0.006583	2.07	230.90	1.79

in which we choose $l(x) = \min\left\{0.41y, 0.082Re^{-1/2}\right\}$ and y is the wall-normal distance. According to the derivation of the $1/2$-equation model, we choose:

$$k(1) = \frac{1}{|\Omega|}\frac{1}{2\tau^2}\int_\Omega l(x)^2\, dx.$$

The turbulent viscosity ν_T is zero for $t < 1$ and $t \geq 1$ is:

$$\nu_T = \sqrt{2}\mu k(t)(\kappa y/L)^2\tau, \quad \kappa = 0.41.$$

We adopt the BE scheme to discrete the momentum equation in time. The famous Taylor–Hood element ($P2 - P1$) will be used to approximate the pressure and velocity field. We generate the unstructured meshes using GMSH [28].

Order of accuracy in time. We set target mesh size $lc = 1/36$. Choose a very small $dt = 0.001$ to provide an approximation taken to be the true solution. The successive time steps are $dt = 0.002, 0.004, 0.006$, and 0.008. We calculate the rate with the data from $t = 1$ to $t = 1.3$. The rates fluctuate about 1 suggesting these time step values are not yet in the asymptotic regime (see Table 1).

Order of accuracy in space. We look at the ratios of differences between $\tilde{u}_h$ computed for different h. We compare the solutions for the grid sizes, e.g. $h, \alpha h, \alpha^2 h$ gives

$$\frac{\tilde{u}_h - \tilde{u}_{\alpha h}}{\tilde{u}_{\alpha h} - \tilde{u}_{\alpha^2 h}} = \alpha^{-p} + O(h).$$

where p is the order of the method [29]. In our test, we take $\alpha = 3/4$. We set $dt = 0.005$ for all simulations, $h = 1/60, 1/60 \cdot (3/4), /60 \cdot (3/4)^2, 1/60 \cdot (3/4)^3, 1/60 \cdot (3/4)^4$. We calculate the rates with the data from $t = 1$ to $t = 1.5$. The rates in Table 2 fluctuate about 2. Our heuristic idea is the fluctuation in rates is related to solution complexity. From the above two tables, we observe the first order of accuracy in time and on average second order in space, which verifies our theoretical results.

7. Conclusions

Limited computation evidence in [2] indicates that volume averaged statistics predicted by 1-equation URANS models can be well approximated from the $1/2$ equation model. This reduces computational costs provided the coupled system can be reliably and accurately approximated. We show herein that this is possible by giving a complete convergence analysis of a fundamental method and delineating how to treat the eddy viscosity nonlinearity in the numerical analysis.

CRediT authorship contribution statement

Wei-Wei Han: Visualization, Formal analysis. **Rui Fang:** Writing – review & editing, Methodology, Formal analysis, Conceptualization. **William Layton:** Writing – original draft, Supervision, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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