Journal of Scientific Computing (2024) 100:80
https://doi.org/10.1007/s10915-024-02626-x

A Hodge Decomposition Finite Element Method for the Quad-Curl Problem on Polyhedral Domains

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Received: 29 March 2024 / Revised: 10 July 2024 / Accepted: 12 July 2024 /
Published online: 1 August 2024

Abstract

We design a finite element method for the quad-curl problem on three dimensional Lipschitz polyhedral domains with general topology that is based on the Hodge decomposition for divergence-free vector fields. Error estimates and corroborating numerical results are presented.

Keywords Quad-curl · Hodge decomposition · Finite elements · Polyhedral domain with general topology

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded connected polyhedral domain with a Lipschitz boundary and the space $\mathbb{E}$ be defined by

$$\mathbb{E} = \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \mathbf{curl}\, \boldsymbol{v} \in [H_0^1(\Omega)]^3, \text{div}\, \boldsymbol{v} = 0 \text{ and } \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$

where $\boldsymbol{n}$ is the unit outer normal on $\partial\Omega$.

Remark 1.1 Here and below we follow the standard notation for differential operators, function spaces and norms that can be found for example in [12, 22, 27].

The quad-curl problem is to find $\boldsymbol{u} \in \mathbb{E}$ such that

$$\begin{aligned} \big(\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}), \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v})\big)_{L^2(\Omega)} + \beta(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{L^2(\Omega)} \\ + \gamma(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)} = (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in \mathbb{E}, \end{aligned} \tag{1.1}$$

where β, γ are nonnegative constants and $\boldsymbol{f} \in [L^2(\Omega)]^3$. We assume $\gamma > 0$ if $\partial\Omega$ is not connected.

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The space $\mathbb{E}$ is a Hilbert space under the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathbb{E}} = (\boldsymbol{v}, \boldsymbol{w})_{L^2(\Omega)} + (\mathbf{curl}\, \boldsymbol{v}, \mathbf{curl}\, \boldsymbol{w})_{H^1(\Omega)} \tag{1.2}$$

where (componentwise)

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\mathbf{grad}\, v, \mathbf{grad}\, w)_{L^2(\Omega)} \qquad \forall\, v, w \in H^1(\Omega),$$

and we have a norm equivalence (cf. Appendix A and [20])

$$\|\mathbf{curl}\,(\mathbf{curl}\, \boldsymbol{v})\|^2_{L^2(\Omega)} + \beta\|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)} + \gamma\|\boldsymbol{v}\|^2_{L^2(\Omega)} \approx (\boldsymbol{v}, \boldsymbol{v})_{\mathbb{E}} \qquad \forall\, \boldsymbol{v} \in \mathbb{E}. \tag{1.3}$$

It follows from (1.3) and the Riesz representation theorem that (1.1) has a unique solution. Our goal is to solve (1.1) numerically through a Hodge decomposition of $\boldsymbol{u}$.

The quad-curl problem appears in the Maxwell transmission eigenvalue problem (cf. [15, 16, 34, 38]) and mathematical models for magnetohydrodynamics with hyperresistivity (cf. [5, 18, 36]). Numerical methods for (1.1) have been investigated in [17, 19–21, 26, 28, 37, 39, 41, 44, 46, 48] and methods for its analog on two dimensional domains can be found in [13, 29, 42, 45, 47].

In this paper we treat the quad-curl problem on three dimensional domains with general topology by extending the Hodge decomposition approach for two dimensional quad-curl problems in [13]. The general topology of Ω leads to additional challenges in the construction of the Hodge decomposition and corresponding finite element methods. Fortunately many of the complications have been addressed in [1] and we are able to solve (1.1) by standard simple finite elements and obtain error estimates solely based on the given Ω and $\boldsymbol{f}$ without any assumed regularity on the solution $\boldsymbol{u}$.

The rest of the paper is organized as follows. The Hodge decomposition of divergence-free vector fields is presented in Sect. 2, followed by the reduction of (1.1) into second order problems in Sect. 3. Finite element methods for these second order problems are introduced in Sect. 4 with convergence analysis given in Sect. 5. Improved error estimates for the case where $\boldsymbol{f} \in [H^1(\Omega)]^3$ are presented in Sect. 6, followed by numerical results in Sect. 7. We end the paper with some concluding remarks in Sect. 8. The appendices contain the proofs of some technical results.

Below we recall some notation that will be used throughout the paper.

- $L^2_0(\Omega)$ is the space of functions $v \in L^2(\Omega)$ that satisfy

$$\int_\Omega v\, dx = 0. \tag{1.4}$$

- $H(\mathbf{curl}\,; \Omega)$ is the space of vector fields defined by

$$H(\mathbf{curl}\,; \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \mathbf{curl}\, \boldsymbol{v} \in [L^2(\Omega)]^3\}$$

  with $\|\boldsymbol{v}\|_{H(\mathbf{curl}\,;\Omega)} = \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}$.
- $H_0(\mathbf{curl}\,; \Omega) = \{\boldsymbol{v} \in H(\mathbf{curl}\,; \Omega) : \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\}$ is the subspace of vector fields in $H(\mathbf{curl}\,; \Omega)$ with vanishing tangential components.
- $H(\mathbf{curl}^0; \Omega) = \{\boldsymbol{v} \in H(\mathbf{curl}\,; \Omega) : \mathbf{curl}\, \boldsymbol{v} = \mathbf{0}\}$ is the space of irrotational vector fields in $H(\mathbf{curl}\,; \Omega)$.
- $H(\text{div}\,; \Omega)$ is the space of vector fields defined by

$$H(\text{div}\,; \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^3 : \text{div}\, \boldsymbol{v} \in L^2(\Omega)\}$$

  with $\|\boldsymbol{v}\|_{H(\text{div}\,;\Omega)} = \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\text{div}\, \boldsymbol{v}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}$.

- $H_0(\mathrm{div}\,;\Omega)$ is the subspace of $H(\mathrm{div}\,;\Omega)$ defined by

$$H_0(\mathrm{div}\,;\Omega) = \{\boldsymbol{v} \in H(\mathrm{div}\,;\Omega) : \boldsymbol{n}\cdot\boldsymbol{v} = 0 \quad \text{on} \quad \partial\Omega\}.$$

- $H(\mathrm{div}^0;\Omega) = \{\boldsymbol{v} \in H(\mathrm{div}\,;\Omega) : \mathrm{div}\,\boldsymbol{v} = 0\}$ is the space of divergence-free vector fields in $H(\mathrm{div}\,;\Omega)$.
- $H_0(\mathrm{div}^0;\Omega) = \{\boldsymbol{v} \in H_0(\mathrm{div}\,;\Omega) : \mathrm{div}\,\boldsymbol{v} = 0\}$ is the space of divergence-free vector fields in $H_0(\mathrm{div}\,;\Omega)$.
- Given a vector field $\boldsymbol{v} = [v_1, v_2, v_3]^t \in [H^1(\Omega)]^3$, the matrix function $\mathbf{Grad}\,\boldsymbol{v}$ is given by

$$\mathbf{Grad}\,\boldsymbol{v} = \begin{bmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 & \partial v_1/\partial x_3 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 & \partial v_2/\partial x_3 \\ \partial v_3/\partial x_1 & \partial v_3/\partial x_2 & \partial v_3/\partial x_3 \end{bmatrix}$$

so that

$$\begin{aligned}(\mathbf{Grad}\,\boldsymbol{v}, \mathbf{Grad}\,\boldsymbol{w})_{L^2(\Omega)} &= \int_\Omega \mathbf{Grad}\,\boldsymbol{v} : \mathbf{Grad}\,\boldsymbol{w}\,dx \\ &= \sum_{i=1}^{3} (\mathbf{grad}\,v_i, \mathbf{grad}\,w_i)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in [H^1(\Omega)]^3\end{aligned}$$

and $\|\mathbf{Grad}\,\boldsymbol{v}\|_{L^2(\Omega)} = (\mathbf{Grad}\,\boldsymbol{v}, \mathbf{Grad}\,\boldsymbol{v})^{\frac{1}{2}}_{L^2(\Omega)} = |\boldsymbol{v}|_{H^1(\Omega)}$.

We end the introduction with a fundamental result on two subspaces of $H(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$ (cf. [1, Proposition 3.7]).

Lemma 1.2 *There exist two positive numbers α_T and α_N in $(\frac{1}{2}, 1]$ such that*

$$\|\boldsymbol{v}\|_{H^{\alpha_T}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)}\big) \tag{1.5}$$

for all $\boldsymbol{v} \in H(\mathbf{curl}\,;\Omega) \cap H_0(\mathrm{div}\,;\Omega)$,

$$\|\boldsymbol{v}\|_{H^{\alpha_N}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)}\big) \tag{1.6}$$

for all $\boldsymbol{v} \in H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega)$. We can take $\alpha_T = \alpha_N = 1$ if Ω is convex.

Here and throughout the paper we use C to denote a generic positive constant independent of the mesh sizes that can take different values at different occurrences.

2 Hodge Decomposition for Divergence-Free Vector Fields

The Hodge decomposition has to take into account the topology of the domain Ω, which may be multiply connected and the boundary of Ω may have multiple components, i.e., the Betti numbers β_1 and β_2 of Ω may be greater than 0.

More precisely we assume that Ω becomes simply connected after m cuts $\Sigma_1, \ldots, \Sigma_m$ have been removed and that $\partial\Omega$ has $n+1$ components $\Gamma_0, \ldots, \Gamma_n$, where $m = \beta_1$ and $n = \beta_2$. Each cut Σ_i, which is the intersection of a plane with Ω, is simply connected and the closures of Σ_i are pairwise disjoint. The open subset $\Omega \setminus \big(\bigcup_{i=1}^m \Sigma_i\big)$ of Ω, denoted by Ω°, is simply connected. We take Γ_0 to be the outer component of $\partial\Omega$ and $\Gamma_1, \ldots, \Gamma_n$ to be the inner components of $\partial\Omega$.

As an illustration, the domain Ω in the center of Fig. 1 with $m = 2$ and $n = 1$ is generated by removing two rectangular columns (shaded in blue) and a cube (shaded in brown) from the

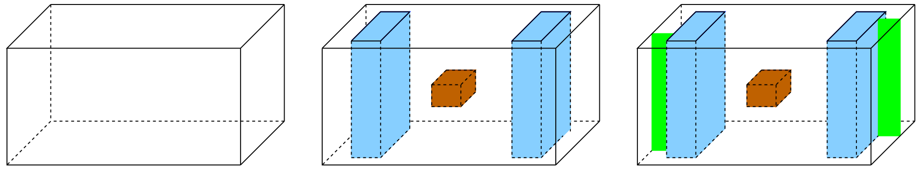

Fig. 1 Rectangular box (left), domain Ω (center), domain Ω with cuts (right)

rectangular box on the left of Fig. 1, where the boundary of the cube is the inner component Γ_1 of $\partial\Omega$. The two cuts Σ_1 and Σ_2 (shaded in green) are depicted on the right of Fig. 1.

2.1 Vector Potentials

We will use the vector potentials for divergence-free vector fields in [1, Section 3] in the construction of the Hodge decomposition and the finite element methods (cf. also [2, Section 3.5 and Section 3.6]).

Let the subspace $\mathscr{P}_T$ of $H(\mathbf{curl}\,;\Omega)\cap H_0(\mathrm{div}^0;\Omega)$ be defined by

$$\mathscr{P}_T=\{\boldsymbol{v}\in H(\mathbf{curl}\,;\Omega)\cap H_0(\mathrm{div}^0;\Omega):\int_{\Sigma_i}\boldsymbol{v}\cdot\boldsymbol{n}_{\Sigma_i}\,dS=0\quad\text{for}\quad 1\le i\le m\},\tag{2.1}$$

where $\boldsymbol{n}_{\Sigma_i}$ is a unit normal of Σ_i.

The following lemma is Theorem 3.12 in [1].

Lemma 2.1 *The operator* **curl** *is a surjection from* $H(\mathbf{curl}\,;\Omega)$ *onto the space*

$$\mathscr{D}_F=\{\boldsymbol{v}\in H(\mathrm{div}^0;\Omega):\int_{\Gamma_j}\boldsymbol{v}\cdot\boldsymbol{n}_{\Gamma_j}\,dS=0\quad\textit{for}\quad 1\le j\le n\},$$

where $\boldsymbol{n}_{\Gamma_j}$ $(1\le j\le n)$ *is the unit outer normal on the inner component* Γ_j *of* $\partial\Omega$*, and the restriction of the operator* **curl** *to* $\mathscr{P}_T$ *is an isomorphism between* $\mathscr{P}_T$ *and* $\mathscr{D}_F$*.*

Remark 2.2 Since $\mathscr{P}_T$ is a Hilbert space under the inner product

$$(\boldsymbol{v},\boldsymbol{w})_{H(\mathbf{curl}\,;\Omega)}=(\boldsymbol{v},\boldsymbol{w})_{L^2(\Omega)}+(\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{w})_{L^2(\Omega)}$$

and **curl** is a bounded linear operator from $\mathscr{P}_T$ onto the closed subspace $\mathscr{D}_F$ of $[L^2(\Omega)]^3$, it follows from the open mapping theorem that

$$\|\boldsymbol{v}\|_{L^2(\Omega)}\le C\|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)}\qquad\forall\,\boldsymbol{v}\in\mathscr{P}_T.$$

Hence $\mathscr{P}_T$ is also a Hilbert space under the inner product $(\boldsymbol{v},\boldsymbol{w})_{\mathscr{P}_T}=(\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{w})_{L^2(\Omega)}$.

While the potentials in $\mathscr{P}_T$ appear in the Hodge decomposition, another space of vector potentials also plays a role in the construction of the finite element methods and their analysis. Let the subspace $\mathscr{P}_N$ of $H_0(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}^0;\Omega)$ be defined by

$$\mathscr{P}_N=\{\boldsymbol{v}\in H_0(\mathbf{curl}\,;\Omega)\cap H(\mathrm{div}^0;\Omega):\int_{\Gamma_j}\boldsymbol{v}\cdot\boldsymbol{n}_{\Gamma_j}\,dS=0\quad\text{for}\quad 1\le j\le n\}.\tag{2.2}$$

Remark 2.3 Note that $\mathscr{P}_N$ is a subspace of $\mathscr{D}_F$.

The following lemma is Theorem 3.17 in [1].

Lemma 2.4 *The operator* **curl** *is a surjection from* $H_0(\mathbf{curl}\,;\Omega)$ *onto the space*

$$\mathscr{D}_F^0 = \{\boldsymbol{v} \in H_0(\mathrm{div}^0;\Omega) : \int_{\Sigma_i} \boldsymbol{v}\cdot\boldsymbol{n}_{\Sigma_i}\,dS = 0 \quad \textit{for} \quad 1 \le i \le m\},$$

and the restriction of the operator **curl** *to* $\mathscr{P}_N$ *is an isomorphism between* $\mathscr{P}_N$ *and* $\mathscr{D}_F^0$.

2.2 Harmonic Functions $\varphi_1, \ldots, \varphi_n$

The harmonic functions $\varphi_1, \ldots, \varphi_n \in H^1(\Omega)$ are defined by

$$\int_\Omega \mathbf{grad}\,\varphi_j \cdot \mathbf{grad}\,\varphi\,dx = 0 \qquad \forall\,\varphi \in H_0^1(\Omega), \tag{2.3a}$$

$$\varphi_j\Big|_{\Gamma_0} = 0, \tag{2.3b}$$

and for $1 \le j, k \le n$,

$$\varphi_j\big|_{\Gamma_k} = \begin{cases} 1 & \text{if}\, k = j \\ 0 & \text{if}\, k \ne j \end{cases}. \tag{2.3c}$$

Remark 2.5 The harmonic functions $\varphi_1, \ldots, \varphi_n$ belong to the Sobolev space $H^{1+\alpha_{\mathrm{Dir}}}(\Omega)$ for some $\alpha_{\mathrm{Dir}} \in (1/2, 1]$ by the elliptic regularity theory for polyhedral domains in [24, 32], where α_{Dir} depends on the conic angles at the vertices of Ω and the dihedral angles of the edges of Ω. We can take α_{Dir} to be 1 if Ω is convex.

Lemma 2.6 *Given any* $\boldsymbol{v} \in H(\mathrm{div}^0;\Omega)$*, there exist unique constants* $c_1, \ldots, c_n$ *such that* $\boldsymbol{w} = \boldsymbol{v} - \sum_{j=1}^n c_j\,\mathbf{grad}\,\varphi_j$ *satisfies*

$$\int_{\Gamma_j} \boldsymbol{w}\cdot\boldsymbol{n}_{\Gamma_j}\,dS = 0 \quad \textit{for} \quad 1 \le j \le n. \tag{2.4}$$

Proof It suffices to show that the $n \times n$ matrix

$$\mathbf{M} = \begin{bmatrix} \int_{\Gamma_1} \mathbf{grad}\,\varphi_1 \cdot \boldsymbol{n}_1 dS & \ldots & \int_{\Gamma_1} \mathbf{grad}\,\varphi_n \cdot \boldsymbol{n}_1 dS \\ \vdots & \ddots & \vdots \\ \int_{\Gamma_n} \mathbf{grad}\,\varphi_1 \cdot \boldsymbol{n}_n dS & \ldots & \int_{\Gamma_n} \mathbf{grad}\,\varphi_n \cdot \boldsymbol{n}_n dS \end{bmatrix}$$

is nonsingular.

Let $[a_1, \ldots, a_n]^t$ belong to the null space of $\mathbf{M}$ and $\varphi = \sum_{k=1}^n a_k\varphi_k$. Then

$$\int_{\Gamma_j} \mathbf{grad}\,\varphi\cdot\boldsymbol{n}_j dS = \sum_{k=1}^n \Big(\int_{\Gamma_j} \mathbf{grad}\,\varphi_k \cdot \boldsymbol{n}_j dS\Big) a_k = 0 \qquad \text{for} \quad 1 \le j \le n$$

and we have

$$\int_\Omega \mathbf{grad}\,\varphi\cdot\mathbf{grad}\,\varphi\,dx = \int_{\partial\Omega} (\mathbf{grad}\,\varphi\cdot\boldsymbol{n})\varphi\,dS = \sum_{j=1}^n a_j \int_{\Gamma_j} \mathbf{grad}\,\varphi\cdot\boldsymbol{n}_{\Gamma_j} dS = 0$$

by (2.3a)–(2.3c) and integration by parts.

Consequently φ is a constant that must be zero because $\varphi = 0$ on Γ_0, and hence $a_1 = \cdots = a_n = 0$ because $\varphi_1, \ldots, \varphi_n$ are linearly independent by (2.3c). □

Remark 2.7 The vector fields $\mathbf{grad}\, \varphi_j$ belongs to $\mathbb{E}$ by construction. Furthermore, in view of Lemma 2.1, (2.3b) and (2.3c) we have

$$(\mathbf{curl}\, \boldsymbol{\psi}, \mathbf{grad}\, \varphi_j)_{L^2(\Omega)} = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathscr{P}_T.$$

Remark 2.8 The constants $c_1, \ldots, c_n$ in Lemma 2.6 are bounded by $\max_{1\le j\le n} \left| \int_{\Gamma_j} \boldsymbol{v}\cdot\boldsymbol{n}_j dS \right|$ and hence by $\|\boldsymbol{v}\|_{H(\mathrm{div}\,;\Omega)} = \|\boldsymbol{v}\|_{L^2(\Omega)}$.

2.3 Hodge Decomposition for $H(\mathrm{div}^0; \Omega)$

Let $\boldsymbol{v}$ belong to $H(\mathrm{div}^0; \Omega)$. It follows from Lemma 2.6 that there exist unique constants $c_1, \ldots, c_n$ such that (2.4) is satisfied, where $\boldsymbol{w} = \boldsymbol{v} - \sum_{j=1}^n c_j \mathbf{grad}\, \varphi_j$ belongs to $H(\mathrm{div}^0; \Omega)$ by (2.3a). We can then apply Lemma 2.1 to conclude that there exists a unique $\boldsymbol{\psi} \in \mathscr{P}_T$ such that

$$\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{\psi},$$

or equivalently

$$\boldsymbol{v} = \mathbf{curl}\, \boldsymbol{\psi} + \sum_{j=1}^n c_j \mathbf{grad}\, \varphi_j, \tag{2.5}$$

which is the Hodge decomposition of $\boldsymbol{v}$.

3 Reduction of the Quad-Curl Problem

Let $\boldsymbol{u} \in \mathbb{E}$ be the solution of (1.1) and

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\phi} + \sum_{j=1}^n \tau_j \mathbf{grad}\, \varphi_j \tag{3.1}$$

be the Hodge decomposition of $\boldsymbol{u}$. Our goal is to find $\boldsymbol{u}$ by finding $\boldsymbol{\phi}$ and $\tau_1, \ldots, \tau_n$.

Remark 3.1 It follows from (2.3b), (2.3c) and (3.1) that $\boldsymbol{n} \times \mathbf{curl}\, \boldsymbol{\phi} = 0$ on $\partial\Omega$, which will be posed as a natural boundary condition in Sect. 3.1.

In the case where $n \ge 1$, the boundary of Ω is not connected and $\gamma > 0$ by assumption. In view of Remark 2.7 and (3.1), we can take $\boldsymbol{v} = \mathbf{grad}\, \varphi_k$ in (1.1) to obtain

$$\sum_{j=1}^n \tau_j (\mathbf{grad}\, \varphi_j, \mathbf{grad}\, \varphi_k)_{L^2(\Omega)} = (\boldsymbol{u}, \mathbf{grad}\, \varphi_k)_{L^2(\Omega)} = \frac{1}{\gamma}(\boldsymbol{f}, \mathbf{grad}\, \varphi_k)_{L^2(\Omega)} \tag{3.2}$$

for $1 \le k \le n$. Therefore the coefficients $\tau_1, \ldots, \tau_n$ are determined by the system (3.2), which is symmetric positive definite by (2.3b).

Below we will show that $\boldsymbol{\phi} \in \mathscr{P}_T$ is determined by several second order saddle point problems. The following observation is useful.

Lemma 3.2 *The operator* **curl** *maps* $\mathbb{E}$ *into the space* $\mathscr{S}$ *defined by*

$$\mathscr{S} = \left\{ \boldsymbol{\eta} \in [H_0^1(\Omega)]^3 \cap H(\mathrm{div}^0; \Omega) : \int_{\Sigma_i} \boldsymbol{\eta} \cdot \boldsymbol{n}_{\Sigma_i} dS = 0 \quad \textit{for} \quad 1 \le i \le m \right\}, \quad (3.3)$$

which is the divergence-free subspace of

$$[H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}} = \left\{ \boldsymbol{\eta} \in [H_0^1(\Omega)]^3 : \int_{\Sigma_i} \boldsymbol{\eta} \cdot \boldsymbol{n}_{\Sigma_i} dS = 0 \quad \textit{for} \quad 1 \le i \le m \right\}. \quad (3.4)$$

Proof This is a direct consequence of Lemma 2.4 since $\mathbb{E}$ is a subspace of $H_0(\mathbf{curl}\,; \Omega) \cap H(\mathrm{div}^0; \Omega)$. □

Remark 3.3 Note that $\mathscr{S}$ is a subspace of $\mathscr{D}_F^0$.

Remark 3.4 $[H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}}$ and $\mathscr{S}$ are Hilbert spaces under the inner product $(\cdot, \cdot)_{H^1(\Omega)}$. In the case where Ω is simply connected, $[H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}} = [H_0^1(\Omega)]^3$ and $\mathscr{S} = [H_0^1(\Omega)]^3 \cap H(\mathrm{div}^0; \Omega)$.

3.1 A Second Order Saddle Point Problem for $\boldsymbol{\phi}$

Let $\boldsymbol{\xi} = \mathbf{curl}\, \boldsymbol{u} \in H_0^1(\Omega)$. It follows from $\boldsymbol{n} \times \boldsymbol{u} = 0$ on $\partial\Omega$, Remark 2.7, (3.1) and integration by parts that, for any $\boldsymbol{\psi} \in \mathscr{P}_T$, we have

$$\begin{aligned}(\mathbf{curl}\, \boldsymbol{\phi}, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)} &= (\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)} - \sum_{j=1}^{n} \tau_j (\mathbf{grad}\, \varphi_i, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)} \\ &= (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{\psi})_{L^2(\Omega)} \\ &= (\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)}. \end{aligned} \quad (3.5)$$

In view of Remark 2.2, $\boldsymbol{\phi} \in \mathscr{P}_T$ is the unique solution of the well-posed problem (3.5) and we have

$$\|\boldsymbol{\phi}\|_{H(\mathbf{curl}\,;\Omega)} \le C \|\boldsymbol{\xi}\|_{L^2(\Omega)}. \quad (3.6)$$

The integral constraints in the definition of $\mathscr{P}_T$ in (2.1) are inconvenient for the numerical solution of (3.5). They can be removed through the reformulation of (3.5) as a saddle point problem, for which we will need the subspace Θ of $H^1(\Omega^\circ)$, where $\Omega^\circ = \Omega \setminus \bigcup_{i=1}^m \Sigma_i$.

For any $v \in H^1(\Omega^\circ)$, the jump $[\![v]\!]_{\Sigma_i}$ of v across the cut Σ_i is well-defined by the trace theorem, and

$$\Theta = \{v \in H^1(\Omega^\circ) : [\![v]\!]_{\Sigma_i} = \text{constant for } 1 \le i \le m\} \quad (3.7)$$

is a closed subspace of $H^1(\Omega^\circ)$. We will denote by $\widetilde{\mathbf{grad}}\, v$ the function in $L^2(\Omega)$ that agrees with $\mathbf{grad}\, v$ in Ω°. The subspace $\Theta \cap L_0^2(\Omega)$ of Θ is denoted by Θ^0.

The following result is a simple consequence of integration by parts.

Lemma 3.5 *A vector field* $\boldsymbol{v} \in [L^2(\Omega)]^3$ *satisfies*

$$(\boldsymbol{v}, \widetilde{\mathbf{grad}}\, \theta)_{L^2(\Omega)} = 0 \qquad \forall \theta \in \Theta^0$$

if and only if $\boldsymbol{v} \in H_0(\mathrm{div}^0; \Omega)$ *and*

$$\int_{\Sigma_i} \boldsymbol{v} \cdot \boldsymbol{n}_{\Sigma_i} dS = 0 \quad \textit{for } 1 \le i \le m. \quad (3.8)$$

Remark 3.6 Note that $v \in H^1(\Omega^\circ)$ belongs to Θ if and only if $\widetilde{\mathbf{grad}}\, v \in H(\mathbf{curl}^0; \Omega)$ (cf. [1, Lemma 3.11]).

Remark 3.7 In the case where Ω is simply connected, we have $\Theta = H^1(\Omega)$ and $\widetilde{\mathbf{grad}}\, v = \mathbf{grad}\, v$ for $v \in \Theta$.

The saddle point problem for (3.5) is to find $(\boldsymbol{\phi}, \theta) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \theta) = (\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in H(\mathbf{curl}\,; \Omega), \tag{3.9a}$$

$$b(\boldsymbol{\phi}, \mu) = 0 \qquad \forall\, \mu \in \Theta^0, \tag{3.9b}$$

where

$$a(\boldsymbol{\eta}, \boldsymbol{\psi}) = (\mathbf{curl}\, \boldsymbol{\eta}, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)}, \tag{3.10}$$

$$b(\boldsymbol{\phi}, \mu) = (\boldsymbol{\phi}, \widetilde{\mathbf{grad}}\, \mu)_{L^2(\Omega)}. \tag{3.11}$$

We will establish the well-posedness of (3.9) by the theory of saddle point problems in [3, 14]. The inf-sup condition is satisfied because

$$\sup_{\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega)} \frac{b(\boldsymbol{\psi}, \mu)}{\|\boldsymbol{\psi}\|_{H(\mathbf{curl}\,;\Omega)}} \geq \frac{(\widetilde{\mathbf{grad}}\, \mu, \widetilde{\mathbf{grad}}\, \mu)}{\|\widetilde{\mathbf{grad}}\, \mu\|_{H(\mathbf{curl}\,;\Omega)}} = \|\mathbf{grad}\, \mu\|_{L^2(\Omega^\circ)} \qquad \forall\, \mu \in \Theta^0 \tag{3.12}$$

by Remark 3.6. Moreover it follows from Lemma 3.5 that

$$\operatorname{Ker} b = \{\boldsymbol{v} \in H(\mathbf{curl}\,; \Omega) : (\boldsymbol{v}, \widetilde{\mathbf{grad}}\, \mu)_{L^2(\Omega)} = 0 \quad \forall \mu \in \Theta^0\}$$

is precisely $\mathscr{P}_T$, and hence

$$a(\boldsymbol{\psi}, \boldsymbol{\psi}) = (\mathbf{curl}\, \boldsymbol{\psi}, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)} \geq C\|\boldsymbol{\psi}\|^2_{H(\mathbf{curl}\,;\Omega)} \tag{3.13}$$

by Remark 2.2. The well-posedness of the saddle point problem follows from (3.12) and (3.13).

Next we show that the unique solution of (3.9) is given by $(\boldsymbol{\phi}, 0)$ where $\boldsymbol{\phi}$ is the solution of (3.5). Indeed by taking $\boldsymbol{\psi} = \widetilde{\mathbf{grad}}\, \theta$ in (3.9a), we see that

$$(\widetilde{\mathbf{grad}}\, \theta, \widetilde{\mathbf{grad}}\, \theta)_{L^2(\Omega)} = (\boldsymbol{\xi}, \widetilde{\mathbf{grad}}\, \theta)_{L^2(\Omega)} = (\boldsymbol{\xi}, \mathbf{grad}\, \theta)_{L_2(\Omega^\circ)} = 0$$

by Lemma 3.2 and Lemma 3.5, and hence $\theta \in \Theta^0$ is the constant 0. Furthermore $\boldsymbol{\phi}$ belongs to $\operatorname{Ker} b = \mathscr{P}_T$ by (3.9b) and therefore (3.9a) implies that it is the solution of (3.5).

Remark 3.8 Since $\theta = 0$, it follows from (3.9a) that $\boldsymbol{\phi} \in \mathscr{P}_T \subset H(\mathbf{curl}\,; \Omega) \cap H_0(\mathrm{div}\,; \Omega)$ satisfies the elliptic Maxwell boundary value problem

$$(\mathbf{curl}\, \boldsymbol{\phi}, \mathbf{curl}\, \boldsymbol{\psi})_{L^2(\Omega)} + (\mathrm{div}\, \boldsymbol{\phi}, \mathrm{div}\, \boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)}$$
$$\forall\, \boldsymbol{\psi} \in H(\mathbf{curl}\,; \Omega) \cap H_0(\mathrm{div}\,; \Omega)$$

that has been analyzed in [23].

Remark 3.9 The saddle point problem was investigated in [31] for domains with trivial topology.

We still need to determine the vector field $\boldsymbol{\xi} = \mathbf{curl}\, \boldsymbol{u}$ that appears on the right-hand side of (3.9). Note that $\boldsymbol{\xi}$ belongs to $\mathscr{S}$ by Lemma 3.2.

3.2 Second Order Saddle Point Problems for $\boldsymbol{\xi}$

Let $\boldsymbol{\eta}$ belong to the space $\mathscr{S}$ defined in (3.3). According to Lemma 2.4, there exists a unique vector potential $\boldsymbol{\psi} \in \mathscr{P}_N \subset H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}^0;\Omega)$ such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{\eta}, \tag{3.14}$$

which implies in particular that $\boldsymbol{\psi} \in \mathbb{E}$.

It then follows from (1.1) and (3.14) that

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{\xi}, \mathbf{curl}\,\boldsymbol{\eta})_{L^2(\Omega)} &= (\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}), \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{\psi}))_{L^2(\Omega)} \\ &= (Q\boldsymbol{f}, \boldsymbol{\psi}) - \gamma(\boldsymbol{u}, \boldsymbol{\psi})_{L^2(\Omega)} - \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)},\end{aligned} \tag{3.15}$$

where Q is the orthogonal projection from $[L^2(\Omega)]^3$ onto $H(\mathrm{div}^0;\Omega)$.

On the other hand, by letting $\boldsymbol{v}$ in (1.1) be the gradient of the harmonic functions $\varphi_1, \ldots, \varphi_n$ from Sect. 2.2, we see that

$$0 = \int_\Omega (Q\boldsymbol{f} - \gamma\boldsymbol{u})\mathbf{grad}\,\varphi_j dx = \int_{\Gamma_j} (Q\boldsymbol{f} - \gamma\boldsymbol{u}) \cdot \boldsymbol{n}_{\Gamma_j} dS \quad \text{for} \quad 1 \le j \le n.$$

Therefore $Q\boldsymbol{f} - \gamma\boldsymbol{u}$ belongs to $\mathscr{D}_F$ and there exists a unique $\boldsymbol{\omega} \in \mathscr{P}_T$ such that

$$\mathbf{curl}\,\boldsymbol{\omega} = Q\boldsymbol{f} - \gamma\boldsymbol{u} \tag{3.16}$$

by Lemma 2.1, which together with (3.14) and (3.15) implies

$$(\mathbf{curl}\,\boldsymbol{\xi}, \mathbf{curl}\,\boldsymbol{\eta})_{L^2(\Omega)} + \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)} = (\mathbf{curl}\,\boldsymbol{\omega}, \boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{\omega}, \boldsymbol{\eta})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta} \in \mathscr{S}. \tag{3.17}$$

Since we have (cf. Lemma A.1)

$$(\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{w})_{L^2(\Omega)} = (\mathbf{Grad}\,\boldsymbol{v}, \mathbf{Grad}\,\boldsymbol{w})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in [H_0^1(\Omega)]^3 \cap H(\mathrm{div}^0;\Omega),$$

the problem (3.17) can be rewritten as

$$(\mathbf{Grad}\,\boldsymbol{\xi}, \mathbf{Grad}\,\boldsymbol{\eta})_{L^2(\Omega)} + \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)} = (\boldsymbol{\omega}, \boldsymbol{\eta})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta} \in \mathscr{S}. \tag{3.18}$$

3.2.1 The Case $\gamma = 0$

In this case (3.16) and (3.17) are decoupled. We can first determine $\boldsymbol{\omega} \in \mathscr{P}_T$ by

$$(\mathbf{curl}\,\boldsymbol{\omega}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (Q\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in \mathscr{P}_T, \tag{3.19}$$

which is a well-posed problem by Remark 2.2, and then $\boldsymbol{\xi} \in \mathscr{S}$ is determined by (3.18), which is also a well-posed problem on the closed subspace $\mathscr{S}$ of $[H_0^1(\Omega)]^3$ because of the Poincaré inequality.

Remark 3.10 It follows from Remark 2.2 and (3.19) that

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl}\,;\Omega)} \le C\|\mathbf{curl}\,\boldsymbol{\omega}\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{3.20}$$

We can remove the inconvenient constraints in the definition of $\mathscr{P}_T$ and $\mathscr{S}$ by reformulating (3.18) and (3.19) as saddle point problems.

The saddle point problem for (3.19) is to find $(\boldsymbol{\omega}, \theta) \in H(\mathbf{curl}\,;\Omega) \times \Theta^0$ such that

$$a(\boldsymbol{\omega}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \theta) = (\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega), \tag{3.21a}$$

$$b(\boldsymbol{\omega}, \mu) = 0 \qquad \forall\, \mu \in \Theta^0, \tag{3.21b}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by (3.10) and (3.11).

We already saw that (3.21) is a well-posed problem in Sect. 3.1. Let $(\boldsymbol{\omega},\theta)$ be the solution of (3.21). By taking $\boldsymbol{\psi} = \widetilde{\mathbf{grad}}\,\theta$ in (3.21a) we see that $(\widetilde{\mathbf{grad}}\,\theta, \widetilde{\mathbf{grad}}\,\theta)_{L^2(\Omega)} = 0$ by Remark 3.6 and hence $\theta = 0$. Then we observe that Lemma 3.5 and (3.21b) imply $\boldsymbol{\omega} \in \operatorname{Ker} b = \mathscr{P}_T$ and finally $\boldsymbol{\omega}$ satisfies (3.19) because of (3.21a).

Remark 3.11 Since $\theta = 0$, it follows from (3.21a) that $\boldsymbol{\omega} \in \mathscr{P}_T \subset H(\mathbf{curl}\,;\Omega)\cap H_0(\operatorname{div}^0;\Omega)$ satisfies the elliptic Maxwell boundary value problem

$$
\begin{aligned}
&(\mathbf{curl}\,\boldsymbol{\omega}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} + (\operatorname{div}\boldsymbol{\omega}, \operatorname{div}\boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)}\\
&\quad \forall\,\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega) \cap H_0(\operatorname{div}\,;\Omega)
\end{aligned}
$$

that has been analyzed in [23].

The saddle point problem for (3.18) is to find $(\boldsymbol{\xi}, p) \in [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$ such that

$$
g(\boldsymbol{\xi},\boldsymbol{\eta}) + \beta(\boldsymbol{\xi},\boldsymbol{\eta})_{L^2(\Omega)} + c(\boldsymbol{\eta}, p) = (\boldsymbol{\omega},\boldsymbol{\eta})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\eta} \in [H_0^1(\Omega)]^3_\Sigma, \tag{3.22a}
$$

$$
c(\boldsymbol{\xi}, q) = 0 \qquad \forall\, q \in L_0^2(\Omega), \tag{3.22b}
$$

where

$$
g(\boldsymbol{\tau},\boldsymbol{\eta}) = (\mathbf{Grad}\,\boldsymbol{\tau}, \mathbf{Grad}\,\boldsymbol{\eta})_{L^2(\Omega)}, \tag{3.23}
$$

$$
c(\boldsymbol{\eta}, q) = -(\operatorname{div}\boldsymbol{\eta}, q)_{L^2(\Omega)}. \tag{3.24}
$$

This saddle point problem is well-posed because (cf. Appendix B)

$$
\sup_{\boldsymbol{\eta}\in[H_0^1(\Omega)]^3_\Sigma} \frac{c(\boldsymbol{\eta}, q)}{|\boldsymbol{\eta}|_{H^1(\Omega)}} = \sup_{\boldsymbol{\eta}\in[H_0^1(\Omega)]^3_\Sigma} \frac{(\operatorname{div}\boldsymbol{\eta}, q)_{L^2(\Omega)}}{|\boldsymbol{\eta}|_{H^1(\Omega)}} \ge C\|q\|_{L^2(\Omega)} \quad \forall\, q \in L_0^2(\Omega), \tag{3.25}
$$

and

$$
g(\boldsymbol{\eta},\boldsymbol{\eta}) + \beta(\boldsymbol{\eta},\boldsymbol{\eta})_{L^2(\Omega)} \ge |\boldsymbol{\eta}|^2_{H^1(\Omega)} \quad \forall\,\boldsymbol{\eta} \in \operatorname{Ker} c = \mathscr{S}.
$$

Let $(\boldsymbol{\xi}, p)$ be the solution of (3.22). It follows from (3.22b) that $\boldsymbol{\xi}$ belongs to $\operatorname{Ker} c = \mathscr{S}$ and then (3.22a) implies $\boldsymbol{\xi}$ is the solution of (3.18).

Remark 3.12 It follows from the well-posedness of (3.22) that

$$
\|\boldsymbol{\xi}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)}.
$$

Remark 3.13 The saddle point problem (3.22) is the standard Stokes problem for incompressible flows if $\beta = 0$ and Ω is simply connected. The regularity of Stokes problem in [25] can be extended to (3.22) for a general Ω that is not necessarily simply connected (cf. Appendix C), i.e., we have

$$
\|\boldsymbol{\xi}\|_{H^{1+\alpha_S}(\Omega)} + \|p\|_{H^{\alpha_S}(\Omega^\circ)} \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)} \tag{3.26}
$$

for some $\alpha_S \in (\frac{1}{2}, 1]$ determined by the geometry of Ω, and we can take α_S to be 1 if Ω is convex. (The last inequality in (3.26) follows from (3.20).)

3.2.2 The Case $\gamma > 0$

In this case the problem (3.19) is replaced by

$$(\mathbf{curl}\,\boldsymbol{\omega}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} + \gamma(\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} \quad \forall\,\boldsymbol{\psi} \in \mathscr{P}_T. \tag{3.27}$$

The coupled problems (3.18) and (3.27) can be reformulated as the following saddle point problem without the inconvenient constraints in the definition of $\mathscr{S}$ and $\mathscr{P}_T$.

Find $(\boldsymbol{\zeta}, \theta, \boldsymbol{\xi}, p) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$ such that

$$a(\boldsymbol{\zeta}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \theta) + \gamma^{\frac{1}{2}}(\boldsymbol{\psi}, \boldsymbol{\xi})_{L^2(\Omega)} = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)}, \tag{3.28a}$$

$$b(\boldsymbol{\zeta}, \mu) = 0, \tag{3.28b}$$

$$\gamma^{\frac{1}{2}}(\boldsymbol{\zeta}, \boldsymbol{\eta})_{L^2(\Omega)} - g(\boldsymbol{\xi}, \boldsymbol{\eta}) - \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)} - c(\boldsymbol{\eta}, p) = 0, \tag{3.28c}$$

$$-c(\boldsymbol{\xi}, q) = 0, \tag{3.28d}$$

for all $(\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$, where $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are defined by (3.10), (3.11), (3.23) and (3.24) respectively.

Remark 3.14 Using Lemma 3.5, Remark 3.6 and the arguments in Sect. 3.2.1, it is straightforward to check that if $(\boldsymbol{\zeta}, \theta, \boldsymbol{\xi}, p) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$ satisfies (3.28), then $\theta = 0$ by (3.28a), $\boldsymbol{\omega} = \gamma^{\frac{1}{2}}\boldsymbol{\zeta}$ belongs to $\mathscr{P}_T$ by (3.28b), $\boldsymbol{\xi}$ belongs to $\mathscr{S}$ by (3.28d), and they satisfy (3.18) (resp., (3.27)) by (3.28c) (resp., (3.28a)). It only remains to show that the saddle point problem (3.28) is well-posed.

In view of (3.12) and (3.25), it suffices to observe that the bilinear form

$$\begin{aligned} B\big((\boldsymbol{\psi}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \boldsymbol{\rho})\big) &= a(\boldsymbol{\psi}, \boldsymbol{\tau}) + \gamma^{\frac{1}{2}}(\boldsymbol{\eta}, \boldsymbol{\tau})_{L^2(\Omega)} + \gamma^{\frac{1}{2}}(\boldsymbol{\psi}, \boldsymbol{\rho})_{L^2(\Omega)} \\ &\quad - g(\boldsymbol{\eta}, \boldsymbol{\rho}) - \beta(\boldsymbol{\eta}, \boldsymbol{\rho})_{L^2(\Omega)} \end{aligned} \tag{3.29}$$

induces an isomorphism between $\mathscr{P}_T \times \mathscr{S}$ and $(\mathscr{P}_T \times \mathscr{S})'$ because of the relation

$$\begin{aligned} B\big((\boldsymbol{\psi}, \boldsymbol{\eta}), (\boldsymbol{\psi}, -\boldsymbol{\eta})\big) &= a(\boldsymbol{\psi}, \boldsymbol{\psi}) + g(\boldsymbol{\eta}, \boldsymbol{\eta}) + \beta(\boldsymbol{\eta}, \boldsymbol{\eta})_{L^2(\Omega)} \\ &\geq \|\mathbf{curl}\,\boldsymbol{\psi}\|^2_{L^2(\Omega)} + \|\mathbf{Grad}\,\boldsymbol{\eta}\|^2_{L^2(\Omega)} \end{aligned} \tag{3.30}$$

and Remark 2.2. The well-posedness of (3.28) then follows from Theorem 1.1 in [14].

Remark 3.15 The well-posedness of (3.28) implies that

$$\|\boldsymbol{\zeta}\|_{H(\mathbf{curl}\,;\Omega)} + \|\boldsymbol{\xi}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{3.31}$$

It then follows from Remark 3.13, (3.28c), (3.28d) and (3.31) that

$$\|\boldsymbol{\xi}\|_{H^{1+\alpha_S}(\Omega)} + \|p\|_{H^{\alpha_S}(\Omega^\circ)} \leq C\|\gamma^{\frac{1}{2}}\boldsymbol{\zeta}\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)} \tag{3.32}$$

for some $\alpha_S \in (\frac{1}{2}, 1]$, where $\alpha_S = 1$ if Ω is convex.

Remark 3.16 We can write (3.28) concisely as

$$\mathcal{A}((\boldsymbol{\zeta}, \theta, \boldsymbol{\xi}, p), (\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q)\big) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{[L^2(\Omega)]^3} \tag{3.33}$$

for all $(\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]_{\Sigma}^3 \times L_0^2(\Omega)$, where

$$\begin{aligned}\mathcal{A}\big((\boldsymbol{\zeta}, \theta, \boldsymbol{\xi}, p), (\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q)\big) &= a(\boldsymbol{\zeta}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \theta) + b(\boldsymbol{\zeta}, \mu) + \gamma^{\frac{1}{2}}(\boldsymbol{\psi}, \boldsymbol{\xi})_{L^2(\Omega)} \\ &\quad + \gamma^{\frac{1}{2}}(\boldsymbol{\zeta}, \boldsymbol{\eta})_{L^2(\Omega)} - g(\boldsymbol{\xi}, \boldsymbol{\eta}) - \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)} \\ &\quad - c(\boldsymbol{\eta}, p) - c(\boldsymbol{\xi}, q), \end{aligned} \tag{3.34}$$

and we have

$$\begin{aligned}\mathcal{A}\big((\boldsymbol{\zeta}, \theta, \boldsymbol{\xi}, p), (\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q)\big) &\le C\big(\|\boldsymbol{\zeta}\|_{H(\mathbf{curl}\,;\Omega)} + |\theta|_{H^1(\Omega^\circ)} + |\boldsymbol{\xi}|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)}\big) \\ &\quad \times \big(\|\boldsymbol{\psi}\|_{H(\mathbf{curl}\,;\Omega)} + |\mu|_{H^1(\Omega^\circ)} + |\boldsymbol{\eta}|_{H^1(\Omega)} + \|q\|_{L^2(\Omega)}\big). \end{aligned} \tag{3.35}$$

3.3 Summary of the Hodge Decomposition Approach

Given $\boldsymbol{f} \in L^2(\Omega)$, we determine $\boldsymbol{\xi}$ through (3.21) and (3.22) if $\gamma = 0$ and through (3.28) if $\gamma > 0$, and then we determine $\boldsymbol{\phi}$ through (3.9). The solution of (1.1) is given by (3.1), where $\varphi_1, \ldots, \varphi_n$ and $\tau_1, \ldots, \tau_n$ are determined by (2.3) and (3.2) respectively. We can find a numerical solution of (1.1) by solving each second order problem numerically.

The Hodge decomposition approach also yields information on the regularity of the solution $\boldsymbol{u}$ of (1.1). If $\partial\Omega$ is smooth, then we can apply classical regularity results for elliptic boundary value problems. This means the harmonic functions $\varphi_1, \ldots, \varphi_n$ are smooth, and $\boldsymbol{\omega} \in H(\mathbf{curl}\,; \Omega) \cap H_0(\mathrm{div}^0; \Omega) \subset [H^1(\Omega)]^3$ for $\boldsymbol{f} \in [L^2(\Omega)]^3$. In the case where $\gamma = 0$, we then have $\boldsymbol{\xi} \in [H^3(\Omega)]^3$ by the regularity of the Stokes problem (cf. Remark 3.13), which implies $\boldsymbol{\phi} \in [H^5(\Omega)]^3$ by elliptic regularity (cf. Remark 3.8). Consequently we have $\boldsymbol{u} \in [H^4(\Omega)]^3$ by (3.1). If $\boldsymbol{f} \in [H^1(\Omega)]^3$, then $\boldsymbol{\omega} \in [H^2(\Omega)]^3$ (cf. Remark 3.11), $\boldsymbol{\xi} \in [H^4(\Omega)]^3$, $\boldsymbol{\phi} \in [H^6(\Omega)]^3$ and $\boldsymbol{u} \in [H^5(\Omega)]^3$. These observations are also valid when γ is positive.

On the other hand, if Ω is a polyhedral domain, then the regularity of $\boldsymbol{\phi}$ from the elliptic Maxwell boundary value problem in Remark 3.8 is limited. Indeed for a convex polyhedron we have in general $\boldsymbol{\phi} \in [H^2(\Omega)]^3$ (cf. [23]) and hence $\boldsymbol{u} \in [H^1(\Omega)]^3$, which is the same regularity satisfied by any vector fields in $\mathbb{E}$ when Ω is convex. Consequently we only considered lower order finite element methods in Sect. 4.

4 Finite Element Methods

We will design finite element methods for the second order subproblems in the Hodge decomposition approach that are mentioned at the beginning of Sect. 3.3.

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω such that

$$\text{each cut } \Sigma_i \, (1 \le i \le m) \text{ is the union of the faces of } \mathcal{T}_h \text{ and every tetrahedron in } \mathcal{T}_h \text{ has a vertex interior to } \Omega^\circ. \tag{4.1}$$

4.1 Finite Element Method for (2.3)

Let $\Phi_h \subset H^1(\Omega)$ be the P_1 Lagrange finite element space associated with $\mathcal{T}_h$ and $\mathring{\Phi}_h = \Phi_h \cap H_0^1(\Omega)$. The finite element method for (2.3) is to find $\varphi_1^h, \ldots, \varphi_n^h \in \Phi_h$ such that

$$\int_{\Omega} \mathbf{grad}\, \varphi_j^h \cdot \mathbf{grad}\, \varphi^h \, dx = 0 \quad \forall \varphi^h \in \mathring{\Phi}_h, \tag{4.2a}$$

$$\varphi_j^h \Big|_{\Gamma_0} = 0, \tag{4.2b}$$

and for $1 \le j, k \le n$,

$$\varphi_j^h|_{\Gamma_k} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \ne j \end{cases}. \tag{4.2c}$$

The approximations $\tau_1^h, \dots, \tau_n^h$ are then determined by the SPD system

$$\sum_{j=1}^{n} \tau_j^h (\mathbf{grad}\, \varphi_j^h, \mathbf{grad}\, \varphi_k^h)_{L^2(\Omega)} = \frac{1}{\gamma}(\boldsymbol{f}, \mathbf{grad}\, \varphi_k^h)_{L^2(\Omega)} \quad \text{for} \quad 1 \le k \le n. \tag{4.3}$$

4.2 Finite Element Methods for the Approximation of ξ

First we introduce the following finite element spaces:

- $\boldsymbol{V}_h \subset H(\mathbf{curl}\,; \Omega)$ is the lowest order edge element space associated with $\mathcal{T}_h$ (cf. [35]).
- $\Theta_h \subset \Theta$ is the P_1 Lagrange finite element subspace associated with $\mathcal{T}_h$ and $\Theta_h^0 = \Theta_h \cap L_0^2(\Omega)$.
- $\boldsymbol{W}_h \subset [H_0^1(\Omega)]^3$ is the P_2 Lagrange vector finite element space associated with $\mathcal{T}_h$ and $\boldsymbol{W}_h^{\boldsymbol{\Sigma}} = \boldsymbol{W}_h \cap [H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}}$.
- $Q_h \subset H^1(\Omega^\circ)$ is the P_1 Lagrange finite element space associated with $\mathcal{T}_h$ and $Q_h^0 = Q_h \cap L_0^2(\Omega)$.

Remark 4.1 In the case where Ω is simply connected, $\Theta_h = Q_h \subset H^1(\Omega)$ is the standard P_1 Lagrange finite element space associated with $\mathcal{T}_h$. In the case where Ω is not simply connected, the functions in Θ_h can have a constant jump across the cuts $\Sigma_1, \dots, \Sigma_m$ and the dimension of Θ_h is the dimension of the standard P_1 finite element space for $H^1(\Omega)$ plus m, while the functions in Q_h can be discontinuous across the cuts and the dimension of Q_h is the dimension of the standard P_1 finite element space for $H^1(\Omega)$ plus the total number of vertices of $\mathcal{T}_h$ that belong to the cuts $\Sigma_1, \dots, \Sigma_m$.

Remark 4.2 The space Θ_h^0 is a subspace of Q_h^0 (cf. Remark 3.6).

Remark 4.3 Note that $\widetilde{\mathbf{grad}}\, \theta_h$ belongs to $\boldsymbol{V}_h$ for any $\theta_h \in \Theta_h^0$.

The following approximation results for $\boldsymbol{V}_h$ (cf. Theorem 5.41 and Remark 5.42 in [33]) are useful for the error analysis in Sect. 5.

Lemma 4.4 *Let* $\boldsymbol{v} \in H^{s_1}(\Omega)$ *and* $\mathbf{curl}\, \boldsymbol{v} \in H^{s_2}(\Omega)$ *for* $s_1, s_2 \in (\frac{1}{2}, 1]$. *Then we have*

$$\inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{H(\mathbf{curl}\,;\Omega)} \le C h^{\min(s_1, s_2)} \big(\|\boldsymbol{v}\|_{H^{s_1}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{H^{s_2}(\Omega)} \big),$$

$$\inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\mathbf{curl}\,(\boldsymbol{v} - \boldsymbol{v}_h)\|_{L^2(\Omega)} \le C h^{s_2} \|\mathbf{curl}\, \boldsymbol{v}\|_{H^{s_2}(\Omega)}.$$

4.2.1 The Case $\gamma = 0$

The finite element method for (3.21) is to find $(\boldsymbol{\omega}_h, \theta_h) \in \boldsymbol{V}_h \times \Theta_h^0$ such that

$$a(\boldsymbol{\omega}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, \theta_h) = (\boldsymbol{f}, \mathbf{curl}\, \boldsymbol{\psi}_h)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{V}_h, \tag{4.4a}$$

$$b(\boldsymbol{\omega}_h, \mu_h) = 0 \qquad \forall\, \mu_h \in \Theta_h^0. \tag{4.4b}$$

Remark 4.5 The stability of the bilinear form for the saddle point problem (4.4) has been established in [1, Section 4.1, Page 854]. It was shown there that

$$\sup_{\boldsymbol{\psi}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{\psi}_h, \mu_h)}{\|\boldsymbol{\psi}\|_{H(\mathbf{curl};\Omega)}} = \sup_{\boldsymbol{\psi}_h \in \boldsymbol{V}_h} \frac{(\boldsymbol{\psi}_h, \widetilde{\mathbf{grad}\, \mu_h})_{L^2(\Omega)}}{\|\boldsymbol{\psi}_h\|_{H(\mathbf{curl};\Omega)}} \geq |\mu_h|_{H^1(\Omega^\circ)} \qquad \forall\, \mu_h \in \Theta_h^0, \tag{4.5}$$

and there exists a positive constant C independent of h such that

$$a(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) = (\mathbf{curl}\, \boldsymbol{\psi}_h, \mathbf{curl}\, \boldsymbol{\psi}_h)_{L^2(\Omega)} \geq C\|\boldsymbol{\psi}_h\|^2_{H(\mathbf{curl};\Omega)} \qquad \forall \boldsymbol{\psi}_h \in \mathrm{Ker}_h b, \tag{4.6}$$

where $\mathrm{Ker}_h b = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : (\boldsymbol{v}_h, \widetilde{\mathbf{grad}\, \mu_h})_{L^2(\Omega)} = 0\ \forall\, \mu_h \in \Theta_h^0\}$.

Remark 4.6 In view of Remark 3.6 and Remark 4.3, it is easy to see that $\theta_h = 0$ for the solution of (4.4).

The discrete problem for (3.22) is then to find $(\boldsymbol{\xi}_h, p_h) \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}} \times Q_h^0$ such that

$$g(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h) + \beta(\boldsymbol{\xi}_h, \boldsymbol{\eta}_h)_{L^2(\Omega)} + c(\boldsymbol{\eta}_h, p_h) = (\boldsymbol{\omega}_h, \boldsymbol{\eta}_h)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta}_h \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}}, \tag{4.7a}$$

$$c(\boldsymbol{\xi}_h, q_h) = 0 \qquad \forall\, q_h \in Q_h^0. \tag{4.7b}$$

Remark 4.7 The bilinear form for the discrete saddle point problem (4.7) is stable because there exists a positive constant C independent of h such that

$$\sup_{\boldsymbol{\eta}_h \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}}} \frac{(\mathrm{div}\, \boldsymbol{\eta}_h, q_h)}{|\boldsymbol{\eta}_h|_{H^1(\Omega)}} \geq C\|q_h\|_{L^2(\Omega)} \qquad \forall\, q_h \in Q_h^0. \tag{4.8}$$

Under the assumption (4.1) on $\mathcal{T}_h$, the discrete inf-sup condition (4.8) can be established by the macro-element techniques in [6] if we ensure that the faces of $\mathcal{T}_h$ on the cuts Σ_i $(1 \leq i \leq m)$ only appear on the boundaries of the macro elements.

Remark 4.8 By introducing Lagrange multipliers to remove the constraints (1.4) and (3.8), we can use the unconstrained space Θ_h, $\boldsymbol{W}_h$ and Q_h in computing the solutions of (4.4) and (4.7).

Remark 4.9 In the case where Ω is simply connected and $\beta = 0$, the saddle point problem (4.7) is nothing but the Taylor-Hood finite element method (cf. [7, 40]) for the Stokes problem.

4.2.2 The Case $\gamma > 0$

The finite element method for (3.28) is to find $(\boldsymbol{\zeta}_h, \theta_h, \boldsymbol{\xi}_h, p_h) \in \boldsymbol{V}_h \times \Theta_h^0 \times \boldsymbol{W}_h^{\boldsymbol{\Sigma}} \times Q_h^0$ such that

$$\mathcal{A}\big((\boldsymbol{\zeta}_h, \theta_h, \boldsymbol{\xi}_h, p_h), (\boldsymbol{\psi}_h, \mu_h, \boldsymbol{\eta}_h, q_h)\big) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\, \boldsymbol{\psi}_h)_{L^2(\Omega)} \tag{4.9}$$

for all $(\boldsymbol{\psi}_h, \mu_h, \boldsymbol{\eta}_h, q_h) \in \boldsymbol{V}_h \times \Theta_h^0 \times \boldsymbol{W}_h^{\boldsymbol{\Sigma}} \times Q_h^0$, where the bilinear form $\mathcal{A}(\cdot, \cdot)$ is defined in (3.34).

Remark 4.10 The stability of the discrete saddle point problem (4.9) follows from (4.5), (4.6), (4.8) and the discrete analog of (3.30) for $\boldsymbol{\psi}_h \in \text{Ker}_h b$ and $\boldsymbol{\eta}_h \in \text{Ker}_h c$.

Remark 4.11 Using the equations

$$
\begin{aligned}
a(\boldsymbol{\zeta}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, \theta_h) + \gamma^{\frac{1}{2}}(\boldsymbol{\psi}_h, \boldsymbol{\xi}_h)_{L^2(\Omega)} = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\, \boldsymbol{\psi}_h)_{L^2(\Omega)} \qquad & \forall\, \boldsymbol{\psi}_h \in \boldsymbol{V}_h, \\
b(\boldsymbol{\zeta}_h, \mu_h) = 0 \qquad & \forall\, \mu_h \in \Theta_h^0, \\
-c(\boldsymbol{\xi}_h, q_h) = 0 \qquad & \forall\, q_h \in Q_h^0,
\end{aligned}
$$

that are part of (4.9), Remarks 3.6, 4.2, 4.3 and integration by parts, it is easy to check that $\theta_h = 0$ for the solution of (4.9).

Remark 4.12 The well-posedness of (4.9) implies that

$$
\|\boldsymbol{\zeta}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{\xi}_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{4.10}
$$

Remark 4.13 Again, by introducing Lagrange multipliers to remove the constraints in (1.4) and (3.8), we can use the unconstrained spaces Θ_h, $\boldsymbol{W}_h$ and Q_h in computing the solution of (4.9).

4.3 Finite Element Method for (3.9)

The finite element method for (3.9) is to find $(\boldsymbol{\phi}_h, \theta_h) \in \boldsymbol{V}_h \times \Theta_h^0$ such that

$$
a(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, \theta_h) = (\boldsymbol{\xi}_h, \boldsymbol{\psi}_h)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{V}_h, \tag{4.11a}
$$

$$
b(\boldsymbol{\phi}_h, \mu_h) = 0 \qquad \forall\, \mu_h \in \Theta_h^0, \tag{4.11b}
$$

where $\boldsymbol{\xi}_h \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}}$ is the approximation of $\boldsymbol{\xi}$ obtained in Sect. 4.2.

Remark 4.14 The bilinear form for (4.11) is stable since it is identical to the bilinear form for (4.4).

Remark 4.15 By introducing a Lagrange multiplier to remove the constraint (1.4), we can use the unconstrained space Θ_h in computing the solution of (4.11).

4.4 Final Output

The final output of the finite element method based on the Hodge decomposition is the approximation $\boldsymbol{u}_h$ of $\boldsymbol{u}$ given by

$$
\boldsymbol{u}_h = \mathbf{curl}\, \boldsymbol{\phi}_h + \sum_{j=1}^{n} \tau_j^h\, \mathbf{grad}\, \varphi_j^h. \tag{4.12}
$$

5 Convergence Analysis

We will analyze the finite element methods in Sect. 4 one by one.

5.1 Error Estimates for $\varphi_1^h, \ldots, \varphi_n^h$ and $\tau_1^h, \ldots, \tau_n^h$

It follows from (2.3a) and (4.2a) that

$$\begin{aligned}|\varphi_j - \varphi_j^h|^2_{H^1(\Omega)} &= \int_\Omega \mathbf{grad}\,(\varphi_j - \varphi_j^h)\cdot \mathbf{grad}\,(\varphi_j - \varphi_j^h)dx \\ &= \int_\Omega \mathbf{grad}\,(\varphi_j - \varphi_j^h)\cdot \mathbf{grad}\,(\varphi_j - \Pi_h\varphi)dx \qquad \forall\,\varphi^h \in \Phi_h,\end{aligned}$$

where Π_h is the nodal interpolation operator for Φ_h, and hence

$$|\varphi_j - \varphi_j^h|_{H^1(\Omega)} \le |\varphi_j - \Pi_h\varphi_j|_{H^1(\Omega)} \le Ch^{\alpha_{\mathrm{Dir}}} \tag{5.1}$$

by Remark 2.5 and a standard interpolation error estimate (cf. [12, 22]).

Since $\varphi_j - \varphi_j^h \in H_0^1(\Omega)$ for $1 \le j \le n$, we have

$$\begin{aligned}&(\mathbf{grad}\,\varphi_j, \mathbf{grad}\,\varphi_k)_{L^2(\Omega)} - (\mathbf{grad}\,\varphi_j^h, \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)} \\ &\quad= (\mathbf{grad}\,\varphi_j - \mathbf{grad}\,\varphi_j^h, \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)} + (\mathbf{grad}\,\varphi_j, \mathbf{grad}\,\varphi_k - \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)} \\ &\quad= (\mathbf{grad}\,\varphi_j - \mathbf{grad}\,\varphi_j^h, \mathbf{grad}\,\varphi_k^h - \mathbf{grad}\,\varphi_k)_{L^2(\Omega)}\end{aligned} \tag{5.2}$$

by (2.3a).

It follows from (5.1) and (5.2) that

$$|(\mathbf{grad}\,\varphi_j, \mathbf{grad}\,\varphi_k)_{L^2(\Omega)} - (\mathbf{grad}\,\varphi_j^h, \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)}| \le Ch^{2\alpha_{\mathrm{Dir}}}, \tag{5.3}$$

i.e., the differences of the components of the matrices in (3.2) and (4.3) are $O(h^{2\alpha_{\mathrm{Dir}}})$.

Furthermore the estimate (5.1) implies

$$|(\boldsymbol{f}, \mathbf{grad}\,\varphi_k)_{L^2(\Omega)} - (\boldsymbol{f}, \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)}| \le \|\boldsymbol{f}\|_{L^2(\Omega)}|\varphi_k - \varphi_k^h|_{H^1(\Omega)} \le Ch^{\alpha_{\mathrm{Dir}}}, \tag{5.4}$$

i.e., the differences of the components of the right-hand sides in (3.2) and (4.3) are $O(h^{\alpha_{\mathrm{Dir}}})$.

A perturbation argument based on (3.2), (4.3), (5.3) and (5.4) then yields the estimate (cf. [8, Lemma 4.8])

$$|\tau_j - \tau_j^h| \le Ch^{\alpha_{\mathrm{Dir}}}\|\boldsymbol{f}\|_{L^2(\Omega)} \qquad \text{for} \quad 1 \le j \le n. \tag{5.5}$$

5.2 Error Estimates for ξ_h

We will treat the two cases $\gamma = 0$ and $\gamma > 0$ separately.

5.2.1 The Case $\gamma = 0$

From the stability of the discrete saddle point problems (4.4), we have

$$\begin{aligned}\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{H(\mathbf{curl};\Omega)} &\le C\Big[\inf_{\boldsymbol{\psi}_h\in \boldsymbol{V}_h}\|\boldsymbol{\omega} - \boldsymbol{\psi}_h\|_{H(\mathbf{curl};\Omega)} + \inf_{\mu_h\in\Theta_h^0}\|\theta - \mu_h\|_{L^2(\Omega)}\Big] \\ &= C\inf_{\boldsymbol{\psi}_h\in \boldsymbol{V}_h}\|\boldsymbol{\omega} - \boldsymbol{\psi}_h\|_{H(\mathbf{curl};\Omega)}\end{aligned} \tag{5.6}$$

since $\theta = 0$.

Remark 5.1 Under the assumption that $\boldsymbol{f} \in [L^2(\Omega)]^3$, we can only conclude from (3.20) and (5.6) that

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{H(\mathbf{curl}\,;\Omega)} \le C\|\boldsymbol{\omega}\|_{H(\mathbf{curl}\,;\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.7}$$

Moreover it follows from (3.21a) where $\theta = 0$ and (4.4a) where $\theta_h = 0$ that

$$\big(\mathbf{curl}\,(\boldsymbol{\omega} - \boldsymbol{\omega}_h), \mathbf{curl}\,\boldsymbol{\psi}_h\big)_{L^2(\Omega)} = 0 \qquad \forall\,\boldsymbol{\psi}_h \in \boldsymbol{V}_h. \tag{5.8}$$

We will use the Galerkin orthogonality (5.8) to derive a duality estimate for $\boldsymbol{\omega} - \boldsymbol{\omega}_h$ that is useful for the analysis of $\boldsymbol{\xi} - \boldsymbol{\xi}_h$.

The proof of the following result is provided in Appendix D.

Lemma 5.2 *Let* $\boldsymbol{\chi} \in [H_0^1(\Omega)]^3_\Sigma$ *and* $\tilde{\boldsymbol{\xi}} \in \mathscr{P}_T$ *be defined by*

$$(\mathbf{curl}\,\tilde{\boldsymbol{\xi}}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{\chi}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in \mathscr{P}_T. \tag{5.9}$$

Then $\mathbf{curl}\,\tilde{\boldsymbol{\xi}}$ *belongs to* $H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}^0;\Omega)$ *and we have*

$$\|\mathbf{curl}\,(\mathbf{curl}\,\tilde{\boldsymbol{\xi}})\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{H^1(\Omega)}. \tag{5.10}$$

Remark 5.3 Let $\boldsymbol{\chi}$ and $\tilde{\boldsymbol{\xi}}$ be as in Lemma 5.2. Then $\tilde{\boldsymbol{\xi}}$ belongs to $\mathscr{P}_T \subset H(\mathbf{curl}\,;\Omega) \cap H_0(\mathrm{div}^0;\Omega)$ and $\mathbf{curl}\,\tilde{\boldsymbol{\xi}}$ belongs to $H_0(\mathbf{curl}\,;\Omega) \cap H(\mathrm{div}^0;\Omega)$. It follows from Lemma 1.2 that

$$\|\tilde{\boldsymbol{\xi}}\|_{H^{\alpha_T}(\Omega)} \le C(\|\tilde{\boldsymbol{\xi}}\|_{L^2(\Omega)} + \|\mathbf{curl}\,\tilde{\boldsymbol{\xi}}\|_{L^2(\Omega)}) \le C\|\mathbf{curl}\,\tilde{\boldsymbol{\xi}}\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{L^2(\Omega)} \tag{5.11}$$

by Remark 2.2 and (5.9), and, in view of (5.10),

$$\|\mathbf{curl}\,\tilde{\boldsymbol{\xi}}\|_{H^{\alpha_N}(\Omega)} \le C(\|\mathbf{curl}\,\tilde{\boldsymbol{\xi}}\|_{L^2(\Omega)} + \|\mathbf{curl}\,(\mathbf{curl}\,\tilde{\boldsymbol{\xi}})\|_{L^2(\Omega)}) \le C\|\boldsymbol{\chi}\|_{H^1(\Omega)}. \tag{5.12}$$

Our goal is to obtain an estimate for $(\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{\chi})_{L^2(\Omega)}$ where $\boldsymbol{\chi} \in [H_0^1(\Omega)]^3_\Sigma$ is arbitrary. Let $\tilde{\boldsymbol{\xi}} \in \mathscr{P}_T$ be defined by (5.9). In view of Lemma 3.5, we can rewrite (5.9) as the following saddle point problem: Find $(\tilde{\boldsymbol{\xi}}, \tilde{\theta}) \in H(\mathbf{curl}\,;\Omega) \times \Theta^0$ such that

$$a(\tilde{\boldsymbol{\xi}}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \tilde{\theta}) = (\boldsymbol{\chi}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega), \tag{5.13a}$$

$$b(\tilde{\boldsymbol{\xi}}, \mu) = 0. \qquad \forall\,\mu \in \Theta^0. \tag{5.13b}$$

Let $(\tilde{\boldsymbol{\xi}}_h, \tilde{\theta}_h) \in \boldsymbol{V}_h \times \Theta_h^0$ be defined by

$$a(\tilde{\boldsymbol{\xi}}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, \tilde{\theta}_h) = (\boldsymbol{\chi}, \boldsymbol{\psi}_h)_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi}_h \in \boldsymbol{V}_h, \tag{5.14a}$$

$$b(\tilde{\boldsymbol{\xi}}_h, \mu_h) = 0 \qquad \forall\,\mu \in \Theta_h^0. \tag{5.14b}$$

We have

$$(\mathbf{curl}\,\tilde{\boldsymbol{\xi}}, \mathbf{curl}\,\boldsymbol{\omega})_{L^2(\Omega)} = (\boldsymbol{\chi}, \boldsymbol{\omega})_{L^2(\Omega)} \tag{5.15}$$

by (5.9),

$$(\mathbf{curl}\,\tilde{\boldsymbol{\xi}}_h, \mathbf{curl}\,\boldsymbol{\omega}_h)_{L^2(\Omega)} = (\boldsymbol{\chi}, \boldsymbol{\omega}_h)_{L^2(\Omega)} \tag{5.16}$$

by (4.4b) and (5.14a), and

$$(\mathbf{curl}\,\tilde{\boldsymbol{\xi}}, \mathbf{curl}\,\boldsymbol{\omega}_h)_{L^2(\Omega)} + (\boldsymbol{\omega}_h, \widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)} = (\boldsymbol{\chi}, \boldsymbol{\omega}_h)_{L^2(\Omega)} \tag{5.17}$$

by (5.13a).

It follows from (5.15)–(5.17) that

$$
\begin{aligned}
(\boldsymbol{\omega}-\boldsymbol{\omega}_h,\boldsymbol{\chi})_{L^2(\Omega)} &= (\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,\boldsymbol{\omega})_{L^2(\Omega)}-(\boldsymbol{\chi},\boldsymbol{\omega}_h)_{L^2(\Omega)} \\
&= \big(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}+(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,\boldsymbol{\omega}_h)_{L^2(\Omega)}-(\boldsymbol{\chi},\boldsymbol{\omega}_h)_{L^2(\Omega)} \\
&= \big(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}-(\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)}.
\end{aligned}
\tag{5.18}
$$

In view of (5.8), we can estimate the first term on the right-hand side of (5.18) by

$$
\begin{aligned}
\big|\big(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}\big| &= \big|\big(\mathbf{curl}\,(\tilde{\boldsymbol{\zeta}}-\boldsymbol{\psi}_h),\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}\big| \\
&\le \|\mathbf{curl}\,(\tilde{\boldsymbol{\zeta}}-\boldsymbol{\psi}_h)\|_{L^2(\Omega)}\|\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\|_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi}_h\in\boldsymbol{V}_h
\end{aligned}
$$

and hence

$$
\big|\big(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}\big| \le \Big[\inf_{\boldsymbol{\psi}_h\in\boldsymbol{V}_h}\|\mathbf{curl}\,(\tilde{\boldsymbol{\zeta}}-\boldsymbol{\psi}_h)\|_{L^2(\Omega)}\Big]\|\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\|_{L^2(\Omega)}.
\tag{5.19}
$$

It follows from Lemma 1.2, Lemma 4.4, Remark 5.3 and (5.19) that

$$
\big|\big(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}},\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\big)_{L^2(\Omega)}\big| \le Ch^{\alpha_N}\|\boldsymbol{\chi}\|_{H^1(\Omega)}\|\mathbf{curl}\,(\boldsymbol{\omega}-\boldsymbol{\omega}_h)\|_{L^2(\Omega)}.
\tag{5.20}
$$

For the second term on the right-hand side of (5.18), we first note that Remark 3.6, (5.13a) and integration by parts imply

$$
(\mathbf{grad}\,\mu,\mathbf{grad}\,\tilde{\theta})_{L^2(\Omega^\circ)} = (\widetilde{\mathbf{grad}}\,\mu,\widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)} = (-\mathrm{div}\,\boldsymbol{\chi},\mu)_{L^2(\Omega^\circ)} \qquad \forall\,\mu\in\Theta^0,
\tag{5.21}
$$

which implies by elliptic regularity for the Neumann problem that

$$
\|\tilde{\theta}\|_{H^{1+\alpha_{\mathrm{Neu}}}(\Omega^\circ)} \le C\|\mathrm{div}\,\boldsymbol{\chi}\|_{L^2(\Omega)} \qquad \text{for some } \alpha_{\mathrm{Neu}}\in(1/2,1].
\tag{5.22}
$$

Remark 5.4 If Ω is convex, then $\Omega^\circ=\Omega$ and we can take α_{Neu} to be 1.

In view of (3.21b) and (4.4b) we have, for any $\mu_h\in\Theta_h^0$,

$$
-(\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)} = -\big(\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,(\tilde{\theta}-\mu_h)\big)_{L^2(\Omega)} = \big(\boldsymbol{\omega}-\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,(\tilde{\theta}-\mu_h)\big)_{L^2(\Omega)}
$$

and hence

$$
\big|(\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)}\big| \le \Big[\inf_{\mu_h\in\Theta_h^0}\|\widetilde{\mathbf{grad}}\,(\tilde{\theta}-\mu_h)\|_{L^2(\Omega)}\Big]\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{L^2(\Omega)}.
\tag{5.23}
$$

It follows from (5.22), (5.23) and a standard interpolation error estimate for the P_1 finite element (cf. [12, 22]) that

$$
\big|(\boldsymbol{\omega}_h,\widetilde{\mathbf{grad}}\,\tilde{\theta})_{L^2(\Omega)}\big| \le Ch^{\alpha_{\mathrm{Neu}}}\|\boldsymbol{\chi}\|_{H^1(\Omega)}\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{L^2(\Omega)}.
\tag{5.24}
$$

Putting (5.18), (5.20) and (5.24) together, we arrive at the following result.

Lemma 5.5 *We have*

$$
|(\boldsymbol{\omega}-\boldsymbol{\omega}_h,\boldsymbol{\chi})_{L^2(\Omega)}| \le Ch^{\min(\alpha_N,\alpha_{\mathrm{Neu}})}\|\boldsymbol{\chi}\|_{H^1(\Omega)}\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\|_{H(\mathbf{curl};\Omega)} \qquad \forall\,\boldsymbol{\chi}\in[H_0^1(\Omega)]^3_{\Sigma}.
\tag{5.25}
$$

We are now ready to tackle the error estimates for $\boldsymbol{\xi}_h$. The key is to introduce $(\tilde{\boldsymbol{\xi}}_h, \tilde{p}_h) \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}} \times Q_h^0$ defined by

$$g(\tilde{\boldsymbol{\xi}}_h, \boldsymbol{\eta}_h) + \beta(\tilde{\boldsymbol{\xi}}_h, \boldsymbol{\eta}_h)_{L^2(\Omega)} + c(\boldsymbol{\eta}_h, \tilde{p}_h) = (\boldsymbol{\omega}, \boldsymbol{\eta}_h)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta}_h \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}}, \tag{5.26a}$$

$$c(\tilde{\boldsymbol{\xi}}_h, q_h) = 0 \qquad \forall\, q_h \in Q_h^0. \tag{5.26b}$$

On one hand, the saddle point problem (5.26) is the finite element approximation of the continuous saddle point problem (3.22) and hence we have

$$|\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}_h|_{H^1(\Omega)} \le C\Big(\inf_{\boldsymbol{w}_h \in \boldsymbol{W}_h^{\Sigma}} |\boldsymbol{\xi} - \boldsymbol{w}_h|_{H^1(\Omega)} + \inf_{q_h \in Q_h^0} \|p - q_h\|_{L^2(\Omega)}\Big) \tag{5.27}$$

by the stability of the discrete problem. According to Remark 3.13, we have

$$\|\boldsymbol{\xi}\|_{H^{1+\alpha_S}(\Omega)} + \|p\|_{H^{\alpha_S}(\Omega^\circ)} \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)} \tag{5.28}$$

for some $\alpha_S \in (1/2, 1]$, where $\alpha_S = 1$ if Ω is convex.

It follows from (5.27), (5.28) and standard interpolation error estimates (cf. [12, 22]) that

$$|\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}_h|_{H^1(\Omega)} \le Ch^{\alpha_S}\|\boldsymbol{\omega}\|_{L^2(\Omega)}. \tag{5.29}$$

On the other hand, $(\tilde{\boldsymbol{\xi}}_h - \boldsymbol{\xi}_h, \tilde{p}_h - p_h) \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}} \times Q_h^0$ is the finite element approximation for the following saddle point problem: Find $(\boldsymbol{\xi}^*, p^*) \in [H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}} \times L_0^2(\Omega)$ such that

$$g(\boldsymbol{\xi}^*, \boldsymbol{\eta}) + \beta(\boldsymbol{\xi}^*, \boldsymbol{\eta})_{L^2(\Omega)} + c(\boldsymbol{\eta}, p^*) = (\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{\eta})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta} \in [H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}}, \tag{5.30a}$$

$$c(\boldsymbol{\xi}^*, q) = 0 \qquad \forall\, q \in L_0^2(\Omega). \tag{5.30b}$$

It then follows from Lemma 5.5 and the stability of (5.30) that

$$\begin{aligned}
|\tilde{\boldsymbol{\xi}}_h - \boldsymbol{\xi}_h|_{H^1(\Omega)} &\le C\big(|\boldsymbol{\xi}^*|_{H^1(\Omega)} + \|p^*\|_{L^2(\Omega)}\big)\\
&\le C \sup_{\boldsymbol{\eta} \in [H_0^1(\Omega)]^3_{\boldsymbol{\Sigma}}} (\boldsymbol{\omega} - \boldsymbol{\omega}_h, \boldsymbol{\eta})_{L^2(\Omega)}/\|\boldsymbol{\eta}\|_{H^1(\Omega)}\\
&\le Ch^{\min(\alpha_N, \alpha_{\mathrm{Neu}})}\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{H(\mathbf{curl};\Omega)}.
\end{aligned} \tag{5.31}$$

Combining (3.20), (5.7), (5.29) and (5.31), we have

$$|\boldsymbol{\xi} - \boldsymbol{\xi}_h|_{H^1(\Omega)} \le |\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}_h|_{H^1(\Omega)} + |\tilde{\boldsymbol{\xi}}_h - \boldsymbol{\xi}_h|_{H^1(\Omega)} \le Ch^{\min(\alpha_N, \alpha_{\mathrm{Neu}}, \alpha_S)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.32}$$

Note that

$$\|\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}_h\|_{L^2(\Omega)} \le Ch^{2\alpha_S}\|\boldsymbol{\omega}\|_{L^2(\Omega)} \tag{5.33}$$

by a standard duality argument. But there is no improvement for $\|\tilde{\boldsymbol{\xi}}_h - \boldsymbol{\xi}_h\|_{L^2(\Omega)}$ under the assumption that $\boldsymbol{f} \in [L^2(\Omega)]^3$ and we can only conclude from (5.32) and the Poincaré inequality that

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \le Ch^{\min(\alpha_N, \alpha_{\mathrm{Neu}}, \alpha_S)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.34}$$

5.2.2 The Case $\gamma > 0$

Let $\chi \in [H_0^1(\Omega)]^3_\Sigma$ be arbitrary and $(\tilde{\zeta}, \tilde{\theta}, \tilde{\xi}, \tilde{p}) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$ be defined by

$$\mathcal{A}\big((\tilde{\zeta}, \tilde{\theta}, \tilde{\xi}, \tilde{p}), (\psi, \mu, \eta, q)\big) = (\chi, \psi)_{L^2(\Omega)} \tag{5.35}$$

for all $(\psi, \mu, \eta, q) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$, where the bilinear form $\mathcal{A}(\cdot, \cdot)$ is defined in (3.34).

We have the following regularity result (cf. Appendix E):

$$\begin{aligned} &\|\tilde{\zeta}\|_{H^{\alpha_T}(\Omega)} + \|\mathbf{curl}\,\tilde{\zeta}\|_{H^{\alpha_N}(\Omega)} + \|\tilde{\theta}\|_{H^{1+\alpha_{\text{Neu}}}(\Omega^\circ)} + \|\tilde{\xi}\|_{H^{1+\alpha_S}(\Omega)} + \|\tilde{p}\|_{H^{\alpha_S}(\Omega^\circ)} \\ &\quad \le C\|\chi\|_{H^1(\Omega)}. \end{aligned} \tag{5.36}$$

It follows from (3.33), (4.9) and (5.35) that

$$\begin{aligned} (\chi, \zeta - \zeta_h)_{L^2(\Omega)} &= \mathcal{A}\big((\tilde{\zeta}, \tilde{\theta}, \tilde{\xi}, \tilde{p}), (\zeta - \zeta_h, \theta - \theta_h, \xi - \xi_h, p - p_h)\big) \\ &= \mathcal{A}\big((\tilde{\zeta} - \psi_h, \tilde{\theta} - \mu_h, \tilde{\xi} - \eta_h, \tilde{p} - q_h), \\ &\qquad (\zeta - \zeta_h, \theta - \theta_h, \xi - \xi_h, p - p_h)\big) \end{aligned}$$

for any $(\psi_h, \mu_h, \eta_h, q_h) \in V_h \times \Theta_h^0 \times W_h^\Sigma \times Q_h^0$, and hence, in view of Remark 3.14, (3.31), (3.35), Remark 4.11, (4.10) and (5.36),

$$\begin{aligned} &\big|(\chi, \zeta - \zeta_h)_{L^2(\Omega)}\big| \\ &\quad \le Ch^{\min(\alpha_N, \alpha_{\text{Neu}}, \alpha_S)}\|\chi\|_{H^1(\Omega)} \\ &\quad\quad \big(\|\zeta - \zeta_h\|_{H(\mathbf{curl}\,;\Omega)} + |\xi - \xi_h|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)}\big) \\ &\quad \le Ch^{\min(\alpha_N, \alpha_{\text{Neu}}, \alpha_S)}\|\chi\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)}, \end{aligned} \tag{5.37}$$

where we have also used Lemma 4.4 and standard interpolation error estimates for Lagrange finite elements.

In view of the equations

$$g(\xi, \eta) + \beta(\xi, \eta)_{L^2(\Omega)} + c(\eta, p) = \gamma^{\frac{1}{2}}(\zeta, \eta)_{L^2(\Omega)} \qquad \forall\, \eta \in [H_0^1(\Omega)]^3_\Sigma, \tag{5.38a}$$

$$c(\xi, q) = 0 \qquad \forall\, q \in L_0^2(\Omega), \tag{5.38b}$$

that come from (3.28), and the equations

$$g(\xi_h, \eta_h) + \beta(\xi_h, \eta_h)_{L^2(\Omega)} + c(\eta_h, p_h) = \gamma^{\frac{1}{2}}(\zeta_h, \eta_h)_{L^2(\Omega)} \qquad \forall\, \eta_h \in W_h^\Sigma, \tag{5.39a}$$

$$c(\xi_h, q_h) = 0 \qquad \forall\, q_h \in Q_h^0, \tag{5.39b}$$

that come from (4.9), we can use (5.37) and the arguments in Sect. 5.2.1 to obtain the estimate (5.32) and hence also (5.34) by the Poincaré inequality.

In summary we have the following result on the approximation of $\xi = \mathbf{curl}\, u$ by ξ_h.

Theorem 5.6 *We have*

$$\|\mathbf{curl}\, u - \xi_h\|_{L^2(\Omega)} \le C|\mathbf{curl}\, u - \xi_h|_{H^1(\Omega)} \le Ch^{\min(\alpha_N, \alpha_{\text{Neu}}, \alpha_S)}\|f\|_{L^2(\Omega)}.$$

5.3 Error Estimate for $\boldsymbol{\phi}_h$

Let $(\tilde{\boldsymbol{\phi}}_h, \tilde{\theta}_h) \in \boldsymbol{V}_h \times \Theta_h^0$ be defined by

$$a(\tilde{\boldsymbol{\phi}}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, \tilde{\theta}_h) = (\boldsymbol{\xi}, \boldsymbol{\psi}_h)_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{V}_h, \tag{5.40a}$$

$$b(\tilde{\boldsymbol{\phi}}_h, \mu_h) = 0 \qquad \forall\, \mu_h \in \Theta_h^0. \tag{5.40b}$$

On one hand, $(\tilde{\boldsymbol{\phi}}_h, \tilde{\theta}_h)$ is a stable approximation of the solution $(\boldsymbol{\phi}, \theta)$ of the saddle point problem (3.9) and therefore

$$\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}_h\|_{H(\mathbf{curl}\,;\Omega)} \le C \inf_{\boldsymbol{\psi}_h \in \boldsymbol{V}_h} \|\boldsymbol{\phi} - \boldsymbol{\psi}_h\|_{H(\mathbf{curl}\,;\Omega)} \tag{5.41}$$

because $\theta = 0$.

Note that $\boldsymbol{\phi}$ belongs to $\mathscr{P}_T \subset H(\mathbf{curl}\,;\Omega) \cap H_0(\operatorname{div}^0;\Omega) \subset H^{\alpha_T}(\Omega)$ and, according to (3.5) and Lemma 5.2, we have

$$\mathbf{curl}\,\boldsymbol{\phi} \in H_0(\mathbf{curl}\,;\Omega) \cap H(\operatorname{div}^0;\Omega) \text{ and } \|\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{\phi})\|_{L^2(\Omega)} \le C\|\boldsymbol{\xi}\|_{H^1(\Omega)}. \tag{5.42}$$

It follows from Lemma 1.2, (3.6), Lemma 4.4, (5.41) and (5.42) that

$$\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}_h\|_{H(\mathbf{curl}\,;\Omega)} \le Ch^{\min(\alpha_T,\alpha_N)}\|\boldsymbol{\xi}\|_{H^1(\Omega)}. \tag{5.43}$$

On the other hand, $(\tilde{\boldsymbol{\phi}}_h - \boldsymbol{\phi}_h, \tilde{\theta}_h - \theta_h)$ is a stable finite element approximation for $(\boldsymbol{\phi}^*, \theta^*) \in H(\mathbf{curl}\,;\Omega) \times \Theta^0$ defined by

$$a(\boldsymbol{\phi}^*, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \theta^*) = (\boldsymbol{\xi} - \boldsymbol{\xi}_h, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega), \tag{5.44a}$$

$$b(\boldsymbol{\phi}^*, \mu) = 0 \qquad \forall\, \mu \in \Theta^0. \tag{5.44b}$$

It follows from the stability of the finite element method, Theorem 5.6 and the well-posedness of (5.44) that

$$\|\tilde{\boldsymbol{\phi}}_h - \boldsymbol{\phi}_h\|_{H(\mathbf{curl}\,;\Omega)} \le C\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \le Ch^{\min(\alpha_N,\alpha_{\mathrm{Neu}},\alpha_S)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.45}$$

Combining (3.31), (5.43) and (5.45), we have

$$\begin{aligned}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{H(\mathbf{curl}\,;\Omega)} &\le \|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}_h\|_{H(\mathbf{curl}\,;\Omega)} + \|\tilde{\boldsymbol{\phi}}_h - \boldsymbol{\phi}_h\|_{H(\mathbf{curl}\,;\Omega)} \\ &\le Ch^{\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Neu}},\alpha_S)}\|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned} \tag{5.46}$$

5.4 Error Estimate for $\boldsymbol{u}_h$

Putting (3.1), (5.1), (5.5) and (5.46) together, we have the following result on the approximation of $\boldsymbol{u}$ by $\boldsymbol{u}_h$.

Theorem 5.7 *The approximate solution $\boldsymbol{u}_h$ given by (4.12) satisfies*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \le Ch^{\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Dir}},\alpha_{\mathrm{Neu}},\alpha_S)}\|\boldsymbol{f}\|_{L^2(\Omega)}.$$

Remark 5.8 It follows from Theorem 5.7 that in general

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \le Ch^{\alpha}\|\boldsymbol{f}\|_{L^2(\Omega)}$$

for some $\alpha \in (\frac{1}{2}, 1]$ and $\alpha = 1$ if Ω is convex.

6 Improved Error Estimates for the Case where $\boldsymbol{f} \in [H^1(\Omega)]^3$

Under the assumption that $\boldsymbol{f} \in [H^1(\Omega)]^3$, we can improve some of the error estimates in Sect. 5.

First of all the estimate (5.4) can be improved to

$$|(\boldsymbol{f}, \mathbf{grad}\,\varphi_k)_{L^2(\Omega)} - (\boldsymbol{f}, \mathbf{grad}\,\varphi_k^h)_{L^2(\Omega)}| = |(\mathrm{div}\,\boldsymbol{f}, \varphi_k - \varphi_k^h)_{L^2(\Omega)}| \\ \leq \|\boldsymbol{f}\|_{H^1(\Omega)}\|\varphi_k - \varphi_k^h\|_{L^2(\Omega)} \leq Ch^{2\alpha_{\mathrm{Dir}}}, \quad (6.1)$$

where we have used the fact that $\varphi_k - \varphi_k^h \in H_0^1(\Omega)$ and the $O(h^{2\alpha_{\mathrm{Dir}}})$ estimate for $\|\varphi_k - \varphi_k^h\|_{L^2(\Omega)}$ that comes from (5.1) and a standard duality argument. Consequently we can improve the estimate (5.5) to

$$|\tau_j - \tau_j^h| \leq Ch^{2\alpha_{\mathrm{Dir}}}\|\boldsymbol{f}\|_{H^1(\Omega)}. \quad (6.2)$$

We can also improve the estimate (5.34) by examining the two cases $\gamma = 0$ and $\gamma > 0$ separately.

6.1 The Case $\gamma = 0$

We begin by improving the estimate (3.20).

Lemma 6.1 *In the case where $\boldsymbol{f} \in [H^1(\Omega)]^3$, we have*

$$\|\mathbf{curl}\,\boldsymbol{\omega}\|_{H^{\alpha_{\mathrm{Dir}}}(\Omega)} \leq C\|\boldsymbol{f}\|_{H^1(\Omega)}. \quad (6.3)$$

Proof Let $\rho \in H_0^1(\Omega)$ be defined by

$$(\mathbf{grad}\,\rho, \mathbf{grad}\,v)_{L^2(\Omega)} = (\mathrm{div}\,\boldsymbol{f}, v) \qquad \forall\, v \in H_0^1(\Omega).$$

Then we have

$$\boldsymbol{f} + \mathbf{grad}\,\rho \in H(\mathrm{div}^0; \Omega), \quad (6.4)$$

$$\|\rho\|_{H^{1+\alpha_{\mathrm{Dir}}}(\Omega)} \leq C\|\mathrm{div}\,\boldsymbol{f}\|_{L^2(\Omega)}. \quad (6.5)$$

According to Lemma 2.6 and Remark 2.8, there exist constants $c_1, \ldots, c_n$ such that

$$\int_{\Gamma_j} \Big[\boldsymbol{f} + \mathbf{grad}\,\rho - \sum_{k=1}^{n} c_k\,\mathbf{grad}\,\varphi_k\Big] \cdot \boldsymbol{n}_{\Gamma_j}\, dS = 0 \qquad \text{for} \quad 1 \leq j \leq n, \quad (6.6)$$

and

$$\max_{1\leq k\leq n} |c_k| \leq C\|\boldsymbol{f} + \mathbf{grad}\,\rho\|_{L^2(\Omega)}. \quad (6.7)$$

Note that

$$(\mathbf{grad}\,\rho, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = 0 \qquad \forall\,\boldsymbol{\psi} \in H(\mathbf{curl}; \Omega)$$

by integration by parts and hence, in view of Remark 2.7 and (3.19),

$$(\mathbf{curl}\,\boldsymbol{\omega}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi}) \\ = \Big(\boldsymbol{f} + \mathbf{grad}\,\rho - \sum_{k=1}^{n} c_k\,\mathbf{grad}\,\varphi_k, \mathbf{curl}\,\boldsymbol{\psi}\Big)_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in \mathscr{P}_{\mathcal{T}}. \quad (6.8)$$

From Lemma 2.1 and (6.8) we have

$$\mathbf{curl}\,\boldsymbol{\omega} = \boldsymbol{f} + \mathbf{grad}\,\rho - \sum_{k=1}^{n} c_k\,\mathbf{grad}\,\varphi_k \tag{6.9}$$

because $\boldsymbol{f} + \mathbf{grad}\,\rho - \sum_{j=1}^{n} c_j\,\mathbf{grad}\,\varphi_j$ belongs to $\mathscr{D}_F$ by (2.3a), (6.4) and (6.6).

The estimate (6.3) follows from Remark 2.5, (6.5), (6.7), and (6.9). □

From Lemma 1.2, Lemma 4.4, (5.6) and Lemma 6.1 we have

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{H(\mathbf{curl};\Omega)} \le Ch^{\min(\alpha_T,\alpha_{\mathrm{Dir}})}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{6.10}$$

We can then improve the estimate in (5.31) to

$$|\tilde{\boldsymbol{\xi}}_h - \boldsymbol{\xi}_h|_{H^1(\Omega)} \le Ch^{\min(\alpha_N,\alpha_{\mathrm{Neu}})+\min(\alpha_T,\alpha_{\mathrm{Dir}})}\|\boldsymbol{f}\|_{H^1(\Omega)} \tag{6.11}$$

by exploiting (6.10). Consequently we have an improved L^2 error estimate

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \le Ch^{\min(2\alpha_S,\min(\alpha_N,\alpha_{\mathrm{Neu}})+\min(\alpha_T,\alpha_{\mathrm{Dir}}))}\|\boldsymbol{f}\|_{H^1(\Omega)} \tag{6.12}$$

through (5.33), (6.11) and the Poincaré inequality.

6.2 The Case $\gamma > 0$

It follows from Lemma 3.5, (3.28a) and (3.28b) that $\boldsymbol{\zeta} \in \mathscr{P}_T$ satisfies

$$(\mathbf{curl}\,\boldsymbol{\zeta}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} + (-\gamma^{\frac{1}{2}}\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in \mathscr{P}_T,$$

and consequently

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 \tag{6.13}$$

where $\boldsymbol{\zeta}_1 \in \mathscr{P}_T$ is defined by

$$(\mathbf{curl}\,\boldsymbol{\zeta}_1, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (\gamma^{-\frac{1}{2}}\boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in \mathscr{P}_T, \tag{6.14}$$

and $\boldsymbol{\zeta}_2 \in \mathscr{P}_T$ is defined by

$$(\mathbf{curl}\,\boldsymbol{\zeta}_2, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} = (-\gamma^{\frac{1}{2}}\boldsymbol{\xi}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\,\boldsymbol{\psi} \in \mathscr{P}_T. \tag{6.15}$$

Note that (6.14) is just (3.19) with $(\boldsymbol{\omega}, \boldsymbol{f})$ replaced by $(\boldsymbol{\zeta}_1, \gamma^{-\frac{1}{2}}\boldsymbol{f})$. Therefore we can apply Lemma 6.1 to obtain the estimate

$$\|\mathbf{curl}\,\boldsymbol{\zeta}_1\|_{H^{\alpha_{\mathrm{Dir}}}(\Omega)} \le C\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{6.16}$$

Similarly (6.15) is just (5.9) with $(\tilde{\boldsymbol{\zeta}}, \chi)$ replaced by $(\boldsymbol{\zeta}_2, -\gamma^{\frac{1}{2}}\boldsymbol{\xi})$. Hence we can conclude from (3.31), Lemma 5.2, Remark 5.3 that

$$\|\boldsymbol{\zeta}_2\|_{H^{\alpha_T}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\zeta}_2\|_{H^{\alpha_N}(\Omega)} \le C\|\boldsymbol{\xi}\|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{6.17}$$

From the stability of the discrete saddle point problem (4.9) we have

$$\begin{aligned} &\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ &\quad \le \Big(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h}\|\boldsymbol{\zeta} - \boldsymbol{v}_h\|_{H(\mathbf{curl};\Omega)} + \inf_{\boldsymbol{w}_h \in \boldsymbol{W}_h^{\boldsymbol{\Sigma}}}\|\boldsymbol{\xi} - \boldsymbol{w}_h\|_{H^1(\Omega)} + \inf_{q_h \in Q_h^0}\|p - q_h\|_{L^2(\Omega)}\Big) \end{aligned} \tag{6.18}$$

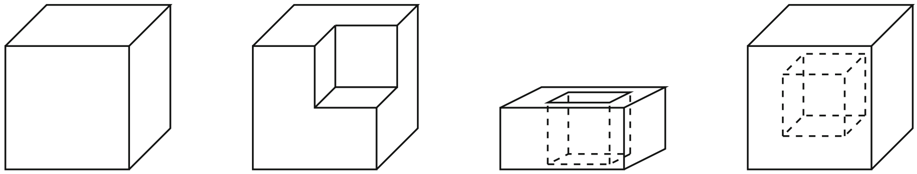

Fig. 2 Domains used for numerical tests: **a** unit cube, **b** Fichera chair, **c** rectangular torus, and **d** hollowed unit cube

because $\theta = 0$.

It then follows from Lemma 1.2, (3.32), Lemma 4.4, (6.16)–(6.18) and standard interpolation error estimates for $\boldsymbol{W}_h$ and Q_h that

$$\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{H(\mathbf{curl};\Omega)} + \|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ \leq Ch^{\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Dir}},\alpha_S)}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{6.19}$$

In view of (6.19), the estimate (5.37) can be improved to

$$\left|(\boldsymbol{\chi}, \boldsymbol{\zeta} - \boldsymbol{\zeta}_h)_{L^2(\Omega)}\right| \leq Ch^{\min(\alpha_N,\alpha_{\mathrm{Neu}},\alpha_S)+\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Dir}},\alpha_S)}\|\boldsymbol{\chi}\|_{H^1(\Omega)}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{6.20}$$

We can then apply (6.20) and the arguments in Sect. 5.2 to conclude that

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \leq Ch^{\min\left(2\alpha_S,\min(\alpha_N,\alpha_{\mathrm{Neu}},\alpha_S)+\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Dir}},\alpha_S)\right)}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{6.21}$$

Putting (6.12) and (6.21) together, we have the following result on the approximation of $\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{u}$ by $\boldsymbol{\xi}_h$ that improves the L^2 error estimate in Theorem 5.6.

Theorem 6.2 *In the case where $\boldsymbol{f} \in [H^1(\Omega)]^3$, we have*

$$\|\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \leq Ch^{2\min(\alpha_T,\alpha_N,\alpha_{\mathrm{Dir}},\alpha_{\mathrm{Neu}},\alpha_S)}\|\boldsymbol{f}\|_{H^1(\Omega)}.$$

Remark 6.3 It follows from Theorem 6.2 that in general

$$\|\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \leq Ch^{2\alpha}\|\boldsymbol{f}\|_{H^1(\Omega)}$$

for some $\alpha \in (\frac{1}{2}, 1]$. Moreover we have $\alpha = 1$ if Ω is convex.

7 Numerical Results

We solved the quad-curl problem (1.1) for four examples on domains with different characteristics (cf. Fig. 2).

The initial triangulations on these domains are obtained by the following procedure. First we create a uniform rectangular grid so that $\partial\Omega$ and the cuts $\Sigma_1, \ldots, \Sigma_m$ are unions of the faces of this uniform grid, and then we triangularize each cube by using the center of the cube as a common vertex of all the tetrahedrons as demonstrated by the figure on the left of Fig. 3. This guarantees that the assumption (4.1) is satisfied, which implies the stability of the Taylor-Hood method for the Stokes problem (cf. [6]). Finer meshes are generated using the nested uniform refinement method presented in [4], where each coarse tetrahedral element is broken into 8 fine elements in each refinement. The refinement process on a single element

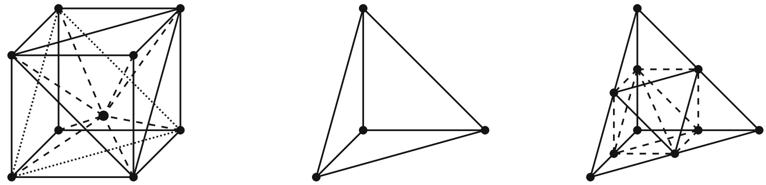

Fig. 3 Triangulation of a cube in the initial uniform grid (left); a coarse tetrahedral element (center); a tetrahedral element after one refinement (right)

Table 1 Number of elements for the domains sketched in Fig. 2

h	Unit cube	Fichera chair	Rectangular torus	Hollowed cube
1	12	–	–	–
1/2	96	84	–	–
1/4	768	672	144	672
1/8	6144	5376	1152	5376
1/16	49,152	43,008	9216	43,008
1/32	393,216	344,064	73,728	344,064
1/64	3,145,728	2,752,512	589,824	2,752,512

is illustrated by the figures in the center and the left of Fig. 3. The number of elements in different levels of refinements are displayed in Table 1.

The computations were carried out in MATLAB.

7.1 Unit Cube with Positive β and γ

Convergence results for the coupled quad-curl problem solved on the unit cube $(0, 1)^3$ are presented for $h = 1$ to $h = 1/64$ with known vector potential

$$\boldsymbol{\phi}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 4\pi \sin^4(\pi x_1)\sin^4(\pi x_2)\sin^3(\pi x_3)\cos(\pi x_3) \\ -4\pi \sin^4(\pi x_1)\sin^3(\pi x_2)\sin^4(\pi x_3)\cos(\pi x_2) \end{bmatrix}$$

and solution

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{bmatrix} -2\pi^2 \sin^4(\pi x_1)\sin^2(\pi x_2)\sin^2(\pi x_3)\left(\cos(2\pi x_2) - 2\cos(2\pi(x_2 - x_3)) + \cos(2\pi x_3) + 4\sin^2(\pi(x_2 + x_3))\right) \\ 16\pi^2 \sin^3(\pi x_1)\sin^3(\pi x_2)\sin^4(\pi x_3)\cos(\pi x_1)\cos(\pi x_2) \\ 16\pi^2 \sin^3(\pi x_1)\sin^4(\pi x_2)\sin^3(\pi x_3)\cos(\pi x_1)\cos(\pi x_3) \end{bmatrix}.$$

The right-hand side $\boldsymbol{f}$ can be analytically computed. The coefficients are

$$\beta = \frac{7}{2} \quad \text{and} \quad \gamma = \frac{3}{2}. \tag{7.1}$$

The problem is solved iteratively to a 10^{-8} relative residual using MINRES.

As seen in Table 2, the convergence rate of $\boldsymbol{u}_h$ in the L^2 norm is almost exactly $O(h)$ once h is sufficiently small, verifying Theorem 5.7 and Remark 5.8. Similarly, the convergence rate of $\boldsymbol{\xi}_h$ in the L^2 norm is $O(h^2)$ once the asymptotic region of convergence is reached, demonstrating Theorem 6.2 and Remark 6.3.

Table 2 Convergence results on the unit cube domain

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}$	Rate	$\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}$	Rate
1/2	1.74×10^{1}	–	1.44×10^{2}	–
1/4	2.63×10^{1}	0.59	2.05×10^{2}	0.51
1/8	5.38×10^{0}	2.29	4.35×10^{1}	2.24
1/16	2.61×10^{0}	1.04	1.06×10^{1}	2.04
1/32	1.28×10^{0}	1.03	2.66×10^{0}	1.99
1/64	6.36×10^{-1}	1.01	6.66×10^{-1}	2.00

Table 3 Convergence results on the Fichera chair domain

h	$\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i-1}\|_{L^2(\Omega)}$	Rate	$\|\boldsymbol{\xi}_{h,i}-\boldsymbol{\xi}_{h,i-1}\|_{L^2(\Omega)}$	Rate
1/4	5.08×10^{-5}	–	1.54×10^{-4}	–
1/8	2.90×10^{-5}	0.81	3.10×10^{-5}	2.31
1/16	1.72×10^{-5}	0.75	8.41×10^{-6}	1.88
1/32	1.04×10^{-5}	0.73	3.17×10^{-6}	1.41
1/64	6.42×10^{-6}	0.70	1.32×10^{-6}	1.27

7.2 Fichera Chair with Positive β and $\gamma = 0$

Next, we examine the convergence on the Fichera chair domain, $(0,1)^3 \setminus [1/2,1]^3$ (cf. figure (b) in Fig. 2). In order to test the effects of the nonconvex edges and corners, we avoid a manufactured solution and just use a smooth right-hand side given by

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix}\sin x_1\\ \sin x_2\\ \sin x_3\end{bmatrix}, \tag{7.2}$$

with coefficients

$$\beta = \frac{7}{2} \quad \text{and} \quad \gamma = 0. \tag{7.3}$$

The problem is solved iteratively to a 10^{-8} relative residual using MINRES.

To estimate $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}$ and $\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}$, the error is measured between the numerical solutions on consecutive mesh refinements, $\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i-1}\|_{L^2(\Omega)}$ and $\|\boldsymbol{\xi}_{h,i}-\boldsymbol{\xi}_{h,i-1}\|_{L^2(\Omega)}$, where i indicates the number of refinements from the coarsest mesh.

Since the domain is not convex, we expect $\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i-1}\|_{L^2(\Omega)}$ to converge at a rate greater than $\mathcal{O}(h^{1/2})$ but less than $\mathcal{O}(h)$, and $\|\boldsymbol{\xi}_{h,i}-\boldsymbol{\xi}_{h,i-1}\|_{L^2(\Omega)}$ to converge at a rate greater than $\mathcal{O}(h)$ but less than $\mathcal{O}(h^2)$. This is verified by the results in Table 3.

Remark 7.1 The results in Table 3 indicate that the dominant singularity is due to the nonconvex edges and the asymptotic convergence rate should be $2/3$ for the approximation of $\boldsymbol{u}$ and $4/3$ for the approximation of $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{\xi}$.

Table 4 Convergence results for the rectangular torus

h	$\|\boldsymbol{u}_{h,i} - \boldsymbol{u}_{h,i-1}\|_{L^2(\Omega)}$	Rate	$\|\boldsymbol{\xi}_{h,i} - \boldsymbol{\xi}_{h,i-1}\|_{L^2(\Omega)}$	Rate
1/8	3.07×10^{-7}	–	4.07×10^{-6}	–
1/16	1.59×10^{-7}	0.95	1.50×10^{-6}	1.44
1/32	9.26×10^{-8}	0.78	2.60×10^{-7}	2.53
1/64	5.02×10^{-8}	0.88	7.60×10^{-8}	1.77

Table 5 Convergence results for $\boldsymbol{u}_h = \mathbf{curl}\,\boldsymbol{\phi}_h$ and $\boldsymbol{\xi}_h = \mathbf{curl}\,\boldsymbol{u}_h$ on the hollowed cube

h	$\|\boldsymbol{u}_{h,i} - \boldsymbol{u}_{h,i-1}\|_{L^2(\Omega)}$	Rate	$\|\boldsymbol{\xi}_{h,i} - \boldsymbol{\xi}_{h,i-1}\|_{L^2(\Omega)}$	Rate
1/8	2.66×10^{-5}	–	3.08×10^{-5}	–
1/16	1.80×10^{-5}	0.56	1.00×10^{-5}	1.62
1/32	1.15×10^{-5}	0.65	3.56×10^{-6}	1.49

7.3 Rectangular Torus with Positive β and $\gamma = 0$

The same data in example 7.2 is used for the rectangular torus domain, $((0, 1) \times (0, 1) \times (0, 1/4)) \setminus ([1/4, 3/4] \times [1/4, 3/4] \times [0, 1/4])$ (cf. figure (c) in Fig. 2). The solid rectangular torus has Betti numbers $\beta_1 = 1$ and $\beta_2 = 0$. To make the domain simply connected, we introduce the cut $\{(x_1, x_2, x_3) : x_1 = \frac{1}{4},\ x_2, x_3 \in (0, \frac{1}{4})\}$ which is also a face of the initial uniform rectangular grid. The problem is solved with MINRES to a relative residual tolerance of 10^{-6}.

The numerical results in Table 4 are similar to the ones in Table 3, indicating that Remark 7.1 is also applicable for this example.

7.4 Hollowed Cube with Positive β and γ

For the final test, the domain is the hollowed cube $(0, 1)^3 \setminus (1/4, 3/4)^3$ (cf. figure (d) in Fig. 2), which has Betti numbers $\beta_1 = 0$ and $\beta_2 = 1$. The solution $\boldsymbol{u}$ to the quad-curl problem now admits a Hodge decomposition with a harmonic component. Since there is a single hole in the domain, the numerical solution takes the form of,

$$\boldsymbol{u}_h = \mathbf{curl}\,\boldsymbol{\phi}_h + \tau_h \varphi_h.$$

We use the right-hand side (7.2) with coefficients β and γ given by (7.1), and the problem is solved with MINRES to a relative residual tolerance of 10^{-6}.

The results in Table 5 for the approximations of $\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u} = \boldsymbol{\xi}$ are similar to the ones in Table 3 and Table 4.

The results for the approximations of the harmonic function φ and the coefficient τ are presented in Table 6. The order of convergence for φ_h in $|\cdot|_{H^1(\Omega)}$ is $2/3$, which is due to the edge singularity. The order of convergence for τ_h is better than $4/3$ stated in Sect. 6.2. This is likely due to the fact that the asymptotic region of convergence has not been reached at $h = 1/64$.

Table 6 Convergence results for the harmonic problem on the hollowed cube

h	$\lvert\varphi_{h,i} - \varphi_{h,i-1}\rvert_{H^1(\Omega)}$	Rate	$\lvert\tau_{h,i} - \tau_{h,i-1}\rvert$	Rate
1/8	2.22×10^0	–	1.26×10^{-2}	–
1/16	1.29×10^0	0.79	3.23×10^{-3}	1.97
1/32	7.90×10^{-1}	0.70	7.01×10^{-4}	2.21
1/64	5.10×10^{-1}	0.63	1.63×10^{-4}	2.10

8 Concluding Remarks

We have developed a numerical scheme based on the Hodge decomposition of divergence-free vector fields for the quad-curl source problem posed on polyhedral domains with general topology. Our approach only employs standard finite elements in $H(\mathbf{grad})$ and $H(\mathbf{curl})$, and it is also relevant for the quad-curl eigenvalue problem (cf. [21, 43]).

For simplicity we have used the P_2–P_1 Taylor-Hood method for the Stokes problem that appears in the Hodge decomposition approach. Since we are only concerned with the approximation of the displacement, it may be advantageous to use a pressure robust mixed finite element method for the Stokes problem (cf. [30]).

As in the two dimensional case (cf. [9–11]), one can also develop adaptive and fast solvers for the individual saddle point problems that appear in the Hodge decomposition approach.

The result for the quad-curl source problem in this paper can be the basis of a Hodge decomposition approach to the quad-curl eigenvalue problem.

Appendix A The Norm Equivalence (1.3)

We begin with a simple observation.

Lemma A.1 *We have*

$$\|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} = |\boldsymbol{v}|_{H^1(\Omega)} \qquad \forall\,\boldsymbol{v} \in [H_0^1(\Omega)]^3 \cap H(\mathrm{div}^0;\Omega). \tag{A.1}$$

Proof Let $\boldsymbol{v}_n \in [C_c^\infty(\Omega)]^3$ converge to $\boldsymbol{v}$ in $[H^1(\Omega)]^3$ as $n \to \infty$. It follows from integration by parts that

$$\begin{aligned}
|\boldsymbol{v}|^2_{H^1(\Omega)} = \|\mathbf{Grad}\,\boldsymbol{v}\|^2_{L^2(\Omega)} &= \lim_{n\to\infty} \|\mathbf{Grad}\,\boldsymbol{v}_n\|^2_{L^2(\Omega)} \\
&= \lim_{n\to\infty} \int_\Omega (-\Delta \boldsymbol{v}_n)\cdot \boldsymbol{v}_n dx \\
&= \lim_{n\to\infty} \left[(\mathbf{curl}\,\boldsymbol{v}_n, \mathbf{curl}\,\boldsymbol{v}_n)_{L^2(\Omega)} + (\mathrm{div}\,\boldsymbol{v}_n, \mathrm{div}\,\boldsymbol{v}_n)_{L^2(\Omega)}\right] \\
&= \|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)} \\
&= \|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)}.
\end{aligned}$$

□

It is clear from the definition of $(\cdot,\cdot)_{\mathbb{E}}$ in (1.2) and Lemma A.1 that

$$\|\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v})\|^2_{L^2(\Omega)} + \beta\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)} + \gamma\|\boldsymbol{v}\|^2_{L^2(\Omega)} \le \max(1,\beta,\gamma)(\boldsymbol{v},\boldsymbol{v})_{\mathbb{E}} \quad \forall \boldsymbol{v} \in \mathbb{E}.$$

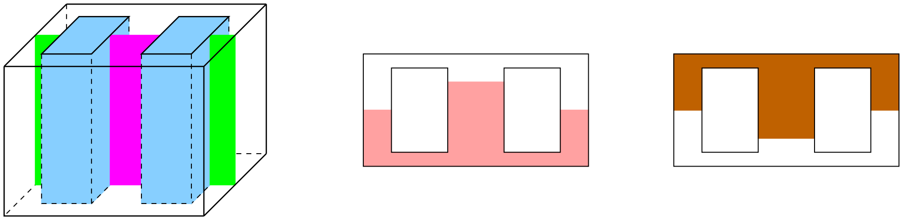

Fig. 4 (Left) the domain Ω together with the cuts Σ_0 (shaded in magenta) and Σ_1, Σ_2 (shaded in green), (Center) the top view of Ω_1^* (shaded in pink) and (Right) the top view of Ω_2^* (shaded in brown)

In the other direction, we have, by Lemma A.1 and a Poincaré-Friedrichs inequality,

$$\|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} \le C_\sharp|\mathbf{curl}\,\boldsymbol{v}|_{H^1(\Omega)} = C_\sharp\|\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v})\|_{L^2(\Omega)} \qquad \forall\,\boldsymbol{v} \in \mathbb{E} \tag{A.2}$$

because $\mathbf{curl}\,\boldsymbol{v} \in [H_0^1(\Omega)]^3 \cap H(\mathrm{div}^0;\Omega)$ for $\boldsymbol{v} \in \mathbb{E}$.

In the case where $\gamma > 0$, it follows from (A.1) and (A.2) that

$$(\boldsymbol{v},\boldsymbol{v})_{\mathbb{E}} \le \max(1 + C_\sharp^2, 1/\gamma)\big(\|\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v})\|_{L^2(\Omega)}^2 + \gamma\|\boldsymbol{v}\|_{L^2(\Omega)}^2\big) \quad \forall\,\boldsymbol{v} \in \mathbb{E}.$$

In the case where $\gamma = 0$ and hence $\partial\Omega$ is connected, we have (cf. [1, Corollary 3.19])

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \le C_\flat\|\mathbf{curl}\,\boldsymbol{v}\|_{L^2(\Omega)} \qquad \forall\,\boldsymbol{v} \in \mathbb{E},$$

which together with (A.1) and (A.2) implies

$$(\boldsymbol{v},\boldsymbol{v})_{\mathbb{E}} \le (1 + C_\sharp^2 + C_\flat^2 C_\sharp^2)\|\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{v})\|_{L^2(\Omega)}^2 \qquad \forall\,\boldsymbol{v} \in \mathbb{E}.$$

Appendix B The Inf-Sup Condition (3.25)

If $m = 0$ and Ω is simply connected, then $[H_0^1(\Omega)]_\Sigma^3 = [H_0^1(\Omega)]^3$ and (3.25) is a standard result (cf. [27, Page 24, Corollary 2.4]), which is equivalent to the statement (cf. [14, Theorem 0.1]) that there exists a positive constant C such that for every $q \in L_0^2(\Omega)$ we can find $\boldsymbol{\eta} \in [H_0^1(\Omega)]^3$ that satisfies

$$\mathrm{div}\,\boldsymbol{\eta} = q \quad \text{and} \quad |\boldsymbol{\eta}|_{H^1(\Omega)} \le C\|q\|_{L^2(\Omega)}.$$

In the case where $m \ge 1$, we can introduce an additional cut Σ_0 so that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, where the disjoint polyhedrons Ω_1 and Ω_2 are the open subsets of Ω separated by the cuts $\Sigma_0, \ldots, \Sigma_m$. By extending Ω_1 and Ω_2 across Σ_0 for a small amount, we have two overlapping open polyhedrons Ω_1^* and Ω_2^* such that $\Omega^\circ \subset \Omega_1^* \cup \Omega_2^*$.

The situation is illustrated in Fig. 4. On the left we have Ω which is obtained by removing two rectangular columns (shaded in blue) from a rectangular box, together with the cuts Σ_1, Σ_2 (shaded in green) and Σ_0 (shaded in magenta). In the middle we have the top view of Ω_1^* (shaded in pink), and on the right we have the top view of Ω_2^* (shaded in brown).

We can construct a continuous partition of unity ϕ_1 and ϕ_2 so that

$$\begin{cases}\phi_1 \ge 0 & \text{on } \Omega^0\\ \phi_1 = 1 & \text{on } \Omega_1 \setminus \Omega_2^*\end{cases}, \qquad \begin{cases}\phi_2 \ge 0 & \text{on } \Omega^0\\ \phi_2 = 1 & \text{on } \Omega_2 \setminus \Omega_1^*\end{cases}, \quad \text{and} \quad \phi_1 + \phi_2 = 1 \quad \text{on} \quad \Omega^\circ.$$

Let χ be a continuous function with compact support in $\Omega_1^* \cap \Omega_2^*$ such that

$$\int_\Omega \chi \, dx = 1.$$

Given any $q \in L_0^2(\Omega)$, we define $q_1, q_2 \in L^2(\Omega)$ by

$$q_1 = \phi_1 q - \chi \int_\Omega \phi_1 q \, dx \quad \text{and} \quad q_2 = \phi_2 q - \chi \int_\Omega \phi_2 q \, dx.$$

Then we have

$$q = q_1 + q_2 \quad \text{and} \quad \|q_1\|_{L^2(\Omega_1^*)} + \|q_2\|_{L^2(\Omega_2^*)} \le C_\sharp \|q\|_{L^2(\Omega)}. \tag{B.1}$$

Note that

$$\int_{\Omega_i^*} q_i \, dx = 0 \qquad \text{for} \quad i = 1, 2.$$

According to the standard result for Ω_1^* and Ω_2^*, there exist $\boldsymbol{\eta}_1 \in [H_0^1(\Omega_1^*)]^3$ and $\boldsymbol{\eta}_2 \in [H_0^1(\Omega_2^*)]^3$ such that

$$\operatorname{div} \boldsymbol{\eta}_i = q_i \quad \text{in } \Omega_i^* \quad \text{and} \quad |\boldsymbol{\eta}_i|_{H^1(\Omega_i^*)} \le C_\flat \|q_i\|_{L^2(\Omega_i^*)} \qquad \text{for} \quad i = 1, 2. \tag{B.2}$$

By extending $\boldsymbol{\eta}_1$ to be $\mathbf{0}$ on $\Omega \setminus \Omega_1^*$, we have a vector field in $[H_0^1(\Omega)]_{\boldsymbol{\Sigma}}^3$, still denoted by $\boldsymbol{\eta}_1$, such that $\operatorname{div} \boldsymbol{\eta}_1 = q_1$ in Ω. Similarly, the trivial extension of $\boldsymbol{\eta}_2$ to Ω, still denoted by $\boldsymbol{\eta}_2$, is a vector field in $[H_0^1(\Omega)]_{\boldsymbol{\Sigma}}^3$ that satisfies $\operatorname{div} \boldsymbol{\eta}_2 = q_2$ in Ω.

In view of (B.1) and (B.2), we have the estimate

$$\frac{(\operatorname{div}(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2), q)_{L^2(\Omega)}}{|\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2|_{H^1(\Omega)}} \ge \frac{\|q\|_{L^2(\Omega)}^2}{|\boldsymbol{\eta}_1|_{H^1(\Omega)} + |\boldsymbol{\eta}_2|_{H^1(\Omega)}} \ge (C_\sharp C_\flat)^{-1} \|q\|_{L^2(\Omega)}$$

that implies (3.25).

Appendix C The Generalized Stokes Problem (3.22)

First we note that $\boldsymbol{\xi} \in [H_0^1(\Omega)]_{\boldsymbol{\Sigma}}^3$ for any collection of acceptable cuts $\boldsymbol{\Sigma} = \{\Sigma_1, \ldots, \Sigma_m\}$ with the property that $\Omega_{\boldsymbol{\Sigma}}^\circ = \Omega \setminus \bigcup_{i=1}^m \Sigma_i$ is simply connected (cf. Remark 3.13 in [1]). Therefore $\boldsymbol{\xi}$ satisfies (3.22) for any such $\boldsymbol{\Sigma}$, where $p = p_{\boldsymbol{\Sigma}}$ depends on $\boldsymbol{\Sigma}$.

Let $\rho_1, \ldots, \rho_N$ be a C^∞ partition of unity for $\bar{\Omega}$ such that the diameter of the support of ρ_k is sufficiently small for $1 \le k \le N$. Then for each k we can choose a collection of acceptable cuts $\boldsymbol{\Sigma}_k$ such that the support of ρ_k is disjoint from $\bar{\Sigma}_{k,1}, \ldots, \bar{\Sigma}_{k,m}$.

Let $\boldsymbol{\eta} \in [H_0^1(\Omega)]^3$ be arbitrary. We have $\rho_k \boldsymbol{\eta} \in [H_0^1(\Omega)]_{\boldsymbol{\Sigma}_k}^3$ and it follows from integration by parts and (3.22a) that

$$\begin{aligned}
\big(\mathbf{Grad}\,(\rho_k \boldsymbol{\xi}), \mathbf{Grad}\,\boldsymbol{\eta}\big)_{L^2(\Omega)} &= \big(\boldsymbol{\xi}(\mathbf{grad}\,\rho_k)^t + \rho_k \mathbf{Grad}\,\boldsymbol{\xi}, \mathbf{Grad}\,\boldsymbol{\eta}\big)_{L^2(\Omega)} \\
&= \big(\mathbf{Grad}\,\boldsymbol{\xi}, \mathbf{Grad}\,(\rho_k \boldsymbol{\eta})\big)_{L^2(\Omega)} - \big(\operatorname{div}(\boldsymbol{\xi}(\mathbf{grad}\,\rho_k)^t), \boldsymbol{\eta}\big)_{L^2(\Omega)} \\
&\quad - \big((\mathbf{Grad}\,\boldsymbol{\xi})\mathbf{grad}\,\rho_k, \boldsymbol{\eta}\big)_{L^2(\Omega)} \\
&= -\beta(\boldsymbol{\xi}, \rho_k \boldsymbol{\eta})_{L^2(\Omega)} + \big(\operatorname{div}(\rho_k \boldsymbol{\eta}), p_{\boldsymbol{\Sigma}_k}\big)_{L^2(\Omega)} + (\boldsymbol{\omega}, \rho_k \boldsymbol{\eta})_{L^2(\Omega)} \\
&\quad - \big(\operatorname{div}(\boldsymbol{\xi}(\mathbf{grad}\,\rho_k)^t), \boldsymbol{\eta}\big)_{L^2(\Omega)} - \big((\mathbf{Grad}\,\boldsymbol{\xi})\mathbf{grad}\,\rho_k, \boldsymbol{\eta}\big)_{L^2(\Omega)},
\end{aligned}$$

and hence

$$(\mathbf{Grad}\,(\rho_k\boldsymbol{\xi}), \mathbf{Grad}\,\boldsymbol{\eta})_{L^2(\Omega)} - (\mathrm{div}\,\boldsymbol{\eta}, \rho_k p_{\boldsymbol{\Sigma}_k})_{L^2(\Omega)} = (\boldsymbol{\kappa}, \boldsymbol{\eta})_{L^2(\Omega)}, \tag{C.1}$$

where

$$\boldsymbol{\kappa} = -\beta\rho_k\boldsymbol{\xi} + p_{\boldsymbol{\Sigma}_k}\mathbf{grad}\,\rho_k + \rho_k\boldsymbol{\omega} - \mathrm{div}\,\big(\boldsymbol{\xi}(\mathbf{grad}\,\rho_k)^t\big) - (\mathbf{Grad}\,\boldsymbol{\xi})\mathbf{grad}\,\rho_k \in [L^2(\Omega)]^3. \tag{C.2}$$

Let $q \in L^2_0(\Omega)$ be arbitrary. From (3.22b) we have

$$-\big(\mathrm{div}\,(\rho_k\boldsymbol{\xi}), q\big)_{L^2(\Omega)} = \big(-(\mathbf{grad}\,\rho_k)\cdot\boldsymbol{\xi}, q\big)_{L^2(\Omega)}, \tag{C.3}$$

where

$$-(\mathbf{grad}\,\rho_k)\cdot\boldsymbol{\xi} \in H^1_0(\Omega). \tag{C.4}$$

It follows from (C.1)–(C.4) that $(\rho_k\boldsymbol{\xi}, \rho_k p_{\boldsymbol{\Sigma}_k})$ is the solution of a standard Stokes problem analyzed in [25]. Consequently we have

$$\|\rho_k\boldsymbol{\xi}\|_{H^{1+\alpha_S}(\Omega)} \le C\big(\|\boldsymbol{\kappa}\|_{L^2(\Omega)} + |(\mathbf{grad}\,\rho_k)\cdot\boldsymbol{\xi}|_{H^1(\Omega)}\big) \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)}, \tag{C.5}$$

where $\alpha_S \in (\frac{1}{2}, 1]$ is determined by the geometry of Ω and the last inequality comes from the well-posedness of (3.22) (cf. Remark 3.12). Since (C.5) is valid for every k, we conclude that

$$\boldsymbol{\xi} \in H^{1+\alpha_S}(\Omega) \quad \text{and} \quad \|\boldsymbol{\xi}\|_{H^{1+\alpha_S}(\Omega)} \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)}. \tag{C.6}$$

Let $\boldsymbol{\Sigma}$ be any collection of acceptable cuts and $\boldsymbol{\eta} \in [C_c^\infty(\Omega^\circ_{\boldsymbol{\Sigma}})]^3$ be arbitrary. Since $\boldsymbol{\eta} \in [H^1_0(\Omega)]^3_{\boldsymbol{\Sigma}}$, we have, by (3.22a),

$$\begin{aligned}\big(\mathrm{div}\,\boldsymbol{\eta}, p_{\boldsymbol{\Sigma}}\big)_{L^2(\Omega^\circ_{\boldsymbol{\Sigma}})} &= \big(\mathrm{div}\,\boldsymbol{\eta}, p_{\boldsymbol{\Sigma}}\big)_{L^2(\Omega)} \\ &= (\mathbf{Grad}\,\boldsymbol{\xi}, \mathbf{Grad}\,\boldsymbol{\eta})_{L^2(\Omega)} + \beta(\boldsymbol{\xi}, \boldsymbol{\eta})_{L^2(\Omega)} - (\boldsymbol{\omega}, \boldsymbol{\eta})_{L^2(\Omega)},\end{aligned}$$

which together with (C.6) implies

$$p_{\boldsymbol{\Sigma}} \in H^{\alpha_S}(\Omega^\circ_{\boldsymbol{\Sigma}}) \quad \text{and} \quad \|p_{\boldsymbol{\Sigma}}\|_{H^{\alpha_S}(\Omega^\circ_{\boldsymbol{\Sigma}})} \le C\|\boldsymbol{\omega}\|_{L^2(\Omega)}. \tag{C.7}$$

Appendix D Proof of Lemma 5.2

Let $\rho \in H^1(\Omega) \cap L^2_0(\Omega)$ be defined by

$$(\mathbf{grad}\,\rho, \mathbf{grad}\,\mu)_{L^2(\Omega)} = (\mathrm{div}\,\boldsymbol{\chi}, \mu)_{L^2(\Omega)} \qquad \forall\, \mu \in H^1(\Omega) \cap L^2_0(\Omega).$$

We have $\boldsymbol{\chi} + \mathbf{grad}\,\rho \in H_0(\mathrm{div}^0; \Omega)$ and

$$\|\mathbf{grad}\,\rho\|_{H(\mathrm{div}\,;\Omega)} \le C\|\mathrm{div}\,\boldsymbol{\chi}\|_{L^2(\Omega)}. \tag{D.1}$$

According to Proposition 3.14 in [1], there exists a harmonic function z on Ω° such that (i) $\boldsymbol{n}\cdot\mathbf{grad}\,z = 0$ on $\partial\Omega$, (ii) $\boldsymbol{n}_{\Sigma_i}\cdot\mathbf{grad}\,z$ is continuous across the cut Σ_i for $1 \le i \le m$, (iii) the jump of z over the cut Σ_i is a constant for $1 \le i \le m$, (iv)

$$\int_{\Sigma_i} \mathbf{grad}\,(\rho + z)\cdot\boldsymbol{n}_{\Sigma_i}\,dS = 0 \qquad \text{for} \quad 1 \le i \le m,$$

and (v)

$$\|\widetilde{\mathbf{grad}}\, z\|_{L^2(\Omega)} \le C\|\mathbf{grad}\,\rho\|_{H(\mathrm{div}\,;\Omega)}. \tag{D.2}$$

Then $\boldsymbol{\eta} = \boldsymbol{\chi} + \mathbf{grad}\,\rho + \widetilde{\mathbf{grad}}\, z$ belongs to $H_0(\mathrm{div}^0; \Omega)$ and $\boldsymbol{\eta}$ satisfies the constraints in (3.8). Therefore $\boldsymbol{\eta}$ belongs to $\mathscr{D}_F^0$ and it follows from Lemma 2.4 that

$$\boldsymbol{\eta} = \mathbf{curl}\,\boldsymbol{w} \tag{D.3}$$

for a unique vector field $\boldsymbol{w} \in \mathscr{P}_N$.

Note that

$$(\mathbf{grad}\,\rho, \boldsymbol{\psi})_{L^2(\Omega)} = 0 = (\widetilde{\mathbf{grad}}\, z, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in \mathscr{P}_T$$

by integration by parts. Consequently we have

$$\begin{aligned}(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} &= (\boldsymbol{\chi}, \boldsymbol{\psi})_{L^2(\Omega)}\\ &= (\boldsymbol{\chi} + \mathbf{grad}\,\rho + \widetilde{\mathbf{grad}}\, z, \boldsymbol{\psi})_{L^2(\Omega)}\\ &= (\boldsymbol{\eta}, \boldsymbol{\psi})_{L^2(\Omega)}\\ &= (\mathbf{curl}\,\boldsymbol{w}, \boldsymbol{\psi})_{L^2(\Omega)} = (\boldsymbol{w}, \mathbf{curl}\,\boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in \mathscr{P}_T,\end{aligned}$$

which implies $\mathbf{curl}\,\tilde{\boldsymbol{\zeta}} = \boldsymbol{w}$ by Lemma 2.1 because $\boldsymbol{w} \in \mathscr{D}_F$ by Remark 2.3.

Finally we have

$$\|\mathbf{curl}\,(\mathbf{curl}\,\tilde{\boldsymbol{\zeta}})\|_{L^2(\Omega)} = \|\mathbf{curl}\,\boldsymbol{w}\|_{L^2(\Omega)} = \|\boldsymbol{\eta}\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{H^1(\Omega)}$$

by (D.1)–(D.3).

Appendix E Derivation of (5.36)

The explicit form of (5.35) is given by

$$a(\tilde{\boldsymbol{\zeta}}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \tilde{\theta}) + \gamma^{\frac{1}{2}}(\boldsymbol{\psi}, \tilde{\boldsymbol{\xi}})_{L^2(\Omega)} = (\boldsymbol{\chi}, \boldsymbol{\psi})_{L^2(\Omega)}, \tag{E.1a}$$

$$b(\tilde{\boldsymbol{\zeta}}, \mu) = 0, \tag{E.1b}$$

$$\gamma^{\frac{1}{2}}(\tilde{\boldsymbol{\zeta}}, \boldsymbol{\eta})_{L^2(\Omega)} - g(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta}) - \beta(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta})_{L^2(\Omega)} - c(\boldsymbol{\eta}, \tilde{p}) = 0, \tag{E.1c}$$

$$-c(\tilde{\boldsymbol{\xi}}, q) = 0, \tag{E.1d}$$

for all $(\boldsymbol{\psi}, \mu, \boldsymbol{\eta}, q) \in H(\mathbf{curl}\,; \Omega) \times \Theta^0 \times [H_0^1(\Omega)]^3_\Sigma \times L_0^2(\Omega)$.

From the well-posedness of the saddle point problem (E.1) we have

$$\|\tilde{\boldsymbol{\zeta}}\|_{H(\mathbf{curl}\,;\Omega)} + \|\tilde{\theta}\|_{H^1(\Omega^\circ)} + \|\tilde{\boldsymbol{\xi}}\|_{H^1(\Omega)} + \|\tilde{p}\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{L^2(\Omega)}, \tag{E.2}$$

and it follows from (E.1a) and (E.1b) that

$$a(\tilde{\boldsymbol{\zeta}}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, \tilde{\theta}) = (\tilde{\boldsymbol{\chi}}, \boldsymbol{\psi})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\psi} \in H(\mathbf{curl}\,; \Omega), \tag{E.3a}$$

$$b(\tilde{\boldsymbol{\zeta}}, \mu) = 0 \qquad \forall\, \mu \in \Theta^\circ, \tag{E.3b}$$

where $\tilde{\boldsymbol{\chi}} = \boldsymbol{\chi} - \gamma^{\frac{1}{2}}\tilde{\boldsymbol{\xi}}$.

Since (E.3) is just (5.13) with $\boldsymbol{\chi}$ replaced by $\tilde{\boldsymbol{\chi}}$, we deduce from Lemma 5.2 that $\mathbf{curl}\,\tilde{\boldsymbol{\zeta}} \in H_0(\mathbf{curl}\,; \Omega) \cap H(\mathrm{div}^0; \Omega)$ and hence

$$\|\tilde{\boldsymbol{\zeta}}\|_{H^{\alpha_T}(\Omega)} + \|\mathbf{curl}\,\tilde{\boldsymbol{\zeta}}\|_{H^{\alpha_N}(\Omega)} \le C\|\tilde{\boldsymbol{\chi}}\|_{H^1(\Omega)} \le C\|\boldsymbol{\chi}\|_{H^1(\Omega)}.$$

by Remark 5.3 and (E.2).

From (E.3a) we also have the following analog of (5.21):

$$(\mathbf{grad}\,\mu, \mathbf{grad}\,\tilde{\theta})_{L^2(\Omega^\circ)} = (-\mathrm{div}\,\tilde{\boldsymbol{\chi}}, \mu)_{L^2(\Omega^\circ)} \qquad \forall\, \mu \in \Theta^0,$$

and hence, in view of (5.22) and (E.2),

$$\|\tilde{\theta}\|_{H^{1+\alpha_{\mathrm{Neu}}}(\Omega^\circ)} \le C\|\mathrm{div}\,\tilde{\boldsymbol{\chi}}\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{H^1(\Omega)}.$$

Observe that (E.1c) and (E.1d) imply

$$g(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta}) + \beta(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta})_{L^2(\Omega)} + c(\boldsymbol{\eta}, \tilde{p}) = (\gamma^{\frac{1}{2}}\tilde{\boldsymbol{\zeta}}, \boldsymbol{\eta})_{L^2(\Omega)} \qquad \forall\, \boldsymbol{\eta} \in [H_0^1(\Omega)]^3_\Sigma, \tag{E.4a}$$

$$c(\tilde{\boldsymbol{\xi}}, q) = 0 \qquad \forall\, q \in L_0^2(\Omega). \tag{E.4b}$$

We conclude from the results in Appendix C and (E.2) that

$$\|\tilde{\boldsymbol{\xi}}\|_{H^{1+\alpha_S}(\Omega)} + \|\tilde{p}\|_{H^{\alpha_S}(\Omega^\circ)} \le C\|\gamma^{\frac{1}{2}}\tilde{\boldsymbol{\zeta}}\|_{L^2(\Omega)} \le C\|\boldsymbol{\chi}\|_{L^2(\Omega)}.$$

Funding This work was supported in part by the National Science Foundation under Grant No. DMS-22-08404.

Data Availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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