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Exponential Stability of Stochastic Functional Differential Equations with Delayed Impulses

Ky Q. Tran and George Yin

***Abstract*—We focus on exponential stability of stochastic functional differential equations with delayed impulses. One of the notable characteristics of this paper is the dependence of both the given impulsive-free stochastic differential equation and impulsive perturbations on the past state of the system. Compared with the existing literature, we introduce new criteria for establishing exponential stability in mean square and almost surely. We provide two examples in this paper to showcase the effectiveness of our criteria.**

***Index Terms*—stochastic functional differential equation; exponential stability; delayed impulse.**

I. Introduction

In this paper, we focus on exponential stability, both in the mean square and in the almost sure sense, for stochastic functional differential equations (SFDEs) when they are influenced by delayed impulses. Impulsive systems have attracted significant interest over recent years with a rich body of literature that spans impulsive differential equations [6], [9], [15] and stochastic processes [11], [12], [21]. An important consideration for such systems is the stability. Stability has been the focus of enormous research with thorough investigation; see for example, Chapter 4 of [15]. The past effort showed a breadth of studies on the exponential stability in various forms of impulsive SFDEs [2], [7], [13], [14], and the development of the numerical approximations for SFDEs involving impulses [18], [19], [22].

In this paper, our study distinguishes with the existing literature featuring the dynamics of system states and impulsive effects are based on historical data from finite past intervals. As a result, following an impulse at time t, the state $X(t)$ is influenced not only by the immediate pre-impulse state $X(t^-)$ but also by states with historical past within a set $\{X(t+u) : u \in [-r_*, 0)\}$, where r_* is a positive constant. This approach provides a broader perspective than previous studies [1], [16], [17], which limited their focus to the state's immediate pre-impulse conditions, thereby offering a more realistic stochastic model. Note that SFDEs with delayed impulses have been the subject of earlier consideration [3], [4], [5], [8], [10], [20], where Lyapunov functions and Razumikhin techniques were applied, alongside concepts like average dwell-time [10] and average impulsive delays [5]. Nevertheless, the stability criteria established through these methods are characterized by complex conditions, and that are difficult to verify for applications.

We propose an effective treatment in this paper for addressing SFDEs impacted by delayed impulses, aiming to establish new and easily verifiable criteria for their exponential stability. We summarize our main contributions in this paper as follows. First, we study a broader class of SFDEs influenced by past state-dependent impulses. Second, we introduce an innovative method to assess the exponential stability of SFDEs with delayed impulses using a comparison procedure. Next, we establish several new stability criteria. It should be highlighted that the category of impulsive SFDEs being examined is significantly complex and abstract, showcasing a range of interactions between delays within the SFDEs themselves and those appeared in the impulses. The stability criteria established in this paper, in particular Theorem III.1, Corollary III.3, and Corollary III.4, have not been documented in the existing literature to the best of our knowledge.

We organize the rest of the paper as follows. Section II begins with the problem formulation. Section III presents the main results of the exponential stability of impulsive SFDEs. Section IV provides two examples for illustration. The paper is concluded with several remarks in Section V.

II. Formulation

We first list the notation to be used in this paper. Denote $\mathbb{R}_+ = [0, \infty)$ and let $\mathbb{N}$ be the set of positive integers. Denote by $c_1 \vee c_2 = \max\{c_1, c_2\}$ for two real numbers $c_1, c_2 \in \mathbb{R}$, and $A^\top$ the transpose and $|A| = \sqrt{\mathrm{tr}(A^\top A)}$ the trace norm of a matrix $A \in \mathbb{R}^{d_1 \times d_2}$ with $d_1, d_2 \in \mathbb{N}$. Denote by $|x| = \big(\sum_{i=1}^d x_i^2\big)^{1/2}$ the Euclidean norm of $x = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$. For $r > 0$, denote by $\mathsf{PC}([-r, 0], \mathbb{R}^d)$ the space of all piecewise continuous functions $\phi : [-r, 0] \to \mathbb{R}^d$

K. Tran is with the Department of Applied Mathematics and Statistics, State University of New York - Korea Campus, Yeonsu-Gu, Incheon, Korea 21985, <ky.tran@stonybrook.edu>. The research of this author was supported by the National Research Foundation of Korea grant funded by the Korea Government (MIST) NRF-2021R1F1A1062361.

G. Yin is with the Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA, <gyin@uconn.edu>. The research of this author was supported in part by the National Science Foundation under Grants DMS-2229108.

that are right continuous with left limits, endowed with the norm $\|\phi\| = \sup_{s\in[-r,0]} |\phi(s)|$. We work with a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ with $\{\mathcal{F}_t\}$ satisfying the usual condition in that it is right-continuous and $\mathcal{F}_0$ contains all the null sets. Assume that the $\widetilde{m}$-dimensional standard Brownian motion $w(\cdot)$ is defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, where $\widetilde{m} \in \mathbb{N}$. For each $t \geq 0$, let $L^2_{\mathcal{F}_t}([-r,0],\mathbb{R}^d)$ be the family of $\mathcal{F}_t$-measurable $\mathsf{PC}([-r,0],\mathbb{R}^d)$-valued random variables ξ such that $\mathbb{E}\|\xi\|^2 < \infty$.

For $r > 0$ and $r_* > 0$, let $f : \mathsf{PC}([-r,0],\mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}^d$, $g : \mathsf{PC}([-r,0],\mathbb{R}^d) \times \mathbb{R}_+ \to \mathbb{R}^{d\times\widetilde{m}}$, and $I_k : \mathsf{PC}([-r_*,0],\mathbb{R}^d) \to \mathbb{R}^d$ ($k \in \mathbb{N}$) be Borel measurable functions. Let $\{t_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a strictly increasing sequence of nonnegative numbers satisfying $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$. Consider the impulsive SFDE

$$\begin{aligned} dX(t) &= f\big(X_t,t\big)dt + g\big(X_t,t\big)dw(t) \\ &\qquad \text{for} \quad t \geq 0, \quad t \notin \{t_k\}_k, \\ X(t_k) &= I_k(X_t^*) \quad \text{for} \quad k \in \mathbb{N}, \end{aligned} \tag{II.1}$$

where $X_t : [-r,0] \to \mathbb{R}^d$ is defined by $X_t(u) = X(t+u)$ for $u \in [-r,0)$ and $X_t(0) = X(t^-)$. Meanwhile, $X_t^* : [-r_*,0] \to \mathbb{R}^d$ is defined by $X_t^*(u) = X(t+u)$ for $u \in [-r_*,0)$ and $X_t^*(0) = X(t^-)$. The initial condition is $\xi \in L^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^d)$, i.e.,

$$X(u) = \xi(u) \quad \text{for any} \quad u \in [-r,0]. \tag{II.2}$$

Thus, for each $k \in \mathbb{N}$,

$$\begin{aligned} X(t) &= X(t_{k-1}) + \int_{t_{k-1}}^t f(X_s,s)ds \\ &\quad + \int_{t_{k-1}}^t g(X_s,s)dw(s) \text{ for } t \in [t_{k-1},t_k), \\ X(t_k) &= I_k(X^*_{t_k}). \end{aligned}$$

It can be seen that the impulses depend on a finite-time segment of past states. Throughout this work, we suppose the following assumption (H1) holds.

(H1) (a) For any $t \geq 0$,

$$f(0,t) = 0 \in \mathbb{R}^d, \quad g(0,t) = 0 \in \mathbb{R}^{d\times\widetilde{m}}.$$

(b) The functions $f(\cdot,t)$ and $g(\cdot,t)$ satisfy the local Lipschitz condition; that is, for each $n \in \mathbb{N}$, there exists a constant $\widetilde{K}_n > 0$ such that

$$\begin{aligned} &|f(\phi_1,t) - f(\phi_2,t)| + |g(\phi_1,t) - g(\phi_2,t)| \\ &\qquad \leq \widetilde{K}_n\|\phi_1 - \phi_2\|, \end{aligned} \tag{II.3}$$

whenever $t \geq 0$ and $\|\phi_1\| \vee \|\phi_2\| \leq n$. Also, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfy the linear growth condition; that is, there exists a constant $\widetilde{K}_0 > 0$ such that

$$|f(\phi,t)| + |g(\phi,t)| \leq \widetilde{K}_0\big(1 + \|\phi\|\big)$$

for any $(\phi,t) \in \mathsf{PC}([-r,0],\mathbb{R}^d) \times \mathbb{R}_+$.

(c) There exist a sequence of positive constants $\{\widehat{K}_k\}_{k\in\mathbb{N}}$ such that

$$\big|I_k(\phi)\big| \leq \widehat{K}_k\|\phi\| \tag{II.4}$$

for any $(\phi,k) \in \mathsf{PC}([-r_*,0],\mathbb{R}^d) \times \mathbb{N}$.

Remark II.1. Assumption (H1) (a) indicates that the process $X(t) \equiv 0$ is a trivial solution of Eq. (II.1) (or $x = 0$ is an equilibrium point). If $I_k(\phi) = \phi(0)$ for any $(\phi,k) \in \mathsf{PC}([-r_*,0],\mathbb{R}^d) \times \mathbb{N}$, Eq. (II.1) is simply a SFDE studied in [11].

The existence and uniqueness theorem is given below. The proof is routine. Hence, we omit it for brevity.

Theorem II.2. *Assume* (H1). *Then for each* $\xi \in L^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^d) \times \mathcal{M}$, *Eq.* (II.1) *has a unique global solution* $X^\xi(\cdot)$ *satisfying* (II.2). *Moreover, for each* $k \in \mathbb{N}$, $X^\xi(\cdot)$ *has continuous sample paths on the interval* $[t_{k-1},t_k)$ *almost surely and*

$$\mathbb{E}\Big(\sup_{-r\leq s\leq T} |X^\xi(s)|^2\Big) < \infty \quad \textit{for any} \quad T > 0.$$

The definitions of the exponential stability in mean square and almost sure exponential stability of impulsive SFDEs are recalled below.

Definition II.3.

1) The equilibrium point $x = 0$ of Eq. (II.1) is
   (a) exponentially stable in mean square if there exist constants $K > 0$ and $\lambda > 0$ such that
   $$\mathbb{E}|X^\xi(t)|^2 \leq Ke^{-\lambda t}\mathbb{E}\|\xi\|^2$$
   for any $\xi \in L^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^d)$ and $t \geq 0$;
   (b) almost surely exponentially stable if there exists a constant $\lambda > 0$ such that
   $$\limsup_{t\to\infty} \frac{1}{t}\ln|X^\xi(t)| \leq -\lambda \quad \text{almost surely}$$
   for each $\xi \in L^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^d)$.
2) Eq. (II.1) is said to be exponentially stable in mean square (resp., almost surely) if its equilibrium point $x = 0$ is exponentially stable in mean square (resp. almost surely).

III. Main results

For the stability analysis in the rest of this paper, we use the following conditions.

(H2) (a) There exist a sequence of positive numbers $\{\gamma_k\}_{k\in\mathbb{N}}$ such that

$$\mathbb{E}|I_k(\zeta)|^2 \leq \gamma_k \sup_{u\in[-r_*,0)} \mathbb{E}|\zeta(u)|^2 \tag{III.1}$$

for any $k \in \mathbb{N}$ and $\zeta \in L^2_{\mathcal{F}_{t_k}}([-r_*,0],\mathbb{R}^d)$. Moreover,

$$\sup_{s\geq 0}\Big(\sum_{s-r<t_k\leq s}\big|\ln\gamma_k\big|\Big)<\infty \quad\text{and}\quad t_{k-1}<t_k-r_*$$

for any $k\in\mathbb{N}$.
(b) There exist real numbers $\lambda_1\in\mathbb{R}$ and $\lambda_2\geq 0$ such that

$$\begin{aligned}&\mathbb{E}\Big(2(\zeta(0))^\top f(\zeta,t)+|g(\zeta,t)|^2\Big)\\ &\quad\leq \lambda_1\mathbb{E}|\zeta(0)|^2+\lambda_2\sup_{u\in[-r,0)}\mathbb{E}|\zeta(u)|^2\end{aligned}\tag{III.2}$$

for any $t\geq 0$ and $\zeta\in L^2_{\mathcal{F}_t}([-r,0],\mathbb{R}^d)$.

Under assumptions (H1) and (H2), we define $\varphi:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ and $\psi:\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}\varphi(c,t)&=ct+\sum_{t_m\leq t}(\ln\gamma_m+c^-r_*),\\ \psi(c)&=\sup_{s\geq 0,\,u\in[-r,0)}\Big(cu-\sum_{s+u<t_m\leq s}(\ln\gamma_m+c^-r_*)\Big),\end{aligned}\tag{III.3}$$

where $c^-=\max\{-c,0\}$. We are now in a position to state the main result of this paper.

Theorem III.1. *Suppose that conditions* (H1) *and* (H2) *are satisfied and that there exists a $\beta\in\mathbb{R}$ satisfying*

$$\lambda_1+\lambda_2 e^{\psi(\beta)}\leq\beta.\tag{III.4}$$

Then we have the following results.
(a) *There exists a $\widetilde{K}>0$ such that*

$$\mathbb{E}|X^\xi(t)|^2\leq\widetilde{K}\mathbb{E}\|\xi\|^2e^{\varphi(\beta,t)}\tag{III.5}$$

for any $\xi\in L^2_{\mathcal{F}_0}([-r,0],\mathbb{R}^d)$ and $t\geq 0$.
(b) *The mean square Lyapunov exponent of Eq.* (II.1) *is not greater than $\limsup_{t\to\infty}\dfrac{\varphi(\beta,t)}{t}$. If $\limsup_{t\to\infty}\dfrac{\varphi(\beta,t)}{t}<0$, then Eq.* (II.1) *is exponentially stable in mean square.*
(c) *If $\limsup_{t\to\infty}\dfrac{\varphi(\beta,t)}{t}<0$ and there is a positive constant $\widehat{K}$ such that*

$$\mathbb{E}\Big(|f(\zeta,t)|^2+|g(\zeta,t)|^2\Big)\leq\widehat{K}\sup_{u\in[-r,0]}\mathbb{E}|\zeta(u)|^2\tag{III.6}$$

for any $t\geq 0$ and $\zeta\in L^2_{\mathcal{F}_t}([-r,0],\mathbb{R}^d)$, then Eq. (II.1) *is almost surely exponentially stable.*

Proof. For notational simplicity, denote $X(\cdot)=X^\xi(\cdot)$. Without loss of generality, suppose $\mathbb{E}\|\xi\|^2>0$. Using standard arguments for SFDEs as discussed in [11, Chapter 5], we can demonstrate that there exists a positive constant $K=K(\widetilde{K}_0,t_1)$ independent of ξ such that

$$\sup_{t\in[0,t_1)}\Big[e^{-\varphi(\beta,t)}\mathbb{E}|X(t)|^2\Big]<K_1,\tag{III.7}$$

where $K_1=K\mathbb{E}\|\xi\|^2$.

(a) We prove by induction that for any $n\in\mathbb{N}$,

$$\sup_{t\in[0,t_n)}\Big[e^{-\varphi(\beta,t)}\mathbb{E}|X(t)|^2\Big]<K_1.\tag{III.8}$$

In view of (III.7), (III.8) holds for $n=1$. Now suppose (III.8) holds for $n\leq k$; that is,

$$\sup_{t\in[0,t_k)}\Big[e^{-\varphi(\beta,t)}\mathbb{E}|X(t)|^2\Big]<K_1.\tag{III.9}$$

We proceed to show that (III.8) holds for $n=k+1$.

Consider the real-valued function $\Phi(\cdot)$ defined by

$$\Phi(t)=e^{-\varphi(\beta,t)}\mathbb{E}|X(t)|^2\quad\text{for}\quad t\in[0,t_{k+1})$$

and $\Phi(t_{k+1})=e^{-\varphi(\beta,t^-_{k+1})}\mathbb{E}|X(t^-_{k+1})|^2$. By (III.9), $\sup_{t\in[0,t_k)}\Phi(t)<K_1$. We also have

$$\begin{aligned}&\sup_{u\in[-r_*,0)}\mathbb{E}|X^*_{t_k}(u)|^2\\ &\leq e^{\varphi(\beta,t_k^-)+\beta^-r_*}\sup_{u\in[-r_*,0)}\Big[e^{-\varphi(\beta,t_k+u)}\mathbb{E}|X^*_{t_k}(u)|^2\Big]\\ &=e^{\varphi(\beta,t_k^-)+\beta^-r_*}\sup_{u\in[-r_*,0)}\Phi(t_k+u)\\ &<K_1e^{\varphi(\beta,t_k^-)+\beta^-r_*}.\end{aligned}$$

This together with (III.1) implies

$$\begin{aligned}&e^{-\varphi(\beta,t_k)}\mathbb{E}|X(t_k)|^2\\ &=e^{-\varphi(\beta,t_k)}\mathbb{E}|I_k(X^*_{t_k})|^2\\ &\leq\gamma_ke^{-\varphi(\beta,t_k)}\sup_{u\in[-r_*,0)}\mathbb{E}|X^*_{t_k}(u)|^2\\ &<\gamma_ke^{-\varphi(\beta,t_k)}K_1e^{\varphi(\beta,t_k^-)+\beta^-r_*}\\ &=K_1;\end{aligned}\tag{III.10}$$

that is, $\Phi(t_k)<K_1$. To show that (III.8) holds for $n=k+1$, it is sufficient to verify that

$$\sup_{t\in(t_k,t_{k+1}]}\Phi(t)<K_1.\tag{III.11}$$

If this statement were false, by the continuity of the function $\Phi(\cdot)$ on $[t_k,t_{k+1}]$, there would exist a number $t_*\in(t_k,t_{k+1}]$ such that

$$\Phi(t)<K_1\quad\text{for}\quad t\in[t_k,t_*),\quad \Phi(t_*)=K_1.\tag{III.12}$$

We assume $t_*\in(t_k,t_{k+1})$. The case $t_*=t_{k+1}$ can be treated similarly. Fix $\lambda>|\lambda_1|$. By the Dynkin formula,

$$\begin{aligned}&e^{\lambda t_*}\mathbb{E}|X(t_*)|^2\\ &=e^{\lambda t_k}\mathbb{E}|X(t_k)|^2+\mathbb{E}\int_{t_k}^{t_*}e^{\lambda s}\Big(\lambda|X(s)|^2\\ &\quad+2(X(s))^\top f(X_s,s)+|g(X_s,s)|^2\Big)ds.\end{aligned}\tag{III.13}$$

By (III.2),

$$\begin{aligned}&\mathbb{E}\int_{t_k}^{t_*} e^{\lambda s}(\lambda|X(s)|^2+2(X(s))^\top f(X_s,s)\\&\qquad\qquad\qquad+|g(X_s,s)|^2)ds\\&\le\int_{t_k}^{t_*} e^{\lambda s}\Big((\lambda+\lambda_1)\mathbb{E}|X(s)|^2\\&\quad+\lambda_2\sup_{u\in[-r,0)}\mathbb{E}|X(s+u)|^2\Big)ds.\end{aligned}\tag{III.14}$$

In view of (III.12), we have

$$\mathbb{E}|X(s)|^2<K_1e^{\varphi(\beta,s)}\quad\text{for any}\quad s\in[t_k,t_*).\tag{III.15}$$

For $s\in[t_k,t_*)$ and $u\in[-r,0)$, we have

$$\begin{aligned}&\varphi(\beta,s+u)-\varphi(\beta,s)\\&=\beta u-\sum_{s+u<t_m\le s}(\ln\gamma_m+\beta^-r_*)\\&\le\sup_{s\ge0,\,v\in[-r,0)}\Big(\beta v-\sum_{s+v<t_m\le s}(\ln\gamma_m+\beta^-r_*)\Big)\\&=\psi(\beta).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}|X(s+u)|^2&<K_1e^{\varphi(\beta,s+u)}\\&=K_1e^{\varphi(\beta,s+u)-\varphi(\beta,s)}e^{\varphi(\beta,s)}\\&\le K_1e^{\psi(\beta)}e^{\varphi(\beta,s)}\ \text{for any}\ s\in[t_k,t_*).\end{aligned}\tag{III.16}$$

Putting the estimates in (III.14)-(III.16) together yields

$$\begin{aligned}&\mathbb{E}\int_{t_k}^{t_*} e^{\lambda s}\Big(\lambda|X(s)|^2+2(X(s))^\top f(X_s,s)\\&\qquad\qquad\qquad+|g(X_s,s)|^2\Big)ds\\&<K_1\int_{t_k}^{t_*} e^{\lambda s+\varphi(\beta,s)}\Big(\lambda+\lambda_1+\lambda_2e^{\psi(\beta)}\Big)ds\\&=K_1e^{\sum_{t_m\le t_k}(\ln\gamma_m+\beta^-r_*)}\int_{t_k}^{t_*}e^{(\lambda+\beta)s}(\lambda+\beta)ds\\&=K_1e^{\sum_{t_m\le t_k}(\ln\gamma_m+\beta^-r_*)}\big(e^{(\lambda+\beta)t_*}-e^{(\lambda+\beta)t_k}\big).\end{aligned}\tag{III.17}$$

It follows from (III.10), (III.13), and (III.17) that

$$\begin{aligned}&e^{\lambda t_*}\mathbb{E}|X(t_*)|^2\\&<K_1e^{\lambda t_k+\varphi(\beta,t_k)}\\&\quad+K_1e^{\sum_{t_m\le t_k}(\ln\gamma_m+\beta^-r_*)}\big(e^{(\lambda+\beta)t_*}-e^{(\lambda+\beta)t_k}\big)\\&=K_1e^{\sum_{t_m\le t_k}(\ln\gamma_m+\beta^-r_*)}e^{(\lambda+\beta)t_*}.\end{aligned}$$

Hence

$$\mathbb{E}|X(t_*)|^2<K_1e^{\beta t_*+\sum_{t_m\le t_k}(\ln\gamma_m+\beta^-r_*)}=K_1e^{\varphi(\beta,t_*)}.$$

That is, $\Phi(t_k)<K_1$, which contradicts the second statement in (III.12). As a result, (III.11) holds. Consequently,

$$e^{-\varphi(\beta,t)}\mathbb{E}|X(t)|^2<K_1\quad\text{for any}\quad t\ge0,$$

which implies (III.5).

(b) By (III.5),

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(\mathbb{E}|X^\xi(t)|^2\right)\le\limsup_{t\to\infty}\frac{\varphi(\beta,t)}{t}.$$

Hence, the mean square Lyapunov exponent of Eq. (II.1) is not greater than $\limsup_{t\to\infty}\frac{\varphi(\beta,t)}{t}$. It can be seen from (III.5) that if $\limsup_{t\to\infty}\frac{\varphi(\beta,t)}{t}<0$, Eq. (II.1) is exponentially stable in mean square.

(c) The proof of this part is a slight modification of that of [1, Theorem 3.2]. We omit the verbatim detailed proof for brevity. □

Remark III.2. Let β be a constant satisfying (III.4). Define

$$\theta=\limsup_{t\to\infty}\frac{1}{t}\sum_{t_m\le t}\ln\gamma_m,\quad\theta_*=\limsup_{t\to\infty}\frac{1}{t}\sum_{t_m\le t}(\beta^-r_*).$$

It follows from (III.3) that

$$\limsup_{t\to\infty}\frac{\varphi(\beta,t)}{t}\le\beta+\theta+\theta_*.$$

By virtue of Theorem III.1, $\beta+\theta+\theta_*$ is an upper bound of the mean square Lyapunov exponent of Eq. (II.1). The value of θ can be regarded as a constant, which describes the contribution of the impulses to the exponential stability in mean square of Eq. (II.1). Meanwhile, θ_* describes the additional contribution of the delayed impulses. If $r_*=0$, the impulses do not depend on the past states of the given system. In such a case, $\theta_*=0$ and

$$\limsup_{t\to\infty}\frac{\varphi(\beta,t)}{t}\le\beta+\theta.$$

We refer to [17] for a similar estimate for SFDEs with Markovian switching and impulsive perturbations.

Corollary III.3. *Suppose that* (H1) *and* (H2) *hold and that*

$$H(c)=\lambda_1+\lambda_2e^{\psi(c)}-c\quad\textit{for}\quad c\in\mathbb{R}.\tag{III.18}$$

Furthermore, assume

$$\theta=\limsup_{t\to\infty}\frac{1}{t}\sum_{t_m\le t}\ln\gamma_m<0\quad\textit{and}\quad H(-\theta)<0.\tag{III.19}$$

Then the following assertions hold.

(a) *Eq.* (II.1) *is exponentially stable in mean square.*
(b) *If condition* (III.6) *is satisfied, then Eq.* (II.1) *is almost surely exponentially stable.*

Proof. (a) We first observe that for $c\in(0,-\theta)$, we have $c^-=0$ and

$$\psi(c)=\sup_{s\ge0,\,u\in[-r,0)}\Big(cu-\sum_{s+u<t_m\le s}\ln\gamma_m\Big).$$

Hence, the function $\psi(\cdot)$ is decreasing and continuous on $(0, -\theta]$. Consequently, the function $H(\cdot)$ is strictly decreasing and continuous on $(0, -\theta]$. Since $H(-\theta) < 0$, there exists a constant $\beta \in (0, -\theta)$ such as $H(\beta) \leq 0$. That is, β satisfies condition (III.4). Moreover, since $\beta \in (0, -\theta)$, then

$$\limsup_{t\to\infty} \frac{\varphi(\beta,t)}{t} = \limsup_{t\to\infty} \frac{1}{t}\Big(\beta t + \sum_{t_m \leq t} \ln \gamma_m\Big) = \beta + \theta < 0.$$

By Theorem III.1, Eq. (II.1) is exponentially stable in mean square.

(b) follows from the result in part (a) and Theorem III.1. This completes the proof. $\square$

Corollary III.4. *Suppose that* (H1), (III.1), (III.2) *hold, and that there exists a constant* $\beta \in \mathbb{R}$ *satisfying*

$$\lambda_1 + \lambda_2 e^{\psi(\beta)} \leq \beta.$$

Moreover, suppose that there exist positive constants ε *and* γ *satisfying*

$$t_k = k\varepsilon \quad \text{and} \quad \gamma_k = \gamma \quad \text{for any} \quad k \in \mathbb{N}. \tag{III.20}$$

Then the following assertions hold.

(a) *The mean square Lyapunov exponent of Eq.* (II.1) *is not greater than*

$$\beta + \frac{\ln\gamma}{\varepsilon} + \frac{\beta^- r_*}{\varepsilon}.$$

(b) *If*

$$\beta + \frac{\ln\gamma}{\varepsilon} + \frac{\beta^- r_*}{\varepsilon} < 0,$$

then Eq. (II.1) *is exponentially stable in mean square.*

(c) *If*

$$\beta + \frac{\ln\gamma}{\varepsilon} + \frac{\beta^- r_*}{\varepsilon} < 0$$

and condition (III.6) *is satisfied, then Eq.* (II.1) *is almost surely exponentially stable.*

Proof. By using (III.20), it is readily seen that

$$\limsup_{t\to\infty} \frac{\varphi(\beta,t)}{t} = \beta + \frac{\ln\gamma}{\varepsilon} + \frac{\beta^- r_*}{\varepsilon}.$$

Thus, the conclusion follows from Theorem III.1. $\square$

IV. Examples

Example IV.1. We consider a scalar impulsive SFDE

$$\begin{aligned} dX(t) &= f\big(X_t,t\big)dt + g\big(X_t,t\big)dw(t) \\ &\qquad \text{for} \quad t \geq 0, \quad t \notin \{t_k\}_k, \\ X(t_k) &= I_k(X_t^*) \quad \text{for} \quad k \in \mathbb{N}, \end{aligned} \tag{IV.1}$$

where $r = 0.3$, $r_* = 0.2$, $t_k = \varepsilon k$ with $\varepsilon = 0.5$,

$$\begin{aligned} f(\phi,t) &= -1.4\phi(0) + 0.5\phi(0)\sin\phi(0) + 0.6\phi(-0.3), \\ g(\phi,t) &= 0.8\phi(0)\cos\phi(-0.2) \end{aligned}$$

for any $(\phi,t) \in \mathsf{PC}([-0.3,0],\mathbb{R}) \times \mathbb{R}_+$, and $I_k(\phi) = \sqrt{20}\int_{-0.2}^{0} \phi(s)ds$ for any $\phi \in \mathsf{PC}([-0.2,0],\mathbb{R})$. We investigate the exponential stability in mean square of Eq. (IV.1). It is readily seen that assumption (H1) is satisfied. Moreover,

$$\begin{aligned} &2\phi(0)f(\phi,t) + |g(\phi,t)|^2 \\ &= 2\phi(0)\Big(-1.4\phi(0) + 0.5\phi(0)\sin\phi(0) + 0.6\phi(-0.3)\Big) \\ &\quad + 0.8^2|\phi(0)|^2\cos^2\phi(-0.2) \\ &\leq -0.56|\phi(0)|^2 + 0.6|\phi(-0.3)|^2 \end{aligned}$$

for any $(\phi,t) \in \mathsf{PC}([-0.3,0],\mathbb{R}) \times \mathbb{R}_+$. In addition, $|I_k(\phi)|^2 \leq 4\int_{-0.2}^{0}|\phi(s)|^2 ds$ for any $(\phi,t) \in \mathsf{PC}([-0.2,0],\mathbb{R}) \times \mathbb{R}_+$. Thus, assumptions (H1) and (H2) are satisfied with $\lambda_1 = -0.56$, $\lambda_2 = 0.6$, $\gamma_k = \gamma = 0.8$ for any $k \in \mathbb{N}$. In view of (III.3), we have

$$\begin{aligned} \psi(c) &= \sup_{s\geq 0,\, u\in[-r,0)} \Big(cu - \sum_{s+u<t_m\leq s} (\ln\gamma_m + c^- r_*)\Big) \\ &= \sup_{s\geq 0,\, u\in[-0.3,0)} \Big(cu - \sum_{s+u<t_m\leq s} (\ln 0.8 + 0.2c^-)\Big) \\ &\leq 0.3c^- + (\ln 0.8 + 0.2c^-)^-. \end{aligned}$$

We can check that $\beta = 0.3$ satisfies $\lambda_1 + \lambda_2 e^{\psi(\beta)} \leq \beta$ and

$$\beta + \frac{\ln\gamma}{\varepsilon} + \frac{\beta^- r_*}{\varepsilon} = 0.3 + \frac{\ln 0.8}{0.5} < 0.$$

Hence, by Corollary III.4, Eq. (IV.1) is exponentially stable in mean square. Since condition (III.6) is satisfied, Eq. (IV.1) is also almost surely exponentially stable.

Example IV.2. We consider another scalar impulsive SFDE

$$\begin{aligned} dX(t) &= f(X_t,t)dt + g(X_t,t)dw(t), \;\; t \geq 0, t \notin \{t_k\}_k, \\ X(t_k) &= I_k(X_{t_k}^*) \quad \text{for} \quad k \in \mathbb{N}, \end{aligned} \tag{IV.2}$$

where $r = 1$, $r_* = 0.3$,

$$\begin{aligned} f(\phi,t) &= -1.55\phi(0) + \phi(-1)\sin t + 0.3\int_{-1}^{0} u^2\phi(u)du, \\ g(\phi,t) &= 0.9\phi(0)\sin\phi(0) \end{aligned}$$

for any $(\phi,t) \in \mathsf{PC}([-1,0],\mathbb{R}) \times \mathbb{R}_+$. Moreover, $t_k = 0.5k$ for any $k \in \mathbb{N}$ and

$$I_k(\phi) = \sqrt{0.25}e^{-0.1}\phi(-0.3) + 2e^{-0.1}\int_{-0.25}^{0} \phi(s)ds$$

for any $(k, \phi) \in \mathbb{N} \times \mathsf{PC}([-0.3, 0], \mathbb{R})$. We have

$$\begin{aligned} & 2\phi(0)f(\phi,t) + |g(\phi,t)|^2 \\ & \le -3.1\phi^2(0) + 2\phi(0)\phi(-1)\sin t \\ & \quad +0.6\int_{-1}^{0} u^2\phi(0)\phi(u)du + 0.81\phi^2(0) \\ & \le -1.29\phi^2(0) + \phi^2(-1) \\ & \quad +0.3\int_{-1}^{0} u^2\phi^2(0)du + 0.3\int_{-1}^{0} u^2\phi^2(u)du \\ & = -1.19\phi^2(0) + \phi^2(-1) + 0.3\int_{-1}^{0} u^2\phi^2(u)du \end{aligned} \tag{IV.3}$$

for any $(\phi, t) \in \mathsf{PC}([-1, 0], \mathbb{R}) \times \mathbb{R}_+$. Thus, $\lambda_1 = -1.19$ and $\lambda_2 = 1 + 0.3\int_{-1}^{0} u^2 du = 1.1$. In addition, it can be verified that

$$|I_k(\phi)|^2 \le 0.5e^{-0.2}|\phi(-0.3)|^2 + 2e^{-0.2}\int_{-0.25}^{0} |\phi(s)|^2 ds$$

for any $(k, \phi) \in \mathbb{N} \times \mathsf{PC}([-0.3, 0], \mathbb{R})$. That is, $\gamma_k = e^{-0.2}$ for any $k \in \mathbb{N}$. Consequently,

$$\theta = \limsup_{t\to\infty} \frac{\sum_{t_m \le t} \ln \gamma_m}{t} = \frac{\ln \gamma_1}{t_1} = -0.4.$$

We also have

$$\begin{aligned} \psi(-\theta) &= \sup_{s\ge 0,\, u\in[-1,0)} \Big(-\theta u - \sum_{s+u<t_m\le s} \ln \gamma_m \Big) \\ &= \sup_{s\ge 0,\, u\in[-1,0)} \Big(0.4u + \sum_{s+u<t_m\le s} 0.2 \Big) \\ &\le 0.2. \end{aligned}$$

In view of (III.18), let

$$H(c) = -1.19 + 1.1e^{\psi(c)} - c \quad \text{for} \quad c \in \mathbb{R}.$$

Then $H(-\theta) = H(0.4) = -1.19 + 1.1e^{0.2} - 0.4 < 0$. Hence, by Corollary III.3, Eq. (IV.2) is exponentially stable in mean square. Since condition (III.6) is satisfied, Eq. (IV.2) is also almost surely exponentially stable.

V. Concluding Remarks

We have focused on examining exponential stability in mean square and in the almost sure sense of a class of SFDEs with delayed impulses. By a comparison procedure involving a contradiction argument, we have established new criteria for proving the exponential stability. It is worth noting that in the models of interest, the impulses are predicated on a finite historical period of system development. It is conceivable that the results and approach can be extended to hybrid SFDEs involving a Markovian switching. This together with developing numerical approximations and stabilization problems for impulsive SFDEs are potential areas of exploration for future research.

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