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A new error analysis for finite element methods for elliptic Neumann boundary control problems with pointwise control constraints☆

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ABSTRACT

We present a new error analysis for finite element methods for a linear-quadratic elliptic optimal control problem with Neumann boundary control and pointwise control constraints. It can be applied to standard finite element methods when the coefficients in the elliptic operator are smooth and also to multiscale finite element methods when the coefficients are rough.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a Lipschitz polyhedral domain with boundary Γ, $y_d, f \in L_2(\Omega)$, $\phi_1, \phi_2 \in L_2(\Gamma)$, $\phi_1 \le \phi_2$ on Γ, $\gamma \in (0, 1]$ be a positive constant, and

$$a(y, z) = \int_\Omega A\nabla y \cdot \nabla z \, dx + \int_\Omega \kappa yz \, dx, \tag{1.1}$$

where the $d \times d$ symmetric coefficient matrix $A(x)$ is positive definite, and κ is a nonnegative bounded measurable function such that $\|\kappa\|_{L_1(\Omega)} > 0$. More precisely, we assume that the components of $A(x)$ are Lebesgue measurable functions and there exist two positive numbers $\alpha \le \beta$ such that

$$\alpha|\xi|^2 \le \xi^t A(x)\xi \le \beta|\xi|^2 \qquad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^d. \tag{1.2}$$

The model linear-quadratic Neumann boundary control problem (cf. [1,2]) is to find

$$(\bar{y}, \bar{u}) = \operatorname*{argmin}_{(y,u)\in\mathbb{K}} \frac{1}{2}\left[\|y - y_d\|^2_{L_2(\Omega)} + \gamma\|u\|^2_{L_2(\Gamma)}\right], \tag{1.3}$$

where the subset $\mathbb{K}$ of $H^1(\Omega) \times L_2(\Gamma)$ consists of ordered pairs (y, u) that satisfy

$$a(y, z) = \int_\Omega fz \, dx + \int_\Gamma uz \, ds \qquad \forall\, z \in H^1(\Omega), \tag{1.4}$$

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and

$$\phi_1 \le u \le \phi_2 \quad \text{a.e. on} \quad \Gamma. \tag{1.5}$$

Here ds is the infinitesimal arc length ($d = 2$) or the infinitesimal surface area ($d = 3$).

Remark 1.1. We follow the standard notation for differential operators, function spaces and norms that can be found for example in [3–5].

Remark 1.2. It is clear that the bilinear form $a(\cdot,\cdot)$ is bounded on $H^1(\Omega)$. From a generalized Poincaré–Friedrichs inequality (cf. [6]) we also have

$$\|v\|^2_{H^1(\Omega)} \le C_{\rm PF}\big(|v|^2_{H^1(\Omega)} + \|\sqrt{\kappa}v\|^2_{L_2(\Omega)}\big),$$

which together with (1.2) implies the coercivity of $a(\cdot,\cdot)$:

$$\|v\|^2_{H^1(\Omega)} \le C_{\rm PF}\max(\alpha^{-1},1)a(v,v) \qquad \forall\, v \in H^1(\Omega). \tag{1.6}$$

Therefore (1.4) is uniquely solvable.

It follows from (1.6) that

$$\|v\|_{H^1(\Omega)} \le \sqrt{C_{\rm PF}\max(\alpha^{-1},1)}\|v\|_a \qquad \forall\, v \in H^1(\Omega), \tag{1.7}$$

where $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$.

Remark 1.3. We can interpret (1.4) as a steady state diffusion–reaction model for a chemical substance in a heterogeneous media with the source f and the boundary control (in a weak sense)

$$(A\nabla y)\cdot \boldsymbol{n} = u \qquad \text{on} \quad \Gamma,$$

where $\boldsymbol{n}$ is the unit outer normal on Γ.

There is a substantial literature (cf. the monographs [7–9] and the references therein) on the analysis of finite element methods for elliptic optimal control problems beginning with the pioneering work of Falk in [10]. Nevertheless the analyses in the literature are not directly applicable to multiscale finite element methods for problems with rough coefficients, i.e., when (1.2) is the only assumption on the coefficient matrix $A(x)$.

Recently a new error analysis for distributed elliptic optimal control problems with pointwise control constraints was developed in [11,12] that is applicable to standard finite element methods (when A is smooth) and also to multiscale finite element methods (when A is rough). Our goal in this paper is to develop a similar error analysis for the model Neumann boundary control problem defined by (1.3)–(1.5).

The rest of the paper is organized as follows. We recall the properties of the continuous problem in Section 2 and introduce the discrete problem in Section 3. An abstract error analysis is given in Section 4, followed by application to problems with smooth coefficients in Section 5 and application to problems with rough coefficients in Section 6. We end with some concluding remarks in Section 7. Some estimates for the continuous problem are given in the Appendix.

Throughout the paper we use C (with or without subscript) to denote a generic positive constant that can take different values at different occurrences.

2. The continuous problem

The convex minimization problem defined by (1.3)–(1.5) has a unique solution $(\bar{y},\bar{u})$ characterized by the first order optimality condition (cf. [13])

$$\int_\Omega (\bar{y}-y_d)(y-\bar{y})dx + \gamma\int_\Gamma \bar{u}(u-\bar{u})ds \ge 0 \qquad \forall\,(y,u)\in\mathbb{K}. \tag{2.1}$$

Let the adjoint state $\bar{p}\in H^1(\Omega)$ be defined by

$$a(q,\bar{p}) = \int_\Omega (\bar{y}-y_d)q\,dx \qquad \forall\, q\in H^1(\Omega). \tag{2.2}$$

We can deduce from (1.4) and (2.2) that

$$\begin{aligned}\int_\Omega (\bar{y}-y_d)(y-\bar{y})dx &= a(y-\bar{y},\bar{p})\\ &= \Big(\int_\Omega f\bar{p}\,dx + \int_\Gamma u\bar{p}\,ds\Big) - \Big(\int_\Omega f\bar{p}\,dx + \int_\Gamma \bar{u}\bar{p}\,ds\Big)\\ &= \int_\Gamma (u-\bar{u})\bar{p}\,ds \qquad\qquad \forall\,(y,u)\in\mathbb{K}.\end{aligned}$$

Consequently we have, by (2.1),

$$\int_\Gamma (\bar{p}+\gamma\bar{u})(u-\bar{u})ds \ge 0 \qquad \forall\, u\in U_{ad}, \tag{2.3}$$

where

$$U_{ad} = \{u \in L_2(\Gamma) : \phi_1 \le u \le \phi_2\}. \tag{2.4}$$

It follows from (2.3) that $\bar{u}$ is the $L_2(\Gamma)$ orthogonal projection of $-(1/\gamma)\bar{p}$ on U_{ad} and hence

$$\bar{u} = \max\big(\phi_1, \min(\phi_2, -(1/\gamma)\bar{p})\big). \tag{2.5}$$

Let the functions $\lambda_1, \lambda_2 \in L_2(\Gamma)$ be defined by

$$\lambda_1 = \begin{cases} \bar{p} + \gamma\phi_1 & \text{if } \phi_1 \ge -(1/\gamma)\bar{p} \\ 0 & \text{if } \phi_1 < -(1/\gamma)\bar{p} \end{cases} = \max(\bar{p} + \gamma\phi_1, 0), \tag{2.6}$$

and

$$\lambda_2 = \begin{cases} \bar{p} + \gamma\phi_2 & \text{if } \phi_2 \le -(1/\gamma)\bar{p} \\ 0 & \text{if } \phi_2 > -(1/\gamma)\bar{p} \end{cases} = \min(\bar{p} + \gamma\phi_2, 0). \tag{2.7}$$

Then it follows from (2.5)–(2.7) (cf. [2, Section 2.8.4]) that

$$\bar{p} + \gamma\bar{u} = \lambda_1 + \lambda_2, \tag{2.8}$$

$$\lambda_1 \ge 0 \quad \text{and} \quad \lambda_2 \le 0, \tag{2.9}$$

$$\int_\Gamma \lambda_1(\bar{u} - \phi_1)ds = 0 = \int_\Gamma \lambda_2(\bar{u} - \phi_2)ds. \tag{2.10}$$

3. The discrete problem

Let $\mathcal{T}_\rho$ be a regular triangulation of Γ with mesh size ρ and V_* be a finite dimensional subspace of $H^1(\Omega)$. The space of piecewise constant functions with respect to $\mathcal{T}_\rho$ is denoted by W_ρ, and the orthogonal projection operator from $L_2(\Gamma)$ onto W_ρ is denoted by Q_ρ.

The discrete problem for (1.3)–(1.5) is to find

$$(\bar{y}_{*,\rho}, \bar{u}_{*,\rho}) = \operatorname*{argmin}_{(y_*,u_\rho)\in\mathbb{K}_{*,\rho}} \frac{1}{2}\big[\|y_* - y_d\|^2_{L_2(\Omega)} + \gamma\|u_\rho\|^2_{L_2(\Gamma)}\big]. \tag{3.1}$$

Here the subset $\mathbb{K}_{*,\rho}$ of $V_* \times W_\rho$ consists of ordered pairs (y_*, u_ρ) such that $u_\rho \in U^\rho_{ad}$, where

$$U^\rho_{ad} = \{w_\rho \in W_\rho : Q_\rho\phi_1 \le w_\rho \le Q_\rho\phi_2 \quad \text{a.e. on } \Gamma\}, \tag{3.2}$$

and

$$y_* = y_*^\dagger - B_*u_\rho, \tag{3.3}$$

where $y_*^\dagger \in V_*$ satisfies

$$a(y_*^\dagger, z_*) = \int_\Omega f z_* dx + \int_\Gamma u_\rho z_* ds \qquad \forall\, z_* \in V_*. \tag{3.4}$$

The operator B_* is a bounded linear operator from $L_2(\Gamma)$ to $H^1(\Omega)$ whose image is orthogonal to V_* with respect to $a(\cdot,\cdot)$, i.e.,

$$a(B_*v, z_*) = 0 \qquad \forall\, v \in L_2(\Gamma),\ z_* \in V_*. \tag{3.5}$$

Remark 3.1. The operator $B_* = 0$ for standard finite element methods. For multiscale finite element methods B_* is a correction operator for the Neumann boundary data.

Remark 3.2. It follows from (2.4) and (3.2) that Q_ρ maps U_{ad} onto U^ρ_{ad}. In particular U^ρ_{ad} is nonempty.

The convex minimization problem defined by (3.1)–(3.4) has a unique solution $(\bar{y}_{*,\rho}, \bar{u}_{*,\rho})$ in $\mathbb{K}_{*,\rho}$ characterized by the first order optimality condition

$$\int_\Omega (\bar{y}_{*,\rho} - y_d)(y_* - \bar{y}_{*,\rho})dx + \gamma\int_\Gamma \bar{u}_{*,\rho}(u_\rho - \bar{u}_{*,\rho})ds \ge 0 \qquad \forall\,(y_*, u_\rho) \in \mathbb{K}_{*,\rho}. \tag{3.6}$$

The discrete adjoint state $\bar{p}_{*,\rho} \in V_*$ is defined by

$$a(q_*, \bar{p}_{*,\rho}) = \int_\Omega (\bar{y}_{*,\rho} - y_d)q_* dx \qquad \forall\, q_* \in V_*. \tag{3.7}$$

The following simple stability result is useful for the error analysis of the discrete problem.

Lemma 3.3. *Let V be a closed subspace of $H^1(\Omega)$, $r \in L_2(\Omega)$ and $g \in L_2(\Gamma)$. If $v \in V$ is defined by*

$$a(v,w) = \int_\Omega rw\,dx + \int_\Gamma gw\,ds \qquad \forall\, w \in V, \tag{3.8}$$

then we have

$$\|v\|_{H^1(\Omega)} \le C_{\rm PF}\max(\alpha^{-1},1)\big(\|r\|_{L_2(\Omega)} + C_{\rm Tr}\|g\|_{L_2(\Gamma)}\big), \tag{3.9}$$

$$\|v\|_a \le \sqrt{C_{\rm PF}\max(\alpha^{-1},1)}\big(\|r\|_{L_2(\Omega)} + C_{\rm Tr}\|g\|_{L_2(\Gamma)}\big), \tag{3.10}$$

where $C_{\rm Tr}$ is the positive constant that appears in the trace inequality

$$\|\zeta\|_{L_2(\Gamma)} \le C_{\rm Tr}\|\zeta\|_{H^1(\Omega)} \qquad \forall\,\zeta \in H^1(\Omega). \tag{3.11}$$

Proof. It follows from (1.6), (3.8) and (3.11) that

$$\begin{aligned}
\|v\|^2_{H^1(\Omega)} &\le C_{\rm PF}\max(\alpha^{-1},1)a(v,v)\\
&= C_{\rm PF}\max(\alpha^{-1},1)\Big(\int_\Omega rv\,dx + \int_\Gamma gv\,ds\Big)\\
&\le C_{\rm PF}\max(\alpha^{-1},1)\big(\|r\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma)}C_{\rm Tr}\|v\|_{H^1(\Omega)}\big),
\end{aligned}$$

which implies (3.9).

Similarly we have, by (1.7), (3.8) and (3.11),

$$\begin{aligned}
\|v\|^2_a &= \int_\Omega rv\,dx + \int_\Gamma gv\,ds\\
&\le (\|r\|_{L_2(\Omega)} + C_{\rm Tr}\|g\|_{L_2(\Gamma)})\|v\|_{H^1(\Omega)}\\
&\le (\|r\|_{L_2(\Omega)} + C_{\rm Tr}\|g\|_{L_2(\Gamma)})\sqrt{C_{\rm PF}\max(\alpha^{-1},1)}\|v\|_a,
\end{aligned}$$

which implies (3.10). □

4. An abstract error analysis

We will derive error estimates in terms of the correction operator $B_* : L_2(\Gamma) \longrightarrow H^1(\Omega)$, the L_2 projection operator $Q_\rho : L_2(\Gamma) \longrightarrow W_\rho$, and the Ritz projection operator $R_* : H^1(\Omega) \longrightarrow V_*$ defined by

$$a(R_*\zeta, v_*) = a(\zeta, v_*) \qquad \forall\, v_* \in V_*. \tag{4.1}$$

Theorem 4.1. *There exists a positive constant $C_\flat$ depending only on α^{-1} and γ^{-1} such that*

$$\begin{aligned}
&\|\bar y - \bar y_{*,\rho}\|^2_{L_2(\Omega)} + \|\bar u - \bar u_{*,\rho}\|^2_{L_2(\Gamma)}\\
&\quad\le C_\flat\Big(\|\bar y - (R_*\bar y - B_*\bar u)\|^2_{L_2(\Omega)} + (1+\|B_*\|^2)\|\bar u - Q_\rho\bar u\|^2_{L_2(\Gamma)} + \|\bar p - R_*\bar p\|^2_{L_2(\Gamma)}\\
&\qquad + \|B_*\|^2\|\bar p - R_*\bar p\|^2_a + \|\lambda_1 - Q_\rho\lambda_1\|^2_{L_2(\Gamma)} + \|\lambda_2 - Q_\rho\lambda_2\|^2_{L_2(\Gamma)}\\
&\qquad + \|\phi_1 - Q_\rho\phi_1\|^2_{L_2(\Gamma)} + \|\phi_2 - Q_\rho\phi_2\|^2_{L_2(\Gamma)}\Big),
\end{aligned}$$

where $\|B_*\|$ is the operator norm of $B_* : L_2(\Gamma) \longrightarrow H^1(\Omega)$ with respect to the norms $\|\cdot\|_{L_2(\Gamma)}$ and $\|\cdot\|_a$.

Proof. From (2.2), (3.3)–(3.5) and (4.1), we find, for any $(y_*,u_\rho) \in \mathbb{K}_{*,\rho}$,

$$\begin{aligned}
\int_\Omega (\bar y - y_d)(y_* - \bar y_{*,\rho})dx &= a(y_* - \bar y_{*,\rho}, \bar p)\\
&= a(y^\dagger_* - \bar y^\dagger_{*,\rho}, \bar p) + a\big(B_*(u_\rho - \bar u_{*,\rho}), \bar p\big)\\
&= a(y^\dagger_* - \bar y^\dagger_{*,\rho}, R_*\bar p) + a\big(B_*(u_\rho - \bar u_{*,\rho}), \bar p - R_*\bar p\big)\\
&= \Big(\int_\Omega f R_*\bar p\,dx + \int_\Gamma u_\rho R_*\bar p\,ds\Big) - \Big(\int_\Omega f R_*\bar p\,dx + \int_\Gamma \bar u_{*,\rho}R_*\bar p\,ds\Big)\\
&\qquad + a\big(B_*(u_\rho - \bar u_{*,\rho}), \bar p - R_*\bar p\big)\\
&= \int_\Gamma (u_\rho - \bar u_{*,\rho})R_*\bar p\,ds + a\big(B_*(u_\rho - \bar u_{*,\rho}), \bar p - R_*\bar p\big).
\end{aligned} \tag{4.2}$$

Let $\tilde u_\rho \in W_\rho$ be defined by

$$\tilde u_\rho = Q_\rho\bar u. \tag{4.3}$$

Then $\tilde{u}_\rho$ belongs to U_{ad}^ρ by Remark 3.2 and we define

$$\tilde{y}_* = \tilde{y}_*^\dagger - B_* \tilde{u}_\rho, \tag{4.4}$$

where $\tilde{y}_{*,\rho}^\dagger \in V_*$ satisfies

$$a(\tilde{y}_*^\dagger, z_*) = \int_\Omega f z_* dx + \int_\Gamma \tilde{u}_\rho z_* ds \qquad \forall\, z_* \in V_* \tag{4.5}$$

so that $(\tilde{y}_*, \tilde{u}_\rho) \in \mathbb{K}_{*,\rho}$.

We can write

$$\begin{aligned}
&\|\bar{y} - \bar{y}_{*,\rho}\|_{L_2(\Omega)}^2 + \gamma \|\bar{u} - \bar{u}_{*,\rho}\|_{L_2(\Gamma)}^2 \\
&\qquad = \int_\Omega (\bar{y} - \bar{y}_{*,\rho})(\bar{y} - \tilde{y}_*) dx + \gamma \int_\Gamma (\bar{u} - \bar{u}_{*,\rho})(\bar{u} - \tilde{u}_\rho) ds \\
&\qquad\qquad + \int_\Omega (\bar{y} - \bar{y}_{*,\rho})(\tilde{y}_* - \bar{y}_{*,\rho}) dx + \gamma \int_\Gamma (\bar{u} - \bar{u}_{*,\rho})(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds.
\end{aligned} \tag{4.6}$$

It follows from (2.8), (3.6), (4.2) and (4.3) that

$$\begin{aligned}
&\int_\Omega (\bar{y} - \bar{y}_{*,\rho})(\tilde{y}_* - \bar{y}_{*,\rho}) dx + \gamma \int_\Gamma (\bar{u} - \bar{u}_{*,\rho})(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \\
&\qquad = \int_\Omega \bar{y}(\tilde{y}_* - \bar{y}_{*,\rho}) dx + \gamma \int_\Gamma \bar{u}(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \\
&\qquad\qquad - \Big[\int_\Omega \bar{y}_{*,\rho}(\tilde{y}_* - \bar{y}_{*,\rho}) dx + \gamma \int_\Gamma \bar{u}_{*,\rho}(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \Big] \\
&\qquad \le \int_\Omega (\bar{y} - y_d)(\tilde{y}_* - \bar{y}_{*,\rho}) dx + \gamma \int_\Gamma \bar{u}(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \\
&\qquad = \int_\Gamma (R_* \bar{p} + \gamma \bar{u})(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds + a\big(B_*(\tilde{u}_\rho - \bar{u}_{*,\rho}), \bar{p} - R_* \bar{p}\big) \\
&\qquad = \int_\Gamma (\lambda_1 + \lambda_2)(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds + \int_\Gamma (R_* \bar{p} - \bar{p})(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \\
&\qquad\qquad + a\big(B_*(Q_\rho \bar{u} - \bar{u}_{*,\rho}), \bar{p} - R_* \bar{p}\big).
\end{aligned} \tag{4.7}$$

We have

$$\int_\Gamma (R_* \bar{p} - \bar{p})(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \le \|R_* \bar{p} - \bar{p}\|_{L_2(\Gamma)} \big(\|Q_\rho \bar{u} - \bar{u}\|_{L_2(\Gamma)} + \|\bar{u} - \bar{u}_{*,\rho}\|_{L_2(\Gamma)} \big) \tag{4.8}$$

by (4.3), the Cauchy–Schwarz inequality and the triangle inequality, and also

$$a\big(B_*(Q_\rho \bar{u} - \bar{u}_{*,\rho}), \bar{p} - R_* \bar{p}\big) \le \|B_*\| \big(\|\bar{u} - Q_\rho \bar{u}\|_{L_2(\Gamma)} + \|\bar{u} - \bar{u}_{*,\rho}\|_{L_2(\Gamma)} \big) \|\bar{p} - R_* \bar{p}\|_a. \tag{4.9}$$

We can estimate the first term on the right-hand side of (4.7) by (2.9), (2.10), (3.2), (4.3) and the Cauchy–Schwarz inequality as follows:

$$\begin{aligned}
&\int_\Gamma (\lambda_1 + \lambda_2)(\tilde{u}_\rho - \bar{u}_{*,\rho}) ds \\
&\qquad = \int_\Gamma \lambda_1(\tilde{u}_\rho - \bar{u}) ds + \int_\Gamma \lambda_2(\tilde{u}_\rho - \bar{u}) ds \\
&\qquad\qquad + \int_\Gamma \lambda_1(\bar{u} - \phi_1) ds + \int_\Gamma \lambda_2(\bar{u} - \phi_2) ds \\
&\qquad\qquad\qquad + \int_\Gamma \lambda_1(\phi_1 - Q_\rho \phi_1) ds + \int_\Gamma \lambda_2(\phi_2 - Q_\rho \phi_2) ds \\
&\qquad\qquad\qquad\qquad + \int_\Gamma \lambda_1(Q_\rho \phi_1 - \bar{u}_{*,\rho}) ds + \int_\Gamma \lambda_2(Q_\rho \phi_2 - \bar{u}_{*,\rho}) ds \\
&\qquad \le \int_\Gamma \lambda_1(\tilde{u}_\rho - \bar{u}) ds + \int_\Gamma \lambda_2(\tilde{u}_\rho - \bar{u}) ds \\
&\qquad\qquad + \int_\Gamma \lambda_1(\phi_1 - Q_\rho \phi_1) ds + \int_\Gamma \lambda_2(\phi_2 - Q_\rho \phi_2) ds \\
&\qquad = \int_\Gamma (\lambda_1 - Q_\rho \lambda_1)(Q_\rho \bar{u} - \bar{u}) ds + \int_\Gamma (\lambda_2 - Q_\rho \lambda_2)(Q_\rho \bar{u} - \bar{u}) ds \\
&\qquad\qquad + \int_\Gamma (\lambda_1 - Q_\rho \lambda_1)(\phi_1 - Q_\rho \phi_1) ds + \int_\Gamma (\lambda_2 - Q_\rho \lambda_2)(\phi_2 - Q_\rho \phi_2) ds \\
&\qquad \le \big(\|\lambda_1 - Q_\rho \lambda_1\|_{L_2(\Gamma)} + \|\lambda_2 - Q_\rho \lambda_2\|_{L_2(\Gamma)} \big) \|\bar{u} - Q_\rho \bar{u}\|_{L_2(\Gamma)} \\
&\qquad\qquad + \|\lambda_1 - Q_\rho \lambda_1\|_{L_2(\Gamma)} \|\phi_1 - Q_\rho \phi_1\|_{L_2(\Gamma)}
\end{aligned} \tag{4.10}$$

$$+\|\lambda_2-Q_\rho\lambda_2\|_{L_2(\Gamma)}\|\phi_2-Q_\rho\phi_2\|_{L_2(\Gamma)}.$$

Note that (1.4), (4.1), (4.3) and (4.5) imply

$$\begin{aligned}a(R_*\bar{y}-\tilde{y}_*^\dagger,z_*)&=a(\bar{y}-\tilde{y}_*^\dagger,z_*)\\&=\Big(\int_\Omega fz_*dx+\int_\Gamma\bar{u}z_*ds\Big)-\Big(\int_\Omega fz_*dx+\int_\Gamma\tilde{u}_\rho z_*ds\Big)\\&=\int_\Gamma(\bar{u}-Q_\rho\bar{u})z_*ds\qquad\forall\,z_*\in V_*,\end{aligned}$$

and hence

$$\|R_*\bar{y}-\tilde{y}_*^\dagger\|_{L_2(\Omega)}\le C_{\mathrm{Tr}}C_{\mathrm{PF}}\max(\alpha^{-1},1)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}$$

by Lemma 3.3. Consequently

$$\begin{aligned}\|\bar{y}-\tilde{y}_*\|_{L_2(\Omega)}&=\|\bar{y}-(\tilde{y}_*^\dagger-B_*\tilde{u}_\rho)\|_{L_2(\Omega)}\\&\le\|\bar{y}-(R_*\bar{y}-B_*\bar{u})\|_{L_2(\Omega)}+\|R_*\bar{y}-\tilde{y}_*^\dagger\|_{L_2(\Omega)}+\|B_*(\bar{u}-\tilde{u}_\rho)\|_{L_2(\Omega)}\\&\le\|\bar{y}-(R_*\bar{y}-B_*\bar{u})\|_{L_2(\Omega)}+C_{\mathrm{Tr}}C_{\mathrm{PF}}\max(\alpha^{-1},1)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\\&\qquad+\|B_*\|(C_{\mathrm{PF}}\max(\alpha^{-1},1))^{\frac12}\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)},\end{aligned}\tag{4.11}$$

where we have used (1.7) and (4.3).

Therefore we have, by (4.3), (4.11) and the Cauchy–Schwarz inequality,

$$\begin{aligned}&\int_\Omega(\bar{y}-\bar{y}_{*,\rho})(\bar{y}-\tilde{y}_*)dx+\gamma\int_\Gamma(\bar{u}-\bar{u}_{*,\rho})(\bar{u}-\tilde{u}_\rho)ds\\&\le\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}(C_{\mathrm{Tr}}C_{\mathrm{PF}}\max(\alpha^{-1},1)+(C_{\mathrm{PF}}\max(\alpha^{-1},1))^{\frac12}\|B_*\|)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\\&\quad+\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}\|\bar{y}-(R_*\bar{y}-B_*\bar{u})\|_{L_2(\Omega)}+\gamma\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}.\end{aligned}\tag{4.12}$$

Putting (4.6)–(4.10) and (4.12) together, we find

$$\begin{aligned}&\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}^2+\gamma\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}^2\\&\le\|R_*\bar{p}-\bar{p}\|_{L_2(\Gamma)}\big(\|Q_\rho\bar{u}-\bar{u}\|_{L_2(\Gamma)}+\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}\big)\\&\quad+\|B_*\|\big(\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}+\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}\big)\|\bar{p}-R_*\bar{p}\|_a\\&\quad+\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}(C_{\mathrm{Tr}}C_{\mathrm{PF}}\max(\alpha^{-1},1)+(C_{\mathrm{PF}}\max(\alpha^{-1},1))^{\frac12}\|B_*\|)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\\&\quad+\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}\|\bar{y}-(R_*\bar{y}-B_*\bar{u})\|_{L_2(\Omega)}+\gamma\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\\&\quad+\big(\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Gamma)}+\|\lambda_2-Q_\rho\lambda_2\|_{L_2(\Gamma)}\big)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\\&\quad+\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Gamma)}\|\phi_1-Q_\rho\phi_1\|_{L_2(\Gamma)}+\|\lambda_2-Q_\rho\lambda_2\|_{L_2(\Gamma)}\|\phi_2-Q_\rho\phi_2\|_{L_2(\Gamma)}\end{aligned}$$

which together with the inequality of arithmetic and geometric means implies

$$\begin{aligned}&\|\bar{y}-\bar{y}_{*,\rho}\|_{L_2(\Omega)}^2+\gamma\|\bar{u}-\bar{u}_{*,\rho}\|_{L_2(\Gamma)}^2\\&\le C_\diamond\Big[\|\bar{y}-(R_*\bar{y}-B_*\bar{u})\|_{L_2(\Omega)}^2+\gamma^{-1}\|\bar{p}-R_*\bar{p}\|_{L_2(\Gamma)}^2+\gamma^{-1}\|B_*\|^2\|\bar{p}-R_*\bar{p}\|_a^2\\&\quad+\big(1+[C_{\mathrm{Tr}}C_{\mathrm{PF}}\max(\alpha^{-1},1)]^2+[C_{\mathrm{PF}}\max(\alpha^{-1},1)+1]\|B_*\|^2\big)\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}^2\\&\quad+\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Gamma)}^2+\|\lambda_2-Q_\rho\lambda_2\|_{L_2(\Gamma)}^2\\&\quad+\|\phi_1-Q_\rho\phi_1\|_{L_2(\Gamma)}^2+\|\phi_2-Q_\rho\phi_2\|_{L_2(\Gamma)}^2\Big],\end{aligned}$$

where $C_\diamond$ is a universal positive constant. □

5. Application to problems with smooth coefficients

In the case of smooth $A(x)$ and $\kappa(x)$, we have more information on $\bar{y}$, $\bar{p}$ and $\bar{u}$. For simplicity we only consider the case where Ω is convex.

5.1. Regularity results for the continuous problem

From the elliptic regularity result in [14, Theorem 3.2.1.3] for homogeneous Neumann boundary value problems, we know from (2.2) that

$$\|\bar{p}\|_{H^2(\Omega)}\le C_\Omega\|\bar{y}-y_d\|_{L_2(\Omega)}.\tag{5.1}$$

Let Σ be the set of the sides (edges for $d = 2$ and faces for $d = 3$) of Γ. It follows from the trace theorem (cf. [4]) that

$$\bar{p}\Big|_{\sigma} \in H^{3/2}(\sigma) \qquad \forall\, \sigma \in \Sigma. \tag{5.2}$$

We assume that the control constraints

$$\phi_i \ (i = 1, 2) \text{ belong to } H^1(\sigma) \text{ for every } \sigma \in \Sigma. \tag{5.3}$$

We can then obtain from (2.5), (5.2), (5.3) and [15, Lemma 7.6] that

$$|\bar{u}|_{H^1(\sigma)} \leq \max\big(|\phi_1|_{H^1(\sigma)}, |\phi_2|_{H^1(\sigma)}, \gamma^{-1}|\bar{p}|_{H^1(\sigma)}\big) \qquad \forall\, \sigma \in \Sigma. \tag{5.4}$$

Similarly, we have

$$|\lambda_1|_{H^1(\sigma)} \leq \max\big(|\bar{p}|_{H^1(\sigma)}, |\phi_1|_{H^1(\sigma)}\big) \qquad \forall\, \sigma \in \Sigma, \tag{5.5}$$

$$|\lambda_2|_{H^1(\sigma)} \leq \max\big(|\bar{p}|_{H^1(\sigma)}, |\phi_2|_{H^1(\sigma)}\big) \qquad \forall\, \sigma \in \Sigma, \tag{5.6}$$

by (2.6), (2.7) and the assumption that $\gamma \leq 1$.

Now we turn to the regularity of $\bar{y}$, which is the solution of the nonhomogeneous Neumann boundary value problem (1.4) (with $u = \bar{u}$). It follows from (5.4) that

$$\|\bar{y}\|_{H^2(\Omega)} \leq C_{\Omega}\big(\|f\|_{L_2(\Omega)} + \sum_{\sigma\in\Sigma} \|\bar{u}\|_{H^{1/2}(\sigma)}\big). \tag{5.7}$$

For $d = 2$ this regularity result follows from Theorem 1.4.3, Theorem 2.4.3 and Remark 2.4.5 in [16]. For $d = 3$ it follows from Theorem 8.1.7 in [17] and the eigenvalue estimate in [18], and it also follows from the results in [19].

5.2. Results for the P_1 finite element method

We take $V_* = V_h$ to be the standard P_1 finite element space associated with a regular triangulation $\mathcal{T}_h$ of Ω with mesh size h. We then take $\mathcal{T}_\rho$ to be a refinement of the triangulation on Γ induced by $\mathcal{T}_h$ and denote the solution for the optimal control problem by $(\bar{y}_{h,\rho}, \bar{u}_{h,\rho}, \bar{p}_{h,\rho})$. The Ritz projection operator $R_* : H^1(\Omega) \longrightarrow V_*$ is now denoted by $R_h : H^1(\Omega) \longrightarrow V_h$, i.e.,

$$a(R_h\zeta, v_h) = a(\zeta, v_h) \qquad \forall\, \zeta \in H^1(\Omega), v_h \in V_h. \tag{5.8}$$

In this case we take $B_* = 0$.

It follows from (5.1), (5.4)–(5.6), (5.7) and standard finite element estimates (cf. [3,5]) that

$$\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} \leq Ch^2\|\bar{y}\|_{H^2(\Omega)}, \tag{5.9}$$

$$\|\bar{p} - R_h\bar{p}\|_{H^1(\Omega)} \leq Ch\|\bar{p}\|_{H^2(\Omega)}, \tag{5.10}$$

$$\|\bar{p} - R_h\bar{p}\|_{L_2(\Omega)} \leq Ch^2\|\bar{p}\|_{H^2(\Omega)}, \tag{5.11}$$

$$\|\bar{u} - Q_\rho\bar{u}\|_{L_2(\Gamma)} \leq C\rho \sum_{\sigma\in\Sigma} |\bar{u}|_{H^1(\sigma)}, \tag{5.12}$$

$$\|\lambda_1 - Q_\rho\lambda_1\|_{L_2(\Gamma)} \leq C\rho \sum_{\sigma\in\Sigma} |\lambda_1|_{H^1(\sigma)}, \tag{5.13}$$

$$\|\lambda_2 - Q_\rho\lambda_2\|_{L_2(\Gamma)} \leq C\rho \sum_{\sigma\in\Sigma} |\lambda_2|_{H^1(\sigma)}, \tag{5.14}$$

$$\|\phi_1 - Q_\rho\phi_1\|_{L_2(\Gamma)} \leq C\rho \sum_{\sigma\in\Sigma} |\phi_1|_{H^1(\sigma)}, \tag{5.15}$$

$$\|\phi_2 - Q_\rho\phi_2\|_{L_2(\Gamma)} \leq C\rho \sum_{\sigma\in\Sigma} |\phi_2|_{H^1(\sigma)}, \tag{5.16}$$

where the positive constant C only depends on the shape regularity of $\mathcal{T}_h$ and $\mathcal{T}_\rho$.

Note that (5.10) and (5.11) imply

$$\|\bar{p} - R_h\bar{p}\|_{L_2(\Gamma)} \leq Ch^{3/2}\|\bar{p}\|_{H^2(\Omega)} \tag{5.17}$$

through the trace inequality (cf. [5, Theorem 1.6.6])

$$\|v\|_{L_2(\Gamma)} \leq C_\Omega\|v\|^{\frac{1}{2}}_{L_2(\Omega)}|v|^{\frac{1}{2}}_{H^1(\Omega)} \qquad \forall\, v \in H^1(\Omega).$$

Combining Theorem 4.1, (5.4)–(5.6), (5.9), (5.12)–(5.17) and Remark A.1, we conclude that

$$\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} \leq C_\ddagger(h^{3/2} + \rho), \tag{5.18}$$

where the positive constant $C_\ddagger$ only depends on α^{-1}, γ^{-1}, $\|f\|_{L_2(\Omega)}$, $\|y_d\|_{L_2(\Omega)}$, $\sum_{\sigma\in\Sigma}\|\phi_i\|_{H^1(\sigma)}$ $(i = 1, 2)$, and the shape regularity of $\mathcal{T}_h$ and $\mathcal{T}_\rho$.

In the case where $\mathcal{T}_\rho$ is the triangulation on Γ induced by $\mathcal{T}_h$, the L_2 error estimate becomes

$$\|y-\bar{y}_{h,\rho}\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_{h,\rho}\|_{L_2(\Gamma)}\leq Ch,$$

and we have recovered the two dimensional result in [20] for the model problem.

5.3. Variational discretization concept

If we adopt the variational discretization concept of Hinze (cf. [9, Chapter 3]) and do not discretize the control variable, then we can replace W_ρ by $L_2(\Gamma)$ and $Q_\rho : L_2(\Gamma)\longrightarrow W_\rho$ by the identity operator on $L_2(\Gamma)$.

Let the solution of the optimal control problem be denoted by $(\bar{y}_h,\bar{u}_h,\bar{p}_h)$. According to Theorem 4.1, the estimate for the L_2 errors becomes

$$\|\bar{y}-\bar{y}_h\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_h\|_{L_2(\Gamma)}\leq C_b\big(\|\bar{y}-R_h\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_h\bar{p}\|_{L_2(\Gamma)}\big)$$

which implies the estimate in [21, display (10)].

The estimate corresponding to (5.18) now reads

$$\|\bar{y}-\bar{y}_{h,\rho}\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_{h,\rho}\|_{L_2(\Gamma)}\leq Ch^{3/2}$$

and we have recovered the result in [22] for the model problem in two dimensions.

6. Application to problems with rough coefficients

We consider the optimal control problem (1.3)–(1.5) on an arbitrary polygonal (resp., polyhedral) domain in $\mathbb{R}^2$ (resp., $\mathbb{R}^3$). In the case of rough coefficients, where (1.2) is the only assumption on the matrix $A(x)$, we have $\bar{y},\bar{p}\in H^1(\Omega)$ and nothing more. Therefore we can only assert that the trace of $\bar{p}$ on Γ belongs to $H^{1/2}(\Gamma)$.

Under the assumption that the control constraints

$$\phi_1 \text{ and } \phi_2 \text{ belong to } H^{1/2}(\Gamma), \tag{6.1}$$

we have the following regularity estimates from (2.5)–(2.7):

$$\|\bar{u}\|_{H^{1/2}(\Gamma)}\leq C_\Omega\max\big(\|\phi_1\|_{H^{1/2}(\Gamma)},\|\phi_2\|_{H^{1/2}(\Gamma)},\gamma^{-1}\|\bar{p}\|_{H^{1/2}(\Gamma)}\big), \tag{6.2}$$

$$\|\lambda_1\|_{H^{1/2}(\Gamma)}\leq C_\Omega\max\big(\|\bar{p}\|_{H^{1/2}(\Gamma)},\|\phi_1\|_{H^{1/2}(\Gamma)}\big), \tag{6.3}$$

$$\|\lambda_2\|_{H^{1/2}(\Gamma)}\leq C_\Omega\max\big(\|\bar{p}\|_{H^{1/2}(\Gamma)},\|\phi_2\|_{H^{1/2}(\Gamma)}\big). \tag{6.4}$$

It follows from (6.1)–(6.4) and standard finite element estimates that

$$\|\phi_1-Q_\rho\phi_1\|_{L_2(\Gamma)}\leq C\rho^{\frac{1}{2}}\|\phi_1\|_{H^{1/2}(\Gamma)}, \tag{6.5}$$

$$\|\phi_2-Q_\rho\phi_2\|_{L_2(\Gamma)}\leq C\rho^{\frac{1}{2}}\|\phi_2\|_{H^{1/2}(\Gamma)}, \tag{6.6}$$

$$\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Gamma)}\leq C\rho^{\frac{1}{2}}\|\bar{u}\|_{H^{1/2}(\Gamma)}, \tag{6.7}$$

$$\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Gamma)}\leq C\rho^{\frac{1}{2}}\|\lambda_1\|_{H^{1/2}(\Gamma)}, \tag{6.8}$$

$$\|\lambda_2-Q_\rho\lambda_2\|_{L_2(\Gamma)}\leq C\rho^{\frac{1}{2}}\|\lambda_2\|_{H^{1/2}(\Gamma)}, \tag{6.9}$$

where the positive constant C only depends on the shape regularity of $\mathcal{T}_\rho$.

6.1. A standard P_1 finite element method

If we solve the optimal control problem by taking V_* to be a standard P_1 finite element space V_h associated with a triangulation $\mathcal{T}_h$ of Ω and denote the solution of the discrete problem by $(\bar{y}_{h,\rho},\bar{u}_{h,\rho},\bar{p}_{h,\rho})$, we have

$$\|\bar{y}-\bar{y}_{h,\rho}\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_{h,\rho}\|_{L_2(\Omega)}\leq C_\spadesuit\big(\|\bar{y}-R_h\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_h\bar{p}\|_{L_2(\Gamma)}+\rho^{\frac{1}{2}}\big) \tag{6.10}$$

by Theorem 4.1 and (6.5)–(6.9), where, in view of (6.2)–(6.4) and Remark A.1, the positive constant $C_\spadesuit$ only depends on α^{-1}, γ^{-1}, $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^{1/2}(\Gamma)}$, $\|\phi_2\|_{H^{1/2}(\Gamma)}$ and the shape regularity of $\mathcal{T}_\rho$.

Since we only have $\bar{p}\in H^1(\Omega)$ and nothing more, we may as well replace $\|\bar{p}-R_h\bar{p}\|_{L_2(\Gamma)}$ by $C_{\rm Tr}\|\bar{p}-R_h\bar{p}\|_{H^1(\Omega)}\leq C_{\rm Tr}\sqrt{C_{\rm PF}\max(\alpha^{-1},1)}\|\bar{p}-R_h\bar{p}\|_a$ so that (6.10) is replaced by

$$\|\bar{y}-\bar{y}_{h,\rho}\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_{h,\rho}\|_{L_2(\Omega)}\leq C_\spadesuit\big(\|\bar{y}-R_h\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_h\bar{p}\|_a+\rho^{\frac{1}{2}}\big). \tag{6.11}$$

We can also bound the energy norm error of the discrete adjoint state as follows. We have

$$\|\bar{p}-\bar{p}_{h,\rho}\|_a\leq\|\bar{p}-R_h\bar{p}\|_a+\|R_h\bar{p}-\bar{p}_{h,\rho}\|_a, \tag{6.12}$$

and

$$a(q_h, R_h\bar{p} - \bar{p}_{h,\rho}) = a(q_h, \bar{p} - \bar{p}_{h,\rho}) = \int_\Omega (\bar{y} - \bar{y}_{h,\rho}) q_h \, dx \qquad \forall\, q_h \in V_h$$

by (2.2), (3.7) and (5.8) so that

$$\|R_h\bar{p} - \bar{p}_{h,\rho}\|_a \le \sqrt{C_{\rm PF}\max(\alpha^{-1}, 1)}\,\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} \tag{6.13}$$

by Lemma 3.3.

It follows from (6.11)–(6.13) that

$$\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a \le C_{\clubsuit}\big(\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_a + \rho^{\frac{1}{2}}\big), \tag{6.14}$$

where the positive constant $C_{\clubsuit}$ only depends on α^{-1}, γ^{-1}, $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^{1/2}(\Gamma)}$, $\|\phi_2\|_{H^{1/2}(\Gamma)}$ and the shape regularity of $\mathcal{T}_\rho$.

By a density argument, we have

$$\lim_{h\downarrow 0} \|\zeta - R_h\zeta\|_{H^1(\Omega)} = 0 \qquad \text{for any } \zeta \in H^1(\Omega),$$

and hence

$$\lim_{h,\rho\downarrow 0}\big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} + \|\bar{p} - R_h\bar{p}\|_a\big) = 0$$

by (6.14). However the convergence in h can be arbitrarily slow (cf. [23]) so that a good approximate solution may require a very small h. This is computationally expensive, especially if the optimal control problem (1.3)–(1.5) has to be solved repeatedly for different f. Therefore it is desirable to improve the performance of the finite element method on coarse meshes by taking V_* to be a multiscale finite element space.

The result below is a sort of converse to (6.14). It is useful for the analysis of the multiscale finite element methods in Section 6.2.

Lemma 6.1. *There exists a positive constant $C_{\natural}$ depending only on α^{-1} such that*

$$\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_a \le C_{\natural}\big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a\big). \tag{6.15}$$

Proof. We begin with a triangle inequality

$$\begin{aligned}&\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_a \\ &\qquad \le \|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{y}_{h,\rho} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a + \|\bar{p}_{h,\rho} - R_h\bar{p}\|_a.\end{aligned} \tag{6.16}$$

Observe that

$$a(\bar{y}_{h,\rho} - R_h\bar{y}, z_h) = a(\bar{y}_{h,\rho} - \bar{y}, z_h) = \int_\Gamma (\bar{u}_{h,\rho} - \bar{u}) z_h \, ds \qquad \forall\, z_h \in V_h$$

by (1.4), (3.4) (where $\bar{y}_{h,\rho} = \bar{y}^\dagger_{h,\rho}$ because $B_* = 0$) and (5.8), and hence

$$\|\bar{y}_{h,\rho} - R_h\bar{y}\|_{H^1(\Omega)} \le C_{\rm Tr}C_{\rm PF}\max(\alpha^{-1}, 1)\|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} \tag{6.17}$$

by Lemma 3.3.

The estimate (6.15) follows from (6.13), (6.16) and (6.17). □

6.2. Multiscale finite element methods

Let $\mathcal{T}_H$ be a triangulation of Ω with mesh size H so that $\mathcal{T}_h$ ($h \ll H$) is a refinement of $\mathcal{T}_H$. The P_1 finite element space associated with $\mathcal{T}_H$ (resp., $\mathcal{T}_h$) is denoted by V_H (resp., V_h).

Let $V_* = V^{ms}_{H,h} \subset V_h$ be the ideal multiscale finite element space in [24] obtained by the orthogonal decomposition methodology. The construction of $V^{ms}_{H,h}$ involves a quasi-local interpolation operator $I_H : H^1(\Omega) \longrightarrow V_H$ that satisfies

$$H_T^{-1}\|v - I_H v\|_{L_2(T)} + \|\nabla(v - I_H v)\|_{L_2(T)} \le C_{I_H}\|\nabla v\|_{L_2(\omega_T)} \qquad \forall\, T \in \mathcal{T}_H, \tag{6.18}$$

where H_T is the diameter of T, ω_T is the star of T and the positive constant C_{I_H} only depends on the shape regularity of $\mathcal{T}_H$.

Let $W_h = \{w_h \in V_h : I_H w_h = 0\}$ be the kernel of I_H on V_h. The space $V^{ms}_{H,h} \subset V_h$ is defined to be the orthogonal complement of W_h with respect to the bilinear form $a(\cdot,\cdot)$, i.e.,

$$V^{ms}_{H,h} = \{v_h \in V_h : a(v_h, w_h) = 0 \quad \forall\, w_h \in W_h\}. \tag{6.19}$$

Remark 6.2. The dimension of $V^{ms}_{H,h}$ is identical to the dimension of V_H. The construction of the basis of $V^{ms}_{H,h}$, which involves solving problems associated with $\mathcal{T}_h$, can be carried out off-line and the on-line computation involving the space $V^{ms}_{H,h}$ with a small dimension is fast.

The operator $B_* : L_2(\Gamma) \longrightarrow W_h$ is defined by

$$a(B_* q, w_h) = -\int_\Gamma q w_h ds \qquad \forall\, q \in L_2(\Gamma),\ w_h \in W_h. \tag{6.20}$$

It follows from (6.19) that (3.5) is satisfied.

We deduce from Lemma 3.3 that

$$\|B_* q\|_a \le \sqrt{C_{\rm PF}\max(\alpha^{-1},1)}\, C_{\rm Tr}\|q\|_{L_2(\Gamma)} \qquad \forall\, q \in L_2(\Gamma)$$

so that

$$\|B_*\| \le C_{\rm Tr}\sqrt{C_{\rm PF}\max(\alpha^{-1},1)}. \tag{6.21}$$

The following is the salient property of $V_{H,h}^{ms}$ that follows from (6.18)–(6.20) (cf. [24, Section 3.3]).

Lemma 6.3. *Let $v_h \in V_h$ and $v_{H,h}^{ms} \in V_{H,h}^{ms}$ satisfy*

$$a(v_h, w_h) = \int_\Omega F w_h dx + \int_\Gamma G w_h ds \qquad \forall\, w_h \in V_h,$$
$$a(v_{H,h}^{ms}, w_{H,h}^{ms}) = \int_\Omega F w_{H,h}^{ms} dx + \int_\Gamma G w_{H,h}^{ms} ds \qquad \forall\, w_{H,h}^{ms} \in V_{H,h}^{ms},$$

where $F \in L_2(\Omega)$ and $G \in L_2(\Gamma)$. Then we have

$$\|v_h - (v_{H,h}^{ms} - B_* G)\|_a \le C_\diamond H\big(\|F\|_{L_2(\Omega)} + \|G\|_{L_2(\Gamma)}\big),$$
$$\|v_h - (v_{H,h}^{ms} - B_* G)\|_{L_2(\Omega)} \le (C_\diamond H)^2\big(\|F\|_{L_2(\Omega)} + \|G\|_{L_2(\Gamma)}\big),$$

where the positive constant $C_\diamond$ is independent of the mesh sizes.

Let the solution of the discrete problem with $V_* = V_{H,h}^{ms}$ be denoted by $(\bar{y}_{H,h,\rho}^{ms}, \bar{u}_{H,h,\rho}^{ms}, \bar{p}_{H,h,\rho}^{ms})$.

Theorem 6.4. *There exist two positive constants C_1 and C_2 independent of the mesh sizes such that*

$$\|\bar{y} - \bar{y}_{H,h,\rho}^{ms}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,h,\rho}^{ms}\|_{L_2(\Gamma)} \tag{6.22}$$
$$\le C_1\big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a\big) + C_2(H + \rho^{\frac{1}{2}}),$$

where $(\bar{y}_{h,\rho}, \bar{u}_{h,\rho}, \bar{p}_{h,\rho})$ is the approximate solution from Section 6.1 computed by the P_1 finite element method on the fine mesh $\mathcal{T}_h$.

Proof. According to Theorem 4.1, (1.7), (3.11), (6.5)–(6.9) and (6.21), we have

$$\|\bar{y} - \bar{y}_{H,h,\rho}^{ms}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,h,\rho}^{ms}\|_{L_2(\Gamma)} \tag{6.23}$$
$$\le C_\square\big(\|\bar{y} - (R_{H,h}^{ms}\bar{y} - B_*\bar{u})\|_{L_2(\Omega)} + \|\bar{p} - R_{H,h}^{ms}\bar{p}\|_a + \rho^{\frac{1}{2}}\big),$$

where $R_{H,h}^{ms} : H^1(\Omega) \longrightarrow V_{H,h}^{ms}$ is the Ritz projection operator, i.e,

$$a(R_{H,h}^{ms}\zeta, v_{H,h}^{ms}) = a(\zeta, v_{H,h}^{ms}) \qquad \forall\, \zeta \in H^1(\Omega), v_{H,h}^{ms} \in V_{H,h}^{ms}, \tag{6.24}$$

and the positive constant $C_\square$ only depends on α^{-1}, γ^{-1}, $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^{1/2}(\Gamma)}$, $\|\phi_2\|_{H^{1/2}(\Gamma)}$ and the shape regularity of $\mathcal{T}_\rho$.

We can estimate the terms on the right-hand side of (6.23) as follows.

In view of (1.4), (5.8) and (6.24), we have

$$a(R_h\bar{y}, w_h) = a(\bar{y}, w_h) = \int_\Omega f w_h dx + \int_\Gamma \bar{u} w_h ds \qquad \forall\, w_h \in V_h,$$
$$a(R_{H,h}^{ms}\bar{y}, w_{H,h}^{ms}) = a(\bar{y}, w_{H,h}^{ms}) = \int_\Omega f w_{H,h}^{ms} dx + \int_\Gamma \bar{u} w_{H,h}^{ms} ds \qquad \forall\, w_{H,h}^{ms} \in V_{H,h}^{ms}.$$

It then follows from Lemma 6.3 that

$$\|R_h\bar{y} - (R_{H,h}^{ms}\bar{y} - B_*\bar{u})\|_{L_2(\Omega)} \le (C_\diamond H)^2\big(\|f\|_{L_2(\Omega)} + \|\bar{u}\|_{L_2(\Gamma)}\big)$$

and hence

$$\|\bar{y} - (R_{H,h}^{ms}\bar{y} - B_*\bar{u})\|_{L_2(\Omega)} \le \|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + (C_\diamond H)^2\big(\|f\|_{L_2(\Omega)} + \|\bar{u}\|_{L_2(\Gamma)}\big). \tag{6.25}$$

Similarly from (2.2), (3.7), (5.8) and (6.24), we see that

$$a(z_h, R_h\bar{p}) = a(z_h, \bar{p}) = \int_\Omega (\bar{y} - y_d) z_h dx \qquad \forall\, z_h \in V_h,$$
$$a(z_{H,h}^{ms}, R_{H,h}^{ms}\bar{p}) = a(z_{H,h}^{ms}, \bar{p}) = \int_\Omega (\bar{y} - y_d) z_{H,h}^{ms} dx \qquad \forall\, z_{H,h}^{ms} \in V_{H,h}^{ms},$$

and hence

$$\|R_h\bar{p} - R_{H,h}^{ms}\bar{p}\|_a \leq C_\diamond H\|\bar{y} - y_d\|_{L_2(\Omega)}$$

by Lemma 6.3. Consequently we have

$$\|\bar{p} - R_{H,h}^{ms}\bar{p}\|_a \leq \|\bar{p} - R_h\bar{p}\|_a + (C_\diamond H)\|\bar{y} - y_d\|_{L_2(\Omega)}. \tag{6.26}$$

Combining (6.23), (6.25) and (6.26), we find, by Lemma 6.1,

$$\begin{aligned}
&\|\bar{y} - \bar{y}_{H,h,\rho}^{ms}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,h,\rho}^{ms}\|_{L_2(\Gamma)} \\
&\qquad \leq C_\square\big(\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_a\big) + C_\heartsuit(H + \rho^{\frac{1}{2}}) \\
&\qquad \leq C_\square C_\natural\big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a\big) + C_\heartsuit(H + \rho^{\frac{1}{2}}).
\end{aligned}$$

where the positive constant $C_\heartsuit$ is independent of the mesh sizes. □

From (6.22) we can say that, up to an $O(H + \rho^{\frac{1}{2}})$ error, the solution obtained by the discretization based on $V_{H,h}^{ms} \times W_\rho$ on a coarse mesh $\mathcal{T}_H$ is qualitatively similar to the error of the approximate solution obtained by the standard P_1 finite element method based on $V_h \times W_\rho$ for a potentially very small h.

Alternatively, for a chosen tolerance tol, one can choose (cf. (6.14))

$$\rho^{\frac{1}{2}} \leq \frac{\text{tol}}{2C_\clubsuit} \tag{6.27}$$

and a sufficiently small h such that

$$\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_a \leq \frac{\text{tol}}{2C_\clubsuit}. \tag{6.28}$$

It follows from (6.14), (6.27) and (6.28) that

$$\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Gamma)} + \|\bar{p} - \bar{p}_{h,\rho}\|_a \leq \text{tol}. \tag{6.29}$$

We then derive from Theorem 6.4, (6.27) and (6.29) that

$$\|\bar{y} - \bar{y}_{H,h,\rho}^{ms}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,h,\rho}^{ms}\|_{L_2(\Gamma)} \leq \Big(C_1 + \frac{C_2}{2C_\clubsuit}\Big)\text{tol} + C_2 H.$$

Hence we can also say that up to a multiple of the tolerance, the multiscale finite element method based on a coarse mesh $\mathcal{T}_H$ has $O(H)$ convergence.

Therefore once a fine mesh has been determined that can resolve the multiscale features in the partial differential equation constraint (1.4) to provide an acceptable approximate solution for the optimal control problem, one can switch to the multiscale finite element method on coarse meshes to obtain good approximate solutions efficiently by solving problems of much smaller dimensions.

The computation of the basis functions of $V_{H,h}^{ms}$, which involves solving problems on V_h, may still be too expensive. This can be remedied by the Local Orthogonal Decomposition (LOD) approach (cf. [25,26]), where the computation is performed on patches around the vertices of $\mathcal{T}_H$. The difference between the solution computed by the LOD method and the solution computed by the ideal multiscale finite element method based on $V_{H,h}^{ms}$ is $O(H^2)$ in $\|\cdot\|_{L_2(\Omega)}$ and $O(H)$ in $\|\cdot\|_a$ (cf. [24, Lemma 3.6 and Conclusion 3.9]), provided that the number of layers in the local patches is sufficiently large. Therefore Theorem 6.4 remains valid for the solution obtained by the LOD method in [24].

7. Concluding remarks

We have developed an abstract error analysis of finite element methods for a linear-quadratic elliptic Neumann boundary control problem purely in terms of the Ritz projection operator for $H^1(\Omega)$, the L_2 projection operator for $L_2(\Gamma)$ and a correction operator for the Neumann boundary data, which makes the analysis applicable to standard finite element methods when the coefficients of the elliptic operator are smooth and to multiscale finite element methods when the coefficients are rough.

For simplicity we have assumed that the bilinear form $a(\cdot,\cdot)$ in (1.1) is symmetric. But the estimates in Section 4 can be extended to a nonsymmetric $a(\cdot,\cdot)$ by replacing the term $R_*\bar{p}$ with the term $S_*\bar{p}$, where $S_* : H^1(\Omega) \longrightarrow V_*$ is defined by

$$a(q_*, S_*\zeta) = a(q_*, \zeta) \qquad \forall\, q_* \in V_*.$$

The results in this paper can also be applied to other multiscale finite element methods such as the one in the recent work [27].

Finally we note that it is more challenging to analyze multiscale finite element methods for elliptic Dirichlet boundary control problems (cf. [28–30]) due to the ultra weak formulation of the state equation and the fact that the adjoint state only belongs to $H^1(\Omega)$ under the assumption of rough coefficients.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Some estimates for $\bar{y}$, $\bar{u}$ and $\bar{p}$

We can take $u_1 = \phi_1$ and $y_1 \in H^1(\Omega)$ defined by (1.4) so that $(y_1, u_1) \in \mathbb{K}$ and

$$a(y_1, z) = \int_\Omega f z \, dx + \int_\Gamma \phi_1 z \, ds \qquad \forall\, z \in H^1(\Omega),$$

which, in view of Lemma 3.3 with $V = H^1(\Omega)$, implies

$$\|y_1\|_{H^1(\Omega)} \le C_{\rm PF} \max(\alpha^{-1}, 1)\big(\|f\|_{L_2(\Omega)} + C_{\rm Tr}\|\phi_1\|_{L_2(\Gamma)}\big). \tag{A.1}$$

It then follows from (A.1) and the relation

$$\|\bar{y} - y_d\|^2_{L_2(\Omega)} + \gamma\|\bar{u}\|^2_{L_2(\Gamma)} \le \|y_1 - y_d\|^2_{L_2(\Omega)} + \gamma\|u_1\|^2_{L_2(\Gamma)} \le \|y_1 - y_d\|^2_{L_2(\Omega)} + \|\phi_1\|^2_{L_2(\Gamma)}$$

that

$$\begin{aligned}\|\bar{y} - y_d\|^2_{L_2(\Omega)} + \gamma\|\bar{u}\|^2_{L_2(\Gamma)} \le 2\|y_d\|^2_{L_2(\Omega)} &+ 4\big[C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2\|f\|^2_{L_2(\Omega)} \\ &+ \big(4\big[C_{\rm Tr}C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2 + 1\big)\|\phi_1\|^2_{L_2(\Gamma)}.\end{aligned} \tag{A.2}$$

Similarly we have

$$\begin{aligned}\|\bar{y} - y_d\|^2_{L_2(\Omega)} + \gamma\|\bar{u}\|^2_{L_2(\Gamma)} \le 2\|y_d\|^2_{L_2(\Omega)} &+ 4\big[C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2\|f\|^2_{L_2(\Omega)} \\ &+ \big(4\big[C_{\rm Tr}C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2 + 1\big)\|\phi_2\|^2_{L_2(\Gamma)}.\end{aligned} \tag{A.3}$$

We conclude from (A.2) and (A.3) that

$$\|\bar{y} - y_d\|_{L_2(\Omega)} \le C_\sharp^{\frac{1}{2}}, \tag{A.4}$$

$$\|\bar{u}\|_{L_2(\Gamma)} \le \gamma^{-\frac{1}{2}} C_\sharp^{\frac{1}{2}}, \tag{A.5}$$

where

$$\begin{aligned}C_\sharp = 2\|y_d\|^2_{L_2(\Omega)} &+ 4\big[C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2\|f\|^2_{L_2(\Omega)} \\ &+ \big(4\big[C_{\rm Tr}C_{\rm PF}\max(\alpha^{-1}, 1)\big]^2 + 1\big)\min\big(\|\phi_1\|^2_{L_2(\Gamma)}, \|\phi_2\|^2_{L_2(\Gamma)}\big).\end{aligned} \tag{A.6}$$

From (1.4), Lemma 3.3 (with $V = H^1(\Omega)$) and (A.5), we immediately have

$$\|\bar{y}\|_{H^1(\Omega)} \le C_{\rm PF}\max(\alpha^{-1}, 1)\|\bar{u}\|_{L_2(\Gamma)} \le C_{\rm PF}\max(\alpha^{-1}, 1)\gamma^{-\frac{1}{2}} C_\sharp^{\frac{1}{2}}. \tag{A.7}$$

Similarly, it follows from (2.2), Lemma 3.3 (with $V = H^1(\Omega)$) and (A.4) that

$$\|\bar{p}\|_{H^1(\Omega)} \le C_{\rm PF}\max(\alpha^{-1}, 1)\|\bar{y} - y_d\|_{L_2(\Omega)} \le C_{\rm PF}\max(\alpha^{-1}, 1) C_\sharp^{\frac{1}{2}}. \tag{A.8}$$

Remark A.1. One can see from (A.4)–(A.8) that the quantities $\|\bar{y} - y_d\|_{L_2(\Omega)}$, $\|\bar{u}\|_{L_2(\Gamma)}$, $\|\bar{y}\|_{H^1(\Omega)}$ and $\|\bar{p}\|_{H^1(\Omega)}$ are bounded by constants that only depend on α^{-1}, γ^{-1}, $\|f\|_{L_2(\Omega)}$, $\|y_d\|_{L_2(\Omega)}$, $\|\phi_1\|_{L_2(\Gamma)}$ and $\|\phi_2\|_{L_2(\Gamma)}$.

Data availability

No data was used for the research described in the article.

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