

Graph Classes with Few Minimal Separators. II. A Dichotomy

Peter Gartland*

Daniel Lokshtanov*

Abstract

A class $\mathcal{F}$ of graphs is called *tame* if every graph in $\mathcal{F}$ on n vertices contains at most $n^{O(1)}$ minimal separators, *quasi-tame* if every graph in $\mathcal{F}$ on n vertices contains at most $2^{\log^{O(1)}(n)}$ minimal separators, and *feral* if there exists a constant $c > 1$ so that $\mathcal{F}$ contains n -vertex graphs with at least c^n minimal separators for arbitrarily large n . The classification of graph classes into (quasi-) tame or feral has numerous algorithmic consequences, and has recently received considerable attention.

In this paper we precisely characterize the structure of graphs which have few minimal separators. Specifically we show that every graph which excludes certain graphs called *k-creatures* and *k-critters* as induced subgraphs has at most quasi-polynomially many minimal separators. We then demonstrate that this sufficient condition for having few minimal separators is the “right” one. In particular we show that every hereditary graph class $\mathcal{F}$ definable in CMSO logic that contains *k-creatures* or *k-critters* for every k is feral.

1 Introduction

Let G be a graph and u and v be distinct vertices in G . A vertex set S is a *u,v-separator* if u and v are in distinct components of $G - S$. The set S is a *u,v-minimal separator* if S is a *u,v-separator*, but no proper subset of S is a *u,v-separator*. Finally, S is a *minimal separator* if S is a *u,v-minimal separator* for some pair of vertices u and v . Building on the terminology of Milanič and Pivač [29], we will say that a graph class $\mathcal{F}$ is *tame* if there exists an integer c so that every graph in $\mathcal{F}$ on n vertices has at most $O(n^c)$ minimal separators.

Minimal separators are a fundamental combinatorial object that show up naturally both in structural arguments [6, 28], as well as in algorithmic applications [27]. Many graph problems including TREEWIDTH, MINIMUM FILL IN, TREELENGTH, INDEPENDENT SET, FEEDBACK VERTEX SET, and others [5, 17, 26] can be solved in time polynomial in the number n of vertices plus the number of minimal separators in the input graph. These algorithms run in time polynomial in n precisely for the graphs that have at most $n^{O(1)}$ minimal separators, motivating the question we address in this paper - *which graph classes are tame?*

Since this paper is the second in a series we will refrain from a more in-depth discussion on the importance of minimal separators, or the history of study of tame graph classes. Such a discussion may be found in the first paper in the series [21]. We will simply mention that a substantial body of work has been devoted to identifying tame graph classes [1, 8, 7, 5, 6, 24, 25, 29, 30, 19].

All of the aforementioned previous work essentially gives different sufficient conditions for a graph to only have polynomially many separators. This naturally leads to the question *is there a “right” sufficient condition for tameness?* That is - a condition that on one hand is easy to state and verify, while on the other hand captures all interesting tame graph classes. One Theorem to tame them all, so to speak.

Abrishami et al. [1] conjectured that the presence or absence of *k-creatures* (more or less) completely dictates whether a graph has many or few minimal separators. To properly state their conjecture we first need to define *k-creatures*.

DEFINITION 1.1. (*k-CREATURES*) (see Figure 1) A graph G is said to be a *k-creature* if its vertices can be partitioned into sets A , $X = \{x_1, x_2, \dots, x_k\}$, $Y = \{y_1, y_2, \dots, y_k\}$, and B satisfying the following conditions:

- (i) $G[A]$ and $G[B]$ are connected.
- (ii) A and $Y \cup B$ are anti-complete (i.e. $N[A] \cap (Y \cup B) = \emptyset$) and B and $A \cup X$ are anti-complete.
- (iii) A dominates X (every vertex in X has a neighbor in A) and B dominates Y .

*University of California, Santa Barbara, USA. Emails: petergartland@ucsb.edu, daniello@ucsb.edu Supported by BSF award 2018302 and NSF award CCF-2008838.

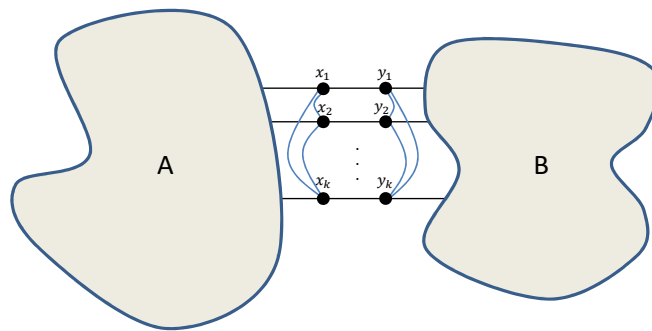


Figure 1: A graph induced by the vertices of a k -creature. The blue edges indicate that x_i (y_i) may or may not be a neighbor of x_j (y_j)

(iv) $x_i y_j$ is an edge if and only if $i = j$.

When identifying a k -creature in a graph, we will typically denote it as a tuple of vertex sets (V_1, V_2, V_3, V_4) such that the graph $G[V_1 \cup V_2 \cup V_3 \cup V_4]$ is a k -creature with the set V_1 corresponding to A , V_2 corresponding to X , V_3 corresponding to Y , and V_4 corresponding to B .

We will say that a graph G is k -creature free if G does not contain a k -creature as an induced subgraph. It is quite easy to see that for a vertex $a \in A$ and $b \in B$ there are precisely 2^k minimal separators S which are disjoint from A and B . Such a separator S must pick precisely one vertex from each of $\{x_i, y_i\}$, and it can make each one of these k choices independently.

Abrishami et al. [1] conjectured that for every integer k there exists a k' such that if an n -vertex graph G does not contain any k -creature as an induced subgraph, then G has at most $n^{k'}$ minimal separators. It turns out, as we showed in the first paper in this series [21], that this conjecture is false: for arbitrarily large n there exist n -vertex graphs that exclude 100-creatures and yet have $2^{\Omega(n)}$ minimal separators. However, before discarding the conjecture of Abrishami et al. [1], let us discuss why it would have been the “right” sufficient condition for polynomially many minimal separators if it had been true.

Towards this we need to ask, which graph families $\mathcal{F}$ would have been tame, but whose tameness would not be captured by the conjecture? It would be precisely families $\mathcal{F}$ that are tame, but that contain for every k an n -vertex graph G that contains a k -creature as an induced subgraph. Since k -creatures have 2^k minimal separators, and $\mathcal{F}$ is tame it must hold that $2^k \leq n^{O(1)}$, meaning that $k = O(\log n)$. In other words the conjecture fails to capture tame graph classes that contain k -creatures in graphs whose number of vertices is at least exponential in k .

This could manifest itself in two different ways. One option, that we call Type 1, is that whenever a graph $G \in \mathcal{F}$ contains a k -creature then G also contains some different piece of size exponential in k which is completely unrelated to the k -creature. The other option, which we call Type 2, is that whenever a graph $G \in \mathcal{F}$ contains a k -creature, then this k -creature itself has size exponential in k . Families of either one of these two types would have to be rather strange, although it is perfectly possible to construct artificial graph families of either type. For an example, most interesting graph families are *hereditary*, that is, closed under vertex deletion. A hereditary family $\mathcal{F}$ cannot possibly be Type 1. Indeed whenever $\mathcal{F}$ contains a graph G that contains a k -creature we can simply delete all the vertices not in the k -creature to obtain a k -creature which is in the family. Thus, the conjecture of Abrishami et al. [1] if true, would only fail to capture tameness of hereditary classes of graphs whose every k -creature has size exponential in k .

As previously mentioned, the conjecture of Abrishami et al. [1] is false. In [21] the authors gave a counterexample, and showed that a weaker version of the conjecture of Abrishami et al. is true. To state this result we need three definitions. First, a family $\mathcal{F}$ is *quasi-tame* if every n -vertex graph in the family has at most $2^{\log^{O(1)} n}$ minimal separators. Second, a k -skinny ladder is a graph G consisting of two anti-complete paths $P_l = \ell_1 \ell_2 \dots \ell_k$ and $P_r = r_1 r_2 \dots r_k$ and a set $\{s_1, s_2, \dots, s_k\}$ of vertices such that for every i , s_i is adjacent to ℓ_i and r_i and to no other vertices. Third, an *induced minor* of a graph G is a graph that can be obtained from G by deleting vertices and contracting edges. The main result of [21] is the following weakening of the conjecture of

Abrishami et al.

THEOREM 1.1. ([21]) *For every integer k , the family of k -creature free graphs that exclude k -skinny ladders as induced minors is quasi-tame.*

Theorem 1.1 fails to be “the one theorem to tame them all” in two different ways. The first is the quasi-polynomial upper bound on the number of minimal separators. The second is that Theorem 1.1 fails to capture the tameness of some perfectly reasonable hereditary graph classes. For an example, the class of all induced subgraphs of k -skinny ladders (for all $k \in \mathbb{N}$) can easily be checked to be tame, yet Theorem 1.1 fails to conclude anything at all about this family.

The first shortcoming of Theorem 1.1 was very recently rectified by Gajarsky et al. [19] who, building heavily upon the work of [21], showed a version of Theorem 1.1, but with the conclusion of quasi-tame replaced by tame. In this paper we rectify the second shortcoming. To properly state our main result we need to introduce another kind of graph, called t -critters. We first need a small generalization of minimal separators which applies to vertex sets instead of just vertices. Given $A, B \subset V(G)$ we define S to be an A, B -separator if $A \cap S = B \cap S = \emptyset$ and no component of $G - S$ contains a vertex from both A and B . An A, B -minimal separator is an A, B -separator such that no proper subset of S is an A, B -separator.

DEFINITION 1.2. (t -CRITTER PARTITION, t -CRITTER) (see Figure 2) *A t -critter partition of a graph G is a partition of the vertex set of G into sets $A_1, A_2, \dots, A_{t+1}, B_1, B_2, \dots, B_{t+1}, X_1, X_2, \dots, X_t$, such that the following conditions are satisfied.*

- (i) *For all $1 \leq i, j \leq t+1$ with $i \neq j$, A_i is anti-complete with A_j , B_i is anti-complete with B_j , and A_i is anti-complete with B_j .*
- (ii) *For all $1 \leq i \leq t+1$ A_i and B_i is connected.*
- (iii) *The vertices of A_i, A_{i+1}, B_i , and B_{i+1} are the only vertices outside of X_i that have a neighbor in X_i .*
- (iv) *There are (at least) two distinct $(A_i \cup A_{i+1}, B_i \cup B_{i+1})$ -minimal separators in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i]$, S_1^i and S_2^i , such that there is a path from A_i to A_{i+1} through both $X_i - S_1^i$ and $X_i - S_2^i$ and there is a path from B_i to B_{i+1} through both $X_i - S_1^i$ and $X_i - S_2^i$.*

A graph G is a t -critter if G has a t -critter partition. A graph G is t -critter free if G does not contain a t -critter as an induced subgraph.

The definition of t -critters is arguably technical and unappealing. After staring at the definition for a few minutes the reader should be able to convince themselves that, just as for k -creatures, for every vertex $a \in A = \bigcup_i A_i$ and vertex $b \in B = \bigcup_i B_i$ there are at least 2^t minimal a, b -separators in G disjoint from $A \cup B$. In particular, in order to separate a from b we may for every $i \leq t$ choose to delete either S_1^i or S_2^i , and for every i the choice between S_1^i or S_2^i can be made independently of the other choices (see Lemma 6.5). We are now ready to state our main theorem.

THEOREM 1.2. *For every pair of integers t, k the family of k -creature free and t -critter free graphs is quasi-tame.*

The upper bound in Theorem 1.2 on the number of minimal separators is $n^{k' \log^{17} n}$ where k' is a constant that depends only on k and t . The proof of Theorem 1.2 contains some interesting ingredients, from the VC-dimension-based lemma of [21], to a “greedy branching” procedure inspired by the recent quasi-polynomial time algorithm for Independent Set on P_k -free graphs [20], to covering-packing dualities [22] and Ramsey- and Erdős-Szekers [16] type arguments.

Theorem 1.2 is yet another sufficient condition for a graph to have few minimal separators. To boot, the condition is very technical and the upper bound in the number of minimal separators is an ugly quasi-polynomial function. What makes *this* sufficient condition for an upper bound for the number of minimal separators special? What makes it special is that it is the *right* sufficient condition, in the way that the conjecture of Abrishami et al. [1] would have been right if only it had been correct. But don’t take our word for it - we actually prove this in a precise and technical sense.

Let us apply the same litmus test for Theorem 1.2 as we did for the conjecture of Abrishami et al. [1], and ask for which tame graph families $\mathcal{F}$ are not captured by Theorem 1.2? Again there could be families of Type 1,

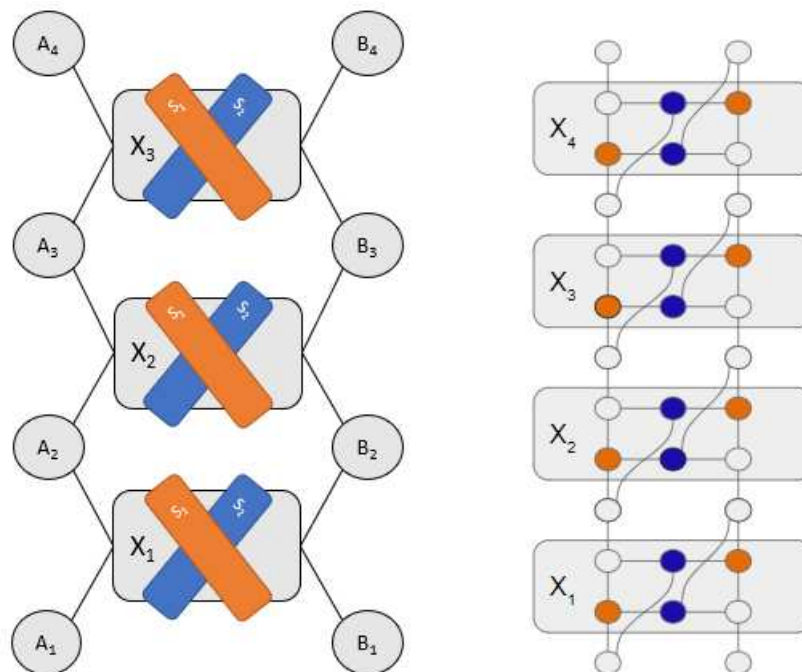


Figure 2: The left hand graph is a general visualization of a 3-critter, the orange blocks representing S_1^i and the blue blocks representing S_2^i . The right hand graph is specific instance of a 4-critter. The orange vertices represent the sets S_1^i and the blue vertices represent the sets S_2^i

namely families that do contain graphs G that contain k -critters or k -creatures for every k , but such graphs G always also contain at least $2^{\Omega(k)}$ additional unrelated vertices. Such families cannot be hereditary, and so, if we restrict attention to hereditary families the only tame families that are not captured by Theorem 1.2 are Type 2 families, that do contain k -critters or k -creatures for every k , but these k -critters or k -creatures have at least $2^{\Omega(k)}$ vertices.

It is in fact possible to construct such hereditary families. It is even possible to construct tame families that contain k -critters for every k , and yet are closed under induced minors, disproving Conjecture 4 of the authors [21] in the process¹ However all such families are pretty artificial. The next theorem shows that they *have* to be artificial.

Monadic Second Order Logic (MSO) and their extension, Counting Monadic Second Order Logic (CMSO), (see Section 6 for a definition) can be viewed as formal languages to express families of graphs. In graph algorithms their main claim to fame probably comes from Courcelle's Theorem [9], which states that every MSO-definable family of graphs can be recognized in linear time on graphs of bounded treewidth. The overwhelming majority of interesting graph families can be expressed in Counting Monadic Second Order Logic, this includes all graph classes with a finite number of forbidden minors, induced minors, topological minors, induced subgraphs or subgraphs, and a number of other classes such as bipartite, or perfect. We show that if we restrict attention to CMSO-definable hereditary properties then the sufficient condition of Theorem 1.2 is also necessary.

THEOREM 1.3. *Let $\mathcal{F}$ be a CMSO-definable hereditary graph family. If there exists an integer k such that $\mathcal{F}$ neither contains a k -creature nor a k -critter then $\mathcal{F}$ is quasi-tame. Otherwise $\mathcal{F}$ is feral.*

The proof of Courcelle's theorem [9] establishes that CMSO-definable graph classes have many properties in common with regular languages. This has been exploited with great success in graph algorithms [15, 23], however, to the best of our knowledge, it has never been used to prove a purely structural result such as Theorem 1.3

¹We do not give such a construction in this paper, as it is long enough as it is.

The proof of Theorem 1.3 is based on a “pumping lemma” style argument that shows that a k -creature or k -critter on n vertices can be “pumped” to a $k \cdot x$ -critter on $n \cdot x$ vertices, thereby demonstrating that in every CMSO-definable hereditary family that contains k -creatures or k -critters for arbitrarily large k , there exist k -creatures or k -critters in the family with only $O(k)$ vertices.

Paper Organization. In Section 2 we review basic definitions and notations. In Section 3 we give an overview of our proofs of Theorems 1.2 and 1.3.

The proof of Theorem 1.2 cleanly breaks into two parts, where the output of the first part is then fed into the second part. The first part of the proof is given in Section 4, while the second part of the proof is given in Section 5. Finally we prove Theorem 1.3 in Section 6. It is worth noting that while the proof of Theorem 1.2 is a bit of a monstrosity, the proof of Theorem 1.3 is relatively short and slick. If the reader is willing to assume the statement of Theorem 1.2 on face value, the proof of Theorem 1.3 may be read independently of the proof of Theorem 1.2.

2 Preliminaries

Unless otherwise stated, graphs in this paper are assumed to be simple, undirected graphs. We denote the edge set of a graph G by $E(G)$ and the vertex set of a graph by $V(G)$. If $v \in V(G)$, then we use $N_G[v]$ to denote the closed neighborhood of v in the graph G , i.e. the set of all neighbors v has in G together with v itself. We use $N_G(v)$ to denote the set $N_G[v] - \{v\}$. If $X \subseteq V(G)$, then $N_G[X] = \bigcup_{x \in X} N_G[x]$ and $N_G(X) = N_G[X] - X$. When the graph G is clear from the context, we will use $N[v]$, $N(v)$, $N[X]$, and $N(X)$. If $X \subseteq V(G)$, then we use $G[X]$ to denote the induced subgraph of G with vertex set X and $G - X$ denotes $G[V(G) - X]$. Additionally, for a natural number i , we inductively define $N_G^i[X]$ to equal $N_G[X]$ if $i = 1$ and $N_G[N_G^{i-1}[X]]$ for $i > 1$. Given a graph G and disjoint sets $X, Y \subseteq V(G)$ we define the *distance* between X and Y to be the lowest integer i such that $N_G^i[X] \cap Y \neq \emptyset$.

Given a graph G , a non-empty set $S \subseteq V(G)$ is called a *separator* if there are at least two distinct components L and R of $G - S$. If $u \in L$ and $v \in R$ then we call S a *u - v -separator* or a *u, v -separator*. S is a *u, v -minimal separator* if S is a u, v -separator and no proper subset of S is a u, v -separator, or equivalently, if $N_G(L) = N_G(R) = S$. This equivalence is folkloric and easy to show. If C is a component of $G - S$ such that $N_G(C) = S$, then we say that C is an *S -full component*. Similarly, given $A, B \subseteq V(G)$ we define S to be an *A, B -separator* if $A \cap S = B \cap S = \emptyset$ and no component of $G - S$ contains a vertex from both A and B . An *A, B -minimal separator* is an A, B -separator such that no proper subset of S is an A, B -separator.

Let G be a graph. A *vertex list* (or simply a *list*), $\mathcal{S}$, is an ordered tuple of vertex sets of G , that is, $\mathcal{S}$ is a collection of vertex sets which allow multiple instances of its vertex sets and gives an index to each element it contains. Let $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ be a vertex list and let S be a vertex set. We define $\mathcal{S} \cup S$ to be the list $\{S_1, S_2, \dots, S_m, S_{m+1}\}$ where $S_{m+1} = S$. Given a vertex list $\mathcal{S}$ we define $N_G[\mathcal{S}]$ to be the vertex list where each element $S \in \mathcal{S}$ is replaced with $N_G[S]$. Lastly, for a set $A \subseteq V(G)$, we define $\mathcal{S} - A$ to be the vertex list $\{S_1 - A, S_2 - A, \dots, S_m - A\}$.

Given a graph G with n vertices, a set $S \subseteq V(G)$ is called a *δ -balanced separator* if no component of $G - S$ contains over δ vertices.

Given a graph G , we say a vertex set $C \subseteq V(G)$ is a *connected vertex set* if $G[C]$ is a connected graph. A *walk* in a graph G is a sequence $v_1, v_2, \dots, v_\ell$ of vertices in G such that each pair of consecutive vertices in the sequence are adjacent. The *length* of a walk $v_1, v_2, \dots, v_\ell$ is the number ℓ of *vertices* in the walk. A walk whose first vertex is v_1 and last is v_ℓ is a walk *from v_1 to v_ℓ* . The vertex v_1 is called the first vertex of the walk, v_ℓ the last. All other vertices are *internal* vertices. A walk $v_1, v_2, \dots, v_\ell$ where all vertices are distinct is a *path*. A path $P = v_1, v_2, \dots, v_\ell$ is an *induced path* if there are no edges between v_i and v_j whenever $|i - j| > 1$. For three disjoint vertex sets A, B, C a walk (or path, or induced path) *from A to B through C* is a walk (or path, or induced path) whose first vertex is in A , last vertex is in B , and all internal vertices (if any) are in C .

We define contracting an edge. Let G be a graph. For an edge $uv \in E(G)$ we define the *contraction of uv* to result in a new graph G' with vertex set $V(G') = (V(G) - \{u, v\}) \cup \{w\}$ and there is an edge between two vertices $x, y \in V(G') - \{w\}$ in G' if there is an edge between x and y in G , and there is an edge between $x \in V(G') - \{w\}$ and w in G' if there is an edge x and u in G or an edge between x and v in G . Given a connected vertex set $A \subseteq V(G)$ we define the *contraction of A* to be the graph that results from contracting all edges between every pair of vertices of A . It can easily be seen that the contractions may be performed in any order without changing the final result.

Given a graph G we define a function $\mathcal{CC}(G)$ that returns a set which contains the vertex sets of all connected components of G .

Given a graph G we say that two vertex sets $A, B \subseteq V(G)$ are *anti-complete* if A and B are disjoint and there is no edge in G between any vertex of A and any vertex of B . Let $X, Y \subseteq V(G)$ be disjoint sets. We define a *semi-induced matching* in G of size m between X and Y to be two subsets $X' \subseteq X, Y' \subseteq Y$ with $X' = \{x'_1, x'_2, \dots, x'_m\}$ and $Y' = \{y'_1, y'_2, \dots, y'_m\}$ where x'_i is neighbors with y'_j in G if and only if $i = j$.

Let A and B be sets. The Cartesian product of A and B , denoted as $A \times B$, is the set with consists of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

3 Overview

3.1 Generalized ω -Creatures In order to prove Theorem 1.2 we will show that a graph that is k -creature free but still has a large number of minimal separators must have a t -critter. t -critters are highly structured objects and thus it is difficult to extract such objects from a graph directly, so in order to find a t -critter in a k -creature free graph with many minimal separators, we first construct an intermediate structure that we call a *generalized ω -creature* which we will define shortly. Generalized ω -creatures are less structured than t -critters and are thus easier to find in a graph, but they share some essential features with t -critters. In fact, the last main component of our proof is that for any natural numbers t , every generalized ω -creature, for ω large enough, contains a t -critter.

It is not necessary to have a full understanding of the definition of generalized ω -creatures until the end Subsection 4.2. We nevertheless give the definition here as it can help put the prior work of Section 4 into context which prepares the ground for extracting a generalized ω -creature from a k -creature free graph with many minimal separators.

Bistars, Bistar Partitions, and Generalized ω -Creatures. We define an ω -bistar, H , to be a graph with two *central vertices*, denoted by c_A and c_B , and ω independent vertices, called the *peripheral vertices*, that are neighbors with c_A and c_B (so H is a complete bipartite graph with 2 vertices on one side and ω on the other).

Let G be a graph. We define an ω -bistar partition of G to be an ω' -bistar graph H , with $\omega' \geq \omega$, along with a function φ from $V(G)$ to $V(H)$ such that for every edge $uv \in V(G)$ either $\varphi(u) = \varphi(v)$ or one of $\varphi(u)$ and $\varphi(v)$ is either c_A or c_B and the other is a peripheral vertex. We will use A_φ to denote the vertices of G in $\varphi^{-1}(c_A)$ and B_φ to denote the vertices of G in $\varphi^{-1}(c_B)$.

We now give the definition for generalized ω -creatures.

DEFINITION 3.1. (GENERALIZED ω -CREATURE) A generalized ω -creature is a tuple $W = (G, H, \varphi, S_1, S_2)$ where G is a graph, (H, φ) is an ω -bistar partition for G , and $S_1, S_2 \subseteq V(G)$ with the following properties:

- (i) There exists $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ such that for every peripheral vertex, u , of H , $\varphi^{-1}(u) \cap S_1^*$ and $\varphi^{-1}(u) \cap S_2^*$ are distinct A_φ, B_φ -minimal separators in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$.
- (ii) A_φ is entirely contained in one component of $G - S_1$ and B_φ is entirely contained in a different component of $G - S_1$ and similarly A_φ is entirely contained in one component of $G - S_2$ and B_φ is entirely contained in a different component of $G - S_2$.
- (iii) For every peripheral vertex u of H and all pair of components C_1 and C_2 of $G[A_\varphi \cup B_\varphi]$ there is a path from C_1 to C_2 through $\varphi^{-1}(u) - S_1$ in G if and only if there is a path from C_1 to C_2 through $\varphi^{-1}(u) - S_2$ in G .
- (iv) For all peripheral vertices, u , of H , if a component, X_A , of $G[A_\varphi]$ has a neighbor in $\varphi^{-1}(u)$ then there is at least one component, X_B , of $G[B_\varphi]$ such that X_A has a path through $\varphi^{-1}(u)$ to X_B in G . Similarly, if a component, X_B , of $G[B_\varphi]$ has a neighbor in $\varphi^{-1}(u)$ then there is at least one component X_A of $G[A_\varphi]$ such that X_B has a path through $\varphi^{-1}(u)$ to X_A in G .

Note that by property (i) that for every $u \in H$, $\varphi^{-1}(u)$ must be non-empty. Sets $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ satisfying property (i) of W are called witness separators for W .

For a generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$ we define the sets $A_1(W)$, $A_2(W)$, $B_1(W)$ and $B_2(W)$ as the component of $G - S_1$ that contains A_φ , the component of $G - S_2$ that contains A_φ , the component of $G - S_1$ that contains B_φ , and the component of $G - S_2$ that contains B_φ respectively. The vertex sets $A_1^*(W)$, $A_2^*(W)$, $B_1^*(W)$ and $B_2^*(W)$ are defined analogously, but with S_1^* and S_2^* in place of S_1 and S_2 .

Just like how for each X_i of a t -critter we can independently pick either S_i^1 or S_i^2 to build a minimal separator for the t -critter (giving 2^t minimal separators), it can be shown that conditions (i)-(iii) together allow us, for each peripheral vertex $u \in H$, to independently choose the vertex set $S_1 \cap \varphi^{-1}(u)$ or $S_2 \cap \varphi^{-1}(u)$ in order to build a minimal separator, S , for the graph G of the generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$, giving 2^ω such minimal separators. In such a minimal separator, the vertex sets A_φ will be in one S -full component of $G - S$ and B_φ will be in a different S -full component. We will prove this later on in the paper.

3.2 Proof of Theorem 1.2 The following two lemmas show how generalized ω -creatures help us to prove Theorem 1.2. Proving the first lemma is the main goal of Section 4 and proving the second lemma is the main goal of Section 5. We will actually need a generalized ω -creature with a some additional structure, in particular we will need the generalized ω -creature to be a connected, good, full generalized ω -creature. We will introduce these definitions later in the paper and the reader does not need to concern themselves with them at the moment.

LEMMA 3.1. *Let G be a k -creature free graph with n vertices, let $\omega > 1$ and $\delta = 3\omega$, let c be an integer large enough so that $400k^3\delta^2 \log^4(c) < c/6$, let $x = 400k^3\delta^2 \log^4(n)$, and let G have at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators. Then there exists an induced minor G' of G , an ω -bistar partition H and φ of G' , and sets $S_1, S_2 \subseteq V(G')$ such that $(G', H, \varphi, S_1, S_2)$ is a connected, good, full generalized ω -creature.*

LEMMA 3.2. *Let $k \geq 2$ and G be a k -creature free graph, and $W = (G, H, \varphi, S_1, S_2)$ be a connected, good, full generalized ω -creature. Then there exists an induced subgraph G' of G which is a t -critter for $t \geq \frac{(\log \log(\omega))^{1/4k}}{96k^4} - 4$.*

This paper natural breaks up into two parts, first part where we prove Lemma 3.1 and the second where we prove Lemma 3.2. We will provide an outline of how we prove Lemmas 3.1 and 3.2 later on in this section, right now we give a proof of Theorem 1.2 using Lemmas 3.1 and 3.2. Additionally, since Lemma 3.1 deals with an induced minor G' of G we need to show that k -creature free graphs and t -critter free graphs are closed under taking induced minors. We state the lemmas here and prove them shortly.

LEMMA 3.3. *Let G be graph and let G' be an induced minor of G . If G is k -creature free then G' is k -creature free.*

LEMMA 3.4. *Let G be a graph and let G' be an induced minor of G . If G is t -critter free then G' is t -critter free.*

Proof. [proof of Theorem 1.2] Let G be a k -creature free graph with n vertices, let ω be large enough to satisfy the inequality $t \leq \frac{(\log \log(\omega))^{1/4k}}{96k^4} - 4$ (note that the size of ω only depends on t and k), let $\delta = 3\omega$, let c be large enough to satisfy the inequality $400k^3\delta^2 \log^4(c) < c/6$, and let $x = 400k^3\delta^2 \log^4(n)$, so $x = k' \log^4(n)$ where k' only depends on k and t . We will show that if G has at least $2^c(12n)^{6k^2x^4 \log(n)} = n^{O(\log(n)^{17})}$ minimal separators then G must contain a t -critter. It follows from Lemma 3.1 that there is an induced minor G' of G , an ω -bistar partition H and φ of G' , and sets $S_1, S_2 \subseteq V(G')$ such that $W = (G', H, \varphi, S_1, S_2)$ is a connected, good, full generalized ω -creature. By Lemma 3.3 G' is k -creature free. Then by Lemma 3.2 it follows that G' contains a t' -critter for $t' \geq \frac{(\log \log(\omega))^{1/4k}}{96k^4} - 4$. By how ω was chosen we have $t' \geq t$, so G' contains a t -critter. Hence, by (the contra-positive of) Lemma 3.4, G contains a t -critter. $\square$

We now give proofs for Lemmas 3.3 and 3.4. They are fairly straightforward and just involve checking a few cases.

Proof. [proof of Lemma 3.3] Let G be a k -creature free graph. We first show that the deletion of a single vertex or the contraction of a single edge leaves us with a graph that is also k -creature free. Let $v \in G$. If $G - v$ contains a k -creature, (A, X, Y, B) , then clearly G contains the same k -creature, (A, X, Y, B) , hence $G - v$ is k -creature free. Now let uv be an edge of G , let G' be the graph that results from contracting uv , and let w denote the vertex that is creature from this contraction, so $V(G') = (V(G) - \{u, v\}) \cup \{w\}$. Assume for a contradiction that G' contains a k -creature, (A, X, Y, B) . We will break the proof into cases when w either belongs to A, X, Y, B or none of these sets. By the symmetry of k -creatures, the cases where $w \in A$ and $w \in B$ are identical and the cases where $w \in X$ and $w \in Y$ are identical, so we will only prove the cases for $w \in A$, $w \in X$ and $w \notin A \cup X \cup Y \cup B$.

If $w \notin A, X, Y, B$, then we can see that (A, X, Y, B) is a k -creature in G , a contradiction. If $w \in A$, then we can see that $((A - \{w\}) \cup \{u, v\}, X, Y, B)$ is a k -creature in G , a contradiction. Lastly, if $w \in X$, say $w = x_i$,

then, in G , at least one of u, v is neighbors with y_i and at least one of u, v has a neighbor in A . If one of these two vertices, say u , is both neighbors with y_i and has a neighbor in A in G , then $(A, (X - \{w\}) \cup \{u\}, Y, B)$ is a k -creature in G . So we may assume that in G exactly one of u, v is neighbors with y_i , say u , and does not have a neighbor in A , and that the other one, v , has a neighbor in A (but is not neighbors with y_i). It then follows that $(A \cup \{v\}, (X - \{w\}) \cup \{u\}, Y, B)$ is a k -creature in G .

A straightforward application of induction then shows that if G'' is any induced minor of G then since G is k -creature free, so is G'' . $\square$

Proof. [proof of Lemma 3.4] Let G be a graph that is t -critter free. We first show that the deletion of a single vertex or the contraction of a single edge leaves us with a graph that is also t -critter free. Let $v \in G$. If $G - v$ contains a t -critter, then clearly G contains the same t -critter, hence $G - v$ is t -critter free. Now, let uv be an edge in G , let G' be the graph that results in from contracting uv , and let w denote the vertex that is creature from this contraction, so $V(G') = (V(G) - \{u, v\}) \cup \{w\}$. Assume for a contradiction that G' contains a t -critter, T , and let $A_1, A_2, \dots, A_{t+1}, B_1, B_2, \dots, B_{t+1}, X_1, X_2, \dots, X_t$ be the partitioning of the vertices of T given in Definition 1.2. We will break the proof into cases when w either belongs to one of the A_i 's, one of the B_i 's, one of the X_i 's, or none of these sets. By the symmetry of k -critters, the cases when $w \in A_i$ for some i and when $w \in B_i$ for some i are identical, so we will only prove the cases where $w \in A_i$ for some i , $w \in X_i$ for some i , and $w \notin T$.

If $w \notin T$ then T is an induced subgraph of G and hence G contains a t -critter as an induced subgraph, a contradiction. If $w \in X_i$ for some i , then let T' be the induced subgraph of G with vertex set $(T - \{w\}) \cup \{u, v\}$. It is straightforward to verify that the partitioning $A_1, A_2, \dots, A_{t+1}, B_1, B_2, \dots, B_{t+1}, X_1, X_2, \dots, (X_i - \{w\}) \cup \{u, v\}, \dots, X_{t+1}$ of $V(T')$ satisfies properties (i)-(iii) of Definition 1.2. Now, since T is a t -critter, there are subsets S_1^i and S_2^i of X_i that satisfy property (iv) from Definition 1.2. Then, in T' , it can be seen that there exists $Z_1, Z_2 \subseteq \{u, v\}$ such that $(S_1^i - \{w\}) \cup Z_1$ and $(S_2^i - \{w\}) \cup Z_2$ satisfy property (iv) and for all other sets $X_j, j \neq i$ it can be seen (using the same S_1^j and S_2^j as in T) that property (iv) is satisfied in T' . This again contradicts the assumption that G is t -critter free. Lastly, if $w \in A_i$ for some i , it is straightforward to verify that the partitioning $A_1, A_2, \dots, (A_i - \{w\}) \cup \{u, v\}, \dots, A_{t+1}, B_1, B_2, \dots, B_{t+1}, X_1, X_2, \dots, X_{t+1}$ satisfies properties (i)-(iv) of Definition 1.2. This contradicts the assumption that G is t -critter free.

A straightforward application of induction then show show that if G'' is an induced minor of G , then G'' is t -critter free. $\square$

3.3 An Overview of Lemma 3.1 In order to find a generalized ω -creature in a k -creature free graph G with many minimal separators, we will “grow” sets $A, B, C \subseteq V(G)$, where initially A and B only contain a single vertex and C is the empty set. The goal is that in the end we will be able to find a connected, good, full generalized ω -creature, $W = (G', H, \varphi, S_1, S_2)$, where A and B correspond to the sets A_φ and B_φ and $G' = G - C$. Intuitively, we use the minimal separators of G as a resource in order to “grow” the sets A, B , and C and slowly obtain all of the properties we will require of them. We try to motivate the key properties of A, B , and C now.

We have, from the definition of an ω -bistar partition, that for any two peripheral vertices $u, v \in H$ there can be no edge between the sets $\varphi^{-1}(u)$ and $\varphi^{-1}(v)$. Since, as noted as a consequence of property (i) in the definition of ω -creatures, for all $u \in H$, $\varphi^{-1}(u) \neq \emptyset$, the first property that we will need A, B , and C to satisfy is that $G - (A \cup B \cup C)$ has at least ω components (actually for technical reasons we will need that no component of $G - (A \cup B \cup C)$ contains many vertices which will imply $G - (A \cup B \cup C)$ has at least ω components).

We now try to motivate the second property that we will require of A, B , and C . As we mentioned before (when we introduced the definition of a generalized ω -creature) that since W is a generalized ω -creature we can construct a minimal separator, S , of G' by selecting sets of the form $S_1 \cap \varphi^{-1}(u)$ or $S_2 \cap \varphi^{-1}(u)$ for each peripheral vertex u in H and unioning them together and S will have the property that A_φ and B_φ will be contained in different S -full components of $G - S$. It turns out that in order to be able to find sets S_1 and S_2 that have this nice property, we must be very careful to ensure that as we grow A, B , and C that G has many minimal separators (at least $\frac{\mu}{n^{\text{polylog}(n)}}$ where μ is the number of minimal separators of G) that are *consistent with A, B , and C* . We will formally define this property later, but roughly speaking, a minimal separator S is consistent with A, B , and C if A is contained in one S -full component of $G - S$ and B is contained in a different S -full component of $G - S$. This requirement isn't surprising, since as we just saw, the minimal separator S that was previously described has the property that A_φ and B_φ will be contained in different S -full components of $G - S$.

A third and closely related property of A, B , and C that we will require is that $G[A]$ and $G[B]$ have few

components (where few means $\log^{O(1)}(n)$). This requirement is partially (but though not entirely) due to property (iii) of generalized ω -creatures. Intuitively, to get property (iii) to work out, we filter the minimal separators we consider in such a way so that for all remaining separators under consideration, if we take any pair, say S_1 and S_2 , it holds that for every component X of $G - (A \cup B \cup C)$ and every pair of components P, Q of $G[A \cup B]$ there is a path from P to Q through $X - S_1$ if and only if there is a path from P to Q through $X - S_2$. As long as the number of components of $G[A]$ and $G[B]$ stay small then by the pigeon hole principle this filtering step will not lose too many minimal separators.

Thus, we have three apposing requirements we want A , B , and C to meet. On the one hand, we want $G - (A \cup B \cup C)$ to have no components with many vertices (and therefore $G - (A \cup B \cup C)$ will have at least ω components). This can be trivially satisfied by just picking A , B , and C to be really large sets. But, our second and third properties require us to be very careful with how we grow A , B , and C . Picking A , B , and C without much care would probably result in G not having many minimal separators that are consistent with A , B , and C and/or $G[A]$ and $G[B]$ having too many components.

It turns out that if we can find sets A , B , and C with these properties, then we would be able to construct a generalized ω -creature. Finding A , B , and C with these three properties is the focus of Subsection 4.1. Recall though that we do not just need a generalized ω -creature, but a connected, good, full generalized ω -creature. Further enriching A , B , and C to allow us to make a connected, good, full generalized ω -creature is the goal of Subsection 4.2.

So, how do we find these sets A , B , and C that have the three previously described properties? It involves a rather surprising combination and extension of two ideas.

The first idea comes from a quasi-polynomial time branching algorithm for independent set on P_k -free graphs of Gartland and Lokshtanov [20]. In the algorithm used in [20], when a vertex, v , is branched on the algorithm is recursively called on the inputs $G - v$ and on $G - N[v]$. The algorithm works by using $n/2$ -balanced separators dominated by few vertices (it is a critical property of P_k -free graph that they always have $n/2$ -balanced separators dominated by few vertices, we prove that k -creature free graphs also have this property in Lemma 4.4) to guide the selection of a vertex v to branch on. This branching “efficiently” breaks up the graph into small connected components (similar to what want the set A , B , and C , to do, but where “efficient” now does not refer to run time, but the fact that G has many minimal separators that are consistent with A , B , and C . It turns out that these two notions of “efficient” are strongly connected). Because we are working with k -creature free graphs though instead of P_k -free graphs, our process of selecting which vertex v to branch on becomes more complicated and requires some new ideas.

The second idea we use allows us to bridge this gap between independent set branching used in [20] and our goal of “growing” the sets A , B , and C . This idea is a lemma which appears in [21] where the authors prove that if G is a k -creature free graph with n vertices, then for every $v \in V(G)$, if $S^v = \{N(v) \cap S \mid S \in \mathcal{S} \text{ and } v \notin S\}$ then $|S^v| \leq n^k$. Intuitively, when a vertex, v , has been selected, instead of branching on the vertex and removing the set $N[v]$ from G , this lemma allows us to allocate the vertices of $N[v]$ to the set A , B , and C in such a way that many minimal separators of G remain consistent with A , B , and C .

3.4 An Overview of Lemma 3.2 In the second part of the paper we extract a k -critter from an ω -creature as long as ω is sufficiently large compared to k , and satisfies some additional structural properties (namely is *good*, *connected*, and *full*). For the purposes of this overview we will not need the precise definition of these notions.

The first thing a reader should notice is that a k -critter is a generalized k -creature. Consider a k -critter partition $(A_1, \dots, A_{k+1}, B_1, \dots, B_{k+1}, X_1, \dots, X_k)$ of a graph G . Set $A = \bigcup_i A_i$ and $B = \bigcup_i B_i$. Letting H be a k -bistar we define the function φ that maps all vertices in A to c_A , all vertices in B to c_B and each X_i to the i^{th} peripheral vertex of H . We then set $S_1 = \bigcup_i S_1^i$ and $S_2 = \bigcup_i S_2^i$. It can now be verified that $(G, H, \varphi, S_1, S_2)$ is in fact a generalized k -creature.

This fact is never stated or used explicitly in the proof, but it should help the reader understand what is going on. We have at hand a generalized ω -creature for some gigantic value of ω . Our goal is to extract from it a highly structured generalized k -creature. Here gigantic means that ω is bigger than any given function of k , but does not depend on n .

The key difference between a k -critter and a generalized ω -creature is that in a k -critter, every X_i has neighbors in precisely 4 components of $G[A \cup B]$, namely A_i , A_{i+1} , B_i and B_{i+1} . On the other hand, the (pre-image $\varphi^{-1}(v)$ of) peripheral vertices v of a generalized ω -creature (which correspond to the X_i 's of a k -critter)

may have neighbors in any number of components of $G[A \cup B]$.

Now the proof for extracting a k -critter from the generalized- ω creature goes like a typical “Ramsey Theory” style proof. We are given a large and unstructured object (here the largeness of the object is the number ω of peripheral vertices) and the goal is to extract from it a smaller, but still large, structured sub-object. We proceed in several steps. First we extract a generalized ω -creature in which the *adhesion size* is upper bounded by $2k$. The adhesion size is the maximum number of components of $G[A \cup B]$ that a peripheral vertex $\varphi^{-1}(v)$ can have neighbors into. From the generalized ω -creature of bounded adhesion size we then extract a “path-like” generalized ω' -creature, for ω' roughly equal to $\log \omega$. After some additional steps we extract a “critter-like” generalized ω' -creature, which basically is an ω' -critter.

It would seem that we are done. But no! We are not able to actually execute the arguments in the way we described above. Instead we are able to get each of the properties listed above on “one side”, say the A side, of the generalized ω' -creature. After the A -side is critter-like we need to turn around and re-do all of the arguments again for the B -side. However, this time when we are cleaning the B -side we need to make sure that we don’t break the nice properties that we worked so hard for on the A side. After both sides of the generalized ω' -creature are critter-like, we inspect its properties and observe that it is in fact an ω' -critter. In several places of the argument we may stumble on a k -creature instead of the more structured generalized ω -creature that we are looking for. In that case we halt and declare a win.

In order to execute this Ramsey-Theory style argument we need to define ways in which we can remove a few pieces of our generalized ω -creature while still keeping it a generalized ω -creature. For this we need to define three operations – dissolve, absorb and erase. One can think of these operations as analogues standard graph operations like vertex deletion, edge deletion and edge contraction for graphs, but for operating on generalized ω -creatures instead.

Because generalized ω -creatures are somewhat brittle and have a long and technical definition, every change to them requires us to go over and verify that the definitions are maintained. This makes many of these proofs excruciatingly long and wordy.

3.5 An Overview of Theorem 1.3 Let $\mathcal{F}$ be a CMSO-definable hereditary graph family. Theorem 1.3 consists of two main statements regarding $\mathcal{F}$ namely (i) If there exists an integer k such that $\mathcal{F}$ neither contains a k -creature nor a k -critter then $\mathcal{F}$ is quasi-tame, and (ii) otherwise $\mathcal{F}$ is feral. Statement (i) follows directly from Theorem 1.2. The premise of the “otherwise” case of Theorem 1.3 is that the family $\mathcal{F}$ contains, for every k , a k -creature or a k -critter. Because a k -creature contains a $(k-1)$ -creature as an induced subgraph, and a k -critter contains a $(k-1)$ -critter as an induced subgraph, it follows that $\mathcal{F}$ either contains a k -creature for every k , or a k -critter for every k (or both). Thus, to prove Theorem 1.3 we are left with proving two implications, namely

- (i) if $\mathcal{F}$ contains a k -creature for every k then $\mathcal{F}$ is feral, and
- (ii) if $\mathcal{F}$ contains a k -critter for every k then $\mathcal{F}$ is feral.

The approach for proving both implications is to use the fact that $\mathcal{F}$ is hereditary and CMSO-definable to show that if $\mathcal{F}$ contains a k -creature (k -critter) for every k then $\mathcal{F}$ contains for every k a k -creature (k -critter) with $O(k)$ vertices. Here the constant hidden in the big-oh depends on the CMSO formula describing $\mathcal{F}$. Since k -creatures and k -critters both contain 2^k minimal separators this proves that $\mathcal{F}$ is feral.

We now sketch the proof of (i). The authors showed in [21] that for every k there exists a k' such that every k' -creature contains one of six highly structured k -creatures as an induced subgraph. Since our premise is that $\mathcal{F}$ contains a k -creature for *every* k , we may now assume without loss of generality that $\mathcal{F}$ contains one of these six structured k -creatures for every k .

All of the the six structured k -creatures have the property that they can be partitioned into at most $5k$ paths, such that only the endpoints of these paths have neighbours on the outside (this is not quite true, but close enough to the truth for this overview, see Section 6). Therefore, if the structured k -creature $G \in \mathcal{F}$ has more than $c \cdot 5k$ vertices, then it contains an induced path on at least c vertices, such that only the endpoints of the path have neighbors on the outside of the path.

We now use the “regular-like” properties of CMSO-definable families: From the perspective of being a member of $\mathcal{F}$ there exists some universal constant γ , such that for every path P of length more than γ there exists a path P' of length at most γ such that “replacing” P with P' in G (which amounts to shortening the relevant path by $|V(P)| - |V(P')|$ vertices) does not affect whether the graph is in $\mathcal{F}$ or not.

In particular if G was in $\mathcal{F}$ before the path shortening, then it is also in $\mathcal{F}$ after. But shortening a path in G does not destroy the property of G being a k -creature. Thus, as long as $c > \gamma$ and the number of vertices in the k -creature is at least $c \cdot 5k$ we can prove that $\mathcal{F}$ contains a structured k -creature strictly smaller than G . Hence $\mathcal{F}$ must contain a k -creature on at most $5ck$ vertices.

The proof for k -critters is based on the same idea, but now things are more technical because k -critters do not necessarily contain long paths. Instead we prove that a k -critter contains a cut of size 3 in each X_i , and use these cuts to drive our “pumping” argument.

4 Finding A Generalized ω -Creature

In order to find a generalized ω -creature in a k -creature free graph G with many minimal separators, we will “grow” sets $A, B, C \subseteq V(G)$, where initially A and B only contain a single vertex and C is the empty set. The goal is that in the end we will be able to find a generalized ω -creature, $W = (G', H, \varphi, S_1, S_2)$ where A and B correspond to the sets A_φ and B_φ and $G' = G - C$. Intuitively, we use the minimal separators of G as a resource in order to “grow” the sets A, B , and C and slowly obtain all of the properties we require of them. As we mentioned before, when we construct a minimal separator, S , by selecting sets of the form $S_1 \cap \varphi^{-1}(u)$ or $S_2 \cap \varphi^{-1}(u)$ for $u \in H$, A_φ and B_φ will be contained in different S -full components of $G - S$. Hence, as we are growing the sets A, B , and C we should try and keep track of the minimal separators, S , such that A is contained in one S -full component of $G - S$, B is contained in some different S -full component, and the vertices of C are “irrelevant” in some sense (as we wind up just removing the vertices of C from G to get G'). It turns out the the correct notion of “irrelevant” is that these vertices are in neither of the S -full components A and B are contained in. This motivates the following definition.

DEFINITION 4.1. Let G be a graph, let $A, B, C \subseteq V(G)$, and let S be a minimal separator of G . We say that S is consistent with A, B , and C if the vertices of A all belong to one S -full component of $G - S$, the vertices of B all belong to a different S -full component of $G - S$, and no vertex of C belongs to the same component as a vertex in $A \cup B$ does in $G - S$.

We make the following two observations about this definition which we will use frequently without explicit reference.

OBSERVATION 4.1. Let G be a graph and let $A, B, C \subseteq V(G)$. If there is at least one minimal separator of G that is consistent with A, B , and C , then A and B are anti-complete and C is disjoint from $A \cup B$.

Proof. Let G be a graph and let $A, B, C \subseteq V(G)$ and assume S is a minimal separator of G that is consistent with A, B , and C . Since A and B are contained in different components of $G - S$, then A and B must be anti-complete. Since no vertex of C belongs to the same component that a vertex in $A \cup B$ does in $G - S$, C must be disjoint from $A \cup B$. $\square$

OBSERVATION 4.2. Let G be a graph, let $A, B, C \subseteq V(G)$, and let S be a minimal separator that is consistent with A, B , and C . If $v \in S$, then S is consistent with A, B , and $C \cup \{v\}$.

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$, and let S be a minimal separator that is consistent with A, B , and C . If $v \in S$, then v will not belong to the same component that a vertex in $A \cup B$ does in $G - S$, hence S is consistent with A, B , and $C \cup \{v\}$. $\square$

We now formalize this notion of C being a set of “irrelevant” vertices in the following lemma.

LEMMA 4.1. Let G be a graph, let $A, B, C \subseteq V(G)$ with $A, B \neq \emptyset$, and let S be a minimal separator that is consistent with A, B , and C . Then $S - C$ is a minimal separator of $G - C$ that is consistent with A, B , and $\emptyset$. Furthermore, if S' is a minimal separator of G that is consistent with A, B , and C such that $S \neq S'$, then $S - C \neq S' - C$.

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$ with $A, B \neq \emptyset$, and let S be a minimal separator of G that is consistent with A, B , and C . Let A_S and B_S be the S -full components of $G - S$ that contain A and B respectively, so $C \cap (A_S \cup B_S) = \emptyset$. It follows that A_S and B_S are then $(S - C)$ -full components of $(G - C) - (S - C)$, hence $S - C$ is a minimal separator of $G - C$ that is consistent with A, B , and $\emptyset$.

Now let S' be another minimal separator of G that is consistent with A, B , and C such that $S' \neq S$. By the preceding paragraph, if $A_{S'}$ and $B_{S'}$ are the S' -full components of $G - S'$ that contain A and B , then $A_{S'}$ and $B_{S'}$ are then $(S' - C)$ -full components of $(G - C) - (S' - C)$. Now, if $S' - C = S - C$, then the $(S - C)$ -full components are the same as the $(S' - C)$ -full components, hence $A_S = A_{S'}$ and $B_S = B_{S'}$, but this then implies that $S = S'$. $\square$

The previous lemma more or less shows that we can remove the vertices of C from G without having much of an effect on the minimal separators of G that are consistent with A, B , and C , so the reader may intuitively think of these vertices as being irrelevant.

Lastly, we prove a lemma which shows that as far as minimal separators of G that are consistent with A, B , and C are concerned, we can intuitively think of the components of $G[A]$ and $G[B]$ as actually being just single vertices of the graph.

LEMMA 4.2. *Let G be a graph, let $A, B, C \subseteq V(G)$, let S be a minimal separator of G that is consistent with A, B , and C , and let G' be the graph that results from contracting each component of $G[A]$ and $G[B]$ in G . Then S is a minimal separator of G' .*

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$ and let S be a minimal separator of G that is consistent with A, B , and C , it follows that $S \cap (A \cup B) = \emptyset$. Let A_S and B_S be the S -full components of $G - S$ that contain A and B respectively. Let G' be the graph that results from contracting each component of $G[A]$ and $G[B]$ in G , and let A'_S and B'_S be the vertex sets of G' that correspond to A_S and B_S . Then A'_S and B'_S are S -full components of $G' - S$, hence S is a minimal separator in G' . $\square$

We could have actually strengthened the result of the previous lemma by also showing that if A' and B' are the vertices of G' that correspond to the components of $G[A]$ and $G[B]$ in G , then S is a minimal separator of G' that is consistent with A', B' , and C . We have no need for such a strengthening though so we do not prove this. In most cases, we will not explicitly perform the contractions that Lemma 4.2 allows us to do as this would create some additional technical complications in our proofs. But, the reader should note that since the components of $G[A]$ and $G[B]$ can all be contracted to single vertices without destroying much structure in G (as far as minimal separators that are consistent with A, B , and C are concerned), these components can often intuitively be thought of as if they were single vertices.

In order to construct a generalized ω -creature from a k -creature free graph G with many minimal separators we will combine structural properties that k -creature free graphs have and use the minimal separators of G as a resource, which if given enough of (more than some amount that is quasi-polynomial in the number of vertices) will allow us to make a generalized ω -creature for some large value of ω . Our goal of creating a generalized ω -creature can be split into two main smaller goals.

The first goal is to find (or “grow”) vertex sets A, B and C so that no component in $G - (A \cup B \cup C)$ has many vertices (say no more than n/δ for some large value of δ) while maintaining that a large fraction of the minimal separators of G are consistent with A, B , and C as well as that $G[A]$ and $G[B]$ have few (poly-log(n)) components. This process of growing A, B and C shares many similarities with the quasi-polynomial time algorithm for independent set on P_k -free graphs by [20], although more complex as P_k -free graphs are in many ways more structurally simple when compared to k -creature free graphs. Additionally, we borrow tools from [21] which allow us to translate independent set branching techniques into branching techniques that can be applied to minimal separators.

The second goal is to leverage the facts that $G[A]$ and $G[B]$ have few components (which from our previous discussion we can pretend each component is just a single vertex), $G - (A \cup B \cup C)$ has many small components, and G still has many minimal separators that are consistent with A, B , and C in order to “grow” A, B , and C into sets A', B' , and C' and find a generalized ω -creature, $W = (G', H, \varphi, S_1, S_2)$, where G' is $G - C'$, $A_\varphi = A'$, $B_\varphi = B'$, and for each peripheral vertex $u \in H$ the set $\varphi^{-1}(u)$ corresponds to a component of $G - (A' \cup B' \cup C')$. Additionally, the generalized ω -creature we will build will have a little bit more structure than is required from a regular generalized ω -creature, which will allow us to prove in Section 5.1 that for all peripheral vertices $u \in H$, $\varphi^{-1}(u)$ will have neighbors in less than k components of $G'[A_\varphi]$ and $G'[B_\varphi]$.

4.1 Breaking up G

LEMMA 4.3. Let G be a k -creature free graph (assume $k \geq 2$) with $n \geq 2$ vertices and μ minimal separators and let $\delta > 1$. Then there exists sets $A, B, C \subseteq V(G)$ with $A, B \neq \emptyset$ where the following properties hold:

- (i) No component of $G - (A \cup B \cup C)$ contains over n/δ vertices.
- (ii) G has at least $\mu/(12n^{(k+1)})^{400k^3\delta^2 \log^4(n)}$ minimal separators that are consistent with A, B , and C .
- (iii) The number of components of $G[A \cup B]$ is at most $400k^3\delta^2 \log^4(n)$.

This section is devoted to proving Lemma 4.3 and Lemma 4.3 is the only lemma from this section that is used outside of this section.

We find (or “grow”) A , B , and C of Lemma 4.3 using techniques similar to those used by Gartland and Lokshtanov [20] where they give a quasi-polynomial time branching algorithm for independent set on P_k -free graphs. A crucial step of the algorithm is proving that P_k -free graphs have balanced separators that are dominated by few vertices. As we now show, k -creature free graphs also have balanced separators dominated by few vertices.

Let G be a graph with n vertices and let $S \subseteq V(G)$ such that S is *not* an $n/2$ -balanced separator, hence $G - S$ has a unique component with over $n/2$ vertices. In the following proof we will refer to this component as the *large component* of $G - S$, all other components of $G - S$ will be referred to as *small components*.

LEMMA 4.4. Let G be a k -creature free graph with n vertices, then there is a set $S \subseteq V(G)$ such that S is an $n/2$ -balanced separator and S can be dominated by $2k$ vertices.

Proof. Let G be a k -creature free graph with n vertices. Assume for a contradiction that G does not contain an $n/2$ -balanced separator dominated by at most $2k$ vertices. Consider all vertex sets, S , such that S is dominated by at most k vertices (hence S is not an $n/2$ -balanced separator and therefore $G - S$ has a large component) and S is dominated by some small component, A , of $G - S$. The open neighborhood of a single vertex meets these conditions, so at least one set satisfies this property. Now, among all such sets, let S' be a set such that the size of the large component of $G - S'$ is smallest. Let C denote the large component of $G - S'$, let A denote some small component of $G - S'$ that dominates S' , and let X be a vertex set of size at most k that dominates S' . Note by how S' was chosen (to minimize the size of the large component, C), S' is not anti-complete with C , since if it was, C would be a component of G . But then $c \in C$ would have the property that the largest component of $G - N_G(c)$ would have fewer vertices than C , which would contradict how S' was chosen.

Let $Y \subseteq V(G)$ be a set of size at most k . Since C is the only component of $G - S'$ that has over $n/2$ vertices, if no component of $G[C - N_G[Y]]$ has over $n/2$ vertices then no component of $G - (S' \cup N_G[Y])$ has over $n/2$ vertices. Since X dominates S' it follows that no component of $G - (N_G[X] \cup N_G[Y])$ has over $n/2$ vertices, but since X and Y both have size at most k this implies G has an $n/2$ balanced separator of size at most $2k$, contrary to assumption. So, for any set $Y \subseteq V(G)$, the largest component of $G[C - N_G[Y]]$ has over $n/2$ vertices and hence this largest component is unique.

For all sets $Z \subseteq S'$ of size at most k , let C_Z denote the component of $G[C - N_G[Z]] = G[C - (N_G[Z] \cap C)]$ that has over $n/2$ vertices (which must exist by the previous paragraph), let γ_Z denote the number of neighbors that C_Z has in $S' - Z$, and let γ denote the minimum over all γ_Z . Let $Z' \subseteq S'$ be a set of size at most k where $C_{Z'}$ has exactly γ neighbors in $S' - Z'$. We study two cases now, one where $\gamma = 0$ and the other where $\gamma > 0$ and get a contradiction in both cases, which will allow us to conclude that our assumption that G does not contain an $n/2$ -balanced separator dominated by $2k$ vertices is impossible.

Case 1: Assume $\gamma = 0$. We will show that $N_G[Z'] \cap C$ is dominated by at most k vertices (namely the vertices of Z'), $N_G[Z'] \cap C$ is dominated by a small component of $G - (N_G[Z'] \cap C)$, and the large component of $G - (N_G[Z'] \cap C)$, is strictly smaller than the large component of $G - S'$, contradicting how S' was chosen. Since $|Z'| \leq k$, clearly any subset of $N_G[Z'] \cap C$ is dominated by at most k vertices, which establishes the first condition we must show.

Now, since C is a component of $G - S'$ and $C_{Z'}$ is a component of $G[C - (N_G[Z'] \cap C)]$, $C_{Z'}$ is a component of $G - (S' \cup (N_G[Z'] \cap C))$. Because $C_{Z'}$ is a component of $G - (S' \cup (N_G[Z'] \cap C))$, $N_G(C_{Z'}) \subseteq S' \cup (N_G[Z'] \cap C)$, furthermore by assumption $\gamma = 0$ so no vertex of $C_{Z'}$ has a neighbor in $S' - Z'$, it follows that $N_G(C_{Z'}) \subseteq N_G[Z'] \cap C$. Hence $C_{Z'}$ is a component of $G - (N_G[Z'] \cap C)$, and therefore the large component of $G - (N_G[Z'] \cap C)$ since by definition (two paragraphs above), $C_{Z'}$ has over $n/2$ vertices.

Now, if $N_G[Z'] \cap C = \emptyset$ then since $\gamma = 0$ this implies S' is anti-complete with C , but as noted in the first paragraph, S' is not anti-complete with C , so $N_G[Z'] \cap C \neq \emptyset$. Since $N_G[Z'] \cap C \neq \emptyset$, it holds that C'_Z is a strict subset of C . Lastly, recall that A is a small component of $G - S'$ that dominates S' , so since A and S' are both disjoint from C (and hence no vertices from $A \cup S'$ belong to $C_{Z'}$) and $Z' \subseteq S'$, the vertices of $A \cup S'$ will belong to a small component of $G - (N_G[Z'] \cap C)$ that dominates $N_G[Z'] \cap C$. Hence, $N_G[Z'] \cap C$ is dominated by a set of at most k vertices (namely Z'), $N_G[Z'] \cap C$ is dominated by a small component of $G - (N_G[Z'] \cap C)$, and the large component of $G - (N_G[Z'] \cap C)$, specifically $C_{Z'}$, is strictly smaller than the large component of $G - S'$, specifically C , contradicting how S' was chosen. We may then conclude that $\gamma = 0$ is impossible.

Case 2: Assume $\gamma > 0$. Among all sets $Z \subseteq S'$ of size at most k such that number of neighbors C_Z has in $S' - Z$ is γ , let $\hat{Z}$ be one of smallest size (fewest vertices). If $|\hat{Z}| < k$ then γ must be 0 or else we could add an element of $S' - \hat{Z}$ that has a neighbor in $C_{\hat{Z}}$ to $\hat{Z}$ to get a set $\hat{Z}' \subseteq S'$ of size at most k and $C_{\hat{Z}'}$ would have less than γ neighbors in $S' - \hat{Z}'$ which contradicts the how we chose γ . So, since by assumption $\gamma \neq 0$, we have that $|\hat{Z}| = k$.

Next, we claim that every vertex $z_i \in \hat{Z}$ has a private neighbor $w_i \in N_G[\hat{Z}] \cap C$ which in turn has a neighbor in the large component of $C - N_G[\hat{Z}]$, $C_{\hat{Z}}$. Assume for a contradiction that this claim is false, so there is a vertex $z \in \hat{Z}$ such that for every neighbor $w \in N_G[\hat{Z}] \cap C$ that z has, where w also has a neighbor in $C_{\hat{Z}}$, there is another vertex in $\hat{Z} - z$ that is a neighbor of w . We claim that by how z was chosen, $C_{\hat{Z}} = C_{\hat{Z}-z}$. It follows immediately from the definition of these sets that $C_{\hat{Z}} \subseteq C_{\hat{Z}-z}$.

Next note that since $C_{\hat{Z}}$ is a component of $C - (N_G[\hat{Z}] \cap C)$, $N_C(C_{\hat{Z}}) \subseteq N_G[\hat{Z}] \cap C$. So, if $C_{\hat{Z}-z} \not\subseteq C_{\hat{Z}}$ then since both sets are connected and $C_{\hat{Z}} \subset C_{\hat{Z}-z} \subseteq C$, it must be that there is some $v \in C_{\hat{Z}-z}$ such that $v \in N_C(C_{\hat{Z}}) \subseteq N_G[\hat{Z}] \cap C$. Since no vertex of $N_G[\hat{Z} - z] \cap C$ is in $C_{\hat{Z}-z}$, it must be that $v \in N_G(z) \cap C$ and $v \notin N_G[\hat{Z} - z] \cap C$. Therefore $v \in N_G[\hat{Z}] \cap C$, is a neighbor of z , v has a neighbor in $C_{\hat{Z}}$, and v is not a neighbor of any vertex in $\hat{Z} - z$, contrary to how z was chosen. Hence $C_{\hat{Z}} = C_{\hat{Z}-z}$.

So, there is no vertex in $C_{\hat{Z}-z}$ that is a neighbor of z , hence number of neighbors that $C_{\hat{Z}-z}$ has in $S' - (\hat{Z} - z)$ is γ , which contradicts that $\hat{Z}$ was chosen to be as small as possible since the number of neighbors $C_{\hat{Z}-z}$ has in $S' - (\hat{Z} - z)$ is also γ . We may then let W denote the set that contains these w_i 's from the claim we just proved.

We now show that $(A, \hat{Z}, W, C_{\hat{Z}})$ is a k -creature. By how A and $C_{\hat{Z}}$ were chosen, we have that $G[A]$ and $G[C_{\hat{Z}}]$ are connected. Next, A and C are both vertex sets of components of $G - S'$, so A and $C_{\hat{Z}}$ are anti-complete since $C_{\hat{Z}} \subseteq C$, A and W are anti-complete since $W \subseteq C$, and $\hat{Z}$ and $C_{\hat{Z}}$ are anti-complete since $C_{\hat{Z}} \subseteq C - N_G[\hat{Z}]$. Next, by the definition of A , A dominates S' and therefore dominates $\hat{Z} \subseteq S'$ and by definition of $C_{\hat{Z}}$ we can see that $C_{\hat{Z}}$ dominates W . Lastly, by how the vertices of W were selected, there is a semi-induced matching between $\hat{Z}$ and W and $|\hat{Z}| = |W| = k$. It follows that $(A, \hat{Z}, W, C_{\hat{Z}})$ is a k -creature, a contradiction to the assumption that G is k -creature free.

It now follows that the original assumption that G does not contain an $n/2$ -balanced separator dominated by $2k$ vertices is impossible. $\square$

LEMMA 4.5. *Let G be a k -creature free graph with n vertices and let $\delta > 1$, then there is a set $S \subseteq V(G)$ such that S is an n/δ -balanced separator and S can be dominated by $8k\delta$ vertices.*

Proof. Let G be a k -creature free graph with n vertices and let $\delta > 1$. We will prove by induction on i that G has an $n/2^i$ -balanced separator dominated by $4k2^i$ vertices. By rounding to the nearest multiple of 2, this proves that G has an n/δ -balanced separator that is dominated by $8k\delta$ vertices.

Lemma 4.4 handles the base case where $i = 1$. Assume that for all i less than $j > 1$, G has a $n/2^i$ -balanced separator dominated by $4k2^i$ vertices. We show that G has a $n/2^j$ -balanced separator dominated by $4k2^j$ vertices. By the inductive hypothesis, G has an $n/2^{j-1}$ -balanced separator, S , dominated by $4k2^{j-1}$ vertices. There are at most 2^j components of $G - S$ that have over $n/2^j$ vertices, so for each such component, C_r , apply Lemma 4.4 to get an $|C_r|/2$ -balanced separator S_r for $G[C_r]$ that is dominated by $2k$ vertices. Setting $S' = S \cup (\bigcup_r S_r)$ it follows that S' is an $n/2^j$ -balanced separator for G . Furthermore, since there are at most 2^j S_r 's, each of which is dominated by $2k$ vertices, S' is dominated by $4k2^{j-1} + 2k2^j = 4k2^j$ vertices. $\square$

In the branching algorithm used in [20], balanced separators for a P_k -free graph, G , are collected in order

to guide in the selection of a vertex, v , to branch on, the algorithm is recursively called on the input $G - v$ and on $G - N[v]$ and it can be shown that with well chosen v , the graph G will be efficiently broken up into small components. In our methods here, we do not wish to remove vertices from the graph, but either add v or $N[v]$ to the sets A, B and C to get new A, B , and C sets. We actually will not be branching either, but greedily choosing the branch that roughly corresponds to the one that contains a new A, B , and C that have the most minimal separators that are consistent with the new A, B , and C . We will use the term *refining* on a vertex v when refer to this “greedy branching” process. At each refining step, some vertices are added to A, B , and C , so the number of minimal separators that are consistent with A, B , and C drops by some factor, but because this refining process efficiently breaks up the graph into small components, we will be able to grow A, B , and C into sets where all components of $G - (A \cup B \cup C)$ are small, and additionally there will still be some large fraction of minimal separators of G that are consistent with A, B , and C .

We now discuss refining in a little more in depth. In general, given $v \in G$, there are $3^{|N_G[v]|}$ way to allocate the vertices of $N[v]$ into the sets A, B , and C (and therefore $3^{|N[v]|}$ different refinement options that must be considered), so we cannot guarantee that any of the refinement options correspond to a new A, B , and C such that a large fraction of minimal separators are consistent with A, B , and C . But, if G is k -creature free, then the next three lemmas, the first of which is taken from [21] and the second of which is a slight strengthening of the first, shows that we only need to consider roughly n^k refinement options and this allows us to guarantee that at least one option corresponds to a new A, B , and C such that a large fraction of minimal separators are consistent with A, B , and C .

LEMMA 4.6. ([21]) *Let G be a k -creature-free graph with n vertices and let $\mathcal{S}$ be a set of minimal separators of G . Then for every $v \in V(G)$, if $S^v = \{N(v) \cap S \mid S \in \mathcal{S} \text{ and } v \notin S\}$ then $|S^v| \leq n^k$.*

LEMMA 4.7. *Let G be a k -creature free graph with n vertices, let $U \subseteq V(G)$ be a vertex set such that $G[U]$ has c components, let $\mathcal{S}$ be a set of minimal separators of G , and let $\mathcal{S}^U = \{N(U) \cap S \mid S \in \mathcal{S} \text{ and } U \cap S = \emptyset\}$. Then $|\mathcal{S}^U| \leq n^{kc}$.*

Proof. Let G be a k -creature free graph with n vertices, let $U \subseteq V(G)$ be a vertex set such that $G[U]$ has c components, let G' be the graph that results from contracting each component of $G[U]$ in G , and let U' be the vertices of G' that correspond to the components of $G[U]$, so U' has c vertices and by Lemma 3.3 G' is k -creature free. Observe that if S is a minimal separator of G such that $U \cap S = \emptyset$, then S is also a minimal separator of G' such that $S \cap U' = \emptyset$ and furthermore $N_G(U) \cap S = N_{G'}(U') \cap S$. So to prove this lemma it is sufficient to prove that if $\mathcal{S}^{U'} = \{N(U') \cap S \mid S \in \mathcal{S} \text{ and } U' \cap S = \emptyset\}$ then $|\mathcal{S}^{U'}| \leq n^{kc}$.

For $u_i \in U'$ let $S^{u_i} = \{N(u_i) \cap S \mid S \in \mathcal{S} \text{ and } u_i \notin S\}$ and let $V \in \mathcal{S}^{U'}$. Then we can see that for each $u_i \in U'$ we can select a $V_i \in S^{u_i}$ such that $V = \bigcup V_i$. Since by Lemma 4.6 S^{u_i} has at most n^k elements and U' has c elements, it follows that $\mathcal{S}^{U'}$ has at most n^{kc} elements. $\square$

LEMMA 4.8. *Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let $v \in G$. Then at least one of the following cases apply:*

- (i) *At least $(1 - 1/n)\mu$ minimal separators of G are consistent with A, B , and $C \cup \{v\}$.*
- (ii) *There exists $A', B', C' \subseteq V(G)$ such that at least $(1/3)(1/n^{k+1})\mu$ minimal separators of G are consistent with A', B' , and C' where $A \subseteq A', B \subseteq B'$, and $C \subseteq C'$, $N_G[v] \subseteq (A' \cup B' \cup C')$, and the number of components in $G[A' \cup B']$ is at most one more than the number of components in $G[A \cup B]$.*

Proof. Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let $\mathcal{S}$ denote this set of minimal separators. Let $v \in G$, if at least $(1 - 1/n)\mu$ minimal separators of $\mathcal{S}$ contain v , then since S is consistent with A, B , and $C \cup \{v\}$, case (i) is satisfied. So, we may assume that at least μ/n minimal separators of $\mathcal{S}$ do not contain v . It then follows that for at least $\mu/3n$ minimal separators $S \in \mathcal{S}$, either v belongs to the same component that A does in $G - S$, v belongs to the same component B does in $G - S$, or v does not belong to the same component A does in $G - S$ nor the same component B does in $G - S$.

First, assume that for at least $\mu/3n$ minimal separators $S \in \mathcal{S}$, v belongs to the same component that A does in $G - S$, denote this subset of $\mathcal{S}$ by $\mathcal{S}_A$. By Lemma 4.7 there are at most n^k sets of the form $N_G(v) \cap S$ for

$S \in \mathcal{S}_A$, hence for some set, X , there are at least $\mu/3n^{k+1}$ minimal separators, S , of $\mathcal{S}_A$ such that $N_G(v) \cap S = X$, denote this subset of $\mathcal{S}_A$ as $\mathcal{S}_A^X$. Since for all $S \in \mathcal{S}_A^X$ v belongs to the same component that A does in $G - S$, it follows that all vertices of $N_G[v] - X$ belong to the same component that A does in $G - S$, hence S is consistent with $A \cup N_G[v] - X$, B and C . Furthermore, since $X \subset S$, it then holds that S is consistent with $A \cup N_G[v] - X$, B , and $C \cup X$. It then follows that case (ii) of the lemma statement is satisfied in this case (the additional new component of $G[A' \cup B']$ arises if $N_G[v] - X$ is anti-complete with A , making $N_G[v] - X$ the new component of $G[A' \cup B']$).

The case where for at least $\mu/3n$ minimal separators $S \in \mathcal{S}$, v belongs to the same component that B does in $G - S$ is handled in the exact same way as in the previous paragraph. So, we now consider the case where for at least $\mu/3n$ minimal separators $S \in \mathcal{S}$, v does not belong to S , nor does v belong to the same component A does in $G - S$, nor the same component B does in $G - S$. Denote this subset of $\mathcal{S}$ by $\mathcal{S}'$. Since for $S \in \mathcal{S}'$ $v \notin S$, it follows that no vertex of $N_G(v)$ belongs to the same component A does in $G - S$ nor the same component B does in $G - S$. Hence, for $S \in \mathcal{S}'$, S is consistent with A , B , and $C \cup N_G[v]$. It then follows that case (ii) of the lemma statement is satisfied in this case, completing the proof. $\square$

We now define a function which formalizes this refinement process (greedily choosing the the best branch). This function will play a central role in this section.

DEFINITION 4.2. Let G be a k -creature free graph with n vertices, let $v \in G$, and let $A, B, C \subseteq V(G)$. We define a function **Refine**(G, v, A, B, C). We match the output of **Refine**(G, v, A, B, C) according to the first case of Lemma 4.8 that is met by G, v, A, B , and C . **Refine**(G, v, A, B, C) returns:

- (i) $(A, B, C \cup \{v\})$ if case (i) of Lemma 4.8 is the first case satisfied.
- (ii) (A', B', C') if case (ii) of Lemma 4.8 is the first case satisfied, where A', B' and C' are the sets whose existence is given by case (ii) of Lemma 4.8.

If the return of **Refine**(G, v, A, B, C) comes from case (i) then we call **Refine**(G, v, A, B, C) a failure refinement, else we call **Refine**(G, v, A, B, C) a success refinement.

A simple application of Lemma 4.8 then gives us the following lemma.

LEMMA 4.9. Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , let $v \in G$, and let $A', B', C' = \mathbf{Refine}(G, v, A, B, C)$. Then the following conditions hold:

- (i) If **Refine**(G, v, A, B, C) is a failure refinement then G has at least $(1 - 1/n)\mu$ minimal separators that are consistent with A', B' , and C' and $A' = A$, $B' = B$, and $C' = C \cup \{v\}$.
- (ii) If **Refine**(G, v, A, B, C) is a success refinement then G has at least $(1/3)(1/n^{k+1})\mu$ minimal separators that are consistent with A', B' , and C' , $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$, $N_G[v] \subseteq (A' \cup B' \cup C')$, and the number of components of $G[A' \cup B']$ is at most one more than the number of components of $G[A \cup B]$.

4.1.1 Collecting Separators with Anti-Complete Cores. This subsection is devoted to proving Lemma 4.10, which is the only lemma from this subsection that is used outside of this subsection. Before we state Lemma 4.10 we give a short definition pertaining to vertex lists. Let G be a graph and let $\mathcal{S}$ be a list that contains vertices from G . We will say that $\mathcal{S}$ is an *anti-complete vertex list* if for every two elements $S_i, S_j \in \mathcal{S}$, S_i is anti-complete with S_j . Note that this implies that if $S_i = S_j$ then $S_i = S_j = \emptyset$.

LEMMA 4.10. Let G be a k -creature free graph with $n \geq 2$ vertices and μ minimal separators and let $\delta > 1$. Without loss of generality assume $k \geq 2$. Then there exists sets $A, B, C \in V(G)$ with $A, B \neq \emptyset$ and an anti-complete vertex list $\mathcal{S}$ of size $\log(n) + 1$ such that the following properties hold:

- (i) Let $G' = G - (A \cup B \cup C)$. For all $S_i \in \mathcal{S}$, $N_G[S_i] - (A \cup B \cup C)$ is an n/δ -balanced separator of G' .
- (ii) G has at least $\mu/(4(3n^{(k+1)})^{160k^2\delta^2\log^3(n)+2})$ minimal separators that are consistent with A, B , and C .

(iii) $G[A \cup B]$ has at most $160k^2\delta^2 \log^3(n) + 2$ components.

(iv) No vertex v in $G' = G - (A \cup B \cup C)$ belongs to all sets of $N_{G'}[S]$.

The algorithm from [20] (which the techniques of subsubsection 4.1.1 is based on) works by collecting balanced separators dominated by few vertices while branching on vertices who have a sufficient number of neighbors into the collected balanced separators. Eventually this leads to having a large collection of balanced separators such that no vertex of the remaining graph has neighbors in more than $\log(n)$ of the collected balanced separators. Since the graphs the algorithm of [20] is run on are P_k -free, this is enough to guarantee that the graph now has been efficiently broken up into small components, and the algorithm is then called on each component.

Lemma 4.10 is proved using very similar techniques to the algorithm described in the previous paragraph, even up to determining what vertex to refine (branch) on. This process of collecting balanced separators and refining allows us to gather a vertex list, $\mathcal{S}$, of balanced separators as well as vertex sets A, B , and C which satisfy the properties stated in Lemma 4.10. A key difference is that when working with P_k -free graphs, this would be enough to ensure that $G - (A \cup B \cup C)$ has no large component, thus proving Lemma 4.3, but when we are dealing with graphs that are k -creature free we must do more in order to break up the graph into small components. We cover this additional process in subsubsection 4.1.2. A second key difference here is that in the independent set algorithm for P_k -free graphs of [20], the algorithm continues to recurse on connected components after breaking up the graph. Here in this paper, we stop our process after finding A, B , and C such that $G - (A \cup B \cup C)$ has no large components, we do not do any recursion on the components.

In order to prove Lemma 4.10, we will study a sequence produced from a k -creature free graph G and a number $\delta > 1$, which we denote by $\text{seq}_1(G, \delta)$. $\text{seq}_1(G, \delta)$ is a sequence of tuples $(A_i, B_i, C_i, \mathcal{S}_i)$ where $A_i, B_i, C_i \subseteq V(G)$ and $\mathcal{S}_i$ is a vertex list. Before we can describe how the sequence is created, we need to provide the following definitions.

DEFINITION 4.3. Let G be a k -creature free graph and let $(A_i, B_i, C_i, \mathcal{S}_i)$ be the i^{th} element of $\text{seq}_1(G, \delta)$. For all natural numbers j , the j^{th} level set with respect to A_i, B_i, C_i and $\mathcal{S}_i$, denoted by $\mathcal{L}_j(A_i, B_i, C_i, \mathcal{S}_i)$, is defined as the set of vertices that belong to at least j sets (counting multiplicity) of $N_G[\mathcal{S}_i] - (A_i \cup B_i \cup C_i)$.

DEFINITION 4.4. Let G be a k -creature free graph with n vertices and let $(A_i, B_i, C_i, \mathcal{S}_i)$ be the i^{th} element of $\text{seq}_1(G, \delta)$. A vertex $v \in G'$, $G' = G - (A_i \cup B_i \cup C_i)$, is good for refining with respect to $(A_i, B_i, C_i, \mathcal{S}_i)$ if there exists a j such that $|N_{G'}[v] \cap \mathcal{L}_j(A_i, B_i, C_i, \mathcal{S}_i)| \geq n/2^j$.

We shall use the notion of “good for refining” when determining what the $i + 1^{\text{th}}$ tuple of the sequence will be. Note that this definition implies that any vertex of $G - (A_i \cup B_i \cup C_i)$ that belongs to $\mathcal{L}_{\log(n)}(A_i, B_i, C_i, \mathcal{S}_i)$ is good for refining, meaning any vertex that belongs to $\log(n)$ sets of $N_G[\mathcal{S}_i] - (A_i \cup B_i \cup C_i)$ is good for refining.

Let G be a k -creature free graph with n vertices and μ minimal separators and let $\delta > 1$. We now define $\text{seq}_1(G, \delta)$. Let $a, b \in G$ be two vertices such that at least μ/n^2 minimal separators of G are a, b -minimal separators (since there are $n(n-1)/2$ pairs of vertices in G and every minimal separator is a u, v -minimal separator for some pair $u, v \in G$ such a pair a, b must exist). For the base case of this sequence we define $A_1 = \{a\}$, $B_1 = \{b\}$, $C_1 = \emptyset$, and $\mathcal{S}_1 = \emptyset$. We will maintain throughout the sequence that $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, $C_i \subseteq C_{i+1}$, and $\mathcal{S}_i \subseteq \mathcal{S}_{i+1}$. We recursively define this sequence as follows.

Assume we are given $(A_i, B_i, C_i, \mathcal{S}_i)$, we will refer to the following process of determine the next tuple $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ as the i^{th} step of $\text{seq}_1(G, \delta)$. If there is a vertex $v \in G - (A_i \cup B_i \cup C_i)$ that is good for refining, then we set $A_{i+1}, B_{i+1}, C_{i+1} = \text{Refine}(G, v, A_i, B_i, C_i)$ and $\mathcal{S}_{i+1} = \mathcal{S}_i$ (hence, by the definition of **Refine**, $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, and $C_i \subseteq C_{i+1}$). If **Refine** (G, v, A_i, B_i, C_i) is a failure refinement then we call $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ a *failure tuple*, else **Refine** (G, v, A_i, B_i, C_i) is a success refinement and we call $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ a *success tuple*.

If there is no vertex that is good for refining, then let $G' = G - (A_i \cup B_i \cup C_i)$. Since G' is a subgraph of G , G' has at most n vertices and by Lemma 3.3 G' is a k -creature free, so by Lemma 4.5 there exists a (possibly empty) set of vertices $S \subseteq V(G')$ where $|S| \leq 8k\delta$ and $N_{G'}[S]$ is an n/δ -balanced separator for G' . Set $A_{i+1} = A_i$, $B_{i+1} = B_i$, $C_{i+1} = C_i$, and $\mathcal{S}_{i+1} = \mathcal{S}_i \cup \{S\}$. In this case we call $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ a *separator tuple*.

The sequence terminates once we reach a tuple $(A_j, B_j, C_j, \mathcal{S}_j)$ such that $|\mathcal{S}_j| = 10k\delta \log^2(n)$.

The following observation was noted just before defining $\text{seq}_1(G, \delta)$ and follows directly from the definition of $\text{seq}_1(G, \delta)$. We will use it frequently without explicit reference.

OBSERVATION 4.3. Let G be a k -creature free graph, let $\delta > 1$, and let $(A_i, B_i, C_i, \mathcal{S}_i)$ and $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ be the i^{th} and $i+1^{\text{th}}$ elements of the sequence $\text{seq}_1(G, \delta)$. Then $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, $C_i \subseteq C_{i+1}$, and $\mathcal{S}_i \subseteq \mathcal{S}_{i+1}$.

LEMMA 4.11. Let G be a k -creature free graph and let $\delta > 1$. Then $\text{seq}_1(G, \delta)$ is finite.

Proof. Let G be a k -creature free graph. Consider the i^{th} and $i+1^{\text{th}}$ tuples of $\text{seq}_1(G, \delta)$, $(A_i, B_i, C_i, \mathcal{S}_i)$ and $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$. If $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a success or failure tuple then $\mathcal{S}_i = \mathcal{S}_{i+1}$, and if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a separator tuple, then $|\mathcal{S}_i| + 1 = |\mathcal{S}_{i+1}|$. Since the sequence ends once we reach a tuple with $(A_j, B_j, C_j, \mathcal{S}_j)$ where $\mathcal{S}_j = 10k\delta \log^2(n)$, the sequence will terminate after the $10k\delta \log^2(n)^{\text{th}}$ separator tuple. So all we must show is there is a finite number of success and failure tuples. This follows from three facts. The first is that regardless if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a success, failure, or separator tuple, $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, $C_i \subseteq C_{i+1}$. The second is that if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a success or failure tuple, then at the i^{th} step, **Refine** is called on a vertex that belongs to $G - (A_i \cup B_i \cup C_i)$ and it follows that $(A_i \cup B_i \cup C_i) \subset (A_{i+1} \cup B_{i+1} \cup C_{i+1})$ where the containment is strict. Hence, if there are $|V(G)|$ failure or success tuples that precede $(A_i, B_i, C_i, \mathcal{S}_i)$, then $A_i \cup B_i \cup C_i = V(G)$. The third fact is that if $A_i \cup B_i \cup C_i = V(G)$, then there are no vertices that are good for refining at this step (or any future step) and therefore $(A_{j+1}, B_{j+1}, C_{j+1}, \mathcal{S}_{j+1})$ is a separator tuple. Hence there are at most $|V(G)|$ success and failure tuples. $\square$

It now follows from 4.11 that we may assume there is a last element of $\text{seq}_1(G, \delta)$.

LEMMA 4.12. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ denote the last tuple of $\text{seq}_1(G, \delta)$. Let $G' = G - (A \cup B \cup C)$, then for every $S \in \mathcal{S}$, $N_{G'}[S] - (A \cup B \cup C)$ is an n/δ -balanced separator for G' .

Proof. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ denote the last tuple of $\text{seq}_1(G, \delta)$. Let S_j denote the j^{th} element in the list $\mathcal{S}$. Then there is some tuple $(A_i, B_i, C_i, \mathcal{S}_i)$ of $\text{seq}_1(G, \delta)$ such that if $G' = G - (A_i \cup B_i \cup C_i)$ then $S_j \subseteq V(G')$ and $N_{G'}[S_j] = N_G[S_j] - (A_i \cup B_i \cup C_i)$ is an n/δ -balanced separator for G' . Since $A_i \subseteq A$, $B_i \subseteq B$, and $C_i \subseteq C$, it follows that $N_G[S_j] - (A \cup B \cup C)$ is a balanced separator for G'' where $G'' = G - (A \cup B \cup C)$. $\square$

LEMMA 4.13. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A_i, B_i, C_i, \mathcal{S}_i)$ denote the i^{th} tuple of $\text{seq}_1(G, \delta)$. Then no vertex of $G - (A_i \cup B_i \cup C_i)$ belongs to over $\log(n)$ sets (counting multiplicity) of $N_G[\mathcal{S}_i] - (A_i \cup B_i \cup C_i)$.

Proof. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A_i, B_i, C_i, \mathcal{S}_i)$ denote the i^{th} tuple of $\text{seq}_1(G, \delta)$. We will prove by induction on i that no vertex of $G' = G - (A_i \cup B_i \cup C_i)$ belongs to over $\log(n)$ sets of $N_G[\mathcal{S}_i] - (A_i \cup B_i \cup C_i)$. If $i = 1$ then $(A_i, B_i, C_i, \mathcal{S}_i)$ is the first element of the sequence, and by definition $\mathcal{S}_i = \emptyset$, so the result holds for the base case. Assume the result holds for all i less than $j > 1$, we prove it holds for $i = j$.

Consider the $j-1^{\text{th}}$ and j^{th} elements of $\text{seq}_1(G, \delta)$, $(A_{j-1}, B_{j-1}, C_{j-1}, \mathcal{S}_{j-1})$ and $(A_j, B_j, C_j, \mathcal{S}_j)$. If $(A_j, B_j, C_j, \mathcal{S}_j)$ is a success or failure tuple then $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, $C_i \subseteq C_{i+1}$, and $\mathcal{S}_{j-1} = \mathcal{S}_j$. Then by the induction hypothesis no vertex of $G - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$ belongs to over $\log(n)$ vertices of $N_G[\mathcal{S}_{j-1}] - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$ which implies no vertex of $G - (A_j \cup B_j \cup C_j)$ belongs to over $\log(n)$ vertices of $N_G[\mathcal{S}_j] - (A_j \cup B_j \cup C_j)$. If $(A_j, B_j, C_j, \mathcal{S}_j)$ is a separator tuple then this implies that no vertex of $G - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$ belongs to $\log(n)$ or more set (counting multiplicity) of $N_G[\mathcal{S}_{j-1}] - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$ since such a vertex would good for refining. Hence, no vertex of $G - (A_j \cup B_j \cup C_j)$ belongs to over $\log(n)$ vertices of $N_G[\mathcal{S}_j] - (A_j \cup B_j \cup C_j)$. $\square$

Let d be a natural number and let G be a graph with n vertices. G is said to be d -degenerate if there is a bijective function $f : V(G) \rightarrow [n]$ where for each vertex $v \in G$, v has at most d neighbors $u \in G$ such that $f(u) < f(v)$. The function f is called a *degeneracy ordering* of G . We will need the following easy to prove lemma, which is folklore, and so we omit the proof.

LEMMA 4.14. (folklore) Let G be a d -degenerate graph with n vertices. Then G has an independent set of size $\lceil n/(d+1) \rceil$.

Let (A, B, C, S) be the last tuple of $\text{seq}_1(G, \delta)$, let $(A_{i-1}, B_{i-1}, C_{i-1}, S_{i-1})$ and (A_i, B_i, C_i, S_i) be the $(i-1)^{\text{th}}$ and i^{th} elements of $\text{seq}_1(G, \delta)$, and let $S \in \mathcal{S}$. We say S was added at step $i-1$ if $S_{i-1} \cup \{S\} = S_i$.

LEMMA 4.15. *Let G be a k -creature free graph (assume $k \geq 2$) with n vertices, let $\delta > 1$, and let (A, B, C, S) be the last tuple of $\text{seq}_1(G, \delta)$. There exists an anti-complete sub-list $\mathcal{S}' \subseteq \mathcal{S}$ of size at least $\log(n) + 1$.*

Proof. Let G be a k -creature free graph (assume $k \geq 2$) with n vertices, let $\delta > 1$, and let (A, B, C, S) be the last tuple of $\text{seq}_1(G, \delta)$. Let $G_{\mathcal{S}}$ be a graph with $|\mathcal{S}|$ vertices and let f be a bijective function $f : V(G_{\mathcal{S}}) \rightarrow [n]$. The vertex $v \in G_{\mathcal{S}}$ will correspond to the $f(v)^{\text{th}}$ element of $\mathcal{S}$. We now define the edges of $G_{\mathcal{S}}$. Let $u, v \in G_{\mathcal{S}}$, the edge uv is in $E(G_{\mathcal{S}})$ if and only if the $f(v)^{\text{th}}$ element and $f(u)^{\text{th}}$ element of $G_{\mathcal{S}}$ are not anti-complete. The statement of this lemma is then equivalent to $G_{\mathcal{S}}$ having an independent set of size at least $\log(n) + 1$. In order to establish this, we show $G_{\mathcal{S}}$ is $8k\delta \log(n)$ -degenerate with degeneracy ordering f and apply Lemma 4.14.

Let $v \in G_{\mathcal{S}}$ and assume that $S_{f(v)}$, the $f(v)^{\text{th}}$ element of $\mathcal{S}$, was added at the $(i-1)^{\text{th}}$ step of $\text{seq}_1(G, \delta)$. Let $(A_{i-1}, B_{i-1}, C_{i-1}, S_{i-1})$ and (A_i, B_i, C_i, S_i) be the $(i-1)^{\text{th}}$ and i^{th} elements of $\text{seq}_1(G, \delta)$. It follows that if $G' = G - (A_{i-1} \cup B_{i-1} \cup C_{i-1})$ then $S_{f(v)} \subseteq V(G')$ and for $u \in G_{\mathcal{S}}$ with $f(u) < f(v)$ the $f(u)^{\text{th}}$ element of $\mathcal{S}$ is also the $f(u)^{\text{th}}$ element of S_{i-1} . By Lemma 4.13 each vertex of $S_{f(v)}$ belongs to at most $\log(n)$ sets of $N_{G'}[S_{i-1}]$, and since there are at most $8k\delta$ vertices in $S_{f(v)}$ this implies v has at most $8k\delta \log(n)$ neighbors u where $f(u) < f(v)$, proving $G_{\mathcal{S}}$ is $8k\delta \log(n)$ -degenerate with degeneracy ordering f .

Now since $G_{\mathcal{S}}$ is $8k\delta \log(n)$ -degenerate, and $G_{\mathcal{S}}$ has $|\mathcal{S}| = 10k\delta \log(n)^2$ vertices (by the definition of $\text{seq}_1(G, \delta)$), it follows from Lemma 4.14 that $G_{\mathcal{S}}$ has an independent set of size at least $\log(n) + 1$. $\square$

We now wish to show that $\text{seq}_1(G, \delta)$ does not contain many success tuples. Toward this end we track the sizes of the level sets. Let i be a natural number, we will say that a vertex $v \in G$ is *added to level set $\mathcal{L}_i$ at step $j-1$* if for the j^{th} tuple, (A_j, B_j, C_j, S_j) , of $\text{seq}_1(G, \delta)$ v is in $\mathcal{L}_i(A_j, B_j, C_j, S_j)$, but v is not in $\mathcal{L}_i(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1})$. We say that $v \in G$ is added to level set $\mathcal{L}_i$ if it is added to level set $\mathcal{L}_i$ at step $j-1$ for some j .

Similarly, we will say that a vertex $v \in G$ is *removed from level set $\mathcal{L}_i$ at step j* if for the j^{th} tuple, (A_j, B_j, C_j, S_j) , of $\text{seq}_1(G, A, B, C)$, v is in $\mathcal{L}_i(A_j, B_j, C_j, S_j)$, but v is not in $\mathcal{L}_i(A_{j+1}, B_{j+1}, C_{j+1}, S_{j+1})$. We say that $v \in G$ is removed from level set $\mathcal{L}_i$ if it is removed from level set $\mathcal{L}_i$ at step j for some j .

LEMMA 4.16. *Let G be a k -creature free graph with n vertices and let $\delta > 1$. In the sequence $\text{seq}_1(G, \delta)$, for all natural numbers i , at most $160k^2\delta^2n \log^2(n)/2^i$ vertices are added to level set $\mathcal{L}_i$.*

Proof. Let G be a k -creature free graph with n vertices and let $\delta > 1$. Consider the $(j-1)^{\text{th}}$ and j^{th} element of $\text{seq}_1(G, \delta)$, $(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1})$ and (A_j, B_j, C_j, S_j) . First, assume that (A_j, B_j, C_j, S_j) is a separator tuple. Since (A_j, B_j, C_j, S_j) is a separator tuple, there was no vertex that was good for refining in step $j-1$, in particular this implies that no vertex of $G' = G - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$ has over $n/2^{i-1}$ neighbors in $\mathcal{L}_{i-1}(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1})$. If a vertex v is added to level set $\mathcal{L}_i$ at step $j-1$ then since S_j only has one additional set, call it S , that S_{j-1} does not have, it follows that $v \in \mathcal{L}_{i-1}(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1})$ and $v \in N_G[S] - (A_{j-1} \cup B_{j-1} \cup C_{j-1})$. Since $|S| \leq 8k\delta$ and no vertex of S has over $n/2^{i-1}$ neighbors in $\mathcal{L}_{i-1}(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1})$, it follows that $|\mathcal{L}_{i-1}(A_{j-1}, B_{j-1}, C_{j-1}, S_{j-1}) \cap (N_G[S] - (A_{j-1} \cup B_{j-1} \cup C_{j-1}))| \leq 8k\delta n/2^{i-1} = 16k\delta n/2^i$. Hence, at most $16k\delta n/2^i$ vertices are added to level set $\mathcal{L}_i$ at step j when (A_j, B_j, C_j, S_j) is a separator tuple.

Now, if (A_j, B_j, C_j, S_j) is a success or failure tuple, then $S_{j-1} = S_j$, so no vertices are added to level set $\mathcal{L}_i$ at step j . Since by how $\text{seq}_1(G, \delta)$ was defined, there are $10k\delta \log^2(n)$ separator tuples, therefore at most $160k\delta n \log^2(n)/2^i$ vertices are added to level set $\mathcal{L}_i$. $\square$

Since a vertex v must be added to level set $\mathcal{L}_i$ before it can be removed from level set $\mathcal{L}_i$, and by Lemma 4.16 at most $160k^2\delta^2n \log^2(n)/2^i$ vertices are added to level set $\mathcal{L}_i$, we get the following corollary, at most $160k^2\delta^2n \log^2(n)/2^i$ are removed from level set $\mathcal{L}_i$.

COROLLARY 4.1. *Let G be a k -creature free graph with n vertices and let $\delta > 1$. In the sequence $\text{seq}_1(G, \delta)$, for all natural numbers i , at most $160k^2\delta^2n \log^2(n)/2^i$ vertices are removed from level set $\mathcal{L}_i$.*

Let G be a graph with n vertices, let $\delta > 1$, and let (A_i, B_i, C_i, S_i) be the i^{th} tuple of $\text{seq}_1(G, \delta)$. If there are at least $n/2^c$ vertices that are removed from level set $\mathcal{L}_c$ at step $i-1$, then we say the tuple (A_i, B_i, C_i, S_i) *drains* level set $\mathcal{L}_c$.

LEMMA 4.17. Let G be a graph with n vertices and let $\delta > 1$. If $(A_i, B_i, C_i, \mathcal{S}_i)$ is a success tuple of $\text{seq}_1(G, \delta)$. Then $(A_i, B_i, C_i, \mathcal{S}_i)$ drains at least one level set $\mathcal{L}_j$.

Proof. Let G be a graph with n vertices, let $\delta > 1$, let $(A_{i-1}, B_{i-1}, C_{i-1}, \mathcal{S}_{i-1})$ and $(A_i, B_i, C_i, \mathcal{S}_i)$ be the $i-1^{\text{th}}$ and i^{th} tuples of $\text{seq}_1(G, \delta)$, and assume $(A_i, B_i, C_i, \mathcal{S}_i)$ is a success tuple. There is then some vertex v in $G' = G - (A_{i-1} \cup B_{i-1} \cup C_{i-1})$ that **Refine** was called on to get sets A_i, B_i , and C_i . By the definition of a vertex being good for refining there must be at least one level set, say $\mathcal{L}_j(A_{i-1}, B_{i-1}, C_{i-1}, \mathcal{S}_{i-1})$, such that $|\mathcal{L}_j(A_{i-1}, B_{i-1}, C_{i-1}, \mathcal{S}_{i-1}) \cap N_{G'}[v]| \geq n/2^j$. Hence, since $(A_i, B_i, C_i, \mathcal{S}_i)$ is a success tuple and therefore $N_{G'}[v] \subset A_i \cup B_i \cup C_i$, there are at least $n/2^j$ vertices from level set $\mathcal{L}_j$ that are removed at step $i-1$. Therefore $(A_i, B_i, C_i, \mathcal{S}_i)$ drains level set $\mathcal{L}_j$. $\square$

LEMMA 4.18. Let G be a k -creature free graph with n vertices and let $\delta > 1$. Then $\text{seq}_1(G, \delta)$ contains at most n failure tuples and at most $160k^2\delta^2 \log^3(n)$ success tuples.

Proof. Let G be a k -creature free graph with n vertices and let $\delta > 1$. By Corollary 4.1 at most $160k^2\delta^2 n \log^2(n)/2^j$ vertices are removed from level set $\mathcal{L}_j$, hence at most $160k^2\delta^2 \log^2(n)$ tuples of $\text{seq}_1(G, \delta)$ drain level set $\mathcal{L}_j$. By Lemma 4.13, for any tuple $(A_i, B_i, C_i, \mathcal{S}_i)$ of $\text{seq}_1(G, \delta)$, no vertex of $G - (A_i \cup B_i \cup C_i)$ belongs to over $\log(n)$ sets of $N_G[\mathcal{S}_i] - (A_i \cup B_i \cup C_i)$, hence no vertices are ever added to level set $\mathcal{L}_j$ for $j > \log(n)$, and therefore no tuple of $\text{seq}_1(G, \delta)$ will ever drain a level set $\mathcal{L}_j$ for $j > \log(n)$. Furthermore, By Lemma 4.17 every success tuple of $\text{seq}_1(G, \delta)$ drains at least one level set $\mathcal{L}_i$. It follows that there are at most $160k^2\delta^2 n \log^3(n)$ success tuples of $\text{seq}_1(G, \delta)$.

Now, consider the i^{th} and $i+1^{\text{th}}$ elements of $\text{seq}_1(G, \delta)$, $(A_i, B_i, C_i, \mathcal{S}_i)$ and $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ and assume that $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a failure tuple. Let v be the vertex of $G' = G - (A_i \cup B_i \cup C_i)$ that **Refine** was called on to get the tuple $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$. It follows that $v \notin (A_i \cup B_i \cup C_i)$, but $v \in (A_{i+1} \cup B_{i+1} \cup C_{i+1})$. Additionally, if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is the n^{th} failure tuple, then we must have that $(A_{i+1} \cup B_{i+1} \cup C_{i+1}) = V(G)$. In this case, this forces any tuples after $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ in $\text{seq}_1(G, \delta)$ to be separator tuples. $\square$

COROLLARY 4.2. Let G be a k -creature free graph with $n \geq 2$ vertices and μ minimal separators, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ be the last tuple of $\text{seq}_1(G, \delta)$. Then G has at least $\mu/(4(3n^{(k+1)})^{160k^2\delta^2 \log^3(n)+2})$ minimal separators that are consistent with A, B , and C .

Proof. Let G be a k -creature free graph with $n \geq 2$ vertices and μ minimal separators, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ be the last tuple of $\text{seq}_1(G, \delta)$. Consider the i^{th} and $i+1^{\text{th}}$ elements of $\text{seq}_1(G, \delta)$, $(A_i, B_i, C_i, \mathcal{S}_i)$ and $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$. Assume that there are μ' minimal separators of G that are consistent with A_i, B_i , and C_i . By Lemma 4.9 it follows that if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a failure tuple, then there are at least $\mu'(1-1/n)$ minimal separators of G that are consistent with A_{i+1}, B_{i+1} , and C_{i+1} and if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a success tuple, then there are at least $\mu'(1/3n^{k+1})$ minimal separators of G that are consistent with A_{i+1}, B_{i+1} , and C_{i+1} . Furthermore, if $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a separator tuple, then $A_i = A_{i+1}$, $B_i = B_{i+1}$ and $C_i = C_{i+1}$, and so there are μ' minimal separators of G that are consistent with A_{i+1}, B_{i+1} and C_{i+1} .

Now, the first tuple, $(A_1, B_1, C_1, \mathcal{S}_1)$, of $\text{seq}_1(G, \delta)$ was chosen so that at least μ/n^2 minimal separators of G agree with A_1, B_1 , and C_1 . Furthermore, by Lemma 4.18 there are at most n failure tuples and at most $160k^2\delta^2 \log^3(n)$ success tuples in $\text{seq}_1(G, \delta)$. It then follows that there are at least $\frac{\mu(1-1/n)^n}{n^2(3n^{(k+1)})^{(160k^2\delta^2 \log^3(n)+2)}} \geq \frac{\mu}{4(3n^{(k+1)})^{160k^2\delta^2 \log^3(n)+2}}$ (using the fact that $(1-1/n)^n > 1/4$ when $n \geq 2$) minimal separators of G that are consistent with A, B , and C . $\square$

COROLLARY 4.3. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ be the last tuple of $\text{seq}_1(G, \delta)$. Then $G[A \cup B]$ has at most $160k^2\delta^2 \log^3(n) + 2$ components.

Proof. Let G be a k -creature free graph with n vertices, let $\delta > 1$, and let $(A_1, B_1, C_1, \mathcal{S}_1)$, $(A_i, B_i, C_i, \mathcal{S}_i)$, $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$, and $(A, B, C, \mathcal{S})$ be the 1^{st} , i^{th} , $i+1^{\text{th}}$, and last elements of $\text{seq}_1(G, \delta)$ respectively. If $(A_{i+1}, B_{i+1}, C_{i+1}, \mathcal{S}_{i+1})$ is a separator tuple then $A_{i+1} = A_i$ and $B_{i+1} = B_i$, if it is a failure tuple then $A_{i+1} = A_i$ and $B_{i+1} = B_i$ by Lemma 4.9, and if it is a success tuple then $G[A_{i+1} \cup B_{i+1}]$ has at most one more component than $G[A_i \cup B_i]$ by Lemma 4.9. Since, by Lemma 4.18 there are at most $160k^2\delta^2 \log^3(n)$ success tuples in $\text{seq}_1(G, \delta)$ and since A_1 and B_1 both have one component, we conclude that $G[A \cup B]$ has at most $160k^2\delta^2 \log^3(n) + 2$ components. $\square$

We are now ready to prove Lemma 4.10.

Proof. [Proof of Lemma 4.10] Let G be a k -creature free graph (assume $k \geq 2$) with $n \geq 2$ vertices and μ minimal separators, let $\delta > 1$, and let $(A, B, C, \mathcal{S})$ be the last tuple of $\text{seq}_1(G, \delta)$. Let the sub-list $\mathcal{S}' \subset \mathcal{S}$ be the anti-complete sub-list of size $\log(n) + 1$ promised by Lemma 4.15, we will show that A, B, C , and $\mathcal{S}'$ satisfy the conclusions of Lemma 4.10. Property (i) is established by Lemma 4.12, property (ii) is established by Corollary 4.2, property (iii) is established by Corollary 4.3, and property (iv) is established by Lemma 4.13 combined with the fact that $|\mathcal{S}'| = \log(n) + 1$. $\square$

4.1.2 Separating Balanced Separators Recall that our final goal of this section is to prove Lemma 4.3, that is, to take a k -creature free graph G with n vertices and to find vertex sets A, B , and C such that no component of $G - (A \cup B \cup C)$ contains over n/δ vertices and a large fraction of minimal separators of G are consistent with A, B , and C . We achieve this in this subsection by taking the output from Lemma 4.10, $\mathcal{S}, A, B$, and C , and enriching A, B , and C to get A', B' , and C' so that for each $S_i, S_j \in \mathcal{S}$ and for each vertex $v_i \in S_i$ and $v_j \in S_j$, no component of $G' = G - (A' \cup B' \cup C' \cup V)$ contains both a vertex from $N_{G'}[v_i]$ and $N_{G'}[v_j]$ where $V = N_G[v_i] \cap N_G[v_j]$. We will show later on using Lemma 4.24 that this property is enough to ensure that no component of $G - (A' \cup B' \cup C')$ has over n/δ vertices and prove Lemma 4.3.

We begin by showing that sets A, B , and C can be enriched so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are anti-complete (where $V = N_G[v_i] \cap N_G[v_j]$), then they can be further enriched so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are far apart in $G - (A \cup B \cup C \cup V)$, then finally they can be enriched so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are in different components of $G - (A \cup B \cup C \cup V)$, which finally gives us the sets A', B' , and C' from the previous paragraph.

Since v_i and v_j are anti-complete, if we set $V = N_G[v_i] \cap N_G[v_j]$, then notice that $(v_i, N_G(v_i) - V, N_G(v_j) - V, v_j)$ is nearly a k -creature, it only lacks the requirement of a semi-induced matching of size k between $N_G(v_i) - V, N_G(v_j) - V$. This is an important observation which we use in our “enrichment” process of A, B , and C and motivates the following definition of a *pre-creature*, (Y_1, X_1, X_2, Y_2) , which is just a k -creature but without the requirement of a semi-induced matching of size k between X_1 and X_2 .

DEFINITION 4.5. Let G be a graph. A four-tuple (Y_1, X_1, X_2, Y_2) of vertex sets in G is called a *pre-creature* if the following conditions are satisfied:

- (i) $G[Y_1]$ and $G[Y_2]$ are connected.
- (ii) Y_1 is anti-complete with $X_2 \cup Y_2$ and Y_2 is anti-complete with $X_1 \cup Y_1$.
- (iii) Y_1 dominates and is disjoint from X_1 and Y_2 dominates and is disjoint from X_2 .

Similar to the methods of subsection 4.1.1, the “enrichment” process of A, B , and C used to get the sets A', B' , and C' which satisfy Lemma 4.3 works by taking a carefully chosen vertex and using the **Refine** function on it. We will use the next lemma to guide us on picking a vertex that will be a good choice to use the **Refine** function on.

LEMMA 4.19. Let G be a k -creature free graph and let (Y_1, X_1, X_2, Y_2) be a pre-creature of G where $X_1 \neq \emptyset$. Then there exists a vertex $v \in X_1$ such that v dominates at least $(1/k)|N_G(X_1) \cap X_2|$ vertices of $N_G(X_1) \cap X_2$.

Proof. Let G be a k -creature free graph and let (Y_1, X_1, X_2, Y_2) be a pre-creature of G where $X_1 \neq \emptyset$. Let $D_1 \subset X_1$ be a minimal subset of X_1 that dominates $N_G(X_1) \cap X_2$. Assume for a contradiction that $|D_1| \geq k$. Since D_1 is minimal, it follows that for each $d_i^1 \in D_1$ there is a $d_i^2 \in N_G(X_1) \cap X_2$ such that d_i^1 is the only vertex of D_1 that is neighbors with d_i^2 . Let D_2 denote this set of d_i^2 's. Then since (Y_1, X_1, X_2, Y_2) is a pre-creature, (Y_1, D_1, D_2, Y_2) is a k' -creature for some $k' \geq k$, a contradiction.

Thus, we may assume that $|D_1| < k$. Hence, at least one vertex $v \in D_1$ must have the property that v dominates at least $(1/k)|N_G(X_1) \cap X_2|$ vertices of $N_G(X_1) \cap X_2$. $\square$

Let v_i and v_j be anti-complete vertices. We now prove a lemma that shows we can enrich A , B , and C so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are anti-complete (where $V = N_G[v_i] \cap N_G[v_j]$). We will actually need to prove a slightly more general lemma that is stated in terms of pre-creatures so that it can be used for induction in a later lemma.

LEMMA 4.20. *Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let (Y_1, X_1, X_2, Y_2) be a pre-creature of G . Then there exists sets $A', B', C' \subseteq V(G)$ with the following properties:*

- (i) G has at least $\mu/(4(3n^{(k+1)})^{k \log(n)})$ minimal separators that are consistent with A', B' , and C' .
- (ii) The number of components of $G[A' \cup B']$ is at most the number of components of $G[A \cup B]$ plus $k \log(n)$.
- (iii) $X_1 - (A' \cup B' \cup C')$ is anti-complete with $X_2 - (A' \cup B' \cup C')$ in G .
- (iv) $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$.

In order to prove Lemma 4.20, we will study the following sequence produced from a k -creature free graph G with n vertices along with sets $A, B, C \subseteq V(G)$ and pre-creature $P = (Y_1, X_1, X_2, Y_2)$, which we denote by $\text{seq}_2(G, A, B, C, P)$. $\text{seq}_2(G, A, B, C, P)$ is a sequence of tuples (A_i, B_i, C_i) with $A_i, B_i, C_i \subseteq V(G)$.

We now define the sequence $\text{seq}_2(G, A, B, C, P)$. For the base case of this sequence we set $A_1 = A$, $B_1 = B$, and $C_1 = C$. We recursively define this sequence as follows. Given (A_i, B_i, C_i) let $X'_1 = X_1 - (A_i \cup B_i \cup C_i)$, $X'_2 = X_2 - (A_i \cup B_i \cup C_i)$, and $G' = G - (A_i \cup B_i \cup C_i)$. If $N_{G'}(X'_1) \cap X'_2 = \emptyset$ then we terminate the sequence (so (A_i, B_i, C_i) is the last tuple of the sequence), else since (Y_1, X_1, X_2, Y_2) is a pre-creature of G , (Y_1, X'_1, X'_2, Y_2) is a pre-creature of G (and $X'_1 \neq \emptyset$ since $N_{G'}(X'_1) \cap X'_2 \neq \emptyset$) and so Lemma 4.19 guarantees the existence of $v \in X'_1$ such that v dominates at least $(1/k)|N_{G'}(X'_1) \cap X'_2|$ vertices of $N_{G'}(X'_1) \cap X'_2$ (Note that $N_{G'}(X'_1) \cap X'_2 = N_G(X'_1) \cap X'_2$). We then set $A_{i+1}, B_{i+1}, C_{i+1} = \text{Refine}(G, v, A_i, B_i, C_i)$. If $\text{Refine}(G, v, A_i, B_i, C_i)$ is a failure refinement then we call $(A_{i+1}, B_{i+1}, C_{i+1})$ a *failure tuple*, else $\text{Refine}(G, v, A_i, B_i, C_i)$ is a success refinement and we call $(A_{i+1}, B_{i+1}, C_{i+1})$ a *success tuple*.

LEMMA 4.21. *Let G be a k -creature free graph with n vertices, let $A, B, C \subset V(G)$, and let P be a pre-creature of G . $\text{seq}_2(G, A, B, C, P)$ has at most n failure tuples and at most $k \log(n)$ success tuples.*

Proof. Let G be a k -creature free graph with n vertices, let $A, B, C \subset V(G)$, and let $P = (Y_1, X_1, X_2, Y_2)$ be a pre-creature of G . Let (A_i, B_i, C_i) and $(A_{i+1}, B_{i+1}, C_{i+1})$ be the i^{th} and $i+1^{\text{th}}$ tuples of $\text{seq}_2(G, A, B, C, P)$. We can see from the definition of **Refine** and seq_2 that $(A_i \cup B_i \cup C_i) \subset (A_{i+1} \cup B_{i+1} \cup C_{i+1})$ where the containment is strict, hence if $i = n$ then $(A_{i+1} \cup B_{i+1} \cup C_{i+1}) = V(G)$. Hence, by the definition of seq_2 $(A_{i+1} \cup B_{i+1} \cup C_{i+1})$ would be the final tuple of $\text{seq}_2(G, A, B, C, P)$. Therefore, there are at most $n+1$ tuples of $\text{seq}_2(G, A, B, C, P)$, hence there are at most n failure tuples in $\text{seq}_2(G, A, B, C, P)$ (recall the first tuple is neither a failure nor success tuple).

Now, assume $(A_{i+1}, B_{i+1}, C_{i+1})$ is a success tuple, let $X_1^i = X_1 - (A_i \cup B_i \cup C_i)$, $X_2^i = X_2 - (A_i \cup B_i \cup C_i)$, $G^i = G - (A_i \cup B_i \cup C_i)$, $X_1^{i+1} = X_1 - (A_{i+1} \cup B_{i+1} \cup C_{i+1})$, $X_2^{i+1} = X_2 - (A_{i+1} \cup B_{i+1} \cup C_{i+1})$, and $G^{i+1} = G - (A_{i+1} \cup B_{i+1} \cup C_{i+1})$. If v is the vertex of X_1^i that refine is called on to get the sets A_{i+1}, B_{i+1} , and C_{i+1} , then by how v was selected, v dominates at least $(1-1/k)|N_{G^i}(X_1^i) \cap X_2^i|$ vertices of $|N_{G^i}(X_1^i) \cap X_2^i|$. Then by Lemma 4.9 $N_G[v] \subseteq A_{i+1} \cup B_{i+1} \cup C_{i+1}$ and it follows that $(1-1/k)|N_{G^i}(X_1^i) \cap X_2^i| \geq |N_{G^{i+1}}(X_1^{i+1}) \cap X_2^{i+1}|$. Hence if $(A_{i+1}, B_{i+1}, C_{i+1})$ is the $(k \log(n))^{\text{th}}$ success tuple, then $|N_{G^{i+1}}(X_1^{i+1}) \cap X_2^{i+1}|$ is at most $n(1-1/k)^{k \log(n)} \leq n/e^{\log(n)} < 1$ (recall for our definition of $\log(n)$ that $\log(n) \geq \ln(n)$), which implies $|N_{G^{i+1}}(X_1^{i+1}) \cap X_2^{i+1}| = 0$. $\square$

COROLLARY 4.4. *Let G be a k -creature free graph with n vertices, let $A, B, C \subset V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let P be a pre-creature of G . Let (A', B', C') be the final tuple of $\text{seq}_2(G, A, B, C, P)$. Then G has at least $\mu/(4(3n^{(k+1)})^{k \log(n)})$ minimal separators that are consistent with A', B' , and C' .*

Proof. Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let P be a pre-creature of G . Let (A', B', C') be the final tuple

of $\text{seq}_2(G, A, B, C, P)$. Consider the i^{th} and $i + 1^{\text{th}}$ tuples of $\text{seq}_2(G, A, B, C, P)$, (A_i, B_i, C_i) and $(A_{i+1}, B_{i+1}, C_{i+1})$, and assume there are μ' minimal separators of G that are consistent with A_i, B_i , and C_i . If $(A_{i+1}, B_{i+1}, C_{i+1})$ is a failure tuple, then by Lemma 4.9 there are at least $(1 - 1/n)\mu'$ minimal separators that are consistent with A_{i+1}, B_{i+1} , and C_{i+1} , and if $(A_{i+1}, B_{i+1}, C_{i+1})$ is a success tuple, then by Lemma 4.9 there are at least $(1/3n^{k+1})\mu'$ minimal separators that are consistent with A_{i+1}, B_{i+1} , and C_{i+1} . Since by Lemma 4.21 there are at most n failure tuples and $k \log(n)$ success tuples, it follows that there are at least $\frac{\mu(1-1/n)^n}{(3n^{k+1})^{k \log(n)}} \leq \frac{\mu}{4(3n^{k+1})^{k \log(n)}}$ minimal separators of G that are consistent with A', B' , and C' (using the fact that $(1 - 1/n)^n \geq 1/4$ for $n \geq 2$). $\square$

COROLLARY 4.5. *Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let P be a pre-creature of G , and let (A', B', C') be the final tuple of $\text{seq}_2(G, A, B, C, P)$. Then $G[A' \cup B']$ has at most $k \log(n)$ more components than $G[A \cup B]$.*

Proof. Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let P be a pre-creature of G , and let (A', B', C') be the final tuple of $\text{seq}_2(G, A, B, C, P)$. Consider the i^{th} and $i + 1^{\text{th}}$ tuples of $\text{seq}_2(G, A, B, C, P)$, (A_i, B_i, C_i) and $(A_{i+1}, B_{i+1}, C_{i+1})$. If $(A_{i+1}, B_{i+1}, C_{i+1})$ is a failure tuple, then by Lemma 4.9 $A_i = A_{i+1}$ and $B_i = B_{i+1}$. If $(A_{i+1}, B_{i+1}, C_{i+1})$ is a success sequence, then by Lemma 4.9 $G[A_{i+1} \cup B_{i+1}]$ has at most one more component than $G[A_i \cup B_i]$. Since by Lemma 4.21 there are at most $k \log(n)$ success tuples, it follows that $G[A' \cup B']$ has at most $k \log(n)$ more components than $G[A \cup B]$. $\square$

We are now ready to prove Lemma 4.20.

Proof. [Proof of Lemma 4.20] Let G be a k -creature free graph with n vertices, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , and let $P = (Y_1, X_1, X_2, Y_2)$ be a pre-creature of G . Let (A', B', C') be the final tuple of $\text{seq}_2(G, A, B, C, P)$, we will show the set A', B' , and C' satisfy the conclusions of Lemma 4.20. Property (i) is established by Corollary 4.4, property (ii) is established by Corollary 4.5, property (iii) is established by the termination condition of $\text{seq}_2(G, A, B, C, P)$, and property (iv) is established by the facts in the first tuple of $\text{seq}_2(G, A, B, C, P)$, (A_1, B_1, C_1) , that $A_1 = A$, $B_1 = B$, and $C_1 = C$ and that for the i^{th} and $i + 1^{\text{th}}$ tuples of $\text{seq}_2(G, A, B, C, P)$, (A_i, B_i, C_i) and $(A_{i+1}, B_{i+1}, C_{i+1})$, that $A_i \subseteq A_{i+1}$, $B_i \subseteq B_{i+1}$, and $C_i \subseteq C_{i+1}$. $\square$

Let v_i and v_j be anti-complete vertices. Lemma 4.20 showed how we can enrich A, B , and C so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are anti-complete (where $V = N_G[v_i] \cap N_G[v_j]$). In the next lemma we show how to further enrich A, B , and C so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are far apart in $G - (A \cup B \cup C \cup V)$. Because we will need to apply this lemma in an inductive proof later on, we will need to prove something slightly more general than what was just stated.

LEMMA 4.22. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , let i be a natural number, let Y_1 and Y_2 be anti-complete connected subsets of $V(G)$, and let $Z \subseteq V(G)$ be disjoint from $N(Y_1) \cap N(Y_2)$. Then there exists sets $A', B', C' \subseteq V(G)$ with the following properties:*

- (i) G has at least $\mu/(12n^{(k+1)})^{ik \log(n)}$ minimal separators that are consistent with A', B' , and C' .
- (ii) The number of components of $G[A' \cup B']$ is at most $ik \log(n)$ more than the number of components of $G[A \cup B]$.
- (iii) Every path from Y_1 to Y_2 in $G[Z \cup Y_1 \cup Y_2]$ of length less than $i + 4$ contains an internal vertex from $(A' \cup B' \cup C')$.
- (iv) $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$.

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , let i be a natural number, let Y_1 and Y_2 be anti-complete connected subsets of $V(G)$, and let Z be a subset of $V(G)$ disjoint from $V = N_G(Y_1) \cap N_G(Y_2)$. We now prove by induction on i that there exists sets A', B', C' that satisfy properties (i)-(iv) of this lemma. For the base case, when $i = 0$, since Y_1 and Y_2 are

anti-complete and Z is disjoint from $N_G(Y_1) \cap N_G(Y_2)$, every path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ must be of length at least 4, so $A' = A$, $B' = B$ and $C' = C$ satisfy the conclusion of the lemma. Assume now that $0 < i$ and that the conclusion of the lemma holds for all $i' < i$. We prove that it also holds for i .

We apply the inductive hypothesis to G , A , B , C , Y_1 , Y_2 , Z , and $i - 1$ to get sets A' , B' , and C' that satisfy properties (i)-(iv) of this lemma (for the natural number $i - 1$). In particular, by property (iii) every path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ of length less than $(i - 1) + 4$ contains an internal vertex from $(A' \cup B' \cup C')$. Now, let ℓ be the length of a shortest path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ that does not contain an internal vertex from $(A' \cup B' \cup C')$ and let $\mathcal{P}$ denote the set of all such shortest paths. If $\ell \geq i + 4$ then the sets A' , B' , and C' satisfy properties (i)-(iv) of the lemma (now for the natural number i) and we are done. So we may assume $\ell = (i - 1) + 4$. Since $i \geq 1$, each path in $\mathcal{P}$ contains at least two internal vertices. Let X_1 denote the set of vertices that occur as the first internal vertex (the vertex closest to Y_1) in a path of $\mathcal{P}$, let X_2 denote the set of vertices that occur as the second internal vertex in a path of $\mathcal{P}$, and let $X_{\geq 3}$ denote the set of vertices that occur as the third or later internal vertex of some path in $\mathcal{P}$. It is straightforward to verify that since $\mathcal{P}$ is a set of shortest paths it holds that X_1 , X_2 , and $X_{\geq 3}$ are disjoint and furthermore $P = (Y_1, X_1, X_2, Y_2 \cup X_{\geq 3})$ is a pre-creature of G . In particular one can verify that $G[Y_1]$ and $G[Y_2 \cup X_{\geq 3}]$ are connected, Y_1 is anti-complete with $X_2 \cup (Y_2 \cup X_{\geq 3})$ and $(Y_2 \cup X_{\geq 3})$ is anti-complete with Y_1 , and Y_1 dominates X_1 and $(Y_2 \cup X_{\geq 3})$ dominates X_2 , which establishes that P is a pre-creature.

Since P is a pre-creature of G we may apply Lemma 4.20 to G , A' , B' , C' , and P to get sets A'' , B'' , and C'' that satisfy properties (i)-(iv) of Lemma 4.20. We now verify that the sets A'' , B'' , and C'' satisfy conditions (i)-(iv) of this lemma.

We first verify property (iii) holds. Let Q be a shortest path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ that has no internal vertex contained in $(A'' \cup B'' \cup C'')$, which implies it has no internal vertex contained in $(A' \cup B' \cup C')$. If this path is of length $(i - 1) + 4$ then Q is a path of $\mathcal{P}$, but since $X_1 - (A' \cup B' \cup C')$ is anti-complete with $X_2 - (A' \cup B' \cup C')$ this is impossible. So it must be that Q has length at least $i + 4$ which establishes property (iii).

To verify property (i), note that by the inductive hypothesis G has at least $\mu' = \mu / (12n^{(k+1)})^{(i-1)k \log(n)}$ minimal separators that are consistent with A' , B' , and C' , and by property (i) of Lemma 4.20 G has at least $\mu' / 4(3n^{(k+1)})^{k \log(n)} > \mu' / (12n^{(k+1)})^{k \log(n)}$ minimal separators that are consistent with A'' , B'' and C'' . It follows that G has at least $\mu / (12n^{(k+1)})^{ik \log(n)}$ minimal separators that are consistent with A'' , B'' , and C'' which established property (i).

To verify property (ii) note that by the inductive hypothesis $G[A' \cup B']$ has at most $(i - 1)k \log(n)$ more components than $G[A \cup B]$ and by property (ii) of Lemma 4.20 $G[A'' \cup B'']$ has at most $k \log(n)$ more components than $G[A' \cup B']$. It follows that $G[A'' \cup B'']$ has at most $ik \log(n)$ more components than $G[A \cup B]$ which established property (ii).

Lastly, to verify property (iv) note that by the inductive hypothesis $A \subseteq A'$, $B \subseteq B'$ and $C \subseteq C'$ and by Lemma 4.20 $A' \subseteq A''$, $B' \subseteq B''$ and $C' \subseteq C''$. It then follows that $A \subseteq A''$, $B \subseteq B''$ and $C \subseteq C''$ which established property (iv). $\square$

Let v_i and v_j be anti-complete vertices. Lemma 4.22 showed how we can enrich A , B , and C so that $N_G[v_i] - (A \cup B \cup C \cup V)$ and $N_G[v_j] - (A \cup B \cup C \cup V)$ are far apart in $G - (A \cup B \cup C \cup V)$. This next lemma shows how we can further enrich A , B , and C so that no component of $G - (A \cup B \cup C \cup V)$ contains both a vertex from $N_G[v_i] - (A \cup B \cup C \cup V)$ and from $N_G[v_j] - (A \cup B \cup C \cup V)$. To make the proof easier, we will prove something slightly more general than what was just stated.

LEMMA 4.23. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , let Y_1 and Y_2 be anti-complete connected subsets of $V(G)$, and let $Z \subseteq V(G)$ be disjoint from $N_G(Y_1) \cap N_G(Y_2)$. Then there exists sets $A', B', C' \subseteq V(G)$ with the following properties:*

- (i) G has at least $\mu / (12n^{(k+1)})^{2k \log(n) \log(|Z|)}$ minimal separators that are consistent with A', B' , and C' .
- (ii) The number of components of $G[A' \cup B']$ is at most $2k \log(n) \log(|Z|)$ more than the number of components of $G[A \cup B]$.
- (iii) Every path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ has an internal vertex that belongs to $(A' \cup B' \cup C')$.

(iv) $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$.

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, let G have μ minimal separators that are consistent with A, B , and C , let Y_1 and Y_2 be anti-complete connected subsets of $V(G)$, and let $Z \subseteq V(G)$ be disjoint from $N_G(Y_1) \cap N_G(Y_2)$.

We prove the statement of the lemma by induction on $|Z|$. If $Z = \emptyset$ then A, B and C already satisfy the conclusion of the lemma. Therefore, assume that $Z \neq \emptyset$ and that the statement of the lemma holds for sets Z' such that $|Z'| < |Z|$. We apply Lemma 4.22 to $G, A, B, C, i = 2, Y_1, Y_2$ and Z to get sets A', B' and C' that satisfy properties (i)-(iv) of Lemma 4.22. We set $Z' = Z - (A' \cup B' \cup C')$ and $G' = G[Y_1 \cup Y_2 \cup Z']$. From property (iii) of Lemma 4.22 it follows that every path from Y_1 to Y_2 in G' has length at least six. If there does not exist a path from Y_1 to Y_2 in G' then using properties (i), (ii) and (iv) of Lemma 4.22 it can be verified that sets A', B' , and C' satisfy the conclusions of this lemma. Therefore we may assume that Y_1 and Y_2 are in the same component of G' .

We define $X_1 = N_{G'}(Y_1)$, $X_2 = N_{G'}(Y_1 \cup X_1)$, $X'_1 = N_{G'}(Y_2)$, and $X'_2 = N_{G'}(Y_2 \cup X'_1)$, since every path from Y_1 to Y_2 in G' has length at least six, X_1, X_2, X'_1 , and X'_2 are mutually disjoint. Furthermore, observe that $X_2 = N_{G'}(X_1) - Y_1$ and similarly that $X'_2 = N_{G'}(X'_1) - Y_2$. By definition of X_1, X_2, X'_1 , and X'_2 it holds that every connected component C of $G' - (Y_1 \cup X_1 \cup X_2 \cup Y_2 \cup X'_1 \cup X'_2)$ satisfies $N_{G'}(C) \subseteq X_2 \cup X'_2$ (any vertex outside of Y_1 that has a neighbor in Y_1 would belong to X_1 and any vertex outside of $Y_1 \cup X_1$ that has a neighbor in X_1 would belong to X_2 , similar statements hold for X'_1 and X'_2). Let Z_1 contain $X_1 \cup X_2$ as well as the union of the vertex sets of all connected components, C , of $G' - (Y_1 \cup X_1 \cup X_2 \cup Y_2 \cup X'_1 \cup X'_2)$ that satisfy $\emptyset \subset N_{G'}(C) \subseteq X_2$. Similarly, let Z_2 contain $X'_1 \cup X'_2$ as well as the union of the vertex sets of all connected components, C , of $G' - (Y_1 \cup X_1 \cup X_2 \cup Y_2 \cup X'_1 \cup X'_2)$ that satisfy $\emptyset \subset N_{G'}(C) \subseteq X'_2$. Observe that no component of $G' - (Y_1 \cup X_1 \cup X_2 \cup Y_2 \cup X'_1 \cup X'_2)$ is added to both Z_1 and to Z_2 . Since every path from Y_1 to Y_2 in G' has length at least six, it follows that $X_1 \cup X_2$ and $X'_1 \cup X'_2$ are disjoint. Therefore $Z_1 \cap Z_2 = \emptyset$. Without loss of generality $|Z_1| \leq |Z_2|$. Hence, since $Z_1 \cup Z_2 \subseteq Z' \subseteq Z$ and it follows that $|Z_1| \leq |Z|/2 < |Z|$.

We define Y'_2 to be the connected component of $G' - (Y_1 \cup Z_1)$ that contains Y_2 . We claim that $Q = Y_1 \cup Z_1 \cup Y'_2$ is equal to the component, T , of G' that contains $Y_1 \cup Y_2$ (recall that by the discussion of the second paragraph, we can assume Y_1 and Y_2 belong to the same component of G'). First, to see that every vertex of Q belongs to T note that Y'_2 to be the connected component of $G' - (Y_1 \cup Z_1)$ that contains Y_2 , hence $Y'_2 \subseteq C$, Z_1 is a connected set and since $X_1 \subseteq Z_1$, Z_1 has neighbors in Y_1 , hence $Z_1 \subseteq Y_1$, and clearly $Y_1 \subseteq T$. Next, we verify that no vertex of G' outside of Q belongs to T . Let v be a vertex of G' outside of Q had a neighbor in T . As noted before, Y'_2 contains X'_1 and X'_2 and by definition Z_1 contains X_1 and X_2 , so we may assume that v belongs to a component of $G' - (Y_1 \cup X_1 \cup X_2 \cup Y_2 \cup X'_1 \cup X'_2)$, hence, by the definition of Z_1 , we can see that this implies that if v has a neighbor in Z_1 then v must belong to Z_1 . Next, if v has a neighbor in Y'_2 , then, by definition of Y'_2 , v would be apart of Y'_2 , so this is impossible. Lastly, if v had a neighbor with Y_1 then v would be in $X_1 \subseteq Z_1$ and therefore in T , so this is impossible. It now follows that $Q = Y_1 \cup Z_1 \cup Y'_2$ is equal to the component, T , of G' that contains $Y_1 \cup Y_2$.

Since Y_1, Z_1 , and Y'_2 are disjoint, it follows that these sets partition the component that $Y_1 \cup Y_2$ belong to in G' . We have that Y'_2 is connected, that Y_1 and Y'_2 are anti-complete (because $X_1 \subseteq Z_1$), and that Y_1 and Y'_2 do not have common neighbors in Z_1 (because $X_1 \subseteq Z_1$ and $X_2 \subseteq Z_2$). We may therefore apply the induction hypothesis to G, A', B', C', Y_1, Y'_2 and Z_1 . Let A'', B'' and C'' be the sets that satisfy properties (i)-(iv) of this lemma when applied to G, A', B', C', Y_1, Y'_2 and Z_1 . We prove that A'', B'' and C'' satisfy the conclusion of the lemma (when applied to G, A, B, C, Y_1, Y_2 and Z).

We first prove property (iii), that every path P from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ contains an internal vertex in $(A'' \cup B'' \cup C'')$. Suppose for contradiction that this is not the case, and let P be a path from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ with internal vertices disjoint from $(A'' \cup B'' \cup C'')$. Without loss of generality we assume the first vertex of P is in Y_1 , the last is in Y_2 , and all internal vertices of P are in $Z - (Y_1 \cup Y_2)$. Since P is internally vertex disjoint from $(A'' \cup B'' \cup C'')$ and by the induction hypothesis $A' \subseteq A'', B' \subseteq B'',$ and $C' \subseteq C''$ we have that P is internally vertex disjoint from $(A' \cup B' \cup C')$, and that therefore P is a path from Y_1 to Y_2 in G' .

The first vertex of P lies in Y_1 , while the last vertex of P is in Y_2 , which is in Y'_2 . Define v to be the first vertex on P in Y'_2 and let P' be the sub-path of P that starts in Y_1 and ends in v . We have that P' lies in the component of G' that contains Y_1 and Y_2 . We argue that all internal vertices of P' lie in Z_1 . Indeed, none of the internal vertices of P' lie in Y_1 (since only the first vertex of P is in Y_1), and none of the internal vertices of P' lie in Y'_2 by the choice of v . But, as we claimed just after the definition of Y'_2 , we have that Y_1, Y'_2 and Z_1

partition the component of G' that contain $Y_1 \cup Y_2$ (and therefore P') and hence all internal vertices of P' are in Z_1 . But then P' contradicts property (iii) of the inductive hypothesis, namely that every path from Y_1 to Y'_2 in $G[Y_1 \cup Y'_2 \cup Z_1]$ has an internal vertex that belongs to $(A'' \cup B'' \cup C'')$. We conclude that every path P from Y_1 to Y_2 in $G[Y_1 \cup Y_2 \cup Z]$ contains an internal vertex in $(A'' \cup B'' \cup C'')$, which proves property (iii) in the statement of the lemma.

We check the remaining properties of the statement of the Lemma. Property (iv) follows from the fact that $A \subseteq A'$, $B \subseteq B'$, $C \subseteq C'$ (by Lemma 4.22) and that $A' \subseteq A''$, $B' \subseteq B''$, $C' \subseteq C''$ by the inductive hypothesis.

For property (i), Lemma 4.22 yields that the number of separators in G consistent with A' , B' and C' in G is at least $\mu/(12n^{(k+1)})^{2k \log(n)}$. The induction hypothesis now yields that there are at least

$$\frac{\mu/(12n^{(k+1)})^{2k \log(n)}}{(12n^{(k+1)})^{2k \log(n) \log(|Z_1|)}} \geq \frac{\mu/(12n^{(k+1)})^{2k \log(n)}}{(12n^{(k+1)})^{2k \log(n) \log(|Z|/2)}} = \frac{\mu}{(12n^{(k+1)})^{2k \log(n) \log(|Z|)}}$$

minimal separators in G consistent with A'' , B'' and C'' .

For property (ii) we have that $G[A' \cup B']$ has at most $2k \log n$ more components than $G[A \cup B]$. By the inductive hypothesis $G[A'' \cup B'']$ has at most $2k \log n \log |Z_1|$ more components than $G[A' \cup B']$. However $|Z_1| \leq |Z|$ so $\log(|Z_1|) \leq \log |Z| - 1$ and therefore $G[A'' \cup B'']$ has at most $2k \log n \log |Z|$ more components than $G[A \cup B]$. This concludes the proof. $\square$

If A , B , and C and $\mathcal{S}$ are the outputs of Lemma 4.10 then Lemma 4.23 shows us how we can enrich A , B , and C so that for each $S_i, S_j \in \mathcal{S}$ and for each vertex $v_i \in S_i$ and $v_j \in S_j$, no component of $G' = G - (A' \cup B' \cup C' \cup V)$ contains both a vertex from $N_{G'}[v_i]$ and $N_{G'}[v_j]$ where $V = N_G[v_i] \cap N_G[v_j]$ (A , B , and C are enriched by applying Lemma 4.23 for each pair of vertices v_i and v_j). After doing this the next lemma shows us why this is sufficient to guarantee that $G - (A \cup B \cup C)$ has no large component.

LEMMA 4.24. *Let G be a graph with n vertices, let $\delta > 1$, and let $\mathcal{S}$ be a vertex list of n/δ -balanced separators of G such that no vertex of G belongs to every balanced separator of $\mathcal{S}$. If for every pair $S_i, S_j \in \mathcal{S}$ it holds that no component of $G - (S_i \cap S_j)$ contains a vertex from both $S_i - (S_i \cap S_j)$ and $S_j - (S_i \cap S_j)$, then G has no component of size greater than n/δ .*

Proof. Let G be a graph with n vertices, let $\delta > 1$, let $\mathcal{S}$ be a list of n/δ -balanced separators of G such that no vertex of G belongs to every balanced separator of $\mathcal{S}$, and for every pair $S_i, S_j \in \mathcal{S}$ it holds that no component of $G - (S_i \cap S_j)$ contains a vertex from both $S_i - (S_i \cap S_j)$ and $S_j - (S_i \cap S_j)$. Assume, for a contradiction then G has a component, X , of size greater than n/δ .

Since X is a component of size greater than n/δ , every n/δ -balanced separator must have at least one vertex in X . Let v be a vertex of X that belongs the largest number of sets of $\mathcal{S}$ as possible. By how $\mathcal{S}$ was defined, there is some $S \in \mathcal{S}$ such that $v \notin S$. Let P be a shortest path from v to S and let s be the endpoint of P that belongs to S .

Since v was chosen to be a vertex of X that belongs the largest number of sets of $\mathcal{S}$ as possible, there must be a set $S' \in \mathcal{S}$ such that $v \in S'$ and $s \notin S'$. Since we also have that $v \notin S$, there must be a subpath P^* of P with endpoints s and v' where $v' \in S'$, $v' \neq s$, and no internal vertex of P^* belongs to S' or S (recall P is a shortest path from v to S). It follows that P^* is a path in $G - (S \cap S')$ and therefore since neither v' nor s belong to $S \cap S'$, v' and s are in the same connected component in $G - (S \cap S')$, a contradiction to how $\mathcal{S}$ was defined. $\square$

We are now ready to prove Lemma 4.3. As indicated before, the proof works by taking A , B , and C and $\mathcal{S}$, the outputs of Lemma 4.10, then for each $S_i, S_j \in \mathcal{S}$ and for each vertex $v_i \in S_i$ and $v_j \in S_j$, applying Lemma 4.23 to v_i and v_j . The graph $G - (A \cup B \cup C)$ along with the vertex list $\mathcal{S}$ will then satisfy the conditions of Lemma 4.24.

Proof. [Proof of Lemma 4.3] Let G be a k -creature free graph, $k \geq 2$, with $n \geq 2$ vertices and μ minimal separators and let $\delta > 1$. Apply Lemma 4.10 to G and δ to get sets A , B , and C and an anti-complete list $\mathcal{S}$ that satisfy properties (i)-(vi) of Lemma 4.10.

We wish to “enrich” A , B , and C so that for each unordered pair of vertices x and y where x belongs to some set of $\mathcal{S}$ and y belong to a different set of $\mathcal{S}$, it will hold that no vertex of $N_G[x]$ and $N_G[y]$ will be in the same component of $G - (A \cup B \cup C \cup (N_G[x] \cap N_G[y]))$. To do this we go through each pair of vertices,

x, y , and after considering the i^{th} pair we will have vertex sets A^i, B^i and C^i such that for any pair x, y that have previously been considered it holds that no vertex of $N_G[x]$ and $N_G[y]$ will be in the same component of $G - (A \cup B \cup C \cup (N_G[x] \cap N_G[y]))$.

More formally, we set $A^0 = A$, $B^0 = B$ and $C^0 = C$. Assume that we have already considered $i - 1$ unordered vertex pairs x and y where x belongs to some set of $\mathcal{S}$ and y belong to a different set of $\mathcal{S}$ and we have sets A^{i-1} , B^{i-1} and C^{i-1} . Since $\mathcal{S}$ is an anti-complete vertex list, x and y are anti-complete, so we set $Y_1 = x$, $Y_2 = y$, and $Z = V(G) - (N_G[x] \cap N_G[y])$ and apply Lemma 4.23 to $A^{i-1}, B^{i-1}, C^{i-1}, Y_1, Y_2$, and Z to get the sets A^i, B^i , and C^i which satisfy properties (i)-(iv) of Lemma 4.23.

Assume that there are ℓ unordered vertex pairs x and y where x belongs to some set of $\mathcal{S}$ and y belong to a different set of $\mathcal{S}$. We show that the sets A^ℓ, B^ℓ , and C^ℓ satisfy properties (i)-(iii) of Lemma 4.3.

We first show property (i) is satisfied. We can see that by properties (iii) and (iv) of Lemma 4.23 that for any pair of vertices x and y where x belongs to some set of $\mathcal{S}$ and y belong to a different set of $\mathcal{S}$, it holds that no vertex of $N_G[x] - (A^\ell \cup B^\ell \cup C^\ell \cup V)$ and $N_G[y] - (A^\ell \cup B^\ell \cup C^\ell \cup V)$ will be in the same component of $G - (A^\ell \cup B^\ell \cup C^\ell \cup V)$ where $V = N_G[x] \cap N_G[y]$. It then follows that for any two distinct sets $S_i, S_j \in N_G(\mathcal{S})$ that no vertex of $S_i - (A^\ell \cup B^\ell \cup C^\ell \cup V')$ and $S_j - (A^\ell \cup B^\ell \cup C^\ell \cup V')$ belong to the same component in $G - (A^\ell \cup B^\ell \cup C^\ell \cup V')$ where $V' = S_i \cap S_j$. Since $A \subseteq A^\ell$, $B \subseteq B^\ell$, and $C \subseteq C^\ell$, if $G' = G - (A^\ell \cup B^\ell \cup C^\ell)$ then it follows from property (i) of Lemma 4.10 that for all $S \in \mathcal{S}$ $N_G[S] - (A^\ell \cup B^\ell \cup C^\ell)$ is an n/δ -balanced separator of G' and no vertex of G' belongs to all sets of $N_{G'}[S]$. It follows from Lemma 4.24 that no vertex of G' has over n/δ components.

Next we show property (ii) and (iii) are satisfied. By properties (ii) and (iii) of Lemma 4.10 there are at least $\mu/(4(3n^{(k+1)})^{160k^2\delta^2\log^3(n)+2})$ minimal separators that are consistent with A, B , and C and $G[A \cup B]$ contains at most $260k^2\delta^2\log(n)^3 + 2$ components. Furthermore, by properties (i) and (ii) of Lemma 4.23 if there are μ' minimal separators that are consistent with A^{i-1}, B^{i-1} and C^{i-1} then there are at least $\mu'/(12n^{(k+1)})^{2k\log^2(n)}$ minimal separators that are consistent with A^i, B^i , and C^i and there are at most $2k\log(n)^2$ more minimal separators in $G[A^i \cup B^i]$ than in $G[A^{i-1} \cup B^{i-1}]$. Since each $S \in \mathcal{S}$ has size at most $8k\delta$ and $\mathcal{S}$ has size $\log(n) + 1$, ℓ must be less than $(8k\delta\log(n))^2$. Hence G has at least

$$\begin{aligned} & \frac{\mu}{4(3n^{(k+1)})^{160k^2\delta^2\log^3(n)+2}(12n^{(k+1)})^{(2k\log^2(n))(8k\delta\log(n))^2}} \\ & \geq \frac{\mu}{((12n^{(k+1)})^{160k^2\delta^2\log^3(n)+2}(12n^{(k+1)})^{(128k^3\log^4(n))})} \geq \frac{\mu}{(12n^{(k+1)})^{400k^3\delta^2\log^4(n)}} \end{aligned}$$

minimal separators that are consistent with A^ℓ, B^ℓ, C^ℓ , and $G[A^\ell \cup B^\ell]$ has at most

$$260k^2\delta^2\log(n)^3 + 2 + (2k\log^2(n))(8k\delta\log(n))^2 \leq 400k^3\delta^2\log(n)^4$$

components. $\square$

4.2 Constructing the Generalized ω -Creature

In this section we will prove that if G is a k -creature free graph with n vertices and a sufficient number of minimal separators, then we can find a generalized ω -creature in G . We do this by taking the output of Lemma 4.3 from the previous subsection giving us A, B , and C such that $G - (A \cup B \cup C)$ has no component with over $\approx n/2\omega$ vertices and a large fraction of minimal separators of G are consistent with A, B , and C . This already gives us something close to a generalized ω -creature. Setting H to be an ω -bistar and φ to be a function that maps A and B to c_A and c_B (the two central vertices of H) respectively and ω of the components of $G - (A \cup B \cup C)$ to the ω peripheral vertices of H gives us something close to a generalized ω -creature, we are just missing the special sets S_1 and S_2 . If there are at least ω components, X , of $G - (A \cup B \cup C)$ such that at least two minimal separators, S_X and S'_X (S_X and S'_X can depend on the specific X chosen), of G are consistent A, B , and C and $S_X \cap X \neq S'_X \cap X$ (plus an additional property which will be describe later on), then we can in fact show that we can construct a generalized ω -creature.

On the other hand, if we cannot find such a set of ω components of $G - (A \cup B \cup C)$ then this implies all minimal separators S of G that are consistent with A, B , and C intersect all but ω components $G - (A \cup B \cup C)$ in the exact same “unique” way. We will show that we can allocate the vertices of these “uniquely” intersected components of $G - (A \cup B \cup C)$ to makes sets A', B' , and C' such that any minimal separator of G that was

consistent with A , B , and C will be consistent with A' , B' and C' . Since each component of $G - (A \cup B \cup C)$ has at most $n/2\omega$, this implies that over half of the vertices of G belong to A' , B' , or C' . Lemmas 4.1 and 4.2 then allow us to making an induced minor, G' , of G with at most half the vertices of G but still maintaining a large fraction of G 's minimal separators. Since we only sacrificed a small fraction of minimal separators to drop the number of vertices in the graph by a factor of 2, repeating this process at most $\log(n)$ times then must result in us finding a generalized ω -creature for some large value of ω (or else we would end up with an empty graph with a supposedly large number of minimal separators).

Unfortunately, this isn't quite enough for our purposes. We will require not just any generalized ω -creature, but one with a bit more structure, which will force us to do some more pre-processing before constructing our generalized ω -creature. We call this more structured object a *connected good* generalized ω -creature, which we now define.

Good Components and Connected, Good, Full Generalized ω -Creatures.

DEFINITION 4.6. (FULL GENERALIZED ω -CREATURES) *A generalized ω -creature is full if, for every peripheral vertex, u , of H , $\varphi^{-1}(u) \cap S_1$ and $\varphi^{-1}(u) \cap S_2$ are distinct A_φ, B_φ -minimal separators in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$.*

Note that a generalized ω -creature being full just means that the witness separators S_1^* and S_2^* in property (i) of generalized ω -creatures are precisely S_1 and S_2 .

Let G be a graph, let $A, B, \subseteq V(G)$, and let X be a component of $G - (A \cup B)$. X is called a *non-leaf component of $G - (A \cup B)$ with respect to A and B* if it has at least two distinct neighbors in $G[A \cup B]$. A component of $G[X - N_G^2[A \cup B]]$ is said to be a *sub-component of X with respect to A and B* . A sub-component, Y , of X with respect to A and B is called a *non-leaf sub-component with respect to A and B* if Y has neighbors in at least two distinct components of $G[N_G^2[A \cup B]]$. X is said to be *good with respect to A and B* if X has at most one non-leaf sub-component with respect to A and B and for every pair of components P and Q of $G[A \cup B]$ it holds that $N_G^2[P] \cap X$ is anti-complete with $N_G^2[Q] \cap X$.

DEFINITION 4.7. (CONNECTED, GOOD) *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. If for all peripheral vertices, u , of H it holds that $\varphi^{-1}(u)$ is a connected vertex set (which implies that the vertex sets $\varphi^{-1}(u)$ are the components of $G - (A_\varphi \cup B_\varphi)$) then we call W a connected generalized ω -creature. If W is a connected generalized ω -creature and all components of $G - (A_\varphi \cup B_\varphi)$ are good with respect to A_φ and B_φ , then we call W a connected good generalized ω -creature.*

We will find it useful to make the definitions just given slightly more flexible by allowing us to incorporate an additional set C into these definitions. Let G be a graph, let A, B, C, X, Y be vertex sets, and let $G' = G - C$. We will say that X is a non-leaf component of $G - (A \cup B \cup C)$ if X is a non-leaf component of $G' - (A \cup B)$ with respect to A and B . We say Y is a sub-component of X with respect to A, B and C if Y is a sub-component of X with respect to A and B in G' . We say Y is a non-leaf sub-component of X with respect to A, B and C , if Y is a non-leaf sub-component of X with respect to A and B in G' . We say X is good with respect to A, B , and C if X is good with respect to A and B in G' .

Our main result this subsection is to prove that any graph with a sufficient number of minimal separators will contain a connected, good, full generalized ω -creature for large ω . In particular, we will prove Lemma 3.1, which is the only lemma from this subsection that will be used outside of this subsection. We repeat the statement of Lemma 3.1 here for convenience.

Lemma 3.1. *Let G be a k -creature free graph with n vertices, let $\omega > 1$ and $\delta = 3\omega$, let c be an integer large enough so that $400k^3\delta^2 \log^4(c) < c/6$, let $x = 400k^3\delta^2 \log^4(n)$, and let G have at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators. Then there exists an induced minor G' of G such that $(G', H, \varphi, S_1, S_2)$ is a connected, good, full generalized ω -creature.*

4.2.1 Making the Components of $G - (A \cup B \cup C)$ Good

Let G be a graph with n vertices and a large number of minimal separators. Our first step toward proving Lemma 3.1 is to apply Lemma 4.3 to G with $\delta \approx \omega$ to get the sets A, B , and C . We then enrich A, B , and C to ensure that all components of $G - (A \cup B \cup C)$ are good components. In particular, we prove the following lemma.

LEMMA 4.25. Let G be a k -creature free graph (assume $k \geq 2$) with $n \geq 2$ vertices, let $\delta > 1$, and let G have μ minimal separators. Then there exist $A, B, C \subseteq V(G)$ such that the following conditions hold:

- (i) No component in $G - (A \cup B \cup C)$ has over n/δ vertices.
- (ii) Let $x = 400k^3\delta^2 \log^4(n)$. Then G has at least $\mu/(12n)^{5k^2x^4}$ minimal separators that are consistent with A, B , and C .
- (iii) $G[A \cup B]$ has at most $400k^3\delta^2 \log^4(n)$ components.
- (iv) All components, X , of $G - (A \cup B \cup C)$ are good with respect to A, B , and C .
- (v) There are at most $k(400k^3\delta^2 \log^4(n))^2$ components of $G - (A \cup B \cup C)$.

Notice that properties (i)-(iii) are similar to what Lemma 4.3 can guarantee us. So, in order to prove Lemma 4.25 we assume that we have a k -creature free graph, G , and that we have been given sets A, B , and C (which will come from Lemma 4.3) and we want to find sets A', B' , and C' which contain A, B , and C respectively and satisfy the properties of Lemma 4.25, in particular, some effort is required in order to satisfy properties (iv) and (v). In order to do this we must study what happens as we grow A and B by successively taking their neighborhoods in a specially chosen induced subgraph of G . We will need the following lemma to determine this induced subgraph of G (which will end up being $G - C'$, where C' is a set produced by the following lemma).

LEMMA 4.26. Let G be a graph, let $A, B, C \subseteq V(G)$ such that G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , and let r be a natural number. Then there exists $A', B', C' \subseteq V(G)$ where the following conditions hold:

- (i) Let $G[A \cup B]$ have c components, then G has at least μ/n^{rkc} minimal separators that are consistent with A', B' , and C' .
- (ii) $C' \cap A = \emptyset$ and $C' \cap B = \emptyset$.
- (iii) Let $G' = G - C'$ then $A' = N_{G'}^r[A]$ and $B' = N_{G'}^r[B]$.

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$ such that G has $\mu > 0$ minimal separators that are consistent with A, B , and C , let $G[A \cup B]$ have c components, and let r be a natural number. We will show by induction on r that there exists sets A', B' , and C' that satisfy conditions (i)-(iii) of this lemma. Recall for a set $X \subseteq V(G)$ that we define $N_G^0[X] = X$. It follows that taking $A' = A, B' = B$ and $C' = C$ satisfies conditions (i)-(iii) of this lemma for the base case when $r = 0$. Now assume this holds for all r less than some $r' > 0$. We will show it holds for $r = r'$.

We use the inductive hypothesis to find sets A', B' , and C' such that (i) G has at least $\mu/n^{(r-1)kc}$ minimal separators that are consistent with A', B' and C' (let $\mathcal{S}$ be the set of these minimal separators) (ii) $C' \cap A = \emptyset$ and $C' \cap B = \emptyset$, and (iii) if $G' = G - C'$ then $A' = N_{G'}^{r-1}[A]$ and $B' = N_{G'}^{r-1}[B]$. Since $G[A \cup B]$ has c components, condition (iii) shows that $G[A' \cup B']$ has at most c components. By definition of consistent, for all $S \in \mathcal{S}$ it holds that $A' \cap S = \emptyset$ and $B' \cap S = \emptyset$, so we may then apply Lemma 4.7 with $U = A' \cup B'$ (and using the fact that $U \cap S = \emptyset$ for $S \in \mathcal{S}$ to conclude the set $\mathcal{S}^U = \{N_G(U) \cap S \mid S \in \mathcal{S}\}$ has size $|\mathcal{S}^U| \leq n^{kc}$. Since $|\mathcal{S}^U| \leq n^{kc}$ there is an $X \in \mathcal{S}^U$ such that at least $|\mathcal{S}|/n^{kc} \geq \mu/n^{rkc}$ (by the inductive hypothesis) minimal separators $S \in \mathcal{S}$ have the property that $N_G(U) \cap S = X$. Denote this subset of $\mathcal{S}$ as $\mathcal{S}^X$, so $|\mathcal{S}^X| \geq \mu/n^{rkc}$. We will show that $N_{G'}[A'] - X$, $N_{G'}[B'] - X$, and $C \cup X$ satisfy the properties of this lemma.

We first establish property (i). We have that for all $S \in \mathcal{S}^X$, since S is consistent with A', B' , and C' and $N_G(A' \cup B') \cap S = X$, it follows that S is consistent with $N_G[A'] - X$, $N_G[B'] - X$, and $C \cup X$ and therefore, since $N_{G'}[A'] \subseteq N_G[A']$ and $N_{G'}[B'] \subseteq N_G[B']$, S is consistent with $N_{G'}[A'] - X$, $N_{G'}[B'] - X$, and $C \cup X$. Since $|\mathcal{S}^X| \geq \mu/n^{rkc}$ this established condition (i).

To see that property (ii) holds note that since A' and B' are both disjoint from $S \in \mathcal{S}$ and from C' we have that $X \cap A' = X \cap B' = \emptyset$, it holds that $(C' \cup X) \cap (N_{G'}[A'] - X) = (C' \cup X) \cap (N_{G'}[B'] - X) = \emptyset$. This establishes condition (ii).

Lastly, we establish condition (iii). Since, by the inductive hypothesis, $A' = N_{G'}^{r-1}[A]$, $B' = N_{G'}^{r-1}[B]$, and $X \cap (A' \cup B') = \emptyset$ it follows that if $G'' = G' - X$ then $N_{G'}[A'] - X = N_{G''}^r[A]$ and $N_{G'}[B'] - X = N_{G''}^r[B]$ which establishes condition (iii). $\square$

We now define a sequence which we will study in order to prove Lemma 4.25. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , let A', B' , and C' be the sets returned by applying Lemma 4.26 to G, A, B, C , and $2r = 2(kc^3 + c)$ and let $G' = G - C'$. We define a sequence, $\text{seq}_3(G, A, B, C)$, to be a sequence of tuples where the i^{th} tuple, $1 \leq i \leq r$, is defined to be (A_i, B_i) where $A_i = N_{G'}^{2i}[A]$ and $B_i = N_{G'}^{2i}[B]$. The sets A', B' , and C' will be referred to as the *core sets* of $\text{seq}_3(G, A, B, C)$ and the graph G' will be referred to as the *core graph* of $\text{seq}_3(G, A, B, C)$.

We now prove a series of lemmas which will eventually allow us to show that there exists some tuple (A_j, B_j) of $\text{seq}_3(G, A, B, C)$ such that A_j, B_j , and C' satisfy the properties of Lemma 4.25, assume that the sets A, B , and C were obtained from Lemma 4.3.

LEMMA 4.27. *Let G be a k -creature free graph and let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C . Then for a tuple (A_i, B_i) of $\text{seq}_3(G, A, B, C)$ it holds that A_i is anti-complete with B_i .*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , and let A', B' , and C' be the core sets of $\text{seq}_3(G, A, B, C)$. If $G[A \cup B]$ has c components then by property (i) of Lemma 4.26 G has at least $\mu/n^{2(kc^3+c)kc} > 0$ minimal separators that are consistent with A', B' , and C' , hence G has at least one minimal separator that is consistent with A', B' , and C' . It follows that A' and B' must be anti-complete. By property (iii) of Lemma 4.26 we can see that for a tuple (A_i, B_i) of $\text{seq}_3(G, A, B, C)$ it holds that $A_i \subseteq A'$ and $B_i \subseteq B'$, hence A_i is anti-complete with B_i . $\square$

LEMMA 4.28. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , and let (A_i, B_i) be a tuple of $\text{seq}_3(G, A, B, C)$. Then $|\mathcal{CC}(G[A_i \cup B_i])|$ is a non-increasing sequence in i .*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , let (A_i, B_i) and (A_{i+1}, B_{i+1}) be tuples of $\text{seq}_3(G, A, B, C)$, and let G' be the core graph of $\text{seq}_3(G, A, B, C)$. Then $A_{i+1} = N_{G'}^2[A_i]$ and $B_{i+1} = N_{G'}^2[B_i]$ and it follows that $G[A_{i+1} \cup B_{i+1}]$ has at most as many components as $G[A_i \cup B_i]$ which proves the lemma. $\square$

Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C . We define the i^{th} critical number, c_i , of $\text{seq}_3(G, A, B, C)$ to be the i^{th} number such that the number of components of $G[A_{c_i} \cup B_{c_i}]$ is strictly less than the number of components of $G[A_{(c_i)+1} \cup B_{(c_i)+1}]$ for the tuples (A_{c_i}, B_{c_i}) and $(A_{(c_i)+1}, B_{(c_i)+1})$ of $\text{seq}_3(G, A, B, C)$. In other words, the critical numbers denote indexes of $\text{seq}_3(G, A, B, C)$ where the number of components strictly decreases. For convenience, we also let 0 be a critical number of $\text{seq}_3(G, A, B, C)$ as well as the index, r , of the last tuple of $\text{seq}_3(G, A, B, C)$ in order to have the fact that every index of the sequence $\text{seq}_3(G, A, B, C)$ is either a critical number or lies between two critical numbers. Let $\text{seq}_3(G, A, B, C)$ have t critical numbers other than 0 and r . We will call 0 the 0^{th} critical number and will be denoted by c_0 and we will call r the $t+1^{\text{th}}$ critical number and will be denoted by c_{t+1} . The following corollary follows from the fact that if there are c components of $G[A \cup B]$ then $\text{seq}_3(G, A, B, C)$ has $r = kc^3 + c$ elements, Lemma 4.28, and the fact that no graph can have a negative number of components.

COROLLARY 4.6. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , and let there be c components of $G[A \cup B]$. Then there exists critical numbers c_i and c_{i+1} of $\text{seq}_3(G, A, B, C)$ such that $c_{i+1} - c_i > kc^2 + 1$.*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , and let there be c components of $G[A \cup B]$. Since for each tuple (A_i, B_i) of $\text{seq}_3(G, A, B, C)$ we have that $A_i, B_i \neq \emptyset$ and A_i and B_i are anti-complete by Lemma 4.27 it follows that $G[A_i \cup B_i]$ must have at least two components. Hence there can be at most c critical numbers (including 0 and $r = kc^3 + c$) of $\text{seq}_3(G, A, B, C)$. Since $\text{seq}_3(G, A, B, C)$ has $kc^3 + c$ elements, it follows that there must be some critical numbers c_i and c_{i+1} such that $c_{i+1} - c_i > kc^2 + 1$. $\square$

LEMMA 4.29. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A, B , and C , let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, and let G' be the core graph of $\text{seq}_3(G, A, B, C)$. If (A_j, B_j) is a tuple of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$ then for all pairs of components P, Q of $G[A_j \cup B_j]$ it holds that $N_{G'}^2[P]$ is anti-complete with $N_{G'}^2[Q]$.*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, let G' be the core graph of $\text{seq}_3(G, A, B, C)$, and let (A_j, B_j) and (A_{j+1}, B_{j+1}) be tuples of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$ (in particular, this implies that j is not a critical number). Assume for a contradiction that there are components P, Q of $G[A_j \cup B_j]$ such that $P' = N_{G'}^2[P]$ is not anti-complete with $Q' = N_{G'}^2[Q]$. It follows that $G[P' \cup Q']$ is connected and therefore only contains one component. Since $A_{j+1} = N_{G'}^2[A_j]$ and $B_{j+1} = N_{G'}^2[B_j]$, this implies that the number of components of $G[A_{j+1} \cup B_{j+1}]$ is strictly less than the number of components of $G[A_j \cup B_j]$. But this contradicts the fact that j is not a critical number. $\square$

The following lemma will be needed to prove property (v) of Lemma 4.25.

LEMMA 4.30. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let c denote the number of components of $G[A \cup B]$, let G' be the core graph of $\text{seq}_3(G, A, B, C)$, and let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$. If (A_j, B_j) is a tuple of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$. Then there are less than kc^2 non-leaf components of $G' - (A_j \cup B_j)$ with respect to A_j and B_j .*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let c denote the number of components of $G[A \cup B]$, let G' be the core graph of $\text{seq}_3(G, A, B, C)$, let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, and let (A_j, B_j) be a tuple of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$. Assume for a contradiction that there are at least kc^2 non-leaf components of $G' - (A_j \cup B_j)$ with respect to A_j and B_j . Since there are c components of $G[A \cup B]$, by Lemma 4.28 there are at most c components of $G[A_j \cup B_j]$. It follows there are $(c^2 - c)/2$ pairs of distinct components of $G[A_j \cup B_j]$, so by the pigeon hole principle, there exists components P and Q of $G[A_j \cup B_j]$ such that there are at least k non-leaf components, $X_1, X_2, \dots, X_k$, of $G' - (A_j \cup B_j)$ that have neighbors in both P and Q . Let Y_i denote an induced path from P to Q whose internal vertices are contained in X_i , so the internal vertices of Y_i are anti-complete with the internal vertices of $Y_{t'}$ for $t \neq t'$. By Lemma 4.29 each Y_i must have at least six vertices. So, since P and Q are connected, P is anti-complete with Q , the internal vertices of Y_i are anti-complete with the internal vertices of $Y_{t'}$ for $t \neq t'$, and the Y_i 's are induced paths from P to Q of length at least six, we can see that this implies that G contains a k -creature, which is a contradiction. $\square$

LEMMA 4.31. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let G' be the core graph of $\text{seq}_3(G, A, B, C)$, let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, and let (A_j, B_j) and (A_{j+1}, B_{j+1}) be tuples of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$. Then every non-leaf component of $G' - (A_j \cup B_j)$ with respect to A_j and B_j contains a non-leaf sub-component with respect to A_j and B_j . Furthermore, if X is a component of $G' - (A_j \cup B_j)$ then every non-leaf sub-component of X with respect to A_j and B_j is a non-leaf component of $G' - (A_{j+1} \cup B_{j+1})$ with respect to A_{j+1} and B_{j+1} .*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let G' be the core graph of $\text{seq}_3(G, A, B, C)$, let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, and let (A_j, B_j) and (A_{j+1}, B_{j+1}) be tuples of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$. Assume X is a component of $G' - (A_j \cup B_j)$ and let Y be a non-leaf sub-component of X with respect to A_j and B_j .

First we show that every non-leaf sub-component of X with respect to A_j and B_j is a non-leaf component of $G' - (A_{j+1} \cup B_{j+1})$ with respect to A_{j+1} and B_{j+1} . By definition, Y is a component of $G[X - N_{G'}^2[A_j \cup B_j]] = G[X - (N_{G'}^2[A_j] \cup N_{G'}^2[B_j])] = G[X - (A_{j+1} \cup B_{j+1})]$, and therefore Y is a component of $G' - (A_{j+1} \cup B_{j+1})$. Furthermore, by definition, Y has neighbors in two components of $G[N_{G'}^2[A_j \cup B_j]] = G[N_{G'}^2[A_j] \cup N_{G'}^2[B_j]] = G[A_{j+1} \cup B_{j+1}]$, so Y is a non-leaf component of $G' - (A_{j+1} \cup B_{j+1})$ with respect to A_{j+1} and B_{j+1} .

Next, we show that if X is a non-leaf component of $G' - (A_j \cup B_j)$ then X must contain a non-leaf sub-component with respect to A_j and B_j . Let Z denote the set of vertices that belong to a component of $G[A_j \cup B_j]$ that has at least one neighbor in X , so $G[Z]$ has at least two components since X is a non-leaf component and $G[X - (N_{G'}^2[A_j] \cup N_{G'}^2[B_j])] = G[X - N_{G'}^2[Z]]$. Since $G[X \cup Z]$ is connected, $G[(X - N_{G'}^2[Z]) \cup N_{G'}^2[Z]] = G[X \cup N_{G'}^2[Z]]$ is connected and it follows from Lemma 4.29 that since $G[Z]$ has at least two components, $G[N_{G'}^2[Z]]$ has at least two components. It follows there must be at least one component of $G[X - N_{G'}^2[Z]] = G[X - N_{G'}^2[A_j \cup B_j]]$ that has at least one neighbor in two components of $G[N_{G'}^2[Z]]$ (or else $G[(X - N_{G'}^2[Z]) \cup N_{G'}^2[Z]]$ would not be connected) and therefore has at least one neighbor in two components of $G[N_{G'}^2[A_{j+1} \cup B_{j+1}]]$. $\square$

The next corollary follows immediately from Lemma 4.31

COROLLARY 4.7. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$ where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let G' be the core graph of $\text{seq}_3(G, A, B, C)$, let c_i and c_{i+1} be critical numbers of $\text{seq}_3(G, A, B, C)$, and let (A_j, B_j) be a tuple of $\text{seq}_3(G, A, B, C)$ where $c_i < j < c_{i+1}$. Then the number of non-leaf components of $G' - (A_j \cup B_j)$ with respect to A_j and B_j is a non-decreasing sequence in j (when the possible values of j are restricted to lie between c_i and c_{i+1}).*

The following lemma is the main lemma that we need to prove Lemma 4.25. Namely, that we can find an (A_j, B_j) of $\text{seq}_3(G, A, B, C)$ has the property that all components of $G' - (A_j \cup B_j)$ are good, where G' is the core graph of $\text{seq}_3(G, A, B, C)$.

LEMMA 4.32. *Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , and let G' be the core graph of $\text{seq}_3(G, A, B, C)$. Then there exists an integer j such that the tuple (A_j, B_j) of $\text{seq}_3(G, A, B, C)$ has the property that all components of $G' - (A_j \cup B_j)$ are good with respect to A_j and B_j .*

Proof. Let G be a k -creature free graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, where G has $\mu \geq 1$ minimal separators that are consistent with A , B , and C , let c denote the number of components of $G[A \cup B]$, and let G' be the core graph of $\text{seq}_3(G, A, B, C)$. By Corollary 4.6 there exists two critical numbers c_i and c_{i+1} of $\text{seq}_3(G, A, B, C)$ such that $c_{i+1} - c_i > kc^2 + 1$. By Lemma 4.30 for any (A_j, B_j) in $\text{seq}_3(G, A, B, C)$ with $c_i < j < c_{i+1}$ there are less than kc^2 non-leaf components of $G' - (A_j \cup B_j)$ with respect to A_j and B_j , and by Corollary 4.7 the number of non-leaf components of $G' - (A_j \cup B_j)$ with respect to A_j and B_j is a non-decreasing sequence in j when $c_i < j < c_{i+1}$. It follows that there must exist some j' , $c_i < j' < c_{i+1}$ such that the number of non-leaf components of $G' - (A_{j'} \cup B_{j'})$ is the same as the number of non-leaf components of $G' - (A_{j'+1} \cup B_{j'+1})$.

We now show that all components, X , of $G' - (A_{j'} \cup B_{j'})$ are good with respect to $A_{j'}$ and $B_{j'}$. Let P and Q be two components of $G[A_{j'} \cup B_{j'}]$. By Lemma 4.29 it holds that $N_{G'}^2[P] \cap X$ is anti-complete with $N_{G'}^2[Q] \cap X$. Next, it follows from Lemma 4.31 that if X has two non-leaf sub-components with respect to $A_{j'}$ and $B_{j'}$, then the number of non-leaf components of $G' - (A_{j'} \cup B_{j'})$ with respect to $A_{j'}$ and $B_{j'}$ is strictly less than the number of non-leaf components of $G' - (A_{j'+1} \cup B_{j'+1})$ with respect to $A_{j'+1}$ and $B_{j'+1}$, which is a contradiction. It follows that X is a good component. $\square$

Let G be a k -creature free graph with many minimal separators, and let A , B , and C be the sets obtained from Lemma 4.3 applied to G . We will show that the tuple (A_j, B_j) obtained from Lemma 4.32 almost satisfies the properties of Lemma 4.25. The problem is that $G' - (A_j \cup B_j)$ could have a large number of components, so property (v) of Lemma 4.25 might not be satisfied. But Lemma 4.30 shows that not many components of $G' - (A_j \cup B_j)$ have neighbors in both A_j and B_j . The next lemma show how we get rid of the components of $G' - (A_j \cup B_j)$ that do not have neighbors in both A_j and B_j .

LEMMA 4.33. *Let G be a graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, and let S be a minimal separator of G that is consistent with A , B , and C . Let A^* , B^* , and C^* be the sets of vertices that belong to a component of $G - (A \cup B \cup C)$ that has neighbors in A and not B , that has neighbors in B and not A , and that does not have neighbor $B \cup A$ respectively. Then S is consistent with $A \cup A^*$, $B \cup B^*$, and $C \cup C^*$.*

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, and let S be a minimal separator of G that is consistent with A , B , and C . Let A^* , B^* , and C^* be the sets of vertices that belong to a component of $G - (A \cup B \cup C)$ that has neighbors in A and not B , that has neighbors in B and not A , and that does not have neighbor $B \cup A$ respectively.

First, we show that $S \cap A^* = \emptyset$. Assume for a contradiction that there is an $s \in S \cap A^*$. Since S is consistent with A , B , and C , B belongs to an S -full component, call it B' , of $G - S$ which does not contain any vertex from A nor C . So, there must be some path, P from B to s such that the internal vertices of P are contained in B' and therefore disjoint from A and C . But, since s lies in a component, Q , of $G - (A \cup B \cup C)$ such that $N_G(Q) \subseteq A \cup C$, any path from B to Q must contain a vertex from $A \cup C$, hence such a path P cannot exist.

A symmetric argument shows that $S \cap B^* = \emptyset$. Now since S does not contain any vertex from $A^* \cup B^*$, we can see that A^* and A belong to the same component of $G - S$ and B^* and B belong to the same component of

$G - S$. Hence S is consistent with $A \cup A^*$, $B \cup B^*$ and C . All that needs to be shown to complete the proof is that no vertex of C^* belongs to the same component that A nor B belongs to $G - S$.

So, assume for a contradiction that there is a $c \in C^* - S$ that belongs to the same component that either A or B does in $G - S$, without loss of generality assume that s belongs to the same component A does in $G - S$. So, there must be some path, P from A to c such that the internal vertices of P are contained in the component A belongs to in $G - S$ and therefore is disjoint from B and C . But, since c lies in a component, Q , of $G - (A \cup B \cup C)$ such that $N_G(Q) \subseteq C$, any path from A to Q must contain a vertex from C , hence such a path P cannot exist. $\square$

We are now ready to prove Lemma 4.25

Proof. [Proof of Lemma 4.25] Let G be a k -creature free graph, $k \geq 2$, with $n \geq 2$ vertices, let $\delta > 1$, and let G have μ minimal separators. Apply Lemma 4.3 to G and δ to get sets A, B , and C that satisfy properties (i)-(iii) of Lemma 4.3. Let G' be the core graph of $\text{seq}_3(G, A, B, C)$ and let A', B' , and C' be the core sets of $\text{seq}_3(G, A, B, C)$. By Lemma 4.32 there exists a tuple (A_j, B_j) of $\text{seq}_3(G, A, B, C)$ such that all components of $G' - (A_j \cup B_j)$ are good with respect to A_j and B_j , hence all components of $G - (A_j \cup B_j \cup C')$ are good with respect to A_j, B_j , and C' . Let A^* be the set of all vertices that belong to a component, X , of $G - (A_j \cup B_j \cup C')$ such that X has neighbors in A and no neighbors in B , let B^* be the set of all vertices that belong to a component, X , of $G - (A_j \cup B_j \cup C')$ such that X has neighbors in B and no neighbors in A , and let C^* be the set of all vertices that belong to a component, X , of $G - (A_j \cup B_j \cup C')$ such that X has no neighbors $A \cup B$. We will show that the sets $A'' = A_j \cup A^*$, $B'' = B_j \cup B^*$, and $C'' = C' \cup C^*$ satisfy properties (i)-(v) of this lemma.

We first establish property (i) of this lemma. By property (i) of Lemma 4.3 no component of $G - (A \cup B \cup C)$ has over n/δ vertices. By property (iii) of Lemma 4.26 and the definition of $\text{seq}_3(G, A, B, C)$ it follows that $A \subseteq A_j$, $B \subseteq B_j$, and $C \subseteq C'$, hence no component of $G - (A'' \cup B'' \cup C'')$ has over n/δ vertices. This establishes property (i) of this lemma.

Next, we establish properties (ii). Let $x = 400k^3\delta^2 \log^4(n)$. Then by property (ii) of Lemma 4.3, G has at most $\mu/(12n^{(k+1)})^x$ minimal separators that are consistent with A, B , and C , and property (iii) of Lemma 4.3 $G[A \cup B]$ has at most x components. So by property (i) of Lemma 4.26 and by how A', B' and C' were defined, G has at least

$$\frac{\mu}{(12n^{(k+1)})^x n^{2(kx^3+x)kx}} \geq \frac{\mu}{(12n)^{(k+1)x+2(kx^3+x)kx}} \geq \frac{\mu}{(12n)^{5k^2x^4}}$$

minimal separators that are consistent with A', B' , and C' . Since $A_j \subseteq A'$ and $B_j \subseteq B'$, G has at least $\mu/(12n)^{5k^2x^4}$ minimal separators that are consistent with A_j, B_j , and C' . Lastly, by Lemma 4.33 G has at least $\mu/(12n)^{5k^2x^4}$ minimal separators that are consistent with $A'' = A_j \cup A^*$, $B'' = B_j \cup B^*$, and $C'' = C' \cup C^*$. Hence property (ii) of this lemma is satisfied.

Now, we establish property (iii). By property (iii) of Lemma 4.3, $G[A \cup B]$ has at most $400k^3\delta^2 \log^4(n)$ components. By how A_j and B_j are defined, we have that $A_j = N_{G'}^{2j}[A]$ and $B_j = N_{G'}^{2j}[B]$, so it follows that $G[A_j \cup B_j]$ has less than or equal to $400k^3\delta^2 \log^4(n)$ components. It can then be seen by the definition of A^* and B^* that $G[A'' \cup B'']$ has less than or equal to $400k^3\delta^2 \log^4(n)$ components. This established property (iii).

Next, we prove property (iv). It follows from the definitions of A^*, B^* , and C^* that $\mathcal{CC}(G - (A'' \cup B'' \cup C''))$ is a subset of $\mathcal{CC}(G - (A_j \cup B_j \cup C'))$, and $N_G(A^*) \subseteq A_j \cup C'$ and $N_G(B^*) \subseteq B_j \cup C'$. Let $G'' = G - C''$. Since $C'' \subseteq C'$ we have that $N_{G''}(A^*) \subseteq A_j$ and $N_{G''}(B^*) \subseteq B_j$. Then for a component, X , of $G - (A'' \cup B'' \cup C'')$, we have that $N_{G''}^2[A'' \cup B''] \cap X = N_{G''}^2[A_j \cup B_j] \cap X$, and since X is a component of $G'' - (A_j \cup B_j)$ and of $G' - (A_j \cup B_j)$ it follows that $N_{G''}^2[A_j \cup B_j] \cap X = N_{G'}^2[A_j \cup B_j] \cap X$. Hence $N_{G''}^2[A'' \cup B''] \cap X = N_{G'}^2[A_j \cup B_j] \cap X$.

So, in G , any non-leaf sub-component of X with respect to A'', B'' , and C'' must be a non-leaf sub-component of X with respect to A_j, B_j , and C' and for any components P, Q of $G[A'' \cup B'']$ it holds that $N_{G''}^2[P] \cap X$ is anti-complete with $N_{G''}^2[Q] \cap X$ (or else we could find components P', Q' of $G[A_j \cup B_j]$ such that $N_{G'}^2[P'] \cap X$ is not anti-complete with $N_{G'}^2[Q'] \cap X$, contradicting the fact that X is good with respect to A_j, B_j , and C'). It follows that all components of $G - (A'' \cup B'' \cup C'')$ are good with respect to A'', B'' , and C'' .

Lastly, we verify property (v). Note that by the definition of A^*, B^* and C^* , the components of $G - (A'' \cup B'' \cup C'')$ are precisely the components of $G - (A_j \cup B_j \cup C')$ that have neighbors in both A_j and B_j , and therefore in A'' and B'' . This implies that every component of $G - (A'' \cup B'' \cup C'')$ is a non-leaf component of $G' - (A_j \cup B_j)$ in G' with respect to A_j and B_j , of which there are at most $k|\mathcal{CC}(G[A_j \cup B_j])|^2$ such component

by Lemma 4.30. As noted when proving property (iii) of this lemma, $G[A_j \cup B_j]$ has at most $400k^3\delta^2 \log^4(n)$ components, hence there are at most $k(400k^3\delta^2 \log^4(n))^2$ components of $G - (A'' \cup B'' \cup C'')$, which establishes property (v). $\square$

4.2.2 Finding the Sets S_1 and S_2 for a Generalized ω -Creature Let G be a k -creature free graph with many minimal separators and let A , B , and C be the output of applying Lemma 4.25 to G . We outlined at the start of 4.2 how we can use A , B , and C to construct a connected, good, full generalized ω -creature. The pieces right now that we are missing are the special sets S_1 and S_2 . We claimed that if there are at least ω components, X , of $G - (A \cup B \cup C)$ such that at least two minimal separators, S_X and S'_X (S_X and S'_X can depend on the specific X chosen), of G are consistent with A , B , and C and $S_X \cap X \neq S'_X \cap X$ (plus an additional property), then we can in fact show that we can construct a good connected generalized ω -creature. The following definition is the additional property that we need the minimal separators to satisfy, it essentially forces all pairs of minimal separators that we consider to satisfy property (iii) of Definition 3.1.

Signatures Let G be a graph, let $A, B, C \subseteq V(G)$, and let $\mathcal{M} \subseteq (\mathcal{CC}(G[A]) \times \mathcal{CC}(G[A]) \cup \mathcal{CC}(G[B]) \times \mathcal{CC}(G[B]))$. We call $\mathcal{M}$ a *mark* for A and B . Let S be a minimal separator of G that is consistent with A , B , and C and let X be a component of $G - (A \cup B \cup C)$. If $\mathcal{M}$ is the mark with the property that the pair (U, V) of $(\mathcal{CC}(G[A]) \times \mathcal{CC}(G[A]) \cup \mathcal{CC}(G[B]) \times \mathcal{CC}(G[B]))$ is in $\mathcal{M}$ if and only if $U \neq V$ and there is a path from U to V through $X - S$, then we call $\mathcal{M}$ the *mark of S with respect to X , A , and B* .

DEFINITION 4.8. Let G be a graph, and let $A, B, C \subseteq V(G)$. Define a function T from the set of components of $G - (A \cup B \cup C)$ to the set of marks for A and B , so for each component X of $G - (A \cup B \cup C)$, $T(X)$ is a mark for A and B . We call T a *signature* for G , A , B , and C . We say that a minimal separator S of G agrees with A , B , C , and T if S is consistent with A , B , and C , and for all components X of $G - (A \cup B \cup C)$ it holds that $T(X)$ is equal to the mark of S with respect to X , A , and B .

Let G be a graph, let $A, B, C \subseteq V(G)$, let T be a signature of G , A , B , and C , and let S_1 and S_2 be two minimal separators of G that agree with A , B , C , and T . Notice that for any component X of $G - (A \cup B \cup C)$, for all pair of components C_1 and C_2 of $G[A \cup B]$ there is a path from C_1 to C_2 through $X - S_1$ in G if and only if there is a path from C_1 to C_2 through $X - S_2$ in G which is what property (iii) of Definition 3.1 requires of generalized ω -creatures.

The next few lemmas show that if G has many minimal separators that are consistent with A , B , and C , then there exists a signature T for G , A , B , and C so that G has many minimal separators that agree with A , B , C , and T . We begin with the following observation about signature functions which we will use without explicitly reference. The proof follows easily from the definition of signatures.

OBSERVATION 4.4. Let G be a graph, let $A, B, C \subseteq V(G)$, and let S be a minimal separator of G that is consistent with A , B , and C . Then there exists exactly one signature, T , for G , A , B , and C such that S agrees with A , B , C , and T .

LEMMA 4.34. Let G be a graph, let $A, B, C \subseteq V(G)$ where A is anti-complete with B and let there be x components of $G[A \cup B]$ and y components of $G - (A \cup B \cup C)$. Then there are $2^{x^2 y}$ functions that are signatures for G , A , B , and C .

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$ where A is anti-complete with B and let there be x components of $G[A \cup B]$ and y components of $G - (A \cup B \cup C)$. Let T be a signature for G , A , B , and C . The domain of T is the set of components of $G - (A \cup B \cup C)$, which by assumption has y elements, and the range of T is the power set of $(\mathcal{CC}(G[A]) \times \mathcal{CC}(G[A]) \cup \mathcal{CC}(G[B]) \times \mathcal{CC}(G[B]))$, which by assumption has at most 2^{x^2} elements. So for each of the y elements of the domain there is a choice of 2^{x^2} elements to map it to, hence there are at most $2^{x^2 y}$ possible signatures for G , A , B , and C . $\square$

LEMMA 4.35. Let G be a k -creature free graph (assume $k \geq 2$) with $n \geq 2$ vertices, let $\delta > 1$, and let G have μ minimal separators. Then there exist $A, B, C \subseteq V(G)$ and signature T for G , A , B , and C such that the following conditions hold:

- (i) No component of $G - (A \cup B \cup C)$ has over n/δ vertices.

(ii) Let $x = 400k^3\delta^2 \log^4(n)$. Then G has at least $\mu/(12n)^{6k^2x^4}$ minimal separators that agree with A, B, C , and T .

(iii) $G[A \cup B]$ has at most $400k^3\delta^2 \log^4(n)$ components.

(iv) All components of $G - (A \cup B \cup C)$ are good with respect to A, B , and C .

Proof. Let G be a k -creature free graph (assume $k \geq 2$) with $n \geq 2$ vertices, let $\delta > 1$, and let G have μ minimal separators. We apply Lemma 4.25 using G and δ to get sets A, B , and C that satisfy properties (i)-(v) of Lemma 4.25. We can see that this implies the sets A, B , and C satisfy properties (i), (iii), and (iv) of this lemma, so we are left with finding a suitable signature T to satisfy property (iii).

Let $x = 400k^3\delta^2 \log^4(n)$, then by property (i) of Lemma 4.25, there are at least $\mu/(12n)^{5k^2x^4}$ minimal separators that are consistent with A, B , and C . Furthermore, by properties (iii) and (iv) of Lemma 4.25 there are x components of $G[A \cup B]$ and kx^2 components of $G - (A \cup B \cup C)$, hence, by Lemma 4.34 there are $2^{x^2kx^2} = 2^{x^4k}$ possible signature functions for G, A, B , and C . It follows that there is some signature function T such that there are at least

$$\frac{\mu}{(12n)^{5k^2x^4}} \frac{1}{2^{x^4k}} \geq \frac{\mu}{(12n)^{6k^2x^4}}$$

minimal separators of G that agrees with A, B, C , and T . $\square$

If G is a k -creature free graph with many minimal separators, then by Lemma 4.35 there are sets $A, B, C \subseteq V(G)$ and signature T for G, A, B , and C such that G has many minimal separators that agree with A, B, C , and T . Let S_1 and S_2 be two minimal separators of G that agree with A, B, C , and T . As stated previously, for any component, X , of $G - (A \cup B \cup C)$, for all pair of components C_1 and C_2 of $G[A \cup B]$ there is a path from C_1 to C_2 through $X - S_1$ in G if and only if there is a path from C_1 to C_2 through $X - S_2$ in G which is what property (iii) of Definition 3.1 requires of generalized ω -creatures. The problem we have is that there may be components X of $G - (A \cup B \cup C)$ such that $S_1 \cap X = S_2 \cap X$, and so S_1 and S_2 will fail property (i) of Definition 3.1. The following lemma and corollary will help us fix this problem.

LEMMA 4.36. *Let G be a graph, let $A, B \subseteq V(G)$, let T be a signature for G, A, B , and $C = \emptyset$, let X be a component of $G - (A \cup B)$, and let S_1, S_2 be two minimal separators of G that agree with A, B, C , and T . Let $S_1 \cap X = S'_1$ and $S_2 \cap X = S'_2$. Then $(S_1 - S'_1) \cup S'_2$ is a minimal separator that agrees with A, B, C , and T .*

Proof. Let G be a graph, let $A, B \subseteq V(G)$, let T be a signature for G, A, B , and $C = \emptyset$, let X be a component of $G - (A \cup B)$, and let S_1, S_2 be two minimal separators of G that agree with A, B, C , and T . Let $S_1 \cap X = S'_1$ and $S_2 \cap X = S'_2$. Let $S = (S_1 - S'_1) \cup S'_2$.

We first show that there is some component of $G - S$ that contains A . An identical argument shows that there is some component of $G - S$ that contains B . Let A_1 and A_2 be two components of $G[A]$, and let P be a path from A_1 to A_2 in $G - S_1$. We show how to get a path P' from A_1 to A_2 in $G - S$ by replacing portions of P . Let P^* be a maximal subpath such that no internal vertex of P^* is contained in A , let A'_1 and A'_2 be the components of $G[A]$ that the endpoints of P^* belong to. Since S_1 is an A, B -separator, we have that all internal vertices of P^* are contained in $G - (A \cup B)$. It follows that the internal vertices of P^* belong to some component X' of $G - (A \cup B)$. If $X' = X$ by assumption $T(X)$ is the mark of both S_1 and S_2 for X, A , and B , so P^* can be replaced with some other path from A'_1 to A'_2 with internal vertices contained in $X' - S_2$. If $X' \neq X$, then it follows that P^* is also a path in $G - S$ so it does not need to be replaced. We can see that we can replace the portions of P that do not belong to A in this manner to get a path P' from A_1 to A_2 in $G - S$.

We now show that S separates A from B . Assume for a contradiction that there is a path from A to B in $G - S$, let P be a shortest such path so we may assume that the end points of P are in A and B and all internal vertices are in $G - (S \cup A \cup B)$. It follows that all the internal vertices of P must be contained in some component X' of $G - (A \cup B)$, but if $X' = X$ then such a path must contain a vertex from S_2 , else S_2 would not be consistent with A, B , and C , and if $X' \neq X$ then such a path must contain a vertex from S_1 , else S_1 would not be consistent with A, B , and C . From the definition of S it follows that no such path can exist, hence A and B are contained in different components of $G - S$.

Now we show that A and B belong to two S -full components of $G - S$. We show that the component A is contained in $G - S$ dominates S , an identical argument proves the component B is contained in dominates S .

Let $s \in S$ and let X' be the component that s belongs to in $G - (A \cup B)$. If $X' = X$, then since A is an S_2 -full component of $G - S_2$ there is path P from A to s in $G - S_2$ whose internal vertices belong to $X' - S_2$. If $X' \neq X$, then since A is an S_1 -full component of $G - S_1$ there must be a path from A to s in $G - S_1$ whose internal vertices belong to $X' - S_1$. It follows that the component A belongs to in $G - S$ dominates S .

Together, this proves that S is a minimal separator of G that is consistent with A , B , and $C = \emptyset$. To show that S agrees with A , B , C , and T , we must show that for all components, X' , of $G - (A \cup B)$ the mark of S with respect to X' , A , and B is $T(X')$. This follows from the fact that if $X' = X$, then the mark of S with respect to X' , A , and B is the same as the mark of S_2 with respect to X' , A , and B , which is $T(X')$, and if $X' \neq X$, then the mark of S with respect to X' , A , and B is the same as the mark of S_1 with respect to X' , A , and B , which is again $T(X')$. $\square$

COROLLARY 4.8. *Let G be a graph, let $A, B \subseteq V(G)$, and let T be a signature for G, A, B , and $C = \emptyset$. If for all components, X , of $G - (A \cup B)$ there are minimal separators S_X and S'_X (S_X and S'_X can depend on the choice of X) of G that agree with A , B , C , and T , where $S_X \cap X \neq S_2 \cap X$, then there exist minimal separators S_1 and S_2 of G that agree with A , B , C , and T , such that for all components, X , of $G - (A \cup B)$ $S_1 \cap X \neq S_2 \cap X$.*

Proof. Let G be a graph, let $A, B \subseteq V(G)$, and let T be a signature for G, A, B , and $C = \emptyset$. Assume for all components, X , of $G - (A \cup B)$ there are minimal separators S_X and S'_X (S_X and S'_X can depend on the choice of X) of G that agree with A , B , C , and T , where $S_X \cap X \neq S_2 \cap X$.

Let S_1 and S_2 be two minimal separators of G that agree with A , B , C , and T that maximize the number of components, X , of $G - (A \cup B)$ such that $S_1 \cap X \neq S_2 \cap X$. Assume for a contradiction that there is an X , of $G - (A \cup B)$ such that $S_1 \cap X = S_2 \cap X$. By Assumption, there is an S_3 that agrees with A , B , C , and T and if $S'_3 = S_3 \cap X$ and $S'_2 = S_2 \cap X$ then $S'_3 \neq S'_2$. By Lemma 4.36 $S''_2 = (S_2 - S'_2) \cup S'_3$ is a minimal separator that agrees with A , B , C , and T . But then the number of components, X , of $G - (A \cup B)$ such that $S_1 \cap X \neq S''_2 \cap X$ is greater than the number of components, X , of $G - (A \cup B)$ such that $S_1 \cap X \neq S_2 \cap X$, a contradiction. $\square$

LEMMA 4.37. *Let G be a graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, let $\mathcal{Y}$ be a set of components of $G - (A \cup B \cup C)$, let $\hat{Y} = \bigcup_{Y \in \mathcal{Y}} Y$, let T be a signature for G, A, B , and C , and let S and S' be a minimal separators of G that agrees with A, B, C , and T such that $S \cap \hat{Y} = S' \cap \hat{Y}$. Furthermore, let A^* be the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has a neighbor in A , let B^* be the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has a neighbor in B , and let $C^* = \hat{Y} - (A^* \cup B^*)$. Then there is a signature T' for $G, A' = A \cup A^*, B' = B \cup B^*,$ and $C' = C \cup C^*$ such that S and S' agree with A', B', C' , and T' .*

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, let $\mathcal{Y}$ be a set of components of $G - (A \cup B \cup C)$, let $\hat{Y} = \bigcup_{Y \in \mathcal{Y}} Y$, let T be a signature for G, A, B , and C , and let S and S' be a minimal separators of G that agrees

with A, B, C , and T such that $S \cap \hat{Y} = S' \cap \hat{Y}$. Furthermore, let A^* be the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has a neighbor in A , let B^* be the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has a neighbor in B , and let $C^* = \hat{Y} - (A^* \cup B^*)$.

Now, let $C_Y = S \cap \hat{Y} = S' \cap \hat{Y}$, so S and S' are consistent with A, B , and $C \cup C_Y$ and observe that $\mathcal{CC}(G[\hat{Y} - S]) = \mathcal{CC}(G[\hat{Y} - S'])$ is a subset of $\mathcal{CC}(G - (A \cup B \cup (C \cup C_Y)))$. So, if we let A^{**}, B^{**} , and C^{**} be the sets of vertices that belong to a component of $G - (A \cup B \cup (C \cup C_Y))$ that has neighbors in A and not B , that has neighbors in B and not A , and that does not have neighbor $B \cup A$ respectively, then it holds that $A^* \subseteq A^{**}, B^* \subseteq B^{**}$, and $C^* \subseteq C^{**} \cup C_Y$. Furthermore, by Lemma 4.33, S and S' are both consistent with $A \cup A^{**}, B \cup B^{**}$, and $C \cup C_Y \cup C^{**}$, hence S and S' are consistent with $A' = A \cup A^*, B' = B \cup B^*$, and $C' = C \cup C^*$.

Lastly, we show for each component, X , of $G - (A' \cup B' \cup C')$ that S and S' have the same mark with respect to $X, A',$ and B' . It will then follow that there is a signature T' for $G, A', B',$ and C' such that S and S' agree with A', B', C' , and T' .

Observe that since $A^* \cup B^* \cup C^* = \hat{Y}$ we have that $\mathcal{CC}(G - (A' \cup B' \cup C')) = \mathcal{CC}(G - (A \cup B \cup C)) - \mathcal{Y}$. This implies that A^*, B^* , and C^* must be anti-complete with the component, X , of $G - (A' \cup B' \cup C')$. Hence $N_G[A'] \cap X = N_G[A] \cap X$ and $N_G[B'] \cap X = N_G[B] \cap X$ and that X must also be a component of $G - (A \cup B \cup C)$. It follows that if there are components P, Q of $G[A' \cup B']$ such that there is a path from P to Q through $X - S$, then there are components P', Q' of $G[A \cup B]$ such that $P' \subset P$ and $Q' \subset Q$ and there is a path from P' to Q'

through $X - S$. Since S and S' have the same mark with respect to X , A , and B , namely $T(X)$, it follows that there is a path from P' to Q' through $X - S'$. Therefore, there is a path from P to Q through $X - S'$. Hence S and S' have the same mark with respect to X , A' , and B' . $\square$

LEMMA 4.38. *Let G be a graph with n vertices, let $\delta > 1$, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, such that no component of $G - (A \cup B \cup C)$ has over $n/3\delta$ vertices and $G[A \cup B]$ has at most $n/6$ components, and let G have μ minimal separators that are consistent with A, B , and C . Furthermore, let $\mathcal{X}$ be the set that contains all components, X , of $G - (A \cup B \cup C)$ such that there exists at least two minimal separators S_X and S'_X that are consistent with A, B , and C and $S_X \cap X \neq S'_X \cap X$ (S_X and S'_X may depend on the component X). If $|\mathcal{X}| < \delta$ then there is an induced minor G' of G with at least μ minimal separators and G' has at most $n/2$ vertices.*

Proof. Let G be a graph with n vertices, let $\delta > 1$, let $A, B, C \subseteq V(G)$, $A, B \neq \emptyset$, such that no component of $G - (A \cup B \cup C)$ has over $n/3\delta$ vertices and $G[A \cup B]$ has at most $n/6$ components, and let G have μ minimal separators that are consistent with A, B , and C . Furthermore, let $\mathcal{X}$ be the set that contains all components, X , of $G - (A \cup B \cup C)$ such that there exists at least two minimal separators S_X and S'_X that are consistent with A, B , and C and $S_X \cap X \neq S'_X \cap X$, and assume that $|\mathcal{X}| < \delta$.

Let $\hat{X} = \bigcup_{X \in \mathcal{X}} X$. We first show that we can find sets $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$ such that $\hat{X} = V(G) - (A' \cup B' \cup C')$ and all minimal separators of G that are consistent with A, B , and C are consistent with A', B' , and C' . Let $\mathcal{Y} = \mathcal{CC}(G - (A \cup B \cup C)) - \mathcal{X}$ and let $\hat{Y} = \bigcup_{Y \in \mathcal{Y}} Y$. It follows by assumption that for any two minimal separators S and S' that are consistent with A, B , and C we have that $S \cap \hat{Y} = S' \cap \hat{Y}$. Let A^* denote the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has at least one neighbor in A , let B^* denote the set of all vertices that belong to a component of $G[\hat{Y} - S] = G[\hat{Y} - S']$ that has at least one neighbor in B , and let $C^* = \hat{Y} - (A^* \cup B^*)$. Then by Lemma 4.37 and the fact that $G[\hat{Y} - S] = G[\hat{Y} - S']$, the μ minimal separators of G that are consistent with A, B , and C are consistent with $A' = A \cup A^*$, $B' = B \cup B^*$, and $C' = C \cup C^*$. Additionally, $\hat{Y} = (A^* \cup B^* \cup C^*)$, so we have that $\hat{X} = G - (A' \cup B' \cup C')$, as desired.

Now, by Lemma 4.1 $G - C'$ has μ minimal separators that are consistent with A', B' , and $\emptyset$ and by Lemma 4.2, if G' is the graph that results from contracting each component of $G[A]$ and $G[B]$ in G , then G' has at least μ minimal separators.

Since $\hat{X} = G - (A' \cup B' \cup C')$, $|\hat{X}| \leq n/3$, and $G[A]$ and $G[B]$ together have at most $n/6$ components, it follows that G' has at most $n/3 + n/6 = n/2$ vertices. This completes the lemma. $\square$

The following lemma is a slight strengthening of Lemma 4.1 that we will require.

LEMMA 4.39. *Let G be a graph, let $A, B, C \subseteq V(G)$ with $A, B \neq \emptyset$, let T be a signature for G, A, B , and C , and let S be a minimal separator that agrees with T, A, B , and C . Then $S - C$ is a minimal separator of $G - C$ that agrees with $A, B, \emptyset$, and T .*

Proof. Let G be a graph, let $A, B, C \subseteq V(G)$ with $A, B \neq \emptyset$, let T be a signature for G, A, B , and C , and let S be a minimal separator that agrees with T, A, B , and C .

By Lemma 4.1 we have that $S - C$ is a minimal separator of $G - C$ that is consistent with A, B , and $\emptyset$. Let $G' = G - C$. Since $G - (A \cup B \cup C) = G' - (A \cup B)$ we have that for a component, X , of $G' - (A \cup B)$, the mark of S with respect to X, A , and B in G is $T(X)$. Since $X \cap C = \emptyset$ the mark of $S - C$ with respect to X, A , and B is also $T(X)$. $\square$

The following lemma comes from [18], where the authors prove a tighter bound, for our purposes the following bound is easier to use and sufficient.

LEMMA 4.40. ([18]) *Every graph G on n vertices has at most 2^n minimal separators.*

LEMMA 4.41. *Let G be a k -creature free graph with n vertices, let $\omega \geq 1$, let $\delta = 3\omega$, let c be a natural number large enough to satisfy the inequality $400k^3\delta^2 \log^4(c) < c/6$, let $x = 400k^3\delta^2 \log^4(n)$, and let G have at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators. Then there exists an induced minor G' of G , $A, B \subseteq V(G')$, and a signature T for G', A, B , and $C = \emptyset$, such that the following properties hold:*

- (i) There are at least ω components of $G' - (A \cup B)$.
- (ii) For all components, X , of $G' - (A \cup B)$ there are two minimal separators S_X and S'_X , $S_X \cap X \neq S'_X \cap X$ (S_X and S'_X may depend on the choice of X), that agree with A , B , $C = \emptyset$, and T .
- (iii) All components of $G' - (A \cup B)$ are good with respect to A and B .

Proof. Let $\omega > 1$, let $\delta = 3\omega$, and let c be a natural number large enough to satisfy the inequality $400k^3\delta^2 \log^4(c) < c/6$. Note that for all $c' > c$ that $400k^3\delta^2 \log^4(c') < c'/6$ also holds. We first show, by a proof by contradiction, that for any k -creature free graph G with n vertices and at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators, where $x = 400k^3\delta^2 \log^4(n)$, that there exists an induced minor G' of G , sets $A, B, C \subseteq V(G')$, and signature T for G' , A, B , and C such that (1) there are at least ω components, X , of $G' - (A \cup B \cup C)$ such that there are minimal separators S_X and S'_X that agree with A, B, C , and T (the minimal separators may depend on X) where $S_X \cap X \neq S'_X \cap X$, and (2) all components of $G' - (A \cup B \cup C)$ are good with respect to A, B , and C .

So, assume for a contradiction, that G is a k -creature free graph with n vertices and at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators where G is chosen with as few vertices as possible so that no induced minor G' of G and sets $A, B, C \subseteq V(G')$, and signature T for G' , A, B , and C satisfy (1) and (2). Since G has at least 2^c minimal separators, by Lemma 4.40 we have that $n \geq c$, hence n is large enough to satisfy the inequality $400k^3\delta^2 \log^4(n) < n/6$.

We now apply Lemma 4.35 to G and $\delta = 3\omega$ to get sets $A, B, C \subseteq V(G)$ and signature T for G , A, B , and C that satisfy properties (i) – (iv) of Lemma 4.35. Let $\mathcal{X}$ be the set of components, X , of $G - (A \cup B \cup C)$ such that there exists at least two minimal separators S and S' that agree with A, B, C , and T and $S \cap X \neq S' \cap X$. Since (2) holds by property (iv) of Lemma 4.35 it must be that property (1) fails, hence $|\mathcal{X}| < \omega$. So, since $400k^3\delta^2 \log^4(n) < n/6$, by property (iii) of Lemma 4.35 $G[A \cup B]$ has less than $n/6$ vertices so we may apply Lemma 4.38 to find an induced minor G' of G which has $n' \leq n/2$ vertices at and least

$$\frac{2^c(12n)^{6k^2x^4 \log(n)}}{(12n)^{6k^2x^4}} = 2^c(12n)^{6k^2x^4(\log(n)-1)} \geq 2^c(12n')^{6k^2x^4 \log(n')}$$

minimal separators (because any minimal separator that agrees with T, A, B , and C is consistent with A, B , and C). Since G was chosen as small as possible so that no induced minor of G satisfies (1) and (2), $|V(G')| < |V(G)|$, and G' has at least $2^c(12n')^{6k^2x^4 \log(n')}$ minimal separators, it follows that there must be an induced minor of G' that satisfies (1) and (2). But then this implies that there is an induced minor of G that satisfied (1) and (2), a contradiction.

We may then assume that G has an induced minor G' and sets $A, B, C \subseteq V(G')$, and signature T for G' , A, B , and C such that (1) and (2) hold. Again, let $\mathcal{X}$ be the set of components, X , of $G' - (A \cup B \cup C)$ such that there exists at least two minimal separators S and S' of G' that agree with A, B, C , and T and $S \cap X \neq S' \cap X$. By assumption $|\mathcal{X}| \geq \omega$. Let $\hat{Y}$ denote the set of vertices that belong to a component, Y , of $G' - (A \cup B \cup C)$ such that $Y \notin \mathcal{X}$, hence for all minimal separators S_Y and S'_Y that agree with A, B, C , and T it holds that $S_Y \cap Y = S'_Y \cap Y$. It follows that we may apply Lemma 4.37 to show that there exists a partition of $\hat{Y}$ into sets A^*, B^* , and C^* and a signature T' for G' , $A' = A \cup A^*, B' = B \cup B^*$, and $C' = C \cup C^*$ such that if S agrees with A, B, C , and T , then S agrees with A', B', C' , and T' . Let $G'' = G' - C'$. We now show that G'', A', B' , and T' satisfies properties (i) – (iii) of this lemma.

First, we make two needed observations which follow from how we defined $\hat{Y}$ and the fact that A^*, B^* , and C^* are a partition of $\hat{Y}$. The first is that $\mathcal{X} = \mathcal{CC}(G' - (A' \cup B' \cup C')) = \mathcal{CC}(G'' - (A' \cup B'))$. The second is that for all components, X , of $G'' - (A' \cup B')$ it holds that $N_{G''}^2[A' \cup B'] \cap X = N_{G''}^2[A \cup B] \cap X = N_{G'}^2[A \cup B] \cap X$ (the first equality holds because $N_{G'}(A^*)$ and $N_{G'}(B^*)$ are contained in $A \cup B \cup C$, and $C' \subseteq C$ hence $N_{G'}(A^*)$ and $N_{G'}(B^*)$ are contained in $A \cup B$. The second equality holds because X is a component of both $G'' - (A \cup B)$ and of $G' - (A \cup B)$).

We now verify properties (i)–(iii). Property (i) holds since by assumption, $|\mathcal{X}| \geq \omega$.

Next, we establish property (ii). Let X be a component of $G'' - (A' \cup B')$, so $X \in \mathcal{X}$. It follows that in G' there are two minimal separators, S_X and S'_X where $S_X \cap X \neq S'_X \cap X$ and S_X and S'_X agree with A, B, C , and T and therefore S_X and S'_X agree with A', B', C' , and T' . By Lemma 4.39 $S_X - C'$ and $S'_X - C'$ are minimal

separators of G'' that agree with A' , B' , $\emptyset$, and T' . Since $X \subseteq V(G'')$ we have that $C' \cap X = \emptyset$ and it follows that $(S_X - C') \cap X \neq (S'_X - C') \cap X$. This established property (ii).

Lastly, we show property (iii) holds. Let X be a component of $G'' - (A' \cup B')$, so X is a component of $G' - (A \cup B \cup C)$ and therefore X is good with respect to A , B , and C in G' . Since $N_{G''}^2[A' \cup B'] \cap X = N_{G'}[A \cup B] \cap X$ it follows that X is good with respect to A' and B' in G'' . $\square$

LEMMA 4.42. *Let G be a graph with n vertices and let $A, B, C \subseteq V(G)$. If $S \subset V(G)$ is a minimal separator that is consistent with A , B , and C then for each component, X , of $G - (A \cup B \cup C)$, $S \cap X$ is a minimal A, B -separator of $G[A \cup B \cup X]$.*

Proof. Let G be a graph with n vertices and let $A, B, C \subseteq V(G)$, let S be a minimal separator of G that is consistent with A , B , and C , and let X be a component of $G - (A \cup B \cup C)$. If $S \cap X$ is not an A, B -separator of $G[A \cup B \cup X]$, then there is a path, P from A to B through $X - S$, hence there is a path from A to B in $G - S$, so which contradicts the assumption that S is consistent with A , B , and C , hence $S \cap X$ is an A, B -separator of $G[A \cup B \cup X]$.

Now let $s \in S \cap X$ and assume for a contradiction that $(S \cap X) - s$ is an A, B -separator of $G[A \cup B \cup X]$. Let A' and B' be the S -full components of $G - S$ that contain A and B respectively. Let P_A and P_B be shortest paths from A to s and B to s such that all internal vertices are contained in A' and B' respectively. It follows that all internal vertices of P_A and P_B belong to $G - (A \cup B \cup C \cup S)$. Therefore all internal vertices of P_A and P_B belong to $X - S$. It follows that $(S \cap X) - s$ is not an A, B -separator of $G[A \cup B \cup X]$. Hence $S \cap X$ is an A, B -minimal separator of $G[A \cup B \cup X]$. $\square$

We are now ready to prove Lemma 3.1.

Proof. [Proof of Lemma 3.1] Let G be a k -creature free graph with n vertices, let $\omega > 1$ and $\delta = 3\omega$, let c be an integer large enough so that $400k^3\delta^2 \log^4(c) < c/6$, let $x = 400k^3\delta^2 \log^4(n)$, and let G have at least $2^c(12n)^{6k^2x^4 \log(n)}$ minimal separators. We may then apply Lemma 4.41 to G and ω to get an induced minor G' of G sets $A, B \subseteq V(G')$ an signature T for G' , A , B , and $C = \emptyset$ that satisfy properties (i)-(iv) of Lemma 4.41. Let ω' denote the number of components of $G' - (A \cup B)$, so $\omega' \geq \omega$ by property (i) of Lemma 4.41. Let H be an ω' -bistar with central vertices c_A and c_B and exactly ω' peripheral vertices, and for each component X of $G' - (A \cup B)$, let v_x denote a unique peripheral vertex of H . Let φ be the function that maps the vertices of A to c_A , the vertices of B to c_B and the vertices of $X \in \mathcal{CC}(G' - (A \cup B))$ to v_x . Lastly, for each component X of $G' - (A \cup B)$, by property (ii) of Lemma 4.41 there are two minimal separators S_X, S'_X , $S_X \cap X \neq S'_X \cap X$ that agree with A , B , $C = \emptyset$ and T . Hence, by Lemma 4.36 there exists minimal separators S_1 and S_2 that agree with A , B , $C = \emptyset$, and T and for each component, X , of $G' - (A \cup B)$, it holds that $S_1 \cap X \neq S_2 \cap X$. We will show that $W = (G', H, \varphi, S_1, S_2)$ is a connected, good, full generalized ω -creature.

To see that property (i) of Definition 3.1 holds let $S_1^* = S_1$ and $S_2^* = S_2$ and let u be a peripheral vertex of H . By how S_1 and S_2 were defined $\varphi^{-1}(u) \cap S_1 \neq \varphi^{-1}(u) \cap S_2$ and by Lemma 4.42 they are A_φ , B_φ -minimal separators in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$. Note that this implies that W is a full generalized ω -creature.

Property (ii) of Definition 3.1 follows from the fact that S_1 and S_2 are minimal separators that agree with A_φ , B_φ , and $C = \emptyset$.

That property (iii) of Definition 3.1 holds follows directly from the fact that S_1 and S_2 are both agree with A , B , $C = \emptyset$, and T , in particular, for every component X of $G' - (A \cup B) = G' - (A_\varphi \cup B_\varphi - \varphi)$, the mark of S_1 and S_2 with respect to X , A_φ and B_φ is $T(X)$.

Next, we show that property (iv) of Definition 3.1 is satisfied. If there was a peripheral vertex, u , such that $\varphi^{-1}(u)$ did not have neighbors in both $A_\varphi = A$ and $B_\varphi = B$, then we can see that no minimal separator that is consistent with A , B , and $C = \emptyset$ would contain a vertex from $\varphi^{-1}(u)$. But S_1 and S_2 are minimal separators that are consistent with A , B , and $C = \emptyset$ and for every peripheral vertex, u , of H , we have by property (i) that $\varphi^{-1}(u) \cap S_1 \neq \varphi^{-1}(u) \cap S_2$, therefore $\varphi^{-1}(u) \cap S_1$ and $\varphi^{-1}(u) \cap S_2$ are not both empty sets. It follows that $\varphi^{-1}(u)$ must have neighbors in both A_φ and B_φ .

Lastly, that for every peripheral vertex $u \in H$, $\varphi^{-1}(u)$ is a connected vertex set follows directly from how we defined φ , hence W is connected. The fact that each component of $G' - (A_\varphi \cup B_\varphi) = G' - (A \cup B)$ is good follows from property (iii) of Lemma 4.41, hence W is good. That W is full was observed when proving property (i).

It now follows that W is a connected, good, full generalized ω -creature. $\square$

5 Extracting Critters from Generalized Creatures

5.1 Generalized ω -Creatures and Their Properties We have already seen the definition of generalized ω -creatures, and three properties that they might or might not have; being *full*, *connected* and *good*. In this subsection we introduce two more properties of generalized ω -creatures, being *disjoint* and *adhesion size* α for some integer $\alpha > 0$. We will also prove some basic properties of generalized ω -creatures that will be used in order to extract a sufficiently large critter from them.

DEFINITION 5.1. (*Adhesion Size*) Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. For each peripheral vertex u of H the *adhesion size* of u is the number of distinct connected components of $G[A_\varphi \cup B_\varphi]$ containing at least one neighbor of $\varphi^{-1}(u)$. The *adhesion size* of the generalized ω -creature $(G, H, \varphi, S_1, S_2)$ is the maximum adhesion size of its peripheral vertices.

DEFINITION 5.2. (*Disjoint Generalized ω -Creatures*) Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. If $S_1 \cap S_2 = \emptyset$ then W is called a *disjoint generalized ω -creature*.

The definition of generalized ω -creatures would lead a reader to have some expectation of what a “typical” generalized ω -creature would look like, and also imagine some strange corner cases that could occur. Over the next few lemmas we rule out some corner cases; e.g. we show (lemmas that imply) that in a generalized ω -creature A_φ and B_φ are non-empty and that every peripheral vertex u of H satisfies that there exists a path from A_φ to B_φ through $\varphi^{-1}(u)$.

OBSERVATION 5.1. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. Then, S_1 and S_2 are disjoint from A_φ and B_φ .

Proof. The statement follows directly from property (ii) of generalized ω -creatures. $\square$

LEMMA 5.1. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. Then, for every peripheral vertex $u \in H$, $\varphi^{-1}(u)$ has a neighbor in A_φ and a neighbor in B_φ .

Proof. By property (i) exists $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ such that $\varphi^{-1}(u) \cap S_1^*$ and $\varphi^{-1}(u) \cap S_2^*$ are distinct A_φ, B_φ -minimal separators in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$. Thus both $\varphi^{-1}(u) \cap S_1^*$ and $\varphi^{-1}(u) \cap S_2^*$ are non-empty. Since $\varphi^{-1}(u) \cap S_1^* \neq \emptyset$, minimality of $\varphi^{-1}(u) \cap S_1^*$ implies that there is a path from A_φ to B_φ in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$. Let P be a shortest such path, then the first vertex of P is in A_φ , the last is in B_φ , and all internal vertices of P are in $\varphi^{-1}(u)$. Since S_1^* is disjoint from $A_\varphi \cup B_\varphi$ it follows that P has at least one internal vertex. Thus the first and last vertices of P are neighbors of $\varphi^{-1}(u)$ in A_φ and B_φ respectively. $\square$

OBSERVATION 5.2. For every generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$, peripheral vertex u of H , and component C of $G[\varphi^{-1}(u)] - (S_1 \cup S_2)$, either $N(C) \cap A_\varphi$ is empty or $N(C) \cap B_\varphi$ is empty.

Proof. Suppose that $N(C) \cap A_\varphi \neq \emptyset$ and $N(C) \cap B_\varphi \neq \emptyset$ for some component C of $G[\varphi^{-1}(u)] - (S_1 \cup S_2)$. This contradicts property (i) of generalized ω -creatures. $\square$

LEMMA 5.2. Let G be a graph and (H, φ) be an ω -bistar partition of G . Let X be a vertex set in G disjoint from $A_\varphi \cup B_\varphi$. Then X is a A_φ - B_φ -separator in G if and only if $X \cap \varphi^{-1}(u)$ is a A_φ - B_φ -separator in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$ for every peripheral vertex u of H .

Proof. We prove instead the equivalence of the negations: that there is a path from A_φ to B_φ in $G - X$ if and only if there exists a peripheral vertex u of H and a path from A_φ to B_φ in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)] - X$.

The backward direction is trivial. For the forward direction, let P be a shortest path from A_φ to B_φ in $G - X$. Then none of the internal vertices of P lie in $A_\varphi \cup B_\varphi$. Since $N_G(\varphi^{-1}(u)) \subseteq A_\varphi \cup B_\varphi$ for every peripheral vertex u , it follows that there exists a peripheral vertex u such that $V(P) \subseteq A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)$. This concludes the proof. $\square$

DEFINITION 5.3. (FLIPPING) Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature, and v be a peripheral vertex of H . Flipping W at v results in the tuple $W' = (G, H, \varphi, S'_1, S'_2)$ where

$$S'_1 = (S_1 - \varphi^{-1}(v)) \cup (S_2 \cap \varphi^{-1}(v)) \text{ and } S'_2 = (S_2 - \varphi^{-1}(v)) \cup (S_1 \cap \varphi^{-1}(v)).$$

Let S_1^* and S_2^* be witness separators for W . Flipping S_1^* and S_2^* at v results in

$$S_1'^* = (S_1^* - \varphi^{-1}(v)) \cup (S_2^* \cap \varphi^{-1}(v)) \text{ and } S_2'^* = (S_2^* - \varphi^{-1}(v)) \cup (S_1^* \cap \varphi^{-1}(v)).$$

LEMMA 5.3. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature, v be a peripheral vertex of H , and $W' = (G, H, \varphi, S_1', S_2')$ be the result of flipping W at v . Then: Furthermore,

- W' is a generalized ω -creature.
- If S_1^* and S_2^* are witness separators for W and $S_1'^*$ and $S_2'^*$ are the result of flipping S_1^* and S_2^* at v , then $S_1'^*$ and $S_2'^*$ are witness separators for W' .
- If W is a disjoint generalized ω -creature then W' is a disjoint generalized ω -creature.

Proof. Let $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ be witness separators for W and $S_1'^*$ and $S_2'^*$ be the result of flipping S_1^* and S_2^* at v .

We observe that for every peripheral vertex u of H we have that

$$\{\varphi^{-1}(u) \cap S_1, \varphi^{-1}(u) \cap S_2\} = \{\varphi^{-1}(u) \cap S_1', \varphi^{-1}(u) \cap S_2'\}$$

and that

$$\{\varphi^{-1}(u) \cap S_1^*, \varphi^{-1}(u) \cap S_2^*\} = \{\varphi^{-1}(u) \cap S_1'^*, \varphi^{-1}(u) \cap S_2'^*\}.$$

From this it immediately follows that $S_1'^*$ and $S_2'^*$ satisfy property (i) for W' , and that W' additionally satisfies properties (iii), and (iv). For property (ii), Lemma 5.2 implies that S_1' and S_2' are both A_φ, B_φ -separators.

We argue that all of A_φ is in the same connected component of $G - S_1'$. By assumption all of A_φ is in the same connected component of $G - S_1$. Consider an arbitrary pair a, a' of vertices in A . Since all of A_φ is in the same connected component of $G - S_1$ there exists a sequence $a_1, a_2, \dots, a_t$ of vertices in A such that $a_1 = a$, $a_t = a'$, and for every pair a_i, a_{i+1} of consecutive vertices in the sequence it holds that $a_i a_{i+1}$ is an edge or there exists a peripheral vertex u of H such that there is a path from a_i to a_{i+1} through $\varphi^{-1}(u) - S_1$. In the second case, by property (iii) it holds that $a_i a_{i+1}$ there is a path from a_i to a_{i+1} through $\varphi^{-1}(u) - S_2$. Thus, since $\varphi^{-1}(u) \cap S_1' \in \{\varphi^{-1}(u) \cap S_1, \varphi^{-1}(u) \cap S_2\}$ we have that there is a path from a_i to a_{i+1} through $\varphi^{-1}(u) - S_1'$. It follows that a and a' are in the same component of $G - S_1'$. But then all of A_φ is in the same component of $G - S_1'$. Identical proofs show that A_φ is in the same component of $G - S_2'$, and that B_φ is in one component of $G - S_1'$ and of $G - S_2'$. Hence W' satisfies property (ii) and is a generalized ω -creature.

Finally we show that if W is a disjoint generalized ω -creature then W' is a disjoint generalized ω -creature. Suppose that W is a disjoint generalized ω -creature. We then have that

$$S_1' \cap S_2' = ((S_1 - \varphi^{-1}(v)) \cup (S_2 \cap \varphi^{-1}(v))) \cap ((S_2 - \varphi^{-1}(v)) \cup (S_1 \cap \varphi^{-1}(v)))$$

But $(S_1 - \varphi^{-1}(v))$ is disjoint with $(S_2 - \varphi^{-1}(v))$ and $(S_2 \cap \varphi^{-1}(v))$ is disjoint with $(S_1 \cap \varphi^{-1}(v))$ because S_1 is disjoint with S_2 , and $(S_1 - \varphi^{-1}(v))$ is disjoint with $(S_1 \cap \varphi^{-1}(v))$ and $(S_2 \cap \varphi^{-1}(v))$ is disjoint with $(S_2 - \varphi^{-1}(v))$ because $\varphi^{-1}(v)$ is disjoint from its complement. So S_1' and S_2' are disjoint and hence W' is a disjoint generalized ω -creature. $\square$

5.2 Properties of Good Connected Generalized ω -Creatures Let $(G, H, \varphi, S_1, S_2)$ be a good connected generalized ω -creature and let X be a component of $G - (A_\varphi \cup B_\varphi)$. Since X is good with respect to A and B we have that X has at most one non-leaf sub-component. We now show that X has precisely one non-leaf sub-component.

LEMMA 5.4. Let $W = (G, H, \varphi, S_1, S_2)$ be a good connected generalized ω -creature and let X be a component of $G - (A_\varphi \cup B_\varphi)$. Then X has precisely one non-leaf sub-component Y .

Proof. By definition of good, X has at most one non-leaf sub-component. By Lemma 5.1 X has a neighbor in A_φ and a neighbor in B_φ . Let $X_A = N_G^2[A_\varphi] \cap X$ and $X_B = N_G^2[B_\varphi] \cap X$. We have that X_A and X_B are non-empty, and since W is good it follows that X_A and X_B are anti-complete. Since W is connected there exists a path P from X_A to X_B through $X - (X_A \cup X_B)$. Let P' be the sub-path of P obtained by removing the endpoints of P . Then P' is a connected set in $X - N_G^2[A_\varphi \cup B_\varphi]$ with neighbors in two distinct components of $G[X \cap N_G^2[A_\varphi \cup B_\varphi]]$. Thus P' is contained in a non-leaf sub-component of X . $\square$

In light of Lemma 5.4 we can give the unique non-leaf sub-component of X a name.

DEFINITION 5.4. Let $(G, H, \varphi, S_1, S_2)$ be a good connected generalized ω -creature and let X be a component of $G - (A_\varphi \cup B_\varphi)$. Then the kernel of X is the unique non-leaf sub-component of X .

LEMMA 5.5. Let $(G, H, \varphi, S_1, S_2)$ be a good generalized ω -creature, and X be a component of $G - (A_\varphi \cup B_\varphi)$. Then, for each connected component C of $G[X \cap N_G^2[A_\varphi \cup B_\varphi]]$ there is precisely one component Y in $G[A_\varphi \cup B_\varphi]$ such that $N_G[X] \cap Y \neq \emptyset$.

Proof. We first argue that there exist at least one component Y in $G[A_\varphi \cup B_\varphi]$ such that $N_G[X] \cap Y \neq \emptyset$. Let $x \in X$. Since $x \notin (A_\varphi \cup B_\varphi)$ we have that $x \in N_G(A_\varphi \cup B_\varphi)$ or $x \in N_G^2(A_\varphi \cup B_\varphi)$. If $x \in N_G^2(A_\varphi \cup B_\varphi)$ then x has a neighbor y in $N_G(A_\varphi \cup B_\varphi)$, and $y \in X$. We may then choose y as x instead, and therefore, without loss of generality $x \in N_G(A_\varphi \cup B_\varphi)$. x has a neighbor in $A_\varphi \cup B_\varphi$ establishing the existence of at least one component Y in $G[A_\varphi \cup B_\varphi]$ such that $N_G[X] \cap Y \neq \emptyset$.

We now prove that Y is unique. Suppose for contradiction that there exists a component $Y' \neq Y$ of $G[A_\varphi \cup B_\varphi]$ such that $N_G[X] \cap Y' \neq \emptyset$. Let P be a path from $N_G(Y) \cap X$ to $N_G(Y') \cap X$. Let q be the first vertex on P such that $q \in N_G^2[Y']$ for some component $Y'' \neq Y$ of $G[A_\varphi \cup B_\varphi]$. If q is the first vertex of P then $q \in N_G(Y)$ contradicting that $N_G^2[Y] \cap N_G^2[Y'] \cap X$ is empty (because the generalized ω -creature is good). Otherwise, let p be the predecessor of q on P . We have that $p \in N_G^2[Y]$ because $p \in N_G^2[A_\varphi \cup B_\varphi]$ and q is the first on P such that $q \in N_G^2[Y']$ for some component $Y'' \neq Y$ of $G[A_\varphi \cup B_\varphi]$. But then p and q contradict that there is no edge from $N_G^2[Y] \cap X$ to $N_G^2[Y'] \cap X$ (which should have been true, because the generalized ω -creature is good). $\square$

LEMMA 5.6. Let $(G, H, \varphi, S_1, S_2)$ be a good generalized ω -creature and X be a component of $G - (A_\varphi \cup B_\varphi)$. For every pair C_1, C_2 of distinct components of $G[A_\varphi \cup B_\varphi]$ and path P from C_1 to C_2 through X , P contains an internal vertex in the kernel of X .

Proof. Let s and t be the first and last vertex of P , respectively. Since $N_G^2[C_1]$ and $N_G^2[C_2]$ are disjoint and anti-complete, P has at least 7 vertices. Let s' and t' be the successor of s on P and the predecessor of t on P , respectively. Let x be the last vertex of P in $N_G^2[C_1] \cap X$. The vertex x is well defined because s' is in $N_G^2[C_1] \cap X$. Let y be the first vertex in $N_G^2[A_\varphi \cup B_\varphi - C_1] \cap X$ on the sub-path of P from x to t . Since $t' \in N_G^2[C_2] \cap X$, y is well defined. Since $N_G^2[C_1] \cap X$ and $N_G^2[A_\varphi \cup B_\varphi - C_1] \cap X$ are anti-complete we have $y \neq x$ and xy is not an edge of G . Let P' be the subpath of P from x to y . Since xy is not an edge, P' contains at least one internal vertex. Furthermore P' is a path from $N_G^2[C_1] \cap X$ to $N_G^2[A_\varphi \cup B_\varphi - C_1] \cap X$ through $X - N_G^2[A_\varphi \cup B_\varphi]$. Thus all the internal vertices of P' are contained in a non-leaf sub-component of X , namely the kernel of X , as claimed. $\square$

LEMMA 5.7. Let G be a k -creature free graph, $(G, H, \varphi, S_1, S_2)$ be a good generalized ω -creature, and X be a component of $G - (A_\varphi \cup B_\varphi)$. Let C be the kernel of X . Then each connected component Y of $G[X \cap N_G^2[A_\varphi \cup B_\varphi]]$ has at least one neighbor in C .

Proof. Since $G[X]$ is connected there exists a path in $G[X]$ that starts in Y and ends in C . Let P be a shortest such path. In particular, only the first vertex of P is in Y and only the last is in C . If P has no internal vertices then the first and last vertex of P are adjacent, proving the statement of the lemma. We now show that P has no internal vertices.

Suppose for contradiction that P has an internal vertex, let a be the endpoint of P which is in Y , and let q be the internal vertex in P which is adjacent to a . Since $q \in X - Y$ and Y is a connected component of $G[X \cap N_G^2[A_\varphi \cup B_\varphi]]$ it follows that $q \notin N_G^2[A_\varphi \cup B_\varphi]$. Then q is in a connected component Z of $G[X - N_G^2[A_\varphi \cup B_\varphi]]$. Since $q \notin C$ we have that $Z \neq C$ and so Z is not the kernel of X . But then $N_G(Z) \subseteq Y$ and the first vertex on P outside of Z is in Y , contradicting that only the first vertex of P is in Y . We conclude that P has no internal vertices, and this shows the statement of the lemma. $\square$

5.3 Dissolving a Peripheral Vertex

DEFINITION 5.5. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature, and let u be a peripheral vertex of H . Dissolving u in H produces a tuple $W' = (G', H', \varphi', S'_1, S'_2)$ where:

- $G' = G - (\varphi^{-1}(u) - (A_1(W) \cup B_1(W)))$,
- $H' = H - u$.
- For all $v \in (V(G'))$,

$$\varphi'(v) = \begin{cases} c_A & \text{if } v \in \varphi^{-1}(u) \cap A_1(W) \\ c_B & \text{if } v \in \varphi^{-1}(u) \cap B_1(W) \\ \varphi'(v) & \text{otherwise} \end{cases}$$

- $S'_1 = S_1 - \varphi^{-1}(u)$ and $S'_2 = S_2 - \varphi^{-1}(u)$.

LEMMA 5.8. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature and let u be a peripheral vertex of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of dissolving u in W . Then H' is a $(w-1)$ -bistar partition of G' . Furthermore, for every peripheral vertex v of H' it holds that

- $N_{G'}[\varphi'^{-1}(v)] = N_G[\varphi^{-1}(v)]$,
- $N_{G'}[\varphi'^{-1}(v)] \cap A_{\varphi'} = N_G[\varphi^{-1}(v)] \cap A_{\varphi}$,
- and $N_{G'}[\varphi'^{-1}(v)] \cap B_{\varphi'} = N_G[\varphi^{-1}(v)] \cap B_{\varphi}$.

Proof. We first show that H' is a $(w-1)$ -bistar partition of G' . Let $xy \in E(G')$. If neither $\varphi(x)$ nor $\varphi(y)$ is equal to u then $\varphi'(x) = \varphi(x)$ and $\varphi'(y) = \varphi(y)$ so $\varphi'(x) = \varphi'(y)$ or $\varphi'(x)\varphi'(y) \in E(H')$. If both $\varphi(x)$ and $\varphi(y)$ are equal to u then $\{x, y\} \subseteq (A_1(W) \cup B_1(W)) \cap \varphi^{-1}(u)$. But $A_1(W)$ and $B_1(W)$ are disjoint and anticomplete so $\{x, y\} \subseteq A_1(W) \cap \varphi^{-1}(u)$ or $\{x, y\} \subseteq B_1(W) \cap \varphi^{-1}(u)$. In the first case $\varphi'(x) = \varphi'(y) = c_A$, in the second $\varphi'(x) = \varphi'(y) = c_B$. If $\varphi(x) = u$ and $\varphi(y) \neq u$ then $\varphi(y) \in \{c_A, c_B\}$. If $\varphi(y) = c_A$ then $y \in A_{\varphi}$ and hence, since $xy \in E(G)$, $x \in A_1(W)$. But then $\varphi'(x) = \varphi'(y) = c_A$. If $\varphi(y) = c_B$ then $y \in B_{\varphi}$ and hence, since $xy \in E(G)$, $x \in B_1(W)$. But then $\varphi'(x) = \varphi'(y) = c_B$. Finally, $V(H') = V(H) - \{u\}$. Thus we conclude that H' is a $(w-1)$ -bistar partition of G' .

For the second part of the statement, We have that $\varphi'^{-1}(v) = \varphi^{-1}(v)$ and that $\varphi^{-1}(u) = \varphi^{-1}(v)$ are disjoint and anticomplete. Furthermore, since (H', φ') is an $(w-1)$ -bistar partition of G' we have that $N_{G'}(\varphi'^{-1}(v)) \subseteq A_{\varphi'} \cup B_{\varphi'}$. Since $A_{\varphi'} \subseteq A_{\varphi} \cup \varphi^{-1}(u)$ and $B_{\varphi'} \subseteq B_{\varphi} \cup \varphi^{-1}(u)$ it follows that $N_{G'}(\varphi'^{-1}(v)) \subseteq A_{\varphi} \cup B_{\varphi}$. But then $N_{G'}[\varphi'^{-1}(v)] = N_G[\varphi^{-1}(v)]$, $N_{G'}[\varphi'^{-1}(v)] \cap A_{\varphi'} = N_G[\varphi^{-1}(v)] \cap A_{\varphi}$, and $N_{G'}[\varphi'^{-1}(v)] \cap B_{\varphi'} = N_G[\varphi^{-1}(v)] \cap B_{\varphi}$. This completes the proof. $\square$

LEMMA 5.9. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature and let u be a peripheral vertex of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of dissolving u in W . Then

- W' is a generalized $(\omega-1)$ -creature. Furthermore,
- if W is a full generalized ω -creature then W' is a full generalized $(\omega-1)$ -creature,
- if W is a good generalized ω -creature then W' is a good generalized $(\omega-1)$ -creature,
- if W is a disjoint generalized ω -creature then W' is a disjoint generalized $(\omega-1)$ -creature,
- if W is a good connected generalized ω -creature then W' is a good connected generalized $(\omega-1)$ -creature,
- if W has adhesion size α then W' has adhesion size α . For every peripheral vertex v of H' its adhesion size in W' is at most its adhesion size in W .

Proof. By Lemma 5.8 we have that H' is a $(w-1)$ -bistar partition of G' . We now check the properties of generalized $(\omega-1)$ creatures for W' .

- For property (i) let S_1^*, S_2^* be witness separators for W , and set $S_1'^* = S_1^* - \varphi^{-1}(u)$ and $S_2'^* = S_2^* - \varphi^{-1}(u)$. We claim that $S_1'^*$ and $S_2'^*$ are witness separators for W' . Let v' be a peripheral vertex of H' . Let $Z = N_G[\varphi^{-1}(v)]$. By Lemma 5.8 we have $Z = N_{G'}[\varphi'^{-1}(v)]$. Since S_1^* and S_2^* are witness separators for W it follows that $S_1^* \cap \varphi^{-1}(v)$ and $S_2^* \cap \varphi^{-1}(v)$ are distinct minimal A_{φ}, B_{φ} -separators in $G[A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(v)]$. Then $S_1^* \cap \varphi^{-1}(v)$ and $S_2^* \cap \varphi^{-1}(v)$ are distinct minimal $A_{\varphi} \cap Z, B_{\varphi} \cap Z$ -separators in $G[(A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(v)) \cap Z]$. Since $Z = N_{G'}[\varphi'^{-1}(v)]$ and $S_1'^* \cap Z = S_1^* \cap Z$ and $S_2'^* \cap Z = S_2^* \cap Z$ it follows that $S_1'^* \cap \varphi'^{-1}(v)$ and $S_2'^* \cap \varphi'^{-1}(v)$ are distinct minimal $A_{\varphi'} \cap Z, B_{\varphi'} \cap Z$ -separators in $G'[(A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)) \cap Z]$. Since every path from $A_{\varphi'}$ to $B_{\varphi'}$ through $\varphi'^{-1}(v)$ is a path from $A_{\varphi'} \cap Z$ to $B_{\varphi'} \cap Z$ through $\varphi'^{-1}(v) \cap Z$ it follows that $S_1'^* \cap \varphi'^{-1}(v)$ and $S_2'^* \cap \varphi'^{-1}(v)$ are distinct minimal $A_{\varphi'}, B_{\varphi'}$ -separators in $G'[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v) \cap Z]$.

- For property (ii) we observe that

$$A_1(W') \cap \varphi^{-1}(u) = A_{\varphi'} \cap \varphi^{-1}(u) = A_1(W) \cap \varphi^{-1}(u),$$

$$B_1(W') \cap \varphi^{-1}(u) = B_{\varphi'} \cap \varphi^{-1}(u) = B_1(W) \cap \varphi^{-1}(u),$$

$A_{\varphi'} - \varphi^{-1}(u) = A_{\varphi} - \varphi^{-1}(u)$, $B_{\varphi'} - \varphi^{-1}(u) = B_{\varphi} - \varphi^{-1}(u)$, and $G' - \varphi^{-1}(u) = G - \varphi^{-1}(u)$. Thus $A_1(W') = A_1(W)$ and $B_1(W') = B_1(W)$. Since $A_{\varphi'} \subseteq A_1(W)$ and $B_{\varphi'} \subseteq B_1(W)$ we conclude that W' satisfies property (ii).

- For property (iii) let C_1 and C_2 be components of $G[A'_{\varphi} \cup B'_{\varphi}]$ and let v be a peripheral vertex of H' . Let $Z = N'_G[\varphi^{-1}(v)]$, by Lemma 5.8 we have $Z = N_G[\varphi^{-1}(v)]$. Suppose there is a path P in G' from C_1 to C_2 through $\varphi'^{-1}(v) - S'_1$. Then P is a path in G' from $C_1 \cap Z$ to $C_2 \cap Z$ through $(\varphi'^{-1}(v) - S'_1) \cap Z$. Hence P is a path in G from $C_1 \cap Z$ to $C_2 \cap Z$ through $(\varphi^{-1}(v) - S'_1) \cap Z$. But $\varphi^{-1}(v) \cap S'_1 = \varphi^{-1}(v) \cap S_1$, so P is a path in G from C_1 to C_2 through $(\varphi^{-1}(v) - S_1)$.

By Property (iii) applied to W there exists a path P' from C_1 to C_2 through $(\varphi^{-1}(v) - S_2)$. Then P' is a path in G from $C_1 \cap Z$ to $C_2 \cap Z$ through $(\varphi^{-1}(v) - S_2) \cap Z$. Hence P' is a path in G' from $C_1 \cap Z$ to $C_2 \cap Z$ through $(\varphi'^{-1}(v) - S_2) \cap Z$. But $\varphi'^{-1}(v) \cap S_2 = \varphi'^{-1}(v) \cap S'_2$, so P' is a path in G' from C_1 to C_2 through $(\varphi'^{-1}(v) - S'_2)$.

The proof that if there is a path P in G' from C_1 to C_2 through $\varphi'^{-1}(v) - S'_2$ then there exists a path P' in G' from C_1 to C_2 through $\varphi'^{-1}(v) - S'_1$ is symmetric.

- For property (iv) consider a peripheral vertex v of H' and a component C_A of $G[A_{\varphi'}]$ that has a neighbor in $\varphi'^{-1}(v)$ in G' . Let x be a vertex in C_A that has a neighbor in $\varphi'^{-1}(v)$ in G' . Since $\varphi'^{-1}(v)$ and $\varphi^{-1}(u)$ are disjoint and anticomplete, and $x \in A_{\varphi} \cup \varphi^{-1}(u)$ it follows that $x \in A_{\varphi}$. Let C'_A be the component of $G[A_{\varphi}]$ that contains x . Since $\varphi^{-1}(v) = \varphi'^{-1}(v)$ the component C'_A has a neighbor in $\varphi^{-1}(v)$ in G . By Property (iv) applied to W there is a path P from C'_A to B_{φ} through $\varphi^{-1}(v)$.

Since $C'_A \subseteq A_{\varphi} \subseteq A_{\varphi'}$, $B_{\varphi} \subseteq B_{\varphi'}$, and $\varphi^{-1}(v) = \varphi'^{-1}(v)$, it follows that P is a path in G' from C'_A to $B_{\varphi'}$ through $\varphi'^{-1}(v)$. But $C'_A \subseteq A_{\varphi} \subseteq A_{\varphi'}$ implies that $C'_A \subseteq C_A$ so P is a path in G' from C_A to $B_{\varphi'}$ through $\varphi'^{-1}(v)$.

The proof that if a component C_B of $G[B_{\varphi'}]$ has a neighbor in $\varphi'^{-1}(v)$ in G' , then there exists a path in G' from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$ is symmetric.

Next we verify that whenever W is full, or good, or disjoint, or connected, or has adhesion size α then W' has the same property.

- First, suppose that W is full and let $S_1^* = S_1, S_2^* = S_2$ be witness separators for W . In the proof that W' had property (i) we showed that $S_1'^* = S_1^* - \varphi^{-1}(u)$ and $S_2'^* = S_2^* - \varphi^{-1}(u)$ are witness separators for W' . Then $S_1'^* = S_1 - \varphi^{-1}(u) = S'_1$ and $S_2'^* = S_2 - \varphi^{-1}(u) = S'_2$, so W' is full as well.
- Suppose now that W is disjoint. Then $S_1 \cap S_2 = \emptyset$. Since $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$, W' is also disjoint.
- If W is connected then, for every peripheral vertex v of H' , $G'[\varphi'^{-1}(v)] = G[\varphi^{-1}(v)]$ so W' is connected as well.
- For bounding the adhesion size of W' let v be a peripheral vertex of H' . By Lemma 5.8 we have that $N_{G'}(\varphi'^{-1}(v)) = N_G(\varphi^{-1}(v)) \subseteq A_{\varphi} \cup B_{\varphi}$. Since $A_{\varphi} \cup B_{\varphi} \subseteq A_{\varphi'} \cup B_{\varphi'}$, every connected component of $A_{\varphi} \cup B_{\varphi}$ is contained in some connected component of $A_{\varphi'} \cup B_{\varphi'}$. Hence the adhesion size of v in W' is at most v 's adhesion size in W . Therefore, if W has adhesion size α then the adhesion size of W' is at most α .
- If W is a good connected generalized ω -creature we have already shown that W' is connected. We now show that W' is also good. Let v be a peripheral vertex of H' and $X = \varphi'^{-1}(v)$. Since $X = \varphi'^{-1}(v) = \varphi^{-1}(v)$ and W is connected it follows that X is a component of $G - (A_{\varphi} \cup B_{\varphi})$. Since W' is connected X is a component of $G' - (A_{\varphi'} \cup B_{\varphi'})$. By Lemma 5.8 we have that $N_{G'}[X] = N_G[X]$, $N_{G'}X \cap A_{\varphi'} = N_G[X] \cap A_{\varphi}$, and

$N_{G'}[X] \cap B_{\varphi'} = N_G[X] \cap B_{\varphi}$. Thus $N_{G'}^2[A_{\varphi'} \cup B_{\varphi'}] \cap X = N_G^2[A_{\varphi} \cup B_{\varphi}] \cap X$. Hence every sub-component Y of X in G' with respect to $A_{\varphi'}$ and $B_{\varphi'}$ is also a sub-component of X in G with respect to A_{φ} and B_{φ} . Since W is good it follows that X has at most one non-leaf sub-component in G with respect to A_{φ} and B_{φ} . Thus X has at most one non-leaf sub-component in G' with respect to $A_{\varphi'}$ and $B_{\varphi'}$.

Suppose now for contradiction that there are two components C_1 and C_2 of $G'[A_{\varphi'} \cup B_{\varphi'}]$ such that $N_{G'}^2[C_1] \cap X$ is not anticomplete with $N_{G'}^2[C_2] \cap X$. Then there is a path P on at most 6 vertices from C_1 to C_2 through X . Let x be the first vertex of P and y be the last vertex of P . By Lemma 5.8 both x and y are in $A_{\varphi} \cup B_{\varphi}$. But then P is a path on at most 6 vertices from x to y through X in G . Since x and y are in different components of $G'[A_{\varphi'} \cup B_{\varphi'}]$, they are also in different components C'_1 and C'_2 of $G[A_{\varphi} \cup B_{\varphi}]$. But then P is a path on at most 6 vertices from C'_1 to C'_2 through X in G , contradicting that $N_G^2[C'_1] \cap X$ is anticomplete with $N_G^2[C'_2] \cap X$.

□

5.4 Absorbing a Component

DEFINITION 5.6. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature with $\omega \geq 2$. We say a component C of $G[A_{\varphi} \cup B_{\varphi}]$ is absorbable if there exists a peripheral vertex, u , of H where $N_G(C) \subseteq \varphi^{-1}(u)$.

Let C be an absorbable component of $G[A_{\varphi} \cup B_{\varphi}]$ and let $u \in H$ be the vertex such that $N_G(C) \subseteq \varphi^{-1}(u)$. Absorbing C in W produces a tuple $W' = (G, H, \varphi', S_1, S_2)$ where $\varphi'(x) = \varphi(x)$ for all $x \in G - C$ and $\varphi'(x) = v$ for all $x \in C$.

LEMMA 5.10. Let $\omega \geq 2$ and $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature, and C be an absorbable component of $G[A_{\varphi} \cup B_{\varphi}]$. Let $W' = (G, H, \varphi', S_1, S_2)$ be the result of absorbing C in W . Then

- W' is a generalized ω -creature. Furthermore,
- if W is a full generalized ω -creature then W' is a full generalized ω -creature,
- if W is a good generalized ω -creature then W' is a good generalized ω -creature,
- if W is a disjoint generalized ω -creature then W' is a disjoint generalized ω -creature,
- if W is a connected generalized ω -creature then W' is a connected generalized ω -creature,
- if W has adhesion size α then W' has adhesion size α . For every peripheral vertex v of H its adhesion size in W' is at most its adhesion size in W .

Proof. We prove the statement of the lemma for C being a component of $G[A_{\varphi}]$. We show that (H, φ') is an ω -bistar partition of G . Let $xy \in E(G)$. If x and y are both in C then $\varphi'(x) = \varphi'(y) = v$. If neither x nor y are in C then $\varphi'(x) = \varphi(x)$, $\varphi'(y) = \varphi(y)$ and therefore either $\varphi'(x)\varphi'(y)$ is an edge of H or $\varphi'(x) = \varphi'(y)$. If $x \in C$ and $y \notin C$ then $\varphi'(y) = \varphi(y) = v$ and $\varphi'(x) = v$. Hence (H, φ') is an ω -bistar partition of G .

Let u be the peripheral vertex of H such that $N_G(C) \subseteq \varphi^{-1}(u)$. Before proving that W' satisfies the properties of generalized ω -creatures we show that $A_{\varphi'}$ is non-empty. Since $\omega \geq 2$ there exists a peripheral vertex $v \neq u$ of H . By Lemma 5.1 v has a neighbor in A_{φ} . Since $v \neq u$ this neighbor is not in C , and hence $A_{\varphi'}$ is non-empty.

Let $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ be witness separators for W . We now proceed to verify that W' satisfies the properties of generalized ω -creatures.

- For property (i) let v be a peripheral vertex of H . Suppose first $v \neq u$. Then $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)] = G[A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(v)] - C$. Further, C is a connected component of $G[A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(v)]$ since $N_G(C) \subseteq \varphi^{-1}(u)$. So S_1^* and S_2^* are minimal $A_{\varphi'}, B_{\varphi'}$ -separators in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)]$.

We now consider the case that $v = u$. We have that S_1^* and S_2^* separate A_{φ} from B_{φ} . Additionally $A_{\varphi'} \subseteq A_{\varphi}$ and $B_{\varphi'} \subseteq B_{\varphi}$. So S_1^* and S_2^* are $A_{\varphi'}, B_{\varphi'}$ separators in G and hence $S_1^* \cap \varphi'^{-1}(v)$ and $S_2^* \cap \varphi'^{-1}(v)$ are $A_{\varphi'}, B_{\varphi'}$ -separators in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)]$.

We prove that $S_1^* \cap \varphi'^{-1}(v)$ is a minimal $A_{\varphi'}, B_{\varphi'}$ -separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)]$. Suppose for contradiction that a proper subset $\hat{S}_1$ of $S_1^* \cap \varphi'^{-1}(v)$ is also an $A_{\varphi'}, B_{\varphi'}$ -separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)]$. Then there exists a partition of $A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)$ into L , R , and $\hat{S}_1$ such that $A_{\varphi'} \subseteq L$, $B_{\varphi'} \subseteq R$, and there are

no edges from L to R . Since $G[C]$ is connected we have that $C \subseteq L$ or $C \subseteq R$. If $C \subseteq L$ then $\hat{S}_1$ separates $A_{\varphi'} \cup C = A_{\varphi}$ from $B_{\varphi'} = B_{\varphi}$, contradicting that $S_1^* \cap \varphi^{-1}(v)$ is a minimal A_{φ}, B_{φ} -separator in $G[A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(v)]$.

Thus $C \subseteq R$. However, since $A_{\varphi'}$ is non-empty and A_{φ} is contained in a connected component of $G - S_1$ (by property (ii) applied to W), there is a path P' from $A_{\varphi} - C = A_{\varphi'}$ to C in $G[A_1(W)]$. Since $A_1(W)$ is disjoint from B_{φ} and $N_G(C) \subseteq \varphi^{-1}(u) = \varphi^{-1}(v)$ the path P' is a path from $A_{\varphi'}$ to C through $\varphi^{-1}(v) - S_1$. Since $V(P')$ is disjoint from S_1 and $\hat{S}_1 \subseteq S_1$ it follows that all of P' must lie in R . But then the endpoint of P' in $A_{\varphi'}$ must be in R , contradicting that $A_{\varphi'} \subseteq L$. Hence $S_1^* \cap \varphi^{-1}(v)$ is a minimal $A_{\varphi'}, B_{\varphi'}$ -separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi^{-1}(v)]$.

The proof that $S_2^* \cap \varphi^{-1}(v)$ is a minimal $A_{\varphi'}, B_{\varphi'}$ -separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)]$ is symmetric.

- For property (ii) it is sufficient to observe that

$$A_{\varphi'} = A_{\varphi} - C \subseteq A_1(W) \text{ and } A_{\varphi'} = A_{\varphi} - C \subseteq A_2(W),$$

$$B_{\varphi'} = B_{\varphi} \subseteq B_1(W) \text{ and } B_{\varphi'} = B_{\varphi} \subseteq B_2(W).$$

Further $A_1(W)$ and $B_1(W)$ are components of $G - S_1$, while $A_2(W)$ and $B_2(W)$ are components of $G - S_2$.

- For property (iii) let v be a peripheral vertex of H , and C_1 and C_2 be components of $G[A_{\varphi'(v)} \cup B_{\varphi'}]$ such that there is a path P from C_1 to C_2 through $\varphi'^{-1}(v) - S_1$. Note that C_1 and C_2 are also components of $G[A_{\varphi(v)} \cup B_{\varphi}]$ and that $C_1 \neq C$ and $C_2 \neq C$ because they are components of $G[A_{\varphi'} \cup B_{\varphi'}]$.

If $V(P)$ does not intersect C then P is a path in G from C_1 to C_2 through $\varphi^{-1}(v) - S_1$. By Property (iii) applied to W there exists a path P' in G from C_1 to C_2 through $\varphi^{-1}(v) - S_2 \subseteq \varphi'^{-1}(v) - S_2$.

If $V(P)$ intersects C then $C \subseteq \varphi'^{-1}(v)$ so $v = u$. We have that P contains a path P_1 from C_1 to C through $\varphi'^{-1}(v) - (S_1 \cup C)$ and a path P_2 from C to C_2 through $\varphi'^{-1}(v) - (S_1 \cup C)$. But $\varphi'^{-1}(v) - C = \varphi^{-1}(v)$ so P_1 is a path from C_1 to C through $\varphi^{-1}(v) - S_1$ and P_2 is a path from C to C_2 through $\varphi^{-1}(v) - S_1$. By Property (iii) applied to W there exist paths P'_1 from C_1 to C through $\varphi^{-1}(v) - S_2$ and P'_2 from C to C_2 through $\varphi^{-1}(v) - S_2$. But then C_1 and C are in the same component of $G[C_1 \cup C \cup C_2 \cup \varphi^{-1}(v)] - S_2$ and C and C_2 are in the same component of $G[C_1 \cup C \cup C_2 \cup \varphi^{-1}(v)] - S_2$, so C_1 and C_2 are in the same component of $G[C_1 \cup C \cup C_2 \cup \varphi^{-1}(v)] - S_2$. We have that $C \cup \varphi^{-1}(v) = \varphi'^{-1}(v)$ so there exists a path P' from C_1 to C_2 through $\varphi'^{-1}(v) - S_2$.

The proof that if there exists a path from C_1 to C_2 through $\varphi'^{-1}(v) - S_2$ then there exists a path from C_1 to C_2 through $\varphi^{-1}(v) - S_1$ is symmetric.

- For property (iv) let C_B be a component of $G[A_{\varphi'} \cup B_{\varphi'}]$ and v be a peripheral vertex of H such that C_B has a neighbor in $\varphi'^{-1}(v)$.

We first prove that if C_B is a component of $G[A_{\varphi'}]$ then there exists a path from C_B to $B_{\varphi'}$ through $\varphi'^{-1}(v)$. We have that C_B is also a component of $G[A_{\varphi}]$ and, and that C and C_B are anti-complete. Since $\varphi^{-1}(v) = \varphi'^{-1}(v) - C$ it follows that C_B has a neighbor in $\varphi^{-1}(v)$. By property (iv) applied to W there is a path P from C_B to B_{φ} through $\varphi^{-1}(v)$. Since $B_{\varphi'} = B_{\varphi}$ this path P is also a path from C_B to $B_{\varphi'}$ through $\varphi'^{-1}(v)$.

We prove that if C_B is a component of $G[B_{\varphi'}]$ then there exists a path from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$. We have that C_B is also a component of $G[B_{\varphi}]$, and that C and C_B are anti-complete. Since $\varphi^{-1}(v) = \varphi'^{-1}(v) - C$ it follows that C_B has a neighbor in $\varphi^{-1}(v)$. By property (iv) applied to W there is a path P from C_B to A_{φ} through $\varphi^{-1}(v)$. Let x be the endpoint of P in A_{φ} . If $x \notin C$ then $x \in A_{\varphi'}$ and thus P is a path from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$.

If $x \in C$ then $v = u$ (since C has a neighbor in $\varphi^{-1}(v)$) and $C \subseteq \varphi'^{-1}(v)$. Since $A_{\varphi'}$ is non-empty and A_{φ} is contained in a connected component of $G - S_1$ (by property (ii) applied to W), there is a path P' from C to $A_{\varphi} - C = A_{\varphi'}$ in $G[A_1(W)]$. Since $A_1(W)$ is disjoint from B_{φ} and $N_G(C) \subseteq \varphi^{-1}(u) = \varphi^{-1}(v)$ the path P' is a path from C to $A_{\varphi'}$ through $\varphi^{-1}(v)$.

Consider now the walk P'' that starts in C_B , follows P to C , goes through C to the startpoint of P' and then follows P' to $A_{\varphi'}$. Since all internal vertices of P are in $\varphi^{-1}(v) \subseteq \varphi'^{-1}(v)$, C is a subset of $\varphi'^{-1}(v)$,

and all internal vertices of P' are in $\varphi^{-1}(v) \subseteq \varphi'^{-1}(v)$ we have that P'' is a walk from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$. Then $V(P'')$ contains a path from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$.

Next we verify that whenever W is full, or good, or disjoint, or connected, or has adhesion size α then W' has the same property.

- If W is a full generalized ω -creature, then $S_1^* = S_1$ and $S_2^* = S_2$, so W' is also a full generalized ω -creature.
- If W is a disjoint generalized ω -creature, then $S_1 \cap S_2 = \emptyset$ and therefore W' is also disjoint generalized ω -creature.
- If W is a connected generalized ω -creature, then for every peripheral vertex $v \neq u$, $G[\varphi'^{-1}(v)] = G[\varphi^{-1}(v)]$ is connected. Furthermore, $G[\varphi'^{-1}(u)] = G[\varphi^{-1}(u) \cup C]$, $G[\varphi^{-1}(u)]$ is connected because W is connected, $G[C]$ is connected because it is a connected component. Finally C has a neighbor in $G[\varphi^{-1}(u)]$ because $N_G(C) \subseteq \varphi^{-1}(u)$, $A_\varphi - C$ is non-empty, and $A_1(W)$ is a connected subgraph of G that contains A_φ . Hence $G[\varphi'^{-1}(u)]$ is connected and therefore then W' is a connected generalized ω -creature.
- Suppose that W is a good connected generalized ω -creature. Let v be a peripheral vertex of H . We show that $\varphi'^{-1}(v)$ good with respect to $A_{\varphi'}$ and $B_{\varphi'}$. If $v \neq u$ then $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(v)] = G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(v)] - C$. Further, C is a connected component of $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(v)]$ since $N_G(C) \subseteq \varphi^{-1}(u)$. Thus, since $\varphi^{-1}(v)$ is good with respect to A_φ and B_φ , $\varphi'^{-1}(v) = \varphi^{-1}(v)$ is also good with respect to $A_{\varphi'}$ and $B_{\varphi'}$.

We now consider the case when $v = u$. We have that $\varphi'^{-1}(u) = \varphi^{-1}(u) \cup C$. We have already shown that $G[\varphi'^{-1}(u)]$ is connected. First, suppose for contradiction that there exist two components C_1 and C_2 of $G[A_{\varphi'} \cup B_{\varphi'}]$ such that $N_G^2[C_1] \cap \varphi'^{-1}(u)$ is not anti-complete with $N_G^2[C_2] \cap \varphi'^{-1}(u)$. Then there exists a path P on at most six vertices from C_1 to C_2 through $\varphi'^{-1}(u)$. If P does not contain any internal vertices in C then all internal vertices in P are in $\varphi^{-1}(u)$. This contradicts that $N_G^2[C_1] \cap \varphi^{-1}(u)$ is anti-complete with $N_G^2[C_2] \cap \varphi^{-1}(u)$. So P contains an internal vertex in C . But then P contains a sub-path on at most 5 vertices from C_1 to C through $\varphi^{-1}(u)$, contradicting that $N_G^2[C_1] \cap \varphi^{-1}(u)$ is anti-complete with $N_G^2[C] \cap \varphi^{-1}(u)$. Hence every pair C_1 and C_2 of distinct components of $G[A_{\varphi'} \cup B_{\varphi'}]$ satisfy that $N_G^2[C_1] \cap \varphi'^{-1}(u)$ is anti-complete with $N_G^2[C_2] \cap \varphi'^{-1}(u)$.

Suppose now for contradiction that $\varphi'^{-1}(u)$ contains two distinct non-leaf sub-components Z_1 and Z_2 with respect to $A_{\varphi'}$ and $B_{\varphi'}$. Since $A_{\varphi'} \cup B_{\varphi'} \subseteq A_\varphi \cup B_\varphi$ it follows that $N_G^2[A_{\varphi'} \cup B_{\varphi'}] \subseteq N_G^2[A_\varphi \cup B_\varphi]$. Thus $\varphi^{-1}(u) - N_G^2[A_\varphi \cup B_\varphi] \subseteq \varphi'^{-1}(u) - N_G^2[A_{\varphi'} \cup B_{\varphi'}]$. Hence every sub-component of $\varphi^{-1}(u)$ with respect to A_φ and B_φ is contained in a sub-component of $\varphi'^{-1}(u)$ with respect to $A_{\varphi'}$ and $B_{\varphi'}$. Since W is good, Lemma 5.4 yields that $\varphi^{-1}(u)$ has a kernel, namely a unique non-leaf sub-component Z with respect to A_φ and B_φ . Since Z is fully contained in a sub-component of $\varphi'^{-1}(u)$ with respect to $A_{\varphi'}$ and $B_{\varphi'}$, at least one of Z_1 and Z_2 is disjoint from Z . Without loss of generality, $Z_1 \cap Z = \emptyset$.

Since Z_1 is a non-leaf sub-component of $\varphi'^{-1}(u)$ with respect to $A_{\varphi'}$ and $B_{\varphi'}$, there exist two components C_1 and C_2 of $G[A_{\varphi'} \cup B_{\varphi'}]$ such that there is a path P from C_1 to C_2 through $(N_G^2[C_1] \cap \varphi'^{-1}(u)) \cup Z_1 \cup (N_G^2[C_2] \cap \varphi'^{-1}(u))$. Notably, P is disjoint from the kernel Z of $\varphi^{-1}(u)$ with respect to A_φ and B_φ . If P does not contain any vertices of C then P is a path from C_1 to C_2 through $\varphi^{-1}(u)$, contradicting Lemma 5.6 which states that every path from C_1 to C_2 through $\varphi^{-1}(u)$ must intersect the kernel Z of $\varphi^{-1}(u)$. If P does contain a vertex of C then P contains a sub-path P' from C_1 to C through $\varphi^{-1}(u)$, again contradicting Lemma 5.6 which states that every path from C_1 to C through $\varphi^{-1}(u)$ must intersect the kernel Z of $\varphi^{-1}(u)$. Thus W' is a good connected generalized ω -creature

- We now bound the adhesion size of (every peripheral vertex of) W' . For every peripheral vertex v of H with $v \neq u$ we have $N(\varphi'^{-1}(v)) = N(\varphi^{-1}(v))$. For u we have $N(\varphi'^{-1}(u)) = N(\varphi^{-1}(u)) - C$. Since every connected component of $G[A_{\varphi'} \cup B_{\varphi'}]$ is a component of $G[A_\varphi \cup B_\varphi]$ it follows that the adhesion size of every peripheral vertex v of H in W' is at most its adhesion size in W . Hence, if W has adhesion size α then W' has adhesion size α .

The proof for the case when C is a component of $G[B_\varphi]$ is symmetric. $\square$

5.5 Extracting a Generalized ω -Creature with Bounded Adhesion Size. Let G be a k -creature free graph, and $W = (G, H, \varphi, S_1, S_2)$ be a connected good full generalized ω -creature. Our goal this subsection is to show that we can extract from W a connected good full $(\omega/2)$ -creature with adhesion size $2k$. This result is encapsulated in Lemma 5.12, which is the only lemma that will be used outside of this section.

LEMMA 5.11. *Let $W = (G, H, \varphi, S_1, S_2)$ be a good generalized ω -creature, let X be a component of $G - (A_\varphi \cup B_\varphi)$, and let C be a component of $G - X$. Then there do not exist k distinct components $D_1, D_2, \dots, D_k$ of $G[A_\varphi \cup B_\varphi]$ such that $D_1 \cup D_2 \cup \dots \cup D_k \subseteq C$ and $N(X) \cap D_i \neq \emptyset$ for every i .*

Proof. Suppose for contradiction that $D_1, D_2, \dots, D_k$ exist. Let $\hat{A}$ be the kernel of X .

For every $i \leq k$ let Y_i be a connected component of $G[X \cap N_G^2[A_\varphi \cup B_\varphi]]$ that has a neighbor in D_i . By Lemma 5.5 the components $Y_1, \dots, Y_k$ are distinct. By Lemma 5.7 each Y_i has a neighbor a_i in $\hat{A}$. The vertices $a_1, \dots, a_k$ need not be distinct.

For every i , a_i has a neighbor x_i in Y_i . We have that x_i is either in the first or second neighborhood of $A_\varphi \cup B_\varphi$. However, x_i can't be in the first neighborhood of $A_\varphi \cup B_\varphi$ since then a_i would be in the second, and it is not (since $a_i \in \hat{A}$). Thus x_i is in the second neighborhood of $A_\varphi \cup B_\varphi$. Then x_i has a neighbor y_i in $N_G(A_\varphi \cup B_\varphi)$, and therefore $y_i \in Y_i$. Since $y_i \in N_G(A_\varphi \cup B_\varphi)$ it follows that y_i has a neighbor $d_i \in A_\varphi \cup B_\varphi$. By Lemma 5.5, $d_i \in D_i$.

We show that $(\hat{A}, \{x_1, \dots, x_k\}, \{y_1, \dots, y_k\}, C)$ is a k -creature. The sets $G[\hat{A}]$ and $G[C]$ are connected (by definition of $\hat{A}$ and C) and anti-complete. Further $\hat{A}$ and $\{y_1, \dots, y_k\}$ are anti-complete because $\{y_1, \dots, y_k\} \subseteq N_G[A_\varphi \cup B_\varphi]$ while $\hat{A} \cap N_G[A_\varphi \cup B_\varphi] = \emptyset$. Similarly C and $\{x_1, \dots, x_k\}$ are anti-complete because $\{x_1, \dots, x_k\} \subseteq N_G^2(A_\varphi \cup B_\varphi)$ while $C \subseteq (A_\varphi \cup B_\varphi)$. Every x_i has a neighbor a_i in $\hat{A}$, and every y_i has a neighbor $d_i \in C$. Finally $x_i y_i$ is an edge and $x_i y_j$ is not an edge for $i \neq j$ because $x_i \in Y_i$ while $y_j \in Y_j$ there are no edges from Y_i to Y_j since they are distinct components of $N_G^2[A_\varphi \cup B_\varphi]$. Thus $(\hat{A}, \{x_1, \dots, x_k\}, \{y_1, \dots, y_k\}, C)$ is a k -creature, contradicting that G is k -creature free. The statement of the lemma follows. $\square$

LEMMA 5.12. *Let $W = (G, H, \varphi, S_1, S_2)$ be a connected, good, and full generalized ω -creature. Then there exists an induced subgraph G' of G and connected, good and full generalized $(\omega/2)$ -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with max adhesion size $2k$.*

Proof. Without loss of generality, W has no absorbable components. If W has absorbable components, let $\hat{W}$ be the result of absorbing all absorbable components in W . By Lemma 5.10, $\hat{W}$ is a connected, good and full generalized ω -creature, and $\hat{W}$ has no absorbable components. Then $G, \hat{W}$ also satisfy the premise of the lemma. We may therefore assume that W has no absorbable components.

CLAIM 5.1. *For every peripheral vertex u of H and every component C of $G - \varphi^{-1}(u)$, there exists a peripheral vertex v in H such that $\varphi^{-1}(v) \subseteq C$.*

Proof. Suppose for contradiction that there exists a peripheral vertex u of H and a component C of $G - \varphi^{-1}(u)$ such that there does not exist a peripheral vertex v satisfying $\varphi^{-1}(v) \subseteq C$.

For every peripheral vertex v of H we have that $\varphi^{-1}(v) \cap \varphi^{-1}(u) = \emptyset$ and that $G[\varphi^{-1}(v)]$ is connected (since W is connected). Thus $\varphi^{-1}(v) \subseteq C$ or $\varphi^{-1}(v) \cap C = \emptyset$. By our assumption $\varphi^{-1}(v) \cap C = \emptyset$ for every peripheral vertex v other than u . But then C is a component of $G[A \cup B]$ and $N_G(C) \subseteq \varphi^{-1}(u)$, so C is absorbable, contradicting the assumption that W has no absorbable components. $\square$

Let S be the set of peripheral vertices u of H such that $G - \varphi^{-1}(u)$ has at most two connected components.

CLAIM 5.2. $|S| \geq \omega/2$.

Proof. Let $\hat{G}$ be the (bipartite) graph that has, on one side, a vertex x_u for every peripheral vertex u of H , and on the other side a vertex v_C for every connected component C of $G[A \cup B]$. There is an edge from x_u to v_C in $\hat{G}$ if and only if there is an edge from $\varphi^{-1}(u)$ to C in G . Let $\hat{P} = \{x_u : u \in V(G) \setminus \{c_A, c_B\}\}$. That is, $\hat{P}$ is the set of vertices of $\hat{G}$ corresponding to peripheral vertices of H .

Note that $\hat{G}$ is obtained from G by contracting every edge $uv \in E(G)$ such that $\varphi^{-1}(u) = \varphi^{-1}(v)$. Thus $\hat{G}$ is connected. Let $\hat{T}$ be an inclusion minimal connected subgraph of G such that $\hat{P} \subseteq V(\hat{T})$. We have that $\hat{T}$ is a

tree (since removing an edge of a cycle preserves connectivity) and that every leaf of $\hat{T}$ is in $\hat{P}$ (since removing a leaf from a tree preserves connectivity).

We claim that for every peripheral vertex u of H which is *not* in S , the degree of x_u in $\hat{T}$ is at least 3. Suppose not, and let C_1, C_2, C_3 be distinct components of $G - \varphi^{-1}(u)$ (the components C_1, C_2, C_3 are well defined because $u \notin S$). Let v_1, v_2 , and v_3 be peripheral vertices of H such that $\varphi^{-1}(v_1) \subseteq C_1$, $\varphi^{-1}(v_2) \subseteq C_2$, and $\varphi^{-1}(v_3) \subseteq C_3$. The vertices v_1, v_2, v_3 exist by Claim 5.1.

Since the degree of x_u in $\hat{T}$ is at most two, $\hat{T} - x_u$ has at most two connected components. By the pigeon hole principle two of the vertices $\{x_{v_1}, x_{v_2}, x_{v_3}\}$ appear in the same component of $\hat{T} - x_u$. Without loss of generality, this is x_{v_1} and x_{v_2} . But then there is a path from x_{v_1} to x_{v_2} in $\hat{G}$ avoiding x_u , and therefore a path from $\varphi^{-1}(v_1)$ to $\varphi^{-1}(v_2)$ in G avoiding $\varphi^{-1}(u)$. But this contradicts that $\varphi^{-1}(v_1)$ and $\varphi^{-1}(v_2)$ are subsets of different components of $G - \varphi^{-1}(u)$. We conclude that for every peripheral vertex u of H which is not in S the degree of x_u in $\hat{T}$ is at least 3.

Let $\hat{P}_3$ be the set of vertices in $\hat{P}$ that have degree at least 3 in $\hat{T}$. A well known fact about trees is that every tree has more leaves than vertices of degree at least 3 (See e.g. [14], Chapter 1, Exercise 17). Therefore, $\hat{T}$ has at least $|\hat{P}_3|$ leaves. For every leaf x_ℓ of $\hat{T}$, we have that ℓ is a peripheral vertex of H and that $\ell \in S$ because the degree of x_ℓ in $\hat{T}$ is 1. It follows that $|S| \geq |\hat{P}_3|$, while $|\hat{P}_3|$ is at least the number of peripheral vertices of H which are not in S . It follows that $|S| \geq \omega/2$. $\square$

We can now finish the proof of the Lemma. By Lemma 5.11, the adhesion size (in W) of every $u \in S$ is at most $2k$. Let W' be the generalized $|S|$ -creature obtained from W by dissolving every peripheral vertex not in S . By Lemma 5.9 W' is in fact a connected, good and full generalized $|S|$ -creature, by Claim 5.2 we have $|S| \geq \omega/2$, and finally (again by Lemma 5.9) the adhesion size (in W') of every $u \in S$ is at most $2k$. The statement of the lemma follows. $\square$

5.6 Extracting a Disjoint Generalized ω -Creature.

LEMMA 5.13. *Let $W = (G, H, \varphi, S_1, S_2)$ be a full generalized ω -creature, u be a peripheral vertex of H , and C be a component of $G[\varphi^{-1}(u)] - (S_1 \cup S_2)$ such that $N(C) \cap (A_\varphi \cup B_\varphi) \neq \emptyset$ and $N(C) \cap \varphi^{-1}(u) \subseteq S_1 \cap S_2$. Define $\varphi'(v) = \varphi(v)$ for every $v \in V(G) - C$. If $N(C) \cap A_\varphi \neq \emptyset$ let $\varphi'(v) = c_A$ for every $v \in C$. If $N(C) \cap B_\varphi \neq \emptyset$ set $\varphi'(v) = c_B$ for every $v \in C$. Then $W' = (G, H, \varphi', S_1, S_2)$ is a full generalized ω -creature. Further the max adhesion size of W' is at most the max adhesion size of W .*

Before proving Lemma 5.13, observe that by Observation 5.2 precisely one of the cases $N(C) \cap A_\varphi \neq \emptyset$ and $N(C) \cap B_\varphi \neq \emptyset$ in the statement of Lemma 5.13 will apply.

Proof. [Proof of Lemma 5.13] We prove the statement of the lemma for the case that $N(C) \cap A_\varphi \neq \emptyset$. Then, by Observation 5.2 we have $N(C) \cap A_\varphi = \emptyset$. First we show that (H, φ') is an ω -bistar partition of G . Let xy be an edge of G . If neither x nor y are in C then $\varphi'(x) = \varphi(y)$ and $\varphi'(x) = \varphi(y)$, and therefore $\varphi'(x) = \varphi'(y)$ or $\varphi'(x)\varphi'(y)$ is an edge of H . If both x and y are in C then $\varphi'(x) = \varphi'(y) = c_A$. Thus, suppose that $x \in C$ and $y \notin C$. We have that $N(C) \subseteq \varphi^{-1}(u) \cup A_\varphi$ and therefore, $y \in A_\varphi$ or $y \in \varphi^{-1}(u)$. In the former case we have $\varphi'(x) = \varphi'(y) = c_A$, while in the latter case we have $\varphi'(x) = c_A$ while $\varphi'(y) = u$, and $c_A u$ is an edge of H . Next we check that W' satisfies the properties of full generalized ω -creatures.

- Property (i) clearly holds for all peripheral vertices of H other than u (with $S_1^* = S_1$ and $S_2^* = S_2$). For u observe that $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(u)] = [A_\varphi \cup B_\varphi \cup \varphi^{-1}(u)]$ and that $\varphi'^{-1}(u) \cap S_1 = \varphi^{-1}(u) \cap S_1$ and $\varphi'^{-1}(u) \cap S_2 = \varphi^{-1}(u) \cap S_2$. Thus $\varphi'^{-1}(u) \cap S_1$ and $\varphi'^{-1}(u) \cap S_2$ are distinct minimal $A_{\varphi'}, B_{\varphi'}$ separators in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(u)]$.
- For property (ii) observe that G, S_1, S_2 are the same for W and W' . Further, $B_{\varphi'} = B_\varphi$ and $A_{\varphi'} = A_\varphi \cup C$. Thus $B_{\varphi'}$ is entirely contained in a component of $G - S_1$, and A_φ is entirely contained in a different component of $G - S_1$. Since C is disjoint from S_1 and has a neighbor in A_φ , all of $A_{\varphi'}$ is contained in the same component of $G - S_1$ as A_φ . An identical argument shows that $B_{\varphi'}$ is entirely contained in a component of $G - S_2$, and that $A_{\varphi'}$ is entirely contained in a different component of $G - S_2$.
- For property (iii), let C_1 and C_2 be components of $G[A_{\varphi'} \cup B_{\varphi'}]$, and v be a peripheral vertex of H such that there is a path P from C_1 to C_2 through $\varphi'^{-1}(v) - S_1$. Observe now that C does not have any neighbors

in $\varphi'^{-1}(v) - S_1$. Indeed, if $v \neq u$ then $\varphi'^{-1}(v) = \varphi^{-1}(v)$ is anticomplete with $\varphi^{-1}(u)$, and $C \subseteq \varphi^{-1}(u)$. If $v = u$ then $N(C) \cap \varphi'^{-1}(u) \subseteq S_1 \cap S_2$ by assumption. Thus C does not have any neighbors in $\varphi'^{-1}(v) - S_1$. Thus the path P starts and ends in $G[A_\varphi \cup B_\varphi]$. Let C'_1 and C'_2 be the components of $G[A_\varphi \cup B_\varphi]$ containing the first and last vertex of P , respectively. Note that $C'_1 \subseteq C_1$ and that $C'_2 \subseteq C_2$.

Since $\varphi'^{-1}(v) \subseteq \varphi^{-1}(v)$ we have that P is a path from C'_1 to C'_2 through $\varphi^{-1}(v) - S_1$. Therefore (by property (iii) applied to W) there is a path P' from C'_1 to C'_2 through $\varphi^{-1}(v) - S_2$. Since C is a component of $G[\varphi^{-1}(u)] - S_2$, we have that the internal vertices of P' are either entirely inside C or disjoint from C . However all internal vertices of P' cannot be inside C , because then the endpoints of P' are in the same component of $G[A_{\varphi'}]$ contradicting that C_1 and C_2 are distinct components of $G[A_{\varphi'}]$. Hence P' is a path from C'_1 to C'_2 through $\varphi'^{-1}(v) - S_2$.

The proof for the reverse direction of the equivalence, namely that if there is a path P from C_1 to C_2 through $\varphi'^{-1}(v) - S_2$, then there also is a path from C_1 to C_2 through $\varphi'^{-1}(v) - S_1$, is identical.

- For property (iv), let X_A be a component of $G[A_{\varphi'}]$ and v be a peripheral vertex of H such that X_A has a neighbor in $\varphi'^{-1}(v)$. Let xy be an edge of G with $x \in X_A$ and $y \in \varphi'^{-1}(v)$. There are two cases, either $x \in C$ or $x \in A_\varphi$. If $x \in C$ then $v = u$ and $y \in S_1$. By property (i) applied to W and $S_1^* = S_1$, there is a path P from y to B_φ through $\varphi^{-1}(u) \cap B_1(u)$. But $C \subseteq A_1(u)$ and $\varphi'^{-1}(u) = \varphi^{-1}(u) - C$, and therefore P is a path from y to $B_\varphi = B_{\varphi'}$ through $\varphi'^{-1}(u)$.

If $x \in A_\varphi$, let X'_A be the component of $G[A_\varphi]$ that contains x . We have that $X'_A \subseteq X_A$. Furthermore, $y \in \varphi^{-1}(v)$ because $\varphi'^{-1}(v) \subseteq \varphi^{-1}(v)$. Thus X'_A has a neighbor in $\varphi^{-1}(v)$, and hence, by property (iv) applied to W , there is a path P from X'_A to B_φ through $\varphi^{-1}(v)$.

If P has no internal vertices in C then P is a path from X'_A to $B_{\varphi'} = B_\varphi$ through $\varphi'^{-1}(v)$. If P has internal vertices in C , then $v = u$. Since $x \in X_A$, $y \in C$ and xy is an edge, we have that $C \subseteq X_A$. Let x' be the last vertex of X_A on P . The sub-path of P from x' to $B_{\varphi'} = B_\varphi$ is a path from X_A to $B_{\varphi'}$ through $\varphi'^{-1}(v)$.

Consider now a component X_B of $G[B_{\varphi'}]$ and let v be a peripheral vertex of H such that X_B has a neighbor in $\varphi'^{-1}(v)$. X_B is also a component of $G[B_\varphi]$ and $\varphi'^{-1}(v) \subseteq \varphi^{-1}(v)$, so by property (iv) applied to W there exists a path P from X_B to A_φ through $\varphi^{-1}(v)$. Let x' be the first vertex on P from $A_{\varphi'}$. The sub-path of P that ends in x' is a path from X_B to $A_{\varphi'}$ through $\varphi'^{-1}(v)$.

Finally we upper bound the max adhesion size of W . Note that every component of $G[A'_\varphi \cup B'_\varphi]$ is a component of $G[A_\varphi \cup B_\varphi]$, with the exception of the unique component X of $G[A'_\varphi \cup B'_\varphi]$ which contains C . The component X is equal to C plus the union of all components of $G[A_\varphi \cup B_\varphi]$ which contain a neighbor of C . By assumption there exists at least one component of $G[A_\varphi \cup B_\varphi]$ which contain a neighbor of C .

Since C does not have any neighbors in $\varphi^{-1}(v)$ for any peripheral vertex $v \neq u$, the adhesion size of every peripheral vertex $v \neq u$ in W' is at most its adhesion size in W . For u , every component of $G[A'_\varphi \cup B'_\varphi]$ that contains a neighbor of $\varphi'^{-1}(u)$ contains a component of $G[A_\varphi \cup B_\varphi]$ that contains a neighbor of $\varphi^{-1}(u)$. Thus the adhesion size of u in W' is at most its adhesion size in W . This completes the proof for the case that $N(C) \cap A \neq \emptyset$. The proof for the case where $N(C) \cap B \neq \emptyset$ is symmetric. $\square$

LEMMA 5.14. (EXTRACT DISJOINT ω -CREATURE) *Let $W = (G, H, \varphi, S_1, S_2)$ be a full generalized ω -creature of max adhesion size α . Then there exists a full disjoint generalized ω -creature, $W' = (G', H, \varphi', S'_1, S'_2)$, of max adhesion size α such that G' is an induced subgraph of G .*

Proof. Without loss of generality W satisfies the following additional property: there does not exist a peripheral vertex u , and component C of $G[\varphi^{-1}(u)] - (S_1 \cup S_2)$ such that $N(C) \cap (A_\varphi \cup B_\varphi) \neq \emptyset$ and $N(C) \cap \varphi^{-1}(u) \subseteq S_1 \cap S_2$. Indeed, if such a u and C exists then Lemma 5.13 yields a full generalized ω -creature with the same graph G , adhesion size at most α , and strictly larger $|A_\varphi \cup B_\varphi|$. Since $|A_\varphi \cup B_\varphi| \leq |V(G)|$ there must exist a full generalized ω -creature $W^* = (G, H, \varphi^*, S_1, S_2)$ on the same graph G that additionally satisfies the additional property. Since W^* satisfies the premise of the lemma we may assume that $W = W^*$.

We set $G' = G - (S_1 \cap S_2)$, $\varphi'(v) = \varphi(v)$ for every $v \in V(G')$, $S'_1 = S_1 - S_2$ and $S'_2 = S_2 - S_1$. We claim that $W' = (G', H, \varphi', S'_1, S'_2)$ satisfies the conclusion of the lemma. We first prove that it is a full generalized ω -creature. Indeed, G' is an induced subgraph of G , and (H, φ') is an ω -bistar partition of G . Note that $A_{\varphi'} = A_\varphi$ and $B_{\varphi'} = B_\varphi$.

- For property (i) we have that for every peripheral vertex, u , of H , $G'[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(u)] - (\varphi'^{-1}(u) \cap S'_1)$ is equal to $G[A_{\varphi} \cup B_{\varphi} \cup \varphi^{-1}(u)] - (\varphi^{-1}(u) \cap S_1)$, and so S'_1 is a minimal separator in $G'[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(u)]$. Similarly, $\varphi'^{-1}(u) \cap S'_2$ a minimal separator in $G'[A_{\varphi'} \cup B_{\varphi'} \cup \varphi'^{-1}(u)]$. Finally, $\varphi'^{-1}(u) \cap S'_1$ and $\varphi'^{-1}(u) \cap S'_2$ are distinct because $\varphi^{-1}(u) \cap S_1$ and $\varphi^{-1}(u) \cap S_2$ are distinct.
- Property (ii) is satisfied by W' because $G' - S'_1 = G - S_1$, and $G' - S'_2 = G - S_2$.
- Property (iii) is satisfied by W' because every pair of components C_1 and C_2 of $G'[A_{\varphi'} \cup B_{\varphi'}]$ are components of $G[A_{\varphi} \cup B_{\varphi}]$, and for every peripheral vertex u of H we have $\varphi'^{-1}(u) - S'_1 = \varphi^{-1}(u) - S_1$ and $\varphi'^{-1}(u) - S'_2 = \varphi^{-1}(u) - S_2$.
- For property (iv), let u be a peripheral vertex and X_A be a component of $G'[A_{\varphi'}]$ that has a neighbor in $\varphi'^{-1}(u)$.

Let xy be an edge of G' with $x \in X_A$ and $y \in \varphi'^{-1}(u)$. We claim that there exists a path in G' from X_A to $S_1 \cup S_2$ through $\varphi'^{-1}(u)$. If $y \in S_1 \cup S_2$ then xy is the desired path, so suppose $y \in \varphi'^{-1}(u) - S_1 \cup S_2$. Let C be the connected component of $\varphi^{-1}(u) - (S_1 \cup S_2)$ that contains u . The property of W discussed at the start of the proof ensures that C has at least one neighbor z in $\varphi'^{-1}(u) \cap (S_1 \cup S_2) - (S_1 \cap S_2)$. Let P now be a path from x to z through C . P does not contain any vertices of $S_1 \cap S_2$ and hence it is a path in G' from X_A to $S_1 \cup S_2$ through $\varphi'^{-1}(u)$.

Let z be the endpoint of the path P in $S_1 \cup S_2$. If z is in S_1 then, by minimality of $S_1 \cap \varphi^{-1}(u)$ (here we use that W is full), there exists a path P' in G from z to B_{φ} through $\varphi^{-1}(u) - S_1$. But P' does not contain any vertices of $S_1 \cap S_2$ and hence P' is a path from z to $B_{\varphi'}$ through $\varphi'^{-1}(u)$. An identical argument shows that if z is in S_2 then there exists a path P' from z to $B_{\varphi'}$ through $\varphi'^{-1}(u)$. But then P , followed by P' is a path in G' from X_A to $B_{\varphi'}$ through $\varphi'^{-1}(u)$.

The proof of the analogous statement for peripheral vertex u and component X_B of $G'[B_{\varphi'}]$ that has a neighbor in $\varphi'^{-1}(u)$ is identical.

Having shown that $W' = (G', H, \varphi', S'_1, S'_2)$ is a full generalized ω -creature, note that S'_1 and S'_2 are disjoint. Further the adhesion size of every peripheral vertex v of H in W' is at most its adhesion size in W , since $A_{\varphi'} = A_{\varphi}$, $B_{\varphi'} = V_{\varphi}$, and $\varphi'^{-1}(v) \subseteq \varphi^{-1}(v)$. This concludes the proof of the lemma. $\square$

5.7 Connectivity Graphs and Long Induced Paths in them

DEFINITION 5.7. (REALIZE) Let W be a generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$. A peripheral vertex u of H realizes an (unordered) pair of distinct components $\{C_1, C_2\}$ of $G[A_{\varphi} \cup B_{\varphi}]$ if there is a path in G from C_1 to C_2 through $\varphi^{-1}(u) - S_1$.

Note that by property (i) of generalized ω -creatures, if u realizes $\{C_1, C_2\}$, then C_1 and C_2 are either both components of $G[A_{\varphi}]$ or both components of $G[B_{\varphi}]$.

DEFINITION 5.8. (CONNECTIVITY GRAPH) The A -connectivity graph of a generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$ is a graph $\mathcal{C}_A$. The vertices of $\mathcal{C}_A$ are the connected components of $G[A_{\varphi}]$. Two components C_1 and C_2 of $G[A_{\varphi}]$ are connected by an edge in $\mathcal{C}_A$ if there exists a peripheral vertex u of H that realizes $\{C_1, C_2\}$.

The B -connectivity graph $\mathcal{C}_B$ of W is defined similarly, with vertices of $\mathcal{C}_B$ being components of $G[B_{\varphi}]$, and two components C_1 and C_2 are connected by an edge in $\mathcal{C}_B$ if there exists a peripheral vertex u of H that realizes $\{C_1, C_2\}$.

The A -connectivity graphs tracks which pairs $\{C_1, C_2\}$ of components of $G[A_{\varphi}]$ are realized by some peripheral vertex of H . We will (towards the end of the proof) also be interested in precisely which peripheral vertices realize a given pair. We encapsulate this in the notion of labeled connectivity graphs.

DEFINITION 5.9. (LABELED CONNECTIVITY GRAPH) The labeled A -connectivity graph of a generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$ is a pair $(\mathcal{C}_A, \lambda)$ where $\mathcal{C}_A$ is the A -connectivity graph of W , and $\lambda : E(\mathcal{C}_A) \rightarrow 2^{V(H)}$ takes as input an edge $\{C_1, C_2\}$ and outputs the subset of peripheral vertices u of H that realize $\{C_1, C_2\}$.

The labeled B -connectivity graph W is a pair (C_B, λ) where C_B is the B -connectivity graph of W , and $\lambda : E(C_B) \rightarrow 2^{V(H)}$ takes as input an edge $\{C_1, C_2\}$ and outputs the subset of peripheral vertices u of H that realize $\{C_1, C_2\}$.

We will occasionally be interested in the subgraph of the A -connectivity graph C_A of a generalized ω -creature W induced by a vertex set Z . Just as for normal graphs, we will denote the induced subgraph by $C_A[Z]$. We can treat such an induced subgraph as a labeled induced subgraph by dropping from the domain of the labeling λ all edges that do not appear in the considered induced subgraph. We will denote this labeled induced subgraph by $(C_A, \lambda)[Z]$.

LEMMA 5.15. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature. Then the A -connectivity graph C_A and the B -connectivity graph C_B of W are connected.*

Proof. This follows immediately from Property (ii) of generalized ω -creatures. $\square$

LEMMA 5.16. *Let G be k -creature free, $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature and C be a component of $G[A_\varphi \cup B_\varphi]$. Let Z be the set of peripheral vertices u of H such that $\varphi^{-1}(u)$ contains a neighbor of C in G . Then $|Z| < k$.*

Proof. Suppose for contradiction that $|Z| \geq k$, and let $z_1, \dots, z_k$ be k distinct vertices in Z . Let $S_1^* \subseteq S_1$ and $S_2^* \subseteq S_2$ be witness separators for W . For each $i \leq k$ proceed as follows. Since C has a neighbor in $\varphi^{-1}(z_i)$, by property (iv) of generalized ω -creatures, there is a path from C to B_φ through $\varphi^{-1}(z_i)$. By property (i) this path contains at least one vertex of S_1^* (and at least one vertex of S_2^*). Thus, there exists a path from C to $S_1^* \cup S_2^*$ in $G[C \cup \varphi^{-1}(z_i)]$. Let P_i^1 be a shortest such path.

Let x_i be the last vertex of P_i^1 , without loss of generality x_i is in S_1^* . If it is in S_2^* we instead consider the generalized ω -creature resulting from flipping W (and S_1^* and S_2^*) at z_i (see Lemma 5.3). Since this changes $S_1 \cap \varphi^{-1}(z_i)$ to $S_2 \cap \varphi^{-1}(z_i)$, and $S_1^* \cap \varphi^{-1}(z_i)$ to $S_2^* \cap \varphi^{-1}(z_i)$, and vice versa (and changes nothing else), we may now assume that x_i is in S_1^* .

By property (i) of generalized ω -creatures we have that $\varphi^{-1}(z_i) \cap S_1^*$ is an A_φ, B_φ -minimal separator in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(z_i)]$. Thus there exists an induced path P_i^2 from x_i to B_φ in $G[B_\varphi \cup \varphi^{-1}(z_i)]$ that does not contain any vertices of S_1^* . We select P_i^2 such that only the last vertex of P_i^2 is in B_φ .

Notice now that P_i^1 followed by P_i^2 is a path from A_φ to B_φ in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(z_i)]$. Let y_i be the vertex immediately after x_i on the path P_i^2 . The vertex y_i can not be the last vertex of P_i^2 , since then $P_i^1 P_i^2$ is a path from A_φ to B_φ in $G[A_\varphi \cup B_\varphi \cup \varphi^{-1}(z_i)]$ disjoint from S_2^* .

We define

$$A = C \cup \bigcup_{i \leq k} P_i^1 - \{x_i\}$$

$$B = B_2^*(W) \cup \bigcup_{i \leq k} P_i^2 - \{x_i, y_i\}$$

and claim that $(A, B, \{x_1, \dots, x_k\}, \{y_1, \dots, y_k\})$ forms a k -creature. We check the properties of k -creatures one by one:

- A is disjoint from $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\} \cup B$ because $A \subseteq A_1^*(W)$ while $\{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\} \cup B \subseteq S_1^* \cup B_1^*(W)$.
- $\{x_1, \dots, x_k\}$ is disjoint from $\{y_1, \dots, y_k\}$ because y_i is the successor of x_i on P_i^2 . For $j \neq i$ we have that $x_i \in \varphi^{-1}(z_i)$ while $y_j \in \varphi^{-1}(z_j)$, which are disjoint and anticomplete. This shows not only that $\{x_1, \dots, x_k\}$ is disjoint from $\{y_1, \dots, y_k\}$, but also that $x_i y_j$ is an edge if and only if $i = j$.
- $\{y_1, \dots, y_k\}$ is disjoint from B because $\{y_1, \dots, y_k\} \subseteq A_2^*(W) \cup S_2$, while $B = B_2^*(W) \cup \bigcup_{i \leq k} P_i^2 - \{x_i, y_i\}$.
- $G[A]$ is connected because C is connected and each P_i^1 is a path that starts from C .
- $G[B]$ is connected because $B_2^*(W)$ is connected and each $P_i^2 - \{x_i, y_i\}$ is a path that ends in $B_\varphi \subseteq B_2^*(W)$.

- A and $B \cup \{y_1, \dots, y_k\}$ are anticomplete: indeed $A \subseteq A_1^*(W)$ while $B \cup \{y_1, \dots, y_k\} \subseteq B_1^*(W)$, and $A_1^*(W)$ and $B_1^*(W)$ are anticomplete.
- B and $A \cup \{x_1, \dots, x_k\}$ are anticomplete: since we have already shown that A and B are anticomplete it suffices to show that B and $\{x_1, \dots, x_k\}$ are anticomplete. Suppose for contradiction that $x_j b$ is an edge with $b \in B$. We have that $\{x_1, \dots, x_k\} \subseteq A_2^*(W)$, so $\{x_1, \dots, x_k\}$ is anticomplete with $B_2^*(W)$. We conclude that $b \in (P_2^i - \{x_i, y_i\})$. Since $b \notin B_2^*(W)$ we have that b cannot be the last vertex of $P_2^i - \{x_i, y_i\}$, and therefore $b \in \varphi^{-1}(z_i)$. But $x_j \in \varphi^{-1}(z_j)$ and $\varphi^{-1}(z_i)$ and $\varphi^{-1}(z_j)$ are anticomplete unless $i = j$, contradicting that $x_i b$ is an edge. So $i = j$ and b is a vertex on $P_2^i - \{x_i, y_i\}$. However y_i is the only vertex on P_2^i adjacent to x_i , and $b \neq y_i$ yielding a contradiction. Thus B and $\{x_1, \dots, x_k\}$ are anticomplete.
- Each x_i has a neighbor in A , namely its predecessor in P_1^i , and each y_i has a neighbor in B , namely its successor in P_1^i .

We conclude that $(A, B, \{x_1, \dots, x_k\}, \{y_1, \dots, y_k\})$ forms a k -creature in G , contradicting that G is k -creature free.

□

LEMMA 5.17. *Let G be a k -creature free graph and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature with adhesion size at most α . Then the A -connectivity graph $\mathcal{C}_A$ of W (and the B -connectivity graph $\mathcal{C}_B$ of W) contains an induced path of length at least $\log_{k\alpha}(\omega/k) - 1$.*

Proof. We prove the statement for the A -connectivity graph $\mathcal{C}_A$ of W . First, by Lemma 5.15 $\mathcal{C}_A$ is connected. Next we show that $\mathcal{C}_A$ has at least $\omega/k\alpha$ vertices. By Lemma 5.1 each peripheral vertex v of H satisfies that $\varphi^{-1}(v)$ has a neighbor in A_φ . On the other hand, for every component C of $G[A_\varphi]$, Lemma 5.16 yields that there are at most $k - 1$ peripheral vertices v such that $\varphi^{-1}(v)$ has a neighbor in C . Thus, $G[A_\varphi]$ has at least ω/k components, and so the A -connectivity graph $\mathcal{C}_A$ of W has at least ω/k vertices.

Next, we show that the maximum degree of $\mathcal{C}_A$ is at most $k\alpha$. Indeed, consider a component C of $G[A_\varphi]$, and another component C' of $G[A_\varphi]$ such that C and C' are adjacent in $\mathcal{C}_A$. Then there exists a peripheral vertex v of H such that $\varphi^{-1}(v)$ has a neighbor both in C and in C' . By Lemma 5.16 there are at most $k - 1$ vertices u such that $\varphi^{-1}(u)$ has a neighbor in C . For each such peripheral vertex u , there are at most α components C'' of $G[A_\varphi]$ that have a neighbor in $\varphi^{-1}(u)$. Thus there are at most $k\alpha$ components C' adjacent to C in $\mathcal{C}_A$.

We have that $\mathcal{C}_A$ is a connected graph with at least ω/k vertices and maximum degree at most $k\alpha$. Pick any vertex C of $\mathcal{C}_A$. For every $d \geq 1$, the number of vertices at distance exactly d from C in $\mathcal{C}_A$ is at most $(k\alpha)^d$, and therefore the number of vertices at distance at most d is at most $(k\alpha)^{d+1}$. Let C' be the vertex of $\mathcal{C}_A$ furthest away from C in $\mathcal{C}_A$, and let d be the distance from C to C' in $\mathcal{C}_A$. We have that $\omega/k \leq (k\alpha)^{d+1}$, and therefore $\log_{k\alpha}(\omega/k) \leq d + 1$. Thus a shortest path from C to C' in $\mathcal{C}_A$ satisfies the conclusion of the lemma. The proof for the B -connectivity graph $\mathcal{C}_B$ of W is identical. □

5.8 Erasing Components We will work towards extracting from W a generalized ω' -creature whose A -connectivity graph is a path. Towards this we will identify a long induced path P in the A -connectivity graph of G , and delete all components of $G[A_\varphi]$ that are not on the path P . However, when we delete components of $G[A_\varphi]$ we need to appropriately modify the generalized ω -creature in the peripheral vertices in order to ensure that the result of path-filtering is still a generalized ω -creature.

Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. A component C of $G[A_\varphi]$ is *erasable* if $\mathcal{C}_A - \{C\}$ is connected. Similarly, a component C of $G[B_\varphi]$ is *erasable* if $\mathcal{C}_B - \{C\}$ is connected. For an induced path P in the A -connectivity graph $\mathcal{C}_A$ of W , or the B -connectivity graph $\mathcal{C}_B$ of W , a component C of $G[A_\varphi \cup B_\varphi]$ is *P -erasable* if C is erasable and $C \notin V(P)$.

LEMMA 5.18. *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature. If P is an induced path in the A -connectivity graph $\mathcal{C}_A$ of W and there exists a component C of $G[A_\varphi]$ not in $V(P)$ then there exists a P -erasable component C' of $G[A_\varphi]$. If P is an induced path in the B -connectivity graph $\mathcal{C}_B$ of W and there exists a component C of $G[B_\varphi]$ not in $V(P)$ then there exists a P -erasable component C' of $G[B_\varphi]$.*

Proof. We prove the statement for P being a path in $\mathcal{C}_A$. By Lemma 5.15 $\mathcal{C}_A$ is connected. Therefore it has a spanning tree T that contains all the edges of P . Since there exists a component C of $G[A_\varphi]$ not in $V(P)$, T has

a leaf C' not in $V(P)$. Since $T - \{C'\}$ is connected, $\mathcal{C}_A - \{C'\}$ is connected as well. Hence C' is erasable. The proof for statement for P being a path in $\mathcal{C}_B$ is identical. $\square$

We are aiming at an operation that will delete all of the vertices of a P -erasable component C from the graph G of a generalized ω -creature. Simply deleting all vertices of C from G does not immediately work, because some peripheral vertices of H might violate property (iv) of generalized ω -creatures after such a deletion. The next definitions aim to highlight the peripheral vertices for which such a problem could occur.

DEFINITION 5.10. (CHUNK) *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature and v be a peripheral vertex of H . A chunk of v is a connected component of $G[\varphi^{-1}(v)]$. A chunk of W is a chunk of v for some peripheral vertex v of H .*

Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature, and C be a component of $G[A_\varphi \cup B_\varphi]$. A peripheral vertex u of H is C -dependent if $C \subseteq A_\varphi$ and C contains $N(\varphi^{-1}(u)) \cap A_\varphi$ or $C \subseteq B_\varphi$ and C contains $N(\varphi^{-1}(u)) \cap B_\varphi$. The set $D(C)$ denotes the set of all C -dependent peripheral vertices in H . Similarly, a chunk Z of W is C -dependent if $C \subseteq A_\varphi$ and C contains $N(\varphi^{-1}(u)) \cap A_\varphi$ or $C \subseteq B_\varphi$ and C contains $N(\varphi^{-1}(u)) \cap B_\varphi$. The set $\mathcal{D}_S(C)$ denotes the set of all C -dependent chunks of W . Note that despite the similar names $D(C)$ and $\mathcal{D}_S(C)$ are objects of different types. More concretely $D(C)$ is a set of vertices of H while $\mathcal{D}_S(C)$ is a set of vertex sets. We are now ready to define the operation of erasing a component.

DEFINITION 5.11. (ERASING COMPONENT) *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature, and let C be an erasable component of $G[A_\varphi \cup B_\varphi]$. We define X to be the union of all C -dependent chunks of W . If C is a component of $G[A_\varphi]$ we define Y to be the union of all C -dependent chunks of W that have at least one neighbor in B_φ . If C is a component of $G[B_\varphi]$ we define Y to be the union of all C -dependent chunks of W that have at least one neighbor in B_φ . We set*

- $G' = (G - C) - (X - Y)$, and
- $H' = H - D(C)$.
- For every $v \in Y$ we set $\varphi'(v) = c_B$ if $C \subseteq A_\varphi$ and $\varphi'(v) = c_A$ if $C \subseteq B_\varphi$. For every $v \in V(G') - Y$ we set $\varphi'(v) = \varphi(v)$.
- We set $S'_1 = S_1 - X$ and $S'_2 = S_2 - X$.

LEMMA 5.19. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature, and let C be an erasable component of $G[A_\varphi]$ or of $G[B_\varphi]$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be obtained from W by erasing C . Then W' is a disjoint generalized ω' -creature, where $\omega' = \omega - |D(C)|$. Further for every peripheral vertex u of H' its adhesion size in W' is at most its adhesion size in W .*

Proof. We prove the statement of the lemma for the case when C is a component of $G[A_\varphi]$. First we show that (H', φ') is a ω' -bistar partition of G' . Note that every peripheral vertex of H , except the vertices in $D(C)$, is a peripheral vertex of H' . Thus H' has ω' peripheral vertices. Consider an arbitrary edge $uv \in E(G')$. If $\varphi(u) \notin Y$ and $\varphi(v) \notin Y$ then $\varphi'(u) = \varphi(u)$, $\varphi'(v) = \varphi(v)$, so $\varphi'(u) = \varphi'(v)$ or $\varphi'(u)\varphi'(v) \in E(H')$. If $\varphi(u) \in Y$ and $\varphi(v) \in Y$ then $\varphi'(u) = \varphi'(v) = c_B$. Suppose now that $\varphi(u) \in Y$ and $\varphi(v) \notin Y$. But then u is in a C -dependent chunk of W , and therefore $v \in B_\varphi$. But then $\varphi'(u) = \varphi'(v) = c_B$. Thus (H', φ') is an ω' -bistar partition of G' .

Next observe that S'_1 and S'_2 are disjoint because $S'_1 \subseteq S_1$, $S'_2 \subseteq S_2$, and S_1 and S_2 are disjoint. Both S'_1 and S'_2 are disjoint from $A_{\varphi'} \cup B_{\varphi'}$ because $A_{\varphi'} \cup B_{\varphi'} \subseteq A_\varphi \cup B_\varphi \cup X$ while $S'_1 = S_1 - X$ and $S'_2 = S_2 - X$. We now verify that $W' = (G', H', \varphi', S'_1, S'_2)$ satisfies the properties of generalized ω' -creatures.

- For property (i) we first show that S'_1 and S'_2 separate $A_{\varphi'}$ from $B_{\varphi'}$ in G' . Suppose for contradiction that there exists a path P from $A_{\varphi'}$ to $B_{\varphi'}$ in $G' - S'_1$. Since (H', φ') is an ω' -bistar partition of G' the path P has at least three vertices. Further we may select P such that none of the internal vertices of P are in $A_{\varphi'} \cup B_{\varphi'}$. Thus, since (H', φ') is an ω' -bistar partition of G' there exists a peripheral vertex u of H' such that P is a path in G' from $A_{\varphi'}$ to $B_{\varphi'}$ through $\varphi'^{-1}(u)$. But $\varphi'^{-1}(u) \subseteq \varphi^{-1}(u)$ and G' is an induced subgraph of G , so

$$N_{G'}(\varphi'^{-1}(u)) \subseteq N_G(\varphi^{-1}(u)) \subseteq A_\varphi \cup B_\varphi$$

It follows that P is a path in G from A_φ to B_φ through $\varphi^{-1}(u)$. Let Q be the chunk of u that contains P . Since P is a path in G' its last endpoint is in $A_\varphi - C$, and therefore Q is not C -dependent. But then $Q \cap S_1 = Q \cap S'_1$ and thus P is a path in G from A_φ to B_φ through $\varphi^{-1}(u) - S_1$, contradicting property (i) applied to W . The proof that S'_2 is a $A_{\varphi'}, B_{\varphi'}$ -separator in G' is symmetric.

Next we show that for every peripheral vertex $u \in H'$ there exists a path from $A_{\varphi'}$ to $B_{\varphi'}$ through $\varphi'^{-1}(u)$. Since u is a peripheral vertex of H' it is not C -dependent in A , and therefore there exists a component C' of $G[A_\varphi]$ other than C with a neighbor in A_φ . By property (iv) applied to W there is a path P from C' to B_φ through $\varphi^{-1}(u)$. The internal vertices of P are contained in a chunk Q of u , and this chunk is not C -dependent because Q has a neighbor in C' . But then $Q \subseteq \varphi'^{-1}(u)$ so P is a path in G' from $A_{\varphi'}$ to $B_{\varphi'}$ through $\varphi'^{-1}(u)$. This implies that every minimal $A_{\varphi'}, B_{\varphi'}$ separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi^{-1}(u)]$ is non-empty.

Since S'_1 is an $A_{\varphi'}, B_{\varphi'}$ -separator in G it follows that for every peripheral vertex u of H' , $S'_1 \cap \varphi^{-1}(u)$ is an $A_{\varphi'}, B_{\varphi'}$ -separator in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi^{-1}(u)]$. Thus $S'_1 \cap \varphi^{-1}(u)$ contains a minimal $A_{\varphi'}, B_{\varphi'}$ -separator $S'^*_{1,u}$ in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi^{-1}(u)]$. Similarly, $S'_2 \cap \varphi^{-1}(u)$ contains a minimal $A_{\varphi'}, B_{\varphi'}$ -separator $S'^*_{2,u}$ in $G[A_{\varphi'} \cup B_{\varphi'} \cup \varphi^{-1}(u)]$. Since S'_1 and S'_2 are disjoint while $S'^*_{1,u} \subseteq S'_1$ and $S'^*_{2,u} \subseteq S'_2$ are nonempty, $S'^*_{1,u}$ and $S'^*_{2,u}$ are distinct. Let $S'^*_1 = \bigcup_u S'^*_{1,u}$ and $S'^*_2 = \bigcup_u S'^*_{2,u}$, where the union is taken over all peripheral vertices u of H' . Then S'^*_1 and S'^*_2 are witness separators for W' .

- For property (ii) we first prove that $A_{\varphi'}$ is contained in a connected component of $G' - S'_1$.

Observe that the A -connectivity graph $\mathcal{C}_A$ of W satisfies that $\mathcal{C}_A - \{C\}$ is connected, and that $A_{\varphi'} = A_\varphi - C$. Thus, for every pair of components C_1, C_2 of $G'[A_{\varphi'}]$ there is a sequence $R_1, R_2, \dots, R_t$ of components of $G'[A_{\varphi'}]$, such that $C_1 = R_1$, $C_2 = R_t$, and for every $i < t$ there exists a peripheral vertex u of H a path P in G from R_i to R_{i+1} through $\varphi^{-1}(u) - S_1$. The internal vertices of the path P are contained in a chunk $Q \subseteq \varphi^{-1}(u)$ of W . Since Q has a neighbor in R_i it is not C -dependent, and therefore $Q \subseteq \varphi'^{-1}(u)$. But then P is a path from R_i to R_{i+1} through $\varphi'^{-1}(u) - S_1$. Further $S'_1 \subseteq S_1$ and therefore P is a path from R_i to R_{i+1} through $\varphi'^{-1}(u) - S'_1$. So R_i and R_{i+1} are contained in the same component of $G' - S'_1$, and therefore so are C_1 and C_2 . But C_1 and C_2 were arbitrarily chosen components in $G'[A_{\varphi'}]$, hence $A_{\varphi'}$ is contained in a connected component of $G' - S'_1$. The proof that $A_{\varphi'}$ is contained in a connected component of $G' - S'_2$ is identical.

We now prove that $B_{\varphi'}$ is contained in a connected component of $G' - S'_1$. First note that $B_\varphi \subseteq B_{\varphi'}$ and therefore every connected component of $G'[B_{\varphi'}]$ contains a connected component of $G[B_\varphi]$. Further, by Lemma 5.15 for every pair of components C_1, C_2 of $G[B_\varphi]$ there is a sequence $R_1, R_2, \dots, R_t$ of components of $G[B_\varphi]$, such that $R_1 = C_1$, $R_t = C_2$, and for every $i < t$ there exists a peripheral vertex u of H and a path P in G from R_i to R_{i+1} through $\varphi^{-1}(u) - S_1$. The internal vertices of the path P are contained in a chunk $Q \subseteq \varphi^{-1}(u)$ of W . Since Q has a neighbor in R_i , Q is not C -dependent, and therefore $Q \subseteq \varphi'^{-1}(u)$. But then P is a path from R_i to R_{i+1} through $\varphi'^{-1}(u) - S_1$. Further $S'_1 \subseteq S_1$ and therefore P is a path from R_i to R_{i+1} through $\varphi'^{-1}(u) - S'_1$. So R_i and R_{i+1} are contained in the same component of $G' - S'_1$, and therefore so are C_1 and C_2 . But C_1 and C_2 were arbitrarily chosen components in $G[B_{\varphi'}]$, hence $B_{\varphi'}$ is contained in a connected component of $G' - S'_1$. But every component of $G[B_\varphi]$ contains a component of $G[B_{\varphi'}]$, so all of $B_{\varphi'}$ is contained in one connected component of $G' - S'_1$. The proof that $B_{\varphi'}$ is contained in one connected component of $G' - S'_2$ is identical.

- For property (iii) let u be a peripheral vertex of H' and let C_1 and C_2 be components of $G'[A_{\varphi'} \cup B_{\varphi'}]$. Suppose there is a path P from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1$ in G' . Since $\varphi'^{-1}(u) \subseteq \varphi^{-1}(u)$ and $N_{G'}(\varphi'^{-1}(u)) \subseteq N_G(\varphi^{-1}(u)) \subseteq A_\varphi \cup B_\varphi$ the first and last vertices of P are in $A_\varphi \cup B_\varphi$. Let C'_1 be the connected component of $G[A_\varphi \cup B_\varphi]$ that contains the first vertex of P and C'_2 be the connected component of $G[A_\varphi \cup B_\varphi]$ that contains the last vertex of P . Note that every component of $G[A_\varphi \cup B_\varphi]$, with the exception of C , is contained in a component of $G[A_{\varphi'} \cup B_{\varphi'}]$, and that therefore $C'_1 \subseteq C_1$ and $C'_2 \subseteq C_2$. By property (iii) applied to W there is a path P' in G from C'_1 to C'_2 through $\varphi^{-1}(u) - S_2$ in G . The internal vertices of the path P' are contained in a chunk $Q \subseteq \varphi^{-1}(u)$ of W . Since Q has a neighbor in C'_1 , Q is not C -dependent, and therefore $Q \subseteq \varphi'^{-1}(u)$. Thus P' is a path in G' from C'_1 to C'_2 through $\varphi'^{-1}(u) - S_2$. Since $S'_2 \subseteq S_2$ it follows that P' is a path in G' from C'_1 to C'_2 in $\varphi'^{-1}(u) - S'_2$. Since $C'_1 \subseteq C_1$ and $C'_2 \subseteq C_2$, P' is a path in G' from C_1 to C_2 through $\varphi'^{-1}(u) - S'_2$. An identical proof shows that if there is

a path in G' from C_1 and C_2 through $\varphi'^{-1}(u) - S'_2$, then there is also a path in G' from C_1 and C_2 through $\varphi'^{-1}(u) - S'_1$.

- For property (iv) let u be a peripheral vertex of H' and C_A be a component of $G'[A_{\varphi'}]$ with a neighbor in $\varphi'^{-1}(u)$. Then, by property (iv) applied to W there is a path P in G from C_A to B_{φ} through $\varphi^{-1}(u)$. The internal vertices of the path P are contained in a chunk $Q \subseteq \varphi^{-1}(u)$ of W . Since Q has a neighbor in C_A , Q is not C -dependent, and therefore $Q \subseteq \varphi'^{-1}(u)$. Since $B_{\varphi} \subseteq B_{\varphi'}$ it follows that P is a path in G' from C_A to $B_{\varphi'}$ through $\varphi'^{-1}(u)$.

Consider now a peripheral vertex u of H' and a component C_B of $G'[B_{\varphi'}]$ with a neighbor y in $\varphi'^{-1}(u)$. Let Q be the connected component of $G'[\varphi'^{-1}(u)]$ that contains y . Then Q is a chunk of u in W , and furthermore, because $Q \subseteq \varphi'^{-1}(u)$, Q is not C -dependent. Therefore Q has a neighbor in $A_{\varphi} - C = A_{\varphi'}$, and so there is a path in G' from C_B to $A_{\varphi'}$ through $\varphi'^{-1}(u)$.

The proof for the case where C is a component of $G[B_{\varphi}]$ is symmetric.

For the upper bound on the adhesion size of every peripheral vertex u it is sufficient to observe that every connected component of $G'[A_{\varphi'} \cup B_{\varphi'}]$ that contains a neighbor of $\varphi^{-1}(u)$ contains a connected component of $G[A_{\varphi'} \cup B_{\varphi'}]$ that contains a neighbor of $\varphi^{-1}(u)$. $\square$

Next we track what erasing a component of $G[A_{\varphi} \cup B_{\varphi}]$ does to the labeled connectivity graph. We will only track the effect on the connectivity graph for the side of the component C that we erase.

LEMMA 5.20. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature, and let C be an erasable component of $G[A_{\varphi}]$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized ω' -creature resulting from erasing C . Let (C_A, λ) be the labeled A -connectivity graph of W , and let (C'_A, λ') be the labeled A -connectivity graph of W' . Then $(C'_A, \lambda') = (C_A, \lambda) - \{C\}$.*

Proof. The components of $G'[A_{\varphi'}]$ are precisely the components of $G[A_{\varphi}]$, except for C . Thus the vertex sets of (C'_A, λ') and (C_A, λ) are equal.

We first prove that for every pair of components C_1, C_2 of $G'[A_{\varphi'}]$, if u is a peripheral vertex of H that realizes $\{C_1, C_2\}$ (in W) then u is also a peripheral vertex of H' that realizes $\{C_1, C_2\}$ in W' . Towards this aim, suppose that there exists a path P in G from C_1 to C_2 through $\varphi^{-1}(u) - S_1$. Then the internal vertices of P are contained in a chunk Q of u . The chunk Q has a neighbor in C_1 and therefore it is not C -dependent. Thus $u \notin D(C)$, so u is a peripheral vertex in H' and P is a path from C_1 to C_2 through $\varphi'^{-1}(u)$. Since $S'_1 \subseteq S_1$ we conclude that P is a path from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1$, and u realizes $\{C_1, C_2\}$ in W' .

Next we show that for every pair of components C_1, C_2 of $G'[A_{\varphi'}]$, if u is a peripheral vertex of H that realizes $\{C_1, C_2\}$ in W' then u also realizes $\{C_1, C_2\}$ in W . Towards this goal suppose that there exists a path P in G' from C_1 to C_2 through $\varphi'^{-1}(u')$. Then all internal vertices of P lie in a chunk Q of u' . Then Q is also a chunk of W , and since Q has a neighbor in C_1 it is not C -dependent. Thus $S_1 \cap Q = S'_1 \cap Q$ and so P is a path in G from C_1 to C_2 through $\varphi^{-1}(u')$. Hence u realizes $\{C_1, C_2\}$ in W . This concludes the proof. $\square$

Finally we track how erasing a component in $G[A_{\varphi}]$ affects which peripheral vertices have neighbors in which components of $G[A_{\varphi}]$.

LEMMA 5.21. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature, and let C be an erasable component of $G[A_{\varphi}]$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized ω' -creature resulting from erasing C . Then, for every peripheral vertex u in H and every component C' in $G[A_{\varphi}]$ with $C' \neq C$ it holds C' has a neighbor in $\varphi^{-1}(u)$ in G if and only if u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' .*

Proof. For the forward direction, suppose $\varphi'^{-1}(u)$ has a neighbor in C' in G' . Then u is not C -dependent and therefore u is a peripheral vertex of H' . Furthermore, $\varphi^{-1}(u)$ contains a chunk Q that has a neighbor in C' . Q is not C -dependent, and therefore $Q \subseteq \varphi'^{-1}(u)$. Thus $\varphi'^{-1}(u)$ has a neighbor of C' in G' .

For the reverse direction suppose that u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' . Since $\varphi'^{-1}(u) \subseteq \varphi^{-1}(u)$ it follows that C' has a neighbor in $\varphi^{-1}(u)$ in G . $\square$

5.9 Path Filtering: Extracting an A -Path-Like ω -creature.

DEFINITION 5.12. We will say that a generalized ω -creature W is A -path-like if the A -connectivity graph of W is a path.

DEFINITION 5.13. A disjoint ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ is an A -descendant of a disjoint generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$ if W' can be obtained from W by any sequence of dissolving peripheral vertices, erasing erasable components of $G[\varphi_A]$, absorbing absorbable components of $G[\varphi_A]$.

At a later stage in the proof we will show that if W has some nice properties (to be defined later) and W' is a descendant of W , then W' also has these properties. Note that the erasing component operation only applies to disjoint generalized ω -creatures, so the notion of descendant is only well defined for disjoint generalized ω -creatures. For now we will make the following simple observation.

LEMMA 5.22. If a disjoint ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ is an A -descendant of a disjoint ω' -creature $W = (G, H, \varphi, S_1, S_2)$ then G' is an induced subgraph of G . Furthermore, if W has adhesion size α then W' has adhesion size α .

Proof. Dissolving peripheral vertices and erasing erasable components of a disjoint generalized ω -creature $W = (G, H, \varphi, S_1, S_2)$ produces a generalized $\hat{\omega}$ -creature $\hat{W} = (\hat{G}, \hat{H}, \hat{\varphi}, \hat{S}_1, \hat{S}_2)$ where $\hat{G}$ is an induced subgraph of G . Absorbing an absorbable component of $G[\varphi_A]$ produces a generalized $\hat{\omega}$ -creature $\hat{W} = (\hat{G}, \hat{H}, \hat{\varphi}, \hat{S}_1, \hat{S}_2)$ where $\hat{G}$ is equal to G . The fact that G' is an induced subgraph of G now follows by induction on the number of operations used to obtain W' from W .

For the bound on the adhesion size, by Lemma 5.9 dissolving a peripheral vertex does not increase adhesion size, by Lemma 5.10 absorbing a component does not increase adhesion size, and by Lemma 5.19 erasing an erasable component does not increase adhesion size. The bound on the adhesion size of W' now follows by induction on the number of operations used to obtain W' from W . $\square$

LEMMA 5.23. (PATH FILTERING) Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature with adhesion size α , let $(\mathcal{C}_A, \lambda)$ be the labeled A -connectivity graph of W . Let (P, λ) be an induced path in $(\mathcal{C}_A, \lambda)$. Then there exists a disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ such that

- W' is A -path-like.
- W' is an A -descendant of W .
- $A_{\varphi'} = \bigcup_{C \in V(P)} C$
- Every peripheral vertex u of H' is a peripheral vertex of H .
- The labeled A -connectivity graph of W' is equal to (P, λ) .
- For every peripheral vertex u of H and component $C' \in V(P)$ it holds that C' has a neighbor in $\varphi^{-1}(u)$ in G if and only if u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' .

Proof. We prove the lemma by induction on the number of components in $G[A_\varphi]$. For the base case, if $P = \mathcal{C}_A$, then W satisfies the conclusion of the lemma. So suppose that $|V(\mathcal{C}_A)| > |V(P)|$. Then, by Lemma 5.18, $G[A_\varphi]$ has a P -erasable component C . By Lemma 5.19, erasing C from W yields a disjoint generalized $\hat{\omega}$ -creature $\hat{W} = (\hat{G}, \hat{H}, \hat{\varphi}, \hat{S}_1, \hat{S}_2)$ of adhesion size at most α . By Lemma 5.20 the labeled A -connectivity graph $\hat{\mathcal{C}}_A$ of $\hat{W}$ is equal to $(\mathcal{C}_A, \lambda) - \{C\}$. Therefore (P, λ) is an induced subgraph of $(\hat{\mathcal{C}}_A, \lambda)$. Since the number of components of $\hat{G}[\hat{A}_{\hat{\varphi}}]$ is less than the number of components in $G[A_\varphi]$, the induction hypothesis implies that there exists a disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ such that

- W' is A -path-like,
- W' is an A -descendant of $\hat{W}$ (and therefore also of W),
- $A_{\varphi'} = \bigcup_{C \in V(P)} C$,
- every peripheral vertex u of H' is a peripheral vertex of $\hat{H}$ (and therefore also of H),
- the labeled A -connectivity graph of W' is equal to (P, λ) , and

- for every peripheral vertex u of $\hat{H}$ and component $C' \in V(P)$ it holds that C' has a neighbor in $\hat{\varphi}^{-1}(u)$ in $\hat{G}$ if and only if u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' .

For the last point, by Lemma 5.21 we have that for every peripheral vertex u of H and component $C' \in V(P)$ it holds that C' has a neighbor in $\varphi^{-1}(u)$ in G if and only if u is a peripheral vertex of $\hat{H}$ and C' has a neighbor in $\hat{\varphi}^{-1}(u)$ in $\hat{G}$. We conclude that For every peripheral vertex u of H and component $C' \in V(P)$ it holds that C' has a neighbor in $\varphi^{-1}(u)$ in G if and only if u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' . Thus W' satisfies the conclusion of the lemma. $\square$

LEMMA 5.24. *Let G be a k -creature free graph and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature with adhesion size α . Then there exists an A -path-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that W' is an A -descendant of G . Further, $\omega' \geq \frac{\log_{k\alpha}(\omega) - 3}{\alpha}$.*

Proof. By Lemma 5.17, the $\mathcal{A}$ -connectivity graph of W contains a path P on at least $\log_{k\alpha}(\omega/k) - 1$ vertices, and thus at least $\log_{k\alpha}(\omega/k) - 2$ edges. By Lemma 5.23 there exists an A -path-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that W' is an A -descendant of G . For every edge $C_i C_j$ of P there is a peripheral vertex u of H' that realizes the edge. Then $\varphi'^{-1}(u)$ has neighbors in C_i and C_j (in G'). Hence, for every integer x , if u realizes x edges of P then u has neighbors in at least $x + 1$ vertices of P (recall that vertices of P are components of $G'[A_{\varphi'}]$). But W' has adhesion size α and therefore each peripheral vertex of H' realizes at most $\alpha - 1$ edges of P . But then the number of peripheral vertices of H' is at least $\frac{\log_{k\alpha}(\omega/k) - 2}{\alpha - 1} \geq \frac{\log_{k\alpha}(\omega) - 3}{\alpha}$. $\square$

5.10 Effect of Dissolve on the Connectivity Graph We will use Lemma 5.24 to extract an A -path-like generalized ω -creature W . This gets us quite far towards making W a critter, but there are still many irregularities to clean up. For this we will use the “dissolve” operation, but now we need to be careful not to destroy the progress that we have already made. Since this progress is in the connectivity graph (in particular the connectivity graph is a path), we need to track how dissolving a peripheral vertex affects the connectivity graph.

At this point we will need to make a small detour and analyze how dissolving a peripheral vertex affects the (labeled) connectivity graph.

LEMMA 5.25. *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature with labeled A -connectivity graph $(\mathcal{C}_A, \lambda_A)$ and labeled B -connectivity graph $(\mathcal{C}_B, \lambda_B)$, and let u be a peripheral vertex of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized $(\omega - 1)$ -creature resulting from dissolving u in W . Then,*

- for every connected component C of $G[A_\varphi]$ there is a connected component C' of $G[A_{\varphi'}]$ such that $C \subseteq C'$.
- Every connected component C' of $G[A_{\varphi'}]$ contains a connected component C of $G[A_\varphi]$.
- For every pair of connected components C_1, C_2 of $G[A_\varphi]$ there exists a connected component C' of $G[A_{\varphi'}]$ such that $C_1 \cup C_2 \subseteq C'$ if and only if there exists a path P from C_1 to C_2 in $(\mathcal{C}_A, \lambda_A)$ such that every edge $\{Z, Z'\}$ of P satisfies $u \in \lambda_A(\{Z, Z'\})$.
- For every pair of distinct connected components C'_1 and C'_2 of $G[A_{\varphi'}]$ and peripheral vertex v of H' , v realizes $\{C'_1, C'_2\}$ in W' if and only if there exist components C_1 and C_2 of $G[A_\varphi]$ such that $C_1 \subseteq C'_1$, $C_2 \subseteq C'_2$ and v realizes $\{C_1, C_2\}$ in W .
- For every connected component C' of $G[A_{\varphi'}]$ and peripheral vertex v of H' , $\varphi'^{-1}(v)$ has a neighbor in C' in G' if and only if C' contains a component C of $G[A_\varphi]$ such that $\varphi^{-1}(v)$ has a neighbor in C .

Furthermore, all of the above statements hold with A replaced by B (and A_φ by B_φ and $A_{\varphi'}$ by $B_{\varphi'}$).

Proof. Since $A_\varphi \subseteq A_{\varphi'}$ it follows immediately that for every connected component C of $G[A_\varphi]$ there is a connected component C' of $G[A_{\varphi'}]$ such that $C \subseteq C'$.

Consider now a connected component C' of $G[A_{\varphi'}]$. If C' does not contain a vertex from $A_{\varphi'} - A_\varphi$ then $C' \subseteq A_\varphi$ and therefore C' is a component of $G[A_\varphi]$. If C' contains a vertex a of $A_{\varphi'} - A_\varphi$, then $a \in A_1(W) \cap \varphi^{-1}(u)$. Since $G[A_1(W)]$ is connected, contains A_φ , and is disjoint from B_φ , the connected component of $G[A_1(W) \cap \varphi^{-1}(u)]$ that contains a has a neighbor a' in A_φ . But then C' contains a' and therefore also the connected component C of $G[A_\varphi]$ that contains a' .

Let C_1 and C_2 be connected components of $G[A_\varphi]$. Suppose there exists a connected component C' of $G[A'_{\varphi}]$ such that $C_1 \cup C_2 \subseteq C'$. Let Q be a path from C_1 to C_2 through $C' - (C_1 \cup C_2)$. Let $q_1, q_2, \dots, q_\ell$ be the vertices of Q that are also vertices of A_φ , in the order that they appear on Q . For every vertex $q_i \in \{q_1, \dots, q_\ell\}$ define Z_i to be the connected component of $G[A_\varphi]$ that contains q_i . Then, for every $i < \ell$ it holds that either $Z_i = Z_{i+1}$ or the subpath of Q from q_i to q_{i+1} is a path from Z_i to Z_{i+1} through $A_{\varphi'} - A_\varphi$. But $A_{\varphi'} - A_\varphi \subseteq \varphi^{-1}(u) - S_1$. Thus, if $Z_i \neq Z_{i+1}$ then $\{Z_i, Z_{i+1}\}$ is an edge of $\mathcal{C}_A$ and u realizes $\{Z_i, Z_{i+1}\}$, so $u \in \lambda_A(\{Z_i, Z_{i+1}\})$. Hence there exists a walk from $Z_1 = C_1$ to $Z_\ell = C_2$ in $\mathcal{C}_A$ such that every edge $\{Z_i, Z_{i+1}\}$ of the walk satisfies $u \in \{Z_i, Z_{i+1}\}$. But then there also exists such a path from C_1 to C_2 .

For the reverse direction let C_1 and C_2 be connected components of $G[A_\varphi]$ such that there exists a path P in $\mathcal{C}_A$ from C_1 to C_2 such that every edge $\{Z, Z'\}$ of P satisfies $u \in \lambda_A(\{Z, Z'\})$. For each edge $\{Z, Z'\}$ of P there exists a path Q from Z to Z' through $\varphi^{-1}(u) - S_1$. The component of $G[\varphi^{-1}(u) - S_1]$ that contains Q is in $A_1(W)$ and therefore $Q \subseteq A_{\varphi'}$. Hence Z and Z' are contained in the same component of $G[A_{\varphi'}]$. But then all vertices of P (including C_1 and C_2) are contained in the same component of $G[A_{\varphi'}]$.

Next, let C'_1 and C'_2 be a pair of distinct components of $G[A'_{\varphi}]$, and v be a peripheral vertex of H' . We prove the fourth statement.

For the forward direction, suppose v realizes $\{C'_1, C'_2\}$ in W' . Let P be a path in G' from C'_1 to C'_2 through $\varphi'^{-1}(v) - S'_1$. Let s be the first vertex of P and t be the last vertex of P . Since $\varphi'^{-1}(v) = \varphi^{-1}(v)$, $A'_{\varphi} \subseteq A_\varphi \cup \varphi^{-1}(u)$ and $\varphi^{-1}(v)$ and $\varphi^{-1}(u)$ are anti-complete, it follows that $N_{G'}(\varphi'^{-1}(v)) \cap A_{\varphi'} \subseteq N_G(\varphi^{-1}(v)) \cap A_\varphi$. Hence s and t are both elements of A_φ . Let C_1 and C_2 be the components of $G[A_\varphi]$ that contain s and t respectively. We have that $C_1 \subseteq C'_1$ and $C_2 \subseteq C'_2$. Then P is a path in G from C_1 to C_2 through $\varphi'^{-1}(v) - S'_1 = \varphi^{-1}(v) - S_1$, so v realizes $\{C_1, C_2\}$ in W .

For the reverse direction, suppose there exist components C_1 and C_2 of $G[A_\varphi]$ such that $C_1 \subseteq C'_1$, $C_2 \subseteq C'_2$ and v realizes $\{C_1, C_2\}$ in W . Let P be a path in G from C_1 to C_2 through $\varphi^{-1}(v) - S_1$. Then P is a path in G' from C_1 to C_2 through $\varphi^{-1}(v) - S_1 = \varphi'^{-1}(v) - S'_1$. Since $C_1 \subseteq C'_1$, $C_2 \subseteq C'_2$ it follows that v realizes $\{C'_1, C'_2\}$ in W' .

Now we show the fifth property. Let C' be a connected component of $G[A_{\varphi'}]$ and v be peripheral vertex v of H' . For the forward direction suppose that $\varphi'^{-1}(v)$ contains a neighbor x in C' . We have that $C' \subseteq A_\varphi \cup \varphi^{-1}(u)$. On the other hand $\varphi'^{-1}(v) = \varphi^{-1}(v)$ and $\varphi^{-1}(u)$ are anticomplete, so $x \in A_\varphi$. Let C be the component of $G[A_\varphi]$ that contains x . By property (i) $C \subseteq C'$ and C .

For the reverse direction suppose that C' contains a component C of $G[A_\varphi]$ such that $\varphi^{-1}(v)$ has a neighbor in C . Then $\varphi'^{-1}(v) = \varphi^{-1}(v)$ has a neighbor in $C \subseteq C'$ in G' .

The proofs of the corresponding statements for B_φ and B'_{φ} are symmetric. $\square$

Next we need a lemma that tracks the effect on the connectivity graph if we dissolve many peripheral vertices instead of just one. To avoid a (slightly) technical induction we do not fully generalize Lemma 5.25 to dissolving sets of peripheral vertices, and instead prove a slightly weaker set of statements that are still sufficient for our needs.

LEMMA 5.26. *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature with labeled A -connectivity graph $(\mathcal{C}_A, \lambda_A)$ and labeled B -connectivity graph $(\mathcal{C}_B, \lambda_B)$, and let U be a set of peripheral vertices of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized $(\omega - |U|)$ -creature resulting from dissolving all peripheral vertices $u \in U$ in W . Then,*

- (i) *for every connected component C of $G[A_\varphi]$ there is a connected component C' of $G[A_{\varphi'}]$ such that $C \subseteq C'$.*
- (ii) *Every connected component C' of $G[A_{\varphi'}]$ contains a connected component C of $G[A_\varphi]$.*
- (iii) *For every pair of connected components C_1, C_2 of $G[A_\varphi]$ if there exists a path P from C_1 to C_2 in $(\mathcal{C}_A, \lambda_A)$ such that for every edge $\{Z, Z'\}$ of P there exists a $u \in U$ such that $u \in \lambda_A(\{Z, Z'\})$, then there exists a connected component C' of $G[A_{\varphi'}]$ such that $C_1 \cup C_2 \subseteq C'$.*

Furthermore, all of the above statements hold with A replaced by B (and A_φ by B_φ and $A_{\varphi'}$ by $B_{\varphi'}$).

Proof. We prove the statements (i) and (ii) of the lemma by induction on $|U|$. If $|U| = 1$ the statements follows by Lemma 5.25. So suppose that the $|U| \geq 2$ and that the statement of the lemma holds for all smaller values of $|U|$.

Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω -creature with labeled A -connectivity graph $(\mathcal{C}_A, \lambda_A)$ and labeled B -connectivity graph $(\mathcal{C}_B, \lambda_B)$, and let U be a set of peripheral vertices of H . Let v be a vertex in U and $U' = U - \{v\}$. Let $\hat{W} = (\hat{G}, \hat{H}, \hat{\varphi}, \hat{S}_1, \hat{S}_2)$ be the generalized $(\omega - 1)$ -creature resulting from dissolving u in W . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be generalized $(\omega - |U|)$ -creature resulting from dissolving all peripheral vertices $u \in U$ in W . Observe that W' is also the $(\omega - |U|)$ -creature resulting from dissolving all peripheral vertices $u \in U'$ in $\hat{W}$.

Statement (i): Let C be a connected component C of $G[A_\varphi]$. By Lemma 5.25 there exists connected component $\hat{C}$ of $\hat{G}[\hat{A}_{\hat{\varphi}}]$ such that $C \subseteq \hat{C}$. By the induction hypothesis there exists a connected component C' of $G[A_{\varphi'}]$ such that $\hat{C} \subseteq C'$. But then $C \subseteq C'$.

Statement (ii): Let C' be a connected component of $G[A_{\varphi'}]$. By the induction hypothesis C' contains a connected component $\hat{C}$ of $\hat{G}[\hat{A}_{\hat{\varphi}}]$. By Lemma 5.25 $\hat{C}$ contains a connected component C of $G[A_\varphi]$. But then $C \subseteq C'$.

Statement (iii): Let C_1 and C_2 be connected components of $G[A_\varphi]$ such that there exists a path P in $\mathcal{C}_A$ from C_1 to C_2 such that every edge $\{Z, Z'\}$ of P satisfies $u \in \lambda_A(\{Z, Z'\})$. For each edge $\{Z, Z'\}$ of P there exists a $u \in U$ and a path Q from Z to Z' through $\varphi^{-1}(u) - S_1$. The component of $G[\varphi^{-1}(u) - S_1]$ that contains Q is in $A_1(W)$ and therefore (since $u \in U$) we have that $Q \subseteq A_{\varphi'}$. Hence Z and Z' are contained in the same component of $G[A_{\varphi'}]$. But then all vertices of P (including C_1 and C_2) are contained in the same component of $G[A_{\varphi'}]$.

The proofs of the corresponding statements for B_φ and B'_{φ} are symmetric. $\square$

The next lemma shows that if we have already made an A -path-like or B -path-like creature then dissolving peripheral vertices will not break this property.

LEMMA 5.27. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature, U be a set of peripheral vertices of H , and $W' = (G, H, \varphi, S_1, S_2)$ be the disjoint generalized $(\omega - 1)$ -creature resulting from dissolving u in W . If W is A -path-like then W' is A -path-like. If W is B -path-like then W' is B -path-like.*

Proof. We show that if W is A -path-like then W' is A -path-like. We only show the statement for $U = \{u\}$. The full statement for arbitrary sets U then follows immediately by induction on $|U|$.

Let $\mathcal{C}_A$ and $\mathcal{C}'_A$ be the A -connectivity graphs of W and W' respectively. We define a function f that assigns to every component C' of $G[A_{\varphi'}]$ the set of components C of $G[A_\varphi]$ such that $C \subseteq C'$. By statement (i) of Lemma 5.25 $f(C')$ is non-empty. The definition of f immediately implies that for two distinct components C'_1 and C'_2 of $G[A_{\varphi'}]$ the output of $f(C'_1)$ and of $f(C'_2)$ are disjoint. By statement (iii) of Lemma 5.25, $f(C')$ induces a connected subgraph of $\mathcal{C}_A$. By statement (iv) of Lemma 5.25, if $\{C'_1, C'_2\}$ is an edge of $\mathcal{C}'_A$ then there exists an edge from some vertex $C_1 \in f(C'_1)$ to a vertex $C_2 \in f(C'_2)$ in $\mathcal{C}_A$. Hence f is a minor model of $\mathcal{C}'_A$ in $\mathcal{C}_A$. Since W is path-like, $\mathcal{C}_A$ is a path. Since $\mathcal{C}'_A$ is a minor of $\mathcal{C}_A$, every connected component of $\mathcal{C}'_A$ is a path as well. Since $\mathcal{C}'_A$ is connected (by Lemma 5.15), $\mathcal{C}'_A$ is a path, and we conclude that W' is A -path-like.

The proof that if W is B -path-like then W' is B -path-like is symmetric. $\square$

5.11 From Path-Like to Irreducible

Let $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like, disjoint generalized ω -creature, and $(\mathcal{C}_A, \lambda)$ be the labeled A -connectivity graph of W . Recall that vertices in the connectivity graph $\mathcal{C}_A$ are components of $G[A_\varphi]$. We may therefore talk about vertices of $\mathcal{C}_A$ being adjacent to vertices in G , or having neighbors in vertex sets in G . A path P in $\mathcal{C}_A$ *reduces* a peripheral vertex u of H if P has at least two vertices, $\varphi^{-1}(u)$ has a neighbor in the first vertex of P and a neighbor in the last vertex of P , and for every edge $\{C_1, C_2\}$ of P there is a peripheral vertex $y \neq x$ that realizes $\{C_1, C_2\}$. An *A -reduction pair* of W is a pair (u, P) where u is a peripheral vertex of H and P is a path in $\mathcal{C}_A$ that reduces u . When W is clear from context we will simply say that (u, P) is a A -reduction pair.

We say that a path-like, disjoint generalized ω -creature is *A -irreducible* if there does not exist a reduction pair (u, P) of W . An *A -reduction packing* is a set $\mathcal{P}$ of reduction pairs such that for every pair $(u_1, P_1), (u_2, P_2)$ of distinct A -reduction pairs in $\mathcal{P}$, $V(P_1) \cap V(P_2) = \emptyset$. In other words an A -reduction packing is a set of A -reduction pairs whose paths are pairwise vertex-disjoint. An *A -reduction hitting set* for W is a set $X \subseteq V(\mathcal{C}_A)$ such that for every A -reduction pair (u, P) of W we have $X \cap V(P) \neq \emptyset$.

LEMMA 5.28. Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature, and (C_A, λ) be the labeled A -connectivity graph of W . For every integer $p \geq 1$ there either exists an A -reduction packing $\mathcal{P}$ of size p or an A -reduction hitting set X of size $p - 1$.

Proof. The lemma follows directly from the well-known fact (see e.g. [22]) that for every set $\mathcal{S}$ of intervals on the real line and integer p , there either exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ of p pairwise disjoint intervals, or there exists a set $X \subseteq \mathbb{R}$ such that $|X| < p$ and every interval in $\mathcal{S}$ contains an element of X .

Indeed we associate every vertex of C_A with an integer, namely the position of this vertex in the path C_A . Every A -reduction pair (u, P) is associated with the interval from the first to the last vertex of P . A set of disjoint intervals now corresponds to an A -reduction packing of the same cardinality. Similarly, since every interval starts and ends at an integer position, every set of reals that intersect all intervals can be assumed without loss of generality to be a set of integers, and therefore corresponds to an A -reduction hitting set of the same size. $\square$

An A -reduction pair (u_1, P_1) conflicts with another A -reduction pair (u_2, P_2) if $\varphi^{-1}(u_1)$ has a neighbor in a vertex of P_2 or $\varphi^{-1}(u_2)$ has a neighbor in a vertex of P_1 . Note that the conflict relation is symmetric - if (u_1, P_1) conflicts with (u_2, P_2) then (u_2, P_2) conflicts with (u_1, P_1) .

LEMMA 5.29. Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature with adhesion size α , and $\mathcal{P}$ be an A -reduction packing of W . Then there exists an A -reduction pair (u, P) in $\mathcal{P}$ that conflicts with at most $2\alpha - 1$ A -reduction pairs in $\mathcal{P}$.

Proof. Given the A -reduction packing $\mathcal{P}$, define for every peripheral vertex u of H the set

$$Q(u) = \{(u', P') \in \mathcal{P} : \varphi^{-1}(u) \text{ has a neighbor in a vertex of } P'\}$$

Note that in an A -reduction packing $\mathcal{P}$ the paths in the A -reduction pairs are all vertex disjoint. Therefore, since u has adhesion size at most α it follows that $|Q(u)| \leq \alpha$ for every peripheral vertex u of H . For every path P in the A -connectivity graph C_A of W , define

$$Q(P) = \{(u', P') \in \mathcal{P} : \varphi^{-1}(u') \text{ has a neighbor in a vertex of } P\}$$

We have that

$$\begin{aligned} \sum_{(u, P) \in \mathcal{P}} |Q(P)| &= \sum_{(u, P) \in \mathcal{P}} \sum_{(u', P') \in \mathcal{P}} \begin{cases} 1 & \text{if } \varphi^{-1}(u') \text{ has a neighbor in a vertex of } P \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{(u', P') \in \mathcal{P}} |Q(u')| \\ &\leq |\mathcal{P}| \alpha \end{aligned}$$

Thus there exists a $(u, P) \in \mathcal{P}$ such that $|Q(P)| \leq \alpha$. Every A -reduction pair (u', P') that conflicts with (u, P) is in $Q(u) \cup Q(P)$. Indeed, if $\varphi^{-1}(u)$ has a neighbor in a vertex of P' then $(u', P') \in Q(u)$. If $\varphi^{-1}(u')$ has a neighbor in a vertex of P then $(u', P') \in Q(P)$. But $|Q(u) \cup Q(P)| \leq 2\alpha$, and $(u, P) \in Q(u)$, so (u, P) conflicts with at most $2\alpha - 1$ A -reduction pairs in $\mathcal{P}$, as claimed. $\square$

An A -reduction packing $\mathcal{P}$ is *conflict free* if no A -reduction pair in $\mathcal{P}$ conflicts with another A -reduction pair in $\mathcal{P}$.

LEMMA 5.30. Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature of adhesion size α and $\mathcal{P}$ be an A -reduction packing of W . Then there exists a conflict free A -reduction packing $\mathcal{P}' \subseteq \mathcal{P}$ such that $|\mathcal{P}'| \geq |\mathcal{P}|/2\alpha$.

Proof. We prove the lemma by induction on $|\mathcal{P}|$. For $|\mathcal{P}| = 0$ the statement trivially holds, so suppose that $|\mathcal{P}| > 0$. By Lemma 5.29 there exists an A -reduction pair $(u, P) \in \mathcal{P}$ that conflicts with at most $2\alpha - 1$ A -reduction pairs in $\mathcal{P}$. Let $\mathcal{P}^*$ be the subset of all A -reduction pairs in $\mathcal{P} - \{(u, P)\}$ that do not conflict with (u, P) . We have that $|\mathcal{P}| - 2\alpha \leq |\mathcal{P}^*| < |\mathcal{P}|$. By the induction hypothesis $\mathcal{P}^*$ contains a conflict free A -reduction packing $\mathcal{P}'$ of size at least $|\mathcal{P}^*|/2\alpha \geq (|\mathcal{P}|/2\alpha) - 1$. Since (u, P) does not conflict with any A -reduction pair in $\mathcal{P}'$ it follows that $\mathcal{P}' \cup \{(u, P)\}$ is a conflict packing of size at least $|\mathcal{P}|/2\alpha$, and that $\mathcal{P}' \cup \{(u, P)\} \subseteq \mathcal{P}$. This concludes the proof. $\square$

LEMMA 5.31. Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature of adhesion size α and $\mathcal{P}$ be a conflict free A -reduction packing of W . Then there exists a path-like disjoint, generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size $\alpha - 1$ and $\omega' = |\mathcal{P}|$. Furthermore, W' is an A -descendant of W .

Proof. Define R to be the set of all peripheral vertices u of H such that there exists an A -reduction pair $(u, P) \in \mathcal{P}$. Since $\mathcal{P}$ is conflict free we have that for every $u \in R$ there is precisely one A -reduction pair (u, P) in $\mathcal{P}$. Indeed, if there were two such pairs (u, P_1) and (u, P_2) then (u, P_1) would conflict with (u, P_2) because $\varphi^{-1}(u)$ has a neighbor in the first vertex of P_2 . Hence $|R| = |\mathcal{P}|$. We set U to be the set of all peripheral vertices of H that are not in R , and obtain $W' = (G', H', \varphi', S'_1, S'_2)$ by dissolving all peripheral vertices in u . By Lemma 5.9 W' is a disjoint generalized $|R|$ -creature. By definition of A -descendants, W' is an A -descendant of W . By Lemma 5.27 W' is A -path-like.

It remains to show that the adhesion size of W' is at most $\alpha - 1$. Let v be a peripheral vertex of H' with maximum adhesion size in W' , and let $(\mathcal{C}_A, \lambda)$ be the labeled A -connectivity graph of W . Let $C_1, C_2, \dots, C_\ell$ be the connected components of $G[A_\varphi \cup B_\varphi]$ that contain $N_G[\varphi^{-1}(v)]$. Since W has adhesion size α we have that $\ell \leq \alpha$. By Lemma 5.26 there exist components $C'_1, C'_2, \dots, C'_\ell$ of $G'[A_{\varphi'} \cup B_{\varphi'}]$ such that $C_i \subseteq C'_i$ for every i . Furthermore $N_G[\varphi^{-1}(v)] \subseteq A_\varphi \cup B_\varphi$, G' is an induced subgraph of G , and $\varphi'^{-1}(v) = \varphi^{-1}(v)$. Thus $N_{G'}[\varphi'^{-1}(v)] \subseteq N_G[\varphi^{-1}(v)]$. Hence $C'_1, C'_2, \dots, C'_\ell$ contain $N_{G'}[\varphi'^{-1}(v)]$.

Since $v \in R$ there is a A -reduction pair $(v, P) \in \mathcal{P}$. The endpoints of P are components of $G[A_\varphi]$ that contain neighbors of $\varphi^{-1}(v)$ in G . Without loss of generality, C_1 and C_2 are the two endpoints of P . Since (v, P) is a A -reduction pair we have that for every edge $\{Z, Z'\}$ of P there exists a peripheral vertex $u \neq v$ of H such that $u \in \lambda(\{Z, Z'\})$.

We claim that $u \in U$. Suppose not, then there exists an A -reduction pair $(u, P') \in \mathcal{P}$. But $\varphi^{-1}(u)$ has a neighbor in Z (since u realizes $\{Z, Z'\}$) and therefore (u, P') conflicts with (v, P) , contradicting that $\mathcal{P}$ is conflict free. We conclude that $u \in U$.

But then, for every edge $\{Z, Z'\}$ of P there exists a peripheral vertex $u \in U$ such that $u \in \lambda(\{Z, Z'\})$. By statement (iii) of Lemma 5.26 C_1 and C_2 are contained in the same component of $G[A_{\varphi'}]$. But then $C'_1 = C'_2$ and therefore there are at most $\ell - 1 = \alpha - 1$ components of $G[A_{\varphi'} \cup B_{\varphi'}]$ that contain neighbors of $\varphi'^{-1}(v)$ in G' . Hence the adhesion size of v in W' is at most $\alpha - 1$. Since v was the vertex in H' with maximum adhesion size, the adhesion size of W' is at most $\alpha - 1$ as claimed. $\square$

LEMMA 5.32. Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature of adhesion size α , $X \subseteq V(P)$, X be an A -reduction hitting set for W and P be a path in the A -connectivity graph $\mathcal{C}_A$ of W such that $V(P)$ is disjoint from X . Then there exists an A -irreducible path-like disjoint generalized ω' creature W' which is A -descendant of W , with adhesion size α and $\omega' \geq |V(P)|/\alpha$.

Proof. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be disjoint the ω' -creature obtained from W and P by the Path Filtering Lemma (Lemma 5.23). We claim that W' satisfies the conclusion of the lemma. From Lemma 5.23 we have that:

- W' is A -path-like.
- W' is an A -descendant of W .
- $A_{\varphi'} = \bigcup_{C \in V(P)} C$
- Every peripheral vertex u of H' is a peripheral vertex of H .
- The labeled A -connectivity graph of W' is equal to (P, λ) , where $(\mathcal{C}_A, \lambda)$ is the labeled A -connectivity graph of W .
- For every peripheral vertex u of H and component $C' \in V(P)$ it holds that C' has a neighbor in $\varphi^{-1}(u)$ in G if and only if u is a peripheral vertex of H' and C' has a neighbor in $\varphi'^{-1}(u)$ in G' .

By Lemma 5.22, the adhesion size of W' is at most α .

All that remains to show is that W' is A -irreducible, and that $\omega' \geq |V(P)|/\alpha$. To see that W' is A -irreducible, suppose for contradiction that u, P' is an A -reduction pair in W' . Then u is also a peripheral vertex of H , P' is also an induced path in G , $\varphi^{-1}(u)$ has a neighbor in the first and last vertex of P' in G , and every edge $\{C_1, C_2\}$ of p satisfies that $\lambda(\{C_1, C_2\}) - \{u\}$ is non-empty. But then (u, P') is an A -reduction pair in W . Since P' is a sub-path of P and $V(P)$ is disjoint from X it follows that $V(P')$ is disjoint from X . This contradicts that X is an A -reduction hitting set for W .

We now show that $\omega' \geq |V(P)|/\alpha$. Indeed, for every edge $\{C_1, C_2\}$ of P some peripheral vertex u of H' realizes $\{C_1, C_2\}$ in W' . Then $\varphi^{-1}u$ has a neighbor both in C_1 and in C_2 .

Since every vertex of P is incident to an edge of P it holds that for every vertex C of P there exists a peripheral vertex u of H' such that $\varphi^{-1}u$ has a neighbor in C . Thus, since W' has adhesion size α , the number of peripheral vertices of H' is at least $|V(P)|/\alpha$. Hence W' is a generalized ω' -creature with $\omega' \geq |V(P)|/\alpha$. $\square$

LEMMA 5.33. *Let G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature of adhesion size α . Then there exists a path-like disjoint, A -irreducible generalized ω' creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{\omega^{1/\alpha}}{4k\alpha^2}$. Furthermore W' is an A -descendant of W .*

Proof. We prove the statement of the lemma by induction on α . If $\alpha \leq 1$ then no A -reduction pair (u, P) can exist. In particular in every A -reduction pair (u, P) , $\varphi^{-1}(u)$ has a neighbor in the first and last vertex of P , and these two vertices are distinct. Thus u has adhesion size at least 2. Thus, if $\alpha \leq 1$ then W already satisfies the conclusion of the lemma.

Suppose now that $\alpha \geq 2$ and that the statement of the lemma holds for all lower values of α . Let $W = (G, H, \varphi, S_1, S_2)$ be a path-like, disjoint generalized ω -creature of adhesion size α . If W is A -irreducible it already satisfies the conclusion of the lemma. We consider the case where W is not A -irreducible, and set $p = \omega^{1-\frac{1}{\alpha}} \cdot 2\alpha$. By Lemma 5.28 there either exists an A -reduction packing $\mathcal{P}$ of size p or an A -reduction hitting set X of size $p - 1$.

We first consider the case that there exists an A -reduction packing $\mathcal{P}$ of size p . Then, by Lemma 5.30 there exists a conflict-free A -reduction packing $\mathcal{P}' \subseteq \mathcal{P}$ of size at least $|\mathcal{P}|/2\alpha$. Then, by Lemma 5.31 there exists a path-like disjoint generalized $\hat{\omega}$ -creature $\hat{W}$, A -descendant of W , with adhesion size $\alpha - 1$ and $\hat{\omega} = |\mathcal{P}'| \geq p/2\alpha \geq \omega^{1-\frac{1}{\alpha}}$. By the induction hypothesis applied to $\hat{W}$ there exists an A -irreducible path-like disjoint, generalized ω' creature $W' = (G', H', \varphi', S'_1, S'_2)$, A -descendant of $\hat{W}$ (and therefore of W) with adhesion size $\alpha - 1$ and

$$\omega' \geq \frac{\hat{\omega}^{\frac{1}{\alpha-1}}}{4k(\alpha-1)^2} \geq \frac{(\omega^{1-\frac{1}{\alpha}})^{\frac{1}{\alpha-1}}}{4k\alpha^2} = \frac{\omega^{\frac{1}{\alpha}}}{4k\alpha^2}$$

We now consider the case that there exists an A -reduction hitting set X of size p . On one hand, by Lemma 5.1, for every peripheral vertex u of H , $\varphi^{-1}(u)$ has a neighbor in A_φ . On the other hand, By Lemma 5.16, for every a component C of $G[A_\varphi]$ there are fewer than k peripheral vertices u of H such that $\varphi^{-1}(u)$ has a neighbor in C . Therefore there are at least ω/k vertices in $\mathcal{C}_A$. Since $\mathcal{C}_A$ is a path, it contains a sub-path P on at least $\frac{(\omega/k)-|X|}{|X|+1} > \frac{\omega}{k(p+1)} - 1$ vertices disjoint from X . Since the number of vertices is an integer, P has at least $\frac{\omega}{k(p+1)}$ vertices. Thus, by Lemma 5.32 there exists an A -irreducible, path-like disjoint generalized ω' creature W' which is A -descendant of W , with adhesion size α and $\omega' \geq \left(\frac{\omega}{k(p+1)} - 1\right)/\alpha > \frac{\omega}{k(p+1)\alpha} - 1$. Since ω' is an integer it follows that

$$\omega' \geq \frac{\omega}{k(p+1)\alpha} \geq \frac{\omega}{2kp\alpha} \geq \frac{\omega^{\frac{1}{\alpha}}}{4k\alpha^2}.$$

This concludes the proof. $\square$

LEMMA 5.34. *Let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature with adhesion size α . Then there exists a path-like disjoint, A -irreducible generalized ω' creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{(\log_{k\alpha}(\omega)-3)^{1/\alpha}}{8k\alpha^2}$. Furthermore W' is an A -descendant of W .*

Proof. Let G be a k -creature free graph and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature with adhesion size α . Then, by Lemma 5.24 there exists an A -path-like disjoint generalized ω'' -creature $W'' = (G'', H'', \varphi'', S''_1, S''_2)$ with adhesion size α , such that W'' is an A -descendant of G . Further, $\omega'' \geq \frac{\log_{k\alpha}(\omega/k)-2}{\alpha-1}$. By Lemma 5.22, G'' is an induced subgraph of G and therefore also k -creature free.

By Lemma 5.33 applied to W'' there exists a path-like disjoint, A -irreducible generalized ω' creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that

$$\omega' \geq \frac{\omega''^{1/\alpha}}{4k\alpha^2} \geq \frac{(\frac{\log_{k\alpha}(\omega)-3}{\alpha})^{1/\alpha}}{4k\alpha^2} \geq \frac{(\log_{k\alpha}(\omega)-3)^{1/\alpha}}{8k\alpha^2}.$$

Thus W' satisfies the conclusion of the lemma. $\square$

5.12 From A -Path-like and A -Irreducible to A -Critter-like

DEFINITION 5.14. (A -CRITTER-LIKE AND B -CRITTER-LIKE) Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized disjoint ω -creature, and let $(\mathcal{C}_A, \lambda_A)$ and $(\mathcal{C}_B, \lambda_B)$ be the labeled A -connectivity graph and the labeled B -connectivity graph of W , respectively. W is A -critter-like if:

- $\mathcal{C}_A$ is a path,
- Every edge $\{C_1, C_2\}$ of $\mathcal{C}_A$ satisfies $|\lambda_A(\{C_1, C_2\})| = 1$.
- For every peripheral vertex u of H and pair $\{C_1, C_2\}$ of components of $G[\varphi_A]$, if $\varphi^{-1}(u)$ has a neighbor in C_1 and a neighbor in C_2 then u realizes $\{C_1, C_2\}$ in W .

Similarly, W is B -critter-like if:

- $\mathcal{C}_B$ is a path,
- Every edge $\{C_1, C_2\}$ of $\mathcal{C}_B$ satisfies $|\lambda_B(\{C_1, C_2\})| = 1$.
- For every peripheral vertex u of H and pair $\{C_1, C_2\}$ of components of $G[\varphi_B]$, if $\varphi^{-1}(u)$ has a neighbor in C_1 and a neighbor in C_2 then u realizes $\{C_1, C_2\}$ in W .

In this section we show how to extract an A -critter-like disjoint generalized ω' -creature from an A -path-like disjoint generalized ω -creature.

We will need to get rid of peripheral vertices of H that do not realize any edges of $\mathcal{C}_A$. More formally, let $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like generalized ω -creature, and let $\mathcal{C}_A$ be the A -connectivity graph of W . We say that a peripheral vertex u of H is A -useless if u does not realize any pair $\{C_1, C_2\}$ of distinct components of $G[A_\varphi]$, and $\varphi^{-1}(u)$ has no neighbors in the endpoints of $\mathcal{C}_A$. We would like to dissolve such peripheral vertices, thus we need to track what dissolving them does to the connectivity graph.

LEMMA 5.35. Let $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like, A -irreducible disjoint generalized ω -creature of adhesion size α , and let $(\mathcal{C}_A, \lambda)$ be the labeled A -connectivity graph of W . Let u be an A -useless peripheral vertex of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized ω -creature resulting from dissolving u in W . Let $(\mathcal{C}'_A, \lambda')$ be the labeled A -connectivity graph of W' . Then W' is an A -path-like, A -irreducible, and disjoint generalized $(\omega - 1)$ -creature of adhesion size α . Furthermore, for every peripheral vertex v of H , if v is not A -useless in W then v is not A -useless in W' .

Proof. By Lemma 5.9 we have that W' is a disjoint generalized $(\omega - 1)$ -creature of adhesion size α . By Lemma 5.25 statement (i) every connected component of $G[A_\varphi]$ is contained in a component of $G[A_{\varphi'}]$. By Lemma 5.25 statements (ii) and (iii), every connected component C' of $G[A_{\varphi'}]$ contains precisely one component of $G[A_\varphi]$. By Lemma 5.25 statement (iv) a peripheral vertex v of H' realizes a pair $\{C'_1, C'_2\}$ of components of $G[A_{\varphi'}]$ in W' if and only if v realizes $\{C_1, C_2\}$ in W , where C_1 and C_2 are the unique components of $G[A_\varphi]$ contained in C'_1 and C'_2 respectively.

Hence the bijection ψ that maps a component of $G[A_\varphi]$ to the unique component of $G[A_{\varphi'}]$ that contains it is an isomorphism from $(\mathcal{C}_A, \lambda)$ to $(\mathcal{C}'_A, \lambda')$, in the sense that $\{C, C'\} \in E(\mathcal{C}_A)$ if and only if $\{\psi(C), \psi(C')\} \in E(\mathcal{C}_A)$ and $\lambda(\{C, C'\}) = \lambda'(\{\psi(C), \psi(C')\})$.

Hence W' is A -path-like, and every peripheral vertex v which realizes an edge of $\mathcal{C}_A$ also realizes the same edge of $\mathcal{C}'_A$. Similarly, every peripheral vertex v with a neighbor in an endpoint Q of $\mathcal{C}_A$ has a neighbor in $\psi(Q)$. Hence, if v is not A -useless in W it is not A -useless in W' either. Furthermore, if W' has an A -reduction pair (v, P) then $(v, \psi^{-1}(P))$ is an A -reduction pair in W . Thus W' is A -irreducible. $\square$

For this we will absorb absorbable components of $G[A_\varphi]$. We therefore need to track how absorbing a component of $G[A_\varphi]$ affects the labeled A -connectivity graph.

LEMMA 5.36. Let $\omega \geq 2$ and $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like, A -irreducible disjoint generalized ω -creature of adhesion size α such that H has no A -useless peripheral vertices. Let C be an absorbable component of $G[\varphi_A]$, and u be the peripheral vertex of H such that $N(C) \subseteq \varphi^{-1}(u)$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the generalized ω -creature resulting from absorbing C in W . Then W' is an A -path-like, A -irreducible, and disjoint generalized ω -creature of adhesion size α , and H' has no A -useless peripheral vertices.

Proof. By Lemma 5.10 W' is a disjoint generalized ω -creature of adhesion size α . By definition of absorb, $A_{\varphi'} = A_{\varphi} - C$ so the set of connected components of $G[A_{\varphi'}]$ is precisely the set of connected components of $G[A_{\varphi}]$ minus $\{C\}$.

We observe that $G[A_{\varphi}]$ has at least 2 components. Otherwise C is the only component of $G[A_{\varphi}]$. Then Lemma 5.1 yields that for every peripheral vertex v of H , $\varphi^{-1}(v)$ has a neighbor in C , contradicting that C is absorbable.

Let $(\mathcal{C}_A, \lambda)$ be the labeled A -connectivity graph of W and $(\mathcal{C}'_A, \lambda')$ be the labeled A -connectivity graph of W' . By assumption $\mathcal{C}_A$ is a path. We have that $S'_1 = S_1$ and for every peripheral vertex $v \neq u$ of H' we have that $\varphi'^{-1}(v) = \varphi^{-1}(v)$. Hence for every pair of components C_1, C_2 of $G[A_{\varphi'}]$ and peripheral vertex $v \neq u$ we have that v realizes $\{C_1, C_2\}$ in W' if and only if v realizes $\{C_1, C_2\}$ in W .

There are two cases, C is an endpoint of $\mathcal{C}_A$ or not. Suppose C is an endpoint of $\mathcal{C}_A$. We claim that for every pair $\{C_1, C_2\}$ of components of $G[A_{\varphi'}]$, u realizes $\{C_1, C_2\}$ in W' if and only if u realizes $\{C_1, C_2\}$ in W . For the reverse direction, let P be a path from C_1 to C_2 through $\varphi^{-1}(u) - S_1$. Then P is also a path from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1$, because $\varphi'^{-1}(u) = \varphi^{-1}(u) \cup C$, $S'_1 = S_1$ and S_1 is disjoint from C . Hence u realizes $\{C_1, C_2\}$ in W' .

For the forward direction, suppose that u realizes an $\{C_1, C_2\}$ in W' . Then there exists a path P from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1 = (\varphi^{-1}(u) \cup C) - S_1$. Since C only has one neighbor in $\mathcal{C}_A$, without loss of generality C_1 is not a neighbor of C in $\mathcal{C}_A$. If P contains a vertex of C , let P' be a shortest sub-path of P with one endpoint in C_1 and the other in C . Then P' is a path from C to C_1 through $\varphi^{-1}(u) - S_1$, contradicting that C_1 is non-neighbor of C in $\mathcal{C}_A$. Therefore P is disjoint from C . Then P is a path from C_1 to C_2 through $\varphi^{-1}(u) - S_1$, so u realizes $\{C_1, C_2\}$ in W .

Thus, when C is an endpoint of $\mathcal{C}_A$ we get that $(\mathcal{C}'_A, \lambda') = (\mathcal{C}_A - \{C\}, \lambda)$. Hence W' is path-like. Furthermore every A -reduction pair (v, P) in W' is an A -reduction pair also in W . Hence W' is irreducible. It remains to show that no vertices of H' are A -useless. Consider now a peripheral vertex $v \neq u$ of H . Since v is not A -useless in W , v realizes an edge of $\mathcal{C}_A$ or $\varphi^{-1}(v)$ has a neighbor in an endpoint of $\mathcal{C}_A$. Since $v \neq u$ and $N(C) \subseteq \varphi^{-1}(u)$ we have that v has no neighbors in C . Thus if v realizes an edge $\{C_1, C_2\}$ of $\mathcal{C}_A$ in W then v realizes $\{C_1, C_2\}$ in W' . If $\varphi^{-1}(v)$ has a neighbor in an endpoint C' of $\mathcal{C}_A$ then $C' \neq C$ and v has a neighbor in C' in W' as well. Hence v is not A -useless in W' . Finally, let C_1 be the unique neighbor of C in $\mathcal{C}_A$. Since u realizes the edge $\{C, C_1\}$ in $\mathcal{C}_A$, $\varphi^{-1}(u)$ has a neighbor in C_1 , and C_1 is an endpoint of $\mathcal{C}'_A$. Therefore u is not A -useless in W' . We conclude that in the case when u is an endpoint of $\mathcal{C}_A$ the statement of the lemma holds.

Suppose now that C is not an endpoint of $\mathcal{C}_A$ and let X and Y be the predecessor and successor of C on the path $\mathcal{C}_A$, respectively. We claim that u realizes a pair $\{C_1, C_2\}$ of components of $G[A_{\varphi'}]$ if and only if u realizes $\{C_1, C_2\}$ in W or $\{C_1, C_2\} = \{X, Y\}$. For the reverse direction, suppose u realizes $\{C_1, C_2\}$ in W . Let P be a path from C_1 to C_2 through $\varphi^{-1}(u) - S_1$. Then P is also a path from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1$, because $\varphi'^{-1}(u) = \varphi^{-1}(u) \cup C$, $S'_1 = S_1$ and S_1 is disjoint from C . Hence u realizes $\{C_1, C_2\}$ in W' .

Suppose now that $C_1 = X$ and $C_2 = Y$. Then there is a path P_1 from C_1 to C through $\varphi^{-1}(u) - S_1$, and a path P_2 from C to C_2 through $\varphi^{-1}(u) - S_1$. Since $S_1 = S'_1$ and $\varphi^{-1}(u) \cup C = \varphi'^{-1}(u)$ there is a walk from C_1 to C_2 through $\varphi'^{-1}(u)$, namely P_1 , followed by a walk from the end of P_1 to the start of P_2 in $G[C]$ and then by P_2 from C to C_2 . Hence u realizes $\{C_1, C_2\} = \{X, Y\}$ in W' .

For the forward direction, suppose that u realizes an $\{C_1, C_2\}$ in W' . Then there exists a path P from C_1 to C_2 through $\varphi'^{-1}(u) - S'_1 = (\varphi^{-1}(u) \cup C) - S_1$. If P is disjoint from C then P is a path from C_1 to C_2 through $\varphi^{-1}(u) - S_1$, so u realizes $\{C_1, C_2\}$ in W . If $V(P)$ intersects with C then P contains a sub-path P_1 from C_1 to C through $\varphi'^{-1}(u) - C = \varphi^{-1}(u)$ and a path P_2 from C_2 to C through $\varphi'^{-1}(u) - C = \varphi^{-1}(u)$. But then C_1 and C_2 are both adjacent to C in $\mathcal{C}_A$ and therefore $\{C_1, C_2\} = \{X, Y\}$. Thus, when C is not an endpoint of $\mathcal{C}_A$ we get that $(\mathcal{C}'_A, \lambda')$ and $(\mathcal{C}_A - \{C\}, \lambda)$ are equal, except that $\{X, Y\}$ is a non-edge of $\mathcal{C}_A - \{C\}$ and an edge of $\mathcal{C}'_A$ with label $\lambda'(\{X, Y\}) = \{u\}$. Hence W' is path-like.

We show that W' is A -irreducible. For every A -reduction pair (v, P) in W' , if $\{X, Y\}$ is not an edge of P then (v, P) is also an A -reduction pair in W . If $\{X, Y\}$ is an edge of P then (v, P') is an A -reduction pair in W , where P' is the path obtained from P by removing the edge $\{X, Y\}$, adding the vertex C and the edges $\{X, C\}$ and $\{C, Y\}$. In particular $\lambda(\{X, C\}) = \lambda\{C, Y\} = \lambda'\{X, Y\} = \{u\}$. Thus, since W is A -irreducible, W' is A -irreducible as well.

It remains to show that no vertices of H' are A -useless. Consider now a peripheral vertex $v \neq u$ of H . Since v is not A -useless in W , v realizes an edge of $\mathcal{C}_A$ or $\varphi^{-1}(v)$ has a neighbor in an endpoint of $\mathcal{C}_A$. Since $v \neq u$ and

$N(C) \subseteq \varphi^{-1}(u)$ we have that v has no neighbors in C . Thus if v realizes an edge $\{C_1, C_2\}$ of $\mathcal{C}_A$ in W then v realizes $\{C_1, C_2\}$ in W' . If $\varphi^{-1}(v)$ has a neighbor in an endpoint C' of $\mathcal{C}_A$ then $C' \neq C$ and v has a neighbor in C' in W' as well. Hence v is not A -useless in W' . Finally observe that u realizes $\{X, Y\}$ in W' and therefore is not A -useless in W' . This concludes the proof. $\square$

LEMMA 5.37. *If W is an A -path-like, A -irreducible, and disjoint generalized ω -creature that does not have A -useless peripheral vertices or absorbable components in $G[A_\varphi]$, then W is A -critter-like.*

Proof. We prove that W satisfies each of the three properties of A -critter-like generalized ω -creatures. Let $(\mathcal{C}_A, \lambda)$ be the A -connectivity graph of W . First, $\mathcal{C}_A$ is a path because W is A -path-like. For the second property, let $\{C_1, C_2\}$ be an edge of the A -connectivity graph $\mathcal{C}_A$. Since every edge of $\mathcal{C}_A$ is realized by at least one peripheral vertex u of H we have that $|\lambda_A(\{C_1, C_2\})| \geq 1$. We now show that $|\lambda_A(\{C_1, C_2\})| \leq 1$. Suppose for contradiction that there exists a peripheral vertex $v \neq u$ such that $v \in \lambda_A(\{C_1, C_2\})$. Since u realizes $\{C_1, C_2\}$ we have that $\varphi^{-1}(u)$ has neighbors both in C_1 and in C_2 . But then (u, C_1C_2) is an A -reduction pair, contradicting that W is A -irreducible.

For the third property, suppose for contradiction that there exists a peripheral vertex u such that $\varphi^{-1}(u)$ has neighbors in two distinct components C_1, C_2 of $G[A_\varphi]$ and u does not realize $\{C_1, C_2\}$. Let P be the path from C_1 to C_2 in $\mathcal{C}_A$. We consider two cases, either at least one edge of P is labeled with $\{v\}$ for some peripheral vertex $v \neq u$, or all edges of P are labeled with $\{u\}$.

Suppose first that at least one edge $\{Z, Z'\}$ of P is labeled with $\{v\}$ for some peripheral vertex $v \neq u$. Let P' be the shortest sub-path of P that contains the edge $\{Z, Z'\}$ such that the first component C'_1 of P' and the last component C'_2 have a neighbor in $\varphi^{-1}(u)$. Since P itself satisfies the two properties, P' is well defined. Since P' is shortest, no internal vertex of P' can have a neighbor in $\varphi^{-1}(u)$. But then u does not realize any edge of P' and therefore (u, P') is an A -reduction pair, contradicting that W is A -irreducible.

Suppose now that all edges of P are labeled with $\{u\}$. Since u does not realize $\{C_1, C_2\}$ the path P has at least one internal vertex C . Since C is not absorbable there exists a peripheral vertex $v \neq u$ such that C has a neighbor in $\varphi^{-1}(v)$. Since v is not A -useless, v realizes at least one edge of $\mathcal{C}_A$ or $\varphi^{-1}(v)$ has a neighbor in an endpoint of $\mathcal{C}_A$. If v realizes at an edge of $\mathcal{C}_A$ then this edge can not be incident to C , because all edges incident to C are labeled $\{u\}$. Hence $\varphi^{-1}(v)$ has a neighbor in at least one vertex of $\mathcal{C}_A$ other than C . If $\varphi^{-1}(v)$ has a neighbor in an endpoint of $\mathcal{C}_A$ then this endpoint is not equal to C , because C is an internal vertex of P . In either case $\varphi^{-1}(v)$ has a neighbor in at least one vertex of $\mathcal{C}_A$ other than C .

Let P' be a shortest path in $\mathcal{C}_A$ from C to another vertex of $\mathcal{C}_A$ that contains a neighbor of $\varphi^{-1}(v)$. None of the edges of P' are labeled $\{v\}$ because only the endpoints of P' contain neighbors of $\varphi^{-1}(v)$, and one of the endpoints, namely C , is not incident to any edges labeled $\{v\}$. But then (v, P') is an A -reduction pair, contradicting that W is A -irreducible. We conclude that W is A -critter-like. $\square$

LEMMA 5.38. *Let G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like, A -irreducible disjoint generalized ω creature of adhesion size α . Then there exists an A -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{\omega}{k\alpha} - 1$. Furthermore W' is an A -descendant of W .*

Proof. Let G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be an A -path-like, A -irreducible disjoint generalized ω creature of adhesion size α .

We claim that $G[A_\varphi]$ has at least ω/k components. Indeed, by Lemma 5.1 for every peripheral vertex v of H , $\varphi^{-1}(v)$ has a neighbor in A_φ , and by Lemma 5.16 for each component C of $G[A_\varphi]$ there are at most $k - 1$ peripheral vertices v such that $\varphi^{-1}(v)$ has a neighbor in C . Thus $G[A_\varphi]$ has at least ω/k components.

Let Z be the set of all peripheral vertices of H that are not A -useless. We claim that $|Z| \geq \omega/k\alpha - 1$. Indeed, $\mathcal{C}_A$ has at least $\omega/k - 1$ edges, and each of these edges is realized by a peripheral vertex $v \in Z$. Since each peripheral vertex v can realize no more than α edges of $\mathcal{C}_A$ it follows that $|Z| \geq \frac{\omega}{k\alpha} - 1$.

Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of repeatedly dissolving A -useless peripheral vertices of H , as long as any are present. By Lemma 5.35 no vertices of Z get dissolved by this process. Furthermore, by Lemma 5.35 W' is an A -path-like, A -irreducible, and disjoint generalized ω' -creature of adhesion size α , where $\omega' = |Z| \geq \frac{\omega}{k\alpha} - 1$.

Let $W'' = (G'', H'', \varphi'', S''_1, S''_2)$ be the result of repeatedly absorbing absorbable components of $G'[A_{\varphi'}]$, as long as any are present. By Lemma 5.36 W'' is an A -path-like, A -irreducible, and disjoint generalized ω' -creature of adhesion size α , and H'' has no A -useless peripheral vertices. Furthermore, $G''[A_{\varphi''}]$ has no absorbable

components. Since W'' was obtained by dissolving peripheral vertices and absorbing absorbable components of $G[A_\varphi]$, W'' is an A -descendant of W . By Lemma 5.37 W'' satisfies the conclusion of the lemma. $\square$

LEMMA 5.39. *Let G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω creature of adhesion size α . Then there exists an A -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{(\log_{k\alpha}(\omega)-3)^{1/\alpha}}{8k^2\alpha^3} - 1$. Furthermore W' is an A -descendant of W .*

Proof. By Lemma 5.34 applied to W there exists an A -path-like, A -irreducible disjoint generalized ω' creature $W'' = (G'', H'', \varphi'', S''_1, S''_2)$ with adhesion size α , such that $\omega'' \geq \frac{(\log_{k\alpha}(\omega)-3)^{1/\alpha}}{8k\alpha^2}$. Furthermore W'' is an A -descendant of W . By Lemma 5.38 applied to W'' there exists an A -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{\omega''}{k\alpha} - 1 \geq \frac{(\log_{k\alpha}(\omega)-3)^{1/\alpha}}{8k^2\alpha^3} - 1$. Furthermore W' is an A -descendant of W'' , and hence of W . This completes the proof. $\square$

5.13 A-Descendants stay B-Critter-Like

LEMMA 5.40. *Let $W = (G, H, \varphi, S_1, S_2)$ be a B -critter-like disjoint generalized ω -creature and u be a peripheral vertex in H . Then either there exists a unique component C of $G[B_\varphi]$ such that $\varphi^{-1}(u)$ has a neighbor in C , or there exist precisely two components C_1, C_2 of $G[B_\varphi]$ such that u realizes $\{C_1, C_2\}$. In this case no other peripheral vertices of H realize $\{C_1, C_2\}$. Furthermore all of the statements above hold with B replaced by A .*

Proof. Let $(\mathcal{C}_B, \lambda)$ be the B -connectivity graph of W . By Lemma 5.1 there exists at least one component C_1 such that $\varphi^{-1}(u)$ has a neighbor in C_1 . If there exist two components C_1, C_2 such that $\varphi^{-1}(u)$ has a neighbor in C_1 and a neighbor in C_2 , then by the third property of critter-like generalized ω -creatures u realizes $\{C_1, C_2\}$. Then $\{C_1, C_2\}$ is an edge of $\mathcal{C}_B$. Further, if another peripheral vertex v also realizes $\{C_1, C_2\}$ then $\{u, v\} \subseteq \lambda(\{C_1, C_2\})$, contradicting the second property of critter-like generalized ω -creatures. If there are exist three components C_1, C_2, C_3 such that $\varphi^{-1}(u)$ has a neighbor in each of them then, by the second property, $\{C_1, C_2\}, \{C_2, C_3\}$ and $\{C_1, C_3\}$ are all edges of $\mathcal{C}_B$. But this contradicts the first property, namely that $\mathcal{C}_B$ is a path. $\square$

LEMMA 5.41. *Let $W = (G, H, \varphi, S_1, S_2)$ be a B -critter-like disjoint generalized ω -creature of adhesion size α , and u be a peripheral vertex of H . Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of dissolving u in W . Then W' is a B -critter-like disjoint generalized $(\omega - 1)$ -creature of adhesion size α . Furthermore, if W is A -critter-like then W' is A -critter-like.*

Proof. By Lemma 5.9 W' is a disjoint generalized $(\omega - 1)$ -creature of adhesion size α . It remains to show that it is B -critter-like. Let $(\mathcal{C}_B, \lambda)$ and $(\mathcal{C}'_B, \lambda')$ be the labeled B -connectivity graphs of W and W' respectively. Let $C_1, C_2, \dots, C_t$ be the components of $G[B_\varphi]$ in the order they are visited by the path $\mathcal{C}_B$. By Lemma 5.25, statement (i) there exists a sequence of (not necessarily distinct) components $C'_1, C'_2, \dots, C'_t$ of $G'[B_{\varphi'}]$ such that $C_i \subseteq C'_i$ for every i . Lemma 5.25, statement (ii) every component of $G'[B_{\varphi'}]$ appears in the sequence $C'_1, C'_2, \dots, C'_t$ at least once. By Lemma 5.40 there are two cases. Either there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r , or there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r and in C_{r+1} .

We first consider the case that there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r . Because u does not realize any edge of $\mathcal{C}_B$, Lemma 5.25, statement (iii) yields that $C'_i \neq C'_j$ for every pair of distinct integers i, j . Lemma 5.25, statement (iv) then yields that the edge set of $\mathcal{C}'_B$ is equal to $\{\{C'_i, C'_{i+1}\} : i < t\}$ and every $\{C'_i, C'_{i+1}\}$ satisfies that $\lambda'(\{C'_i, C'_{i+1}\}) = \lambda(\{C_i, C_{i+1}\})$. Thus $\mathcal{C}_B$ is path-like and $|\lambda'(\{C'_i, C'_{i+1}\})| \leq 1$ for every i . By Lemma 5.25, statement (v) every peripheral vertex v of H' and every component C'_i satisfies that $\varphi'^{-1}(v)$ has a neighbor in C'_i if and only if $\varphi^{-1}(v)$ has a neighbor in C_i . Therefore, if $\varphi'^{-1}(v)$ has a neighbor in C'_i and in C'_j then $\{C'_i, C'_j\}$ is an edge of $\mathcal{C}'_B$ and $\lambda'(\{C'_i, C'_j\}) = \{v\}$.

Next we consider the case that there exists precisely one r such that $\varphi^{-1}(u)$ has a neighbor in C_r and in C_{r+1} . Because u realizes the edge $\{C_r, C_{r+1}\}$ in $\mathcal{C}_B$ and no other edges, Lemma 5.25, statement (iii) yields that $C'_r = C'_{r+1}$ and $C'_i \neq C'_j$ for every pair of distinct integers i, j such that $\{i, j\} \neq \{r, r+1\}$.

Lemma 5.25, statement (iv) then yields that the edge set of $\mathcal{C}'_B$ is equal to $\{\{C'_i, C'_{i+1}\} : i < t \text{ and } i \neq r\}$. Note here that the edge $\{C'_{r+1}, C'_{r+2}\}$, if $r+2 \leq t$, is equal to $\{C'_r, C'_{r+2}\}$, because $C_r = C'_{r+1}$. Thus $\mathcal{C}'_B$ is path-like. Furthermore Lemma 5.25, statement (iv) implies that $\lambda'(\{C'_i, C'_{i+1}\}) = \lambda(\{C_i, C_{i+1}\})$ for every $i \neq r$.

(for $i = r$ we have $C'_i = C'_{i+1}$ and there is no self loop in C'_B). Thus $|\lambda'(\{C'_i, C'_j\})| \leq 1$ for every edge $\{C'_i, C'_j\}$ of C'_B .

By Lemma 5.25, statement (v) every peripheral vertex v of H' and every component C'_i with $i \notin \{r, r+1\}$ satisfies that $\varphi'^{-1}(v)$ has a neighbor in C'_i if and only if $\varphi^{-1}(v)$ has a neighbor in C_i . Furthermore Lemma 5.25, statement (v) yields that for every peripheral vertex v of H' , $\varphi'^{-1}(v)$ has a neighbor in C'_r if and only if $\varphi^{-1}(v)$ has a neighbor in C_r or in C_{r+1} .

For the final property of B -critter-like generalized $(\omega - 1)$ -creatures, suppose that $\varphi'^{-1}(v)$ has a neighbor in two distinct components C'_i and in C'_j of $G'[B_{\varphi'}]$. Without loss of generality $j \notin \{r, r+1\}$. Then $\varphi^{-1}(v)$ has a neighbor in C_j . If $C'_i \neq C'_r$ then $\varphi^{-1}(v)$ has a neighbor in C_i in G . Then, since W is B -critter-like it follows that v realizes $\{C_i, C_j\}$ in W . But by Lemma 5.25, statement (iv) v realizes $\{C'_i, C'_j\}$ in W' . Suppose now that $C'_i \neq C'_r$. Then exists $q \in \{r, r+1\}$ such that $\varphi^{-1}(v)$ has a neighbor in C_q in G . Since $q \neq j$ and W is B -critter-like it follows that v realizes $\{C_q, C_j\}$ in W . But then Lemma 5.25, statement (iv) implies that v realizes $\{C'_r, C'_j\}$ in W' , completing the proof.

The proof that if W is A -critter-like then W' is A -critter-like is symmetric. $\square$

LEMMA 5.42. *Let $W = (G, H, \varphi, S_1, S_2)$ be a B -critter-like disjoint generalized ω -creature of adhesion size α , and C be an absorbable component of $G[A_\varphi]$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of absorbing C in W . Then W' is a B -critter-like disjoint generalized ω -creature of adhesion size α .*

Proof. By Lemma 5.10 W' is a disjoint generalized $(\omega - 1)$ -creature of adhesion size α . It remains to show that it is B -critter-like. Let $(\mathcal{C}_B, \lambda)$ and $(\mathcal{C}'_B, \lambda')$ be the labeled B -connectivity graphs of W and W' respectively. Let $C_1, C_2, \dots, C_t$ be the components of $G[B_\varphi]$ in the order they are visited by the path $\mathcal{C}_B$. By Lemma 5.40 for every $i < t$ there exists a unique peripheral vertex v_i that realizes $\{C_i, C_{i+1}\}$ in W . Again by Lemma 5.40 $\varphi^{-1}(v_i)$ has no neighbors in any components of $G[B_\varphi]$ other than C_i, C_{i+1} , and therefore the vertices $v_1, \dots, v_{t-1}$ are all distinct. Let $Q = \{v_1, \dots, v_{t-1}\}$. By Lemma 5.40 every peripheral vertex u of H not in Q satisfies that $\varphi^{-1}(u)$ has neighbors in precisely one component of $G[B_\varphi]$.

By definition of absorbing, $B_{\varphi'} = B_\varphi$ so $C_1, C_2, \dots, C_t$ are also the components of $G[B_{\varphi'}]$. Since $C \in A_\varphi$, and A_φ and B_φ are anti-complete, it follows that for every peripheral vertex v of H and component C_i , $\varphi'^{-1}(v)$ has a neighbor in C_i if and only if $\varphi^{-1}(v)$ has a neighbor in C_i .

Consider an edge $\{C_i, C_j\}$ of $\mathcal{C}'_B$, and let v be the peripheral vertex that realizes this edge. Then $\varphi'^{-1}(v)$ has neighbors in C_i and C_j , and therefore $\varphi^{-1}(v)$ has neighbors in C_i and C_j . Since W is B -critter-like it follows that v realizes $\{C_i, C_j\}$ in $\mathcal{C}_B$.

Hence $\mathcal{C}'_B$ is a sub-graph of $\mathcal{C}_B$. However $\mathcal{C}_B$ is a path and $\mathcal{C}'_B$ is connected by Lemma 5.15, and therefore $\mathcal{C}'_B = \mathcal{C}_B$. Further, for every edge $\{C_i, C_{i+1}\}$ of $\mathcal{C}_B$ the peripheral vertex v_i is the only peripheral vertex of $H' = H$ such that $\varphi'^{-1}(v)$ has neighbors in $\{C_i, C_{i+1}\}$. Since every edge of $\mathcal{C}_B$ is realized by at least one peripheral vertex, $\{C_i, C_{i+1}\}$ is realized by v_i and no other peripheral vertices in W' .

But then W' is path-like, $|\lambda'(\{C_i, C_{i+1}\})| \leq 1$ for every i and every peripheral vertex v such that $\varphi'^{-1}(v)$ has neighbors in two distinct components $\{C_i, C_j\}$ realizes the pair $\{C_i, C_j\}$ in W' . Hence W' is B -critter-like. $\square$

LEMMA 5.43. *Let $W = (G, H, \varphi, S_1, S_2)$ be a B -critter-like disjoint generalized ω -creature of adhesion size α , and C be an erasable component of $G[A_\varphi]$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of erasing C in W . Then W' is a B -critter-like disjoint generalized ω' -creature of adhesion size α .*

Proof. By Lemma 5.19 W' is a disjoint generalized ω' -creature of adhesion size α . It remains to show that it is B -critter-like. Let $(\mathcal{C}_B, \lambda)$ and $(\mathcal{C}'_B, \lambda')$ be the labeled B -connectivity graphs of W and W' respectively. Let $C_1, C_2, \dots, C_t$ be the components of $G[B_\varphi]$ in the order they are visited by the path $\mathcal{C}_B$. By Lemma 5.40 for every $i < t$ there exists a unique peripheral vertex v_i that realizes $\{C_i, C_{i+1}\}$ in W . Again by Lemma 5.40 $\varphi^{-1}(v_i)$ has no neighbors in any components of $G[B_\varphi]$ other than C_i, C_{i+1} , and therefore the vertices $v_1, \dots, v_{t-1}$ are all distinct. Let $Q = \{v_1, \dots, v_{t-1}\}$. By Lemma 5.40 every peripheral vertex u of H not in Q satisfies that $\varphi^{-1}(u)$ has neighbors in precisely one component of $G[B_\varphi]$. We conclude that $G - A_\varphi$ satisfies the following property: every path from C_i to C_j (with $i \leq j$) in $G - A_\varphi$ intersects all components $C_i, C_{i+1}, \dots, C_j$.

We will make use of the following properties of the erase operation, all of which are easily observed directly from the definition of erase.

- (i) $B_\varphi \subseteq B_{\varphi'} \subseteq V(G) - A_\varphi$,

- (ii) Every connected component of $G'[B_{\varphi'} - B_{\varphi}]$ has a neighbor in B_{φ}
- (iii) Every peripheral vertex v of H' satisfies $\varphi'^{-1}(v) \subseteq \varphi^{-1}(v)$

By (i) every connected component of $G[B_{\varphi}]$ is contained in a connected component of $G'[B_{\varphi'}]$. By (ii) every connected component of $G'[B_{\varphi'}]$ contains at least one connected component of $G[B_{\varphi}]$.

We claim that if a peripheral vertex v of H' satisfies that $\varphi'^{-1}(v)$ has a neighbor in a component C' of $G'[B_{\varphi'}]$, then C' contains a component C_i of $G[B_{\varphi}]$ such that $\varphi^{-1}(v)$ has a neighbor in C_i . Indeed, suppose that $\varphi'^{-1}(v)$ has a neighbor x in a component C' of $G'[B_{\varphi'}]$. Then, by (iii) $x \in B_{\varphi}$ or $x \in \varphi^{-1}(v)$. If $x \in B_{\varphi}$ then C' contains the component C_i of $G[B_{\varphi}]$ that contains x , and $\varphi^{-1}(v)$ has a neighbor in C_i , namely x . If $x \in \varphi^{-1}(v)$, let Z be the component of $C' - B_{\varphi}$ that contains x . Since $x \in \varphi^{-1}(v)$ and Z is disjoint from $A_{\varphi} \cup B_{\varphi}$ it follows that $Z \subseteq \varphi^{-1}(v)$. By (ii), Z has a neighbor y in B_{φ} . Then C' contains the component C_i of $G[B_{\varphi}]$ that contains y , and $\varphi^{-1}(v)$ has a neighbor in C_i , namely y .

Let C' be a component of $G'[B_{\varphi'}]$. If C' contains C_i and C_j it also contains a path from C_i to C_j in $G - A_{\varphi}$. But this path intersects all components $C_i, C_{i+1}, \dots, C_j$, and therefore

$$C_i \cup C_{i+1} \cup \dots \cup C_j \subseteq C'.$$

Thus there exists an ordering of the components of $G'[B_{\varphi'}]$ into $C'_1, C'_2, \dots, C'_{t'}$ and sequence $0 = j_0 < j_1 < j_2 < \dots, j_{t'} = t$ of integers such that for every $i \in \{1, \dots, t\}$ it holds that

$$C_{j_{i-1}+1} \cup C_{j_{i-1}+2}, \dots, C_{j_i} \subseteq C'_i$$

Consider now a peripheral vertex v of H' and suppose that $\varphi'^{-1}(v)$ has neighbors both in C'_i and C'_j . Without loss of generality $i < j$. Then C'_i contains a component $C_{i'}$ of $G[B_{\varphi}]$ and C'_j contains a component $C_{j'}$ of $G[B_{\varphi}]$ such that $\varphi^{-1}(v)$ has a neighbor in $C_{i'}$ and in $C_{j'}$. But then $i' = j_{i'}$, $j' = j_{j'} + 1$ and $v = v_{j_{i'}}$. Hence $j = i + 1$. Thus, every edge of the B -connectivity graph $\mathcal{C}'_B$ of W' goes from a component C'_i to a component C'_{i+1} , and if the edge $\{C'_i, C'_{i+1}\}$ is present then $\lambda'(\{C'_i, C'_{i+1}\}) = v_{j_{i'}}$. By Lemma 5.15 $\mathcal{C}'_B$ is connected and therefore every pair $\{C'_i, C'_{i+1}\}$ is an edge of $\mathcal{C}'_B$.

Hence we have proved that if, for a peripheral vertex v of H' , $\varphi'^{-1}(v)$ has neighbors both in C'_i and C'_j , then $j = i + 1$ and $\{C'_i, C'_{i+1}\}$ is an edge of $\mathcal{C}'_B$ (so v realizes $\{C'_i, C'_j\}$ and W' is B -path-like), and $v = v_{j_{i'}}$ (so $|\lambda'(\{C'_i, C'_{i+1}\})| = 1$). This concludes the proof. $\square$

LEMMA 5.44. *Let $W = (G, H, \varphi, S_1, S_2)$ be a B -critter-like disjoint generalized ω -creature of adhesion size α , and $W' = (G', H', \varphi', S'_1, S'_2)$ be an A -descendant of W . Then W' is a B -critter-like disjoint generalized ω' -creature of adhesion size α .*

Proof. Since $W' = (G', H', \varphi', S'_1, S'_2)$ is an A -descendant of W , W' is obtained from G by a sequence of dissolving peripheral vertices, absorbing absorbable components of $G[A_{\varphi}]$, and erasing components of $G[A_{\varphi}]$. By Lemma 5.41, Lemma 5.42, and Lemma 5.43, dissolving peripheral vertices, absorbing absorbable components of $G[A_{\varphi}]$, and erasing components of $G[A_{\varphi}]$ in a B -critter-like disjoint generalized ω -creature of adhesion size α results in a B -critter-like disjoint generalized ω' -creature of adhesion size α . The statement of the lemma now follows by induction on the number of operations in the sequence used to obtain W' from W . $\square$

5.14 Making the Generalized Creature Critter-Like on Both Sides

LEMMA 5.45. *Let G be a k -creature free graph and $W = (G, H, \varphi, S_1, S_2)$ be an A -critter-like disjoint generalized ω -creature with adhesion size α . Then there exists a B -critter-like disjoint generalized ω -creature $W' = (G, H, \varphi', S_1, S_2)$ with adhesion size α .*

Proof. We set

$$\varphi'(v) = \begin{cases} c_B & \text{if } \varphi(v) = c_A \\ c_A & \text{if } \varphi(v) = c_B \\ \varphi(v) & \text{otherwise.} \end{cases}$$

Then $A_{\varphi'} = B_{\varphi}$ and $B_{\varphi'} = A_{\varphi}$. Since A and B are interchangeable in the definitions of (disjoint) generalized ω -creatures, the A - and B -connectivity graphs $(\mathcal{C}_A, \lambda_A)$ and $(\mathcal{C}_B, \lambda_B)$, A/B -critter-like and adhesion size, W' is a B -critter-like disjoint generalized ω -creature with adhesion size α . $\square$

We are now ready to extract a generalized ω -creature that is both A -critter-like and B -critter-like.

LEMMA 5.46. *Let $k \geq 2$ and G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω creature of adhesion size α . Then there exists an induced subgraph G' of G and an A -critter-like and B -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$, where $\omega' \geq \frac{(\log_{k\alpha} \log_{k\alpha}(\omega))^{1/\alpha}}{16k^2\alpha^3} - 12$.*

Proof. By Lemma 5.39 there exists an A -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ with adhesion size α , such that $\omega' \geq \frac{(\log_{k\alpha}(\omega) - 3)^{1/\alpha}}{8k^2\alpha^3} - 1$. Furthermore W' is an A -descendant of W , so G' is an induced subgraph of G . By Lemma 5.45 there exists a B -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi^*, S'_1, S'_2)$ with adhesion size α . By Lemma 5.39 applied to W' there exists an A -critter-like disjoint generalized ω'' -creature $W'' = (G'', H'', \varphi'', S''_1, S''_2)$ with adhesion size α , such that

$$\begin{aligned} \omega'' &\geq \frac{(\log_{k\alpha}(\omega') - 3)^{1/\alpha}}{8k^2\alpha^3} - 1 \\ &\geq \frac{(\log_{k\alpha} \left(\frac{(\log_{k\alpha}(\omega) - 3)^{1/\alpha}}{8k^2\alpha^3} - 1 \right) - 3)^{1/\alpha}}{8k^2\alpha^3} \\ &\geq \frac{(\log_{k\alpha} \log_{k\alpha}(\omega))^{1/\alpha}}{16k^2\alpha^3} - 12 \end{aligned}$$

Furthermore W'' is an A -descendant of W' , so G'' is an induced subgraph of G' , and therefore of G . Since W'' is an A -descendant of W' and W' is B -critter-like, by Lemma 5.44 W'' is B -critter-like. This concludes the proof. $\square$

5.15 Coordinating The Orderings We are almost done, but not quite. In particular the A and B side of W are now both critter-like, but the peripheral vertices may come in different order on the two sides. We fix this by finding a large common sub-sequence using the famous Erdős-Szekers theorem [16], and dissolve all of the peripheral vertices that are not in the sequence. By Lemma 5.41 this maintains the critter-like property both on the A and the B side. However we do still need to prove that when we dissolve the peripheral vertices that are out of order, we don't re-order the ones that are in order.

$W = (G, H, \varphi, S_1, S_2)$ be an A -critter-like a disjoint generalized ω creature of adhesion size α .

An ordering $v_1, v_2, \dots, v_w$ of the peripheral vertices of H is an A -critter ordering if the following condition is satisfied: if $\varphi^{-1}(v_i)$ and $\varphi^{-1}(v_j)$ both have a neighbor in a component C of $G[A_\varphi]$, then all $v_r \in \{v_i, v_{i+1} \dots v_j\}$ also satisfy that $\varphi^{-1}(v_r)$ has a neighbor in C . A B -critter ordering is defined similarly, using components C of $G[B_\varphi]$.

LEMMA 5.47. *Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω creature. If W is A -critter-like then W has an A -critter ordering. If W is B -critter-like then W has a B -critter ordering.*

Proof. Suppose W is A -critter-like. Let (C_A, λ) be the labeled A -connectivity graph of W . Let $C_1, C_2, \dots, C_t$ be the order in which the components of $G[A_\varphi]$ appear on the path C_A .

By Lemma 5.40 every peripheral vertex v of H satisfies that either there is a unique C_i such that $\varphi^{-1}(u)$ has a neighbor in C_i and in no other components, or there is a unique C_i such that $\varphi^{-1}(u)$ has a neighbor in C_i and C_{i+1} and in no other components. In the second case no other peripheral vertex v also satisfies that $\varphi^{-1}(v)$ has a neighbor both in C_i and C_{i+1} .

Consider the following ordering: first take all peripheral vertices u such that C_1 is the only component such that $\varphi^{-1}(u)$ has a neighbor in it, then take the unique peripheral vertex u such that $\varphi^{-1}(u)$ has a neighbor in C_1 and C_2 , then take all peripheral vertices u such that C_2 is the only component such that $\varphi^{-1}(u)$ has a neighbor in it, and so on. It is easily verified that this ordering is an A -critter ordering. The proof that if W is B -critter-like then W has a B -critter ordering is symmetric. $\square$

Let X be a set and $\sigma = x_1, x_2, \dots, x_t$ be an ordering of X . Let Y be a subset of X . Then the sub-ordering of σ induced by Y is denoted by $\sigma[Y]$ and is the ordering of Y in which all elements of Y come in the same order as in σ .

LEMMA 5.48. Let $W = (G, H, \varphi, S_1, S_2)$ be an A -critter-like disjoint generalized ω creature and u be a peripheral vertex in H . Let σ be an A -critter ordering of W . Let $W' = (G, H, \varphi, S_1, S_2)$ be the A -critter-like disjoint generalized ω creature resulting from dissolving u . Then $\sigma[V(H') - \{c_A, c_B\}]$ is an A -critter ordering of W' . Similarly, if W is B -critter-like and σ is a B -critter ordering of W then $\sigma[V(H') - \{c_A, c_B\}]$ is a B -critter ordering of W' .

Proof. By Lemma 5.41 W' is an A -critter-like generalized $(\omega - 1)$ -creature. Let $(\mathcal{C}_A, \lambda)$ and $(\mathcal{C}'_A, \lambda')$ be the labeled A -connectivity graphs of W and W' respectively. Let $C_1, C_2, \dots, C_t$ be the components of $G[A_\varphi]$ in the order they are visited by the path $\mathcal{C}_A$. By Lemma 5.25, statement (i) there exists a sequence of (not necessarily distinct) components $C'_1, C'_2, \dots, C'_t$ of $G'[A_{\varphi'}]$ such that $C_i \subseteq C'_i$ for every i . Lemma 5.25, statement (ii) every component of $G'[A_{\varphi'}]$ appears in the sequence $C'_1, C'_2, \dots, C'_t$ at least once. By Lemma 5.40 there are two cases. Either there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r , or there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r and in C_{r+1} .

We first consider the case that there exists precisely one component C_r such that $\varphi^{-1}(u)$ has a neighbor in C_r . Because u does not realize any edge of $\mathcal{C}_B$, Lemma 5.25, statement (iii) yields that $C'_i \neq C'_j$ for every pair of distinct integers i, j . By Lemma 5.25, statement (v) every peripheral vertex v of H' and every component C'_i satisfies that $\varphi'^{-1}(v)$ has a neighbor in C'_i if and only if $\varphi^{-1}(v)$ has a neighbor in C_i . Therefore $\sigma[V(H) - \{u\}]$ is an A -critter ordering of W .

Next we consider the case that there exists precisely one r such that $\varphi^{-1}(u)$ has a neighbor in C_r and in C_{r+1} . Because u realizes the edge $\{C_r, C_{r+1}\}$ in $\mathcal{C}_B$ and no other edges, Lemma 5.25, statement (iii) yields that $C'_r = C'_{r+1}$ and $C'_i \neq C'_j$ for every pair of distinct integers i, j such that $\{i, j\} \neq \{r, r+1\}$.

By Lemma 5.25, statement (v) every peripheral vertex v of H' and every component C'_i with $i \notin \{r, r+1\}$ satisfies that $\varphi'^{-1}(v)$ has a neighbor in C'_i if and only if $\varphi^{-1}(v)$ has a neighbor in C_i . Furthermore Lemma 5.25, statement (v) yields that for every peripheral vertex v of H' , $\varphi'^{-1}(v)$ has a neighbor in C'_r if and only if $\varphi^{-1}(v)$ has a neighbor in C_r or in C_{r+1} . Therefore $\sigma[V(H) - \{u\}]$ is an A -critter ordering of W .

The proof that if W is B -critter-like and σ is a B -critter ordering of W then $\sigma[V(H') - \{c_A, c_B\}]$ is a B -critter ordering of W' is symmetric. $\square$

For an ordering $\sigma = x_1, x_2, \dots, x_t$ of a set X the *reverse* ordering of σ is the ordering denoted by σ^R and defined as $\sigma^R = x_t, x_{t-1}, \dots, x_1$.

OBSERVATION 5.3. Let $W = (G, H, \varphi, S_1, S_2)$ be a generalized ω creature and σ be an A -critter ordering of W . Then σ^R is an A -critter ordering of W .

Proof. Let $\sigma = v_1, \dots, v_\omega$ and $\sigma^R = v_t, \dots, v_1$. If $\varphi^{-1}(v_i)$ and $\varphi^{-1}(v_j)$ both have a neighbor in a component C of $G[A_\varphi]$, then all $v_r \in \{v_i, v_{i+1}, \dots, v_j\}$ also satisfy that $\varphi^{-1}(v_r)$ has a neighbor in C . These are precisely the peripheral vertices between v_i and v_j both in the ordering σ and in the ordering σ^R . $\square$

Given two orderings $\sigma_A = x_1^A, x_2^A, \dots, x_t^A$ and $\sigma_B = x_1^B, x_2^B, \dots, x_t^B$ of a set X we say that σ_A and σ_B *agree* on a subset Y of X if, for every pair of elements $y, y' \in Y$, if $y = y_i^A = y_{i'}^B$ and $y' = y_j^A = y_{j'}^B$ then $i < j$ if and only if $i' < j'$. The following theorem is a re-formulation of the well-known Erdős-Szekers Theorem [16] in terms of orderings.

THEOREM 5.1. (ERDŐS-SZEKERS THEOREM [16]) For any two orderings σ_A, σ_B of a set U there exists a subset X of U such that $|X|^2 \geq |U|$ and either σ_A and σ_B agree on X or the reverse σ_A^R of σ_A agrees with σ_B on X .

LEMMA 5.49. Let $W = (G, H, \varphi, S_1, S_2)$ be an A -critter-like and B -critter-like disjoint generalized ω -creature. Then there exist an A -critter-like and B -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$ and ordering σ of $V(H')$ such that G' is an induced subgraph of G , $\omega' \geq \sqrt{\omega}$ and σ is both an A -critter ordering and a B -critter ordering of W' .

Proof. By Lemma 5.47 there exists an A -critter ordering $\sigma_A = v_1^A, v_2^A, \dots, v_\omega^A$ and a B -critter ordering $\sigma_B = v_1^B, v_2^B, \dots, v_\omega^B$ of W . Let σ_A^R be the reverse of σ_A . By Observation 5.3 σ_A^R is also an A -critter order of W . By Theorem 5.1 there exists a subset X of the peripheral vertices such that $|X| \geq \omega$ and either σ_A or σ_A^R agree with σ_B on X . Without loss of generality σ_A agrees with σ_B on X (otherwise we can simply exchange σ_A with

σ_A^R). Let $\omega' = |X|$, we have that $\omega' \geq \sqrt{\omega}$. Let W' be the generalized ω' -creature resulting from dissolving all peripheral vertices of H not in X . By Lemma 5.41, W' is a disjoint A -critter-like and B -critter-like generalized ω -creature. By Lemma 5.48, $\sigma_A[X]$ is an A -critter-order of W' and $\sigma_B[X]$ is a B -critter-order of W' . But σ_A and σ_B agree on X , so $\sigma_A[X] = \sigma_B[X]$. $\square$

LEMMA 5.50. *Let $k \geq 2$ and G be k -creature free, and $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω creature of adhesion size α . Then there exists an induced subgraph G' of G and an A -critter-like and B -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$, where $\omega' \geq \frac{(\log_{k\alpha} \log_{k\alpha}(\omega))^{1/2\alpha}}{4k\alpha^2} - 12$.*

Furthermore there exists an ordering σ of the peripheral vertices such that σ is both an A -critter ordering and a B -critter ordering of W .

Proof. By Lemma 5.46 there exists an induced subgraph G' of G and an A -critter-like and B -critter-like disjoint generalized ω' -creature $W' = (G', H', \varphi', S'_1, S'_2)$, where $\omega' \geq \frac{(\log_{k\alpha} \log_{k\alpha}(\omega))^{1/2\alpha}}{16k^2\alpha^3} - 12$. By Lemma 5.49 applied to W' there exist an A -critter-like and B -critter-like disjoint generalized ω'' -creature $W'' = (G'', H'', \varphi'', S''_1, S''_2)$ and ordering σ of $V(H'')$ such that G'' is an induced subgraph of G' (and therefore of G),

$$\omega'' \geq \sqrt{\omega'} \geq \frac{(\log_{k\alpha} \log_{k\alpha}(\omega))^{1/2\alpha}}{4k\alpha^2} - 12,$$

and σ is both an A -critter ordering and a B -critter ordering of W'' . Hence W'' satisfies the conclusion of the lemma. $\square$

5.16 Extracting a Critter The conclusion of Lemma 5.49 is almost sufficient for us to directly extract an ω -critter from it. There is still one complication, namely peripheral vertices that on at least one side do not realize any edge of the connectivity graph.

LEMMA 5.51. *Let G be a k -creature free graph, and let $W = (G, H, \varphi, S_1, S_2)$ be a disjoint generalized ω -creature. Let Z_A be the set of all peripheral vertices of H such that there exists a component C_A of $G[A_\varphi]$ such that $N(\varphi^{-1}(v)) \cap A_\varphi \subseteq C_A$, and Z_B be the set of all peripheral vertices of H such that there exists a component C_B of $G[B_\varphi]$ such that $N(\varphi^{-1}(v)) \cap B_\varphi \subseteq C_B$. Then $|Z_A| < k$ and $|Z_B| < k$.*

Proof. We prove that $|Z_A| < k$. Suppose for contradiction that $|Z_A| \geq k$. Let $W' = (G', H', \varphi', S'_1, S'_2)$ be the result of dissolving all peripheral vertices not in Z_A . Then, by Lemma 5.9 W' is a disjoint generalized ω' -creature, where $\omega' = |Z_A| \geq k$, and G' is an induced subgraph of G . By Lemma 5.15 the connectivity graph $\mathcal{C}_A$ of W' is connected. Further no vertex of Z_A realizes an edge of $\mathcal{C}_A$. Therefore, by Lemma 5.26, statement (iii), there exists a component C of $G[A'_\varphi]$ such that $A_\varphi \subseteq C$. But then, by Lemma 5.26, statement (ii), $G[A'_\varphi] = C$. This yields a contradiction. On the one hand, Lemma 5.1 shows that each vertex $v \in Z_A$ satisfies that $\varphi'^{-1}(v)$ has a neighbor in $A_{\varphi'}$. On the other hand, G' is k -creature free and W' is a disjoint generalized ω -creature, and so by Lemma 5.16, there can not be k peripheral vertices v such that $\varphi'^{-1}(v)$ has a neighbor in $C' = A_{\varphi'}$. This contradicts the assumption that $|Z_A| \geq k$. The proof of the upper bound for $|Z_B|$ is symmetric. $\square$

With Lemma 5.51 in hand we are ready to extract a critter!

LEMMA 5.52. *Let G be a k -creature free graph and $W = (G, H, \varphi, S_1, S_2)$ be an A -critter-like and B -critter-like disjoint generalized ω -creature, such that there exists an ordering σ of the peripheral vertices such that σ is both an A -critter ordering and a B -critter ordering of W . Then there exists an induced subgraph G' in G , such that G' is an ω' -critter, where $\omega' \geq \frac{\omega - 2k}{2k + 1}$.*

Proof. Let $(\mathcal{C}_A, \lambda_A)$ and $(\mathcal{C}_B, \lambda_B)$ be the labeled A -connectivity graph and labeled B -connectivity graph of W respectively. Let $A_1, A_2, \dots, A_p$ be the components of $G[A_\varphi]$ in the order that they appear in on the path $\mathcal{C}_A$, and $B_1, B_2, \dots, B_q$ be the components of $G[B_\varphi]$ in the order that they appear in on the path $\mathcal{C}_B$. Let $v_1, \dots, v_\omega$ be an ordering of the peripheral vertices such that σ is both an A -critter ordering and a B -critter ordering of W . Let X_A be the subset of peripheral vertices such that every $v_i \in X_A$ realizes a pair $\{A_j, A_{j+1}\}$. Select integers $i_1 < i_2 < \dots < i_{p-1}$ such that $X_A = \{v_{i_1}, v_{i_2}, \dots, v_{i_{p-1}}\}$.

We claim that either, for every $j \in \{1, \dots, p-1\}$ we have that v_{i_j} realizes $\{A_j, A_{j+1}\}$, or for every $j \in \{1, \dots, p-1\}$ we have that v_{i_j} realizes $\{A_{p-j}, A_{p+1-j}\}$. Suppose v_{i_a} realizes $\{A_j, A_{j+1}\}$ and v_{i_b} realizes $\{A_{j+1}, A_{j+2}\}$. If $|a-b| > 1$ then there exists a v_{i_c} in X_A between v_{i_a} and v_{i_b} in the ordering σ . Since σ is an A -critter ordering $\varphi^{-1}(v_{i_c})$ has a neighbor in A_{j+1} . But then $\varphi^{-1}(v_{i_a})$, $\varphi^{-1}(v_{i_b})$ and $\varphi^{-1}(v_{i_c})$ all have a neighbor in A_{j+1} . But $\{\varphi^{-1}(v_{i_a}), \varphi^{-1}(v_{i_b}), \varphi^{-1}(v_{i_c})\} \subseteq X_A$ so each of them realize an edge of $\mathcal{C}_A$. But then each of these edges is incident to A_{j+1} , contradicting that W is A -critter-like. We conclude that if v_{i_a} realizes $\{A_j, A_{j+1}\}$ and v_{i_b} realizes $\{A_{j+1}, A_{j+2}\}$ then $|a-b| \leq 1$. In other words, the vertices of X_A that realize consecutive edges of $\mathcal{C}_A$ come consecutively in the ordering $v_{i_1}, v_{i_2}, \dots, v_{i_{p-1}}$. Hence for every $j \in \{1, \dots, p-1\}$ we have that v_{i_j} realizes $\{A_j, A_{j+1}\}$, or for every $j \in \{1, \dots, p-1\}$ we have that v_{i_j} realizes $\{A_{p-j}, A_{p+1-j}\}$, as claimed.

Without loss of generality the first of these two cases holds. In the second case we may re-name for every i the component A_i to A_{p+1-i} and vice versa (this corresponds to traversing the path $\mathcal{C}_A$ from right to left, rather than from left to right). After re-naming for every $j \in \{1, \dots, p-1\}$ we have that v_{i_j} realizes $\{A_j, A_{j+1}\}$. Note that this does not affect that σ is an A -critter ordering, because the definition of A -critter orderings is independent of orderings of the components of A . Let X_B be the subset of peripheral vertices such that every $v_i \in X_B$ realizes a pair $\{B_j, B_{j+1}\}$. Select integers $i'_1 < i'_2 < \dots < i'_{q-1}$ such that $X_B = \{v_{i'_1}, v_{i'_2}, \dots, v_{i'_{q-1}}\}$. An identical argument to the one for X_A shows that without loss of generality for every $j \in \{1, \dots, q-1\}$ we have that $v_{i'_j}$ realizes $\{B_j, B_{j+1}\}$.

Let Z_A be the set of all peripheral vertices of H such that there exists a component C_A of $G[A_\varphi]$ such that $N(\varphi^{-1}(v)) \cap A_\varphi \subseteq C_A$, and Z_B be the set of all peripheral vertices of H such that there exists a component C_B of $G[B_\varphi]$ such that $N(\varphi^{-1}(v)) \cap B_\varphi \subseteq C_B$. Then, by Lemma 5.51 we have $|Z_A| < k$ and $|Z_B| < k$.

Note that all peripheral vertices that are not in $Z_A \cup Z_B$ realize an edge in $\mathcal{C}_A$ and an edge in $\mathcal{C}_B$ and therefore are in $X_A \cap X_B$. Since $|Z_A \cup Z_B| < 2k$ it follows that there exists a consecutive sub-sequence $v_\ell, v_{\ell+1}, \dots, v_\rho$ of at least $\frac{\omega-2k}{2k+1}$ vertices of $X_A \cap X_B$. Further, for every $j \leq p-1$ we have that v_{i_j} realizes $\{A_j, A_{j+1}\}$, and for every $j \leq q-1$ we have that $v_{i'_j}$ realizes $\{B_j, B_{j+1}\}$. Thus there exist integers a and b such that for every $1 \leq c \leq \rho+1-\ell$ it holds that $v_{\ell+c-1}$ realizes $\{A_{a+c-1}, A_{a+c}\}$ and $\{B_{b+c-1}, B_{b+c}\}$. Said more plainly the peripheral vertices $v_\ell, v_{\ell+1}, \dots, v_\rho$ realize the subpath $A_a, \dots, A_{a+\rho+1-\ell}$ of $\mathcal{C}_A$ and the subpath $B_b, \dots, B_{b+\rho+1-\ell}$ of $\mathcal{C}_B$.

We now construct a critter. We set $\hat{t} = \rho+1-\ell$ and for every $1 \leq i \leq \hat{t}+1$ we set $\hat{A}_i = A_{a+i-1}$ and $\hat{B}_i = B_{b+i-1}$. For every $1 \leq i \leq \hat{t}$ we set $\hat{X}_i = \varphi^{-1}(v_{\ell+i-1})$. We now verify that $\hat{A}_1, \dots, \hat{A}_{\hat{t}+1}, \hat{B}_1, \dots, \hat{B}_{\hat{t}+1}, \hat{X}_1, \dots, \hat{X}_{\hat{t}}$ satisfy the properties of a $\hat{t}$ -critter. For property (i), $\hat{A}_i$ is anticomplete with $\hat{A}_j$ for $j \neq i$ because $\hat{A}_i$ and $\hat{A}_j$ are distinct components of $G[A_\varphi]$. Similarly $\hat{B}_i$ is anticomplete with $\hat{B}_j$ for $i \neq j$. Finally, $\hat{A}_i$ is anticomplete with $\hat{B}_j$ (even for the case $i=j$) because A_φ is anticomplete with B_φ . For property (ii) every set $\hat{A}_i$ and $\hat{B}_i$ is connected in G because they are components of $G[A_\varphi]$ and $G[B_\varphi]$ respectively.

For property (iii) we observe that for every $1 \leq i \leq \hat{t}$, $\hat{X}_i = \varphi^{-1}(v_{\ell+i-1})$, and $v_{\ell+i-1}$ realizes the edges

$$\{A_{a+i-1}, A_{a+i}\} = \{\hat{A}_i, \hat{A}_{i+1}\} \text{ and}$$

$$\{B_{b+i-1}, B_{b+i}\} = \{\hat{B}_i, \hat{B}_{i+1}\}$$

in $\mathcal{C}_A$ and $\mathcal{C}_B$, respectively. Since W is A -critter-like and B -critter-like it follows that $N(\hat{X}_i) \subseteq \hat{A}_i \cup \hat{A}_{i+1} \cup \hat{B}_i \cup \hat{B}_{i+1}$.

For property (iv), let S_1^*, S_2^* be witness separators for W . By property (i) of generalized ω -creatures we have that $S_1^* \cap \hat{X}_i$ and $S_1^* \cap \hat{X}_i$ are distinct minimal A_φ, B_φ -separators in $G[A_\varphi \cup B_\varphi \cup \hat{X}_i]$. Since $N_G(\hat{X}_i) \subseteq \hat{A}_i \cup \hat{A}_{i+1} \cup \hat{B}_i \cup \hat{B}_{i+1}$ it follows that $S_1^* \cap \hat{X}_i$ and $S_1^* \cap \hat{X}_i$ are distinct minimal $(\hat{A}_i \cup \hat{A}_{i+1})$, $(\hat{B}_i \cup \hat{B}_{i+1})$ -separators in $G[\hat{X}_i \cup \hat{A}_i \cup \hat{A}_{i+1} \cup \hat{B}_i \cup \hat{B}_{i+1}]$.

For the last part of property (iv) note that that $\hat{X}_i = \varphi^{-1}(v_{\ell+i-1})$ and that $v_{\ell+i-1}$ realizes the pairs $\{\hat{A}_i, \hat{A}_{i+1}\}$ and $\{\hat{B}_i, \hat{B}_{i+1}\}$. Thus (from the definition of A -connectivity graphs and B -connectivity graphs) there is a path from $\hat{A}_i$ to $\hat{A}_{i+1}$ through $\hat{X}_i - S_1$ and from $\hat{B}_i$ to $\hat{B}_{i+1}$ through $\hat{X}_i - S_1$. By property (iii) of generalized ω -creatures there is also a path from $\hat{A}_i$ to $\hat{A}_{i+1}$ through $\hat{X}_i - S_2$ and from $\hat{B}_i$ to $\hat{B}_{i+1}$ through $\hat{X}_i - S_2$. Since $S_1^* \subseteq S_1$, and $S_2^* \subseteq S_2$ it follows that there exist paths from $\hat{A}_i$ to $\hat{A}_{i+1}$ both through $\hat{X}_i - S_1^*$ and through $\hat{X}_i - S_2^*$ and from $\hat{B}_i$ to $\hat{B}_{i+1}$ both through $\hat{X}_i - S_1^*$ and through $\hat{X}_i - S_2^*$.

We conclude that $\hat{A}_1, \dots, \hat{A}_{\hat{t}+1}, \hat{B}_1, \dots, \hat{B}_{\hat{t}+1}, \hat{X}_1, \dots, \hat{X}_{\hat{t}}$ is a $\hat{t}$ -critter and $\hat{t} \geq \frac{\omega-2k}{2k+1}$ $\square$

We are now ready to prove Lemma 3.2

Lemma 3.2 Let $k \geq 2$ and G be a k -creature free graph, and $W = (G, H, \varphi, S_1, S_2)$ be a connected, good, full generalized ω -creature. Then there exists an induced subgraph G' of G which is a t -critter for $t \geq \frac{(\log \log(\omega))^{1/4k}}{96k^4} - 4$.

Proof. Since W is a connected, good full generalized ω -creature, by Lemma 5.12 there exists an induced subgraph G' of G and a good, full, connected generalized ω' -creature W' with adhesion size $\alpha \leq 2k$ and $\omega' \geq \omega/2$.

Since W' is a full generalized ω' -creature with adhesion size $2k$, by Lemma 5.14 applied to W' there exists a full disjoint generalized ω'' -creature, $W'' = (G'', H'', \varphi'', S_1'', S_2'')$, of adhesion size $2k$ such that G'' is an induced subgraph of G' (and therefore of G), and $\omega'' = \omega' \geq \omega/2k$.

Since G'' is k -creature free and W'' is a disjoint generalized ω'' -creature with adhesion size $2k$, by Lemma 5.50 applied to W'' there exists an induced subgraph G''' of G'' and an A -critter-like and B -critter-like disjoint generalized ω''' -creature $W''' = (G''', H''', \varphi''', S_1''', S_2''')$, where $\omega''' \geq \frac{(\log_{2k^2} \log_{2k^2} (\omega/2k))^{1/4k}}{16k^3} - 12$. Furthermore there exists an ordering σ of the peripheral vertices such that σ is both an A -critter ordering and a B -critter ordering of W''' .

Therefore, by Lemma 5.52 there exists an induced subgraph G^* of G''' (and therefore of G) that is a t -critter for

$$t \geq \frac{\omega''' - 2k}{2k + 1} \geq \frac{(\log \log(\omega))^{1/4k}}{96k^4} - 4$$

This concludes the proof. $\square$

6 Families with Creatures or Critters are Feral

6.1 Boundaried Graphs, Monadic Second Order Logic, and Finite State Towards the proof of Theorem 1.3 we first review the definitions of CMSO logic, boundaried graphs, gluing and finite state.

DEFINITION 6.1. [Graph Family] A graph family is a set $\mathcal{F}$ of graphs.

DEFINITION 6.2. [Boundaried graph] A boundaried graph is a graph G with a set $\delta(G) \subseteq V(G)$ of distinguished vertices called boundary vertices, and an injective labeling $\lambda_G : \delta(G) \rightarrow \mathbb{N}$. The set $\delta(G)$ is the boundary of G , and the label set of G is $\Lambda(G) = \{\lambda_G(v) \mid v \in \delta(G)\}$.

For ease of presentation, we sometimes abuse notation and treat equally-labeled vertices of different boundaried graphs, as well as the vertex that is the result of the identification of two such vertices, as the same vertex. Given a finite set $I \subseteq \mathbb{N}$, $\mathcal{G}_I$ denotes the class of all boundaried graphs whose label set is I , and $\mathcal{G}_{\subseteq I} = \bigcup_{I' \subseteq I} \mathcal{G}_{I'}$. A boundaried graph in $\mathcal{G}_{\subseteq I}$ is called a t -boundaried graph. Finally, $\mathcal{G}$ denotes the class of all boundaried graphs. The main operation employed to unite two boundaried graphs is the one that glues their boundary vertices together. Formally,

DEFINITION 6.3. [Gluing by $\oplus$] Let G_1 and G_2 be two boundaried graphs. Then, $G_1 \oplus G_2$ is the (not-boundaried) graph obtained from the disjoint union of G_1 and G_2 by identifying equally-labeled vertices in $\delta(G_1)$ and $\delta(G_2)$.

Counting Monadic Second Order Logic The syntax of Monadic Second Order Logic (MSO) of graphs includes the logical connectives $\vee, \wedge, \neg, \Leftrightarrow, \Rightarrow$, variables for vertices, edges, sets of vertices and sets of edges, the quantifiers $\forall$ and $\exists$, which can be applied to these variables, and five binary relations:

1. $u \in U$, where u is a vertex variable and U is a vertex set variable;
2. $d \in D$, where d is an edge variable and D is an edge set variable;
3. $\text{inc}(d, u)$, where d is an edge variable, u is a vertex variable, and the interpretation is that the edge d is incident to u ;
4. $\text{adj}(u, v)$, where u and v are vertex variables, and the interpretation is that u and v are adjacent;
5. equality of variables representing vertices, edges, vertex sets and edge sets.

Counting Monadic Second Order Logic (CMSO) extends MSO by including atomic sentences testing whether the cardinality of a set is equal to q modulo r , where q and r are integers such that $0 \leq q < r$ and $r \geq 2$. That is, CMSO is MSO with the following atomic sentence: $\text{card}_{q,r}(S) = \text{true}$ if and only if $|S| \equiv q \pmod{r}$, where S is a set. We refer to [2, 10, 12] for a detailed introduction to CMSO.

DEFINITION 6.4. [**Family $\mathcal{F}_\psi$**] Given a CMSO-formula ψ , the family $\mathcal{F}_\psi$ is defined as the set of all graphs G such that $G \models \psi$.

For an example the formula

$$\psi = \exists X_1 \subseteq V(G) \exists X_2 \subseteq V(G) \left[\begin{aligned} &(\forall u \in V(G) u \in X_1 \vee u \in X_2) \\ &\wedge \forall u \in V(G) \forall v \in V(G) \\ &(\neg \text{adj}(u, v) \vee (u \in X_1 \wedge v \in X_2) \vee (u \in X_2 \wedge v \in X_1)) \end{aligned} \right]$$

yields the family $\mathcal{F}_\psi$ of all bipartite graphs.

DEFINITION 6.5. [**CMSO-definable family**] A family $\mathcal{F}$ is CMSO-definable if there exists a CMSO-formula ψ such that $\mathcal{F} = \mathcal{F}_\psi$. In this case, we say that ψ defines $\mathcal{F}$.

Finite State The goal of this subsection is to recall a variant of the classical Courcelle's Theorem [10, 11, 12] (see also [13]), which is a central component in the proof of our main result. This statement essentially says that the canonical equivalence relation over boundaried graphs defined below has finite index.

DEFINITION 6.6. [**Canonical equivalence**] Given a graph family $\mathcal{F}$, the canonical equivalence relation $\equiv_{\mathcal{F}}$ on boundaried graphs is defined as follows. For two boundaried graphs G_α and G_β , we say that $G_\alpha \equiv_{\mathcal{F}} G_\beta$ if (i) $\Lambda(G_\alpha) = \Lambda(G_\beta)$ and (ii) for all boundaried graphs G_γ we have

$$G_\alpha \oplus G_\gamma \in \mathcal{F} \Leftrightarrow G_\beta \oplus G_\gamma \in \mathcal{F}$$

It is easy to verify that $\equiv_{\mathcal{F}}$ is indeed an equivalence relation. Given a family $\mathcal{F}$ of graphs and $I \subseteq \mathbb{N}$, we let $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ denote the set of equivalence classes of $\equiv_{\mathcal{F}}$ when restricted to $\mathcal{G}_{\subseteq I}$.

DEFINITION 6.7. [**Finite state**] A graph property σ has finite state if, for every $I \subseteq \mathbb{N}$, $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ is finite.

Given a CMSO sentence ψ , the canonical equivalence relation associated with ψ is $\equiv_{\mathcal{F}_\psi}$, and for the sake of simplicity, we denote this relation by $\equiv_\psi$. We are now ready to state the variant of Courcelle's Theorem which was proven by Bodlaender et al. [3] (see also [10, 11, 12]) and which we use in this paper.

THEOREM 6.1. ([3]) Every CMSO-definable graph property has finite state.

Remark 1. Theorem 6.1 is stated for graphs here, while it is stated and proved for more general structures by Bodlaender et al.[3]. Because we do not need the full power of the theorem of [3], and stating the theorem in its full generality requires an extra page of definitions we only state it here for the special case of graphs.

Remark 2. We would like to remark that neither the notion of "finite state" nor the statement of Theorem 6.1 should in any way be attributed to Bodlaender et al. [3].

The notion of finite state and a theorem very similar to the statement of Theorem 6.1 was stated and proved explicitly by Downey and Fellows [15]. For technical reasons the precise statement of the theorem(s) of Downey and Fellows [15] does not adequately suit our needs (or the needs of Bodlaender et al. [3]), nor is it obvious how to derive Theorem 6.1 as a corollary from the results of Downey and Fellows [15]. However the proof of Theorem 6.1 very closely follows proofs of analogous statements by Downey and Fellows [15].

The fact that every MSO-definable or CMSO-property on graphs has finite state is implicitly used, if (to the best of our knowledge) never explicitly stated, in every proof of (variants of) Courcelle's Theorem [4, 10, 11, 12].

6.2 Pumping Proof Let G be a t -critter with t -critter partition $(A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$. The pair (G, W) is called a *witness-minimal t -critter* if there does not exist a proper induced subgraph G' of G such that

$$W' = \{A_i \cap V(G') : i \leq t+1\}, \{B_i \cap V(G') : i \leq t+1\}, \{X_i \cap V(G') : i \leq t\}$$

is a t -critter partition of G' . A pair (S_1^i, S_2^i) of vertex subsets of X_i satisfy X_i if S_1^i, S_2^i satisfy property (iv) of t -critters for X_i .

LEMMA 6.1. Let G be a t -critter and $W = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$ be a t -critter partition of G such that (G, W) is a witness minimal t -critter. Then, for every $i \leq t$ and pair (S_1^i, S_2^i) that satisfy X_i we have $S_1^i \cap S_2^i = \emptyset$.

Proof. Suppose for contradiction that $S_1^i \cap S_2^i$ contains a vertex v . We claim that

$$W' = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_i - \{v\}, X_{i+1}, \dots, X_t)$$

is a t -critter partition of $G - v$. Since W satisfies properties (i), (ii), (iii) for G , we have that W' satisfies properties (i), (ii), (iii) for $G - v$. Thus it is sufficient to argue that $(S_1^i - \{v\}, S_2^i - \{v\})$ satisfy $X_i - \{v\}$ in $G - v$. Since S_1^i is a minimal $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i]$ it follows that $S_1^i - \{v\}$ is a minimal $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$. Further, since $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}] - (S_1^i - \{v\}) = G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i] - S_1^i$ it follows that there is a path from A_i to A_{i+1} through $(X_i - \{v\}) - (S_1^i - \{v\})$ and from B_i to B_{i+1} through $(X_i - \{v\}) - (S_1^i - \{v\})$. A symmetric argument shows that $S_2^i - \{v\}$ is a minimal $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$ and that there is a path from A_i to A_{i+1} through $(X_i - \{v\}) - (S_2^i - \{v\})$ and from B_i to B_{i+1} through $(X_i - \{v\}) - (S_2^i - \{v\})$. But then $G - v$ is a t -critter, contradicting witness-minimality of (G, W) . $\square$

An immediate corollary to Lemma 6.1 is that A and B can not have any common neighbors.

LEMMA 6.2. Let G be a t -critter and $W = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$ be a t -critter partition of G such that (G, W) is a witness-minimal t -critter. Then, $N(A_1 \cup A_2 \dots, A_{t+1}) \cap N(B_1 \cup B_2 \dots, B_{t+1}) = \emptyset$.

Proof. Suppose for contradiction that there exists a vertex v in the common neighborhood of $A_1 \cup A_2 \dots, A_{t+1}$ and of $B_1 \cup B_2 \dots, B_{t+1}$. Then $x \in X_i$ for some i , so $x \in N(A_i \cup A_{i+1}) \cap N(B_i \cup B_{i+1}) \cap X_i$. Let (S_1^i, S_2^i) satisfy X_i . Since both S_1^i, S_2^i separate $A_i \cup A_{i+1}$ from $B_i \cup B_{i+1}$ it follows that $x \in S_1^i$ and $x \in S_2^i$. This contradicts the conclusion of Lemma 6.1 that S_1^i and S_2^i are disjoint. $\square$

LEMMA 6.3. Let G be a t -critter and $W = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$ be a t -critter partition of G , such that (G, W) is a witness-minimal t -critter. Then, for every $i \leq t$ we have $|N(A_i) \cap X_i| \leq 3$, $|N(B_i) \cap X_i| \leq 3$, $|N(A_{i+1}) \cap X_i| \leq 3$, $|N(B_{i+1}) \cap X_i| \leq 3$.

Proof. Let (S_1^i, S_2^i) be a pair that satisfies X_i . Since S_1^i, S_2^i are disjoint minimal $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separators in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i]$ there exists a path P_{AB} from A to B through X_i . We select P_{AB} to be the shortest such path, in particular P_{AB} contains precisely one vertex in $N(A_i \cup A_{i+1})$ and precisely one vertex in $N(B_i \cup B_{i+1})$.

Since (S_1^i, S_2^i) satisfy X_i there exists a path P_{A1} from A_i to A_{i+1} through $X_i - S_1^i$, a path P_{A2} from A_i to A_{i+1} through $X_i - S_2^i$, a path P_{B1} from B_i to B_{i+1} through $X_i - S_1^i$, and a path P_{B2} from B_i to B_{i+1} through $X_i - S_2^i$. We select P_{A1}, P_{A2}, P_{B1} and P_{B2} to be the shortest such paths, specifically each of P_{A1} and P_{A2} contain precisely one vertex in $N(A_i)$ and precisely one in $N(A_{i+1})$. and each of P_{B1} and P_{B2} contain precisely one vertex in $N(B_i)$ and precisely one in $N(B_{i+1})$. Note that P_{A1} and P_{A2} are disjoint from $N(B_i \cup B_{i+1})$, and Note that P_{B1} and P_{B2} are disjoint from $N(A_i \cup A_{i+1})$ because each of these paths is disjoint from at least one $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator.

We prove that $|N(A_i) \cap X_i| \leq 3$. Let x be the unique vertex in $N(A_i \cup A_{i+1})$ on P_{AB} , y be the unique vertex in $N(A_i)$ on P_{A1} , and z be the unique vertex in $N(A_i)$ on P_{A2} . Suppose for contradiction that $|N(A_i) \cap X_i| > 3$, and select a vertex $v \in (N(A_i) \cap X_i) - \{x, y, z\}$. We claim that

$$W' = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_i - \{v\}, X_{i+1}, \dots, X_t)$$

is a t -critter partition of $G - v$.

Since G is a minimal t -critter, by Lemma 6.1 S_1^i and S_2^i are disjoint. Since they are distinct (by property (iv) of t -critters) they are also non-empty. If $v \in S_1^i$ then let $\hat{S}_1^i = S_1^i - \{v\}$. We have that $\hat{S}_1^i$ is a minimal $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$. If $v \notin S_1^i$ then $S_1^i - \{v\}$ is a (not necessarily minimal) $(A_i \cup A_{i+1}, (B_i \cup B_{i+1}))$ -separator in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$.

Thus S_1^i contains a minimal $(A_i \cup A_{i+1}), (B_i \cup B_{i+1})$ -separator $\hat{S}_1^i$ in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$. Since $\hat{S}_1^i$ contains a vertex of P_{AB} we have that $\hat{S}_1^i$ is non-empty. In either case S_1^i contains a non-empty minimal $(A_i \cup A_{i+1}), (B_i \cup B_{i+1})$ -separator $\hat{S}_1^i$ in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$. An identical argument shows that S_2^i contains a non-empty minimal $(A_i \cup A_{i+1}), (B_i \cup B_{i+1})$ -separator $\hat{S}_2^i$ in $(G - v)[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i - \{v\}]$. Since S_1^i and S_2^i are disjoint, so are $\hat{S}_1^i$ and $\hat{S}_2^i$.

Finally, P_{A1} is a path in $G - v$ from A_i to A_{i+1} through $(X_i - \{v\}) - \hat{S}_1^i$, P_{A2} is a path in $G - v$ from A_i to A_{i+1} through $(X_i - \{v\}) - \hat{S}_2^i$, P_{B1} is a path in $G - v$ from B_i to B_{i+1} through $(X_i - \{v\}) - \hat{S}_1^i$, and P_{B2} is a path in $G - v$ from B_i to B_{i+1} through $(X_i - \{v\}) - \hat{S}_2^i$. Hence W' is a t -critter partition of $G - v$ contradicting minimality of (G, W) . The proofs that $|N(B_i) \cap X_i| \leq 3$, $|N(A_{i+1}) \cap X_i| \leq 3$, and $|N(B_{i+1}) \cap X_i| \leq 3$ are symmetric. $\square$

Let s be a positive integer. A t -critter partition $W = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$ of a graph G is said to be s -size-bounded if $|A_i| \leq s$ and $|B_i| \leq s$ for $i \leq t+1$ and $|X_i| \leq s$ for $i \leq t$. In other words a t -critter partition W is s -size bounded if all parts of the partition have size at most s . An s -size bounded t -critter is a graph G that has an s -size bounded t -critter partition W .

LEMMA 6.4. *For every CMSO-definable hereditary graph family $\mathcal{F}$ there exists an integer T , such that for every integer s , if there exists an s -size-bounded t -critter $G \in \mathcal{F}$ and $t > T$ then there exists an s -size-bounded t -critter $G' \in \mathcal{F}$ and $t' > t$.*

Proof. Let $\mathcal{F}$ be a CMSO-definable graph family, and G be a minimal t -critter such that $G \in \mathcal{F}$. Set $I = \{1, \dots, 6\}$. Since $\mathcal{F}$ is a CMSO-definable graph property, by Theorem 6.1 we have that the number of equivalence classes in $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ is finite. Let γ be the number of equivalence classes in $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$. We prove that $T = 2^6 \gamma$ satisfies the conclusion of the lemma.

Let G be a graph and $W = (A_1, \dots, A_{t+1}, B_1, \dots, B_{t+1}, X_1, \dots, X_t)$ be an s -size bounded t -critter partition for G . Because σ is hereditary we may assume without loss of generality that (G, W) is a witness-minimal t -critter. For every $1 \leq i \leq t$ we define

$$\begin{aligned} D_i &= N(A_i \cup B_i) \cap X_i, \\ Z_i &= D_i \cup \bigcup_{j \leq i} (A_j \cup B_j) \cup \bigcup_{j < i} X_j, \text{ and} \\ Q_i &= (V(G) - Z_i) \cup D_i. \end{aligned}$$

By Lemma 6.3 $|D_i| \leq 6$ for every i , while Lemma 6.2 yields that $D_i \cap N(A_i)$ and $D_i \cap N(B_i)$ are disjoint. For each i we define an injective labeling $\lambda_i : D_i \rightarrow \{1, \dots, 6\}$, such that every $v \in D_i \cap N(A_i)$ satisfies $\lambda_i(v) \in \{1, 2, 3\}$, while every $v \in D_i \cap N(B_i)$ satisfies $\lambda_i(v) \in \{4, 5, 6\}$.

For every $i \leq t$ we define the boundaried graph $G_i^{pre} = G[Z_i]$ with boundary D_i and labeling λ_i and $G_i^{post} = G[Q_i]$ with boundary D_i and labeling λ_i . Note that for every i we have $G = G_i^{pre} \oplus G_i^{post}$. If $t > T = 2^6 \gamma$ then there exist $i < j \leq t$ such that $\Lambda(G_i^{pre}) = \Lambda(G_j^{pre})$ and $G_i^{pre} \equiv_{\mathcal{F}} G_j^{pre}$. Let $G' = G_j^{pre} \oplus G_i^{post}$. We claim that G' satisfies the conclusion of the Lemma. First, note that $G \in \mathcal{F}$ by assumption, so $G = (G_i^{pre} \oplus G_i^{post}) \in \mathcal{F}$. Since $G_j^{pre} \equiv_{\mathcal{F}} G_i^{pre}$ it follows that $(G_j^{pre} \oplus G_i^{post}) \in \mathcal{F}$. However, $G' = G_j^{pre} \oplus G_i^{post}$ and so $G' \in \mathcal{F}$.

We set $\hat{t} = (t + j - i)$ and show that that G' is a $\hat{t}$ -critter by giving a $\hat{t}$ -critter partition $\hat{W} = \hat{A}_1, \dots, \hat{A}_{\hat{t}+1}, \hat{B}_1, \dots, \hat{B}_{\hat{t}+1}, \hat{X}_1, \dots, \hat{X}_{\hat{t}}$ of G' . For $p \leq j$ we set $\hat{A}_p = A_p$, $\hat{B}_p = B_p$, and $\hat{X}_p = X_p$, or rather the copies of A_p , B_p and X_p respectively, in G_j^{pre} . For p from $j+1$ and up to $\hat{t}+1$ we set $\hat{A}_p = A_{p+i-j}$ and $\hat{B}_p = B_{p+i-j}$, more specifically the copies of A_{p+i-j} and B_{p+i-j} in G_i^{post} . For p from $j+1$ and up to p we set $\hat{X}_p$ to be the copy of X_{p+i-j} in G_i^{post} . It follows directly from their definitions that

$$\{\hat{A}_p, \hat{B}_p : 1 \leq p \leq \hat{t}+1\} \cup \{\hat{X}_p : 1 \leq p \leq \hat{t}\}$$

is a partition of $V(G')$ and that it satisfies properties (i), (ii) and (iii) of $\hat{t}$ -critter partitions. We now check property (iv).

For property (iv) we note that for every $p < j$ we have that $G'[\hat{A}_p \cup \hat{A}_{p+1} \cup \hat{B}_p \cup \hat{B}_{p+1} \cup \hat{X}_p] = G[A_p \cup A_{p+1} \cup B_p \cup B_{p+1} \cup X_i]$, and that therefore (iv) is satisfied for all $p < j$. Similarly, for $p > j$ we

have that $G'[\hat{A}_p \cup \hat{A}_{p+1} \cup \hat{B}_p \cup \hat{B}_{p+1} \cup \hat{X}_i] = G[A_{p+i-j} \cup A_{p+i-j+1} \cup B_{p+i-j} \cup B_{p+i-j+1} \cup X_{p+i-j}]$, and that therefore (iv) is satisfied for all $p > j$.

We are left with verifying property (iv) for $p = j$. We have that $G'[\hat{A}_j \cup \hat{B}_j] = G[A_j \cup B_j]$ and that $G'[\hat{A}_{j+1} \cup \hat{B}_{j+1} \cup \hat{X}_j] = G[A_{i+1} \cup B_{i+1} \cup X_i]$. Additionally, for every edge $uv \in E(G_j^{pre})$ such that $u \in A_j \cup B_j$ and $v \in D_j$ the copy u' of u in $\hat{A}_j \cup \hat{B}_j$ and vertex v' of $\hat{X}_j$ corresponding to v are adjacent in G' .

Let $\hat{S}_1^j$ and $\hat{S}_2^j$ be the copies in $\hat{X}_j$ of S_1^i and S_2^i respectively. We claim that $\hat{S}_1^j$ is a $(\hat{A}_j \cup \hat{A}_{j+1})$, $(\hat{B}_j \cup \hat{B}_{j+1})$ -separator in $G'[\hat{A}_j \cup \hat{A}_{j+1} \cup \hat{B}_j \cup \hat{B}_{j+1} \cup \hat{X}_j]$. Indeed, suppose there was a path $\hat{P}$ in $G'[\hat{X}_j] - \hat{S}_1^j$ that starts in a neighbor of $\hat{A}_j \cup \hat{A}_{j+1}$ and ends in a neighbor of $\hat{B}_j \cup \hat{B}_{j+1}$. Then the copy P of $\hat{P}$ in X_i would be a path that starts in a neighbor of $\hat{A}_i \cup \hat{A}_{i+1}$ and ends in a neighbor of $\hat{B}_i \cup \hat{B}_{i+1}$, contradicting that S_1^i separates $\hat{A}_i \cup \hat{A}_{i+1}$ from $\hat{B}_i \cup \hat{B}_{i+1}$ in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i]$. For an identical reason $\hat{S}_2^j$ is a $(\hat{A}_j \cup \hat{A}_{j+1})$, $(\hat{B}_j \cup \hat{B}_{j+1})$ -separator in $G'[\hat{A}_j \cup \hat{A}_{j+1} \cup \hat{B}_j \cup \hat{B}_{j+1} \cup \hat{X}_j]$.

Since S_1^i and S_2^i are disjoint, so are $\hat{S}_1^j$ and $\hat{S}_2^j$. Since X_i contains a path from a neighbor of $A_i \cup A_{i+1}$ to a neighbor of $B_i \cup B_{i+1}$ in G , it follows that $\hat{X}_j$ contains a path from a neighbor of $\hat{A}_j \cup \hat{A}_{j+1}$ to a neighbor of $\hat{B}_j \cup \hat{B}_{j+1}$ in G . Therefore each of $\hat{S}_1^j$ and $\hat{S}_2^j$ contains a non-empty minimal $(\hat{A}_j \cup \hat{A}_{j+1})$, $(\hat{B}_j \cup \hat{B}_{j+1})$ -separator in $G'[\hat{A}_j \cup \hat{A}_{j+1} \cup \hat{B}_j \cup \hat{B}_{j+1} \cup \hat{X}_j]$. Since $\hat{S}_1^j$ and $\hat{S}_2^j$ are disjoint these minimal separators are distinct.

Finally, $X_i - S_1^i$ contains a path P from a neighbor of A_i to a neighbor of A_{i+1} . The copy of P in $\hat{X}$ is a path from a neighbor of $\hat{A}_j$ to a neighbor of $\hat{A}_{j+1}$ in $\hat{X}_j - \hat{S}_1^j$. Identical arguments yield the existence of a paths from a neighbor of $\hat{A}_j$ to a neighbor of $\hat{A}_{j+1}$ in $\hat{X}_j - \hat{S}_2^j$, from a neighbor of $\hat{B}_j$ to a neighbor of $\hat{B}_{j+1}$ in $\hat{X}_j - \hat{S}_1^j$, and from a neighbor of $\hat{B}_j$ to a neighbor of $\hat{B}_{j+1}$ in $\hat{X}_j - \hat{S}_2^j$. We conclude that $\hat{W}$ is a $t + j - i$ -critter partition of G' . Since every part of $\hat{W}$ is a copy of a part of W and W is s -size-bounded, so is $\hat{W}$. $\square$

LEMMA 6.5. *Let G be a t -critter, then G has at least 2^t minimal separators.*

Proof. Let G be a t -critter and let $A_1, A_2, \dots, A_{t+1}$, $B_1, B_2, \dots, B_{t+1}, X_1, X_2, \dots, X_t$ be the partitioning of its vertices given in Definition 1.2. Let S initially be an empty set, and for each i , $1 \leq i \leq t$, choose either S_1^i or S_2^i and add this set to S . Since each choice of adding S_1^i or S_2^i is made independently, there are 2^t choices for S . Let $a_1 \in A_1$ and $b_1 \in B_1$, we claim that S is a a_1, b_1 -minimal separator.

It follows from properties (ii) and (iv) of Definition 1.2 that the vertices of the A_i 's all belong to one component, say A , of $G - S$ and that the vertices of the B_i 's all belong to one component, say B , of $G - S$. It follows from properties (i), (iii), and (iv) that $A \neq B$. Hence S is an a_1, b_1 -separator. To see that it is minimal, take some vertex $v \in S$, say v belongs to S^i where S^i is either S_1^i or S_2^i for $1 \leq i \leq t$. Then since S^i is a minimal $(A_i \cup A_{i+1})$, (B_i, B_{i+1}) -separator in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i]$, there will be a path from either A_i or A_{i+1} to either B_i or B_{i+1} in $G[A_i \cup A_{i+1} \cup B_i \cup B_{i+1} \cup X_i] - (S^i - v)$. Since all vertices of the A_i 's belong to one component of $G - S$ and all vertices of the B_i 's belong to a different component of $G - S$, there will be a path from a_1 to b_1 in $G - (S - v)$. It follows that S is a minimal separator. $\square$

LEMMA 6.6. *For every hereditary CMSO-definable family $\mathcal{F}$, if for every t there exists a t -critter $G \in \mathcal{F}$ then σ is feral.*

Proof. Let $\mathcal{F}$ be a hereditary CMSO-definable family. By Lemma 6.4 there exists an integer T , such that for every integer s , if there exists an s -size-bounded t -critter $G \in \mathcal{F}$ with $t > T$ then there exists an s -size-bounded t -critter $G' \in \mathcal{F}$ with $t' > t$. Select $t = T + 1$, then by assumption there exists a t -critter $G \in \mathcal{F}$. Let $s = |V(G)|$, then G is an s -bounded t -critter with $t > T$. By Lemma 6.4 there exists an s -bounded t' -critter $G' \in \mathcal{F}$, and $t' \geq t + 1$. Induction then yields that for every q there exists an s -bounded n -critter $G' \in \mathcal{F}$, and $n \geq q$. Such an n -critter G' has at least n and at most $5sn$ vertices, and at least 2^n minimal separators (by Lemma 6.5). Thus for every n there exists a graph $G' \in \mathcal{F}$, at least n vertices, and at least $(2^{1/5s})^n$ minimal separators. Therefore $\mathcal{F}$ is feral. $\square$

Lemma 6.6 handle graph properties that contain arbitrarily large critters. We now need to handle properties that contain arbitrarily large creatures.

The authors [21] showed that if G contains a k -creature for sufficiently large k , then G must contain a k' -creature which falls into one out of 6 very structured graph families. Specifically, The authors proved the following.

LEMMA 6.7. ([21]) *Let k be a natural number. Then there exists a natural number k' large enough so that if G is a graph that contains a k' -creature $(A, B, \{x_1, x_2, \dots, x_{k'}\}, \{y_1, y_2, \dots, y_{k'}\})$, then G contains an induced k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism, or a k -ladder.*

We do not re-define the graphs listed in Lemma 6.7 in this paper, because everything we need to know about them is encapsulated in a simple observation which can easily be derived by inspection.

To phrase this observation we need a few definitions. A *boundaried path* is a 2-boundaried graph G such that G is a path and the endpoints of the path are the boundary vertices. An *apex path* is a graph G such that there exists a vertex v such that $G - v$ is a path and v is adjacent to both endpoints of the path $G - v$. The vertex v is then called an *apex*. The apex may or may not have edges to the internal vertices of the path. A *boundaried apex path* is a 3-boundaried graph G such that G is an apex path, and the boundary of G is the apex v as well as the two endpoints of the path $G - v$. A *shortening* of a boundaried path is a boundaried path on fewer vertices. A *shortening* of a boundaried apex path is a boundaried apex path on fewer vertices.

An inspection k -thetas, k -prisms, k -pyramids, k -ladder-thetas, k -ladder-prisms, and k -ladders shows that if G is one of these graphs and G has $n \gg k$ vertices, then G either contains a long induced path P or a long induced apex path $\hat{P}$, such that the internal vertices of P (or $\hat{P}$) do not have any neighbors outside of P (or $\hat{P}$). Further, shortening this path does not destroy the property of G being a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism, and k -ladder. We now formalize this observation in the language of boundaried graphs.

LEMMA 6.8. *Let G be a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder on n vertices. Then $G = P \oplus R$ where P is either a boundaried path or a boundaried apex path on at least $\frac{n}{5k}$ vertices. Furthermore, for every shortening P' of P , $P' \oplus R$ is a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder.*

Proof. [Proof sketch.] The statement of the lemma immediately follows from the observation that for each of the listed graphs, the vertex set can be partitioned into at most $5k$ induced paths and apex paths, such that for each path/apex path in the partition only the endpoints and apex have neighbours outside the path/apex path. Details omitted. $\square$

LEMMA 6.9. *Let $\mathcal{F}$ be a CMSO-definable graph family. Then there exists a constant c such that for every k , if there exists a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder G such that $\sigma(G) = \text{true}$ then there exists a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder $G' \in \mathcal{F}$ such that $|V(G')| \leq ck$*

Proof. Set $I = \{1, 2, 3\}$ and consider the equivalence classes $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ of $\equiv_{\sigma}$ restricted to I -boundaried graphs. Let γ_1 be the maximum, taken over all equivalence classes in $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ that contain at least one boundaried path, of the *minimum* number of vertices of a boundaried path in that equivalence class. Similarly, let γ_2 be the maximum taken over all equivalence classes in $\mathcal{E}_{\equiv_{\mathcal{F}}}[\mathcal{G}_{\subseteq I}]$ that contain at least one boundaried apex path, of the minimum number of vertices of a boundaried apex path in that equivalence class.

We set $\gamma = \max(\gamma_1, \gamma_2)$. From the choice of γ it follows that for every boundaried path P , if $V(P) > \gamma$ then there exists a boundaried path P' such that $P' \equiv_{\mathcal{F}} P$ and $|V(P')| \leq \gamma < |V(P)|$. Similarly, for every boundaried apex path P , if $V(P) > \gamma$ then there exists a boundaried apex path P' such that $P' \equiv_{\mathcal{F}} P$ and $|V(P')| \leq \gamma < |V(P)|$.

We set $c = 5\gamma$ and claim that c satisfies the conclusion of the lemma. Let k be an integer and suppose that there exists a graph $G \in \mathcal{F}$ such that G is a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder. Of all such graphs pick G with the minimum number of vertices. We claim $n = |V(G)| \leq ck = 5\gamma k$.

Suppose not, then by Lemma 6.8 we have that $G = P \oplus R$, where P is either a boundaried path or a boundaried apex path on at least $\frac{n}{5k} > \frac{5\gamma k}{5k} \geq \gamma$ vertices.

By the choice of γ there exists a shortening P' of P such that $P' \equiv_{\mathcal{F}} P$ and $|V(P')| \leq \gamma < |V(P)|$. Since P and P' are both boundaried paths or both boundaried apex paths it follows that the sizes of their boundaries are equal, namely $|\delta(P)| = |\delta(P')|$. We set $G' = P' \oplus R$. We have that

$$|V(G')| = |V(P')| + |V(R)| - |\delta(P')| < |V(P)| + |V(R)| - |\delta(P)| = |V(G)|.$$

By Lemma 6.8, since P' is a shortening of P we have that G' is a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder. Further, since $P' \equiv_{\mathcal{F}} P$ and $\sigma(P \oplus R) \in \mathcal{F}$ it follows that $G' = (P' \oplus R) \in \mathcal{F}$. But

that contradicts the choice of G as the k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder with fewest vertices such that $G \in \mathcal{F}$. $\square$

LEMMA 6.10. *Let G be a k -creature. Then G has at least 2^k minimal separators.*

Proof. Let G be a k -creature, and let A , $X = \{x_1, x_2, \dots, x_k\}$, $Y = \{y_1, y_2, \dots, y_k\}$, and B be the partition of $V(G)$ given in Definition 1.1. Let $a \in A$ and $b \in B$. We can make a minimal a, b -separator by selecting exactly one vertex from each pair x_i and y_i , $1 \leq i \leq k$. There are 2^k choices for such a minimal separator, which proves the lemma. $\square$

LEMMA 6.11. *For every hereditary CMSO-definable graph family $\mathcal{F}$, if for every t there exists a t -creature G such that $G \in \mathcal{F}$ then $\mathcal{F}$ is feral.*

Proof. Let $\mathcal{F}$ be a hereditary CMSO-definable graph family such that for every t there exists a t -creature G such that $G \in \mathcal{F}$.

We first claim that for every integer k there exists a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder G such that $G \in \mathcal{F}$. Towards a proof of this claim let k be given. By Lemma 6.7 there exists a k' such that every k' -creature G contains a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder as an induced subgraph. By our assumption on σ there exists a k' -creature G such that $G \in \mathcal{F}$. Let G' be an induced k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder in G . Since G' is an induced subgraph of G and $\mathcal{F}$ is hereditary it follows that $G' \in \mathcal{F}$. This proves the claim.

The claim, together with Lemma 6.9 yields that there exists a constant c such that for every k there exists a k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder G such that $G \in \mathcal{F}$ and $|V(G)| \leq ck$. However each of k -theta, k -prism, k -pyramid, k -ladder-theta, k -ladder-prism or k -ladder is a k -creature and therefore has at least 2^k minimal separators by Lemma 6.10. Hence, for every n there exists a graph G on at least n and at most cn vertices such that $G \in \mathcal{F}$ and G has at least $2^{n/c}$ minimal separators. Hence $\mathcal{F}$ is feral, as claimed. $\square$

Theorem 1.2, together with Lemmas 6.6 and 6.11 together imply Theorem 1.3

Proof. [Proof of Theorem 1.3] Let $\mathcal{F}$ be a CMSO-definable hereditary graph family. If there exists an integer k such that $\mathcal{F}$ neither contains a k -creature nor a k -critter then, by Theorem 1.2 $\mathcal{F}$ is quasi-tame. If no such integer k exists it follows that $\mathcal{F}$ either contains a t -critter for every t , or a t -creature for every t . In the first case $\mathcal{F}$ is feral by Lemma 6.6, in the second case $\mathcal{F}$ is feral by Lemma 6.11. $\square$

References

- [1] T. ABRISHAMI, M. CHUDNOVSKY, C. DIBEK, S. THOMASSÉ, N. TROTIGNON, AND K. VUSKOVIC, *Graphs with polynomially many minimal separators*, J. Comb. Theory, Ser. B, 152 (2022), pp. 248–280.
- [2] S. ARNBORG, J. LAGERGREN, AND D. SEESE, *Easy problems for tree-decomposable graphs*, Journal of Algorithms, 12 (1991), pp. 308–340.
- [3] H. L. BODLAENDER, F. V. FOMIN, D. LOKSHTANOV, E. PENNINKX, S. SAURABH, AND D. M. THILIKOS, *(meta) kernelization*, J. ACM, 63 (2016), pp. 44:1–44:69.
- [4] R. B. BORIE, R. G. PARKER, AND C. A. TOVEY, *Automatic generation of linear-time algorithms from predicate calculus descriptions of problems on recursively constructed graph families*, Algorithmica, 7 (1992), pp. 555–581.
- [5] V. BOUCHITTÉ AND I. TODINCA, *Treewidth and minimum fill-in: Grouping the minimal separators*, SIAM J. Comput., 31 (2001), pp. 212–232.
- [6] A. BRANDSTÄDT, V. B. LE, AND J. P. SPINRAD, *Graph classes: a survey*, SIAM, 1999.
- [7] M. CHUDNOVSKY, M. PILIPCZUK, M. PILIPCZUK, AND S. THOMASSÉ, *On the maximum weight independent set problem in graphs without induced cycles of length at least five*, SIAM J. Discret. Math., 34 (2020), pp. 1472–1483.
- [8] M. CHUDNOVSKY, S. THOMASSÉ, N. TROTIGNON, AND K. VUSKOVIC, *Maximum independent sets in (pyramid, even hole)-free graphs*, CoRR, abs/1912.11246 (2019).
- [9] B. COURCELLE, *The monadic second-order logic of graphs. i. recognizable sets of finite graphs*, Information and computation, 85 (1990), pp. 12–75.

- [10] B. COURCELLE, *The monadic second-order logic of graphs I: Recognizable sets of finite graphs*, Inform. and Comput., 85 (1990), pp. 12–75.
- [11] B. COURCELLE, *The monadic second-order logic of graphs. III. Tree-decompositions, minors and complexity issues*, RAIRO Inform. Théor. Appl., 26 (1992), pp. 257–286.
- [12] ———, *The expression of graph properties and graph transformations in monadic second-order logic*, in Handbook of graph grammars and computing by graph transformation, Vol. 1, World Sci. Publ, 1997, pp. 313–400.
- [13] B. COURCELLE AND J. ENGELFRIET, *Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach*, Cambridge University Press, 2012.
- [14] R. DIESTEL, *Graph theory 3rd ed*, Graduate texts in mathematics, 173 (2005), p. 33.
- [15] R. G. DOWNEY AND M. R. FELLOWS, *Parameterized Complexity*, Springer, Berlin, 1998.
- [16] P. ERDÖS AND G. SZEKERES, *A combinatorial problem in geometry*, Compositio mathematica, 2 (1935), pp. 463–470.
- [17] F. V. FOMIN, I. TODINCA, AND Y. VILLANGER, *Large induced subgraphs via triangulations and CMSO*, SIAM J. Comput., 44 (2015), pp. 54–87.
- [18] F. V. FOMIN AND Y. VILLANGER, *Treewidth computation and extremal combinatorics*, Combinatorica, 32 (2012), pp. 289–308.
- [19] J. GAJARSKÝ, L. JAFFKE, P. T. LIMA, J. NOVOTNÁ, M. PILIPCZUK, P. RZAŻEWSKI, AND U. S. SOUZA, *Taming graphs with no large creatures and skinny ladders*, arXiv preprint arXiv:2205.01191, (2022).
- [20] P. GARTLAND AND D. LOKSHTANOV, *Independent set on P_k -free graphs in quasi-polynomial time*, in 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), 2020, pp. 613–624.
- [21] ———, *Graph classes with few minimal separators. I. Finite forbidden induced subgraphs*. 2022.
- [22] M. C. GOLUMBIC, *Algorithmic graph theory and perfect graphs*, Elsevier, 2004.
- [23] M. GROHE, K.-I. KAWARABAYASHI, AND B. REED, *A simple algorithm for the graph minor decomposition- logic meets structural graph theory—*, in Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, 2013, pp. 414–431.
- [24] T. KLOKS, D. KRATSCHE, AND C. K. WONG, *Minimum fill-in on circle and circular-arc graphs*, J. Algorithms, 28 (1998), pp. 272–289.
- [25] D. KRATSCHE, *The structure of graphs and the design of efficient algorithms, habilitation*, Friedrich-Schiller-University of Jena, Germany, (1996).
- [26] D. LOKSHTANOV, *On the complexity of computing treelength*, Discret. Appl. Math., 158 (2010), pp. 820–827.
- [27] D. MARX, *Parameterized graph separation problems*, Theor. Comput. Sci., 351 (2006), pp. 394–406.
- [28] K. MENGER, *Zur allgemeinen kurventheorie*, Fundamenta Mathematicae, 10 (1927), pp. 96–115.
- [29] M. MILANIC AND N. PIVAC, *Polynomially bounding the number of minimal separators in graphs: Reductions, sufficient conditions, and a dichotomy theorem*, Electron. J. Comb., 28 (2021), p. 1.
- [30] K. SUCHAN, *Minimal separators in intersection graphs*, Master’s thesis, Akademia Gorniczo-Hutnicza im. Stanislawia Staszica w Krakowie, (2003).