Graph Classes with Few Minimal Separators. I. Finite Forbidden Induced Subgraphs*

Peter Gartland† Daniel Lokshtanov†

Abstract

A vertex set S in a graph G is a *minimal separator* if there exist vertices u and v that are in distinct connected components of $G - S$, but in the same connected component of $G - S'$ for every $S' \subset S$. A class $\mathcal{F}$ of graphs is called *tame* if there exists a constant c so that every graph in $\mathcal{F}$ on n vertices contains at most $O(n^c)$ minimal separators. If there exists a constant c so that every graph in $\mathcal{F}$ on n vertices contains at most $O(n^{c \log n})$ minimal separators the class is *strongly-quasi-tame*. If there exists a constant $c > 1$ so that $\mathcal{F}$ contains n-vertex graphs with at least c^n minimal separators for arbitrarily large n then $\mathcal{F}$ is called *feral*. The classification of graph classes into tame or feral has numerous algorithmic consequences, and has recently received considerable attention.

A key graph-theoretic object in the quest for such a classification is the notion of a *k-creature*. A k-creature consists of 4 disjoint vertex sets $A, B, X = \{x_1, \ldots, x_k\}, Y = \{y_1, \ldots y_k\}$ such that: (a) A and B are connected, (b) there are no edges from A to $Y \cup B$ and no edges from B to $X \cup A$, (c) A dominates X (every vertex in X has a neighbor in A) and B dominates Y and (d) $x_i y_j$ is an edge if and only if $i = j$. It is easy to verify that a k-creature contains at least 2^k minimal separators. On the other hand, in a recent article Abrishami et al. [1] conjecture that every hereditary class $\mathcal{F}$ that excludes k-creatures for some fixed constant k is tame.

In this paper we first give a counterexample to the conjecture of Abrishami et al. Our main result is a proof of a weaker form of their conjecture. More concretely, we prove that a hereditary class $\mathcal{F}$ is strongly quasi-tame if it excludes k-creatures for some fixed constant k and additionally every minimal separator can be dominated by another fixed constant k' number of vertices. The tools developed on the way lead to a number of additional results of independent interest.

(i) We obtain a complete classification of all hereditary graph classes defined by a finite set of forbidden induced subgraphs into strongly quasi-tame or feral. This substantially generalizes a recent result of Milanič and Pivač [18] , who classified all hereditary graph classes defined by a finite set of forbidden induced subgraphs on at most 4 vertices into tame or feral. **(ii)** We show that every hereditary class that excludes k-creatures and additionally excludes all cycles of length at least c, for some constant c, is tame. This generalizes the result of Chudnovsky et al. [6] who obtained the same statement for $c = 5$. **(iii)** We show that every hereditary class that excludes k-creatures and additionally excludes a complete graph on c vertices for some fixed constant c is tame.

1 Introduction

Let G be a graph and u and v be distinct vertices in G. A vertex set S is a *u,v-separator* if u and v are in distinct components of $G - S$. The set S is a *u,v-minimal separator* if S is a u,v-separator, but no proper subset of S is a u,v-separator. Finally, S is a *minimal separator* if S is a u,v-minimal separator for some pair of vertices u and v. Minimal separators have a tremendous role in the design of graph algorithms, both directly, such as in the structural characterization of chordal graphs [5] but also indirectly in optimization algorithms for graph separation and routing problems (for example [17, 16, 21]). The theory of potential maximal cliques, developed by Bouchitté and Todinca [4] implies that a several fundamental graph problems, such as computing the *treewidth* and *minimum fill in* of a graph G can be done in time polynomial in the number of vertices of G and the number of minimal separators in G. Lokshtanov [15] showed that the same result holds for computing the *tree-length* of the graph G, while Fomin et al. [10] proved a general result that showed that a whole class of problems (including e.g. *maximum independent set* and *minimum feedback vertex set*) can be solved in time polynomial in the number of vertices and minimal separators of the graph. All of these algorithms require a list of all the minimal separators of G to be provided as input. However, the listing algorithms for minimal separators of Kloks and Kratsch [12] or Berry et al. [3] can be used to compute such a list in time polynomial in the number of vertices times a factor linear in the number of minimal separators of G.

*The full version of the paper can be accessed at `https://arxiv.org/abs/2007.08761`

†University of California, Santa Barbara, USA. Emails: `petergartland@ucsb.edu, daniello@ucsb.edu` Supported by BSF award 2018302 and NSF award CCF-2008838.

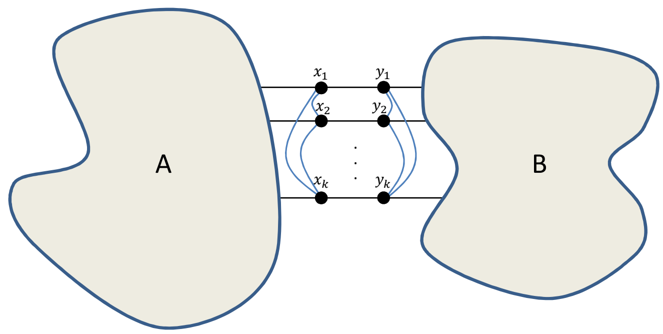

Figure 1: A graph induced by the vertices of a k-creature. The blue edges indicate that x_i (y_i) may or may not be a neighbor of x_j (y_j)

This brings to the forefront the main question asked in this paper — *which classes of graphs have polynomially many minimal separators?* We will say that a graph class $\mathcal{F}$ is *tame* if there exists an integer c such that every graph in $\mathcal{F}$ on n vertices has at most $O(n^c)$ minimal separators. A number of important graph classes have been shown to be tame, such as Chordal [5] (and more generally Weakly Chordal [4]), Permutation (and, more generally d-Trapezoid [14]), Circular Arc [13] and Polygon Circle graphs [23]. Most of these results date back to the late 1990s and early 2000s. Much more recently [1, 7, 6, 19], research has started to focus on a more systematic classification of which graph classes are tame and which are not. Indeed the term *tame* for graph classes with polynomially many minimal separators was defined by Milanič and Pivač [19], who classified all hereditary (closed under vertex deletion) classes defined by a set of forbidden induced subgraphs, all of which have at most 4 vertices, as tame or not tame.

Building on the terminology of Milanič and Pivač [19], we will say that a class of graphs $\mathcal{F}$ is *quasi-tame* if there exist constants c, c' such that every n-vertex graph in the family contains at most $O(n^{c \log^{c'} n})$ minimal separators. Further, $\mathcal{F}$ is *strongly quasi-tame* if it is quasi-tame with $c' \leq 1$. On the opposite side of the spectrum, we will say that $\mathcal{F}$ is *feral* if there exists a constant c such that for every $N \geq 0$ there exists an $n \geq N$ such that $\mathcal{F}$ contains an n-vertex graph with at least c^n minimal separators.

Abrishami et al. [1] define a structure, called a k-creature, the presence of which appears to control, to a large extent, whether a graph has many or few (quasi-polynomially many) separators. A *k-creature* in a graph G is a four-tuple $(A, B, X = \{x_1, x_2, \ldots, x_k\}, Y = \{y_1, y_2, \ldots, y_k\})$ of mutually disjoint vertex subsets of $V(G)$, satisfying the following conditions (see Figure 1).

1. A and B induce connected subgraphs of G,
2. A and $Y \cup B$ are *anti-complete* (i.e., no vertex in A is adjacent to a vertex in $B \cup Y$) and B is anti-complete with $X \cup A$.
3. A dominates X (every vertex in X has a neighbor in A) and B dominates Y, and
4. $x_i y_j$ is an edge if and only if $i = j$.

A graph G is *k-creature-free* if there does not exists a 4-tuple of vertex sets of $V(G)$ that form a k-creature. It is easy to see that a k-creature contains at least 2^k minimal separators (select precisely one of $\{x_i, y_i\}$ for every $i \leq k$). Because deleting a vertex cannot increase the number of minimal separators, a graph G that contains a k-creature contains at least 2^k minimal separators. Thus, a graph family $\mathcal{F}$ that contains n-vertex graphs with k-creatures for arbitrarily large n and with $k = \Omega(n)$ is feral. For $\mathcal{F}$ to not be tame it is sufficient for k to grow super-logarithmically with n (i.e $n \in 2^{o(k)}$). A sort of converse to this observation was conjectured in [1].

Conjecture 1.1. *[1] For every fixed natural number k, the family of graphs that are k-creature-free is tame.*

Even if Conjecture 1.1 were to be true, it would still not give a complete characterization of hereditary graph classes into tame or non-tame. In particular Abrishami et al. [1] give an example of a tame hereditary class $\mathcal{F}$ that contains k-creatures for arbitrarily large k. Their example can also be slightly modified to show that there exist hereditary families that are neither tame nor feral. This makes it appear that, at least for hereditary classes in their full generality, the boundary between tame and non-tame graph classes is so "strange-looking" that a complete dichotomy may be out of reach, and that we therefore have to settle for sufficient conditions for tameness / non-tameness, and possibly complete characterizations for more well-behaved sub-classes of hereditary families. For an example, Conjecture 1.1, if true, would have yielded a complete dichotomy into tame or feral for all classes of graphs closed under *induced minors* (i.e closed under vertex deletion and edge contraction).

Unfortunately it turns out that **Conjecture 1.1 is false**. In particular we give (in Section 4) an example of a feral family $\mathcal{F}$ that excludes 100-creatures. The family $\mathcal{F}$ consists of all k-*twisted ladders* (see Section 4 for a definition). Our main result is nevertheless that Conjecture 1.1 is true "in spirit", in the sense that for large classes of hereditary families, excluding k-creatures does imply few minimal separators. To state Theorem 1.1 we need to define k-*skinny-ladders*. A k-*skinny-ladder* is a graph G consisting of two anti-complete paths $P_l = \ell_1\ell_2\dots\ell_k$ and $P_r = r_1r_2\dots r_k$ and a set $\{s_1, s_2, \dots, s_k\}$ of vertices such that for every i, s_i is adjacent to ℓ_i and r_i and to no other vertices.

THEOREM 1.1. *For every natural number k, the family of graphs that are k-creature-free and do not contain a k-skinny-ladder as an induced minor is strongly-quasi-tame.*

Theorem 1.1 suggests that other counterexamples to Conjecture 1.1 should resemble the counterexample we provide in Section 4. Furthermore, we do not have an an example of a non-tame class for which strong quasi-tameness follows from Theorem 1.1. Therefore we conjecture that the statement of Theorem 1.1 remains true even if strongly quasi-tame is replaced by tame.

Excluding the k-skinny-ladder is closely tied to *domination* of minimal separators. A vertex set X *dominates* S if every vertex in S is either in X or has a neighbor in X. An important ingredient in the proof of Theorem 1.1 (see Lemma 5.13) is that for every k there exists a k' such that if G excludes k-creatures and excludes k-skinny-ladders as an induced minor then every minimal separator S in G is dominated by a set X on at most k' vertices. In fact, because a k-skinny-ladder is itself 5-creature-free and contains a minimal separator (namely the set $\{s_1, s_2, \dots, s_k\}$) which cannot be dominated by $k-1$ vertices, among the hereditary classes $\mathcal{F}$ that exclude k-creatures, the presence or absence of k-skinny-ladders (as induced minors) precisely characterizes whether every minimal separator of every graph in $\mathcal{F}$ can be dominated by a constant size set of vertices.

While the statement of Theorem 1.1 is concise, it is not immediately clear which graph families it applies to. Which families are k-creature-free? What does it mean in terms of forbidden induced subgraphs to exclude a k-skinny-ladder as an induced minor? In the second half of the paper we obtain an equivalent characterization of the premise of Theorem 1.1 in terms of forbidden induced subgraphs. Specifically, we first show that for every $k \geq 1$ there exists a k' such that if G contains a k'-creature then G contains a k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, or a k-ladder as an induced subgraph (see Figure 2, formal definitions in Section 6). Additionally, it is easy to see that every graph that contains a $2k$-skinny-ladder as an induced minor either contains a k-ladder or a k-*contracted-ladder* as an induced subgraph. Here a k-*contracted-ladder* is a graph obtained from a k-ladder by contracting all of the horizontal paths into single vertices (see Section 6 for a formal definition). This leads to the following variant of Theorem 1.1.

THEOREM 1.2. *For every natural number k, the family of graphs that exclude the k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, k-ladder, and the k-contracted-ladder as induced subgraphs, is strongly-quasi-tame.*

Theorems 1.1 and 1.2 are equivalent in the sense that for every k there exists a k' such that the graph family that satisfies the premise of Theorem 1.1 with k also satisfies the premise of Theorem 1.2 with k', and the graph family that satisfies the premise of Theorem 1.2 with k also satisfies the premise of Theorem 1.1 with k'.

To demonstrate the power of Theorem 1.1 (or equivalently, Theorem 1.2) we show that it gives, as a pretty direct consequence, a complete classification of all hereditary graph classes defined by a finite set of forbidden induced subgraphs into strongly quasi-tame or feral. Indeed, it is an easy exercise to show that if a family $\mathcal{F}$ is defined by a finite set of forbidden induced subgraphs and contains k-skinny-ladders for arbitrarily large k as induced minors, then there exists a constant p such that $\mathcal{F}$ either contains all p-subdivisions of 3-regular graphs (an p-*subdivision* of G is the graph obtained from G by replacing each edge of G by a path on $p+1$ edges) or all line graphs (see [8] for a definition) of p-subdivisions of 3-regular graphs. In this case $\mathcal{F}$ is feral. Therefore, Theorem 1.1 proves Conjecture 1.1 for hereditary graph classes defined by a finite set of forbidden induced subgraphs, albeit with strongly quasi-tame instead of tame.

The "strongly quasi-tame" part of the classification of families $\mathcal{F}$ defined by a finite set of forbidden induced subgraphs into strongly quasi-tame or feral follows directly by inspecting the graphs in the statement of Theorem 1.2. The "feral" part follows by observing that if $\mathcal{F}$ contains some of the graphs in the premise of Theorem 1.2 for arbitrarily large k, then $\mathcal{F}$ must also contain such graphs with only $O(k)$ vertices. This part of the proof crucially depends on $\mathcal{F}$ being defined by a *finite* set of forbidden induced subgraphs.

THEOREM 1.3. *Let $\mathcal{F}$ be a graph family defined by a finite number of forbidden induced subgraphs. If there exists a natural number k such that $\mathcal{F}$ forbids all k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, k-claw, and k-paw graphs, then $\mathcal{F}$ is strongly-quasi-tame. Otherwise $\mathcal{F}$ is feral.*

Note that some of the graphs of Figure 2 share a name with graphs that appear in the work of Abrishami et al. [1], but the definitions given here are slightly different. In particular, in some of the places where they require single edges we allow arbitrarily long paths. Abrishami et al. [1] prove that the family of (what they define to be) theta-free, pyramid-free, prism-free, and turtle-free graphs is tame. We remark that our results are incomparable to theirs, in the sense that there are classes of graphs whose tameness follows from their work, but not ours, and vice versa.

Theorem 1.3 substantially generalizes the main result of Milanič and Pivač [19], who obtained a complete classification into tame or feral of hereditary graph classes characterized by forbidden induced subgraphs on at most 4 vertices. The generalization comes at a price - as our upper bounds on the number of minimal separators are quasi-polynomial instead of polynomial.

Next we explore for which classes we are able to improve our quasi-polynomial upper bounds to polynomial ones. Here, again, domination plays a crucial role. We show that for every pair k, k' of integers, every class of graphs that excludes k-creatures and additionally has the property that every minimal separator S is dominated by a vertex set X of size at most k' and *disjoint from* S is tame. We then proceed to show that graphs that exclude k-creatures and all cycles of length at least r for any choice of natural numbers k and r have this property, leading to Theorem 1.4.

THEOREM 1.4. *For every pair of natural numbers k and r, the family of graphs that are $C_{\geq r}$-free, k-theta-free, k-prism-free, and k-pyramid-free is tame.*

Here a graph G is *$C_{\geq r}$-free* if it contains no induced cycles of length at least r. Theorem 1.4 is optimal in the sense that k-theta, k-prism, and k-pyramid graphs have at least 2^{k-2} minimal separators and therefore can have exponentially many minimal separators. Further, it substantially strengthens the results of Chudnovsky et al. [6], who prove the same statement but only for $r = 5$.

Finally we show that graph classes that exclude k-creatures, k-skinny-ladders, as well as k-cliques satisfy the property that every minimal separator S can be dominated by a constant size set X disjoint from S. This implies that this family of graphs is tame as well.

THEOREM 1.5. *For any fixed natural number k, the family of graphs that are k-creature-free, contain no k-skinny-ladder as an induced minor, and contain no minimal separator that has a clique of size k is tame.*

Theorem 1.3 provided a classification of all hereditary graph classes defined by a finite set of forbidden induced subgraphs into strongly quasi-tame or feral. In the same way that Theorem 1.3 is a fairly direct consequence of Theorem 1.2, we can obtain from Theorem 1.5 a complete dichotomy of all hereditary graph classes defined by a finite set of forbidden induced subgraphs, and additionally exclude at least one clique, into *tame* or feral.

THEOREM 1.6. *Let $\mathcal{F}$ be a graph family defined by a finite number of forbidden induced subgraphs. If there exists a natural number k such that $\mathcal{F}$ forbids all k-clique, k-theta, k-ladder-theta, k-claw, and k-paw graphs then $\mathcal{F}$ is tame. Otherwise, $\mathcal{F}$ contains all cliques or $\mathcal{F}$ is feral.*

Subsequent Work. There have been two significant developments since the first version of this manuscript. The first is a manuscript by Gajarský et al. [11] which answers Conjectures 9.1 and 9.2 in the affirmative, that is, they prove that "strongly-quasi-tame" can be replaced by "tame" in the statements of Theorems 1.1, 1.2, and 1.3, respectively.

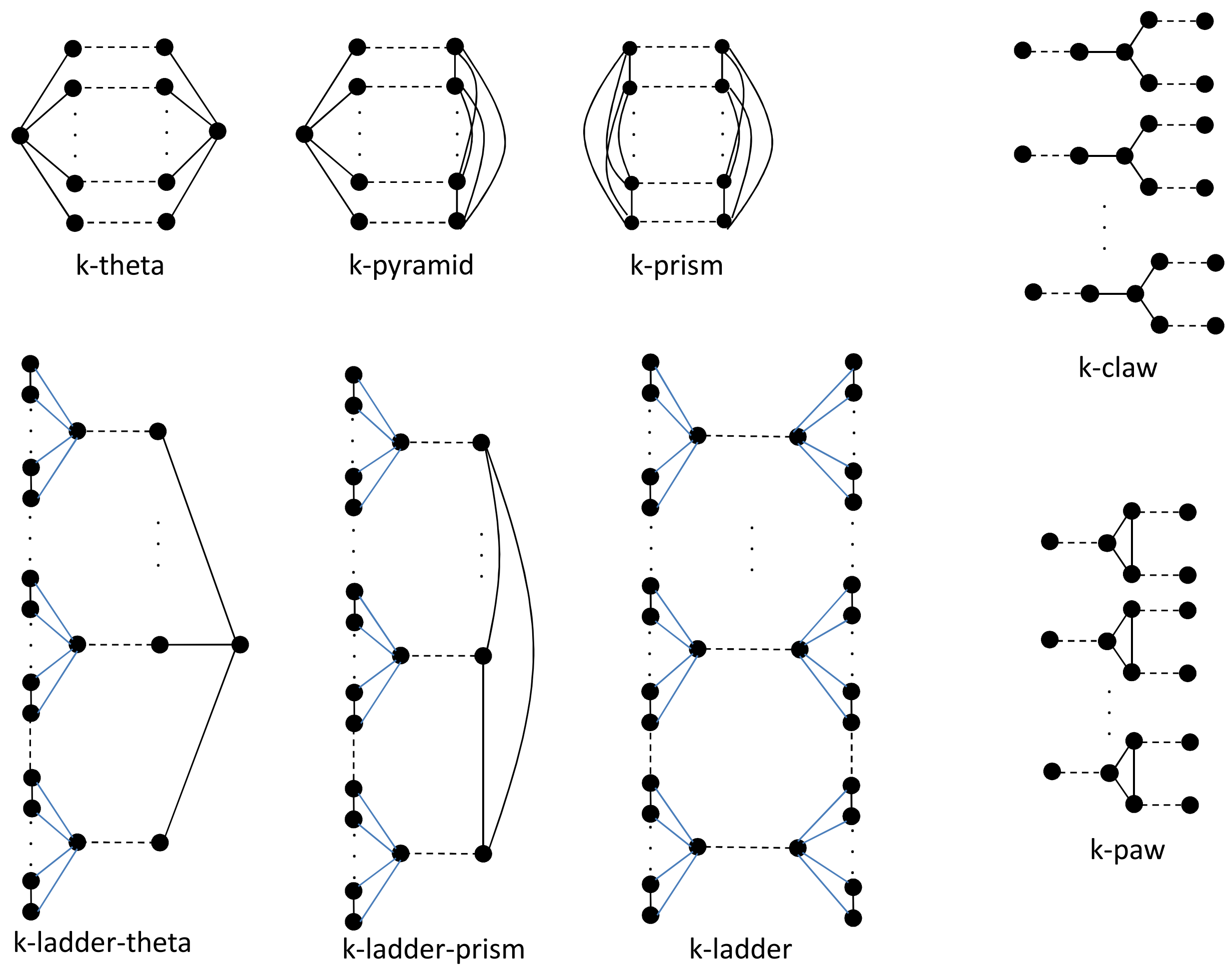

Figure 2: Dashed lines represent the option of having an arbitrary length path or just an edge (except for k-claw and k-paw graphs which the dotted line is always a path of length k.) The blue lines used in the k-ladder-theta, k-ladder-prism, and k-ladder graphs represents the option of either having or not having that edge, but for each vertex incident to more than one of the blue edges, at least one of those blue edges must belong to the graph.

The proof of Gajarský et al. builds heavily on top of the results in this paper. In particular, their proof of Conjecture 9.1 (i.e the strenthening of our Theorem 1.1 from quasi-tame to tame) requires all the tools that we develop in the proof of Theorem 1.1, except that the last crucial piece of our proof, namely Lemma 5.14, is replaced by a remarkably elegant argument that yields a polynomial upper bound on the number of minimal separators, rather than a quasi-polynomial one. Given Theorem 1.1, the proof of Theorem 1.2, which is given in Section 6, amounts to characterizing the inclusion minimal hereditary families $\mathcal{F}$ that contain a k-creature for every integer k. Thus, given that Conjecture 9.1 is true, the truth of Conjecture 9.2 follows directly from our proof of Theorem 1.2 as given in Section 6.

Our Theorems 1.4, 1.5 and 1.6 all give polynomial upper bounds for a subset of the graph classes covered by Theorems 1.1 and 1.2. Thus the proof of Gajarský et al. [11] that Conjectures 9.1 and 9.2 are true completely subsumes Theorems 1.4, 1.5 and 1.6. We nevertheless keep their statements and proofs in this paper, both because their proofs pre-dates the proof of Gajarský et al., and because they retain some (if arguably small) value. Specifically the proofs of Theorems 1.4, 1.5 and 1.6 all work a manner similar to the proof of Conjecture 9.1 by Gajarský et al., namely by replacing Lemma 5.14 by a polynomial upper bound. Our "replacements of Lemma 5.14" are slightly simpler than the one by Gajarský et al. Of course our proofs of Theorems 1.4, 1.5 and 1.6 only replace the quasi-polynomial bound of Lemma 5.14 by a polynomial upper bound for different special cases, while Gajarský et al., (essentially) do it for Lemma 5.14 in its full generality.

The second development concerns Conjecture 9.3, which conjectures that every induced-minor-closed class $\mathcal{F}$ is either tame or feral. It turns out that Conjecture 9.3 is false by a counterexample that combines the features of the counterexample to Conjecture 1.1 given in section 4 of this paper with the construction that shows that there exist hereditary families that are neither feral nor tame.

However, a much more general statement (that avoids the special cases which make Conjecture 9.3 flase) is true. In a yet unpublished follow up article [2] the authors show that every hereditary graph class which is definable in Monodic Second Order Logic (CMSO_2 Logic) is either quasi-tame or feral.

In terms of generality the result of [2] completely subsumes Theorems 1.1 and Theorem 1.2, at a cost of the quasi-polynomial bound on the number of minimal separators being much worse (about $n^{O(\log^{17} n)}$, as opposed to $n^{O(\log n)}$). The proof of [2] requires some, but far from all, tools in the present paper (namely Lemma 5.5 and the entire characterization of the inclusion minimal hereditary families $\mathcal{F}$ that contain a k-creature for every integer k, given in Section 6).

More importantly, the proof of [2] is very complex (spanning close to 100 pages), and appears to be very difficult to strengthen to a polynomial upper bound, leaving the polynomial bound of Gajarský et al. [11] as highly relevant. Therefore all of the main contributions of the present manuscript (the proofs of Theorems 1.1 and 1.2, with exception of Lemma 5.14) are crucial to either the proof Gajarský et al. [11] of Conjectures 9.1 and 9.2, or the proof of the dichotomy for CMSO-definable hereditary classes [2], or both.

Outline of the paper. In Section 2 we give a high level overview of our proofs. In Section 3 we set up the standard definitions and notations used in the paper. In Section 4 we give the counterexample to Conjecture 1.1. In Section 5 we prove our main result, Theorem 1.1. In Section 6 we characterize the premise in the statement of Theorem 1.1 (being k-creature-free and k-skinny-ladder induced minor-free) in terms of forbidden induced subgraphs, and use this characterization to prove Theorems 1.2 and 1.3. In Sections 7 and 8 we prove the polynomial bounds on the number of minimal separators in graphs that are both k-creature-free and long cycle-free, and in graphs that are k-creature-free, k-skinny-ladder induced minor-free, and k-clique-free. We conclude with some open problems in Section 9.

2 Overview

In this section we provide high level overview of our proofs. We will give quite detailed proof sketches of some of the pivotal steps, while skipping technical details of the more cumbersome parts. We start with the main ideas behind the proof of Theorem 1.1.

2.1 Overview of the Proof of Theorem 1.1. Recall that Theorem 1.1 states that for every natural number k, the family of graphs that are k-creature-free and do not contain a k-skinny-ladder as an induced minor is strongly-quasi-tame. There are three key lemmas that lie at the heart of the proof of Theorem 1.1. The first of these states that for a k-creature-free graph G, there are at most n^k distinct ways for the neighborhood of a

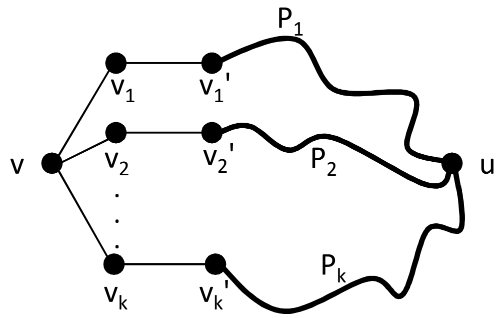

Figure 3: The k-creature formed in Claim 2.1 and Lemma 5.5. Note there may or may not be edges between the v_i's and the P_i's may overlap or have edges between them as well.

vertex v to intersect the minimal separators S of G, where $n = |V(G)|$.

CLAIM 2.1. *Let G be a k-creature-free graph with $n = |V(G)|$, $v \in G$, and let $S^v = \{N(v) \cap S : v \notin S$ and S is a minimal separator of $G\}$. Then $|S^v| \leq n^k$.*

Claim 2.1 is stated as Lemma 5.5 in the formal proof. Note that Claim 2.1 on its own does not imply that the number of minimal separators of G is bounded, only that the number of ways the neighborhood of a vertex can intersect the minimal separators of G is polynomial. In fact the counterexample given in Section 4 shows that the number of minimal separators of a k-creature-free graph of maximum degree at most 3 can be exponential.

The proof of Claim 2.1 is based on VC-dimension (see Definition 5.1) and follows from the Sauer-Shelah Lemma (see Lemma 5.3). In particular, if $|S^v|$ is large then there exists a vertex u not adjacent to v such that there are at least n^{k-1} distinct intersections $N(v) \cap S'$, where S' is a u, v-minimal separator. By the Sauer-Shelah Lemma there is a subset of $N(v)$ of size k that is shattered by the sets of the form $N(v) \cap S'$.

From the definition of shattering it follows that there are vertices $X = \{v_1, \ldots, v_k\}$ in $N(v)$ such that each v_i belongs to a private u, v-minimal separator S_i, i.e., $X \cap S_i = \{v_i\}$. Now, let C_i be the component that u belongs to in $G - S_i$, let v'_i be a neighbor of v_i in C_i with minimum distance to u, and let P_i be a shortest path from v'_i to u in C_i. Notice that no vertex of P_i can be neighbors with v_j for $i \neq j$ or else there would be a u, v path in $G - S_i$, and v_i cannot be neighbors with u, or else there would be a u, v path in $G - S_j$ for $j \neq i$. It then follows that v together with the set X and the P_i's make a k-creature (See Figure 3). So, for any fixed u, v, there are at most n^{k-1} unique sets of the form $N(v) \cap S$ where S is a u, v-minimal separator. Finally, it is easy to check that for every u-w minimal separator S there exists some v-w minimal separator or v-u minimal separator S' such that $N(v) \cap S = N(v) \cap S'$, proving the claim.

The second ingredient in the proof of Theorem 1.1 states that the minimal separators of graphs that are k-creature-free and have no k-skinny-ladder as an induced minor can be dominated by few vertices.

CLAIM 2.2. *Let $\mathcal{F}$ be a graph family that forbids k-creatures and has no k-skinny-ladder as an induced minor, then there exists a constant c such that for all graphs $G \in \mathcal{F}$, every minimal separator of G can be dominated by c vertices.*

The proof of Claim 2.2 (re-stated as Lemma 5.13 in the formal proof) is substantially more involved than the proof of Claim 2.1. Indeed the full proof of Lemma 5.13 takes up the bulk of Section 5. The overall strategy of the proof of Claim 2.2 is to start with the assumption that G is a graph and S is a minimal separator in G that cannot be dominated by c vertices and use this assumption to show the existence of either a k-creature or a k-skinny-ladder in G for sufficiently large k. Here sufficiently large means that k tends to infinity when c tends to infinity.

The proof is carried out in a sequence of steps, where each step "zooms in" on a more structured induced subgraph of G which still has a minimal separator (which is a subset of the original separator S) that cannot be dominated by c' vertices for some sufficiently large c'. As an example step let C_u and C_v be two full components of $G - S$ (a *full component* of $G - S$ is a component C such that $N(C) = S$). Without loss of generality

$V(G) = C_u \cup S \cup C_v$, because if $G - S$ also contains some other component C then S is still a minimal separator in $(G - C) - S$ and S still cannot be dominated by c vertices in $G - C$.

The first step of the proof of Claim 2.2 is to reduce to the case where $G - S$ has precisely two components P_L and P_R and both P_L and P_R induce paths. While this sounds like a pretty strong claim this is actually one the less technical steps in the proof of Claim 2.2. The idea is to look for a k-creature (A, X, Y, B) where $A \cup X \subseteq C_u$, $Y \subseteq S$ and $B = C_v$. If we fail to find a k-creature of this form then one can find $k - 1$ induced paths $P_1 \ldots P_{k-1}$ in C_u that together dominate S (see Lemma 5.6). A symmetric argument shows the existence of induced paths $Q_1 \ldots Q_{k-1}$ in C_v that together dominate S. Since S is completely covered by at most k^2 sets on the form $N(P_i) \cap N(Q_j)$, it follows that there must exist some pair i, j such that $S \cap N(P_i) \cap N(Q_j)$ cannot be dominated by c/k^2 vertices. We now consider $G[P_i \cup (S \cap N(P_i) \cap N(Q_j)) \cup Q_j]$, in this graph the minimal separator $(S \cap N(P_i) \cap N(Q_j))$ cannot be dominated by c/k^2 vertices and the two full components are $P_L = P_i$ and $P_R = Q_j$.

The next sequence of steps (Lemmas 5.7, 5.8, and ultimately 5.9) show that it is possible to zoom in on a sub-path P'_L of P_L, a sub-path P'_R of P_R and a subset $I \subseteq S$ of size at least c' (where c' is lower bounded by an unbounded function of c) such that both P'_L and P'_R dominate I, I is an independent set (no pair of vertices in I are adjacent) and furthermore no vertex in P'_L or P'_R have more than one neighbor in I. Note that I is now a minimal separator in $G[P'_L \cup I \cup P'_R]$. The additional properties of I witness that I cannot be dominated by less than $|I| \geq c'$ vertices, because no vertex of $G[P'_L \cup I \cup P'_R]$ can dominate more than one vertex in I. Thus, Lemmas 5.7, 5.8, and 5.9 allow us to reduce the proof of Claim 2.2 from the general case where S cannot be dominated by few vertices, but we do not know why, to the special case where S cannot be dominated by few vertices because no vertex in G dominates more than one vertex of S. The proofs of Lemmas 5.7, 5.8, and 5.9 are fairly technical, and we skip them in this overview.

Assuming Lemma 5.9 we are in the following setting. Our graph G consists of an independent set S of size at least c, which is much larger than k and two paths P_L and P_R that both dominate S. Further, no two vertices in S have any common neighbor. Our goal is to find a k-skinny-ladder as an induced minor in G. Observe that the graph G already kind of looks like a skinny-ladder. The main problem is that each of the vertices of S can have many neighbors in P_L and in P_R and that these neighbors can "interleave" a lot (see e.g. Figure 8). The next series of lemmas — namely Lemmas 5.10, 5.11, and 5.12, culminating with 5.13 — show that if the neighbors of the vertices in S interleave "too much", then we can find a k-creature in G, while if they do not then G contains a k-skinny-ladder.

Claim 2.1 together with Claim 2.2 are almost enough to prove Theorem 1.1. Suppose that instead of Claim 2.2 we had the stronger statement that for every minimal separator S in every k-creature-free, k-skinny-ladder induced minor-free graph there is a dominating set D of size c such that D is disjoint from S. [1] In this hypothetical scenario we can give a simple proof of a statement stronger than Theorem 1.1 — a polynomial upper bound on the number of minimal separators of k-creature-free, k-skinny-ladder induced minor-free graphs. Suppose for contradiction that the number of minimal separators is super-polynomial. By the dream-claim there is some constant size set D such that there are super-polynomially many minimal separators S that are disjoint from D and dominated by D. By Claim 2.1 each vertex $v \in D$ has only polynomially many options for the intersection $N(v) \cap S$. But then there must be two distinct minimal separators S_1 and S_2 that are disjoint from D, dominated by D, and that satisfy $S_1 \cap N(v) = S_2 \cap N(v)$ for every vertex v in D. But D dominates S_1 and S_2, and therefore we have

$$S_1 = \bigcup_{v \in D} N(v) \cap S_1 = \bigcup_{v \in D} N(v) \cap S_2 = S_2$$

contradicting that S_1 and S_2 are different minimal separators.

The final ingredient of the proof of Theorem 1.1 is a strengthening of this argument that also works for the case when the dominating set D is not necessarily disjoint from S. This strengthening comes at the cost that we are only able to prove a quasi-polynomial upper bound on the number of minimal separators.

Claim 2.3. *There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a graph with n vertices and let k and k' be integers such that for all induced subgraphs G' of G and for all $v \in G'$, if $S^v_{G'} = \{N(v) \cap S : v \notin S$*

[1] This claim is actually false: for any $k \geq 1$ start with a k-skinny-ladder for and turn $\{s_1, \ldots, s_k\}$ into a clique. It is easy to check that this graph does not contain a 5-creature or a 5-skinny-ladder (as an induced minor), while no set of size less than k disjoint from S can dominate S.

and S is a minimal separator of $G'\}$, then $|S^v_{G'}| \leq n^k$ and every minimal separator of every induced subgraph of G can be dominated by k' vertices. Then G has at most $n^{f(k,k')\log(n)}$ minimal separators where $n = |V(G)|$.

Claim 2.3 is stated as Lemma 5.14 in the formal proof (we note that for technical reasons the statement of Lemma 5.14 slightly differs from that of Claim 2.3). Note that by Claims 2.1 and 2.2, graphs that are k-creature-free and forbid k-skinny-ladders as an induced minor satisfy the premise of Claim 2.3.

The basic idea of the proof is to use a recursive branching algorithm that outputs all of the minimal separators of the graph, and upper bound the total number of sets output by this algorithm. The algorithm takes a tuple (G, X) where G is a graph that satisfies the conditions of Claim 2.3, and $X \subseteq V(G)$ and returns all minimal separators of G contained in X (and possibly other sets as well). Initially the algorithm is called with $X = V(G)$. We will measure the running time of the algorithm in terms of n and an upper bound x on the size of X. Initially we have $x = n$.

Fix some minimal separator S of G that is contained in X. We set Q to be the set of vertices of G that have at least $\frac{1}{2k'}|X|$ neighbors in X. The reason for choosing this particular fraction will become apparent shortly. By assumption, for each $q \in Q$ if $q \notin S$ then there are at most n^k options as to what $N[q] \cap S$ is. For each option $Y \in S^q_G$ we call the algorithm on $(G - Y, X - N[Y])$, and for each set S' that is returned by the call $(G - Y, X - N[q])$, we add $S' \cup Y$ to our collection of sets that we will return. If $Y' \in S^v_G$ is equal to $N[q] \cap S$, then $S - Y'$ is a minimal separator of $G - Y'$ contained in $X - N[q]$ and so $(S - Y') \cup Y' = S$ will be included in the list of sets that we return. In each of our branches we are calling the algorithm on $X' = X - N[q]$, and since q has at least $\frac{1}{2k'}|X|$ neighbors in X, X' is a constant fraction smaller than X. Thus the running time of (and the number of sets output by) the algorithm is governed by the recurrence $T(n, x) \leq n^{k+O(1)}T(n, x(1 - \frac{1}{2k'}))$, which solves to $n^{O(\log x)} \leq n^{O(\log n)}$ for fixed k and k'.

But what if $Q \subseteq S$? To handle this case we use that fact that $S - Q$ must then be a minimal separator of $G - Q$ and by assumption there are at most k' vertices of $G - Q$ that dominate $S - Q$. We can now see why the fraction $\frac{1}{2k'}$ was used to define Q; the neighborhood of these k' vertices contain at most $1/2$ the vertices of X. Thus, for every set R of k' vertices of G, we call the algorithm on $(G - Q, (X - Q) \cap N(R))$. For each set S' that is returned from the call $(G - Q, (X - Q) \cap N(R))$, we add $S' \cup Q$ to the list of sets output by the algorithm. Since there is some set R' of k' vertices in $G - Q$ such that R' dominates $S - Q$, S will get added to the list. Each of the recursive calls invoke the algorithm on $X' = X \cap N[R]$, and $|X'| \leq .5|X|$. In this case the running time of (and the number of sets output by) the algorithm is governed by the recurrence $T(n, x) \leq n^{k'+k}T(n, \frac{x}{2})$, which also solves to $n^{O(\log x)} \leq n^{O(\log n)}$ for fixed k and k'. This completes the sketch of the proof of Claim 2.3 and therefore also of Theorem 1.1.

2.2 Overview of the Proof of Theorems 1.2 and 1.3. The conclusion of Theorem 1.1 is simple - G only has a quasi-polynomial number of minimal separators. On the other hand the premise is somewhat opaque. It is not immediately obvious which graphs contain k-creatures for arbitrarily large k, and which graphs contain a k-skinny-ladder as an induced minor. In the second part of the paper we re-formulate the premise of Theorem 1.1 in terms of forbidden induced subgraphs. More concretely, the bulk of the work in Section 6 goes into proving the following statement.

Claim 2.4. *For every natural number k, there is a number k' such that if a graph G contains a k'-creature, then G contains a k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, or k-ladder-prism as an induced subgraph.*

See Figure 2 for a depiction of the graphs in Claim 2.4, and see Section 6 for definitions. This statement appears as Lemma 6.9 in the formal proof. Claim 2.4 is best possible in the sense that each one of the k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, or k-ladder-prism contains a k-creature, and that dropping any one of them from the list would make the conclusion of Claim 2.4 false. The contrapositive of the statement of Claim 2.4 implies that if a hereditary graph family $\mathcal{F}$ excludes the k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, and k-ladder-prism as induced subgraphs, then there exists a k' depending only on k such that $\mathcal{F}$ is k'-creature-free. Therefore, Claim 2.4 together with Theorem 1.1 and the observation that a $2k$-skinny-ladder induced minor either yields a k-creature or a k-contracted ladder as an induced subgraph implies Theorem 1.2.

So, how do we prove Claim 2.4? At a very high level it is just a sequence of structural lemmas, each on the form "if G contains a k-creature that additionally has some property X, then G also contains a k'-creature for some k' which is much smaller than k, but still tends to infinity with k, and the k'-creature has some stronger structural property Y". The next lemma now has as premise "if G contains a k creature that additionally has property Y", continuing the chain. Since we are looking for highly symmetric induced subgraphs in fairly general graphs it should come as no surprise that this sequence of arguments makes frequent use of Ramsey's Theorem.

Slightly more concretely, the proof of Claim 2.4 considers the two "sides" of the k-creature separately. Specifically, suppose that G contains a k'-creature $(A, B, \{x_1, x_2, \ldots, x_{k'}\}, \{y_1, y_2, \ldots, y_{k'}\})$, then we focus in on just one half of the k'-creature, say the side the consists of A and $\{x_1, x_2, \ldots, x_{k'}\}$. Here we can find what amounts to a k-half-theta, k-half-prism, or k-half-ladder (imagine cutting a k-theta, k-prism, or k-ladder in half vertically, see Figure 9). Then on the other side that consists of B and $\{y_1, y_2, \ldots, y_{k'}\}$ we find a k-half-theta, k-half-prism, or k-half-ladder in the same way and we can merge them together to make either a k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, or a k-ladder-prism.

Let us now see how to derive Theorem 1.3 from Theorem 1.2. We can see that for every natural number k there is a k' large enough such that if a graph contains a k'-ladder as an induced subgraph or a k'-skinny-ladder as an induced minor, then it contains a k-paw or k-claw as an induced subgraph. Hence, putting this together with Theorem 1.2 leads us to one half of Theorem 1.3, that if there is a natural number k such that a family of graphs $\mathcal{F}$ forbids k-theta, k-pyramid, k-prism, k-ladder-theta, k-ladder-prism, k-claw, and k-paw graphs, then $\mathcal{F}$ is strongly-quasi-tame.

Next, let's see how we prove the second part of the statement of Theorem 1.3. This states that if $\mathcal{F}$ is a family of graphs defined by a finite number of forbidden induced subgraphs, then if $\mathcal{F}$ contains k-theta, k-pyramid, k-prism, k-ladder-theta, k-ladder-prism, k-claw, or k-paw graphs for arbitrarily large k, then $\mathcal{F}$ is feral. If $\mathcal{F}$ contains k-theta, k-pyramid, k-prism, k-ladder-theta, or k-ladder-prism graphs for arbitrarily large k, then we take some k-theta, k-pyramid, k-prism, k-ladder-theta, or k-ladder-prism graph that is contained in $\mathcal{F}$ and show that if we contract the correct edges, then we can end up with a k-theta, k-pyramid, k-prism, k-ladder-theta, or k-ladder-prism graph that is still contained in $\mathcal{F}$ but now has $O(k)$ vertices. It follows that in this case $\mathcal{F}$ is feral.

It is critical here that our graph family is defined by a *finite* set of forbidden induced subgraphs. In general one cannot just contract an edge of a graph G that belongs to a family $\mathcal{F}$ and expect G to still belong to $\mathcal{F}$ after contraction. However, for a family that is defined by a finite set of forbidden induced subgraphs we can contract edges that are in the middle of sufficiently long paths consisting of vertices of degree 2. In our proofs we only contract such edges. Now, if $\mathcal{F}$ contains k-paw or k-claw graphs for arbitrarily large k, then we show that for large enough k we can essentially glue the claws or paws together on top of each other an create a graph with $O(k)$ vertices, exponentially many minimal separators, and avoid any of the forbidden subgraphs of $\mathcal{F}$ (see Figure 10 of Section 6 for a picture of this. Again it is crucial here that that our graph family is defined by a *finite* set of forbidden induced subgraphs). This concludes our sketch of the proof of Theorem 1.3.

2.3 Overview of the Proofs of Theorems 1.4, 1.5, and 1.6 Recall that Theorems 1.4 and 1.5 give *polynomial* upper bounds on the number of minimal separators for graphs that exclude k-creatures, k-skinny-ladders as induced minors, and additionally long cycles (in the case of Theorem 1.4) or large cliques (in the case of Theorem 1.5).

Given the tools developed on the way to proving Theorems 1.1 and 1.3, Theorems 1.4 and 1.5 follow almost for free. Recall the "dream strengthening" of Claim 2.2 from the proof sketch of Theorem 1.1; every minimal separator S in a graph that excludes a k-creature and a k-skinny-ladder as an induced minor is dominated by a set D of constant size k', disjoint from S. This dream strengthening is false in general, but it turns out to be true (and fairly easy to prove) in graphs that additionally exclude either all long cycles or all sufficiently large cliques. Now Theorems 1.4 and 1.5 follow directly from the argument in the failed proof attempt in Section 2.1 for Theorem 1.1 based on the dream claim. Theorem 1.6 is "extracted" from Theorem 1.5 in exactly the same way Theorem 1.3 is derived from Theorem 1.1.

3 Preliminaries

All graphs in this paper are assumed to be simple, undirected graphs unless otherwise stated. We denote the edge set of a graph G by $E(G)$ and the vertex set of a graph by $V(G)$. If $v \in V(G)$, then we use $N_G[v]$ to denote the closed neighborhood of v in the graph G, i.e., the set of all neighbors v has in G together with v itself. We use $N_G(v)$ to denote the set $N_G[v] - \{v\}$. If $X \subseteq V(G)$, then $N_G[X] = \bigcup_{x \in X} N_G[x]$ and $N_G(X) = N_G[X] - X$. When the graph G is clear from the context, we will use $N[v]$, $N(v)$, $N[X]$, and $N(X)$. If $X \subset V(G)$, then we use $G[X]$ to denote the induced subgraph of G with vertex set X and $G - X$ denotes $G[V(G) - X]$.

Given a graph G, a non-empty set $S \subset G$ is called a *separator* if there are at least two distinct connected components L and R of $G - S$. If $u \in L$ and $v \in R$ then we call S a *u-v-separator* or a *u, v-separator*. S is a *u, v-minimal separator* if S is a u, v-separator and no proper subset of S is a u, v-separator, or equivalently, if $N_G(L) = N_G(R) = S$. This equivalence is folkloric and easy to show. We say a component X of $G - S$ is an S-full component if $N_G(X) = S$.

A family of graphs $\mathcal{F}$ is called *tame* if there exists a constant c such that for all $G \in \mathcal{F}$, G has at most $|V(G)|^c$ minimal separators. A family of graphs $\mathcal{F}$ is called *strongly-quasi-tame* if there exists a constant c such that for all $G \in \mathcal{F}$, G has at most $|V(G)|^{c\log(|V(G)|)}$ minimal separators. A family of graphs $\mathcal{F}$ is called *feral* if there exists a constant $c > 1$ such that for all natural numbers N there exists a $G \in \mathcal{F}$, such that $|V(G)| = n > N$ and G has at least c^n minimal separators.

Given a path $P = v_1, v_2, \ldots, v_k$ we call v_1 and v_k the *endpoints* of P, and all other vertices of P are *internal* vertices of P. The *length* of a path is the number of vertices in the path. Given a graph G and a graph H, G is said to be *H-free* or *G forbids H* if G does not contains H as an induced subgraph. We will sometimes talk about the induced minors of a graph so being H-free should not be confused with G not containing H as an induced minor. If G does not contain H as an induced minor then that is precisely what we will say, that G does not contain H as an induced minor. If $\mathcal{G}$ is a family of graphs such that every $G \in \mathcal{G}$ is H-free, then $\mathcal{G}$ is said to be H-free or that $\mathcal{G}$ forbids H. Similarly, given a graph G and a family of graph $\mathcal{H}$, G is said to be $\mathcal{H}$-free or G forbids $\mathcal{H}$ if G is H-free for all $H \in \mathcal{H}$. If $\mathcal{G}$ is a family of graphs such that every $G \in \mathcal{G}$ is $\mathcal{H}$-free, then $\mathcal{G}$ is said to be $\mathcal{H}$-free or that $\mathcal{G}$ forbids $\mathcal{H}$.

Let $\mathcal{F}$ be a family of graphs. We say that $\mathcal{F}$ is a *family of graphs defined by a finite number of forbidden induced subgraphs* if there exists a finite set of graphs $\mathcal{H}$ such that $G \in \mathcal{F}$ if and only if G if G is $\mathcal{H}$-free. We say that $\mathcal{H}$ is a set of forbidden subgraphs that define $\mathcal{F}$.

Given a graph G let H and K be two subsets of $V(G)$. We say that H is *anti-complete* with K or that H and K are *anti-complete* if H and K are disjoint and every vertex in H is non-adjacent to every vertex in K in G. We extend this definition in an obvious way to allow H (and possibly K) to be a subgraph of G by saying H is anti-complete with K if $V(H)$ is anti-complete with K ($V(K)$ if K is also a subgraph). A set $X \subseteq V(G)$ is said to dominate a set $Y \subseteq V(G)$ if for every $y \in Y$ either $y \in X$ or there is an $x \in X$ such that $yx \in E(G)$.

Given a graph G and an edge $uv \in E(G)$ we denote by G^{uv} the graph that results from *contracting* the edge uv in G, so $V(G^{uv}) = (V(G) - \{u, v\}) \cup \{w\}$ (where w is the new vertex created from the contraction) and for $x, y \in V(G) - \{u, v\}$, $xy \in E(G^{uv})$ if and only if $xy \in E(G)$ and for $x \in V(G) - \{u, v\}$, $xw \in E(G^{uv})$ if and only if x is neighbors with u and/or v in G. Note that as all graphs that we deal with are simple graphs, our definition of contraction does not allow for multiple edges. Given an induced path P of G, we denote by G^P the graph that results from contracting each edge of P one at a time. Note that the resulting graph is independent of the order the edges are contracted in. Given two anti-complete graphs A and B with $a \in A$ and $b \in B$, we define an operation *gluing a to b* which is the graph that results in adding the edge, ab between a and b and then contracting the edge ab.

4 A k-Creature-Free Feral Graph Family

In this section we will show that the graph of Figure 4, which we will refer to as the *k-twisted-ladder*, is a counterexample to Conjecture 1.1. We begin the next paragraph by giving a few definitions, then in the following paragraph we will observe that the k-twisted-ladder has 2^k minimal separators, and finally Lemma 4.1 completes the counterexample by showing that the k-twisted-ladder does not contain a large k-creature.

We define a partition of the vertices as follows, let S denote the set of labeled vertices of the k-twisted ladder that have 1 as their superscript. If we remove S from the k-twisted-ladder we get two induced paths, one on the left side which we will refer to as L and one on the right side which we will refer to as R. We also define the i^{th} *block* of the k-twisted-ladder to be the set of vertices that contains the vertices of the subpath of L that has c^L_{i+1} and c^L_i as its endpoints, the vertices of the subpath of R that has c^R_{i+1} and c^R_i as its endpoints, and the vertices a^1_i and b^1_i. So, the i^{th} block and the $(i+1)^{th}$ block overlap at the vertices c^R_{i+1} and c^L_{i+1}.

To see that the k-twisted-ladder has at least 2^k minimal separators we make the following set, X. For each i with $1 \leq i \leq k$ we choose $j \in \{1, 2\}$ and add a^j_i and b^j_i to X. X is then an x, y-minimal separator, and there are 2^k different choices we had when making X, so the k-twisted-ladder has at least 2^k minimal separators.

To complete the counterexample, we show in the following lemma that this structure does not have a large k-creature. To make the result as easy as possible to verify, we show no k-twisted-ladder has a 100-creature, although a significantly smaller upper bound exists.

LEMMA 4.1. *k-twisted-ladders are 100-creature-free for all k.*

Proof. Let H be a k-twisted-ladder. Assume for a contradiction that H contains a 100-creature $(A, B, \{x_1, x_2, \ldots, x_{100}\}, \{y_1, y_2, \ldots, y_{100}\})$.

Let X_A and X_B denote the highest numbered block that A and B have a vertex in respectively, and let Y_A and Y_B denote the lowest numbered block that A and B have a vertex in respectively. Let $i = max(Y_A, Y_B)+1$ and let $j = min(X_A, X_B)-1$. Let r be an integer such that $i \leq r \leq j$ (if no such r exists, then the only blocks that can contain vertices from both A and B must be two adjacent blocks. Since each block only has 10 vertices, A has size at most 20 and since the max degree of the twisted-ladder is 3, A cannot dominate all 100 vertices of $\{x_1, x_2, \ldots, x_{100}\}$, a contradiction to the definition of k-creature). Then since A and B are connected and both contain vertices in blocks above and below block r we can see by inspection that A must contain one vertex from $\{c_r^L, c_r^R\}$ and one from $\{c_{r+1}^L, c_{r+1}^R\}$ and B must contain one vertex from $\{c_r^L, c_r^R\}$ and one from $\{c_{r+1}^L, c_{r+1}^R\}$. Furthermore, since A is anti-complete with B, we can again see from inspection that if $c_r^L \in A$ then we must have $c_{r+1}^L \in A$, $c_r^R \in B$, and $c_{r+1}^R \in B$ (the removal of the closed neighborhoods of c_r^L and c_{r+1}^R would separate blocks numbered greater than r from blocks numbered less than r, so both c_r^L and c_{r+1}^R cannot belong to A since B is connected and has vertices in blocks above and below r). Similarly if $c_r^R \in A$ then we must have $c_{r+1}^R \in A$, $c_r^L \in B$, and $c_{r+1}^L \in B$.

Therefore, without loss of generality we may assume that for all r with $i \leq r \leq j$ that $c_r^L \in A$ and $c_r^R \in B$. It then follows from this assumption and the fact that A is anti-complete with B that there are only two possibilities for the restriction of A and B to the r^{th} block. Either we have that both the restriction of A to the r^{th} block is the subpath of L with endpoints c_r^L and c_{r+1}^L and the restriction of B is the subpath of R with endpoints c_r^R and c_{r+1}^R or the restriction of A is the induced path made up of c_{r+1}^L along with b_r^1 and b_r^1's two neighbors in L and the restriction of B is the induced path made up of c_r^R along with a_r^1 and a_r^1's two neighbors in R. Note that in either case, we may conclude by inspection of the twisted-ladder that every vertex in block r is either in A, B or $N(A) \cap N(B)$.

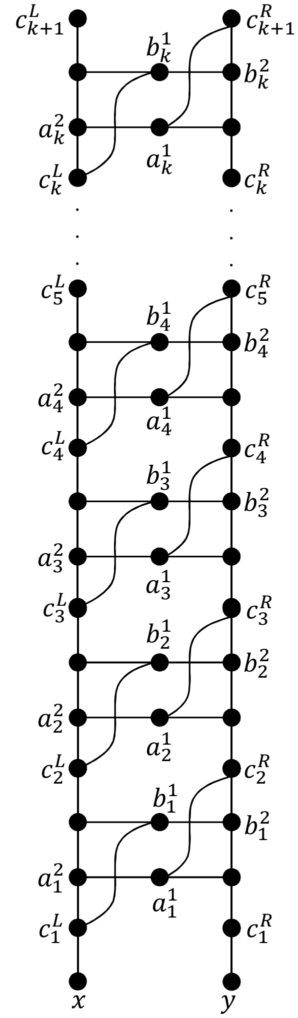

Figure 4: The k-twisted-ladder.

We now show that it is impossible for the vertices of $\{x_1, x_2, \ldots, x_{100}\}$ to be within distance two of both A and B, which would contradict the definition of a k-creature. By the definition of a k-creature, no vertex of $\{x_1, x_2, \ldots, x_{100}\}$ can belong to $N(A) \cap N(B)$. Hence, by the last sentence of the previous paragraph, no vertex of $\{x_1, x_2, \ldots, x_{100}\}$ belongs to blocks i through j. Since no vertex of $\{x_1, x_2, \ldots, x_{100}\}$ belongs to blocks i through j, $i-1 = max(Y_A, Y_B)$ and $j+1 = min(X_A, X_B)$, and the vertices of $\{x_1, x_2, \ldots, x_{100}\}$ must be within distance two of both A and B it follows that all vertices of $\{x_1, x_2, \ldots, x_{100}\}$ must be with distance two of blocks $i-1$ and $j+1$. But we can see by inspection of the twisted-ladder that there do not exists that many vertices within distance two of these two blocks. We can conclude that $(A, B, \{x_1, x_2, \ldots, x_{100}\}, \{y_1, y_2, \ldots, y_{100}\})$ cannot be a 100-creature. ◻

5 k-Creature and k-Skinny-Ladder Induced Minor Free Graphs

In this section we will provide all the lemmas needed for a proof of Theorem 1.1 and conclude this section with a proof of Theorem 1.1. We begin this section by stating some well known results which we will need later on. The three key ideas of this section are Lemma 5.5 which shows that the neighborhood of a vertex v of a k-creature-free graph G can intersect the minimal separators of G that do not contain v in at most n^k different ways, Lemma 5.13 which shows that all minimal separators of graphs that are k-creature-free and do not contain a k-skinny-ladder as an induced minor can be dominated by a constant number vertices, and Lemma 5.14 which uses a branching algorithm to list all minimal separators of its input graph assuming the input graph satisfies certain properties and proves a bound on the number of minimal separators produced by this algorithm. An easy proof combining Lemma 5.5, and Lemma 5.14 is then used to establish Theorem 1.1. Most of the work of this section goes into proving lemmas needed for the proof of Lemma 5.13, in particular, Lemmas 5.6 through 5.12 build up to a proof of Lemma 5.13. In this section and the rest of the paper when we refer to k-creatures and k-skinny-ladders we will assume $k > 1$.

Lemma 5.1. (**Ramsey's Theorem**) *[20]*
For every pair of positive integer k and ℓ there is a least positive integer $R(k,\ell)$ such that every graph with at least $R(k,\ell)$ vertices contains a clique of size k or an independent set of size ℓ.

Throughout this paper we will us the notation $R(k,\ell)$ to denote the least positive integer such that every graph with at least $R(k,\ell)$ vertices contains a clique of size k or an independent set of size ℓ.

Lemma 5.2. (**Erdös-Szekeres Theorem**) *[9]*
For every pair on positive integers r and s, any sequence of distinct real numbers of length at least (r-1)(s-1) + 1 contains a monotone increasing subsequence of length r or a monotone decreasing subsequence of length s.

Definition 5.1. (**VC-Dimension**) *Let $\mathcal{F} = \{S_1, S_2, \ldots\}$ be a finite family of finite sets and let H be a set. $\mathcal{F}$ is said to shatter H if for every subset $H' \subseteq H$ there is a $S_i \in \mathcal{F}$ such that $H' = S_i \cap H$. The VC-dimension of $\mathcal{F}$ is the cardinality of the largest set that it shatters.*

Lemma 5.3. (**Sauer-Shelah Lemma**) *[22] Let $\mathcal{F}$ be a finite family of finite sets such that the VC-dimension of $\mathcal{F}$ is $k > 1$, and let $n = |\bigcup_{S_i \in \mathcal{F}} S_i|$, so n is the number of distinct elements contained in the sets of $\mathcal{F}$. Then the number of sets of $\mathcal{F}$ is at most $\Sigma_{i=0}^{k}\binom{n}{i} \le n^k$.*

Lemma 5.4. *Let S be a u,v-minimal separator and a u,w-separator and let $S' \subset S$ be a u,w-minimal separator. Then $N(w) \cap S' = N(w) \cap S$.*

Proof. Let S be a u,v-minimal separator and a u,w-separator, let $S' \subset S$ be a u,w-minimal separator, and let C_u and C'_u be the connected components that u lies in in $G - S$ and $G - S'$ respectively. Note that $C_u \subseteq C'_u$. Clearly $N(w) \cap S' \subseteq N(w) \cap S$. Now let $y \in N(w) \cap S$. Then there is a path from y to u such that all internal vertices of this path are contained in C_u. This same path has its internal vertices in C'_u, so if $y \notin S'$ then S' does not separate u from w. The result follows. ∎

Lemma 5.5. *Let G be a k-creature-free graph and let $\mathcal{S}$ be a set of minimal separators of G. Then for every $v \in G$, if $S^v = \{N(v) \cap S | \ S \in \mathcal{S}$ and $v \notin S\}$ then $|S^v| \le |V(G)|^k$.*

Proof. Let G be a k-creature-free graph with n vertices, let $\mathcal{S}$ be a set of minimal separators of G, and fix two non-adjacent vertices u, v. Let $S^{v,u} = \{N(v) \cap S | S \in \mathcal{S}$ and S is a u,v-minimal separator of $G\}$. We first show that $|S^{v,u}| \le n^{k-1}$. Assume for a contradiction that $|S^{v,u}| > n^{k-1}$, then by the Sauer-Shelah Lemma there is a subset of size k of $N(v)$ that is shattered by $S^{v,u}$. It follows that there are vertices $V = \{v_1, \ldots, v_k\}$ in $N(v)$ such that each v_i belongs to a private u,v-minimal separator S_i, i.e., $V \cap S_i = \{v_i\}$. Now, let C_i be the component that u belongs to in $G - S_i$, C_i dominates S_i since S_i is a u,v-minimal separator. Let v'_i be a neighbor of v_i in C_i with minimum distance to u (note that v_i is not a neighbor of u or else there would be a u,v path in $G - S_j$ for $j \ne i$, hence $v'_i \ne u$), and let P_i be a shortest path from v'_i to u in C_i. Let $P = \bigcup_{i=1} (V(P_i) - \{v'_i\})$.

We claim that (v, P, $\{v_1, v_2, \ldots, v_k\}$, $\{v'_1, v'_2, \ldots, v'_k\}$) is a k-creature (see Figure 3 to for a visual description of this step of the proof). To see this note that (1) $G[P]$ is connected since it is a set of paths which all contain u.

(2) v is anti-complete with each P_i since v and P_i are both contained in two different S_i-full components hence v is anti-complete with P and $\{v'_1, v'_2, \ldots, v'_k\}$. To see that P is anti-complete with $\{v_1, v_2, \ldots, v_k\}$ note that each $P_i - v'_i$ is anti-complete with v_i since v'_i was chosen to be a neighbor of v_i in C_i that is as close as possible to u and P_i is a shortest path from v'_i to u in C_i. Furthermore, if P_i was not anti-complete with v_j for $i \neq j$ then S_i would not be a u, v-minimal separator since by assumption $v_j \notin S_i$ and no vertex of P_i is in S_i (since $V(P_i) \subseteq C_i$), hence there would be a path from v to u in $G - S_i$. It then follows that P is anti-complete with $\{v_1, v_2, \ldots, v_k\}$. (3) P dominates $\{v'_1, v'_2, \ldots, v'_k\}$ follows from how we defined P and $\{v_1, v_2, \ldots, v_k\} \subseteq N(v)$. (4) v_i is anti-complete with P_j for $i \neq j$ was showing when establishing (2), hence v_i is a neighbor of v'_j only if $i = j$. So, (v, P, $\{v_1, v_2, \ldots, v_k\}$, $\{v'_1, v'_2, \ldots, v'_k\}$) is a k-creature, a contradiction to G being k-creature-free. It follows that $|S^{v,u}| \leq n^{k-1}$.

Now, let $S \in \mathcal{S}$ such that $v \notin S$ and assume that S is an x, y-separator. S must be either a v, x-separator or a v, y-separator, let us assume without loss of generality that S is a v, x-separator, so there is an $S' \subseteq S$ that is a v, x-minimal separator. By Lemma 5.4 we have that $N(v) \cap S' = N(v) \cap S$, hence $N(v) \cap S \in S^{v,x}$, where $S^{v,x} = \{N(v) \cap S'' | S'' \in \mathcal{S}$ and S'' is a v, x-minimal separator of $G\}$. It follows that if $S^v = \{N(v) \cap S'' |\ S'' \in \mathcal{S}$ and $v \notin S''\}$ then $S^v = \bigcup_{x \in V(G)} S^{v,x}$. Then by the conclusion of the previous paragraph, it follows that $|S^v| \leq n^k$. ∎

The following corollary will be needed in Sections 7 and 8.

Corollary 5.1. *If G is a k-creature-free graph and every minimal separator, S, of G can be dominated by k' vertices of G not in S, then G has at most $|V(G)|^{kk'+k'}$ minimal separators.*

Proof. Assume G is a k-creature-free graph and every minimal separator, S, of G can be dominated by k' vertices of G not in S. For every $v \in G$ let $S^v = \{N(v) \cap S | v \notin S$ and S is a minimal separator of $G\}$. By Lemma 5.5 it holds that $|S^v| \leq |V(G)|^k$. Let $X = \bigcup_{v \in G} S^v$. Then $|X| = |V(G)|^{k+1}$ and the assumption that all minimal separators, S, of G can be dominated by k' vertices in G not in S implies that S is the union of at most k' sets in X. It follows there are at most $|V(G)|^{kk'+k'}$ minimal separators in G. ∎

We remark that it is possible to generalize Lemma 5.5 and Corollary 5.1 to the r^{th} neighborhood of a vertex for any fixed positive integer r while still maintaining polynomial bounds by using the fact the family of k-creature-free graphs are closed under contracting edges.

The following lemmas will be building towards a proof of Lemma 5.13, that all minimal separators of a graph that is k-creature-free and has no k-skinny-ladder as an induced minor can be dominated by few vertices. We begin with a proof that minimal separators can be dominated by a few induced paths in k-creature-free graphs (the paths may have edges between them).

Lemma 5.6. *Let G be a graph that is k-creature-free, let S be a minimal separator of G, and let A be an S-full component of $G - S$. Then S is dominated by a set of less than k induced paths of A.*

Proof. Let G, S, and A be as in the statement of this lemma, and let A' be a minimally connected induced subgraph of A such that S is dominated by A'. Let T be a breadth first search tree of A' rooted at some vertex $v \in A'$, and let $L = \{\ell_1, \ell_2, \ldots, \ell_c\}$ be the set of leaves of T. Since A' is minimal each leaf, $\ell_i \in L$, must have a neighbor $s_i \in S$ such that no other vertex of A' is a neighbor of s_i, else $A' - s_i$ would still be connected and dominate S. Then if K is another S-full component different from A we claim that the tuple $(V(A') - L, V(K), L = \{\ell_1, \ell_2, \ldots, \ell_c\}, \{s_1, s_2, \ldots, s_c\})$ forms a c-creature. To see this, note that (1) $G[V(A') - L]$ is still connected since L is a set of leaves of T, hence $T - L$ is a spanning tree of $G[V(A') - L]$. Additionally, K is connected by definition. (2) $V(A') - L$ is anti-complete with $\{s_1, s_2, \ldots, s_c\}$ since ℓ_i is the only vertex of A' that is neighbors of s_i, $V(A')$ is anti-complete with $V(K)$ since they are contained in two different S-full components hence $V(A' - L)$ and $V(K')$ are anti-complete and $V(K')$ and L are anti-complete. (3) That $V(A') - L$ dominates L is straight forward, and K dominates $\{s_1, s_2, \ldots, s_c\}$ since K is an S-full component. (4) By how we chose the s_i's we have have ℓ_i is a neighbor of s_j if and only if $i = j$.

It follows that if G is k-creature-free, then T has at most $k - 1$ leaves. Since T is a breadth first search tree of A', a root to leaf path in T is also an induced path in A', therefore A' is the union of at most $k - 1$ induced paths and the result follows. ∎

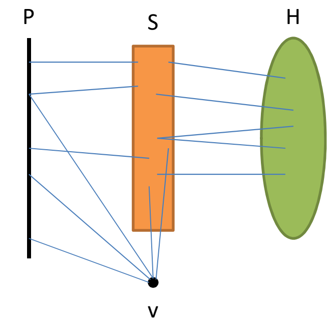

Figure 5: P, S, and H and the vertex v of Lemma 5.7

A key step to proving Lemma 5.13 is to show that if a k-creature-free graph G has a minimal separator, S, that cannot be dominated by $f(k)$ vertices, then using Lemma 5.6 we can find an induced path in an S-full, call the induced path P_L, and another induced paths from another S-full component, call this induced path P_R, such that we can find a large independent set, I, of S where every vertex in I has at least one neighbor in P_L and one neighbor in P_R and furthermore, no pair of vertices in I share a neighbor in either P_L or P_R. This is proven in Lemma 5.9.

The paths P_L and P_R are paths obtained by selecting induced paths such that the set $S_1 = N(P_L) \cap S \cap N(P_R)$ takes at least $f(k)/k^2$ vertices to dominate, such paths must exits by Lemma 5.6. The idea of the proof of Lemma 5.9 is to find a special vertex $v_1 \in S_1$, such that we can find a small set of vertices X_1 such that no vertex in $S_1 - N[X_1]$ shares a neighbor with v_1 in P_L and we find a small set of vertices Y_1 such that no vertex in $S_1 - N[Y_1]$ shares a neighbor with v_1 in P_R. We then add v_1 to I and set $S_2 = S_1 - (N[X_1] \cup N[Y_1] \cup \{v_1\})$. Then we find a special vertex $v_2 \in S_2$ and repeat. If the X_i's and Y_i's are small in size and S_1 cannot be dominated by few vertices then we can create a large set I in this way so that no vertices in P_L and P_R have more than one neighbor in I, so we get our desired P_L, P_R, and I.

The following lemma shows us how to locate X_i given the vertex v_i. The specially chosen vertex v_i will have the property that there exists a connected graph H such that all vertices of S_i have at least one neighbor in H except for v_i, and H is anti-complete with P_L (see Figure 5, where $P_L = P, S_i = S$, and $v_i = v$). In this situation, Lemma 5.7 shows how to obtain the desired set X_i, such that no vertex of $S_i - N[X_i]$ shares a neighbor with v_i in the path P_L.

Lemma 5.7. *Let G be a k-creature-free graph and let (S, H, P, v) be a tuple of disjoint subsets of $V(G)$ with the following properties (see Figure 5): $G[H]$ is connected, $G[P]$ is an induced path, H is anti-complete with P and v, and H dominates S. Then there is a set, X, of size at most k such that $N(S - N[X]) \cap N(v) \cap P = \emptyset$ and no vertex of $S - N[X]$ is a neighbor of v.*

Proof. Let G, S, H, P, and v be as in the statement of this lemma. Let $P' = P \cap N(v)$, let $S' = (S \cap N(P')) - N(v)$, and let $P'' = \{p_1, p_2, \dots, p_c\}$ be a minimal subset of P' that dominates S'. Since P'' is a minimal dominating set each element of P'' has a private neighbor in S', in other words for each $p_i \in P''$ there is an $s_i \in S'$ such that p_i is a neighbor of s_i and p_j is not a neighbor of s_i if $i \neq j$. We claim that the tuple $(\{v\}, H, P'' = \{p_1, p_2, \dots, p_c\}, \{s_1, s_2, \dots, s_c\})$ forms a c-creature (see figure 6). To see this note that (1) by assumption $G[H]$ is connected. (2) By assumption v and H are anti-complete, v and $\{s_1, s_2, \dots, s_c\} \subseteq S'$ are anti-complete by how S' was defined, and by assumption H and $\{p_1, p_2, \dots, p_c\} \subseteq P$ are anti-complete. (3) v dominates $\{p_1, p_2, \dots, p_c\} \subseteq P'$ by how P' is defined and H dominates $\{s_1, s_2, \dots, s_c\}$ since H dominates S by assumption. (4) By how the s_i's were chosen, s_i is a neighbor of p_j if and only if $i = j$.

Hence, since G is k-creature free we may assume that $c \leq k - 1$. By how P'' was chosen then we have that $N(S' - N[P'']) \cap N(v) \cup P = \emptyset$. Then setting $X = P'' \cup \{v\}$ is a set of size at most k that satisfies the conclusion of this lemma.

□

While Lemma 5.7 works for finding X_i such that no vertex of $S_i - N[X_i]$ has a common neighbor with v_i in P_L, it will not work to obtain a corresponding Y_i for P_R, the problem will lie in finding a suitable H with

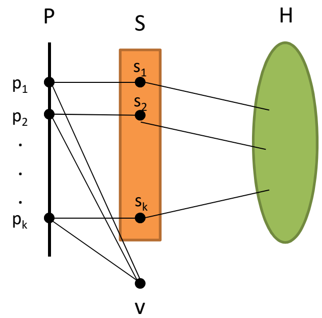

Figure 6: The k-creature formed in Lemma 5.7

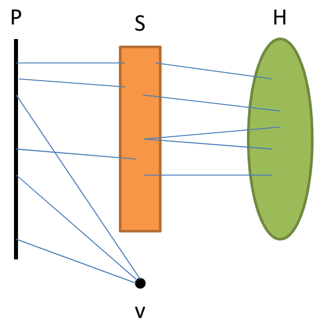

Figure 7: P, S, and H and the vertex v of Lemma 5.8

respect to P_R, S_i, and v_i (v_i is chosen specifically so we can find a suitable H with respect to P_L, S_i, and v_i, the path P_R is not taken into consideration in this selection). This issue is taken care of by Lemma 5.8, which shows that there is a set of at most $k-1$ connected components, $C_1, C_2, \dots, C_{k-1}$ of $P_L - N(v_i)$ that collectively dominated S_i. Then for each C_j clearly all vertices of $S_i \cap N(C_j)$ have a neighbor in C_j, and P_R and v_i are anti-complete with C_j, so we can apply Lemma 5.7 to $(S_i \cap N(C_j),\ C_i,\ P_R,\ v_i)$ to get a set $Y_{i,j}$ such that no vertex of $(S_i \cap N(C_j)) - N[Y_{i,j}]$ shares a neighbor with v_i in P_R. Since collectively all the C_j's dominate S_i, if we take $Y_i = \bigcup\limits_j Y_{i,j}$ then no vertex of $S_i - N[Y_i]$ shares a neighbor with v_i in P_R.

Lemma 5.8. *Let G be a k-creature-free graph and let (S, H, P, v) be a tuple of disjoint subsets of $V(G)$ with the following properties (see Figure 7): $G[H]$ is connected, $G[P]$ is an induced path, H is anti-complete with P and v, v is anti-complete with S, S is dominated by H and S is dominated by P, and $N(S) \cap N(v) \cap P = \emptyset$. Then there is a set of at most $k-1$ connected components of $G[P] - N(v)$ such that every vertex of S has a neighbor in at least one of these connected components.*

Proof. Let G, S, H, P, and v be as in the statement of this lemma. Assume for a contradiction that there does not exists a set of at most $k-1$ connected components of $G[P]-N(v)$ such that every vertex of S has a neighbor in at least one of these connected components. It follows then there is a set of k connected components of $G[P]-N(v)$, say $C_1,\ C_2, \dots, C_k$, such that there exists $s_1,\ s_2, \dots, s_k$ in S where $N(s_i) \cap V(C_j) \neq \emptyset$ if and only if $i = j$. Since $G[P]$ is connected, for every C_i there exists a vertex $c_i \in N(v) \cap P$ such that $c_i \in N(C_i)$ (the c_i's may not be unique even though the C_i's are). Now, for each s_i, let s_i' be the vertex in C_i that s_i is a neighbor of such that there exists an induced path P_i from s_i' to c_i with internal vertices in C_i such that s_i' is the only neighbor of s_i on the path P_i (recall by assumption that $N(S) \cap N(v) \cap V(P) = \emptyset$ so s_i cannot be a neighbor of c_i).

We claim the tuple $(\{v\} \cup \bigcup V(P_i - s_i'), H, \{s_1', s_2' \dots, s_k'\}, \{s_1, s_2 \dots, s_k\})$ is a k-creature, contradicting the assumption G is k-creature-free. To see this note that (1) H is connected by assumption and for all $i, 1 \leq i \leq k$, c_i belongs to $P_i - s_i'$ which is a neighbor of v, so $\{v\} \cup \bigcup V(P_i - s_i')$ induces a connected graph, (2) $\{v\} \cup \bigcup V(P_i - s_i')$ is anti-complete with H by definition, $\{v\} \cup \bigcup V(P_i - s_i')$ is anti-complete with $\{s_1, s_2 \dots, s_k\}$ by how the paths P_i were chosen and by the assumption that v is anti-complete with S, and H is anti-complete with $\{s'_1, s'_2 \ldots, s'_k\} \subseteq V(P)$ by assumption. (3) $\{v\} \cup \bigcup V(P_i - s'_i)$ dominates $\{s'_1, s'_2 \ldots, s'_k\}$ since $s'_i \in P_i$ and H dominates $\{s_1, s_2 \ldots, s_k\} \subseteq S$ by assumption. (4) s_i is a neighbor of s'_i by how s'_i was chosen and for $i \neq j$ s_i is not a neighbor of $s'_j \in C_j$ because s_i has no neighbor in C_j by how the C_j's where chosen. ∎

We are now ready to prove Lemma 5.9.

LEMMA 5.9. *Let S be a minimal separator of a k-creature-free graph G such that S cannot be dominated by $2k^4x$ vertices. Then there exists there exists an independent subset I of S of size x such that there exists two induced paths, P_L and P_R, in two different components of $G - S$ that dominate the vertices of I and no vertex of P_L nor P_R has more than one neighbor in I (see Figure 8).*

Proof. Assume that G is a k-creature-free graph, and let S be a minimal separator of G that cannot be dominated by $2k^4x$ vertices of G, and let L and R be two different S'-full components of G. It follows from Lemma 5.6 that there is a set of less than k induced paths in L that together dominated S and there is a set of less than k induced paths in R that together dominate S. So, since there are less than k^2 pairs of these induced paths with one from R and one from L, it follows there exists two induced paths P_L in L and P_R in R such that $(N(P_L) \cap S \cap N(P_R))$ cannot be dominated by $2k^2x$ vertices of G. Let $S' = (N(P_L) \cap S \cap (N(P_R))$. Number the vertices of P_R sequentially 1 through $|V(P_R)|$.

Assume that we have an independent set of vertices I_{i-1} of size $i-1$, $1 \leq i \leq x$, and a vertex set Z_{i-1} of size at most $2k^2(i-1)$, with the properties that no vertex of $S' - N[Z_{i-1}]$ is a neighbor of a vertex in I_{i-1}, and for all $v \in I_{i-1}$ if $w \in I_{i-1} \cup (S' - N[Z_{i-1}])$, $v \neq w$, then $N(v) \cap N(w) \cap (V(P_L) \cup V(P_R)) = \emptyset$, that is for all $v \in I_{i-1}$, no $w \in I_{i-1} \cup (S' - N[Z_{i-1}])$, $w \neq v$, shares a neighbor with v in P_L nor P_R. We will show how to produce a set I_i of size i and Z_i of size at most $2k^2i$ with the same properties, assuming $i \leq x$. Note that for the base case the empty set satisfies the conditions required of S_0 and Z_0.

Let $S'' = S' - N[Z_{i-1}]$, since $i \leq x$ and since S' cannot be dominated by $4k^2x$ vertices S'' must be non-empty. Label the vertices of S'' according to the lowest numbered neighbor it has in P_R. Let v be a highest labeled vertex in S''. Let w be the lowest numbered neighbor v has in P_R and assume w (and therefore v) is labeled with the number p. Let H denote the subpath of P_R that is made up of the vertices labeled 1 through $p-1$, hence H is anti-complete with v and H dominates all vertices of $S'' - N(w)$ (since v is a highest labeled vertex of S'' and has label p, all vertices of $S'' - N(w)$ must have a neighbor that has a label lower than p and hence in H).

We now wish to apply Lemma 5.7 using $(S'' - N(w), V(H), V(P_L), v)$. To see that this tuple satisfies the assumption of Lemma 5.7 note that H and P_L are paths and therefore connected, H is anti-complete with P_L since they are contained in two different S-full components, that H is anti-complete with v was noted at the end of the previous paragraph, and that H dominates $S'' - N(w)$ was also noted at the end of the previous paragraph. Hence, we may apply Lemma 5.7 using $(S'' - N(w), V(H), V(P_L), v)$ to get a set X of size at most k such that $N((S'' - N[w]) - N[X]) \cap N(v) \cap V(P_L) = \emptyset$ and no vertex of $(S'' - N[w]) - N[X]$ is a neighbor of v. This implies if we set $X' = X \cup \{w\}$ then $N(S'' - N[X']) \cap N(v) \cap V(P_L) = \emptyset$, no vertex of $S'' - N[X']$ is a neighbor of v, and $v \notin S'' - N[X']$.

We now wish to find a set Y of size less than k^2 such that no vertex of $S'' - (N[X'] \cup N[Y])$ shares a neighbor with v in either P_L or P_R. For ease of notation, set $S''' = S'' - N[X']$. It is tempting to try to use Lemma 5.7 on something like (S''', P_L, P_R, v), but this lemma requires that v not have any neighbors in P_L. Instead, we first use Lemma 5.8 on $(S''', V(H), V(P_L), v)$. To see that we can apply this lemma, note that both H and P_L are paths, H is anti-complete with P_L since they are contained in two different S-full components, that H is anti-complete with v was noted at in the last sentence two paragraphs ago as was the fact that H dominates $S''' \subseteq S'' - N(w)$ (recall $w \in X$), that v is anti-complete with $S''' = S'' - N[X']$ was noted at the end of the previous paragraph, and P_L dominates S''' because P_L dominates S' by assumption and $S''' \subseteq S'$.

So, we use Lemma 5.8 on $(S''', V(H), V(P_L), v)$ to get connected components $C_1, C_2, \ldots, C_c$, $c < k$, of $P_L - N(v)$ (hence v has no neighbors in each C_i) such that all vertices of S''' have a neighbor in at least one C_i. Now, for each C_i we apply Lemma 5.7 on $(S''' \cap N(C_i),\ C_i,\ P_R,\ v)$ to get a set Y_i of size at most k such that $N((S''' \cap V(C_i)) - N[Y_i]) \cap N(v) \cap V(P_R) = \emptyset$.

Since the C_i's dominate S''', it follows that if we set $Y = \bigcup Y_i$ then $N(S''' - N[Y]) \cap N(v) \cap V(P_R) = \emptyset$. Since $S''' = S'' - N[X']$ and $N(S'' - N[X']) \cap N(v) \cap V(P_L) = \emptyset$ it follows that if we set $Z_i = Z_{i-1} \cup X' \cup Y$ then no vertex $S' - N[Z_i]$ shares a neighbor with v in P_L nor P_R. We may set $I_i = I_{i-1} \cup \{v\}$ and $Z_i = Z_{i-1} \cup X' \cup Y$.

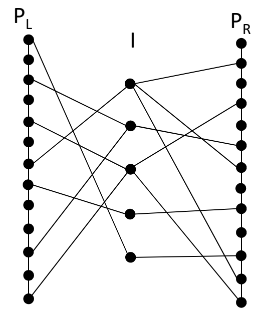

Figure 8: An example of a graph produced by Lemma 5.9

Since each Y_i has at most k vertices we have that $|Y| \leq k^2$ and $|X'| \leq k$ so $|Z_i| \leq |Z_{i-1}| + k + k^2 \leq 2k^2 i$ as required.

The statement of the lemma now follows from the fact that S cannot be dominated by $2k^2x$ vertices so this process can continue until we attain the set I_x, which has the property that for any pair $v, w \in I_x$ $v \neq w$ it holds that $N(v) \cap N(w) \cap (V(P_L) \cup V(P_R)) = \emptyset$. So I_x which is the desired set, along with the paths P_L and P_R.
∎

We now present three straightforward lemmas that will help us in the proof of Lemma 5.13. This first lemma is essentially a quick application of the Erdös-Szekeres Theorem. First we must give the following definition.

We call a graph G a *k-almost-skinny-ladder* if the following conditions hold:

- $V(G) = L \cup S \cup R$ with L, S, and R mutually disjoint and $|S| = k$.
- $G[L]$ and $G[R]$ form induced paths of G and L is anti-complete with R.
- Each $s \in S$ has at least one neighbor in L and at least one neighbor in R.
- For all pairs $x, y \in S$, if a, b are neighbors of x in L, then y has no neighbors on the subpath of $G[L]$ that has a and b as its endpoints. Similarly, if a, b are neighbors of x in R, then y has no neighbors on the subpath of $G[R]$ that has a and b as its endpoints.

The last condition of almost-skinny-ladders requires that no vertex of L or R has more than one neighbor in S. It is a straight forward application of the Erdös-Szekeres Theorem to show that a k-almost-skinny-ladder contains a k-skinny-ladder as an induced minor, as the next lemma shows.

Lemma 5.10. *Let G be a graph that contains a k^2-almost-skinny-ladder as an induced subgraph. Then G contains a k-skinny-ladder as an induced minor.*

Proof. Let G be a graph that has a k^2-almost-skinny-ladder, H, as an induced subgraph. $V(H) = L \cup S \cup R$ where L, S, R each have the same meaning as in the definition of an almost-skinny-ladder. Number the vertices of L sequentially 1 through $|V(L)|$, and similarly, number the vertices of R sequentially 1 through $|V(R)|$.

Next we label each vertex in S with a number 1 through $|S|$ such that for all $s_i, s_j \in S$ $i > j$ if and only if all of s_i's neighbors in L have a higher number than all of s_j's neighbors in L (by the definition of an almost-skinny-ladder such a numbering exists). Let $n(s_i)$ be the number of the highest numbered neighbor s_i has is R. We now apply the Erdös-Szekeres Theorem to the sequence $n(s_1), n(s_2) \ldots, n(s_{k^2})$ to get an increasing or decreasing subsequence of length at least k and set S^* to be the subset of S that corresponds to the subsequence obtained from our application of the Erdös-Szekeres Theorem. If the Erdös-Szekeres Theorem returned a decreasing subsequence then reverse the numbering of R, else leave it unchanged. Then for every $s_i, s_j \in S^*$, if $i > j$ then all of s_i's neighbors in L have a higher number than all of s_j's neighbors in L and all of s_i's neighbors in R have a higher number than all of s_j's neighbors in R. We can now apply the obvious edge contractions to L and R to form a k-skinny-ladder. ∎

LEMMA 5.11. *Let G be a graph, let $a, b \in G$ be two non adjacent vertices of G, and let $P_1, P_2, \dots, P_k$ be k mutually anti-complete induced paths. Assume for all P_i that both a and b have a neighbor in P_i and no vertex of P_i is a neighbor of both a and b. Then G contains a k-creature.*

Proof. Let G, a, b, $P_1, P_2, \dots, P_k$ be as in the statement of the lemma. For each P_i we can then, by assumption, find a subpath of P_i, call it P_i^*, such that P_i^* has endpoints a_i, b_i where a_i is a neighbor of a, b_i is a neighbor of b, no internal vertex is a neighbor of a or b. Since a and b do not share any neighbors in P_i we can see that P_i^* has at least 2 vertices and since the P_i^*'s are anti-complete by assumption, together the P_i^*'s along with a and b make a k-creature. □

LEMMA 5.12. *Let G be a directed graph with maximum out-degree or maximum in-degree at most c, $c > 0$. Then G has an independent set (no vertex is an in-neighbor or out-neighbor of any other vertex in this set) of size at least $\frac{|V(G)|}{2c+1}$. Furthermore, if $|V(G)| \geq 2t$ and the maximum out-degree or maximum in-degree of G is at most $\frac{1}{4t}|V(G)|$, then G has an independent set of size at least t.*

Proof. Let G be a directed graph. We will prove the statements for bounded maximum out-degree (for maximum in-degree the proof is nearly identical). If the maximum out-degree of G is c, $c > 0$, then as long as G has at least one vertex, there must exists a vertex $v \in G$ with in-degree at most c. If we let G' be the subgraph induced by all vertices of $G - v$ that do not have v as an in-neighbor or an out-neighbor, then the size of G' is at least $|V(G)| - 2c - 1$, and G' has maximum out-degree c. It follows by an inductive argument that we can find an independent set of size at least $\frac{|V(G)|}{2c+1}$.

To prove the furthermore statement, assume the maximum out-degree of G is at most $\frac{1}{4t}|V(G)|$, $t > 0$, and $|V(G)| \geq 2t$, so we have that $\frac{|V(G)|}{2t} + 1 \leq \frac{|V(G)|}{t}$. From the first paragraph we have that G contains an independent set of size at least $\frac{|V(G)|}{\frac{2|V(G)|}{4t}+1} = \frac{|V(G)|}{\frac{|V(G)|}{2t}+1} \geq \frac{|V(G)|}{\frac{|V(G)|}{t}} = t$. □

We are now in a position to prove Lemma 5.13, which states that if our graph G is k-creature-free and has a minimal separator that cannot be dominated by few vertices, then G contains a k-skinny-ladder as an induced minor. How do we show this? By Lemma 5.9 we know that if our graph G is k-creature-free and has a minimal separator that cannot be dominated by few vertices, then we can find a large independent set I and paths P_L and P_R such that all vertices of I have neighbors in P_L and P_R and no vertex in P_L and P_R has over one neighbor in I. This is the starting point for the proof of Lemma 5.13

Heuristically, the idea of the proof is that we set $I_1 = I, L_1 = P_L$, and $R_1 = P_R$ and we can either find a large subset $A \subset I_1$ that satisfies certain properties, in which case we use Lemmas 5.12 and 5.10 to show that L_1, R_1 and A contain a k-skinny-ladder as an induced minor, or there is a large subset $I_2 \subset I_1$ such there is a way to divide either L_1 or R_2 into two "halves" such that both halves dominate I_2, for simplicity let us say we can do this with L_1. We then set P_1 to be one half of L_1, we set L_2 to be the other half, and we set $R_2 = R_1$. We now repeat this process with L_2, R_2 and I_2 and so on. In the end we either end up with our desired k-skinny-ladder, or we end up with k anti-complete induced paths all of which dominate some independent set I_k, and no vertices in I_k have a common neighbor in any of these anti-complete paths. But Lemma 5.11 shows this implies the existence of a k-creature in G which is a contradiction, so we must be in the case where this process produces a k-skinny-ladder as an induced minor.

LEMMA 5.13. *Let S be a minimal separator of a k-creature-free graph G such that S cannot be dominated by $2k^4[(8k^2)^{k+1}]$ vertices. Then G contains a k-skinny-ladder as an induced minor.*

Proof. Assume that G is k-creature-free and S is a minimal separator of G such that S cannot be dominated by $2k^4[(8k^2)^{k+1}]$ vertices. It follows from Lemma 5.9 that there is an independent set $I \subseteq S$ of $(8k^2)^{k+1}$ vertices and two induced paths P_L and P_R that dominate I, P_L anti-complete with P_R, and every vertex in $v \in V(P_L) \cup V(P_R)$ has at most one neighbor in I.

Number the vertices of P_L sequentially 1 through $|V(P_L)|$ and number the vertices of P_R sequentially 1 through $|V(P_R)|$. For a vertex x in P_L or P_R we will use the notation $n(x)$ to denote the number it has been given in P_L or P_R. For every $v \in I$ let $\ell(v)$ and $r(v)$ denote the highest numbered vertex v is a neighbor of in P_L and P_R respectively. We now set $L_1 = P_L, R_1 = P_R$, and $I_1 = I$. We will consider the following process to produce a k^2-almost-skinny-ladder. We will show this process cannot go past k iterations do to the fact that G is k-creature-free.

We will ensure the following properties are met at the end of the i^{th} step (and we will assume that these properties hold for the previous steps). At the i^{th} step we produce L_{i+1} which is a subpath of L_i, R_{i+1} which is a subpath of R_i, $I_{i+1} \subseteq I_i$, $|I_{i+1}| \geq (8k^2)^{k-i+1}$, and for every $v \in I_{i+1}$ it holds that $\ell(v) \in L_{i+1}$ and $r(v) \in R_{i+1}$. We will also produce P_i which will be either a subpath of L_i or R_i and is anti-complete with L_{i+1}, anti-complete with R_{i+1}, and anti-complete with P_j for $j < i$, and P_i will dominate I_j if $i < j$ (note by Lemma 5.11 that if we have k such paths then G would have a k-creature, so this process cannot go past k steps).

At the i^{th} step, $1 \leq i \leq k$, we do as follows. Create an auxiliary directed graph, A_i, whose vertex set is I_i and there is an edge from $v \in I_i$ to $w \in I_i$ if at least one of the following two cases hold

1. $n(\ell(v)) > n(\ell(w))$ and v has a neighbor x in L_i such that $n(x) < n(\ell(w))$
2. $n(r(v)) > n(r(w))$ and v has a neighbor x in R_i such that $n(x) < n(r(w))$

If the maximum in-degree of A_i is at most $\frac{1}{4k^2}|I_i|$ then we stop. Since by assumption $|I_i| \geq (8k^2)^{k-i+2}$ this gives an independent set of size at least k^2 by Lemma 5.12. If there is an $s_t \in I_i$ with in-degree over $\frac{1}{4k^2}|I_i|$ then either case 1 or case 2 is satisfied for at least half of s_t's in-neighbors. This means that for at least $\frac{1}{8k^2}$ fraction of the vertices of I_i, call this subset of vertices I_{i+1}, all vertices $s \in I_{i+1}$ must satisfy case 1 with s playing the role of v and s_t playing the role of w, or all vertices $s \in I_{i+1}$ must satisfy case 2 again with s playing the role of v and s_t playing the role of w. For both case 1 and case 2 we now describe what to do if all the vertices of I_{i+1} satisfy that case (if all vertices of I_{i+1} happen to satisfy both cases, then we go with case 1). Each number here corresponds with what to do in that case.

1. In case 1, set P_i to the subpath of L_i that is made up of vertices numbered less than $n(\ell(s_t))$, set $R_{i+1} = R_i$, and set L_{i+1} to be the vertices of L_i numbered greater than $n(\ell(s_t))$.
2. In case 2, set P_i to the subpath of R_i that is made up of vertices numbered less than $n(r(s_t))$, set $L_{i+1} = L_i$, and set R_{i+1} to be the vertices of R_i numbered greater than $n(r(s_t))$.

This concludes the i^{th} step. We now show that in case 1, all properties required of L_{i+1}, R_{i+1}, I_{i+1}, and P_i are met (the argument is identical when we are in case 2). It is straight forward to see that L_{i+1} is a subpath of L_i, R_{i+1} is a subpath of R_i, and $I_{i+1} \subseteq I_i$. Since I_i was assumed to have size at least $(8k^2)^{k-i+2}$ and $|I_{i+1}| \geq \frac{1}{8k}|I_i|$, it holds that $|I_{i+1}| \geq (8k^2)^{k-i+1}$. Since L_{i+1} is made up of vertices of L_i numbered greater than $n(\ell(s_t))$ and for every vertex $s \in I_{i+1}$ it holds that $n(\ell(s)) \geq n(\ell(s_t))$ it follows that $\ell(s) \in L_{i+1}$. Also, since $R_{i+1} = R_i$ it follows that $r(s) \in R_i$. Lastly, we can see that P_i is a subpath of L_i and is anti-complete with L_{i+1} and R_{i+1}. By definition every vertex $s \in I_{i+1}$ must have a neighbor $x \in L_i$ such that $n(x) \leq n(\ell(s_t))$ and therefore $x \in P_i$ so P_i dominates I_{i+1}.

Now we observe that for $a < b$ that P_a dominates I_b. This follows from the fact that P_a dominated I_{a+1} and that $I_b \subseteq I_{a+1}$ (since $I_a \subseteq I_{a+1} \subseteq \ldots I_b$). Also observe that P_a is anti-complete with P_b since P_a is anti-complete with L_{a+1} and R_{a+1} and therefore L_b and R_b, which P_b is a subpath of. Hence the P_a's are pairwise anti-complete. Lastly, observe that if $x, y \in I_{i+1}$ then $x, y \in I$ which means x and y share no neighbors in P_L and P_R. We can now see that the conditions of Lemma 5.11 are satisfied, therefore this process cannot go past the k^{th} step without producing a k-creature.

We conclude there is some step $j \leq k$ such that the auxiliary graph A_j has max in-degree less than $\frac{1}{4k^2}|I_j|$, and since $|I_j| \geq (8k^2)^{k-i+2} \geq 8k^2$ it therefore has an independent set of size k^2 by Lemma 5.12. Let I^* denote such an independent set, we claim that $G[V(L_j) \cup I^* \cup V(R_j)]$ makes an k^2-almost-skinny-ladder. Let $x, y \in I^*$ and let a, b be the highest and lowest numbered neighbors of x in L_j respectively, and assume that y has a neighbor c on the induced path of L_j that has a and b as its endpoints. If y's highest numbered neighbor in L_j is greater than $n(a)$ then y has an edge to x in A_j by case 1. If y's highest numbered neighbor is L_j is less than $n(a)$, then x has an edge to y again by case 1. Both cases yield a contradiction to I^* being an independent set in A_j. A symmetric argument show that if a', b' are x's highest and lowest numbered neighbors R_j respectively, then y cannot have a neighbor in the induced subpath of R that has a', b' as its endpoints. It follows that $G[V(L_j) \cup I^* \cup V(R_j)]$ is a k^2-almost-skinny-ladder. Applying Lemma 5.10 shows that G contains a k-skinny-ladder as an induced minor.

□

The following lemma uses a branching algorithm to produce all of the minimal separators of a graph G and proves a bound on the number of minimal separators produced by this algorithm. See Claim 2.3 of Section 2 and the discussion following the claim for a high level description of Lemma 5.14.

LEMMA 5.14. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a graph and let k and c be integers such that for all induced subgraphs G' of G and for all $v \in G'$, if $S^v_{G'} = \{N(v) \cap S : v \notin S$ and S is a minimal separator of $G'\}$, then $|S^v_{G'}| \leq c$ and every minimal separator of any induced subgraph of G can be dominated by k vertices. Then G has a most $(c + n^k)^{f(k)\log(n)}$ minimal separators where $n = |V(G)|$.*

Proof. Let G, S^v_G, k, c, and n be as in the statement of this lemma. The proof of the bound makes use of a branching algorithm. The algorithm takes as input G and $X \subseteq V(G)$ and the algorithm will use the set K_{ret} to store the vertex sets it will return. It will return K_{ret} which will contain all minimal separators of G contained in X (possibly along with other vertex sets which are not minimal separators). We have no concern about the runtime of the algorithm, but we care about the size of the final set it returns. The algorithm is intended to be used initially on the input $(G, V(G))$.

Assume the the input to the algorithm is (G, X). If X is empty, then the algorithm returns $\{\emptyset\}$ (if G is disconnected then $\emptyset$ is a minimal separator of G). Else, the algorithm determines the set $Q \subseteq V(G)$ where Q contains all vertices $v \in G$ such that $|N[v] \cap X| \geq \frac{1}{2k}|X|$. The algorithm then initializes K_{ret} to $\emptyset$ then branches in the following two ways:

1. For every $q \in Q$ and every $Y \in S^q_G$ the algorithm recursively calls itself on $(G-Y,\, X-N_G[q])$. The recursive call $(G-Y,\, X-N_G[q])$ returns the collection K' of vertex sets and for each set S in K', the algorithm adds the set $S \cup Y$ to K_{ret}.

2. For every set R of k vertices of G such that $R \cap Q = \emptyset$, the algorithm recursively calls itself on $(G - Q,$ $(X - Q) \cap N_G(R))$. The recursive call $(G - Q,\, (X - Q) \cap N_G(R))$ returns a collection K' of vertex sets. Then for each set, S in K' the algorithm adds the set $S \cup Q$ to K_{ret}.

After completing this, the algorithm then returns the set K_{ret}.

This algorithm will terminate since each recursive call is on input $(G',\, X')$ where the size of X' is strictly less than X. Note that since the set R has no vertex in Q and $|R| = k$, the neighborhood of R contains at most $\frac{1}{2}$ of the vertices of X, so in (2) each recursive call made is on input $(G',\, X')$ where $|X| \geq \frac{1}{2}|X'|$, additionally note by how the vertices of Q were chosen, in (1) each recursive call is made on input $(G',\, X')$ where is made on $|X| \geq \frac{1}{2k}|X'|$.

We now show that if this algorithm is called on an instance (G, X) the set returned from this algorithm contains all minimal separators of G contained in X. Let S be a minimal separator of G contained in X. Assume all of the recursive calls $(G',\, X')$ the algorithm makes returns a set that contains all minimal separators of G' contained in X', possibly along with additional vertex sets (note that the base case for when $X = \emptyset$ is handled by returning $\{\emptyset\}$). If $Y = N_G(q) \cap S$ for some $q \in G$ and $q \notin S$, then $S - Y$ is a minimal separator of $G - Y$ that is contained in $X - N_G[q]$. So if there is a $q \in Q$ such that $q \notin S$, then S gets added to K_{ret} in (1). If $Q \subseteq S$, then $S - Q$ is a minimal separator of $G - Q$, and by assumption there exists some collection of at most k vertices, R, in $G - Q$ such that $S - Q \subseteq N_{G-Q}(R)$ and therefore $S - Q \subseteq (X - Q) \cap N_G(R)$. It follows that in this case we also have S gets added to K_{ret} in (2). Induction on the the depth of the recursive call now shows that this algorithm returns all minimal separators.

Let $T(n, x)$ denote the maximum number of minimal separators that a vertex set X of size at most x can contains for any graph G with $|V(G)| \leq n$ and $X \subset V(G)$, such that the graph G satisfies the conditions of the lemma. The algorithm just shown makes at most cn recursive calls in (1) and n^k recursive calls in (2), each on an instance (G', X') where $|X| \geq \frac{1}{2k}|X'|$. Hence, $T(n, x) \leq (cn + n^k)T(n, [1 - \frac{1}{2k}]x)$. Using the fact that $(1 - \frac{1}{y})^y \leq \frac{1}{e} < \frac{1}{2}$ we expand the inequality $T(n, x) \leq (cn + n^k)T(n, [1 - \frac{1}{2k}]x)$ out $2k$ times to get $T(n, x) \leq (cn+n^k)^{2k}T(n, \frac{1}{2}x)$, then expanding this inequality $\log(x)$ times gives $T(n, x) \leq (cn+n^k)^{2k\log(x)}T(n, 1)$. Since $T(n, 1) = 2$ it follows that $T(n, x) \leq 2(cn + n^k)^{2k\log(x)}$. By taking the initial set X to be $V(G)$, it follows that G then contains at most $2(cn + n^k)^{2k\log(n)}$ minimal separator. ∎

We are now ready to prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] Let G be a graph that is k-creature-free and has no k-skinny-ladder as an induced minor and let $n = |V(G)|$. For every induced subgraph G' of G and for every $v \in G'$, let $S^v_{G'} = \{N(v) \cap S : v \notin S$ and S is a minimal separator of $G'\}$. Then $|S^v_{G'}| \leq n^k$ for by Lemma 5.5. By Lemma 5.13, since G is k-creature-free and has no k-skinny-ladder as an induced minor every minimal separator of any induced subgraph of G' is dominated by $k' = 2k^4\lceil(8k^2)^{k+1}\rceil$ vertices. Lemma 5.14 then implies that G has at most $2(n^{k+1} + n^{k'})^{2k' \log(n)}$ minimal separators. It follows that the family of graphs that are k-creature-free and do not contain a k-skinny-ladder as an induced minor are strongly-quasi-tame. ∎

6 Finite Forbidden Induced Subgraphs

In this section we will provide the lemmas needed in the proofs of Theorems 1.2 and 1.3 as well give a proof of these theorems. The majority of the work of this section goes into proving that given an integer k, if G contains a k'-creature for large enough k', then G must contain a k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, or k-ladder as an induced subgraph.

We will require a number of new graph definitions for this section. The following graphs, except for k-ladder graphs, appear in the statement of Theorem 1.3. Figure 2 depicts these graphs. It can be seen that all graphs here except for k-claw and k-paw graphs contains at least 2^{k-2} minimal separators.

- A graph G is a *k-theta* if G consist of two vertices a, b and k induced paths $P_1, P_2, \ldots P_k$. For $1 \leq i \leq k$ the end points of P_i are a and b, every P_i is anti-complete with P_j, and every P_i has length at least 4.

- A graph G is a *k-prism* if G consist of two disjoint cliques $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ along with k induced paths $P_1, P_2, \ldots P_k$ each of length at least 2. For $1 \leq i \leq k$ the end points of P_i are a_i and b_i, every $P_i - \{a_i, b_i\}$ is anti complete with P_j, and a_i is neighbors with b_j if and only if $i = j$ and P_i is a path of length 2.

- A graph G is a *k-pyramid* if G consist of a vertex a and a clique $b_1, b_2, \ldots, b_k$, where a is anti-complete with $b_1, b_2, \ldots, b_k$, along with k induced paths $P_1, P_2, \ldots P_k$ each of length at least 3. For $1 \leq i \leq k$ the end points of P_i are a and b_i and every $P_i - \{a, b_i\}$ is anti complete with $P_j - \{a\}$.

- A graph G is a *k-ladder* if G consists of two anti-complete paths L and R with the vertices of L and R are numbered sequentially from 1 to $|V(L)|$ and 1 to $|V(R)|$, along with k anti-complete induced paths $P_1, P_2, \ldots P_k$ that are also disjoint from L and R, each of length at least 2. For $1 \leq i \leq k$ the end points of P_i are a_i and b_i. Every a_i has at least one neighbor in L, every b_i has at least one neighbor in R. Furthermore if $i > j$, then every neighbor of a_i in L has a higher number then every neighbor of b_j in L and every neighbor of b_i in R has a higher number then every neighbor of b_j in R.

- A graph G is a *k-contracted-ladder* if can be obtained from a k-ladder by contracting each of the the paths $P_1, P_2, \ldots P_k$ into single vertices.

- A graph G is a *k-ladder-theta* if G consists of an induced path L and a vertex b anti-complete with L, along with k induced paths $P_1, P_2, \ldots P_k$ that are also disjoint from L, each of length at least 3. For $1 \leq i \leq k$ the end points of P_i are a_i and b, every $P_i - \{b\}$ is anti-complete with $P_j - \{b\}$, $P_i - \{a_i\}$ is anti-complete with L, every a_i has at least one neighbor in L, and if x, y are neighbors with a_i in L, then no a_j with $i \neq j$ has a neighbor in the induced subpath of L that has x and y as its endpoints.

- A graph G is a *k-ladder-prism* if G consists of an induced path L and clique $b_1, b_2, \ldots, b_k$ where L is anti-complete with $b_1, b_2, \ldots, b_k$, along with k induced paths $P_1, P_2, \ldots P_k$ that are also disjoint from L, each of length at least 2. For $1 \leq i \leq k$ the end points of P_i are a_i and b_i, every $P_i - \{b_i\}$ is anti-complete with P_j, $P_i - \{a_i\}$ is anti-complete with L, every a_i has at least one neighbor in L, and if x, y are neighbors with a_i in L, then no a_j with $i \neq j$ has a neighbor in the induced subpath of L that has x and y as its endpoints.

- A graph G is a *k-claw* if G consists of k anti-complete copies of the following graph which we call a *long-claw of arm length k*: let v be a vertex and P_1, P_2, P_3 be three paths of length k each with v as one of its endpoints and $P_i - \{v\}$ is anti-complete with $P_j - \{v\}$ (i.e., the graph is a claw with each edge subdivide $k - 2$ times)

- A graph G is a *k-paw* if G consists of k anti-complete copies of the following graph which we call a *long-paw of arm length k*: let v_1, v_2, v_3 be a triangle and P_1, P_2, P_3 be three disjoint induced paths of length k each such that P_i has v_i as one of its endpoints and $P_i - \{v_i\}$ is anti-complete with P_j for $1 \leq i \neq j \leq 3$.

It will be useful in this section to define the following graphs as well. These graphs are depicted in Figure 9.

- A graph G is a *k-half-theta* if G consists of a vertex v and k induced paths P_1, $P_2, \ldots, P_k$ of G such that each path has length at least 2, for $1 \leq i \leq k$ it holds that v is one endpoint of P_i, and for $j \neq i$ it hold that $P_i - v$ is anti-complete with $P_j - v$. Let x_i denote the endpoint of P_i that is not v. Then we say the vertices $x_1, x_2, \ldots, x_k$ are the endpoints of the k-half-theta. If X is a vertex set and $x_i \in X$ for all i with $1 \leq i \leq k$, then we say G is a k-half-theta ending in X.

- A graph G is a *k-half-prism* if G consists of a clique of vertices $v_1, v_2, \ldots, v_k$ and k induced paths P_1, P_2, $\ldots$, P_k of G such that each path has length at least 1, for $1 \leq i \leq k$ it holds that v_i is one endpoint of P_i, and for $j \neq i$ it hold that $P_i - v_i$ is anti-complete with P_j. If the length of P_i is greater than 1 then let x_i denote the endpoint of P_i that is not v_i, and if the length of P_i is 1 then let $x_i = v_i$. We say the vertices $x_1, x_2, \ldots, x_k$ are the endpoints of the k-half-prism. If X is a vertex set and $x_i \in X$ for all i with $1 \leq i \leq k$, then we say G is a k-half-prism ending in X.

- A graph G is a *k-half-ladder* if G consists of a path P of G along with k additional paths $P_1, P_2, \ldots, P_k$ of G such that each path has length at least 1. For $1 \leq i \leq k$ let P_i's endpoints be v_i and x_i (with v_i possibly equal to x_i). We call P the backbone path and the P_i's the auxiliary paths. We require that v_i has at least one neighbor in P, P is anti-complete with $P_i - v_i$, and for $j \neq i$ P_i is anti-complete with P_j. Lastly, we also require that if a and b are two neighbors of some v_i in P, then there is no v_j, $i \neq j$ such that v_j has a neighbor in the induced subpath of P with endpoint a and b. We say the vertices $x_1, x_2, \ldots, x_k$ are the endpoints of the k-half-ladder. If X is a vertex set and $x_i \in X$ for all i with $1 \leq i \leq k$, then we say G is a k-half-ladder ending in X.

- A graph G is a *k-half-quasi-ladder* if G consists of a path P of G along with k additional paths $P_1, P_2, \ldots, P_k$ of G such that each path has length at least 1. For $1 \leq i \leq k$ let P_i's endpoints be v_i and x_i (with v_i possibly equal to x_i). We call P the backbone path and the P_i's the auxiliary paths. We require that v_i has at least one neighbor in P, P is anti-complete with $P_i - v_i$, and for $j \neq i$ P_i is anti-complete with P_j. We say the vertices $x_1, x_2, \ldots, x_k$ are the endpoints of the k-half-quasi-ladder. If X is a vertex set and $x_i \in X$ for all i with $1 \leq i \leq k$, then we say G is a k-half-ladder ending in X. Note that a k-half-quasi-ladder is almost the same as a k-half-ladder, but we drop the requirement that if a and b are two neighbors of some v_i in P, then there is no v_j, $i \neq j$ such that v_j has a neighbor in the subpath of P with endpoint a and b.

The following lemmas, culminating with Lemma 6.9, work towards proving that given an integer k, if G contains a k'-creature for large enough k', then G must contain an induced k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, or k-ladder.

Our first goal is to show that if we have a k' creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$) for large enough k', then if we focus in on one half, say that half with A and $\{x_1, x_2, \ldots, x_{k'}\}$, then we can find a k-half-theta, k-half-prism, or k-half-quasi-ladder that that ends in $\{x_1, x_2, \ldots, x_{k'}\}$. This goal is accomplished with Lemmas 6.1 through 6.4. Since if we have a k-clique in the set $\{x_1, x_2, \ldots, x_{k'}\}$ then we have a k-half-prism ending in $\{x_1, x_2, \ldots, x_{k'}\}$ we may assume, by Ramsey's Theorem, that $\{x_1, x_2, \ldots, x_{k'}\}$ is an independent set.

LEMMA 6.1. *Let G be a graph that contains a k-creature (A, B, $\{x_1, x_2, \ldots, x_k\}$, $\{y_1, y_2, \ldots, y_k\}$) where $\{x_1, x_2, \ldots, x_k\}$ is an independent set of G. Let A' be a minimally connected induced subgraph of $G[A]$ such that $\{x_1, x_2, \ldots, x_k\} \subset N(A')$. If A' contains a vertex with degree at least $R(d, d)$ in A', then $G[A \cup \{x_1, x_2, \ldots, x_k\}]$ contains a d-half theta or a d-half-prism ending in $\{x_1, x_2, \ldots, x_k\}$.*

Proof. Let G be a graph that contains a k-creature (A, B, $\{x_1, x_2, \ldots, x_k\}$, $\{y_1, y_2, \ldots, y_k\}$). Let A' be a minimally connected induced subgraph of $G[A]$ such that $\{x_1, x_2, \ldots, x_k\} \subset N_G(A')$. Assume $v \in A'$ has degree at least $\mathrm{R}(d, d)$ in A'. Let $v_1, v_2, \ldots, v_{R(d,d)}$ be distinct neighbors of v in A'. By the minimality of A', for each v_i there must be a vertex x_{v_i} such that every path starting from v and ending at x_{v_i} with internal vertices contained A'

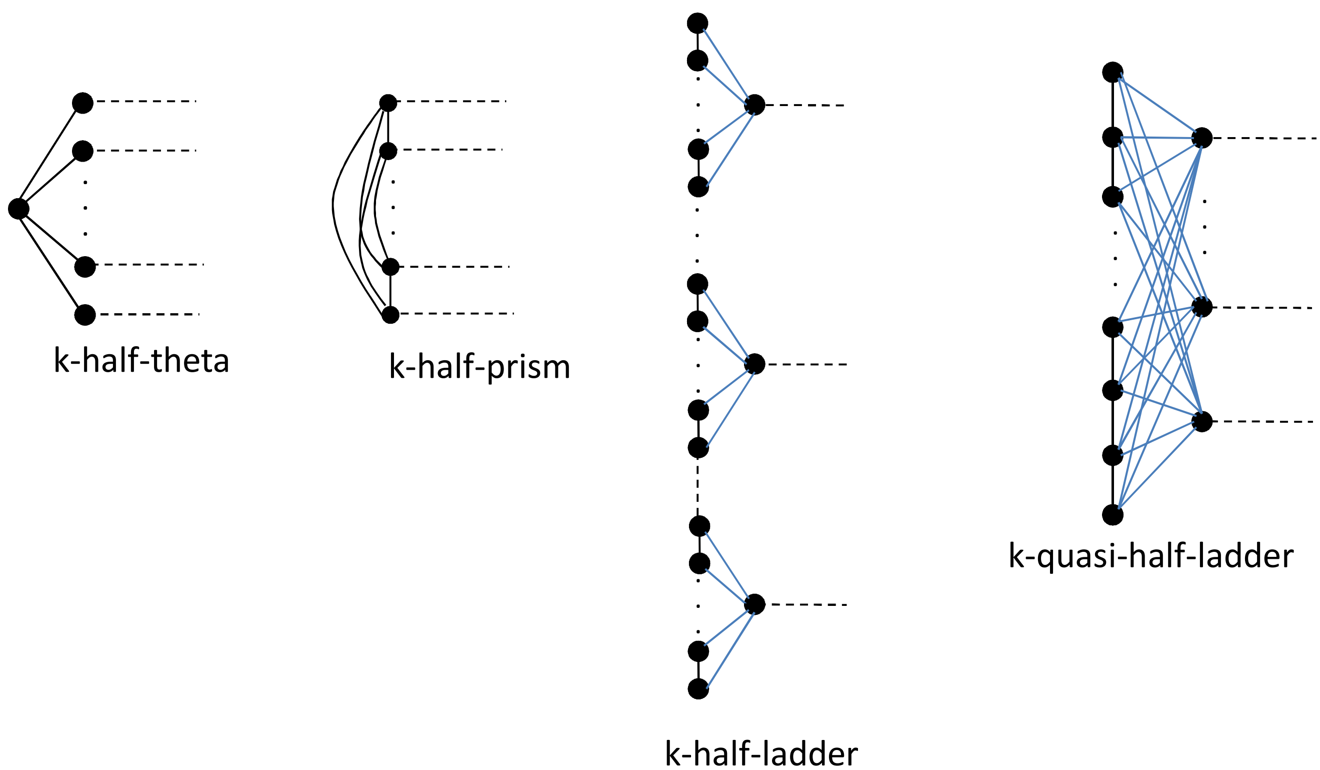

Figure 9: Dashed lines represent the option of having an arbitrary length path (possibly of length 0). The blue lines used in the k-half-ladder and k-almost-half-ladder graphs represents the option of either having or not having that edge, but for each vertex not on the backbone path that is adjacent at least one blue edges, at least one of those blue edges must belong to the graph.

must contain v_i, since if this does not happen for some given v_i then the connected component of $A' - v_i$ that contains v would be a proper induced subgraph of A' that is connected and whose open neighborhood contains $\{x_1, x_2, \dots, x_k\}$. It follows there must exist induced paths $P_1, P_2, \dots, P_{R(d,d)}$ such that $v_i \in P_i$, P_i's endpoints are v_i and x_{v_i}, and $P_i - v_i$ is anti-complete with P_j. We then apply Ramsey's Theorem to the v_i's get a subset of size d of the P_i's that along with v form a d-half theta that ends in $\{x_1, x_2, \dots, x_k\}$ (if Ramsey's Theorem provides an independent set of size d) or a subset of size d of the P_i's that form a d-half prism that ends in $\{x_1, x_2, \dots, x_k\}$ (if Ramsey's Theorem provides a clique of size d) and the result now follows. □

LEMMA 6.2. *Let G be connected graph with maximum degree d and contains at least d^k vertices with degree greater than 2. Then there exists an induced path of G that contains at least k vertices of degree greater than 2.*

Proof. Let G be a connected graph with maximum degree d and contains at least d^k vertices with degree greater than 2. Let T be a breadth first search tree of G rooted at some vertex $v \in G$. We create the desired path as follows. Let v_1 be the first descendent of v in T that has degree greater than 2 in G (v_1 could be v). We begin our path at v_1. We will grow the path $P_i = \{x_1, x_2, \dots x_m\}$ where $x_1 = v_1$, x_j is the parent of x_{j+1} in T, P_i contains at least i vertices of G with degree greater than 2 in G, and the subtree of T rooted at x_m contains at least d^{k-i+1} vertices of G with degree greater than 2 in G.

Assume that we have such a path $P_i = \{x_1, x_2, \dots x_m\}$, $i < k$ (the vertex v_1 satisfies the conditions of P_1). We will show how to attain P_{i+1}. Since the maximum degree in G is d, x_m has at most d children in T, and by assumption the subtree of T rooted at x_m has at least d^{k-i+1} vertices of degree greater than 2 in G, it follows that for at least one child, call it x_{m+1}, the subtree rooted at x_{m+1} has at least d^{k-i} vertices of G with degree greater than 2 in G. Now let v_{i+1} be the first descendant of x_{m+1} with degree different from 2 in G (v_{i+1} could be x_{m+1}) and let P_{i+1} be the path P_i along with the induced path in T from x_{m+1} to v_{i+1}. It follows P_{i+1} satisfies the required conditions.

Hence we can produce a P_k that satisfies the conditions stated before, and we can then see that P_k is an induced path in G with at least k vertices of degree greater than 2. □

LEMMA 6.3. *Let G be a graph that contains a k-creature (A, B, $\{x_1, x_2, \dots, x_k\}$, $\{y_1, y_2, \dots, y_k\}$) where $\{x_1, x_2, \dots, x_k\}$ is an independent set. Let A' be a minimally connected subgraph of $G[A]$ such that $\{x_1, x_2, \dots, x_k\} \subset N(A')$. If A' contains an induced path, P, with at least $R(d, d)$ vertices of degree greater than 2 in A', then there is a d-half-quasi-ladder or a d-half-prism in $G[A \cup \{x_1, x_2, \dots, x_k\}]$ that ends in $\{x_1, x_2, \dots, x_k\}$.*

Proof. Let G, A', $\{x_1, x_2, \dots, x_k\}$, and P be as in the statement of the lemma, let $v_1, v_2, \dots, v_{R(d,d)}$ be vertices of P that have degree greater than 2 in A', and for each v_i let v_i' be a neighbor of v_i in A' that is not in P. By the minimality of A', for each v_i' there must exist a vertex x_{v_i} such that every path from v_i to x_{v_i} with internal vertices contains in A' must contain v_i', since if this does not happen for some given v_i' then the component of $A' - v_i'$ that contains v_i would be a proper induced subgraph of A' that is connected and whose open neighborhood contains $\{x_1, x_2, \dots, x_k\}$. It follows there must exists induced paths $P_1, P_2, \dots, P_{R(d,d)}$ disjoint from P with internal vertices contained in A', P_i's endpoints are v_i' and x_{v_i}, and $P_i - v_i'$ is anti-complete with P_j. We then apply Ramsey's Theorem to the v_i''s to get a subset of size d of the P_i's along with P that form a d-half-quasi-ladder that ends in $\{x_1, x_2, \dots, x_k\}$ (if Ramsey's Theorem provides an independent set of size d) or a subset of size d of the P_i's that yield a d-half-prism that ends in $\{x_1, x_2, \dots, x_k\}$ (if Ramsey's Theorem provides a clique of size d). □

LEMMA 6.4. *Let G be a graph that contains a $k \cdot (d^{c+1} + d)$-creature (A, B, $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}$, $\{y_1, y_2, \dots, y_{k\cdot(d^{c+1}+d)}\}$). Let A' be a minimally connected subgraph of $G[A]$ such that $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\} \subset N(A')$. Assume the max degree in A' is d and that A' contains less than d^c vertices of degree greater than 2 in A'. Then $G[A \cup \{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}]$ contains a k-half-quasi-ladder ending in $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}$.*

Proof. Let G, A' and $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}$ be as in the statement of the lemma. Let T be a breadth first search tree of A' rooted at some vertex v. Then T is a tree in which every vertex except for the root can have at most $d - 1$ children, hence there are at most $d^c + 1$ vertices that have more than one descendent, and the maximum number of descendants any vertex from this set can have is d. It follows that there are at most $d^{c+1} + d$ leaves of T, and therefore A' is the union of at most $d^{c+1} + d$ induced paths in A'. Hence, there exists some induced path P in A' such that P's open neighborhood contains at least k vertices in $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}$, which gives us a k-half-quasi-ladder ending in $\{x_1, x_2, \dots, x_{k\cdot(d^{c+1}+d)}\}$. □

LEMMA 6.5. *Let $k' = k \cdot R(k,k)^{R(k,k)+1} + R(k,k)$, and let G be a graph that contains an $R(k',k')$-creature (A, B, $\{x_1, x_2, \ldots, x_{R(k',k')}\}$, $\{y_1, y_2, \ldots, y_{R(k',k')}\}$). Then $G[A \cup \{x_1, x_2, \ldots, x_{R(k',k')}\}]$ contains an induced k-half-theta, k-half-prism, or a k-half-quasi-ladder, ending in $\{x_1, x_2, \ldots, x_{R(k',k')}\}$.*

Proof. Let $k' = k \cdot R(k,k)^{R(k,k)+1} + R(k,k)$. Assume that G contains a $R(k',k')$-creature (A, B, $\{x_1, x_2, \ldots, x_{R(k',k')}\}$, $\{y_1, y_2, \ldots, y_{R(k',k')}\}$). Apply Ramsey's Theorem to $\{x_1, x_2, \ldots, x_{R(k',k')}\}$. If Ramsey's Theorem returns a clique of size k' or more then we have that $G[A \cup \{x_1, x_2, \ldots, x_{R(k',k')}\}]$ contains a k-half-prism ending in $\{x_1, x_2, \ldots, x_{R(k',k')}\}$, so we can assume that Ramseys theorem returns an independent set of size at least k'. By relabeling the x_i's and y_i's if follows that G contains a k'-creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$) where $\{x_1, x_2, \ldots, x_{k'}\}$ is an independent set.

Let A' be a minimally connected induced subgraph of $G[A]$ such that $\{x_1, x_2, \ldots, x_{k'}\} \subset N(A')$. If A' contains a vertex of degree $\mathrm{R}(k,k)$ in A', then by Lemma 6.1 $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a k-half-theta ending in $\{x_1, x_2, \ldots, x_{k'}\}$. So we may assume max degree of A' is $\mathrm{R}(k,k)$.

If A' contains $\mathrm{R}(k,k)^{R(k,k)}$ vertices of degree greater than two, then there is an induced path of A' that contains $\mathrm{R}(k,k)$ vertices of degree greater than two by Lemma 6.2. Then by Lemma 6.3 $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a k-half-quasi-ladder or a k-half-prism ending in $\{x_1, x_2, \ldots, x_{k'}\}$. So we may assume that A' has maximum degree $\mathrm{R}(k,k)$ and contains fewer than $\mathrm{R}(k,k)^{R(k,k)}$ vertices of degree greater than two. It then follows from Lemma 6.4 that $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a k-half-quasi-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$. ◻

If in the statement of Lemma 6.5 we would replace k-half-quasi-ladder with k-half-ladder, then we would basically be done. If we had a k''-creature (A, B, $\{x_1, x_2, \ldots, x_{k''}\}$, $\{y_1, y_2, \ldots, y_{k''}\}$) for large enough k'' then we could find a k'-half-theta, k'-half-prism, or k'-half-ladder in $A \cup \{x_1, x_2, \ldots, x_{k''}\}$. We could then switch over to the other side with B and $\{y_1, y_2, \ldots, y_{k''}\}$ and restricting our self to the k' vertices of $\{y_1, y_2, \ldots, y_{k''}\}$ that match up with end endpoints of the k'-half-theta, k'-half-prism, or k'-half-ladder in $\{x_1, x_2, \ldots, x_{k''}\}$ we just found, repeat the same process in B and $\{y_1, y_2, \ldots, y_{k''}\}$ to find a k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, k-ladder-prism. Our goal then now is clear, we must clean up a k'-half-quasi-ladder to give use a k-half-ladder (or possible a k-half theta or even a k-theta).

The next three lemmas show how to clean up a half-quasi-ladder into a half-ladder, half-theta, or theta. Their proofs are similar to those of lemmas 5.7 5.9, and 5.13 respectively, although the conclusions we draw from them are somewhat different.

So, we are now in a situation where we have have found a k'-half-quasi-ladder. Let us say that P is the backbone path, and $P_1, P_2, \ldots, P_{k'}$ are the auxiliary paths of the k'-half-quasi-ladder where P_i has endpoints s_i and x_i, and let the x_i's be the endpoint of our half-quasi-ladder. Set S equal to the set of s_i's. To turn our k'-half-quasi-ladder into a k-half-ladder, we first want to find a subset $S' \subset S$ such that no vertices of S' share a neighbor in P. Notice that if any vertex of P has k neighbors in S, then we have a k-half-theta ending in $\{x_1, x_2, \ldots, x_{k'}\}$ and we are done, so we can assume that the vertices of S cannot be dominated by a small set of vertices of P. So, what the next lemma shows that if we take some $v \in S$ then we can either find a k-half-theta ending in V (and therefore W) or find a small set $X \subset V(P)$ such that no vertex of $S - N[X]$ shares a neighbor with v in P. As S cannot be dominated by a small subset of P, this lemma can be repeatedly used to give us a large subset S' of S such that no two vertices of S' share a neighbor in P, which is precisely what we do in Lemma 6.7.

LEMMA 6.6. *Let (G, S, P, v) be a tuple where G is a graph, $v \in G$, $S \subset V(G)$, and P is an induced path of G such that ($S \cup \{v\}$) and $V(P)$ are disjoint. Assume $G[V(P) \cup S \cup \{v\}]$ does not have a k-half-theta ending in S, then there is a set $X \subset S \cup V(P) \cup \{v\}$ of size at most $4k - 1$ such that $N(S - N[X]) \cap N(v) \cap V(P) = \emptyset$, and no vertex of $S - N[X]$ is neighbors with v.*

Proof. Let G, S, P, and v be as in the statement of this lemma. Number the vertices of P 1 through $|V(P)|$ such that the vertex numbered i is neighbors with the vertices numbers $i-1$ and $i+1$. We now consider the following process to build the set desired set X such that $N(S - N[X]) \cap N(v) \cap V(P) = \emptyset$ and $X \subset S \cup V(P) \cup \{v\}$.

We do the following for the first step of the process. Let $X_1 = \{v\}$, and let $S_1 = \{s : s \in S - N(X_1)$ and $N(s) \cap N(v) \cap V(P) \neq \emptyset\}$ (i.e., S_1 is the set of vertices of $S - N(X_1)$ that share a neighbor with v in P). Label the vertices of S_1 by the lowest numbered vertex it is neighbors with in $V(P) \cap N(v)$. Let s_1 be a highest labeled vertex in S_1, and let p_1 be s_1's lowest numbered neighbor in $N(v) \cap V(P)$. This completes the first step.

For the i^{th} step we do the following. Let $X_i = X_{i-1} \cup \{s_{i-1}, p_{i-1}\}$, and let $S_i = S_{i-1} - N[X_i]$ and label the vertices of S_i by the lowest vertex it sees in $V(P) \cap N(v)$ (the vertices of S_i inherit their labels from their labels in S_{i-1}). Let s_i be a highest labeled vertex in S_i and let p_i be s_i's lowest neighbor in $N(v) \cap V(P)$. Note by how we selected v, s_1, p_1, s_2, p_2, $\ldots$ s_i, p_i that s_a, $1 \leq a \leq i$, cannot be neighbors with p_b if $a > b$ since p_b would be in X_a and therefore s_a would not be in S_a, and s_a cannot have a neighbor with p_b if $a < b$ since that would contradict either p_a being s_a's lowest numbered neighbor in $N(v) \cap P$ or s_a being a highest labeled vertex in S_a. Hence, we then have that among these vertices s_j is only neighbors with p_j for $1 \leq j \leq i$, and v is only neighbors with p_j for $1 \leq j \leq i$. p_{2i} could be neighbors with p_{2i+1} and/or p_{2i-1} since they could be consecutive vertices on the path P, but p_{2i} cannot be neighbors with p_{2j}. It follows that the set $\{v\} \cup \{p_2, p_4, \ldots, p_{2c}\} \cup \{s_2, s_4, \ldots, p_{2c}\}$, $2c \leq i$, forms a c-half-theta in $G[V(P) \cup S\{v\}]$ ending in S.

We continue this process until we reach an S_j that is empty. By what we noted in the previous paragraph, this process cannot go past the $2k^{th}$ step if $G[V(P) \cup S \cup \{v\}]$ does not contain a k-half-theta ending in S. Set X to be X_j. Since S_j is empty, it follows $N(S - N[X]) \cap N(v) \cap V(P) = \emptyset$. We also have that no vertex of $S - N[X]$ is neighbors with v since $v \in X$ and $|X| \leq 4k - 1$ since $j \leq 2k$ and since the first step adds a single vertex and each step after that only adds two vertices. ∎

LEMMA 6.7. *Let (G, S, P) be a tuple such that G is a graph, $S \subset V(G)$ such that S cannot be dominated by $4kx$ vertices and P is an induced path disjoint from S that dominates S. Assume $G[V(P) \cup S]$ does not contain a k-half-theta ending in S. Then there exists a subset S' of S of size x such that no vertex of P has more than one neighbor in S'.*

Proof. Let G, S, and P be as in the statement of the lemma. Assume that we have an independent set of vertices vertices S_{i-1} of size $i - 1$, $i \leq k$, and a set Z_{i-1} of size at most $4k(i-1)$, with the properties that no vertex $S - N[Z_{i-1}]$ is neighbors with a vertex in S_{i-1}, and any vertex in P that is neighbor with some vertex in S_{i-1} has no other neighbors in S_{i-1} nor in $S - N[Z_{i-1}]$. We will use this to produce a set S_i of size i and Z_i of size at most $4k^2 i$ with the same properties. Note that the empty set satisfies the conditions of S_0.

Let $S' = S - N[Z_{i-1}]$. Let s be some vertex in S', since $i \leq k$ and S cannot be dominated by $4kx$ vertices, such an s must exists. We can then apply Lemma 6.6 using (G, S', P, s) and to get a set X of size at most $4k - 1$ such that $(S' - N[X]) \cap N(s) \cap V(P) = \emptyset$ and no vertex of $S' - N[X]$ is neighbors with s. We then set $S_i = S_{i-1} \cup \{s\}$ and $Z_i = Z_{i-1} \cup X$ and we can see these sets satisfies the required properties.

Since the empty set satisfies the properties of S_0 and S cannot be dominated by $4kx$ vertices, we can continue the process until we generate the set S_x which has size x and no vertex of P has more than one neighbor in S_x. ∎

The previous two lemmas now give us a k'-half-quasi-ladder with backbone path P, auxiliary paths $P_1, P_2, \ldots, P_{k'}$ where P_i has endpoints s_i and w_i such that no vertex of P is neighbors with more than one of the s_i's. The next lemma now show use how to take such a k'-half-quasi-ladder and produce a k-half-ladder.

LEMMA 6.8. *Let T be an induced $4k[2(4k)^{k+1}]^2$-half-quasi-ladder of a graph G ending in X. Assume T does not have an induced k-half-theta ending in X and assume that G does not contain an induced k-theta. Then T contains a k-half-ladder ending in X.*

Proof. Let G, T, and X be as in the statement of the lemma. Let P be the backbone path of T and $P_1, P_2, \ldots, P_{4k[2(4k)^{k+1}]^2}$ be its auxiliary paths, where the endpoints of P_i are v_i and x_i, and the x_i's are the endpoints of T, so $x_i \in X$. Let $S = \{v_1, v_2, \ldots, v_{4k[2(4k)^{k+1}]^2}\}$. Clearly, if any vertex of P is neighbors with k distinct v_i's, then T contains a k-half-theta ending in X. It follows that since T does not have a k-half-theta ending in X, the vertices of S cannot be dominated by less than $4[2(4k)^{k+1}]^2$ vertices in T. Also, if $G[S \cup V(P)]$ contain a k-half-theta ending in S, then it contains a k-half-theta ending in X, so we can apply Lemma 6.7 with (G, P, S) to get a set $S' \subset S$ of size $2(4k)^{k+1}$ such that no vertex of P is neighbors with more than one vertex in S'. It follows that by only taking the paths P_i such that $v_i \in S'$, that these P_i's together with P, form a $2(4k)^{k+1}$-half-quasi-ladder where no vertex of P has a neighbor with more than one vertex in any of the P_i's. We will call this $2(4k)^{k+1}$-half-quasi-ladder T', we will call its backbone path P' so $P' = P$, and we will call the auxiliary paths $P'_1, P'_2, \ldots, P'_{2(4k)^{k+1}}$ where the endpoints of P'_i are v'_i and x'_i, and the x'_i's are the endpoints of T', so $x'_i \in X$. We use S' as before to denote the set of v'_i's.

Now, number the vertices of P' 1 through $|V(P')|$ such that the vertex numbered i is neighbors with the vertices numbers $i-1$ and $i+1$. For a vertex x in P' we will use the notation $n(x)$ to denote the number it has been given in P'. For every $s_j \in S'$ let $p_j \in P'$ be the highest numbered neighbor s_j has in P. We now set $P_1 = P'$ and $S_1 = S'$. We will consider the following process, where we will try to produce a large independent set in an auxiliary graph related to some P_i and S_i which we will then use to produce a k-half-ladder. We will show this process cannot go past k iterations if T does not have a k-half-theta ending in X. We will ensure that at the i^{th} step that $V(P_i) \subset V(P')$, $S_i \subset S'$, $|S_i| \geq 2(4k)^{k-i+2}$, P_i is an induced path, and if $s_j \in S_i$ then $p_j \in P_i$. We will also produce induced subpaths D_i of P such that the D_i's are anti-complete with respect to one another and the vertices of D_i will dominate S_j if $i < j$.

At the i^{th} step we do as follows. Create an auxiliary directed graph, AUX_i, whose vertex set is S_i and there is an edge from $s_a \in S_i$ to $s_b \in S_i$ if the following condition holds

1. $n(p_a) > n(p_b)$ and s_a has a neighbor x in P' such that $n(x) < n(p_b)$

If the maximum in degree of AUX_i is at most $\frac{1}{4k}|S_i|$ then we stop. If $i \leq k$ (which we will show must happen) then since $|S_i| \geq 2(4k)^{k-i+2}$ this gives an independent set of size at least k by Lemma 5.12. If there is an $s_j \in S_i$ with in degree at least $\frac{1}{4k}|S_i|$ then for at least $\frac{1}{4k}$ fraction of the vertices of S_i must satisfy (1) playing the role of s_a while s_j plays the role of s_b. Call this set of vertices S_{i+1}. If $s_j \in S_i$ with in degree at least $\frac{1}{4k}|S_i|$ then we do as follows. Define D_i to be the subpath of P_i that is made up of vertices with numbers less than $n(p_j)$. Set P_{i+1} to be the vertices of P_i with numbers greater than $n(p_j)$. This concludes the i^{th} step.

It can then be seen that $V(P_{i+1}) \subset V(P)$, $S_{i+1} \subset S$, $|S_{i+1}| \geq 2(4k)^{k-i+1}$, P_{i+1} is an induced path, and if $s_j \in S_{i+1}$ then $p_j \in P_{i+1}$ as required. Furthermore, it can be seen that any of the previously D_j's that have been produced in this process ($j \leq i$) dominate all vertices of S_{i+1}. Since the D_j's are disjoint and anti complete, By Lemma 5.11 then, this process cannot go past the k^{th} iteration without producing a k-theta in G.

We conclude there is some step $j \leq k$ such that the auxiliary graph AUX_j has max in-degree less than $\frac{1}{4k}|S_j|$, and since $|S_j| \geq 8k$ it therefore has an independent set of size k by Lemma 5.12. Let S^* denote such an independent set.

We claim by only taking the paths P'_i such that $v'_i \in S^*$, that these P'_i's together with P', form a k-half-ladder. Let $x, y \in S^*$ and let a, b be the highest and lowest numbered neighbors of x in L respectively, and assume that y has a neighbor c on the induced path of L that has a and b as its endpoints. If y's highest numbered neighbor in L is greater than $n(a)$ then y has an edge to x in AUX_j. If y's highest numbered neighbor in L is less than $n(a)$, then x has an edge to y. It follows that taking the P'_i such that $v'_i \in S^*$ together with P', form a k-half-ladder. ∎

Corollary 6.1. *Let k be a natural number. There exists a natural number k' large enough such that if G is a graph that contains a k'-creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$), then $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains an induced k-half-theta, k-half-prism, or k-half-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$ or G contains an induced k-theta.*

Proof. By Lemma 6.5 there exists a k' large enough such that if G contains a k'-creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$) then $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains an induced $4k[2(4k)^{k+1}]^2$-half-theta, $4k[2(4k)^{k+1}]^2$-half-prism, or a $4k[2(4k)^{k+1}]^2$-half-quasi-ladder, ending in $\{x_1, x_2, \ldots, x_{k'}\}$. If $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a $4k[2(4k)^{k+1}]^2$-half-theta or a $4k[2(4k)^{k+1}]^2$-half-prism ending in $\{x_1, x_2, \ldots, x_{k'}\}$ then we are done. If $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a $4k[2(4k)^{k+1}]^2$-half-quasi-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$ then we may apply Lemma 6.8 to get that either $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a k-half-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$ or G contains a k-theta. ∎

We now know that given a k'-creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$) for large enough k' in each half, $A \cup \{x_1, x_2, \ldots, x_{k'}\}$ and $B \cup \{y_1, y_2, \ldots, y_{k'}\}$ we can find a k-half-theta, k-half-prism, or k-half-ladder, and we can combine them together to make a k-theta, k-prism, k-pyramid, k-ladder, k-ladder-theta, or a k-ladder-prism. The next lemma formalizes this.

Lemma 6.9. *Let k be a natural number. Then there exists a natural number k' large enough such that if G is a graph that contains a k'-creature (A, B, $\{x_1, x_2, \ldots, x_{k'}\}$, $\{y_1, y_2, \ldots, y_{k'}\}$), then G contains an induced k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, or a k-ladder.*

Proof. Let k be a natural number. By Corollary 6.1 there exists a k' large enough such that if G is a graph that contains a k'-creature $(A,\, B,\, \{x_1, x_2, \ldots, x_{k'}\},\, \{y_1, y_2, \ldots, y_{k'}\})$, then $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains an induced k^2-half-theta, k^2-half-prism, or k^2-half-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$ or G contains an induced k^2-theta. It then also follows from Corollary 6.1 there exists a k'' large enough such that if G is a graph that contains a k''-creature $(A,\, B,\, \{x_1, x_2, \ldots, x_{k''}\},\, \{y_1, y_2, \ldots, y_{k''}\})$, then $G[B \cup \{y_1, y_2, \ldots, y_{k''}\}]$ contains an induced k'-half-theta, k'-half-prism, or k'-half-ladder ending in $\{y_1, y_2, \ldots, y_{k''}\}$ or G contains an induced k'-theta.

So, assume that G is a graph that contains an k''-creature $(A,\, B,\, \{x_1, x_2, \ldots, x_{k''}\},\, \{y_1, y_2, \ldots, y_{k''}\})$. If G contains an induced k'-theta then we are done, so assume that $G[B \cup \{y_1, y_2, \ldots, y_{k''}\}]$ contains an induced k'-half-theta, k'-half-prism, or k'-half-ladder ending in $\{y_1, y_2, \ldots, y_{k''}\}$. By relabeling the x_i's and y_i's we can then assume that G contains a k' creature $(A',\, B',\, \{x_1, x_2, \ldots, x_{k'}\},\, \{y_1, y_2, \ldots, y_{k'}\})$ such that $G[B' \cup \{y_1, y_2, \ldots, y_{k'}\}]$ is a k'-half-theta, k'-half-prism, or k'-half-ladder. Then applying Corollary 6.1 gives us that $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains an induced k^2-half-theta, k^2-half-prism, or k^2-half-ladder ending in $\{x_1, x_2, \ldots, x_{k'}\}$. If $G[B' \cup \{y_1, y_2, \ldots, y_{k'}\}]$ is a k'-half-ladder and $G[A \cup \{x_1, x_2, \ldots, x_{k'}\}]$ contains a k^2-half-ladder, then an application of the Erdös-Szekeres Theorem gives us a k-ladder. Otherwise, it follows that G must contain a k^2-theta, a k^2-prism, k^2-pyramid, k^2-ladder-theta, or k^2-ladder-prism. ◻

With Lemma 6.9 in hand we can now provide a proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2] Let G be a graph, $|V(G)| = n$, where G forbids all k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, and k-ladder-prism graphs as well as k-contracted-ladder graphs. By Lemma 6.9 there exists a function $f : \mathbb{N} \to \mathbb{N}$ (f in independent of the choice of k or G) such that G is $f(k)$-creature-free. Furthermore, we can see that any graph that contains a $2k$-skinny-ladder as an induced minor must either contain a k-ladder or a k-contracted-ladder as an induced subgraph, therefore G contains no $2k$-skinny-ladder as an induced minor. Hence by Theorem 1.1 there is a function $f^* : \mathbb{N} \to \mathbb{N}$ (f^* is independent of the choice of k or G) such that G has at most $n^{f^*(k)\log(n)}$ minimal separators. It follows that the family of graphs that forbid all k-theta, k-pyramid, k-prism, k-ladder, k-ladder-theta, and k-ladder-prism graphs as well as k-contracted-ladder graphs are strongly-quasi-tame. ◻

The following two lemmas will be used in Lemma 6.12 to establish that if $\mathcal{F}$ is a family of graphs defined by a finite number of forbidden induced subgraphs and $\mathcal{F}$ allows for at least one of k-thetas, k-prisms, k-pyramids, k-ladder-thetas, or k-ladder-prisms, for arbitrarily large k, then we can ensure it contains these graphs where their number of vertices only grow linearly with respect to k, and therefore have exponentially many minimal separators. These two lemmas achieve this by showing that a graph in $\mathcal{F}$ has certain paths that are too long, then we can contract part of those paths and maintain that the resulting graph remains in $\mathcal{F}$.

LEMMA 6.10. *Let G be a graph and let H be a graph with $|V(H)| \leq h$, where $h > 5$. Assume that G contains an induced path P of length at least $5h$ where all internal vertices of P have degree 2 in G. Then there exists an edge e in G such that if G^e contains H as an induced subgraph, then so does G.*

Proof. Let G be a graph, let H be a graph with $|V(H)| \leq h$ where $h > 5$, and let P be an induced path of G of length at least $5h$ where all internal vertices of P have degree 2, say $P = p_1, p_2, \ldots, p_{5h}$. Let e be the edge between $p_{\lceil \frac{5h-1}{2} \rceil}$ and $p_{\lceil \frac{5h+1}{2} \rceil}$. Let v denote the new vertex $p_{\lceil \frac{5h-1}{2} \rceil}$ and $p_{\lceil \frac{5h+1}{2} \rceil}$ create when e is contracted in G to make G^e, and let P' be what the path P becomes after contracting e in G, so $P' = p_1, p_2, \ldots, p_{\lceil \frac{5h-1}{2} \rceil - 1}, v, p_{\lceil \frac{5h+1}{2} \rceil + 1}, \ldots, p_{5k}$. Assume that G^e contains H as an induced subgraph. We will show that there exists a set $X \subset V(G^e)$ that induces H such that $v \notin X$. It will then follows that G contains an induced H.

Any component of H that is not an induced path can only contain vertices outside of P' or within distance h of either the endpoints of P' since all internal vertices of P' have degree 2 in G^e. For the components of H that are paths, since there are at most h vertices among these components, we can ensure that the vertices of X that we use to induce these components either do not belong to P' or only contain vertices from the subpaths $p_{h+2}, p_{h+3}, \ldots, p_{\lceil \frac{5h-1}{2} \rceil - 1}$ and $p_{\lceil \frac{5h+1}{2} \rceil + 1}, p_{\lceil \frac{5h+1}{2} \rceil}, \ldots, p_{4h-2}$. It follows that $v \notin X$. ◻

LEMMA 6.11. *Let G be a graph and let H be a graph with $|V(H)| \leq h$, where $h > 5$. Assume that G contains an induced path P of length $5h[(h+1)(5h)^{2h+2} + 1]$ such that the only neighbor the vertices of P might have outside of P is a single vertex v. Then there exists a subpath P' of P such that if $G^{P'}$ contains H as an induced subgraph, then so does G.*

Proof. Let G be a graph and let H be a graph with $|V(H)| \leq h$, where $h > 5$. Assume that G contains an induced path P of length $5h[(h+1)(5h)^{2h+2}+1]$ such that the only neighbor the vertices of P might have outside of P is a single vertex v. Let a, b be the endpoints of P. Now divide P into a sequence of subpaths $P_1, P_2, \ldots, P_k$ each of length at least 2 so that all internal vertices of P_i have degree 2 in G, all endpoints of P_i are either a vertex of degree 3 or a or b, P_1 has a and one of its endpoints, P_k has b as one of its endpoints, and P_i shares one of its endpoints with P_{i+1} (i.e., these are subpaths that whose endpoints are a, b, or the vertices that are neighbors with v and are sequenced going from one end of P to the other). We define a second sequence $a_1, a_2, \ldots a_k$ where $a_i = |E(P_i)|$. If any $a_i \geq 5h$ then the result follows from Lemma 6.10, so we can assume for all i that $a_i \leq 5h$. It then follows that k is at least $(h+1)(5h)^{2h+2}+1$, and therefore by the pigeonhole principle there must be a continuous subsequence of length $2h+2$ that is repeated at least $h+2$ times, where none of these continuous subsequences overlap with each other. Let $S = s_0, s_1, \ldots, s_{2h+1}$ be this repeated subsequence. So we have $h+2$ sequences for $1 \leq i \leq h+2$, $A_i = a_{j_i}, a_{j_i+1}, \ldots, a_{j_i+2h+1}$ where for $1 \leq m \leq h+2$ and c, $0 \leq c \leq 2h+1$, $a_{j_m+c} = s_c$ and no part of A_m overlaps with some other A_n (so $|j_n - j_m| \geq 2h+2$) and $j_m > j_n$ if $m > n$. Fix the values denoted by j_m for $1 \leq m \leq h+2$.

We wish to combine the first half of A_1 with the second half of A_2 by contracting a path in P. Let x be the endpoint of P_{j_1+h+1} that it shares with P_{j_1+h}, and let y be the endpoint P_{j_2+h+1} shares with P_{j_2+h}. Let P' be the subpath of P that has x and y as its endpoints. Let w be the vertex that gets created when contracting the path P' in G to get $G^{P'}$ and let all the subpaths P_i of P in G that were not contained in P' retain their labels in $G^{P'}$, so P_{j_1+h} and P_{j_2+h+1} share w as an endpoint, and let the a_i's retain their same meaning as long as P_i was not a subpath of P'. It follows that $G^{P'}$ has h sequences for $3 \leq i \leq h+2$, $A_i = a_{j_i}, a_{j_i+1}, \ldots, a_{j_i+2h+1}$ where for $1 \leq m \leq h+2$ and c, $0 \leq c \leq 2h+1$, $a_{j_m+c} = s_c$ and no part of A_m overlaps with some other A_n (so $|j_n - j_m| \geq 2h+2$) and $j_m > j_n$ if $m > n$. Furthermore, A_1 and A_2 have now been combined to give $A' = a_{j_1}, a_{j_1+1}, \ldots, a_{j_1+h}, a_{j_2+h+1}, a_{j_2+h+2}, \ldots, a_{j_2+2h+1}$ so that $a_{j_1+c} = s_c$ for $0 \leq c \leq h$ and $a_{j_2+c} = s_c$ for $h+1 \leq c \leq 2h+1$. We will show that if there exists a set $X \subset V(G^{P'})$ that induces H in $G^{P'}$ then we can require $w \notin X$. The result then follows since if $w \notin X$ then the vertices that correspond to X in G induced an H in G.

So, assume $X \subset V(G^{P'})$ and induces H. If $w \notin X$ then we are done, so assume $w \in X'$ for some connected component X' of X. For i with $3 \leq i \leq h+1$, let P_i^* denote the path induced by $V(P_{j_i}), V(P_{j_i+1}), \ldots, V(P_{j_i+2h+1})$ in $G^{P'}$, so P_i^* is the path that naturally corresponds to S_i, and let P_1^* denote the path induced by

$$V(P_{j_1}), V(P_{j_1+1}), \ldots, V(P_{j_1+h}), V(P_{j_2+h+1}), V(P_{j_2+h+2}), \ldots, V(P_{j_2+2h+1}),$$

so P_1^* naturally corresponds with A'. Then since X' has at most h vertices there is at least one P_i^* that contains no vertex of X and since X' is connected and contains w, all vertices of $X' \cap P$ must be completely contained in $V(P_1^*)$ since w is at least distance h from either endpoint of P_1^*. It follows that we can replace the vertices of $X' \cap P$, which must be completely contained in the interal vertices of P_1^*, with the corresponding vertices in a P_i^* that contains no vertices of X and still maintain that the vertices of X induce H. Now $w \notin X$ and the result then follows. ∎

LEMMA 6.12. *Let $\mathcal{F}$ be a family of graphs determined by a finite number of forbidden induced subgraphs. Then if $\mathcal{F}$ does not forbid all k-thetas, k-prisms, k-pyramids, k-ladder-thetas, k-ladder-prisms, and k-ladders for arbitrarily large k, then $\mathcal{F}$ is feral.*

Proof. Let $\mathcal{F}$ be a family of graphs determined by a finite number of forbidden induced subgraphs, and let $\mathcal{H}$ be a set of forbidden subgraphs that define $\mathcal{F}$. Let let $h > 5$ be a number such that for any $H \in \mathcal{H}$, $|V(H)| \leq h$. First assume that $\mathcal{F}$ allows for either k-thetas k-prisms, or k-pyramids for arbitrarily large k. Then by Lemma 6.10 we can ensure that all paths with internal vertices all having degree 2 of the k-thetas k-prisms, or k-pyramids are at most $5h$ (we keep on contracting the appropriate edges given by Lemma 6.10 until no path where all internal vertices have degree 2 have length more than $5h$) and therefore $\mathcal{F}$ contains a k-theta k-prism, or k-pyramid with at most $5h \cdot k$ vertices. Since a k-theta, k-prism, or k-pyramid must have at least 2^k minimal separators, it follows that there exists a $c > 1$ such that for every natural number N ther exists a $G \in \mathcal{F}$ such that $|V(G)| = n > N$ and the number of minimal separators in G is at least c^n.

Now assume that $\mathcal{F}$ allows for k-ladder-thetas or k-ladder-prisms for arbitrarily large k. Every k-ladder-theta and k-ladder-prism contains a k-half-ladder and by Lemma 6.10 we can ensure that all paths with internal vertices all having degree 2 of the k-ladder-theta or k-ladder-prism are at most $5h$ and by Lemma 6.11 we can ensure that the backbone path of the corresponding k-half-ladder has length at most $[5h(h+1)(5h)^{2h+1}+1]\cdot k$ by contracting the appropriate edges and paths if necessary while still guaranteeing the resulting graph belongs to $\mathcal{F}$ (Lemma 6.11 gives us that if there is a subpath of length over $[5h(h+1)(5h)^{2h+1}+1]$ of the backbone path that only has one neighbor outside of the backbone path, there there exists a subpath of the backbone path that we can contract and still maintain that the resulting graph is a k-ladder-theta or k-ladder-prism contained in $\mathcal{F}$). Since k-ladder-thetas and k-ladder-prisms have at least 2^k minimal separators it follows that there exists a contains $c > 1$ such that for every natural number N there exists a $G \in \mathcal{F}$ such that the number of minimal separators in G is at least c^n. It follows that $\mathcal{F}$ is feral. ∎

The following lemma shows why it is necessary to forbid k-paw and k-claw graphs for a family of graphs defined by a finite number of forbidden induced subgraphs to be strongly-quasi-tame. Figure 10 gives a picture of the two graphs constructed in the following lemma.

Lemma 6.13. *Let $\mathcal{F}$ be a family of graphs determined by a finite number of forbidden induced subgraphs. Then if $\mathcal{F}$ does not forbid k-claws and k-paws for some natural number k, then $\mathcal{F}$ is feral.*

Proof. Let $\mathcal{F}$ be a family of graphs determined by a finite number of forbidden induced subgraphs, and let $\mathcal{H}$ be a set of forbidden subgraphs that define $\mathcal{F}$. Let $h > 5$ be a number such that for any $H \in \mathcal{H}$, $|V(H)| \leq h$. First we assume that $\mathcal{F}$ allows k-claw for arbitrarily large k. We will construct a graph with many minimal separators. Assume that we have two sets of $2^c - 1$ long-claws, $C_1^1, C_2^1, \ldots C_{2^c}^1$, and $C_1^2, C_2^2, \ldots C_{2^c}^2$ where in both sets each long claw has arm length h. We label the leaves of C_i^1 as a_i^1, b_i^1, c_i^1 and we label the endpoints of C_i^2 as a_i^2, b_i^2, c_i^2. Then for $1 \leq i \leq 2^{c-1} - 1$ we glue a_{2i}^1 to b_i^1, a_{2i+1}^1 to c_i^1, a_{2i}^2 to b_i^2, and a_{2i+1}^2 to c_i^2. Furthermore, for $2^{c-1} \leq i \leq 2^c - 1$ we add an edge between b_i^1 and b_i^2 and between c_i^1 and c_i^2. Note that any collection of $b_i^{j_i}$ and $c_i^{\ell_i}$ with $2^{c-1} \leq i \leq 2^c - 1$ and $j_i, \ell_i = 1$ or 2 is a minimal separator, so there are at least 2^{2^c} minimal separators in this construction. Since the arm length of each long-claw is h, the total number of vertices in this construction is less than $3h \cdot 2^{c+1}$.

If $\mathcal{F}$ allows for k-claws, then forest of paths and subdivided claws cannot be forbidden in $\mathcal{F}$, and it can be seen that any induced subgraph of size at most h of the construction just given is a forest of paths and subdivided claws (i.e., three anti-complete paths where one endpoint of each path are glued together). It follows that this construction must belong to $\mathcal{F}$ and since this construction has at least 2^{2^c} minimal separators and less than $3h \cdot 2^{c+1}$ vertices, the statement of the lemma follows for the case where k-claw graphs for arbitrarily large k are not forbidden.

Now we assume that $\mathcal{F}$ allows k-paw graphs for arbitrarily large k. The construction and analysis we make in this case is nearly identical to the k-claw case. We present it here for completeness. Assume that we have two set of $2^c - 1$ long-paws, $C_1^1, C_2^1, \ldots C_{2^c}^1$, and $C_1^2, C_2^2, \ldots C_{2^c}^2$ where in both sets each long-paw has arm length h. We label the endpoints of C_i^1 as a_i^1, b_i^1, c_i^1 and we label the endpoints of C_i^2 as a_i^2, b_i^2, c_i^2. Then for $1 \leq i \leq 2^{c-1} - 1$ we glue a_{2i}^1 to b_i^1, a_{2i+1}^1 to c_i^1, a_{2i}^2 to b_i^2, and a_{2i+1}^2 to c_i^2. Lastly, for $2^{c-1} \leq i \leq 2^c - 1$ we add an edge between a_i^1 and a_i^2 and between b_i^1 and b_i^2. Note that any collection of $b_i^{j_i}$ and $c_i^{\ell_i}$ with $2^{c-1} \leq i \leq 2^c - 1$ and $j_i, \ell_i = 1$ or 2 is a minimal separator, so there are at least 2^{2^c} minimal separators in this construction. Since the arm length of each long-claw is h, the total number of vertices in this construction is less than $3h \cdot 2^{c+1}$.

Since $\mathcal{F}$ allows for k-paws, a forest of paths and subdivided paws cannot be forbidden in $\mathcal{F}$, and it can be seen that any induced subgraph of size at most h of the construction just given is a forest of paths and subdivided paws. It follows that this construction must belong to $\mathcal{F}$ and since this construction has at least 2^{2^c} minimal separators and less than $3h \cdot 2^{c+1}$ vertices, the statement of the lemma follows for the case where k-paw graphs for arbitrarily large k are not forbidden. ∎

We are now ready to prove Theorem 1.3

Proof. [Proof of Theorem 1.3] Let $\mathcal{F}$ be a family of graphs defined by a finite number of forbidden induced subgraphs. It follows from Lemmas 6.12 and 6.13 that if $\mathcal{F}$ allows for any k-thetas, k-prisms, k-pyramids, k-ladder-thetas, k-ladder-prisms, k-claws, or k-paws for arbitrarily large k, $\mathcal{F}$ is feral.

Now assume that there exists a natural number k such that $\mathcal{F}$ forbids k-thetas, k-prisms, k-pyramids, k-ladder-thetas, k-ladder-prisms, k-claws, and k-paws. Observe that there exists a k' large enough such that if G

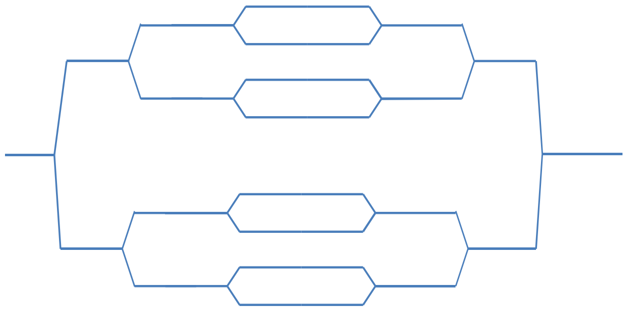

Figure 10: The two graphs in this figure are small versions of the constructions of the graphs given in Lemma 6.13, explicit vertices are omitted in this graph. The left side graph is the construction provided when when the k-claw is not forbidden for arbitrarily large k. The right hand side graph is the construction provided when when the k-paw is not forbidden for arbitrarily large k.

contains an induced k'-ladder, then G contains an induced k-claw or k-paw graph, therefore $\mathcal{F}$ forbids k'-ladders. It then follows from Lemma 6.9 there exists a k'' such that no $G \in \mathcal{F}$ can contain a k''-creature, where the minimum value of k'' is a function of k. Furthermore, it is clear that there exists a k''' large enough such that if G contains a k'''-skinny-ladder as an induced minor, then G contains a k-claw or a k-paw as an induced subgraph. Hence $\mathcal{F}$ forbids k'''-skinny-ladders as an induced minor. It then follows from Theorem 1.1 that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{F}$ the number of minimal separators of G is at most $n^{f(k)\log(n)}$. Hence $\mathcal{F}$ is strongly quasi-tame. ∎

7 Long Cycle-free Graphs

Here we present a proof of Theorem 1.4 which is based on an easy application of Corollary 5.1. We will need the following lemma in order to apply Corollary 5.1.

Lemma 7.1. *Let G be a $C_{\geq r}$-free graph and assume G does not contain a k-creature. Then every minimal separator, S, can be dominated by $r \cdot k^2$ vertices of G not in S.*

Proof. Let G be a $C_{\geq r}$-free graph and assume G does not contain a k-creature. Assume for a contradiction that there exists a minimal separator, S, of G such that S cannot be dominated by $r \cdot k^2$ vertices in G and not in S. Let H be an S-full component of $G - S$, then by Lemma 5.6, S is dominated a subset of H that is the union of k^2 induced paths in H. It follows there must exists some induced path P in H such that $S_P = N(P) \cap S$ cannot be dominated by r vertices in P. There then exists a subpath P' of P such that there are vertices $a, b \in S_P$ that have no neighbor in P', both component of $P - P'$ have vertices that are neighbors with a and/or b. It follows that we can extend the path P' to have endpoints x_a and x_b such that the only neighbors of a in P' is x_a and possible x_b and the only neighbors of b in P' is x_b and possibly x_a. If x_a and x_b are both neighbors with a then P' and a form a cycle of length r, and if x_a and x_b are both neighbors with b then P' and b form a cycle of length r so assume neither of these cases occur. If a and b are neighbors then P' a, b make a cycle of length more than r. Else, there is an induced path, T between a and b with all of its internal vertices contained in some S-full component other than H. It follows that P', and T makes a cycle of length more than r, a contradiction. ∎

Proof. [Proof of Theorem 1.4] Let G be a $C_{\geq k}$-free graph that is k-theta, k-prism, and k-pyramid free. Since G is $C_{\geq k}$-free this implies that G is also k-ladder-theta, k-ladder-prism, and k-ladder free. Lemma 6.9 then implies that there exists a function $f : \mathbb{N} \to \mathbb{N}$ (independent of the choice of k or G) such that G is $f(k)$-creature-free. Lemma 7.1 gives that every minimal separator S of G can be dominated by $kf(k)^2$ vertices not in S. Hence, by Corollary 5.1 G has at most $|V(G)|^{(kf(k)^2)^2+2kf(k)^2}$ minimal separators. It follows that the family of graphs that are $C_{\geq k}$-free, k-theta, k-prism, and k-pyramid free is tame. ∎

8 Graph With Bounded Clique Size

Here we present a proof of Theorems 1.5 and 1.6 which are based on an easy application of Corollary 5.1. We will need the following lemma in order to apply Corollary 5.1.

LEMMA 8.1. *Let $k' = 4[(8k^2)^{k+1}]^7$. If G is k-creature-free, G does not contain a k-skinny-ladder as an induced minor, and no minimal separator of G contains a clique of size k, then every minimal separator S of G can be dominated by at most $(k')^{k+1}$ vertices of $G - S$.*

Proof. Let k', k, and G be as in the statement of the lemma. Let G' be an induced subgraph of G and let S' be a minimal separator of G'. Then G' must be k-creature-free and k-ladder free, so it follow from Lemma 5.13 that S' can be dominated by k' vertices of $G' - S'$.

We will produce a set of $(k')^{k+1}$ vertices of $G - S$ that dominate S by considering the following recursive algorithm. The input to the algorithm is (G',S') where G' is a subgraph of G and S' is a minimal separator of G', and the algorithm returns a set of vertices which will be described shortly. The algorithm finds two vertex sets A and B such that $|A| + |B| \leq k'$, $A \subset V(G')$, $B \subset S'$, and $A \cup B$ dominate S' (such a set must exists by what was established in the previous paragraph). Let B' be a set of vertices in $G' - S'$ such that $|B'| \leq |B|$ and B' dominates B. For each $b \in B$ we recursively call the algorithm on $(G' - (S' - [S' \cap N(b)]), S' \cap N(b))$ (note that $S' \cap N(b)$ is a minimal separator of $G' - (S' - [S' \cap N(b)])$). Let X be the union of the sets returned by each recursive call. Then algorithm then returns $X \cup A \cup B'$.

If we initially call this algorithm on (G, S) for some minimal separator S of G, then it is clear that the set this algorithm returns is a subset of vertices of $G - S$ that dominate S. We can also see the depth of this recursive algorithm cannot go past k without producing a clique of size k in S since the minimal separator we recursively call this algorithm on is always dominated by the open neighborhood of some vertex v of S. So, the depth of the recursion tree is at most $k - 1$ and each node has at most k' children since $|B| \leq k'$. It follows that since each recursive call of the algorithm adds at most k' vertices to the set it returns, the size of the final returned set cannot exceed $k' \cdot k'^k$ ∎

Proof. [Proof of Theorem 1.5] Let G be a graph that is k-creature-free and does not contain a k-skinny-ladder as an induced minor, and furthermore assume that no minimal separator of G has a clique of size k. By Lemma 8.1 there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that all minimal separators, S, of any graph that is k-creature-free, does not contain a k-skinny-ladder as an induced minor, and has no minimal separator that contains a clique of size k, can be dominated by $f(k)$ vertices outside of S. It then follows from Corollary 5.1 that G has at most $|V(G)|^{f(k)^2+2f(k)}$ minimal separators. Hence, the family of graphs that are k-creature-free, do not contain a k-skinny-ladder as an induced minor, and have no minimal separator has a clique of size k is tame. ∎

Proof. [Proof of Theorem 1.6] Let $\mathcal{F}$ be a family of graphs defined by a finite number of forbidden induced subgraphs. Assume that $\mathcal{F}$ forbids the complete graph on k vertices for some natural number k. It follows from Lemmas 6.12 and 6.13 that if $\mathcal{F}$ allows for any k'-thetas, k'-ladder-thetas, k'-claws, or k'-paws for arbitrarily large k', then $\mathcal{F}$ is feral.

Now assume that for some integer k that $\mathcal{F}$ forbids k-thetas, k-ladder-thetas, k-claws, and k-claws. Since $\mathcal{F}$ forbids k-cliques as well, it follows that $\mathcal{F}$ forbids k-prisms, k-pyramids, and k-ladder-prisms. Observe that there exists a k' large enough such that if G contains an induced k'-ladder, then G contains an induced k-claw or k-paw, therefore G does not contain a k'-ladder. It follows from Lemma 6.9 there exists a k'' such that no $G \in \mathcal{F}$ can contain a k''-creature, where the minimum value of k'' is a function of k. Furthermore, it is clear that there exists a k''' large enough such that if G contains a k'''-skinny-ladder as an induced minor, then G contains a k-claw or a k-paw as an induced subgraph. Hence $\mathcal{F}$ forbids k'''-skinny-ladders as an induced minor. Now, if no graph of $\mathcal{F}$ contains a minimal separator with a clique of size k, then it follows by Lemma 8.1 there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{F}$ it holds that all minimal separators S of G can be bounded by $f(k)$ vertices in $G - S$. It then follows from Corollary 5.1 that for all $G \in \mathcal{F}$ has at most $|V(G)|^{f(k)^2+2f(k)}$ minimal separators. Therefore $\mathcal{F}$ is tame. ∎

9 Conclusion

In this paper we disproved a conjecture of Abrishami et al. [1] that for any natural number k, the family of graphs that exclude k-creatures is tame. On the other hand, we proved a weakened form of the conjecture, that every family of graphs that excludes k-creatures and also excludes k-skinny-ladders as induced minors is strongly-quasi-tame. This led to a complete classification of graph families defined by a finite number of forbidden induced subgraphs into strongly-quasi-tame and feral, substantially generalizing the main result of Milanič and Pivač [19]. The tools we develop on the way to prove our main results yield with some additional effort polynomial upper bounds instead of quasi-polynomial, proving tameness instead of strong quasi-tameness, for two interesting special cases. In particular we show that the conjecture of Abrishami et al. [1] is true for $C_{\geq r}$-free graphs for every integer r, as well as for K_r-free graphs excluding an r-skinny-ladder for every integer r. The first of these results generalizes work of Chudnovsky et al. [6], who proved that $C_{\geq 5}$-free, k-creature-free graphs are tame,

Although Theorems 1.1 and 1.3 provide a strongly-quasi-tame bound we have no examples of non-tame families that exclude k-creatures and k-skinny-ladders for some k. We conjecture that these classes of graphs are actually tame.

Conjecture 9.1. *For every natural number k, the family of graphs that are k-creature-free and do not contain a k-skinny-ladder as an induced minor is tame.*

Conjecture 9.1, if true, put together with the proof of Theorem 1.3 would lead to the following classification of hereditary families defined by a finite set of forbidden induced subgraphs.

Conjecture 9.2. *Let $\mathcal{F}$ be a graph family defined by a finite number of forbidden induced subgraphs. If there exists a natural number k such that $\mathcal{F}$ forbids all k-theta, k-prism, k-pyramid, k-ladder-theta, k-ladder-prism, k-claw, and k-paw graphs, then $\mathcal{F}$ is tame. Otherwise $\mathcal{F}$ is feral.*

We remark that Conjecture 9.1 implies Conjecture 9.2, but not the other way around. In particular Conjecture 9.2 might be easier to prove.

We have so far been unsuccessful in identifying other counterexamples to Conjecture 1.1 that look "substantially different" from the k-twisted ladders constructed in Section 4. For this reason it is tempting to conjecture that at least for induced minor closed classes, a "clean" classification of all classes into tame or feral is possible.

Conjecture 9.3. *Every induced-minor-closed class $\mathcal{F}$ is either tame or feral.*

Since removing vertices and contracting edges can not increase the number of minimal separators, Conjecture 9.3, would show (in an informal sense) that both the brittleness of the boundary between tame and non-tame hereditary classes, as well as the existence of non-tame hereditary classes that are not feral is primarily due to "number fiddling" effects such as in the example of Abrishami et al. [1] of a tame family containing k-creatures for arbitrarily large k.

Remark: As mentioned in the introduction, subsequent work [11, 2], has confirmed that Conjectures 9.1 and 9.2 are true, while Conjecture 9.3 is false. We nevertheless keep the statements of these conjectures here, both because they provided guidance and motivation for the subsequent work [11, 2] and to ensure backwards compatibility of citations.

References

[1] T. Abrishami, M. Chudnovsky, C. Dibek, S. Thomassé, N. Trotignon, and K. Vuskovic, *Graphs with polynomially many minimal separators*, J. Comb. Theory, Ser. B, 152 (2022), pp. 248–280.

[2] A. Authors, *Graph classes with few minimal separators. II. A dichotomy.* 2022.

[3] A. Berry, J. P. Bordat, and O. Cogis, *Generating all the minimal separators of a graph*, Int. J. Found. Comput. Sci., 11 (2000), pp. 397–403.

[4] V. Bouchitté and I. Todinca, *Treewidth and minimum fill-in: Grouping the minimal separators*, SIAM J. Comput., 31 (2001), pp. 212–232.

[5] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph classes: a survey*, SIAM, 1999.

[6] M. Chudnovsky, M. Pilipczuk, M. Pilipczuk, and S. Thomassé, *On the maximum weight independent set problem in graphs without induced cycles of length at least five*, SIAM J. Discret. Math., 34 (2020), pp. 1472–1483.

[7] M. Chudnovsky, S. Thomassé, N. Trotignon, and K. Vuskovic, *Maximum independent sets in (pyramid, even hole)-free graphs*, CoRR, abs/1912.11246 (2019).

[8] R. Diestel, *Graph Theory, 4th Edition*, vol. 173 of Graduate texts in mathematics, Springer, 2012.
[9] P. Erdos and G. Szekeres, *A combinatorial problem in geometry*, in Classic Papers in Combinatorics, Springer, 2009, pp. 25–48.
[10] F. V. Fomin, I. Todinca, and Y. Villanger, *Large induced subgraphs via triangulations and CMSO*, SIAM J. Comput., 44 (2015), pp. 54–87.
[11] J. Gajarskỳ, L. Jaffke, P. T. Lima, J. Novotná, M. Pilipczuk, P. Rzażewski, and U. S. Souza, *Taming graphs with no large creatures and skinny ladders*, arXiv preprint arXiv:2205.01191, (2022).
[12] T. Kloks and D. Kratsch, *Finding all minimal separators of a graph*, in STACS 94, 11th Annual Symposium on Theoretical Aspects of Computer Science, vol. 775 of Lecture Notes in Computer Science, Springer, 1994, pp. 759–768.
[13] T. Kloks, D. Kratsch, and C. K. Wong, *Minimum fill-in on circle and circular-arc graphs*, J. Algorithms, 28 (1998), pp. 272–289.
[14] D. Kratsch, *The structure of graphs and the design of efficient algorithms, habilitation*, Friedrich-Schiller-University of Jena, Germany, (1996).
[15] D. Lokshtanov, *On the complexity of computing treelength*, Discret. Appl. Math., 158 (2010), pp. 820–827.
[16] D. Marx, *Parameterized graph separation problems*, Theor. Comput. Sci., 351 (2006), pp. 394–406.
[17] K. Menger, *Zur allgemeinen kurventheorie*, Fundamenta Mathematicae, 10 (1927), pp. 96–115.
[18] M. Milanič and N. Pivač, *Minimal separators in graph classes defined by small forbidden induced subgraphs*, in International Workshop on Graph-Theoretic Concepts in Computer Science, Springer, 2019, pp. 379–391.
[19] M. Milanic and N. Pivac, *Polynomially bounding the number of minimal separators in graphs: Reductions, sufficient conditions, and a dichotomy theorem*, Electron. J. Comb., 28 (2021), p. 1.
[20] F. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society, s2-30 (1930), pp. 264–286.
[21] N. Robertson and P. D. Seymour, *Graph minors .XIII. The disjoint paths problem*, J. Comb. Theory, Ser. B, 63 (1995), pp. 65–110.
[22] N. Sauer, *On the density of families of sets*, Journal of Combinatorial Theory, Series A, 13 (1972), pp. 145 – 147.
[23] K. Suchan, *Minimal separators in intersection graphs*, Master's thesis, Akademia Gorniczo-Hutnicza im. Stanislawa Staszica w Krakowie, (2003).