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New Error Estimates for An Elliptic Distributed Optimal Control Problem with Pointwise Control Constraints

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We derive error estimates for a linear-quadratic elliptic distributed optimal control problem with pointwise control constraints that can be applied to standard finite element methods when the coefficients of the elliptic operator are smooth and to multiscale finite element methods when the coefficients are rough.

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1. Introduction

Let Ω be a polygonal (resp., polyhedral) domain in $\mathbb{R}^d$ for $d = 2$ (resp., 3), $y_d \in L_2(\Omega)$, $f \in L_2(\Omega)$ and $\gamma \leq 1$ be a positive constant. The optimal control problem is to find

$$(\bar{y}, \bar{u}) = \underset{(y,u)\in K}{\operatorname{argmin}} \frac{1}{2}\Big[\|y - y_d\|^2_{L_2(\Omega)} + \gamma\|u\|^2_{L_2(\Omega)}\Big], \tag{1.1}$$

where (y, u) belongs to $K \subset H^1_0(\Omega) \times L_2(\Omega)$ if and only if

$$a(y, z) = \int_\Omega (f + u)z\,dx \qquad \forall\, z \in H^1_0(\Omega) \tag{1.2}$$

and

$$u \in U_{\text{ad}} = \{v \in L_2(\Omega) : \, \phi_1 \leq v \leq \phi_2 \ \text{ in } \ \Omega\}. \tag{1.3}$$

Here the symmetric bilinear form $a(\cdot,\cdot)$ on $H^1(\Omega)$ satisfies

$$\alpha|y|^2_{H^1(\Omega)} \leq a(y, y) \leq \beta|y|^2_{H^1(\Omega)} \qquad \forall\, y \in H^1(\Omega), \tag{1.4}$$

where $\alpha \le \beta$ are positive constants, and we assume that

$$\phi_1, \phi_2 \in H^1(\Omega) \tag{1.5}$$

satisfy

$$\phi_1 \le \phi_2 \ \text{ in } \ \Omega. \tag{1.6}$$

Remark 1.1. Throughout this paper the inequalities and equalities between functions are to be interpreted in the sense of almost everywhere in Ω.

Remark 1.2. We follow the standard notation for function spaces, norms and differential operators that can be found for example in [1, 7].

Remark 1.3. The condition (1.4) is satisfied by many partial differential equation constraints with rough coefficients.

The optimal control problem defined by (1.1)–(1.4) is a model linear-quadratic problem (cf. [23, 33]) and the error analysis of a finite element method for this problem was first given in [14] under additional assumptions on the bilinear form $a(\cdot,\cdot)$. A substantial literature has been developed over the years (cf. the monographs [31, 24, 18] and the references therein). Nevertheless, the existing error analysis cannot be directly applied to multiscale finite element methods under the rough coefficient assumption in (1.4).

Our goal is to develop new abstract error estimates under the assumption (1.4) that are suitable for the error analysis of classical finite element methods and also for multiscale finite element methods. Our results (cf. Theorem 4.1 and Theorem 4.3) reduce the error analysis of finite element methods for the optimal control problem to the error analysis of finite element methods for elliptic boundary value problems. Therefore they can be applied to any finite element methods that have already been analyzed for elliptic boundary value problems. In particular they can be applied to many multiscale finite element methods.

The rest of the paper is organized as follows. We recall the relevant properties of the optimal control problem in Section 2 and introduce the approximation problem in Section 3. We derive the abstract error estimates in Section 4, present several applications in Section 5 and end with some concluding remarks in Section 6.

2. The Continuous Problem

According to the classical theory in [13, 21], the convex minimization problem defined by (1.1)–(1.4) and (1.6) has a unique solution $(\bar{y}, \bar{u}) \in K$ characterized by the first order optimality condition

$$\int_\Omega (\bar{y} - y_d)(y - \bar{y})dx + \gamma \int_\Omega \bar{u}(u - \bar{u})dx \geq 0 \qquad \forall\, (y, u) \in K. \tag{2.1}$$

Let the adjoint state $\bar{p} \in H_0^1(\Omega)$ be defined by

$$a(q, \bar{p}) = \int_\Omega (\bar{y} - y_d)q\, dx \qquad \forall\, q \in H_0^1(\Omega). \tag{2.2}$$

In view of (1.2) and (2.2), we have, for any $(y, u) \in K$,

$$\begin{aligned} \int_\Omega (\bar{y} - y_d)(y - \bar{y})dx + \gamma \int_\Omega \bar{u}(u - \bar{u})dx &= a(\bar{y} - y, \bar{p}) + \gamma \int_\Omega \bar{u}(u - \bar{u})dx \\ &= \int_\Omega \bar{p}(u - \bar{u})dx + \gamma \int_\Omega \bar{u}(u - \bar{u})dx, \end{aligned} \tag{2.3}$$

and hence

$$\int_\Omega (\bar{p} + \gamma \bar{u})(u - \bar{u})dx \geq 0 \qquad \forall\, u \in U_{\mathrm{ad}},$$

which means that $\bar{u}$ is the L_2 projection of the function $-(\bar{p}/\gamma)$ on the closed convex subset U_{ad} of $L_2(\Omega)$. Consequently we have

$$\bar{u} = \max(\phi_1, \min(\phi_2, -(\bar{p}/\gamma))) \tag{2.4}$$

and $(\bar{y}, \bar{u})$ is determined by (1.2), (2.2) and (2.4).

2.1. Bounds for $\|\bar{y} - y_d\|_{L_2(\Omega)}$ and $\|\bar{u}\|_{L_2(\Omega)}$

It follows from (1.1), (1.3), (1.6) and $\gamma \leq 1$ that

$$\|\bar{y} - y_d\|^2_{L_2(\Omega)} + \gamma \|\bar{u}\|^2_{L_2(\Omega)} \leq \|y_1 - y_d\|^2_{L_2(\Omega)} + \|\phi_1\|^2_{L_2(\Omega)}, \tag{2.5}$$

where $y_1 \in H_0^1(\Omega)$ is defined by

$$a(y_1, z) = \int_\Omega (f + \phi_1)z\, dx \qquad \forall\, z \in H_0^1(\Omega). \tag{2.6}$$

From (1.4) and (2.6) we have

$$\alpha |y_1|^2_{H^1(\Omega)} \leq a(y_1, y_1) = \int_\Omega (f + \phi_1)y_1\, dx \leq \|f + \phi_1\|_{L_2(\Omega)} \|y_1\|_{L_2(\Omega)},$$

which together with the Poincaré-Friedrichs inequality

$$\|v\|_{L_2(\Omega)} \le \mathrm{C}_{\mathrm{PF}}|v|_{H^1(\Omega)} \quad \forall\, v \in H_0^1(\Omega) \tag{2.7}$$

implies

$$\|y_1\|_{L_2(\Omega)} \le (\mathrm{C}_{\mathrm{PF}}/\alpha)\|f+\phi_1\|_{L_2(\Omega)}. \tag{2.8}$$

Combining (2.5), (2.8) and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\|\bar{y}-y_d\|^2_{L_2(\Omega)} + \gamma\|\bar{u}\|^2_{L_2(\Omega)} &\le 2\|y_d\|^2_{L_2(\Omega)} + 4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)\|f\|^2_{L_2(\Omega)} \\ &\qquad + [4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)+1]\|\phi_1\|^2_{L_2(\Omega)}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\|\bar{y}-y_d\|^2_{L_2(\Omega)} + \gamma\|\bar{u}\|^2_{L_2(\Omega)} &\le 2\|y_d\|^2_{L_2(\Omega)} + 4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)\|f\|^2_{L_2(\Omega)} \\ &\qquad + [4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)+1]\|\phi_2\|^2_{L_2(\Omega)}\end{aligned}$$

and hence

$$\|\bar{y}-y_d\|_{L_2(\Omega)} \le C_\sharp, \tag{2.9}$$

$$\|\bar{u}\|_{L_2(\Omega)} \le \gamma^{-1}C_\sharp, \tag{2.10}$$

where

$$\begin{aligned}C_\sharp = \Big(&2\|y_d\|^2_{L_2(\Omega)} + 4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)\|f\|^2_{L_2(\Omega)} \\ &+ [4(\mathrm{C}^2_{\mathrm{PF}}/\alpha^2)+1]\min(\|\phi_1\|^2_{L_2(\Omega)}, \|\phi_2\|^2_{L_2(\Omega)})\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.11}$$

2.2. Bounds for $|\bar{u}|_{H^1(\Omega)}$ and $|\bar{p}|_{H^1(\Omega)}$

It follows from (1.4) and (2.2) that

$$\alpha|\bar{p}|^2_{H^1(\Omega)} \le a(\bar{p},\bar{p}) = \int_\Omega (\bar{y}-y_d)\bar{p}\,dx \le \|\bar{y}-y_d\|_{L_2(\Omega)}\|\bar{p}\|_{L_2(\Omega)},$$

which together with (2.7) and (2.9) implies

$$|\bar{p}|_{H^1(\Omega)} \le (\mathrm{C}_{\mathrm{PF}}/\alpha)C_\sharp. \tag{2.12}$$

Since the space $H^1(\Omega)$ is invariant under the max and min operators (cf. [15, Lemma 7.6]), we conclude from (1.5) and (2.4) that $\bar{u} \in H^1(\Omega)$ and

$$|\bar{u}|_{H^1(\Omega)} \le \max\big(|\phi_1|_{H^1(\Omega)}, |\phi_2|_{H^1(\Omega)}, \gamma^{-1}|\bar{p}|_{H^1(\Omega)}\big). \tag{2.13}$$

2.3. The Lagrange Multiplier λ

The function

$$\lambda = \bar{p} + \gamma\bar{u} \in H^1(\Omega), \tag{2.14}$$

which can be interpreted as a Lagrange multiplier for the inequality constraints in (1.3), plays a key role in the error analysis in Section 4.

We can write

$$\lambda = \lambda_1 + \lambda_2, \tag{2.15}$$

where

$$\lambda_1 = \max(\lambda, 0) \ge 0 \quad \text{and} \quad \lambda_2 = \min(\lambda, 0) \le 0, \tag{2.16}$$

and, in view of (2.13), (2.14) (and $\gamma \le 1$),

$$\begin{aligned} |\lambda_1|_{H^1(\Omega)}, |\lambda_2|_{H^1(\Omega)} &\le |\lambda|_{H^1(\Omega)} \\ &\le |\bar{p}|_{H^1(\Omega)} + \max\big(|\phi_1|_{H^1(\Omega)}, |\phi_2|_{H^1(\Omega)}, |\bar{p}|_{H^1(\Omega)}\big). \end{aligned} \tag{2.17}$$

From (1.6) and (2.4) we have

$$\bar{u} = \begin{cases} \phi_2 & \text{if } -(\bar{p}/\gamma) \ge \phi_2 \\ -(\bar{p}/\gamma) & \text{if } \phi_1 < -(\bar{p}/\gamma) < \phi_2 \\ \phi_1 & \text{if } -(\bar{p}/\gamma) \le \phi_1 \end{cases},$$

which implies through (2.14) and (2.16) the following complementarity conditions:

$$\int_\Omega \lambda_1(\bar{u} - \phi_1)dx = 0 = \int_\Omega \lambda_2(\bar{u} - \phi_2)dx. \tag{2.18}$$

Remark 2.1. In view of (2.10), (2.11)–(2.13), and (2.17), $\|\bar{u}\|_{L_2(\Omega)}$, $|\bar{u}|_{H^1(\Omega)}$, $|\lambda_1|_{H^1(\Omega)}$ and $|\lambda_2|_{H^1(\Omega)}$ are bounded by constants that only depend on the numbers $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^1(\Omega)}$, $\|\phi_2\|_{H^1(\Omega)}$, α^{-1} and γ^{-1}.

3. The Approximation Problem

Let V_* (resp. $W_\dagger$) be a closed subspace of $H_0^1(\Omega)$ (resp., $L_2(\Omega)$). The approximation problem for (1.1) is to find

$$(\bar{y}_{*,\dagger}, \bar{u}_{*,\dagger}) = \underset{(y_*,u_\dagger)\in K_{*,\dagger}}{\operatorname{argmin}} \frac{1}{2}\Big[\|y_* - y_d\|^2_{L_2(\Omega)} + \gamma\|u_\dagger\|^2_{L_2(\Omega)}\Big], \tag{3.1}$$

where $(y_*, u_\dagger)$ belongs to $K_{*,\dagger} \subset V_* \times W_\dagger$ if and only if

$$a(y_*, z_*) = \int_\Omega (f + u_\dagger) z_* dx \qquad \forall\, z_* \in V_* \tag{3.2}$$

and

$$Q_\dagger \phi_1 \le u_\dagger \le Q_\dagger \phi_2 \ \text{ in } \ \Omega. \tag{3.3}$$

Here $Q_\dagger : L_2(\Omega) \longrightarrow W_\dagger$ is the L_2 projection operator and we assume that

$$Q_\dagger v \ge 0 \quad \text{if} \quad v \ge 0. \tag{3.4}$$

Again by the classical theory the minimization problem defined by (3.1)–(3.3) has a unique solution $(\bar{y}_{*,\dagger}, \bar{u}_{*,\dagger}) \in K_{*,\dagger}$ characterized by the first order optimality condition

$$\int_\Omega (\bar{y}_{*,\dagger} - y_d)(y_* - \bar{y}_{*,\dagger}) dx + \gamma \int_O \bar{u}_{*,\dagger}(u_\dagger - \bar{u}_{*,\dagger}) dx \ge 0 \tag{3.5}$$

for all $(y_*, u_\dagger) \in K_{*,\dagger}$.

Let $\bar{p}_{*,\dagger} \in V_*$ be defined by

$$a(q_*, \bar{p}_{*,\dagger}) = \int_\Omega (\bar{y}_{*,\dagger} - y_d) q_* dx \qquad \forall\, q_* \in V_*. \tag{3.6}$$

We will provide estimates for $\|\bar{y} - \bar{y}_{*,\dagger}\|_{L_2(\Omega)}$, $\|\bar{u} - \bar{u}_{*,\dagger}\|_{L_2(\Omega)}$, $\|\bar{p} - \bar{p}_{*,\dagger}\|_{L_2(\Omega)}$, $|\bar{y} - \bar{y}_{*,\dagger}|_a$ and $|\bar{p} - \bar{p}_{*,\dagger}|_a$ in Section 4, where

$$|v|_a = \sqrt{a(v, v)} \qquad \forall\, v \in H^1(\Omega).$$

The simple result below is useful for the analysis of the approximation problem.

Lemma 3.1. *Let $g \in L_2(\Omega)$ and $v_* \in V_*$ satisfy*

$$a(v_*, w_*) = \int_\Omega g w_* dx \qquad \forall\, w_* \in V_*. \tag{3.7}$$

We have

$$\|v_*\|_{L_2(\Omega)} \le (\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|g\|_{L_2(\Omega)}, \tag{3.8}$$

$$|v_*|_a \le (\mathrm{C}_{\mathrm{PF}}/\sqrt{\alpha})\|g\|_{L_2(\Omega)}. \tag{3.9}$$

Proof. The estimate (3.8) follows from (1.4), (2.7) and (3.7):

$$\|v_*\|^2_{L_2(\Omega)} \le \mathrm{C}^2_{\mathrm{PF}}|v|^2_{H^1(\Omega)} \le (\mathrm{C}^2_{\mathrm{PF}}/\alpha)\, a(v_*,v_*)$$
$$= (\mathrm{C}^2_{\mathrm{PF}}/\alpha)\int_\Omega g v_* dx \le (\mathrm{C}^2_{\mathrm{PF}}/\alpha)\|g\|_{L_2(\Omega)}\|v_*\|_{L_2(\Omega)}.$$

Similarly we have

$$\begin{aligned}|v_*|^2_a = a(v_*,v_*) &= \int_\Omega g v_* dx \\ &\le \|g\|_{L_2(\Omega)}\|v_*\|_{L_2(\Omega)} \\ &\le \|g\|_{L_2(\Omega)}\mathrm{C}_{\mathrm{PF}}|v_*|_{H^1(\Omega)} \le \|g\|_{L_2(\Omega)}(\mathrm{C}_{\mathrm{PF}}/\sqrt{\alpha})|v_*|_a\end{aligned}$$

by (1.4), (2.7) and (3.7), which implies (3.9). □

4. Error Estimates

We will derive error estimates in terms of the L_2 projection $Q_\dagger : L_2(\Omega) \longrightarrow W_\dagger$ and the Ritz projection $R_* : H^1_0(\Omega) \longrightarrow V_*$ defined by

$$a(R_*\zeta, v_*) = a(\zeta, v_*) \qquad \forall\, v_* \in V_*. \tag{4.1}$$

4.1. Estimate for the L_2 Errors

Theorem 4.1. *There exists a positive constant $C_\flat$ depending only on α^{-1} and γ^{-1} such that*

$$\begin{aligned}&\|\bar y - \bar y_{*,\dagger}\|_{L_2(\Omega)} + \|\bar u - \bar u_{*,\dagger}\|_{L_2(\Omega)} + \|\bar p - \bar p_{*,\dagger}\|_{L_2(\Omega)} \\ &\quad\le C_\flat\big(\|\bar y - R_*\bar y\|_{L_2(\Omega)} + \|\bar p - R_*\bar p\|_{L_2(\Omega)} + \|\bar u - Q_\dagger\bar u\|_{L_2(\Omega)} \\ &\qquad + \|\lambda_1 - Q_\dagger\lambda_1\|_{L_2(\Omega)} + \|\lambda_2 - Q_\dagger\lambda_2\|_{L_2(\Omega)} \\ &\qquad\quad + \|\phi_1 - Q_\dagger\phi_1\|_{L_2(\Omega)} + \|\phi_2 - Q_\dagger\phi_2\|_{L_2(\Omega)}\big).\end{aligned} \tag{4.2}$$

Proof. First we note that (2.2), (3.2) and (4.1) imply

$$\begin{aligned}\int_\Omega(\bar y - y_d)(y_* - \bar y_{*,\dagger})dx &= a(y_* - \bar y_{*,\dagger}, \bar p) \\ &= a(y_* - \bar y_{*,\dagger}, R_*\bar p) = \int_\Omega (u_\dagger - \bar u_{*,\dagger})R_*\bar p\, dx\end{aligned} \tag{4.3}$$

for all $(y_*, u_\dagger) \in K_{*,\dagger}$.

Let $(\tilde{y}_*, \tilde{u}_\dagger) \in V_* \times W_\dagger$ be defined by

$$\tilde{u}_\dagger = Q_\dagger \bar{u} \tag{4.4}$$

and

$$a(\tilde{y}_*, z_*) = \int_\Omega (f + \tilde{u}_\dagger) z_* dx \qquad \forall\, z_* \in V_*. \tag{4.5}$$

Then $\tilde{u}_\dagger$ satisfies the constraint (3.3) by (1.3) and (3.4), and hence $(\tilde{y}_*, \tilde{u}_\dagger)$ belongs to $K_{*,\dagger}$.

We have

$$\begin{aligned}
&\|\bar{y} - \bar{y}_{*,\dagger}\|^2_{L_2(\Omega)} + \gamma \|\bar{u} - \bar{u}_{*,\dagger}\|^2_{L_2(\Omega)} \\
&= \int_\Omega (\bar{y} - \bar{y}_{*,\dagger})(\bar{y} - \tilde{y}_*) dx + \gamma \int_\Omega (\bar{u} - \bar{u}_{*,\dagger})(\bar{u} - \tilde{u}_\dagger) dx \\
&\qquad + \int_\Omega (\bar{y} - \bar{y}_{*,\dagger})(\tilde{y}_* - \bar{y}_{*,\dagger}) dx + \gamma \int_\Omega (\bar{u} - \bar{u}_{*,\dagger})(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx.
\end{aligned} \tag{4.6}$$

Using (2.14), (3.5) and (4.3), we find

$$\begin{aligned}
&\int_\Omega (\bar{y} - \bar{y}_{*,\dagger})(\tilde{y}_* - \bar{y}_{*,\dagger}) dx + \gamma \int_\Omega (\bar{u} - \bar{u}_{*,\dagger})(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad = \int_\Omega \bar{y}(\tilde{y}_* - \bar{y}_{*,\dagger}) dx + \gamma \int_\Omega \bar{u}(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad\qquad - \int_\Omega \bar{y}_{*,\dagger}(\tilde{y}_* - \bar{y}_{*,\dagger}) dx - \gamma \int_\Omega \bar{u}_{*,\dagger}(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad \le \int_\Omega (\bar{y} - y_d)(\tilde{y}_* - \bar{y}_{*,\dagger}) dx + \gamma \int_\Omega \bar{u}(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad = \int_\Omega (R_* \bar{p} + \gamma \bar{u})(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad = \int_\Omega \lambda (\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx + \int_\Omega (R_* \bar{p} - \bar{p})(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&\int_\Omega (R_* \bar{p} - \bar{p})(\tilde{u}_\dagger - \bar{u}_{*,\dagger}) dx \\
&\qquad \le \|\bar{p} - R_* \bar{p}\|_{L_2(\Omega)} \big(\|Q_\dagger \bar{u} - u\|_{L_2(\Omega)} + \|u - \bar{u}_{*,\dagger}\|_{L_2(\Omega)} \big)
\end{aligned} \tag{4.8}$$

by (4.4), the Cauchy-Schwarz inequality and the triangle inequality.

We can estimate the first term on the right-hand side of (4.7) by (2.15), (2.18), (3.3) and (4.4) as follows.

$$\begin{aligned}
\int_\Omega \lambda(\tilde{u}_\dagger - \bar{u}_{*,\dagger})dx &= \int_\Omega \lambda_1(\tilde{u}_\dagger - \bar{u}_{*,\dagger})dx + \int_\Omega \lambda_2(\tilde{u}_\dagger - \bar{u}_{*,\dagger})dx \\
&= \int_\Omega \lambda_1(Q_\dagger\bar{u} - \bar{u})dx + \int_\Omega \lambda_2(Q_\dagger\bar{u} - \bar{u})dx \\
&\quad + \int_\Omega \lambda_1(\bar{u} - \phi_1)dx + \int_\Omega \lambda_2(\bar{u} - \phi_2)dx \\
&\qquad + \int_\Omega \lambda_1(\phi_1 - Q_\dagger\phi_1)dx + \int_\Omega \lambda_2(\phi_2 - Q_\dagger\phi_2)dx \\
&\qquad\quad + \int_\Omega \lambda_1(Q_\dagger\phi_1 - \bar{u}_{*,\dagger})dx + \int_\Omega \lambda_2(Q_\dagger\phi_2 - \bar{u}_{*,\dagger})dx \\
&\le \int_\Omega \lambda_1(Q_\dagger\bar{u} - \bar{u})dx + \int_\Omega \lambda_2(Q_\dagger\bar{u} - \bar{u})dx \\
&\quad + \int_\Omega \lambda_1(\phi_1 - Q_\dagger\phi_1)dx + \int_\Omega \lambda_2(\phi_2 - Q_\dagger\phi_2)dx \\
&= \int_\Omega (\lambda_1 - Q_\dagger\lambda_1)(Q_\dagger\bar{u} - \bar{u})dx + \int_\Omega (\lambda_2 - Q_\dagger\lambda_2)(Q_\dagger\bar{u} - \bar{u})dx \\
&\quad + \int_\Omega (\lambda_1 - Q_\dagger\lambda_1)(\phi_1 - Q_\dagger\phi_1)dx + \int_\Omega (\lambda_2 - Q_\dagger\lambda_2)(\phi_2 - Q_\dagger\phi_2)dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_\Omega \lambda(\tilde{u}_\dagger - \bar{u}_{*,\dagger})dx \\
&\qquad \le \big(\|Q_\dagger\bar{u} - \bar{u}\|_{L_2(\Omega)} + \|\phi_1 - Q_\dagger\phi_1\|_{L_2(\Omega)} + \|\phi_2 - Q_\dagger\phi_2\|_{L_2(\Omega)}\big) \\
&\qquad\quad \times \big(\|\lambda_1 - Q_\dagger\lambda_1\|_{L_2(\Omega)} + \|\lambda_2 - Q_\dagger\lambda_2\|_{L_2(\Omega)}\big).
\end{aligned} \tag{4.9}$$

Putting (4.4) and (4.6)–(4.9) together, we arrive at the estimate

$$\begin{aligned}
&\|\bar{y} - \bar{y}_{*,\dagger}\|^2_{L_2(\Omega)} + \gamma\|\bar{u} - \bar{u}_{*,\dagger}\|^2_{L_2(\Omega)} \\
&\qquad \le \|\bar{y} - \bar{y}_{*,\dagger}\|_{L_2(\Omega)}\|\bar{y} - \tilde{y}_*\|_{L_2(\Omega)} + \gamma\|\bar{u} - \bar{u}_{*,\dagger}\|_{L_2(\Omega)}\|\bar{u} - Q_\dagger\bar{u}\|_{L_2(\Omega)} \\
&\qquad\quad + \|\bar{p} - R_*\bar{p}\|_{L_2(\Omega)}\big(\|Q_\dagger\bar{u} - u\|_{L_2(\Omega)} + \|u - \bar{u}_{*,\dagger}\|_{L_2(\Omega)}\big) \\
&\qquad\qquad + \big(\|Q_\dagger\bar{u} - \bar{u}\|_{L_2(\Omega)} + \|\phi_1 - Q_\dagger\phi_1\|_{L_2(\Omega)} + \|\phi_2 - Q_\dagger\phi_2\|_{L_2(\Omega)}\big) \\
&\qquad\qquad\quad \times \big(\|\lambda_1 - Q_\dagger\lambda_1\|_{L_2(\Omega)} + \|\lambda_2 - Q_\dagger\lambda_2\|_{L_2(\Omega)}\big),
\end{aligned}$$

which together with the inequality of arithmetic and geometric means implies

$$
\begin{aligned}
&\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}^2+\gamma\|\bar{u}-\bar{u}_{*,\dagger}\|_{L_2(\Omega)}^2\\
&\qquad\le C\big(\|\bar{y}-\tilde{y}_*\|_{L_2(\Omega)}^2+\|\bar{u}-Q_\dagger\bar{u}\|_{L_2(\Omega)}^2+\gamma^{-1}\|\bar{p}-R_*\bar{p}\|_{L_2(\Omega)}^2\\
&\qquad\quad+\|\lambda_1-Q_\dagger\lambda_1\|_{L_2(\Omega)}^2+\|\lambda_2-Q_\dagger\lambda_2\|_{L_2(\Omega)}^2\\
&\qquad\qquad+\|\phi_1-Q_\dagger\phi_1\|_{L_2(\Omega)}^2+\|\phi_2-Q_\dagger\phi_2\|_{L_2(\Omega)}^2\big),
\end{aligned}
\tag{4.10}
$$

where C is a universal positive constant.

Note that (1.2), (4.1), (4.4) and (4.5) imply

$$
\begin{aligned}
a(R_*\bar{y}-\tilde{y}_*,z_*)&=a(\bar{y}-\tilde{y}_*,z_*)\\
&=\int_\Omega(\bar{u}-\tilde{u}_\dagger)z_*dx=\int_\Omega(\bar{u}-Q_\dagger\bar{u})z_*dx\qquad\forall\,z_*\in V_*
\end{aligned}
$$

and hence

$$
\|R_*\bar{y}-\tilde{y}_*\|_{L_2(\Omega)}\le(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{u}-Q_\dagger\bar{u}\|_{L_2(\Omega)}
$$

by Lemma 3.1. Therefore we have

$$
\|\bar{y}-\tilde{y}_*\|_{L_2(\Omega)}\le\|\bar{y}-R_*\bar{y}\|_{L_2(\Omega)}+(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{u}-Q_\dagger\bar{u}\|_{L_2(\Omega)}.\tag{4.11}
$$

Similarly (2.2), (3.6) and (4.1) imply

$$
a(q_*,R_*\bar{p}-\bar{p}_{*,\dagger})=a(q_*,\bar{p}-\bar{p}_{*,\dagger})=\int_\Omega(\bar{y}-\bar{y}_{*,\dagger})q_*dx\qquad\forall\,q_*\in V_*,\tag{4.12}
$$

and hence

$$
\|R_*\bar{p}-\bar{p}_{*,\dagger}\|_{L_2(\Omega)}\le(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}\tag{4.13}
$$

by Lemma 3.1. Consequently we have

$$
\|\bar{p}-\bar{p}_{*,\dagger}\|_{L_2(\Omega)}\le\|\bar{p}-R_*\bar{p}\|_{L_2(\Omega)}+(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}\tag{4.14}
$$

by (4.13) and the triangle inequality.

The estimate (4.2) follows from (4.10), (4.11) and (4.14). □

The following result shows that (4.2) is sharp up to the terms involving $Q_\dagger$.

Theorem 4.2. *There exists a positive constant $C_\natural$ depending only on α^{-1} such that*

$$\begin{aligned}&\|\bar{y}-R_*\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_*\bar{p}\|_{L_2(\Omega)}\\&\qquad\le C_\natural\big(\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}+\|\bar{u}-\bar{u}_{*,\dagger}\|_{L_2(\Omega)}+\|\bar{p}-\bar{p}_{*,\dagger}\|_{L_2(\Omega)}\big).\end{aligned}\tag{4.15}$$

Proof. We have

$$\|\bar{y}-R_*\bar{y}\|_{L_2(\Omega)}\le\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}+\|\bar{y}_{*,\dagger}-R_*\bar{y}\|_{L_2(\Omega)},\tag{4.16}$$

and, in view of (1.2), (3.2) and (4.1),

$$a(R_*\bar{y}-\bar{y}_{*,\dagger},z_*)=a(\bar{y}-\bar{y}_{*,\dagger},z_*)=\int_\Omega(\bar{u}-\bar{u}_{*,\dagger})z_*dx\qquad\forall\,z_*\in V_*,\tag{4.17}$$

which implies through Lemma 3.1

$$\|\bar{y}_{*,\dagger}-R_*\bar{y}\|_{L_2(\Omega)}\le(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{u}-\bar{u}_{*,\dagger}\|_{L_2(\Omega)}.\tag{4.18}$$

Similarly we have

$$\|\bar{p}-R_*\bar{p}\|_{L_2(\Omega)}\le\|\bar{p}-\bar{p}_{*,\dagger}\|_{L_2(\Omega)}+(\mathrm{C}_{\mathrm{PF}}^2/\alpha)\|\bar{y}-\bar{y}_{*,\dagger}\|_{L_2(\Omega)}\tag{4.19}$$

by (4.13) and the triangle inequality.

The estimate (4.15) follows from (4.16), (4.18) and (4.19). □

4.2. Estimate for the Energy Errors

Theorem 4.3. *There exists a positive constant $C_{\maltese}$ depending only on α^{-1} and γ^{-1} such that*

$$\begin{aligned}&|\bar{y}-\bar{y}_{*,\dagger}|_a+|\bar{p}-\bar{p}_{*,\dagger}|_a\\&\qquad\le C_{\maltese}\big(|\bar{y}-R_*\bar{y}|_a+|\bar{p}-R_*\bar{p}|_a+\|\bar{u}-Q_\dagger\bar{u}\|_{L_2(\Omega)}\\&\qquad\quad+\|\lambda_1-Q_\dagger\lambda_1\|_{L_2(\Omega)}+\|\lambda_2-Q_\dagger\lambda_2\|_{L_2(\Omega)}\\&\qquad\quad+\|\phi_1-Q_\dagger\phi_1\|_{L_2(\Omega)}+\|\phi_2-Q_\dagger\phi_2\|_{L_2(\Omega)}\big).\end{aligned}\tag{4.20}$$

Proof. We begin with a triangle inequality

$$\begin{aligned}&|\bar{y}-\bar{y}_{*,\dagger}|_a+|\bar{p}-\bar{p}_{*,\dagger}|_a\\&\qquad\le|\bar{y}-R_*\bar{y}|_a+|R_*\bar{y}-\bar{y}_{*,\dagger}|_a+|\bar{p}-R_*\bar{p}|_a+|R_*\bar{p}-\bar{p}_{*,\dagger}|_a.\end{aligned}\tag{4.21}$$

From (4.17) we obtain

$$|R_*\bar{y} - \bar{y}_{*,\dagger}|_a \leq (\mathrm{C}_{\mathrm{PF}}/\sqrt{\alpha})\|\bar{u} - \bar{u}_{*,\dagger}\|_{L_2(\Omega)} \tag{4.22}$$

by Lemma 3.1.

Similarly we have

$$|R_*\bar{p} - \bar{p}_{*,\dagger}|_a \leq (\mathrm{C}_{\mathrm{PF}}/\sqrt{\alpha})\|\bar{y} - \bar{y}_{*,\dagger}\|_{L_2(\Omega)} \tag{4.23}$$

by (4.12) and Lemma 3.1.

Finally the Poincaré-Friedrichs inequality (2.7) and (1.4) imply

$$\|\bar{y}-R_*\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_*\bar{p}\|_{L_2(\Omega)} \leq (\mathrm{C}_{\mathrm{PF}}/\sqrt{\alpha})\big(|\bar{y}-R_*\bar{y}|_a+|\bar{p}-R_*\bar{p}|_a\big), \tag{4.24}$$

and the estimate (4.20) follows from Theorem 4.1 and (4.21)–(4.24). □

5. Applications

We can apply the error estimates in Section 4 to standard finite element methods and multiscale finite element methods.

5.1. Standard Finite Element Methods

We assume that the bilinear form $a(\cdot,\cdot)$ is given by

$$a(y,z) = \int_\Omega \big[A(x)\nabla y\cdot\nabla z + c(x)yz\big]\,dx, \tag{5.1}$$

where the nonnegative function $c(x)$ and the $d\times d$ symmetric matrix function $A(x)$ are sufficiently smooth, and there exists a positive constant μ such that

$$\xi^t A(x)\xi \geq \mu|\xi|^2 \qquad \forall\, x\in\Omega,\ \xi\in\mathbb{R}^d.$$

We can take $V_* = V_h \subset H_0^1(\Omega)$ to be the P_1 Lagrange finite element space (cf. [10, 7]) associated with a regular triangulation $\mathcal{T}_h$ of Ω, and $W_\dagger = W_\rho$ to be the space of piecewise constant functions associated with a regular triangulation $\mathcal{T}_\rho$ of Ω. The optimal state (resp., optimal control and adjoint state) is denoted by $\bar{y}_{h,\rho}$ (resp., $\bar{u}_{h,\rho}$ and $\bar{p}_{h,\rho}$).

For simplicity, we assume Ω is convex. It is known that $\bar{y}$ and $\bar{p}$ belong to $H^2(\Omega)$ (cf. [16, 11, 30]) and we have the following estimates (cf. [10, 7]) for the Ritz projection operator $R_h : H_0^1(\Omega) \longrightarrow V_h$:

$$|\zeta - R_h\zeta|_a \leq C_1 h|\zeta|_{H^2(\Omega)} \qquad \forall\,\zeta\in H^2(\Omega)\cap H_0^1(\Omega), \tag{5.2}$$

$$\|\zeta - R_h\zeta\|_{L_2(\Omega)} \leq C_1 h^2 |\zeta|_{H^2(\Omega)} \qquad \forall\, \zeta \in H^2(\Omega) \cap H_0^1(\Omega), \tag{5.3}$$

where the positive constant C_1 depends only on the coefficients in (5.1) and the shape regularity of $\mathcal{T}_h$.

The L_2 projection $Q_\rho : L_2(\Omega) \longrightarrow W_\rho$ satisfies (3.4) and we have a standard error estimate (cf. [10, 7]):

$$\|\zeta - Q_\rho\zeta\|_{L_2(\Omega)} \leq C_2 h |\zeta|_{H^1(\Omega)} \qquad \forall\, \zeta \in H^1(\Omega), \tag{5.4}$$

where the positive constant C_2 depends only on the shape regularity of $\mathcal{T}_\rho$.

It follows from Remark 2.1, Theorem 4.1, Theorem 4.3, and (5.2)–(5.4) that

$$\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{h,\rho}\|_{L_2(\Omega)} \leq C(h^2 + \rho), \tag{5.5}$$

$$|\bar{y} - \bar{y}_{h,\rho}|_a + |\bar{p} - \bar{p}_{h,\rho}|_a \leq C(h + \rho), \tag{5.6}$$

where the positive constant C is independent of h and ρ, and we have recovered the error estimates in [14] for a convex Ω.

We can also take $W_\dagger$ to be $L_2(\Omega)$, which is the variational discretization concept in [17]. In this case $Q_\dagger$ is the identity map on $L_2(\Omega)$ so that (3.4) is satisfied trivially and we denote the optimal state (resp., optimal control and adjoint state) by $\bar{y}_h$ (resp., $\bar{u}_h$ and $\bar{p}_h$). The estimates (5.5) and (5.6) become

$$\|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L_2(\Omega)} \leq Ch^2, \tag{5.7}$$

$$|\bar{y} - \bar{y}_h|_a + |\bar{p} - \bar{p}_h|_a \leq Ch, \tag{5.8}$$

where C is independent of h, and we have recovered the result in [17].

Remark 5.1. The estimates (5.5)–(5.8) also hold for a general Ω provided the triangulations $\mathcal{T}_h$ and $\mathcal{T}_\rho$ are properly graded around the singular parts of $\partial\Omega$ (cf. [22]).

5.2. Multiscale Finite Element Methods

Under assumption (1.4), the optimal state $\bar{y}$ and adjoint state $\bar{p}$ belong to $H_0^1(\Omega)$ and we cannot claim any additional regularity.

If we take $V_* = V_h \subset H_0^1(\Omega)$ to be the P_1 finite element space associated with Ω and $W_\dagger = W_\rho$ to be the space of piecewise constant functions associated with $\mathcal{T}_\rho$, then Theorem 4.1 implies

$$\lim_{h,\rho\downarrow 0} \big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{h,\rho}\|_{L_2(\Omega)}\big)$$

$$
\begin{aligned}
&\leq \lim_{h\downarrow 0}\big(\|\bar{y}-R_h\bar{y}\|_{L_2(\Omega)}+\|\bar{p}-R_h\bar{p}\|_{L_2(\Omega)}\big)\\
&\qquad+\lim_{\rho\downarrow 0}\big(\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Omega)}+\|\lambda_1-Q_\rho\lambda_1\|_{L_2(\Omega)}+\|\phi_1-Q_\rho\phi_1\|_{L_2(\Omega)}\\
&\qquad\qquad+\|\phi_2-Q_\rho\phi_2\|_{L_2(\Omega)}+\|\bar{u}-Q_\rho\bar{u}\|_{L_2(\Omega)}\big)\\
&=0.
\end{aligned}
$$

Therefore this standard finite element method converges, but the convergence in h can be arbitrarily slow (cf. [4]), and an accurate approximation of $(\bar{y},\bar{u},\bar{p})$ will require a very small mesh size h.

We can remedy this slow convergence by taking V_* to be a multiscale finite element space. For example we can take V_* to be the rough polyharmonic space V_H^{rps} in [32, 25] associated with a triangulation $\mathcal{T}_H$ and $W_\dagger = W_\rho$ remains the space of piecewise constant functions associated with a triangulation $\mathcal{T}_\rho$. The optimal state (resp., optimal control and adjoint state) is denoted by $\bar{y}_{H,\rho}^{rps}$ (resp., $\bar{u}_{H,\rho}^{rps}$ and $\bar{p}_{H,\rho}^{rps}$).

Let $\zeta \in H_0^1(\Omega)$ satisfy

$$
a(\zeta, v_H) = \int_\Omega g v_H\,dx \qquad \forall\, v_H \in V_H^{rps}, \tag{5.9}
$$

where $g\in L_2(\Omega)$. Then we have, by (1.4), (2.7) and the estimates in [32, 25],

$$
\|\zeta - R_H^{rps}\zeta\|_{L_2(\Omega)} \leq (\mathrm{C_{PF}}/\sqrt{\alpha})|\zeta - R_H^{rps}\zeta|_a \leq C_3 H\|g\|_{L_2(\Omega)}, \tag{5.10}
$$

where $R_H^{rps}: H_0^1(\Omega) \longrightarrow V_H^{rps}$ is the Ritz projection operator and the positive constant C_3 depends only on the shape regularity of $\mathcal{T}_H$ and α^{-1}.

It follows from Remark 2.1, Theorem 4.1, Theorem 4.3 and (5.10) that

$$
\begin{aligned}
\|\bar{y}-\bar{y}_{H,\rho}^{rps}\|_{L_2(\Omega)} + \|\bar{u}-\bar{u}_{H,\rho}^{rps}\|_{L_2(\Omega)} + \|\bar{p}-\bar{p}_{H,\rho}^{rps}\|_{L_2(\Omega)}&\\
+|\bar{y}-\bar{y}_{H,\rho}^{rps}|_a + |\bar{p}-\bar{p}_{H,\rho}^{rps}|_a \leq C_\diamond(H+\rho),&
\end{aligned} \tag{5.11}
$$

where the positive constant $C_\diamond$ depends only on the numbers $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^1(\Omega)}$, $\|\phi_2\|_{H^1(\Omega)}$, α^{-1}, γ^{-1}, and the shape regularities of $\mathcal{T}_H$ and $\mathcal{T}_\rho$, and we have recovered the results in [8].

We can also take V_* to be the constraint energy minimizing generalized multiscale finite element space V_H^{gms} in [9] associated with a triangulation $\mathcal{T}_H$. In this case the function $\zeta\in H_0^1(\Omega)$ defined by (5.9) satisfies

$$
\|\zeta - R_H^{gms}\zeta\|_{L_2(\Omega)} \leq (\mathrm{C_{PF}}/\sqrt{\alpha})|\zeta - R_H^{gms}\zeta|_a \leq C_4 H\|g\|_{L_2(\Omega)},
$$

where the positive constant C_4 depends only on α^{-1}, the shape regularity of $\mathcal{T}_H$ and Λ^{-1}. (Λ is a spectral parameter used in the construction of the multiscale finite element space V_H^{gms}.) Therefore (5.11) also holds for the approximate solution $(\bar{y}_{H,\rho}^{gms}, \bar{u}_{H,\rho}^{gms}, \bar{p}_{H,\rho}^{gms})$ obtained by this multiscale finite element method where $C_\diamond$ depends also on Λ^{-1}. This is the result in [2] in the case where $\mathcal{T}_\rho = \mathcal{T}_h$.

Finally we can take V_* to be the local orthogonal decomposition multiscale finite element spaces V_H^{lod} in [28, 29, 5] associated with a triangulation $\mathcal{T}_H$ that incorporates information from a standard finite element space V_h associated with a refinement $\mathcal{T}_h$ of $\mathcal{T}_H$. We denote the optimal state (resp., optimal control and adjoint state) by $\bar{y}_{H,\rho}^{lod}$ (resp., $\bar{u}_{H,\rho}^{lod}$ and $\bar{p}_{H,\rho}^{lod}$), and the Ritz projection operator from $H_0^1(\Omega)$ to V_H^{lod} is denoted by R_H^{lod}.

Let $v_h \in V_h$ and $v_{\scriptscriptstyle H} \in V_H^{lod}$ satisfy

$$
\begin{aligned}
a(v_h, w_h) &= \int_\Omega g w_h \, dx \qquad \forall\, w_h \in V_h, \\
a(v_{\scriptscriptstyle H}, w_{\scriptscriptstyle H}) &= \int_\Omega g w_{\scriptscriptstyle H} \, dx \qquad \forall\, w_{\scriptscriptstyle H} \in V_H^{lod}.
\end{aligned}
$$

Then we have, by the results in [28] and [5],

$$
|v_h - v_{\scriptscriptstyle H}|_a \le C_5 H \|g\|_{L_2(\Omega)} \quad \text{and} \quad \|v_h - v_{\scriptscriptstyle H}\|_{L_2(\Omega)} \le C_5 H^2 \|g\|_{L_2(\Omega)}, \tag{5.12}
$$

where the positive constant C_5 depends only on α^{-1} and the shape regularity of $\mathcal{T}_H$.

According to Remark 2.1, Theorem 4.1 and (5.4), we have

$$
\begin{aligned}
&\|\bar{y} - \bar{y}_{H,\rho}^{lod}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,\rho}^{lod}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{H,\rho}^{lod}\|_{L_2(\Omega)} \\
&\qquad \le C_6 \big(\|\bar{y} - R_H^{lod} \bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_H^{lod} \bar{p}\|_{L_2(\Omega)} + \rho \big),
\end{aligned} \tag{5.13}
$$

where the positive constant C_6 only depends on the numbers $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^1(\Omega)}$, $\|\phi_2\|_{H^1(\Omega)}$, α^{-1}, γ^{-1}, and the shape regularity of $\mathcal{T}_\rho$.

Let $(\bar{v}_{h,\rho}, \bar{u}_{h,\rho}, \bar{p}_{h,\rho})$ be the solution of (3.1) based on the space $V_* = V_h$ and the space $W_\dagger = W_\rho$. Then we have

$$
\begin{aligned}
a(R_h \bar{y}, z_h) &= \int_\Omega (f + \bar{u}) z_h \, dx \qquad \forall\, z_h \in V_h, \\
a(R_H^{lod} \bar{y}, z_{\scriptscriptstyle H}) &= \int_\Omega (f + \bar{u}) z_{\scriptscriptstyle H} \, dx \qquad \forall\, z_{\scriptscriptstyle H} \in V_H^{lod},
\end{aligned}
$$

by (1.2) and (4.1), which together with (5.12) imply

$$\|R_h\bar{y} - R_H^{lod}\bar{y}\|_{L_2(\Omega)} \leq C_5 H^2 \|f + \bar{u}\|_{L_2(\Omega)}. \tag{5.14}$$

Similarly we have

$$\|R_h\bar{p} - R_H^{lod}\bar{p}\|_{L_2(\Omega)} \leq C_5 H^2 \|\bar{y} - y_d\|_{L_2(\Omega)} \tag{5.15}$$

by (2.2), (4.1) and (5.12).

Putting Theorem 4.2 and (5.13)–(5.15) together we arrive at

$$\begin{aligned}
&\|\bar{y} - \bar{y}_{H,\rho}^{lod}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{H,\rho}^{lod}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{H,\rho}^{lod}\|_{L_2(\Omega)} \\
&\quad \leq C_6\big(\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|R_h\bar{y} - R_H^{lod}\bar{y}\|_{L_2(\Omega)} \\
&\qquad\qquad + \|\bar{p} - R_h\bar{y}\|_{L_2(\Omega)} + \|R_h\bar{y} - R_H^{lod}\bar{p}\|_{L_2(\Omega)} + \rho\big) \\
&\quad \leq C_6\big(\|\bar{y} - R_h\bar{y}\|_{L_2(\Omega)} + \|\bar{p} - R_h\bar{p}\|_{L_2(\Omega)} \\
&\qquad\qquad + C_5 H^2\|f + \bar{u}\|_{L_2(\Omega)} + C_5 H^2 \|\bar{y} - y_d\|_{L_2(\Omega)} + \rho\big) \\
&\quad \leq C_6 C_\sharp\big(\|\bar{y} - \bar{y}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{u} - \bar{u}_{h,\rho}\|_{L_2(\Omega)} + \|\bar{p} - \bar{p}_{h,\rho}\|_{L_2(\Omega)}\big) \\
&\qquad\qquad + C_6 C_5 H^2\big(\|f + \bar{u}\|_{L_2(\Omega)} + \|\bar{y} - y_d\|_{L_2(\Omega)}\big) + C_6\rho.
\end{aligned} \tag{5.16}$$

Similarly, we have by Theorem 4.3

$$|\bar{y} - \bar{y}_{H,\rho}^{lod}|_a + |\bar{p} - \bar{p}_{H,\rho}^{lod}|_a \leq C_7\big(|\bar{y} - \bar{y}_{h,\rho}|_a + |\bar{p} - \bar{p}_{h,\rho}|_a + H + \rho\big), \tag{5.17}$$

where the positive constant C_7 only depends on the numbers $\|y_d\|_{L_2(\Omega)}$, $\|f\|_{L_2(\Omega)}$, $\|\phi_1\|_{H^1(\Omega)}$, $\|\phi_2\|_{H^1(\Omega)}$, α^{-1}, γ^{-1}, and the shape regularities of $\mathcal{T}_H$ and $\mathcal{T}_\rho$.

Comparing (5.5)–(5.6) and (5.16)–(5.17), we conclude that up to the error of a fine scale approximation, the performance of the local orthogonal decomposition multiscale finite element method for a problem with rough coefficients on a general Ω is identical to the performance of standard finite element methods for a problem with smooth coefficients on a convex domain. Moreover all the constants in the estimates are independent of the mesh sizes and the contrast β/α.

Numerical results for the local orthogonal decomposition method for (1.1) can be found in [6].

6. Conclusions

We have developed new abstract error estimates for a model linear-quadratic elliptic distributed optimal control problem that reduce the error analysis to the properties of the Ritz projection operator for the finite element space for the state and the L_2 projection operator for the finite element space for the control. They can be applied to standard finite element methods for a classical partial differential equation constraint and multiscale finite element methods when the coefficients in the partial differential equation constraint are rough. Besides the multiscale finite element methods mentioned in Section 5, they can also be applied to many others, such as the ones investigated in [19, 20, 3, 12, 26, 27].

For simplicity we have assumed that $a(\cdot,\cdot)$ is symmetric. But the estimates in Section 4 can be extended to a nonsymmetric $a(\cdot,\cdot)$ by replacing the term $R_*\bar{p}$ with the term $S_*\bar{p}$, where $S_* : H_0^1(\Omega) \longrightarrow V_*$ is defined by

$$a(q_*, S_*\zeta) = a(q_*, \zeta) \qquad \forall\, q_* \in V_*.$$

Finally we note that error estimates for boundary control problems with rough coefficients are still absent.

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