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Almost Sure Stabilization of Markovian Switched Linear Systems with Uncontrollable Subsystems

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Abstract—**This paper develops control design algorithms to achieve almost sure stabilization for Markovian randomly switched linear systems (RSLSs) involving uncontrollable subsystems. Under the conditions of irreducible and aperiodic Markovian switching processes, a controller design method is introduced that utilizes the stationary distribution of the Markov Chain (MC) and stabilizes the overall system almost surely. This proposed method addresses the intricacies arising from uncontrollable subsystems within the context of Markovian RSLSs, and offers a constructive solution for achieving system stability by coordinating subsystem controllers. Almost sure stability of the closed-loop system is established. In addition, a simulation case study on an IEEE 5-Bus system illustrates model development, controller design procedures, and convergence properties.**

Index Terms—**Markov Chain, randomly switched linear system, controller design, almost sure stabilization**

I. Introduction

In recent years, randomly switched linear systems (RSLSs), an important branch of hybrid systems, has garnered considerable interest. This growing attention is primarily prompted by their relevance and impact in representing complex and connected systems in various fields, especially in emerging technologies, including control systems, communication networks, smart grids, intelligent transportation systems, autonomous systems, among many others [1]–[5]. The core characteristic of RSLSs under uncontrollable and unobservable subsystems that distinguishes them from traditional systems in their ability in representing complex systems and diversified scenarios of operation interruptions, attacks, faults, contingencies, control mode switching, system reconfiguration, which often lead to loss of sensors and actuators, and consequently controllability and observability. As a result, control of such RSLSs must coordinate different controllers for subsystems to achieve collectively the common mission of stabilization and optimization. Common examples of switching processes include various operational modes in machinery or devices, fluctuations in the states of communication networks, or changes in environmental conditions. Such variability often leads to unpredictable and rapidly changing system dynamics, posing significant challenges in maintaining system stability and efficiency.

In the context of control systems, RSLSs are particularly relevant as they can represent complex processes with multiple discrete states or modes, each governed by its own set of linear dynamics. The random switching among these modes, often given by a probabilistic rule, can lead to complexities in ensuring system stability and performance. Therefore, it becomes crucial to investigate methods that can ensure the systems overall stability and optimal performance. Not only will research in this area contribute to theoretical advancements in control theory, but also have profound practical implications across various sectors.

The challenges in analyzing and controlling RSLSs stem from their inherent switching behavior. Depending on specific switching patterns and subsystem characteristics, the overall system behavior can exhibit a wide range of dynamic variations. In comparison to traditional linear systems, this complexity makes stability analysis, controller synthesis, and state estimation for RSLSs more complex. Researchers and engineers have pursued to address these complexities, aiming to harness the effectiveness of RSLSs in practical applications. For instance, [6] considers the strong observability and strong detectability of the linear hybrid systems with a periodic jumps restriction. [7] proposes sufficient conditions for robust stability of a class of linear discrete-time switched systems. [8] studies the optimal control problem for a class of linear discrete-time hybrid systems. [9] provides a comprehensive investigation on the stability of stochastic hybrid systems.

This paper aims to develop new control design algorithms for achieving almost sure stabilization for Markovian RSLSs that involve uncontrollable subsystems. Our previous research work explored observability and controllability of RSLSs [10]–[13] under independent and identically distributed (i.i.d.) switching processes. From theoretical and practical viewpoints, Markovian switching processes represent much larger classes of hybrid systems and capture common features of state-dependent conditions in practical systems. In our recent work, we started using Markov Chains (MCs) in RSLSs. Our works [14] and [15] considered observer design for Markovian RSLSs with unobservable subsystems. However, to the best of our knowledge, the control problems for Markovian RSLSs with uncontrollable subsystems have not been studied yet.

Complex systems with many interconnected local dynamics often cannot be controlled by a single input. In such systems, the presence of unforeseen events, such as system failures,

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leads to random variations in controllable subsystem states. In practical scenarios, these stochastic situations are commonly represented using MCs. The main objective of this paper is to explore how characteristics of MCs can be leveraged to design state feedback controllers. This approach aims to tackle the overall control challenge in Markovian RSLSs where subsystems may be uncontrollable.

Our work contributes to the literature on RSLSs in multiple ways.

1) We treat the Markovian RSLSs with uncontrollable subsystems, which extend the i.i.d. switching patterns in [13].
2) We propose an effective controller design algorithm based on the stationary distribution of irreducible and aperiodic MCs and establish almost sure stability of the closed-loop systems.
3) We further demonstrate the capability of the coordinated controller from subsystem controllers for achieving almost sure stabilization of triangular structured network systems.
4) To demonstrate the theoretical results, a practical power system is used in our case study. Although [16] is an application of RSLSs in power systems, this is the first time that a Markovian RSLS has been applied to power systems.

The subsequent sections of this paper are delineated as follows. In Section II, the notation and the problem statement are presented. In Section III, state feedback design algorithms are proposed for RSLSs with uncontrollable subsystems. Section IV is focused on almost sure stabilization properties of the designed controller for RSLSs. In Section V, IEEE 5-Bus system is utilized to exemplify the development of RSLSs in practical systems. Finally, Section VI concludes the paper.

II. Preliminaries

A. Notation

The following standard notations are used in this paper, similar to [10], [13]. The Euclidean norm on a column vector $v \in \mathbb{R}^n$ is $\|v\|$. For a matrix $M \in \mathbb{R}^{n\times m}$, denote M' as its transpose, $\|M\| = \sup_{\|v\|=1} \|Mv\|$ as its operator norm, $\text{Range}(M) = \{y = Mx : x \in \mathbb{R}^m\}$ as its range. A base $M \in \mathbb{R}^{n\times m}$ of a subspace $\mathbb{U} \subseteq \mathbb{R}^n$ of dimension m, written as $M = \text{Base}(\mathbb{U})$, is a matrix whose column vectors are linearly independent, and $\text{Range}(M) = \mathbb{U}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a discrete-time stochastic process $s = \{s_k, k = 0, 1, 2, \ldots\}$ on the probability space, $\mathcal{F}_k$, $k = 0, 1, 2, \ldots$, with $\mathcal{F}_{k-1} \subseteq \mathcal{F}_k$ being the filtration, and $\mathbb{E}_\infty(\cdot)$ the expectation over $\mathcal{F}_\infty = \cup_{k\geq 0}\mathcal{F}_k$. For a subset $\mathcal{S}_0 \subseteq \mathcal{S} = \{1, \ldots, m\}$, the indicator function of $\mathcal{S}_0$ is $\mathbf{1}_{q\in\mathcal{S}_0} = 1$ if $q \in \mathcal{S}_0$; and $\mathbf{1}_{q\in\mathcal{S}_0} = 0$ otherwise. $\mathbb{P}\{\cdot\}$ is the probability.

This paper treats

$$\dot{x}(t) = A(\alpha(t))x(t) + B(\alpha(t))u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^r$ is the control input, and $\alpha(t) \in \mathcal{S} = \{1, \ldots, m\}$ is a finite-state MC.

This paper is motivated by real-world systems in which physical plans are continuous-time systems but data acquisition (sampling), exchange (communication time division), switching (fault detection and protection), and control (digital control) must be done in the discrete-time domain. These are reflected in the following assumptions.

(A1) For a given time interval τ,

(a) the switching process $\alpha(t)$ is piece-wise constant, whose switching may occur only at the instants $k\tau$, $k = 0, 1, 2, \ldots$, generating the skeleton sequence $\{\alpha_k = \alpha(k\tau)\}$.

(b) The MC $\{\alpha_k\}$ is irreducible and aperiodic with probability transition matrix $P = [p_{ij}] \in \mathbb{R}^{m\times m}$, with $p_{ij} = \mathbb{P}\{\alpha_{k+1} = j|\alpha_k = i\} \geq 0$ and $\sum_{j=1}^m p_{ij} = 1$, $i \in \mathcal{S}$. The initial distribution of α_0 is $p_0 = [p_0^1, \ldots, p_0^m]$.

(c) The MC α_k can be observed after its occurrence.

Since α_k is irreducible and aperiodic, its stationary distribution $p_\infty = [p_\infty^1, \ldots, p_\infty^m]$ exists and satisfies $p_\infty P = p_\infty$ and $p_\infty \mathbf{1} = 1$ where $\mathbf{1} = [1, \ldots, 1]'$.

As functions of α_k, $A(\cdot) \in \mathbb{R}^{n\times n}$ and $B(\cdot) \in \mathbb{R}^{n\times r}$ are stochastic and for each $i \in \mathcal{S}$, the corresponding system in (1) with $(A(i), B(i))$ is called the *ith subsystem of the RSLS*. Denote the stochastic matrix sequences

$$A(\alpha_k) = \sum_{i=1}^m A(i)\mathbf{1}_{\{\alpha_k=i\}},$$
$$B(\alpha_k) = \sum_{i=1}^m B(i)\mathbf{1}_{\{\alpha_k=i\}}.$$

The controllability matrix for the ith subsystem is

$$W(i) = [B(i), A(i)B(i), \ldots, (A(i))^{n-1}B(i)] \in \mathbb{R}^{n\times nr}.$$

The combined controllability matrix for $\mathcal{S}$ is

$$W_\mathcal{S} = [W(1), W(2), \ldots, W(m)] \in \mathbb{R}^{n\times mnr}.$$

$W(i)$ and $W_\mathcal{S}$ are constant matrices.

(A2) (a) Subsystems may be uncontrollable, namely, $\text{Rank}(W(i)) = n_i \leq n$, $i \in \mathcal{S}$. (b) $W_\mathcal{S}$ is full row rank.

III. Design Procedures

We first summarize briefly a state feedback design method from [13]. The main steps are: (a) For the ith subsystem, we extract the controllable sub-state $\widetilde{x}_1^i$ by using the Kalman decomposition. (b) The ith controller is a feedback gain L_i on the controllable sub-state $\widetilde{x}_1^i$ when $\alpha_k = i$. (c) The design of L_i utilizes both the system dynamics and switching probabilities. This creates m controllers. (d) During implementation, these controllers are used according to the occurrence of α_k. These steps are detailed next.

Extraction of Controllable Sub-States

Subspace decomposition on each subsystem is based on the Kalman decomposition, see [13]. If the ith subsystem is not controllable[1], then the controllability matrix $W(i)$ has its rank $\text{Rank}(W(i)) = n_i < n$. A non-singular matrix $T_i \in \mathbb{R}^{n\times n}$

[1]This step is needed only if the subsystem is uncontrollable.

can be found to transform the coordinates and obtain the new state variable

$$\widetilde{x}^i = T_i^{-1} x = \begin{bmatrix} \widetilde{x}_1^i \\ \widetilde{x}_2^i \end{bmatrix}.$$

Here $\widetilde{x}_1^i \in \mathbb{R}^{n_i}$ has dynamics $\dot{\widetilde{x}}^i = \widetilde{A}^i \widetilde{x}^i + \widetilde{B}^i u$, where $\widetilde{A}^i = T_i^{-1} A(i) T_i = \begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & A_{22}^i \end{bmatrix}$, $\widetilde{B}^i = T_i^{-1} B(i) = \begin{bmatrix} B_1^i \\ 0 \end{bmatrix}$. Now (A_{11}^i, B_1^i) (with lower dimensions) is controllable.

The matrix T_i has the structure $T_i = [M_i, N_i]$, where $M_i = \mathrm{Base}(\mathrm{Range}(W(i))) \in \mathbb{R}^{n \times n_i}$ and N_i can be any compatible matrix that makes T_i non-singular. It is noted that a simple construction for $r = 1$ is to select M_i as $M_i = [B(i), A(i)B(i), \ldots, (A(i))^{n_i - 1} B(i)]$. The resulting A_{11}^i, B_1^i will be in a controllable canonical form. Decompose $T_i^{-1} = \begin{bmatrix} G_i \\ F_i \end{bmatrix}$. It follows that $\widetilde{x}_1^i = G_i x \in \mathbb{R}^{n_i}$. Collect all G_i in a matrix

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}.$$

Under assumption (A2), G is full rank. As a result, $\Phi = (G'G)^{-1} G'$ exists. This allows us to map all controllable sub-states back to x with a one-to-one mapping $\widetilde{x}_1 = \begin{bmatrix} \widetilde{x}_1^1 \\ \vdots \\ \widetilde{x}_1^m \end{bmatrix} = Gx$ and $x = \Phi \widetilde{x}_1$. Consequently, we may simply concentrate on the convergence of controllable sub-states.

Feedback Gains for Each Subsystem

When $\alpha_k = i$, $\widetilde{x}_2^i$ is internal and run open-loop since it has no control involved. The total dynamics for the ith subsystem are

$$\begin{cases} \dot{\widetilde{x}}_1^i &= A_{11}^i \widetilde{x}_1^i + A_{12}^i \widetilde{x}_2^i + B_1^i u, \\ \dot{\widetilde{x}}_2^i &= A_{22}^i \widetilde{x}_2^i, \end{cases}$$

for $t \in [k\tau, (k+1)\tau)$.

Although $\widetilde{x}_2^i$ runs independently, it affects $\widetilde{x}_1^i$ via $A_{12}^i \widetilde{x}_2^i$. This interaction may result in unstable closed-loop systems. One remedy is to add a decoupling control action to eliminate this interaction at sampling points. This action can be achieved from the solution of $\widetilde{x}_2^i$ in the interval $t \in [k\tau, (k+1)\tau)$

$$\widetilde{x}_2^i(t) = e^{A_{22}^i (t - k\tau)} \widetilde{x}_2^i(k\tau).$$

Substituting it into $\dot{\widetilde{x}}_1^i = A_{11}^i \widetilde{x}_1^i + A_{12}^i \widetilde{x}_2^i + B_1^i u$, then we have the solution of $\widetilde{x}_1^i$,

$$\widetilde{x}_1^i(t) = e^{A_{11}^i (t - k\tau)} \widetilde{x}_1^i(k\tau) + v_1(t) + \int_{k\tau}^{t} e^{A_{11}^i (t - \theta)} B_1^i u(\theta) d\theta.$$

The signal $v_1(t)$ is

$$v_1(t) = \int_{k\tau}^{t} e^{A_{11}^i (t - \theta)} A_{12}^i e^{A_{22}^i (\theta - k\tau)} d\theta \ \widetilde{x}_2^i(k\tau).$$

At the sampling points, $v_1((k+1)\tau)$ is

$$\begin{aligned} & v_1((k+1)\tau) \\ &= \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau - \theta)} A_{12}^i e^{A_{22}^i (\theta - k\tau)} d\theta \ \widetilde{x}_2^i(k\tau) \\ &= \int_0^{\tau} e^{A_{11}^i (\tau - \theta)} A_{12}^i e^{A_{22}^i \theta} d\theta \ \widetilde{x}_2^i(k\tau). \end{aligned}$$

Now, we may add a decoupling input $\widetilde{u}(t)$ first, and design a state feedback, leading to

$$u(t) = \widetilde{u}(t) - L^i \widetilde{x}_1^i(t), \ t \in [k\tau, (k+1)\tau), \tag{2}$$

where $\widetilde{u}(t) = -(B_1^i)' e^{(A_{11}^i)'((k+1)\tau - t)} (\Xi^i)^{-1} v_1((k+1)\tau)$ and Ξ^i is the controllability Gramian

$$\begin{aligned} \Xi^i &= \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau - \theta)} B_1^i (B_1^i)' e^{(A_{11}^i)'((k+1)\tau - \theta)} d\theta \\ &= \int_0^{\tau} e^{A_{11}^i \theta} B_1^i (B_1^i)' e^{(A_{11}^i)' \theta} d\theta. \end{aligned}$$

Since (A_{11}^i, B_1^i) is controllable, Ξ^i (the controllability Gramian) is full rank for any $\tau > 0$.

Consequently, at $t = (k+1)\tau$,

$$\begin{aligned} \widetilde{x}_1^i((k+1)\tau)) = {} & e^{A_{11}^i \tau} \widetilde{x}_1^i(k\tau) + v_1((k+1)\tau) \\ & + \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau - \theta)} B_1^i \widetilde{u}(\theta) d\theta \\ & - \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau - \theta)} B_1^i L^i \widetilde{x}_1^i(\theta) d\theta. \end{aligned}$$

It can be shown that this added control generates a decoupled system

$$\begin{aligned} & \widetilde{x}_1^i((k+1)\tau)) \\ &= e^{A_{11}^i \tau} \widetilde{x}_1^i(k\tau) - \int_{k\tau}^{(k+1)\tau} e^{A_{11}^i ((k+1)\tau - \theta)} B_1^i L^i \widetilde{x}_1^i(\theta) d\theta. \end{aligned}$$

The closed-loop system has $A_c^i = A_{11}^i - B_1^i L^i$, and $\dot{\widetilde{x}}_1^i = A_c^i \widetilde{x}_1^i$. At $t = (k+1)\tau$, its solution has value $\widetilde{x}_1^i((k+1)\tau)) = e^{A_c^i \tau} \widetilde{x}_1^i(k\tau)$. Since (A_{11}^i, B_1^i) is controllable, the poles of A_c^i can be placed arbitrarily and so is its norm. Consequently, the following conclusion is true.

Lemma 3.1: [13] For any selected $0 < \gamma_c < 1$, a feedback gain L_i can be designed such that $\gamma_c^i = \|e^{A_c^i \tau}\| \leq \gamma_c$.

It is noted that v_1 is a function of $\widetilde{x}_2^i(k\tau)$. After we prove the convergence of x (almost surely) in the next section, it will become clear that v_1 is also convergent almost surely.

IV. Almost Sure Stability of Closed-Loop Systems

Since [13] assumes that α_k, its conclusions and proofs cannot be used to establish almost sure stability under Markov chains. One structural property is needed. To explain, we note that when $\alpha_k \neq i$, $\widetilde{x}_1^i$ may not be controllable. It implies that it may grow by itself. In this scenario, certain interactions with other subsystems may lead to instability, regardless how feedback gains are designed. For this reason, we impose the following structural condition.

(A3) When $\alpha_k = j \neq i$, for $t \in [k\tau, (k+1)\tau)$,

$$\dot{\widetilde{x}}_1^i(t) = A_j^i \widetilde{x}_1^i(t) + \sum_{\rho=1}^{i-1} A_{j,\rho}^i \widetilde{x}_1^\rho(t), \quad i \in \mathcal{S}.$$

for some matrices A_j^i and $A_{j,\rho}^i$ that depend on the current discrete state $\alpha_k = j$.

This assumption defines a triangular interaction structure: when $\alpha_k = j \neq i$, $\widetilde{x}_1^i$ runs open-loop and its dynamics are affected by other subsystems (ρth subsystems) only for $\rho < i$. This means that for $\alpha_k = j \neq i$,

$$\widetilde{x}_1^i((k+1)\tau) = e^{A_j^i \tau} \widetilde{x}_1^i(k\tau) + \textstyle\sum_{\rho=1}^{i-1} H_{j,\rho}^i(\tau) \widetilde{x}_1^\rho(k\tau),$$

where the matrices $H_{j,\rho}^i(\tau)$ are functions of $A_{j,\rho}^i$.

Now, by combining the above cases, the sampled values $\widetilde{x}_1^i(k\tau)$ becomes

$$\widetilde{x}_1^i((k+1)\tau) = \Gamma_{ii} \widetilde{x}_1^i(k\tau) + \sum_{\rho=1}^{i-1} \Gamma_{i\rho} \widetilde{x}_1^\rho(k\tau), \tag{3}$$

where

$$\Gamma_{ii} = \mathbf{1}_{\{\alpha_k=i\}} e^{A_c^i \tau} + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} e^{A_j^i \tau},$$

$$\Gamma_{i\rho} = \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} H_{j,\rho}^i(\tau), \quad i, j, \rho \in \mathcal{S}, \ j \neq i, \ \rho < i.$$

This implies that

$$\widetilde{x}_1((k+1)\tau) = \begin{bmatrix} \Gamma_{11} & 0 & \cdots & 0 \\ \Gamma_{21} & \Gamma_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \Gamma_{m1} & \Gamma_{m2} & \cdots & \Gamma_{mm} \end{bmatrix} \widetilde{x}_1(k\tau).$$

For stability analysis, we will concentrate on stability of each sub-system. Denote

$$\gamma_c^i = \|e^{A_c^i \tau}\|, \ \gamma_j^i = \|e^{A_j^i \tau}\|, \ \gamma_o^i = \max_{j\neq i}\{\gamma_j^i\},$$

$$h_{j,\rho}^i = \|H_{j,\rho}^i(\tau)\|, \ h^i = \max_{j\neq i, \rho<i}\{h_{j,\rho}^i\}, \ i, j, \rho \in \mathcal{S}, j \neq i.$$

Also, let $\mu_k^i = \|\widetilde{x}_1^i(k\tau)\|$. From (3), for all $i \in \mathcal{S}$,

$$\begin{aligned} \mu_{k+1}^i &= \Big\|\Gamma_{ii}\mu_k^i + \sum_{\rho=1}^{i-1} \Gamma_{i\rho}\mu_k^\rho\Big\| \\ &\leq \Big(\mathbf{1}_{\{\alpha_k=i\}} \|e^{A_c^i \tau}\| + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} \|e^{A_j^i \tau}\|\Big) \|\mu_k^i\| \\ &\quad + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} \sum_{\rho=1}^{i-1} \|H_{j,\rho}^i(\tau)\| \|\mu_k^\rho\| \\ &\leq \Big(\mathbf{1}_{\{\alpha_k=i\}} \gamma_c^i + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} \gamma_o^i\Big) \mu_k^i + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} h^i \sum_{\rho=1}^{i-1} \mu_k^\rho. \end{aligned}$$

It follows that

$$\mu_{k+1}^i \leq \hat{\gamma}^i(\alpha_k)\mu_k^i + \widetilde{\gamma}^i(\alpha_k) \sum_{\rho=1}^{i-1} \mu_k^\rho, \tag{4}$$

where $\hat{\gamma}^i(\alpha_k) = \mathbf{1}_{\{\alpha_k=i\}}\gamma_c^i + \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}}\gamma_o^i$ and $\widetilde{\gamma}^i(\alpha_k) = \sum_{j\neq i} \mathbf{1}_{\{\alpha_k=j\}} h^i$ for any i.

Our design method uses the stationary distribution p_∞ of the MC, rather than the matrix P itself. By Lemma 3.1, we can always achieve the following design criterion.

Lemma 4.1: Under condition (A1), by choosing the eigenvalues of A_c^i suitably, for any $0 < \gamma_* < 1$, there exists a feedback gain L^i such that

$$(\gamma_c^i)^{p_\infty^i} (\gamma_o^i)^{q_\infty^i} \leq \gamma_* < 1.$$

We are now ready to establish almost sure stability. We first recall that a stochastic sequence $\{s_k\}$ is said to converge to 0 almost surely with exponential rate if its Lyapunov exponent converges with

$$\lim_{k\to\infty} \frac{1}{k} \ln \|s_k\| = \widetilde{d} < 0, \quad \text{almost surely.}$$

Define the continuous-time state norm $\mu^i(t) = \|\widetilde{x}_1^i(t)\|$, which is a scalar stochastic process. The following lemma is a special case of Theorem 17.0.1 of [17] and Theorem 1 of [18], and will be used in convergence analysis. It is a statement of SLLN for MCs.

Lemma 4.2: Suppose that $g_k = g(\alpha_k)$ is a sequence taking values in $\{g(1), \ldots, g(m)\}$. Under condition (A1),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} g_k = \mathbb{E}_\infty(g_k) = \sum_{i=1}^{m} p_\infty^i g(i), \ \text{almost surely.}$$

The following theorem shows that under the designed controller, the system state converges to 0 almost surely.

Theorem 4.1: Under assumptions (A1), (A2), and (A3), and the controller design in Lemma 4.1, (a) μ_k^i converges to 0 almost surely and exponentially, as $k \to \infty$; (b) $\|x(t)\|$ converges to 0 almost surely and exponentially, as $k \to \infty$.

Proof:

(a) We prove this theorem by induction on i. For $i = 1$, by (4),

$$\mu_{k+1}^1 \leq \hat{\gamma}^1(\alpha_k)\mu_k^1 \leq \prod_{\ell=0}^{k-1} \hat{\gamma}^1(\alpha_\ell)\mu_0^1.$$

By Lemma 4.1 and Lemma 4.2, as $k \to \infty$,

$$\begin{aligned} \tfrac{1}{k}\ln \mu_k^1 &\leq \frac{1}{k}\Big(\sum_{\ell=0}^{k-1} \ln \hat{\gamma}^1(\alpha_\ell) + \ln \mu_0^1\Big) \\ &\to \mathbb{E}_\infty(\ln \hat{\gamma}^1(\alpha_\ell)), \quad \text{almost surely} \\ &\leq p_\infty^1 \ln \gamma_c^1 + q_\infty^1 \ln \gamma_o^1 \\ &= \ln(\gamma_c^1)^{p_\infty^1} (\gamma_o^1)^{q_\infty^1} \\ &\leq \ln \gamma_* < 0. \end{aligned}$$

It follows that

$$\mu_k^1 \leq e^{k \ln \gamma_*} \leq (e^{\ln \gamma_*})^k \to 0, \ \text{as } k \to \infty.$$

As a result, μ_k^1 converges to 0 almost surely and exponentially.

Suppose that for $i = 1, \dots, s$, μ_k^i converges. Then for $i = s+1$,

$$\begin{aligned}\mu_{k+1}^{s+1} \leq & \hat{\gamma}^{s+1}(\alpha_k)\mu_k^{s+1} + \widetilde{\gamma}^{s+1}\sum_{\rho=1}^{s}\mu_k^\rho \\ \leq & \hat{\gamma}^{s+1}(\alpha_k)\mu_k^{s+1} + \mathbf{1}_{\{\alpha_k \neq s+1\}} h^{s+1} \max_{\rho=1,\cdots,s}\{\mu_k^\rho\}. \quad (5)\end{aligned}$$

for some $h^{s+1} \geq 0$. When $h^{s+1} = 0$, by Lemma 4.1, similar to the proof of convergence of μ_k^1, L^{s+1} can be designed such that μ_k^{s+1} converges.

Furthermore, when $h^{s+1} > 0$, in (5), μ_k^ρ, $\rho = 1, \dots, s$, act as exponentially decaying inputs. Therefore, as the response of an exponentially stable system to such input, μ_k^{s+1} converges. Hence, for all $i \in \mathcal{S}$, μ_k^i converges.

(b) For $t \in [k\tau, (k+1)\tau)$ with finite length τ, we have $\sup_{t\in[k\tau,(k+1)\tau]} \mu^i(t) \leq c\mu_k^i$ for some constant $c > 0$. This, together with the almost sure convergence and exponential convergence rate of μ_k^i, implies that $\mu^i(t)$ converges to 0 almost surely and exponentially. Then, by $x = \Phi\widetilde{x}_1$. the convergence of controllable sub-states from all subsystems implies the convergence of x. Therefore, (b) is proved. □

V. Case Study

We now use a practical power system to illustrate the theoretical results. This is the first time that Markovian switching RSLSs have been applied to power systems.

Example 5.1: The IEEE 5-Bus system [19] in Fig. 1 is a well-established power grid in power system analysis. In this grid, there are two generators (Buses 1-2) and three loads (Buses 3-5).

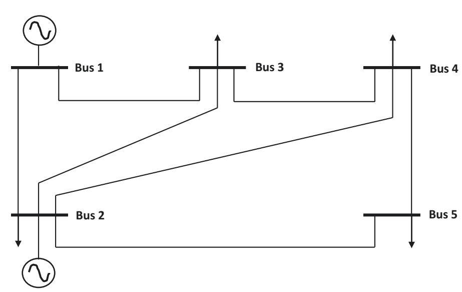

Fig. 1. IEEE 5-Bus System.

The generators are dynamic systems. Their state variables are $\omega_1 = \dot{\delta}_1, z_1^d = [\delta_1, \omega_1], \omega_2 = \dot{\delta}_2, z_2^d = [\delta_2, \omega_2]$. and their models are swing equations

$$\begin{aligned}M_1\dot{\omega}_1 + g_1(\omega_1) =& P_1^{in} - P_1^L + P_1^{21} + P_1^{31}, \\ M_2\dot{\omega}_2 + g_2(\omega_2) =& P_2^{in} - P_2^L + P_2^{12} + P_2^{32} + P_2^{42} + P_2^{52}.\end{aligned}$$

The real power flows in these equations are

$$P_j^{ij} = \frac{V_j^2}{X_{ij}}\cos(\theta_{ij}) - \frac{V_iV_j}{X_{ij}}\cos(\theta_{ij} + \delta_{ij}),$$

and $\delta_{ij} = \delta_i - \delta_j$; the friction part $g_i(w_i)$ has the linear part $b_i\omega_1$ with $b_i > 0$, $i = 1, 2$. The load buses have real-power flows

$$\begin{aligned}P_3^L &= P_3^{13} + P_3^{23} + P_3^{43}, \\ P_4^L &= P_4^{24} + P_4^{34} + P_4^{54}, \\ P_5^L &= P_5^{25} + P_5^{45}.\end{aligned}$$

Define $z^d = [z_1^d, z_2^d], z^{nd} = [\delta_3, \delta_4, \delta_5]$. The total model for the 5-Bus system is $\dot{z}^d = F(z^d, \ell^{nd}) + B_1 v + D_1\ell^d$, where $v = \left[P_1^{in}, P_2^{in}\right], \ell^d = \left[P_1^L, P_2^L\right], \ell^{nd} = \left[P_3^L, P_4^L, P_5^L\right]$, and

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1/M_1 & 0 \\ 0 & 0 \\ 0 & 1/M_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ -1/M_1 & 0 \\ 0 & 0 \\ 0 & -1/M_2 \end{bmatrix}.$$

For stability analysis around the nominal operating condition, we consider the perturbations $x = z^d - \overline{z}^d, u = v - \overline{v}, \zeta = \ell^d - \overline{\ell}^d, \zeta^n = \ell^{nd} - \overline{\ell}^{nd}$.

In our analysis and design, we perform the common small-signal linearization, see [16] for details. Define $x = [\delta_1, \omega_1, \delta_2, \omega_2]$. The linearized system is

$$\dot{x} = Ax + B_1 u + D_1\zeta + D_2\zeta^n,$$

where the Jacobian matrices

$$A = \frac{\partial F(z^d, \ell^{nd})}{\partial z^d}\Big|_{z^d=\overline{z}^d, \ell^{nd}=\overline{\ell}^{nd}},$$

$$D_2 = \frac{\partial F(z^d, \ell^{nd})}{\partial \ell^{nd}}\Big|_{z^d=\overline{z}^d, \ell^{nd}=\overline{\ell}^{nd}}.$$

The IEEE 5-Bus system has the parameters showed in Table I with real power P (MW) and reactive power Q (MVar). The base MVA is $S_B = 100$ MVA and the base voltage is $V_B = 230$ kV (see [19], [20]).

TABLE I
IEEE 5-Bus System Bus Data

Bus	V (p.u. $\angle$ rad)	P	Q	P_L	Q_L
1	$1.06\angle 0$	129	-7.42	0	0
2	$1.0474\angle -2.8063$	40	30	20	10
3	$1.0242\angle -4.997$	0	0	45	15
4	$1.0236\angle -5.3291$	0	0	40	5
5	$1.0179\angle -6.1503$	0	0	60	10

TABLE II
IEEE 5-Bus System Line Parameters

Line	Resistance (p.u.)	Reactance (p.u.)	Z (p.u $X\angle\theta$ rad)
1-2	0.02	0.06	$0.06\angle 1.25$
1-3	0.08	0.24	$0.25\angle 1.25$
2-3	0.06	0.25	$0.26\angle 1.33$
2-4	0.06	0.18	$0.19\angle 1.25$
2-5	0.04	0.12	$0.13\angle 1.25$
3-4	0.01	0.03	$0.03\angle 1.25$
4-5	0.08	0.24	$0.25\angle 1.25$

We use the same generator parameters as in [16], $M_1 = 1.9$ and $b_1 = 0.2$ for Generator 1, and $M_2 = 0.9$, $b_2 = 0.16$ for Generator 2. The linearized system has

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7.7926 & -0.1053 & -7.7926 & 0 \\ 0 & 0 & 0 & 1 \\ -20.3866 & 0 & 20.3866 & -0.1778 \end{bmatrix}.$$

The real-power P_1^{in} is the control input and $B(1) = [0, 1/M_1, 0, 0]$ is the normal operation (control is in effect). Contingencies in power systems can interrupt control randomly, that generates $B(2) = [0, 0, 0, 0]$. The contingency is represented by a Markov process. We will design the controller to stabilize such system, even under such Markovian contingencies.

Suppose that the initial state is $x(0) = [1, 2, 3, 1]$, and the initial distribution is $p_0 = [0.9, 0.1]$. The sampling interval $\tau = 0.2$ second and the probability transition matrix for the MC is $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$. Then,

$$p_\infty = [p_\infty^1, p_\infty^2] = \lim_{k\to\infty} P^k = [0.8333, 0.1667].$$

The pole placement design is used for designing controller feedback gains. When $\alpha_k = 1$, the controller runs close-loop, and when $\alpha_k = 2$, the system can only run open-loop.

We choose the desired closed-loop poles to be $\lambda = [-3.6, -2.7, -3, -3.3]$. Then the Matlab function $L_1 = \text{place}(A, B(1), \lambda)$ yields the suitable feedback gains and the closed-loop error dynamics with $A_c^1 = A - L_1 B(1)$. Fig. 2 shows the switching sequence, control input, and state norm trajectory. The results illustrate the almost sure convergence (one selected sample path).

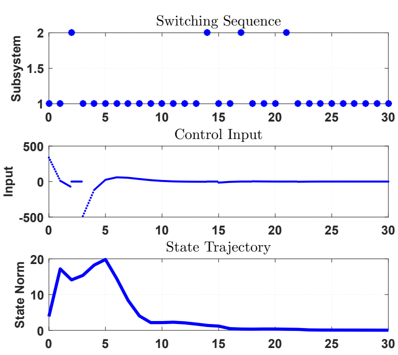

Fig. 2. Switching sequence and state norm trajectory.

VI. Concluding Remarks

This paper presents a control design method for stabilization of Markovian RSLSs. This new design method employs the unique requirement of almost sure stabilization in selecting its feedback gain matrices for subsystems. Unlike the requirement on mean-square state estimation of RSLSs with unobservable subsystems that impose an upper bound on the switching time interval, the design method of this paper is valid for any decision interval, allowing the overall system to achieve almost sure stability even if there are uncontrollable subsystems. There remain many potential open topics worthy of further research, such as robust control, optimal control, and time-delay systems of RSLSs.

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