Polynomial Kernel for Interval Vertex Deletion

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Given a graph G and an integer k, the Interval Vertex Deletion (IVD) problem asks whether there exists a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is an interval graph. This problem is known to be NP-complete (according to Yannakakis at STOC 1978). Originally in 2012, Cao and Marx showed that IVD is fixed parameter tractable: they exhibited an algorithm with running time $10^k n^{O(1)}$. The existence of a polynomial kernel for IVD remained a well-known open problem in parameterized complexity. In this article, we settle this problem in the affirmative.

CCS Concepts: • **Theory of computation → Parameterized complexity and exact algorithms;**

Additional Key Words and Phrases: Interval Vertex Deletion, kernelization, polynomial kernel, parameterized complexity

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1 INTRODUCTION

In a *graph modification problem*, the input consists of an n-vertex graph G and an integer k. The objective is to determine whether k *modification operations*—such as vertex deletions, or edge

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deletions, insertions or contractions—are sufficient to obtain a graph with prescribed structural properties such as being planar, bipartite, chordal, interval, acyclic, or edgeless. Graph modification problems include some of the most basic problems in graph theory and graph algorithms. Unfortunately, most of these problems are NP-complete [45, 53]. Therefore, they have been studied intensively within algorithmic paradigms for coping with NP-completeness [22, 26, 48], including approximation algorithms, parameterized complexity, and algorithms for restricted input classes.

Graph modification problems have played a central role in the development of parameterized complexity (see Section 1.2). Here, the number of allowed modifications, k, is considered a *parameter*. With respect to k, we seek a **Fixed Parameter Tractable (FPT)** algorithm, namely an algorithm whose running time has the form $f(k)n^{O(1)}$ for some computable function f. One way to obtain such an algorithm is to exhibit a *kernelization algorithm*, or *kernel*. A kernel for a graph problem Π is an algorithm that given an instance (G, k) of Π runs in polynomial time and outputs an equivalent instance (G', k') of Π such that $|V(G')|$ and k' are upper bounded by $f(k)$ for some computable function f. The function f is called the *size* of the kernel, and if f is a polynomial function, then we say that the kernel is a *polynomial kernel*. A kernel for a problem immediately implies that it admits an FPT algorithm, but kernels are also interesting in their own right. In particular, kernels allow us to model the performance of polynomial-time pre-processing algorithms. The field of kernelization has received a significant amount of attention, especially after the introduction of methods for showing kernelization lower bounds [6, 15, 16, 19, 25, 30, 31]. We refer to the surveys [24, 29, 41, 46], as well as the books [13, 18, 20, 51], for a detailed treatment of the area of kernelization. In this article, we study the kernelization complexity of modification (using vertex deletions) to interval graphs. A graph is an *interval graph* if it is the intersection graph of intervals on the real line. Formally, we study the following problem.

Interval Vertex Deletion (IVD)	**Parameter:** k
Input: A graph G and an integer k.	
Question: Does there exist a subset $S \subseteq V(G)$ of size at most k such that $G - S$ is an interval graph?	

Due to their intriguing combinatorial properties and many applications in diverse areas, such as industrial engineering and archeology [5, 38], the class of interval graphs is perhaps one of the most studied graph classes [8, 28]. Whether **Interval Vertex Deletion (IVD)** admits an FPT algorithm has been a long-standing open problem in the area until it was resolved by Cao and Marx [11], who gave an algorithm with running time $O(10^k n^9)$. Subsequently, Cao [9] designed an FPT algorithm with linear dependence on the input size, as well as slightly better dependence on the parameter k. More precisely, Cao's algorithm has running time $O(8^k(n+m))$. A natural follow-up question to this work, explicitly asked multiple times in the literature [14, 32, 34], is whether IVD admits a polynomial kernel. In this article, we resolve this question in the affirmative.

Theorem 1. *IVD admits a polynomial kernel.*

1.1 Methods

The first ingredient of our kernelization algorithm is the factor 8 polynomial-time approximation algorithm for IVD by Cao [9] (Theorem 6.1). We use this algorithm to obtain an approximate solution of size at most $8k$, or conclude that no solution of size at most k exists. By re-running the approximation algorithm on the graph with some of the vertices marked as "undeletable," we grow our approximate solution to a 9-*redundant* solution M of size $O(k^{10})$. Here, 9-redundancy

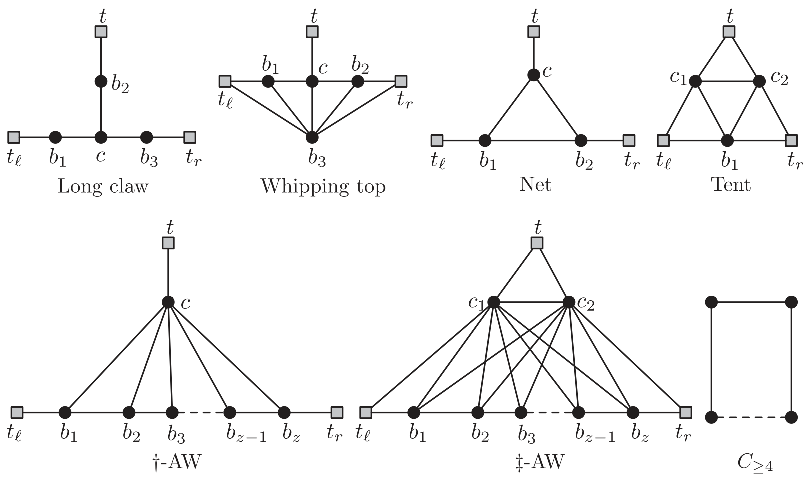

Fig. 1. The set of obstructions for an interval graph.

roughly means that for every subset $W \subseteq M$ of size at most 9, either $M \setminus W$ is also a solution or every solution S' of size at most $k + 2$ has non-empty intersection with W.[1]

Our kernelization heavily uses the characterization of interval graphs in terms of their *forbidden induced subgraphs*, also called *obstructions*. Specifically, a graph H is an obstruction to the class of interval graphs if H is not an interval graph, and for every vertex $v \in V(H)$ we have that $H - \{v\}$ is an interval graph. A graph G is an interval graph if and only if it does not contain any obstruction as an induced subgraph. The set of obstructions to interval graphs have been completely characterized by Lekkeikerker and Boland [44]. It consists of the *long claw*, the *whipping top*, the *net*, and the *tent*, as well as three infinite families of graphs: the *single-dagger asteroidal witness* (†-AW), the *double-dagger asteroidal witnesses* (‡-AW), and the cycle of length at least 4 (Figure 1). (The vertices t_ℓ, t_r, t in a †-AW and a ‡-AW are said to form an *asteroidal triple*.)

Having a 9-redundant solution yields the following advantage. In several places, we remove a carefully chosen vertex $v \notin M$ from G and claim that $G - \{v\}$ has a solution of size at most k if and only if G does. One direction of the equivalence is trivial. The interesting direction is to show that a solution X of size at most k to $G - \{v\}$ implies the existence of a solution of size at most k for G. The starting point for such an analysis is to ask why X is not already a solution for G. The only possible reason is that $G - X$ contains an obstruction $\mathbb{O}$, and $\mathbb{O}$ must contain v. We claim that $\mathbb{O}$ contains at least 10 vertices from M. Suppose not, then let W be the intersection of M and $\mathbb{O}$. We know that $(G - (M \setminus W))$ contains $\mathbb{O}$, and therefore it is not an interval graph. Hence, by the 9-redundancy of M, this implies that X (being a solution of size at most $k + 2$) must intersect $\mathbb{O}$, which contradicts the choice of $\mathbb{O}$. Thus, in this analysis, we only need to care about *large* obstructions that, furthermore, have a large intersection with M. This is crucial throughout the design and analysis of the kernel.

We then proceed to classify the connected components of $G - M$ based on whether they are *modules* in G or not. (Recall that a module is a set X such that all vertices in X have the same neighbors outside X.) For each component C that is not a module, there is an edge (u, v) in C and a vertex w in M such that w is adjacent to u but not to v. Thus, if there are more than $(k + 2)|M|$

[1]The precise definition in Section 3 contains another condition that is not specified in Section 1 for the sake of clarity of exposition.

non-module components in total, then there must exist $k+3$ non-module components and a vertex $w \in M$ such that each of these components has an edge (u, v), where w is adjacent to u but not to v. However, this means that for every subset $S \subseteq V(G)$ of size at most k, either $w \in S$ or $G - S$ contains a long claw (whose center c is w) and hence not interval. It follows that w must belong to every solution of size at most $k + 2$; thus, we can simply remove w and decrease the budget k by 1. Hence, the number of non-module components can be bounded by $(k + 2)|M|$, which is polynomial in k.

As $G-M$ is an interval graph, an obstruction cannot be entirely contained in $G-M$. In particular, if an obstruction contains a vertex from a connected component in $G - M$ that is a module in G, then this obstruction must also contain a vertex from M. From the preceding, we can obtain that every obstruction (with more than four vertices) can intersect every module component in at most one vertex. Furthermore, there is no point in keeping more than $k + 1$ copies of any vertex, and this allows us to reduce the module components to cliques of size $k + 1$.

We are left with the following situation. We have a 9-redundant solution M of size $O(k^{10})$. At most $O(k|M|)$ components of $G - M$ are not modules, but these components could be arbitrarily large. The remaining components are all modules that are cliques of size at most $k + 1$; thus, the module components are structured and small, but there could be arbitrarily many of them. This means that we are left with two tasks: (i) reduce the *number* of module components, and (ii) reduce the *size* of the non-module components. These two tasks can be approached separately, and both turn out to be non-trivial. Since both tasks are quite technically involved, we only give a few highlights in the remainder of this overview.

Bounding the Number of Module Components. Consider first the case where there are no non-module components at all, and every module component is a single vertex. In this case, $G - M$ is edgeless, so M is a *vertex cover* of G. The kernelization complexity of even this very special case was asked as an open problem by Fomin et al. [21].

A key ingredient in our solution to this special case is a new bound for the setting considered in the classic two families theorem of Bollobás [7]. Suppose there are two families of sets over a universe U, $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ such that every set A_i has size p, every set B_j has size q, and for every i, the sets A_i and B_i are disjoint, whereas for every $i \neq j$, the sets A_i and B_j intersect. The two families theorem gives an upper bound of $\binom{p+q}{p}$ for the size m of the family. The upper bound on m is *independent of the universe size*, and this has been extensively used in the design of parameterized algorithms [23, 49]. Further, when p or q is a *constant*, the bound is *polynomial* in $p + q$, and this has been extensively used in kernelization [42].

In our setting, neither the sets $A_1, \ldots, A_m$ nor the sets $B_1, \ldots, B_m$ have constant cardinality. However, we know that for every $i \neq j$, $|A_i \cap B_j| \in \{1, 2\}$. We prove that in this case, the bound is $O(|U|^2)$. More generally, we prove the following.

Lemma 1.1 (Bounded Intersection Two Families Lemma). *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be families over a universe U such that (i) for every $i \leq m$, $A_i \cap B_i = \emptyset$, and (ii) for every $j \neq i$, $|A_i \cap B_j| \in \{1, \ldots, c\}$. Then, $m \leq \sum_{t=0}^{c} \binom{|U|}{t}$.*

Comparing Lemma 1.1 with the two families theorem, the bound in Lemma 1.1 does depend on the universe size $|U|$. However, the exponent of $|U|$ only depends on the maximum cardinality c of the *intersection* between the sets A_i and B_j.

In the setting of kernelizing IVD parameterized by the size of a vertex cover M, the size of the kernel is intimately linked to m for the case where $A_1, \ldots, A_m$ is a collection of cliques in $G[M]$, whereas $B_1, \ldots, B_m$ is a collection of induced paths. Since a clique can only intersect an induced path in at most two vertices, we can apply Lemma 1.1 with $c = 2$, thereby obtaining an $O(|M|^2)$

bound for m and (after a significant amount of additional efforts, which we skip in this overview) a polynomial bound on the kernel size.

The kernel for IVD parameterized by the size of a vertex cover quite simply translates into a procedure that bounds the number, and therefore the total size, of module components of $G-M$. We remark that because the *number* of non-module components is bounded by $\mathcal{O}(k|M|)$, by bounding the number of module components we also bound the total number of components of $G - M$.

Bounding the Size of Non-Module Components. Suppose now that the number of module components has been bounded by $k^{O(1)}$. We can now include all of the module components in M and proceed under the assumption that there are no module components at all.

The size reduction of non-module components proceeds in three phases. In the first phase, we bound the maximum clique size in a component. Our clique-reduction procedure builds upon the clique-reduction procedure of Marx [50], which was used in kernelizations for Chordal Vertex Deletion [2, 35]. Both the procedure of Marx and ours are based on an "irrelevant vertex rule." However, our procedure is necessarily much more involved—our irrelevant vertex rule needs to preserve not only long induced cycles but also large single- and double-dagger asteroidal witnesses.

Having reduced the maximum clique size in the component, we proceed to the second phase, where we reduce the set of vertices that appear in at least two maximal cliques in the component. In this phase, we partition the component into $k^{O(1)}$ "long" and "thin" parts, and prove that an optimal solution will either not touch a part at all or it will cut it into two pieces using a minimal separator. Then, provided that a part is sufficiently large, we identify an edge e whose contraction does not decrease the size of any minimal separator inside the part. Thus, on the one hand, contracting e does not decrease the size of an optimal solution. On the other hand, contracting e—or any edge for that matter—cannot *increase* the size of an optimal solution (since interval graphs are closed under contraction).

After the second phase, the number of vertices appearing in at least two maximal cliques of the component is upper bounded by $k^{O(1)}$. In the third phase, we bound the number of the remaining vertices—these are the vertices that are "private" to some maximal clique of the component. At this point, we can take the set of vertices appearing in at least two components and add them to M. This makes M grow by $k^{O(1)}$ vertices, but now the large component breaks up into components whose size is not larger than that of a maximal clique—that is, $k^{O(1)}$. We can now re-apply the procedure for bounding the number of components, and this bounds the total number of vertices in G by $k^{O(1)}$. *We remark that, for technical reasons, in the actual proof, phases* 2 *and* 3 *as described here are interleaved.*

1.2 Related Work on Parameterized Graph Modification Problems

The $\mathcal{F}$-Vertex Deletion problems corresponding to the families of edgeless graphs, forests, chordal graphs, interval graphs, bipartite graphs, and planar graphs are known as Vertex Cover, Feedback Vertex Set, Chordal Vertex Deletion, IVD, Odd Cycle Transversal/Vertex Bipartization, and Planar Vertex Deletion, respectively. These problems are among the most well studied problems in the field of parameterized complexity. The study of parameterized graph deletion problems together with their various restrictions and generalizations has been an extremely active subarea over the past few years. In fact, just over the course of the past few years, there have been results on parameterized algorithms for Chordal Editing [12], Unit Interval Vertex (Edge) Deletion [10, 37], Interval Vertex (Edge) Deletion [9, 11], Planar $\mathcal{F}$ Deletion [22, 40], Planar Vertex Deletion [33], Block Graph Deletion [1, 39], and Simultaneous Feedback Vertex Set [4]. It is important to note that for many of these problems, polynomial kernels gave rise to several new techniques in the area. However, the problem that is closest to ours is the Chordal Vertex Deletion problem. In a recent breakthrough, Jansen and Pilipczuk [35, 36] gave a polynomial kernel (of size $\mathcal{O}(k^{162})$) for Chordal Vertex Deletion, resolving a more than a decade old open problem. Shortly afterward, Agrawal et al. [2, 3] gave a kernel of size $\mathcal{O}(k^{13})$.

2 PRELIMINARIES

We denote the set of natural numbers by $\mathbb{N}$. For $n \in \mathbb{N}$, we use $[n]$ and $[n]_0$ as shorthands for $\{1, 2, \ldots, n\}$ and $\{0, 1, \ldots, n\}$, respectively. For a set X and an integer $n \in \mathbb{N}$, by X^n we denote the set $\{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in X\}$.

Basic Graph Theory. We refer to standard terminology from the book of Diestel [17] for those graph-related terms that are not explicitly defined here. Consider a graph G. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G and $N_G[v] = N_G(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N_G(S) = (\cup_{v \in S} N_G(v)) \setminus S$. We omit the subscript G from the preceding two notations whenever the context is clear. Given a set $\mathcal{C}$ of connected components of G, denote $V(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} V(C)$. Moreover, when the graph G is clear from context, denote $n = |V(G)|$. Given a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. Accordingly, a graph H is an *induced subgraph* of G if there exists $U \subseteq V(G)$ such that $G[U]$ is isomorphic to H. For a set of vertices $X \subseteq V(G)$, $G - X$ denotes the induced subgraph $G[V(G) \setminus X]$—that is, the graph obtained by deleting the vertices in X from G. For an edge $(u, v) \in E(G)$, $G/(u, v)$ denotes the graph obtained by contracting the edge (u, v)—that is, the graph obtained by introducing a new vertex that is adjacent to all vertices in $N(u) \cup N(v)$ and deleting the vertices $\{u, v\}$. We say that G is a *clique* if for all distinct vertices $u, v \in V(G)$, we have that $(u, v) \in E(G)$, and that $V(G)$ is an *independent set* if for all distinct vertices $u, v \in V(G)$ we have that $(u, v) \notin E(G)$. A subset $U \subseteq V(G)$ is a *module* if for all $u, u' \in U$ and $v \in V(G) \setminus U$ either both u and u' are adjacent to v or both u and u' are not adjacent to v. For the sake of simplicity, we also call $G[U]$ a module (where we mean that it is a module in G) when the graph G is clear from the context.

A *path* $P = (x_1, x_2, \ldots, x_\ell)$ in G is a subgraph of G where $V(P) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(P) = \{(x_i, x_{i+1}) \mid i \in [\ell - 1]\} \subseteq E(G)$, where $\ell \in [n]$. The vertices x_1 and x_ℓ are the *endpoints* of P, and the remaining vertices in $V(P)$ are the *internal vertices* of P. A *cycle* $C = (x_1, x_2, \ldots, x_\ell)$ in G is a subgraph of G where $V(C) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(C) = \{(x_i, x_{i+1}) \mid i \in [\ell - 1]\} \cup \{(x_1, x_\ell)\} \subseteq E(G)$. We say that $(u, v) \in E(G)$ is a *chord* of a path P if $u, v \in V(P)$ but $(u, v) \notin E(P)$. Similarly, we say that $(u, v) \in E(G)$ is a *chord* of a cycle C if $u, v \in V(C)$ but $(u, v) \notin E(C)$. A path P or cycle C is said to be *induced* (or, alternatively, *chordless*) if it has no chords.

Interval Graphs. An *interval graph* is a graph that does not contain any of the following graphs, called *obstructions*, as an induced subgraph (see Figure 1):

- *Long claw*: A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{(t_\ell, b_1), (t_r, b_3), (t, b_2), (c, b_1), (c, b_2), (c, b_3)\}$.
- *Whipping top*: A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{(t_\ell, b_1), (t_r, b_2), (c, t), (c, b_1), (c, b_2), (b_3, t_\ell), (b_3, b_1), (b_3, c), (b_3, b_2), (b_3, t_r)\}$.
- †-AW: A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{(t, c), (t_\ell, b_1), (t_r, b_z)\} \cup \{(c, b_i) \mid i \in [z]\} \cup \{(b_i, b_{i+1}) \mid i \in [z - 1]\}$, and $z \geq 2$. A †-AW where $z = 2$ will be called a *net*.
- ‡-AW: A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c_1, c_2\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{(t, c_1), (t, c_2), (c_1, c_2), (t_\ell, b_1), (t_r, b_z), (t_\ell, c_1), (t_r, c_2)\} \cup \{(c, b_i) \mid i \in [z]\} \cup \{(b_i, b_{i+1}) \mid i \in [z - 1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ will be called a *tent*.
- *Hole*: A chordless cycle on at least four vertices.

We refer to †-AW and ‡-AW as AWs. In each of the first four obstructions, the vertices t_ℓ, t_r, and t are called *terminals*; the vertices c, c_1, and c_2 are called *centers*; and the other vertices are called *base vertices*. Furthermore, the vertex t is called the *shallow terminal* and the vertices t_ℓ and t_r are called the *non-shallow terminals*. In the case where $\mathbb{O}$ is one of the AWs, the induced path on the set of base vertices is called the *base* of the AW, and it is denoted by base($\mathbb{O}$). Moreover, we say that the induced path on the set of base vertices, t_ℓ, and t_r is the *extended base* of the AW, and it is denoted by $P(\mathbb{O})$.

Path Decomposition. A *path decomposition* of a connected graph G is a pair (P, β), where P is a path and $\beta : V(P) \to 2^{V(G)}$ is a function that satisfies the following properties:

(1) $\bigcup_{x \in V(P)} \beta(x) = V(G)$.
(2) For any edge $(u, v) \in E(G)$, there is a node $x \in V(P)$ such that $u, v \in \beta(x)$.
(3) For any $v \in V(G)$, the collection of nodes $P_v = \{x \in V(P) \mid v \in \beta(x)\}$ is a subpath of P.

For $v \in V(P)$, we call $\beta(v)$ the *bag* of v. We refer to the vertices in $V(P)$ as nodes. A *clique path* of a connected graph G is a path decomposition of G where every bag is a distinct maximal clique. If a graph G admits a clique path, then we say that G is a clique path. The following proposition states that the class of interval graphs is exactly the class of graphs where each connected component is a clique path.

Proposition 2.1 ([27], Sections 2 and 3 of [28]). *A graph is an interval graph if and only if each connected component of it is a clique path. Moreover, such a clique path can be found in linear time.*

Parameterized Complexity. Let Π be an NP-hard problem. In the framework of parameterized complexity, each instance of Π is associated with an integer k, which is called the *parameter*. Here, the goal is to confine the combinatorial explosion in the running time of an algorithm for Π to depend only on k. The main concepts defined to achieve this goal are of *fixed-parameter tractability* and *kernelization*. First, we say that Π is FPT if any instance (I, k) of Π is solvable in time $f(k) \cdot |I|^{O(1)}$, where $f(\cdot)$ is an arbitrary (computable) function of k. Second, Π is said to admit a *polynomial kernel* if there is a polynomial-time algorithm (the degree of polynomial is independent of the parameter k), called a *kernelization algorithm*, that transforms the input instance into an equivalent instance of Π whose size is bounded by a polynomial $p(k)$ in k. Here, two instances are equivalent if one of them is a Yes-instance if and only if the other one is a Yes-instance. The reduced instance is called a *$p(k)$-kernel* for Π. For a detailed introduction to the field of kernelization, we refer to the following surveys [41, 46] and the corresponding chapters in the following books [13, 18, 20, 51].

Kernelization algorithms often rely on the design of *reduction rules*. The rules are numbered, and each rule consists of a condition and an action. We always apply the first rule whose condition is true. Given a problem instance (I, k), the rule computes (in polynomial time) an instance (I', k') of the same problem, where $k' \leq k$. Typically, $|I'| < |I|$, where if this is not the case, it should be argued why the rule can be applied only polynomially many times. We say that the rule *safe* if the instances (I, k) and (I', k') are equivalent.

Linear Algebra. For a set A and X, by an *operation of A onto X* we mean a function $f : A \times X \to X$. For an element $(a, x) \in A \times X$ by ax, we denote the element $f(a, x) \in X$. For a field $\mathbb{F}$ with $+$ as the additive operation and $\cdot$ as the multiplicative operation, a commutative group $(V, +)$ with an operation of $\mathbb{F}$ onto V is a *vector space over* $\mathbb{F}$ if for all $a, b \in \mathbb{F}$ and $x, y \in V$, we have (i) $a(bx) = (ab)x$, (ii) $a(x + y) = ax + ay$, (iii) $(a + b)x = ax + bx$, and (iv) $1 \cdot x = x$. Here, 1 is the multiplicative identity of the field $\mathbb{F}$. If V is a vector space over $\mathbb{F}$, then the elements of V are called *vectors*. One of the natural candidates for vector spaces over a field $\mathbb{F}$ is $\mathbb{F}^n$, where $n \in \mathbb{N}$ and the function $f(\cdot)$ is the component-wise multiplication. In this article, we restrict ourselves only to such types of vector spaces.

In the following, consider a field $\mathbb{F}$ and a vector space $V = \mathbb{F}^n$, where $n \in \mathbb{N}$. For a vector $\mathbf{v} = (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$ and an integer $i \in [n]$, by $\mathbf{v}[i]$ we denote the i^{th} element (or entry) of $\mathbf{v}$ (i.e., the element b_i). For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t \in \mathbb{F}^n$, a linear combination of them is a vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_t\mathbf{v}_t$, where $a_1, a_2, \ldots, a_t \in \mathbb{F}$. Furthermore, a *linear relation* among them is exhibited when $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_t\mathbf{v}_t = 0$, for some $a_1, a_2, \ldots, a_t \in \mathbb{F}$. In the preceding, the a_is are called the *coefficients*. A set of vectors is said to be *linearly independent* if there is no linear relation among them except the trivial one, where each of the coefficients is 0. A set of vectors that is not linearly independent is said to be *linearly dependent*. An inclusion-wise maximal set of linearly independent vectors is called a *basis* of the vector space. It is known that for bases B, B' of a vector space, we have $|B| = |B'|$. By $\mathbb{F}_2$, we denote the field with exactly two elements, namely 0 and 1, with the usual addition and multiplication modulo 2 as the field operations. For two vectors $\mathbf{u}, \mathbf{v} \in V'$, $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product of these two vectors. We refer the reader to the work of Lay [43] for more details on linear algebra.

Matroids. A pair $\mathcal{M} = (E, \mathcal{I})$, where E is a set (called *ground set*) and $\mathcal{I}$ is a family of subsets of E (called *independent sets*) is called a *matroid* if the following conditions are satisfied:

- $\emptyset \in \mathcal{I}$;
- If $A \in \mathcal{I}$ and $A' \subseteq A$, then $A' \in \mathcal{I}$;
- If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there is $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{I}$.

An inclusion-wise maximal set in $\mathcal{I}$ is called a *basis* of $\mathcal{M}$. All the bases of a matroid are of same size. The size of a basis is called the *rank of the matroid*. One of the important notions of a matroid that we use is linear representations of matroids.

A matroid is a *linear matroid* (or *representable matroid*) if, for some field $\mathbb{F}$, it can be defined as follows. Let A be a matrix over a field $\mathbb{F}$ and E its set of columns. Then, the matroid $\mathcal{M} = (E, \mathcal{I})$ is defined as follows: a subset $X \subseteq E$ is an independent set in $\mathcal{M}$ if and only if the set of columns in X is linearly independent over $\mathbb{F}$. The matrix A is called a *representation* of $\mathcal{M}$, and $\mathcal{M}$ is said to be *representable over* $\mathbb{F}$. Thus, a matroid is linear (alternatively, representable) if it is representable over some field $\mathbb{F}$. We refer the reader to the work of Oxley [52] for more details on matroids.

For $n, k \in \mathbb{N}$, where $k \leq n$, a pair $\mathcal{M} = (E, \mathcal{I})$, where $|E| = n$ is a k-*uniform matroid* (or simply, a *uniform matroid*) if $\mathcal{I} = \{X \subseteq E \mid |X| \leq k\}$, where $k \in [n]$; such a matroid will be denoted by $U_{n,k}$. The uniform matroid $U_{n,k}$ is representable over any field with at least $n + 1$ elements, and a representation for it can be found in polynomial time (e.g., see Section 12.1.2 [13]).

q-Representative Family. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $\mathcal{B}$ be a family of subsets of size p of E. We say that $\widehat{\mathcal{B}} \subseteq \mathcal{B}$ is a *q-representative for* $\mathcal{B}$ if for every set $Y \subseteq E$ of size q, if there is a set $X \in \mathcal{B}$ such that $X \cap Y = \emptyset$ and $X \cup Y \in \mathcal{I}$, then there is a set $\widehat{X} \in \widehat{\mathcal{B}}$ such that $\widehat{X} \cap Y = \emptyset$ and $\widehat{X} \cup Y \in \mathcal{I}$. If $\widehat{\mathcal{B}} \subseteq \mathcal{B}$ is a q-representative for $\mathcal{B}$, then we use the notation $\widehat{\mathcal{B}} \subseteq^q_{rep} \mathcal{B}$. The following result asserts that small representative families can be computed efficiently.

Proposition 2.2 ([23]). *Let $\mathcal{M} = (E, \mathcal{I})$ be a linear matroid of rank $k = p + q$, and let matrix $A_{\mathcal{M}}$ be a representation of $\mathcal{M}$ over a field $\mathbb{F}$. Additionally, let $\mathcal{B} = \{B_1, B_2, \ldots, B_t\}$ be a family of independent sets of size p over E. Then, there exists $\widehat{\mathcal{B}} \subseteq^q_{rep} \mathcal{B}$ of size at most $\binom{p+q}{p}$. Moreover, such $\widehat{\mathcal{B}}$ can be computed in at most $O(\binom{p+q}{p} tp^{\omega} + t\binom{p+q}{p}^{\omega-1})$ operations over $\mathbb{F}$. Here, ω is the exponent in the running time of matrix multiplication.*

3 COMPUTING A REDUNDANT SOLUTION

Let (G, k) be an instance of IVD. A subset $S \subseteq V(G)$ such that $G - S$ is an interval graph is called a *solution*, and a solution of size at most t is called a *t-solution*. Toward the definition of redundancy, we need to introduce a few simple notions related to hitting and covering. Given a family $\mathcal{W} \subseteq 2^{V(G)}$, we say that a subset $S \subseteq V(G)$ *hits* $\mathcal{W}$ if for all $W \in \mathcal{W}$ we have $S \cap W \neq \emptyset$. A family $\mathcal{W} \subseteq 2^{V(G)}$ is *t-necessary* if every solution of size at most t hits $\mathcal{W}$. Moreover, we say that an obstruction $\mathbb{O}$ is *covered by* $\mathcal{W}$ if there exists $W \in \mathcal{W}$ such that $W \subseteq V(\mathbb{O})$. Now, we are ready to formally define our notion of redundancy.

Definition 3.1. Given a family $\mathcal{W} \subseteq 2^{V(G)}$ and $t \in \mathbb{N}$, a subset $M \subseteq V(G)$ is *t-redundant with respect to* $\mathcal{W}$ if for every obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$ it holds that $|M \cap V(\mathbb{O})| > t$.

The purpose of this section is to prove Lemma 3.2. Intuitively, this lemma asserts that an r-redundant solution M whose size is polynomial in k (for a fixed constant r) can be computed in polynomial time. Such a set M plays a central role in all of our subsequent reduction rules that comprise our kernelization algorithm. We remark that in this statement we use the letter ℓ rather than k to avoid confusion, as we will use this result with $\ell = k + 2$.

Lemma 3.2. *Let $r \in \mathbb{N}$ be a fixed constant, and let (G, ℓ) be an instance of* IVD*. In polynomial time, it is possible to either conclude that (G, ℓ) is a No-instance, or compute an ℓ-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ and a set $M \subseteq V(G)$, such that $\mathcal{W} \subseteq 2^M$ and M is a $(r+1)(8\ell)^{r+1}$-solution that is r-redundant with respect to $\mathcal{W}$.*

A central component in our proof of Lemma 3.2 is an approximation algorithm for IVD, given by Cao [9].

Proposition 3.3 (Theorem 6.1 [9]). IVD *admits a polynomial-time 8-approximation algorithm, called* `ApproxIVD`*.*

In particular, a main idea in our proof is to iteratively grow the redundancy of a solution by making calls to this approximation algorithm. Besides Proposition 3.3, toward the proof of Lemma 3.2, we give a simple definition of a graph on which we will apply the approximation algorithm and hence determine whether a set of vertices should be added to $\mathcal{W}$.

Definition 3.4. Let G be a graph, $U \subseteq V(G)$, and $t \in \mathbb{N}$. Then, $\text{copy}(G, U, t)$ is defined as the graph G' on the vertex set $V(G) \cup \{v^i \mid v \in U, i \in [t]\}$ and the edge set $E(G) \cup \{(u^i, v) \mid (u, v) \in E(G), u \in U, i \in [t]\} \cup \{(u^i, v^j) \mid (u, v) \in E(G), u, v \in U, i, j \in [t]\} \cup \{(v, v^i) \mid v \in U, i \in [t]\} \cup \{(v^i, v^j) \mid v \in U, i, j \in [t], i \neq j\}$.

Informally, $\text{copy}(G, U, t)$ is simply the graph G where for every vertex $u \in U$ we add t twins that (together with u) form a clique. Intuitively, this operation allows us to make a vertex set "undeletable"; in particular, this enables us to test later whether a vertex set is "redundant" and hence we can grow the redundancy of our solution, or whether it is "necessary" and hence we should update $\mathcal{W}$ accordingly. Before we turn to discuss computational issues, let us first assert that the operation in Definition 3.4 does not makes an interval graph become a non-interval graph. This is a basic requirement to verify before turning to design the preceding test.

Lemma 3.5. *Let G be a graph, $U \subseteq V(G)$, and $t \in \mathbb{N}$. If G is an interval graph, then $G' = \text{copy}(G, U, t)$ is an interval graph as well.*

Proof. Suppose that G is an interval graph. Then, by Proposition 2.1, G admits a clique path (P, β). Now, we define (P', β') as follows: $P' = P$, and for all $x \in V(P')$, $\beta'(x) = \beta(x) \cup \{v^i \mid v \in \beta(x) \cap U, i \in [t]\}$. We claim that (P', β') is a clique path for G'. By using the fact that (P, β) is a path decomposition of G, we directly have the following properties. First, it is clear that $\bigcup_{x \in V(P')} \beta'(x) = V(G')$. Second, for any edge $e = (u, v) \in E(G')$ such that $u, v \in V(G)$, there exists $x_e \in V(P')$ such that $u, v \in \beta'(x_e)$. Then, since for all $v \in U$ and $i \in [t]$ it holds that $\beta'^{-1}(v) = \beta'^{-1}(v^i)$, we derive that for any edge $(u', v') \in E(G')$ there is a node $x \in V(P')$ such that $u', v' \in \beta'(x)$. Third, for any $v \in V(G)$, the collection of nodes $P'_v = \{x \in V(P') \mid v \in \beta'(u)\}$ is a subpath of P', and since for any $v \in U$ and $i \in [t]$ it holds that $\beta'^{-1}(v) = \beta'^{-1}(v^i)$, we derive that for any $v' \in V(G')$ the collection of nodes $P'_{v'} = \{x \in V(P') \mid v' \in \beta'(x)\}$ is a subpath of P'. Now, note that for all $x \in V(P')$, $\beta(x)$ is a clique, and for all $u, v \in \beta(x)$ (possibly $u = v$) and $i, j \in [t]$, u^i is adjacent to u, u^j (if $i \neq j$), v and v^j, which implies that $\beta'(x)$ is also a clique path. Hence, (P', β') is indeed clique path for G'. By Proposition 2.1, we derive that G' is an interval graph. □

Now, let us present two simple claims that exhibit relations between the algorithm ApproxIVD and Definition 3.4. After presenting these two claims, we will be ready to give our algorithm for computing a redundant solution. Roughly speaking, the first claim exhibits the meaning of a situation where ApproxIVD returns a "large" solution; intuitively, for the purpose of the design of our algorithm, we interpret this meaning as an indicator to extend $\mathcal{W}$.

Lemma 3.6. *Let G be a graph, $U \subseteq V(G)$, and $\ell \in \mathbb{N}$. If the algorithm* ApproxIVD *returns a set A of size larger than 8ℓ when called with $G' = \text{copy}(G, U, 8\ell)$ as input, then $\{U\}$ is ℓ-necessary.*

Proof. Suppose that ApproxIVD returns a set A of size larger than 8ℓ when called with G' as input. Then, (G', ℓ) is a No-instance. Suppose, by way of contradiction, that $\{U\}$ is not ℓ-necessary. If (G, ℓ) is a No-instance, then trivially we can say that $\{U\}$ is ℓ-necessary (as there is no solution of size at most ℓ, so the statement is vacuously true). Now consider the case when G has an ℓ-solution S such that $S \cap U = \emptyset$. In particular, $\widehat{G} = G - S$ is an interval graph such that $U \subseteq V(\widehat{G})$. However, this means that $\text{copy}(\widehat{G}, U, 8\ell) = G' - S$, which by Lemma 3.5 implies that $G' - S$ is an interval graph. Thus, S is an ℓ-solution for G', which is a contradiction (as (G', ℓ) is a No-instance). □

Complementing our first claim, the second claim exhibits the meaning of a situation where ApproxIVD returns a "small" solution A; we interpret this meaning as an indicator to grow the redundancy of our current solution M by adding A—indeed, this lemma implies that every obstruction is hit one more time when adding A to a subset $U \subseteq M$ (to grow the redundancy of M, every subset $U \subseteq M$ will have to be considered).

Lemma 3.7. *Let G be a graph, $U \subseteq V(G)$, and $\ell \in \mathbb{N}$. If the algorithm* ApproxIVD *returns a set A of size at most 8ℓ when called with $G' = \text{copy}(G, U, 8\ell)$ as input, then for every obstruction $\mathbb{O}$ of G, $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$.*

Proof. Suppose that ApproxIVD returned a set A of size at most 8ℓ when called with G' as input. Let $\mathbb{O}$ be some obstruction of G, and denote $B = V(\mathbb{O}) \cap U$. Since $|A| \leq 8\ell$, for every vertex $v \in B$, we have that $v \in V(G') \setminus A$ or there exists $i(v) = i \in [8\ell]$ such that $v^i \in V(G') \setminus A$. Moreover, we have that the graph obtained from $\mathbb{O}$ by replacing each vertex $v \in B \cap A$ by $v^{i(v)}$ is an obstruction (as v and $v^{i(v)}$ are twins). Thus, as A is a solution for G', there exists $v \in V(G) \setminus B$ such that $v \in A \cap V(\mathbb{O})$. Hence, we have that $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$. □

Now, let us describe our algorithm, RedundantIVD, to compute a redundant solution. First, RedundantIVD initializes M_0 to be the output obtained by calling the algorithm ApproxIVD with G as input, $\mathcal{W}_0 := \emptyset$ and $\mathcal{T}_0 := \{(v) \mid v \in M_0\}$. If $|M_0| > 8\ell$, then RedundantIVD concludes that (G, ℓ) is a No-instance. Otherwise, for $i = 1, 2, \ldots, r$ (in this order), the algorithm executes the following steps:

(1) Initialize $M_i := M_{i-1}$, $\mathcal{W}_i := \mathcal{W}_{i-1}$ and $\mathcal{T}_i := \emptyset$.
(2) For every tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$:
   (a) Let A be the output obtained by calling the algorithm `ApproxIVD` with copy $(G, \{v_0, v_1, \ldots, v_{i-1}\}, 8\ell)$ as input.
   (b) If $|A| > 8\ell$, then insert $\{v_0, v_1, \ldots, v_{i-1}\}$ into $\mathcal{W}_i$.
   (c) Otherwise, insert every vertex in $(A \cap V(G)) \setminus \{v_0, v_1, \ldots, v_{i-1}\}$ into M_i, and for all $u \in (A \cap V(G)) \setminus \{v_0, v_1, \ldots, v_{i-1}\}$, insert $(v_0, v_1, \ldots, v_{i-1}, u)$ into $\mathcal{T}_i$.

Eventually, the algorithm outputs the pair $(M_r, \mathcal{W}_r)$.

Let us comment that in this algorithm, we make use of the sets $\mathcal{T}_{i-1}$ rather than going over all subsets of size i of M_{i-1} to obtain a substantially better algorithm in terms of the size of the produced redundant solution.

The properties of the algorithm `RedundantIVD` that are relevant to us are summarized in the following lemma and observation, which are proved by induction and by making use of Lemmas 3.5, 3.6, and 3.7. Roughly speaking, we first assert that, unless (G, ℓ) is concluded to be a No-instance, we compute sets $\mathcal{W}_i$ that are ℓ-necessary as well as that the tuples in $\mathcal{T}_i$ "hit more vertices" of the obstructions in the input as i grows larger.

Lemma 3.8. *Consider a call to* `RedundantIVD` *with* (G, ℓ, r) *as input that did not conclude that* (G, ℓ) *is a No-instance. For all* $i \in [r]_0$*, the following conditions hold:*

(1) *For any set* $W \in \mathcal{W}_i$*, every solution* S *of size at most* ℓ *satisfies* $W \cap S \neq \emptyset$.
(2) *For any obstruction* $\mathbb{O}$ *of* G *that is not covered by* $\mathcal{W}_i$*, there exists* $(v_0, v_1, \ldots, v_i) \in \mathcal{T}_i$ *such that* $\{v_0, v_1, \ldots, v_i\} \subseteq V(\mathbb{O})$.

Proof. The proof is by induction on i. In the base case, where $i = 0$, Condition 1 trivially holds as $\mathcal{W}_0 = \emptyset$, and thus there are no sets in $\mathcal{W}_0$. Condition 2 holds as M_0 is a solution (so each obstruction must contain at least one vertex from M_0) and $\mathcal{T}_0$ simply contains a 1-vertex tuple for every vertex in M_0. Now, suppose that the claim is true for $i - 1 \geq 0$, and let us prove it for i.

To prove Condition 1, consider some set $W \in \mathcal{W}_i$. If $W \in \mathcal{W}_{i-1}$, then by the inductive hypothesis, every solution of size at most ℓ satisfies $W \cap S \neq \emptyset$. Thus, we next suppose that $W \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}$. Then, there exists a tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$ in whose iteration `RedundantIVD` inserted $W = \{v_0, v_1, \ldots, v_{i-1}\}$ into $\mathcal{W}_i$. In that iteration, `ApproxIVD` was called with copy$(G, W, 8\ell)$ as input and returned a set A of size larger than 8ℓ. Thus, by Lemma 3.6, every solution S of size at most ℓ satisfies $W \cap S \neq \emptyset$.

To prove Condition 2, consider some obstruction $\mathbb{O}$ of G that is not covered by $\mathcal{W}_i$. By the inductive hypothesis and since $\mathcal{W}_{i-1} \subseteq \mathcal{W}_i$, there exists a tuple $(v_0, v_1, \ldots, v_{i-1}) \in \mathcal{T}_{i-1}$ such that $\{v_0, v_1, \ldots, v_{i-1}\} \subseteq V(\mathbb{O})$. Consider the iteration of `RedundantIVD` corresponding to this tuple, and denote $U = \{v_0, v_1, \ldots, v_{i-1}\}$. In that iteration, `ApproxIVD` was called with copy$(G, U, 8\ell)$ as input and returned a set A of size at most 8ℓ. By Lemma 3.7, $|V(\mathbb{O}) \cap U| + 1 \leq |V(\mathbb{O}) \cap (U \cup (A \cap V(G)))|$. Thus, there exists $v_i \in (A \cap V(G)) \setminus U$ such that $U \cup \{v_i\} \subseteq V(\mathbb{O})$. However, by the specification of `ApproxIVD`, this means that there exists $(v_0, v_1, \ldots, v_i) \in \mathcal{T}_i$ such that $\{v_0, v_1, \ldots, v_i\} \subseteq V(\mathbb{O})$. □

Toward showing that the output set M_r is "small," let us upper bound the sizes of the sets M_i and $\mathcal{T}_i$.

Observation 3.9. *Consider a call to* `RedundantIVD` *with* (G, ℓ, r) *as input that did not conclude that* (G, ℓ) *is a No-instance. For all* $i \in [r]_0$, $|M_i| \leq \sum_{j=0}^{i} (8\ell)^{j+1}$, $|\mathcal{T}_i| \leq (8\ell)^{i+1}$*, and every tuple in* $\mathcal{T}_i$ *consists of distinct vertices.*

Proof. The proof is by induction on i. In the base case, where $i = 0$, the correctness follows as `ApproxIVD` returned a set of size at most 8ℓ. Now, suppose that the claim is true for $i - 1 \geq$

0, and let us prove it for i. By the specification of the algorithm and inductive hypothesis, we have that $|M_i| \leq |M_{i-1}| + 8\ell|\mathcal{T}_{i-1}| \leq \sum_{j=1}^{i+1}(8\ell)^j$ and $|\mathcal{T}_i| \leq 8\ell|\mathcal{T}_{i-1}| \leq (8\ell)^{i+1}$. Moreover, by the inductive hypothesis, for every tuple in $\mathcal{T}_i$, the first i vertices are distinct, and by the specification of `ApproxIVD`, the last vertex is not equal to any of them. □

By the specification of `RedundantIVD`, as a corollary to Lemma 3.8 and Observation 3.9, we directly obtain the following result.

Corollary 3.10. *Consider a call to* `RedundantIVD` *with* (G, ℓ, r) *as input that did not conclude that* (G, ℓ) *is a No-instance. For all* $i \in [r]_0$, $\mathcal{W}_i$ *is an* ℓ*-necessary family and* M_i *is a* $\sum_{j=0}^{i}(8\ell)^{j+1}$*-solution that is* i*-redundant with respect to* $\mathcal{W}_i$.

Clearly, `RedundantIVD` runs in polynomial time (as r is a fixed constant), and by the correctness of `ApproxIVD`, if it concludes that (G, ℓ) is a No-instance, then this decision is correct. Thus, since $\sum_{i=0}^{r}(8\ell)^{i+1} \leq (r+1)(8\ell)^{r+1}$, the correctness of Lemma 3.2 now directly follows as a special case of Corollary 3.10. Thus, our proof of Lemma 3.2 is complete.

In light of Lemma 3.2, from now on, we suppose that we have a $(k+2)$-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ along with a $(r+1)(8(k+2))^{r+1}$-solution M that is r-redundant with respect to $\mathcal{W}$ for $r = 9$. Let us note that any obstruction in G that is not covered by $\mathcal{W}$ intersects M in at least 10 vertices. We have the following reduction rule that follows immediately from Lemma 3.8.

Reduction Rule 3.1. Let v be a vertex such that $\{v\} \in \mathcal{W}$. Then, output the instance $(G - \{v\}, k-1)$.

Henceforward, we will assume that each set in $\mathcal{W}$ has size at least 2.

4 HANDLING MODULE COMPONENTS

Let (G, k) be an instance of IVD. We will assume that $k \geq 2$, as otherwise, in polynomial time, we can check whether or not (G, k) is a Yes-instance and accordingly return a trivial kernel of constant size. Let us explicitly recap the steps taken so far and then state our current objective in this context. First, we call Lemma 3.2 with $r = 9$ and $\ell = k+2$,[2] and one of the following holds. If (in polynomial time) we conclude that $(G, k+2)$ is a No-instance, then we can (correctly) conclude that (G, k) is a No-instance as well. Otherwise, in polynomial time, we obtain a $(k+2)$-necessary family $\mathcal{W} \subseteq 2^{V(G)}$ and a set $M \subseteq V(G)$ such that $\mathcal{W} \subseteq 2^M$ and M is a $10(8(k+2))^{10}$-solution that is 9-redundant with respect to $\mathcal{W}$. Furthermore, each set in $\mathcal{W}$ has size at least 2. The main goal of this section is to bound the total number of vertices across all module connected components of $G - M$. We remark that we will prove a slightly more general result, as it will be used later in our algorithm. Before that, we provide a simple reduction rule to bound the number of non-module components.

Bounding the Number of Non-Module Components. Let C denote the set of connected components of $G - M$. Moreover, we let $\mathcal{D}$ denote the set of connected components in C that are modules, and $\overline{\mathcal{D}} = C \setminus \mathcal{D}$. To bound the size of $\overline{\mathcal{D}}$, we apply the following reduction rule.

Reduction Rule 4.1. Suppose that there exist $v \in M$ and a set $\mathcal{A} \subseteq \overline{\mathcal{D}}$ of size $k+3$ such that for each $D \in \mathcal{A}$ there exist $u, w \in V(D)$ such that $u \in N_G(v)$ and $w \notin N_G(v)$. Then, output the instance $(G - \{v\}, k-1)$.

[2]We use Lemma 3.2 with $\ell = k+2$ because at a later stage (particularly, in Section 6) we find an irrelevant edge to contract. With the parameter $k+2$, we are still able to exclude the need to argue about obstructions that are covered by $\mathcal{W}$, as the additional 2 allows us to add the two endpoints of the contracted edge to an assumed solution in our arguments. We use the lemma with $r = 9$ since it helps us to find large obstructions that contain enough vertices from M in base($\mathbb{O}$), for an AW $\mathbb{O}$.

Lemma 4.1. *Reduction Rule 4.1 is safe.*

Proof. In one direction, suppose that (G, k) is a Yes-instance, and let S be a k-solution for G. Since $|\mathcal{A}| \geq k+3$, there exist three connected components $D_1, D_2, D_3 \in \mathcal{A}$ such that $S \cap (V(D_1) \cup V(D_2) \cup V(D_3)) = \emptyset$. However, for each $i \in [3]$, the subgraph of G induced by the vertex set consisting of v, together with an edge e in D_i with one endpoint of e being a neighbor of v and the other endpoint of e being a non-neighbor of v, is a long claw. Here, we relied on the fact that for each $i \in [3]$, D_i is connected. Thus, as $G - S$ is an interval graph, we derive that $v \in S$, and therefore $S \setminus \{v\}$ is a $(k-1)$-solution for $G - \{v\}$.

In the other direction, it is clear that if $(G - \{v\}, k-1)$ is a Yes-instance, then (G, k) is a Yes-instance. □

We now observe that our rule indeed bounds the size of $\overline{\mathcal{D}}$.

Observation 4.2. *After the exhaustive application of Reduction Rule 4.1, $|\overline{\mathcal{D}}| \leq (k+2)|M|$.*

Proof. After the exhaustive application of Reduction Rule 4.1, every vertex in M has at most $k+2$ connected components in C where it has both a neighbor and a non-neighbor. Since for a connected component in $\overline{\mathcal{D}}$ that is not a module there must exist a vertex in M that has both a neighbor and a non-neighbor in that component, we conclude that the observation is correct. □

The Main Lemma of This Section. From now on, we focus on the main goal of this section: bound the total number of vertices in $\mathcal{D}$. As mentioned earlier, the arguments used to derive this bound will also be necessary at a later stage of our kernelization algorithm, and hence we present our goal in the form of a more general statement.

Lemma 4.3. *Let $\widehat{M} \subseteq V(G)$, and let $\widehat{C}$ be some set of connected components of $G - (M \cup \widehat{M})$ that are modules. In polynomial time, it is possible to either output an instance (G', k) equivalent to (G, k) where $|V(G')| < |V(G)|$, or to compute a subset $B \subseteq V(\widehat{C})$ of size at most $8(k+1)^3|M \cup \widehat{M}|^{10}$, such that for any subset $S \subseteq V(G)$ of size at most k, the following property holds: If there exists an obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, then there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}') \cap S = \emptyset$ and $V(\mathbb{O}') \cap (V(\widehat{C}) \setminus B) = \emptyset$.*

Intuitively, the statement of this lemma expands M to $M \cup \widehat{M}$ and zooms into a subset $\widehat{C}$ of the set of connected components in $G - (M \cup \widehat{M})$ that are modules in G. Then, either it enables us to reduce the instance, or it produces a "small" subset $B \subseteq V(\widehat{C})$ and implies that we need not "worry" about obstructions that intersect $V(\widehat{C})$ but not B—if such an obstruction is not hit, then there is an obstruction that does not intersect $V(\widehat{C}) \setminus B$ and which is not hit as well.

Let us now show that having Lemma 4.3 at hand, we can indeed bound the total number of vertices in all module components.

Reduction Rule 4.2. Let X be the output of the algorithm in Lemma 4.3 when called with $\widehat{M} = \emptyset$ and $\widehat{C} = \mathcal{D}$. If X is an instance (G', k), then output X. Otherwise, X is a set $B \subseteq V(\mathcal{D})$, and we output the instance $(G - \{v\}, k)$ for a vertex v arbitrarily chosen from $V(\mathcal{D}) \setminus B$.

By using Lemma 4.3, we derive the safeness of Reduction Rule 4.2.

Lemma 4.4. *Reduction Rule 4.2 is safe.*

Proof. If X is an instance (G', k), then Lemma 4.3 directly implies that the rule is safe. Thus, we next suppose that $X = B$. In one direction, it is clear that if (G, k) is a Yes-instance, then $(G - \{v\}, k)$ is a Yes-instance as well.

In the other direction, suppose that $(G-\{v\},k)$ is a Yes-instance. Let S be a k-solution for $G-\{v\}$. We claim that S is also a k-solution for G. Suppose, by way of contradiction, that this claim is false. Then, there exists an obstruction $\mathbb{O}$ for $G-S$. As $S\cup\{v\}$ is a $(k+1)$-solution for G and $\mathcal{W}$ is $(k+2)$-necessary, we have that $S\cup\{v\}$ hits $\mathcal{W}$. Since $v\notin M$ and $\mathcal{W}\subseteq 2^M$, we derive that S hits $\mathcal{W}$. Thus, since $\mathbb{O}$ is an obstruction for $G-S$, we deduce that $\mathbb{O}$ is not covered by $\mathcal{W}$. Hence, by Lemma 4.3, there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}')\cap S=\emptyset$ and $V(\mathbb{O}')\cap(V(\mathcal{D})\setminus B)=\emptyset$. However, as $v\in V(\mathcal{D})\setminus B$, this implies that $\mathbb{O}'$ is also an obstruction for $(G-\{v\})-S$, which is a contradiction as S is a k-solution for $G-\{v\}$. □

Due to Reduction Rule 4.2, we have the following result.

Observation 4.5. *After the exhaustive application of Reduction Rule 4.2,* $|V(\mathcal{D})|\leq 8(k+1)^3|M|^{10}$.

We now turn to prove Lemma 4.3. In what follows, $\widehat{M}$ and $\widehat{C}$ are as stated in this lemma. We denote $M'=M\cup\widehat{M}$. Note that since M is 9-redundant with respect to $\mathcal{W}$, we have that M' is also 9-redundant with respect to $\mathcal{W}$. We begin our proof by showing that the common neighborhood outside M' of any two non-adjacent vertices, unless these two vertices form a pair in $\mathcal{W}$, induces a clique. This simple claim will come in handy in several arguments later.

Lemma 4.6. *Let $u,v\in V(G)$ be distinct vertices such that $(u,v)\notin E(G)$ and $\{u,v\}\notin\mathcal{W}$. Then, $G[(N_G(u)\cap N_G(v))\setminus M']$ is a clique.*

Proof. Suppose, by way of contradiction, that $G[(N_G(u)\cap N_G(v))\setminus M']$ is not a clique. Then, there exist two vertices $x,y\in(N_G(u)\cap N_G(v))\setminus M'$ that are not neighbors in G. Note that $\mathbb{O}=G[\{u,v,x,y\}]$ is a hole, and that $M\cap V(\mathbb{O})\subseteq\{u,v\}$. Moreover, $\mathbb{O}$ is not covered by $\mathcal{W}$ (because $\{u,v\}\notin\mathcal{W}$ and every set in $\mathcal{W}$ has size at least 2). Since M is 9-redundant, this means that $|M\cap V(\mathbb{O})|>9$. However, $|V(\mathbb{O})|=4$, hence we have reached a contradiction. □

Structure of Obstructions Intersecting Module Components. To reduce our instance or to obtain a set B as required to prove Lemma 4.3, we need to understand how obstructions can intersect module components. For this purpose, we state a simple proposition by Cao and Marx [11]. This proposition asserts that because we are dealing with modules, these intersections are quite restricted.

Proposition 4.7 ([11]). *Let C be a module in G, and let $\mathbb{O}$ be an obstruction. If $|V(\mathbb{O})|>4$, then either $V(\mathbb{O})\subseteq V(C)$ or $|V(\mathbb{O})\cap V(C)|\leq 1$.*

By Proposition 4.7, we directly obtain the following lemma.

Lemma 4.8. *Let C be a module such that $V(C)\cap M'=\emptyset$, and let $\mathbb{O}$ be an obstruction that is not covered by $\mathcal{W}$. Then, $|V(\mathbb{O})\cap V(C)|\leq 1$.*

Proof. Since $\mathbb{O}$ is an obstruction that is not covered by $\mathcal{W}$, it holds that $|M'\cap V(\mathbb{O})|>9$. In particular, as $V(C)\cap M'=\emptyset$, we have that $|V(\mathbb{O})|>4$ and $V(\mathbb{O})\setminus V(C)\neq\emptyset$. Then, as C is a module, by Proposition 4.7, we have that $|V(\mathbb{O})\cap V(C)|\leq 1$. □

Reducing the Size of Module Components. To ensure we have only small module components, we apply the following rule.

Reduction Rule 4.3. Suppose that there exists $C\in\widehat{C}$ such that $|V(C)|>k+1$. Then, output the instance $(G-\{v\},k)$, where v is an arbitrarily chosen vertex of C.

Lemma 4.9. *Reduction Rule 4.3 is safe.*

Proof. In one direction, it is clear that if (G, k) is a Yes-instance, then $(G - \{v\}, k)$ is a Yes-instance as well.

In the other direction, suppose that $(G-\{v\}, k)$ is a Yes-instance. Let S be a k-solution for $G-\{v\}$. We claim that S is also a k-solution for G. Suppose, by way of contradiction, that this claim is false. Then, there exists an obstruction $\mathbb{O}$ for $G - S$. As $S \cup \{v\}$ is a $(k+1)$-solution for G and $\mathcal{W}$ is $(k+2)$-necessary, we have that $S \cup \{v\}$ hits $\mathcal{W}$. Since $v \notin M$ and $\mathcal{W} \subseteq 2^M$, we derive that S hits $\mathcal{W}$. Thus, since $\mathbb{O}$ is an obstruction for $G - S$, we deduce that $\mathbb{O}$ is not covered by $\mathcal{W}$. Hence, by Lemma 4.8, $|V(\mathbb{O}) \cap V(C)| \leq 1$. Thus, $V(\mathbb{O}) \cap V(C) = \{v\}$. Then, as C is a module, for any vertex $u \in V(C)$, it holds that $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is an obstruction. Since $|V(C)| > k+1$, we have that $V(C) \setminus (S \cup \{v\}) \neq \emptyset$. However, this implies that there exists an obstruction $\mathbb{O}'$ for $(G - \{v\}) - S$, which is a contradiction as S is a k-solution for $G - \{v\}$. ◻

Preliminary Marking Scheme. By Lemma 4.6, for all $u, v \in M'$ such that $(u, v) \notin E(G)$ and $\{u, v\} \notin \mathcal{W}$, there exists at most one $C \in \widehat{C}$, denoted by C_{uv}, such that $N_G(u) \cap N_G(v) \cap V(C) \neq \emptyset$. Accordingly, denote

$$C^\star = \{C_{uv} \in \widehat{C} \mid u, v \in M', (u, v) \notin E(G), \{u, v\} \notin \mathcal{W}\}.$$

Moreover, denote $A^\star = V(C^\star)$. From Reduction Rule 4.3, we have the following observation.

Observation 4.10. *The size of $A^\star$ is upper bounded by $(k+1)|M'|^2$.*

Thus, in what follows, we do not need to "worry" about the modules in $C^\star$ since we already know that they contain only few vertices in total. In the following, we proceed to analyze the modules in $\widehat{C} \setminus C^\star$. An important property of every vertex v in the modules in $\widehat{C} \setminus C^\star$, unlike the modules in $C^\star$, is that every pair of vertices in its neighborhood in M' must be adjacent unless they form a set in $\mathcal{W}$.

Observation 4.11. *Consider a vertex $v \in V(\widehat{C} \setminus C^\star)$. For (distinct) vertices $u, w \in N_G(v) \cap M'$, at least one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds.*

Proof. For $v \in V(\widehat{C} \setminus C^\star)$, and (distinct) vertices $u, w \in N_G(v) \cap M'$, if one of $\{u, w\} \in \mathcal{W}$ or $(u, v) \in E(G)$ holds, then the claim trivially holds. Therefore, we assume that $\{u, w\} \notin \mathcal{W}$ and $(u, v) \notin E(G)$. Recall that each set in $\mathcal{W}$ is of size at least 2 (since Reduction Rule 3.1 is not applicable). From the preceding discussions together with Lemma 4.6, we obtain that there is at most one connected component $C_{uw} \in \widehat{C}$ such that $N_G(u) \cap N_G(w) \cap V(C_{uw}) \neq \emptyset$. Since $u, w \in N_G(v)$, it must be the case that $v \in V(C_{uw})$. But by our preliminary marking scheme, $C_{uw} \in C^\star$. This contradicts that $v \in V(\widehat{C} \setminus C^\star)$. ◻

Let us also consider the relation between obstructions and the modules in $\widehat{C} \setminus C^\star$. Roughly speaking, the following lemma already implies that we can focus on AWs of a very specific form. However, handling these obstructions requires a substantive amount of work in the rest of this section.

Lemma 4.12. *Let $C \in \widehat{C} \setminus C^\star$, and let $\mathbb{O}$ be an obstruction that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap V(C) \neq \emptyset$. Then, $|V(\mathbb{O}) \cap V(C)| = 1$ and $\mathbb{O}$ is an AW where the vertex in $V(\mathbb{O}) \cap V(C)$ is a terminal.*

Proof. Consider $C \in \widehat{C} \setminus C^\star$ and an obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap V(C) \neq \emptyset$. First, as C is a module, from Lemma 4.8 we deduce that $|V(\mathbb{O}) \cap V(C)| = 1$. Furthermore, as $\mathbb{O}$ is not covered by $\mathcal{W}$, we have that $|V(\mathbb{O})| > 9$. This means that $\mathbb{O}$ is not a long claw, a whipping top, a net, or a tent. Let v be the unique vertex in $V(C) \cap V(\mathbb{O})$. If $\mathbb{O}$ is an induced cycle on at least four vertices, or one of the AWs where v is not one of the terminals, then $N_G(v) \cap V(\mathbb{O})$

contains a pair of non-adjacent vertices. But from Observation 4.11 together with the facts that $\mathbb{O}$ is not covered by $\mathcal{W}$ and $N_G(v) \subseteq V(C) \cup M$, for each $u, w \in N_G(v) \cap M' \cap V(\mathbb{O})$, we have $(u, v) \in E(G)$. Thus, we conclude that $\mathbb{O}$ is one of the AWs, where v is one of the terminals. □

Marking Scheme to Handle Non-Shallow Terminals. For every two subsets $X, Y \subseteq M'$ such that $|X| \leq 2$ and $|Y| \leq 2$, denote $A_{X,Y} = \{v \in V(\widehat{C} \setminus C^\star) \mid X \subseteq N_G(v), Y \cap N_G(v) = \emptyset\}$. Now, if $|A_{X,Y}| \leq k+1$, then define $A'_{X,Y} = A_{X,Y}$, and otherwise let $A'_{X,Y}$ be an arbitrarily chosen subset of size $k+1$ of $A_{X,Y}$. Let us denote $A' = \bigcup_{X,Y} A'_{X,Y}$, where X, Y range over all subsets $X, Y \subseteq M'$ such that $|X| \leq 2$ and $|Y| \leq 2$. Let us first observe that $|A'|$ is small.

Observation 4.13. *The size of A' is upper bounded by $(k+1)|M'|^4$.*

Proof. Let $t = |M'|$. Note that $|A'| \leq \sum_{i \in \{0,1,2\}} \binom{t}{i} \sum_{j \in \{0,1,2\}} \binom{t-i}{j}(k+1)$. Note that the following holds: (i) $\binom{t}{0}\binom{t}{0} + \binom{t}{0}\binom{t}{1} + \binom{t}{0}\binom{t}{2} = \frac{t^2+t+2}{2}$, (ii) $\binom{t}{1}\binom{t-1}{0} + \binom{t}{1}\binom{t-1}{1} + \binom{t}{1}\binom{t-1}{2} = \frac{t^3-t^2+2t}{2}$, and (iii) $\binom{t}{2}\binom{t-2}{0} + \binom{t}{2}\binom{t-2}{1} + \binom{t}{2}\binom{t-2}{2} = \frac{t^4-4t^3+7t^2-4t}{4}$. Thus, we can obtain that $|A'| \leq (k+1) \cdot \frac{t^4-2t^3+7t^2+2t+4}{4}$. As $t \geq k \geq 2$, we can obtain that $-2t^3+7t^2+2t+4 \leq 3t^4$. Hence, we can obtain that $|A'| \leq (k+1)t^4 = (k+1)|M'|^4$. □

Now, let us verify that we have thus marked a set of vertices that is sufficient to "handle" non-shallow terminals. Roughly speaking, by this we mean that for any vertex v and obstruction $\mathbb{O}$ that satisfy the premise in this lemma, we can find $k+1$ "replacements" of v (so that we still have an obstruction) that belong to our marked set A'.

Lemma 4.14. *Let $C \in \widehat{C} \setminus C^\star$, $v \in V(C) \setminus A'$, and $\mathbb{O}$ be an obstruction that is not covered by $\mathcal{W}$ such that $v \in V(\mathbb{O})$. If $\mathbb{O}$ is an AW where v is a non-shallow terminal, then there exists a subset $\hat{A} \subseteq A'$ of size $k+1$ such that for each $u \in \hat{A}$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ contains an obstruction.*

Proof. First, by Lemma 4.12, we have that $\mathbb{O}$ is an AW such that $V(\mathbb{O}) \cap V(C) = \{v\}$ and v is a terminal of $\mathbb{O}$. Let us also note that $N_G(v) \subseteq M' \cup V(C)$, and therefore $N_G(v) \cap V(\mathbb{O}) \subseteq M'$. Let $\mathbb{O}$ comprise of the base path base($\mathbb{O}$) $= (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and centers c_1 and c_2 (as in the definition in Section 2). Here, if $\mathbb{O}$ is a †-AW, then we let $c = c_1 = c_2$. Suppose that v is not the shallow terminal of $\mathbb{O}$. Then, we have that v is either t_ℓ or t_r. Without loss of generality, suppose that $v = t_\ell$. Let us consider two cases, depending on whether $\mathbb{O}$ is a †-AW or a ‡-AW:

- Suppose that $\mathbb{O}$ is a †-AW. Notice that $b_1 \in M'$ as $(b_1, v) \in E(G)$, $V(\mathbb{O}) \cap V(C) = \{v\}$, and $N_G(v) \subseteq M' \cup V(C)$. From Lemma 4.12, any vertex in $V(\mathbb{O}) \cap V(\widehat{C} \setminus C^\star)$ must be one of the terminals. Thus, we have $V(\widehat{C} \setminus C^\star) \cap (\{b_1, b_2, \ldots, b_z\} \cup \{c\}) = \emptyset$. We also recall that for each $u \in V(\widehat{C} \setminus C^\star)$, we have $N_G(u) \subseteq M' \cup V(\widehat{C} \setminus C^\star)$. In particular, if b_2 (or c) is not in M', no vertex in $V(\widehat{C} \setminus C^\star)$ can be adjacent to b_2 (or c). The preceding discussions together with the construction of A' implies the following: there exists a subset $Q \subseteq A'$ of $k+1$ vertices such that for each $u \in Q$, u is adjacent to b_1, and u is not adjacent to b_2 and c. Indeed, these are the vertices in the set $A'_{\{b_1\}, \{b_2, c\} \cap M'}$ (the size of this set is $k+1$ since otherwise v should have belonged to it, but $v \notin A'$). Furthermore, b_1 is not adjacent to any vertex on $\mathbb{O}$ besides v, c, and b_2. Therefore, for all $u \in Q$, using Observation 4.11 for obstructions not covered by $\mathcal{W}$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap M'$ besides b_1. Furthermore, for all $u \in Q$, since $N_G(u) \subseteq V(\widehat{C} \setminus C^\star) \cup M'$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap V(C^\star)$. Last, because $V(\mathbb{O}) \cap V(C) = \{v\}$, for all $u \in Q$, we have that u is not adjacent to any vertex on $V(\mathbb{O}) \cap V(\widehat{C} \setminus C^\star)$ besides possibly v. Hence, for any vertex $u \in Q$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is also a †-AW.

- Suppose that $\mathbb{O}$ is a ‡-AW. Notice that $b_1, c_1 \in M'$ as $(b_1, v), (c_1, v) \in E(G)$, $V(\mathbb{O}) \cap V(C) = \{v\}$, and $N_G(v) \subseteq M' \cup V(C)$. From Lemma 4.12, any vertex in $V(\mathbb{O}) \cap V(\widehat{C} \setminus C^\star)$ must be one of the terminals. Thus, we have $V(\widehat{C} \setminus C^\star) \cap (\{b_1, b_2, \ldots, b_z\} \cup \{c\}) = \emptyset$. We also recall that for each $u \in V(\widehat{C} \setminus C^\star)$, we have $N_G(u) \subseteq M' \cup V(\widehat{C} \setminus C^\star)$. The preceding discussions together with the construction of A' implies the following: there exists a subset $Q \subseteq A'$ of $k+1$ vertices $u \in A'$ such that u is adjacent to both c_1 and b_1, and u is adjacent to neither c_2 nor b_2. Indeed, these are the vertices in the set $A'_{\{b_1,c_1\},\{b_2,c_2\}\cap M'}$ (as in the previous case, the size of this set is $k+1$ since otherwise v should have belonged to it, but $v \notin A'$). Notice that b_1 is not adjacent to any vertex on $\mathbb{O}$ besides v, c_1, c_2, and b_2. For all $u \in Q$, using Observation 4.11 for obstructions not covered by $\mathcal{W}$ and the facts that $N_G(u) \subseteq V(\widehat{C} \setminus C^\star) \cup M'$ and $V(\mathbb{O}) \cap V(C) = \{v\}$ (using the exact same rationale as in the previous case), we have that u is not adjacent to any vertex on $\mathbb{O} - \{v\}$ besides c_1 and b_1. Hence, for any vertex $u \in Q$, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{u\}]$ is also a ‡-AW.

In both cases, we derived the desired claim, and thus the proof is complete. □

Marking Scheme to Handle Shallow Terminals. For this part in our proof, we require the following notation: we say that a path P is *covered by* $\mathcal{W}$ if there is a set $W \in \mathcal{W}$ such that $W \subseteq V(P)$. Intuitively, we think of P as part of the base of an obstruction, hence the preceding notation is a natural extension of covering to this context.

Before we present our marking scheme, let us explicitly state the following observation, which follows from Observation 4.11 in the same manner as Lemma 4.12.

Observation 4.15. *Let P be an induced path in $G[V(G) \setminus V(C)]$ for some $C \in \widehat{C} \setminus C^\star$ such that P is not covered by $\mathcal{W}$. For all $v \in V(C)$, $|N_G(v) \cap V(P)| \leq 2$, and if $|N_G(v) \cap V(P)| = 2$, then the two vertices in $N_G(v) \cap V(P)$ are adjacent on P.*

Proof. Consider $C \in \widehat{C} \setminus C^\star$, $v \in V(C)$, and an induced path P in $G[V(G) \setminus V(C)]$ that is not covered by $\mathcal{W}$. If $|N_G(v) \cap V(P)| \leq 1$, then the claim trivially follows. Otherwise, we assume that $|N_G(v) \cap V(P)| \geq 2$. Consider (distinct) vertices $u, w \in N_G(v) \cap V(P)$. From Observation 4.11, we have that $(u, w) \in E(G)$. Here, we relied on the fact that P is not covered by $\mathcal{W}$. Since P is an induced path, u and w must be adjacent vertices in P. From the preceding, we can conclude that v cannot have three neighbors in P, as P is an induced path in G. Moreover, if v has two neighbors in P, then they must be adjacent vertices. □

Denote $N = M' \cup A^\star \cup A'$. (Recall that $A^\star = V(C^\star)$ and that A' is the set of vertices marked when we dealt with non-shallow terminals.) For all (not necessarily distinct) vertices $c_1, c_2 \in M'$, denote $A_{\{c_1,c_2\}} = \{v \in V(\widehat{C}) \setminus (A^\star \cup A') \mid \{c_1, c_2\} \subseteq N_G(v)\}$. Intuitively, $A_{\{c_1,c_2\}}$ is the set of vertices among the unmarked vertices in $\widehat{C}$ that are neighbors of both c_1 and c_2 and hence can play the role of shallow terminals in obstructions having c_1 and c_2 as centers. Moreover, let us arbitrarily order N and $E(G[N])$ as follows: $N = \{v_1, v_2, \ldots, v_{|N|}\}$ and $E(G[N]) = \{e_1, e_2, \ldots, e_{|E(G[N])|}\}$. Thus, when we define vectors having $|N|$ or $|E(G[N])|$ entries below, we can work with a natural correspondence between the index of an entry in the vector and an element of N or $E(G[N])$, respectively.

In what follows, we begin the part in our analysis that is based on linear algebra. To this end, we first need to encode our problem in this language, which entails the introduction of appropriate notations. Afterward, we will present a marking scheme based on these notations. The analysis of this scheme is done in a sequence of several lemmas, after which we will be ready to conclude the proof of Lemma 4.3.

First, with every vertex $u \in V(\widehat{C}) \setminus (A^\star \cup A')$, we associate two binary vectors that capture incidence relations between u and the elements (vertices and edges) in $G[N]$:

- ***Vertex incidence relations***: $\mathsf{vinc}(u) = (b_1, b_2, \ldots, b_{|N|})$, where for all $i \in [|N|]$, $b_i = 1$ if and only if $v_i \in N_G(u)$;
- ***Edge incidence relations***: $\mathsf{einc}(u) = (b_1, b_2, \ldots, b_{|E(G[N])|})$, where for all $i \in [|E(G[N])|]$, $b_i = 1$ if and only if u is adjacent to both endpoints of e_i.

Complete Incidence Relations. In addition, we define $\mathsf{inc}(u)$ as the vector that is the concatenation of $\mathsf{vinc}(u)$ and $\mathsf{einc}(u)$, to which we add 1 at the end. Formally, $\mathsf{inc}(u)$ is a binary vector with $q = |N| + |E(G[N])| + 1$ entries, where for all $i \in [|N|]$, the i^{th} entry of $\mathsf{inc}(u)$ equals the i^{th} entry of $\mathsf{vinc}(u)$, for all $i \in [|E(G[N])| + |N|] \setminus [|N|]$, the i^{th} entry of $\mathsf{inc}(u)$ equals the $(i - |N|)^{th}$ entry of $\mathsf{einc}(u)$, and the last entry of $\mathsf{inc}(u)$ is 1. These incidence vectors are associated with the vector space $\mathbb{F}_2^q$, and all calculations related to these vectors are performed accordingly. This completes the description of the notations required to present our marking scheme.

For all (not necessarily distinct) vertices $c_1, c_2 \in M'$, we have the following subprocedure of our marking scheme. First, we define $\mathbf{V}_{\{c_1,c_2\}}$ to be the *multiset* $\{\mathsf{inc}(u) \mid u \in A_{\{c_1,c_2\}}\}$. More precisely, the number of occurrences of a vector in $\mathbf{V}_{\{c_1,c_2\}}$ equals the number of vertices $u \in A_{\{c_1,c_2\}}$ such that $\mathsf{inc}(u)$ equals that vector. Now, we proceed as follows:

(1) Initialize $\widehat{\mathbf{V}}^0_{\{c_1,c_2\}} = \emptyset$.
(2) For $i = 1, 2, \ldots, k+1$, compute some basis $\mathbf{B}^i_{\{c_1,c_2\}}$ for the vector subspace $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$ (with respect to $\mathbb{F}_2^q$),[3] and denote $\widehat{\mathbf{V}}^i_{\{c_1,c_2\}} = \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}} \cup \mathbf{B}^i_{\{c_1,c_2\}}$.
(3) For every occurrence of a vector $\mathbf{v} \in \widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}$, arbitrarily choose a unique vertex $u \in A_{\{c_1,c_2\}}$ such that $\mathsf{inc}(u) = \mathbf{v}$ and denote it by $u_{\mathbf{v}}$ (the existence of sufficiently many such distinct vertices directly follows from the definition of $\mathbf{V}_{\{c_1,c_2\}}$).
(4) Denote $\widehat{A}_{\{c_1,c_2\}} = \{u_{\mathbf{v}} : \mathbf{v} \in \widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}\}$, and note that $\widehat{A}_{\{c_1,c_2\}}$ is a set (rather than a multiset).

Finally, having performed all subprocedures, we denote $\widehat{A} = \bigcup_{c_1,c_2 \in M'} \widehat{A}_{\{c_1,c_2\}}$. Here, union refers to sets—that is, every vertex occurs in $\widehat{A}$ once even if it belongs to more than one set of the form $\widehat{A}_{\{c_1,c_2\}}$. This completes the description of our marking scheme.

We proceed to analyze our marking scheme. Let us first observe that we have not marked "many" vertices—that is, we upper bound $|\widehat{A}|$. Recall that $N = A' \cup A^* \cup M'$ and $k \geq 2$, and thus $|M'| \geq 2$. Hence, using Observations 4.10 and 4.13, we can obtain that $|N| \leq 2(k+1)|M'|^4$.

Lemma 4.16. *The size of* $\widehat{A}$ *is upper bounded by* $(k+1)|M'|^2|N|^2 \leq 4(k+1)^3|M'|^{10}$.

Proof. To show that $|\widehat{A}| \leq (k+1)|M'|^2|N|^2$, it is sufficient to show that for all $c_1, c_2 \in M'$, $|\widehat{A}_{\{c_1,c_2\}}| \leq (k+1)|N|^2$. To this end, consider some $c_1, c_2 \in M'$. Now, observe that the number of entries of the vectors in $\mathbf{V}_{\{c_1,c_2\}}$ is $q = |N| + |E(G[N])| + 1 \leq |N| + \frac{|N|(|N|-1)}{2} + 1 \leq |N|^2$. (In the preceding, we use the assumption that $k \geq 2$, and thus $|N| \geq 2$.) Hence, every basis of $\mathbf{V}_{\{c_1,c_2\}}$ (or of a subset of $\mathbf{V}_{\{c_1,c_2\}}$) is of size at most $|N|^2$. As $\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}$ is a multiset that is the union of $(k+1)$ bases of $\mathbf{V}_{\{c_1,c_2\}}$ (or of subsets of $\mathbf{V}_{\{c_1,c_2\}}$), we have that $|\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}| \leq (k+1)|N|^2$. Since $|\widehat{\mathbf{V}}^{k+1}_{\{c_1,c_2\}}| = |\widehat{A}_{\{c_1,c_2\}}|$, the proof is complete. □

Now, let us verify that we have a set of vertices that is sufficient to "handle" shallow terminals. This will be done in a sequence of two lemmas and a corollary. For this purpose, we need the following notation where we alter incidence vectors by nullifying some of their entries:

[3] Here, note that the subtraction concerns multisets. In particular, if an element occurs x times in a multiset X, and y times in a multiset $Y \subseteq X$, then it occurs $x - y$ times in $X \setminus Y$.

- ***Nullifying subsets of vertices and edges*:** Given a pair (X, Y), where $X \subseteq N$ and $Y \subseteq E(G[N])$, and a vertex $u \in V(\widehat{C}) \setminus (A^\star \cup A')$, we define $\mathsf{inc}^{X,Y}(u)$ to be the vector obtained from $\mathsf{inc}(u)$ by changing all the entries associated with vertices and edges that do not belong to $X \cup Y$ to 0. Formally, $\mathsf{inc}^{X,Y}(u)$ is a binary vector with $|N| + |E(G[N])| + 1$ entries, where for all $i \in [|N|]$, the i^{th} entry of $\mathsf{inc}^{X,Y}(u)$ equals the i^{th} entry of $\mathsf{vinc}(u)$ if $v_i \in X$ and to 0 otherwise, for all $i \in [|E(G[N])|+|N|] \setminus [|N|]$, the i^{th} entry of $\mathsf{inc}^{X,Y}(u)$ equals the $(i-|N|)^{th}$ entry of $\mathsf{einc}(u)$ if $e_{i-|N|} \in Y$ and to 0 otherwise, and the last entry of $\mathsf{inc}^{X,Y}(u)$ is 1.
- ***Nullifying an induced path*:** Furthermore, for an induced path P in $G - (V(\widehat{C}) \setminus (A^\star \cup A'))$ and a vertex $u \in V(\widehat{C}) \setminus (A^\star \cup A')$, we denote $\mathsf{inc}^P(u) = \mathsf{inc}^{X,Y}(u)$, where $X = V(P) \cap N$ and $Y = E(P) \cap E(G[N])$.

Moreover, recall that given a vector $\mathbf{v}$ and an entry index i, $\mathbf{v}[i]$ denotes the i^{th} entry of $\mathbf{v}$.

Lemma 4.17. *Let P be an induced path in $G[V(G) \setminus V(C)]$ for some $C \in \widehat{C} \setminus C^\star$ such that P is not covered by $\mathcal{W}$. For all $u \in V(C)$, $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 1 \mod 2$ if and only if $N_G(u) \cap V(P) = \emptyset$.*

Proof. Consider some vertex $u \in V(C)$. For the reverse direction of the proof, suppose that $N_G(u) \cap V(P) = \emptyset$. Then, all of the entries of $\mathsf{inc}^P(u)$ equal 0, except for the last entry that equals 1. Thus, $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 1 \mod 2$.

For the forward direction of the proof, suppose that $N_G(u) \cap V(P) \neq \emptyset$. Then, by Observation 4.15, $|N_G(u) \cap V(P)|$ is either 1 or 2, and if it is 2, then the two vertices in $N_G(u) \cap V(P)$ are adjacent on P. Furthermore, observe that as $V(P) \cap V(C) = \emptyset$ and $N_G(u) \subseteq V(C) \cup M'$, we have that $N_G(u) \cap V(P) \subseteq M'$. Thus, in case $|N_G(u) \cap V(P)| = 1$, it follows that there exists exactly one entry in $\mathsf{inc}^P(u)$ that equals 1 apart from the last entry, which is the entry corresponding to the vertex in $N_G(u) \cap V(P)$. Moreover, in case $|N_G(u) \cap V(P)| = 2$, it follows that there exist exactly three entries in $\mathsf{inc}^P(u)$ that equal 1 apart from the last entry, which are the two entries corresponding to the two vertices in $N_G(u) \cap V(P)$ and the entry corresponding to the edge between these two vertices. In both cases, we derive that $\sum_{i=1}^{q} \mathsf{inc}^P(u)[i] = 0 \mod 2$ as desired. □

The reason we need Lemma 4.17 is that we make use of it in the proof of the following lemma. Informally, this lemma exhibits the existence of $k + 1$ "replacements" for each unmarked shallow terminal.

Lemma 4.18. *Let $w \in V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})$, and let $\mathbb{O}$ be an AW that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap (V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$ and w is the shallow terminal of $\mathbb{O}$. Let $\{c_1, c_2\}$ be the set of centers of $\mathbb{O}$ (with $c_1 = c_2$ if $\mathbb{O}$ is a †-AW). Then, for all $i \in [k + 1]$, there exists $\mathbf{v} \in \mathbf{B}^i_{\{c_1,c_2\}}$ such that $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u_{\mathbf{v}}\}]$ is an obstruction.*

Proof. Consider some $i \in [k+1]$. Let C be the connected component in $\widehat{C}$ containing w. Notice that $c_1, c_2 \in M'$ as $(c_1, w), (c_2, w) \in E(G)$, $V(\mathbb{O}) \cap (V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$, and $N_G(w) \subseteq M' \cup V(C)$. Let us first argue that there exists an occurrence of $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$. To this end, note that as w is the shallow terminal of $\mathbb{O}$, it is adjacent to c_1 and c_2, and therefore $w \in A_{\{c_1,c_2\}}$. Moreover, because $w \notin \widehat{A}$, there exists an occurrence of $\mathsf{inc}(w)$ that does not belong to $\mathbf{V}^{k+1}_{\{c_1,c_2\}}$, which implies that there exists an occurrence of $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$.

As we have shown that $\mathsf{inc}(w)$ in $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$, the fact that $\mathbf{B}^i_{\{c_1,c_2\}}$ is a basis for $\mathbf{V}_{\{c_1,c_2\}} \setminus \widehat{\mathbf{V}}^{i-1}_{\{c_1,c_2\}}$ implies that there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t$ for some $t \in \mathbb{N}$ (in particular, $t \geq 1$) and non-zero coefficients $\lambda_1, \lambda_2, \ldots, \lambda_t$ such that $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_t\mathbf{v}_t = \mathsf{inc}(w)$ over $\mathbb{F}_2^q$. As the coefficient are from field $\mathbb{F}_2$, they are all necessarily 1. Thus, we have that

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_t = \mathsf{inc}(w) \text{ over } \mathbb{F}_2^q.$$

Denote $u_i = u_{\mathbf{v}_i}$ for all $i \in [t]$. Then, $\mathrm{inc}(u_1)+\mathrm{inc}(u_2)+\cdots+\mathrm{inc}(u_t) = \mathrm{inc}(w)$ over $\mathbb{F}_2^q$. In particular, $\mathrm{inc}^P(u_1) + \mathrm{inc}^P(u_2) + \cdots + \mathrm{inc}^P(u_t) = \mathrm{inc}^P(w)$ over $\mathbb{F}_2^q$, where P is the extended base of $\mathbb{O}$. This implies that $\sum_{i=1}^t \sum_{j=1}^q \mathrm{inc}^P(u_i)[j] = \sum_{j=1}^q \mathrm{inc}^P(w)[j] \mod 2$. (Note that since $V(\mathbb{O}) \cap (V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$, the extended base is completely contained in $G[V(G) \setminus (V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A}))]$, and furthermore P is not covered by $\mathcal{W}$ by the premise of the lemma.) By Lemma 4.17 and since $N_G(w) \cap V(P) = \emptyset$ (because w is the shallow terminal of $\mathbb{O}$), we have that $\sum_{j=1}^q \mathrm{inc}^P(w)[j] = 1 \mod 2$. Thus, $\sum_{i=1}^t \sum_{j=1}^q \mathrm{inc}^P(u_i)[j] = 1 \mod 2$. This implies that there exists $i \in [t]$ such that $\sum_{j=1}^q \mathrm{inc}^P(u_i)[j] = 1 \mod 2$. However, by Lemma 4.17, this means that $N_G(u_i) \cap V(P) = \emptyset$. Moreover, we have that $u_i \in A_{\{c_1,c_2\}}$ because u_i is associated with the vector $\mathbf{v}_i$ which belongs to $\mathbf{B}^i_{\{c_1,c_2\}}$. Hence, $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u_i\}]$ is an AW. This completes the proof. □

Due to the definition of $\widehat{A}$, as a direct corollary to Lemma 4.18 we have the following result.

Corollary 4.19. *Let $w \in V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})$, and let $\mathbb{O}$ be an AW that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap (V(\widehat{C}) \setminus (A^\star \cup A' \cup \widehat{A})) = \{w\}$ and w is the shallow terminal of $\mathbb{O}$. Then, there exists a set $\tilde{A} \subseteq \widehat{A}$ of size $k+1$ such that for each $u \in \tilde{A}$, $G[(V(\mathbb{O}) \setminus \{w\}) \cup \{u\}]$ is an obstruction.*

We are now ready to conclude the proof of Lemma 4.3 and thereby this section.

Proof of Lemma 4.3. Toward the proof, first note that if the condition of Reduction Rule 4.3 applies, then we are clearly done—indeed, in this case, we output an instance (G', k) equivalent to (G, k) where $|V(G') < |V(G)|$. Thus, we next suppose that this rule has been applied exhaustively. Then, our output is the set $B = A^\star \cup A' \cup \widehat{A}$. By Observations 4.10 and 4.13, and by Lemma 4.16, we have that $|B| \le |A^\star| + |A'| + |\widehat{A}| \le (k+1)|M'|^2 + (k+1)|M'|^4 + 4(k+1)^3|M'|^{10} \le 8(k+1)^3|M'|^{10}$ as desired (recall that $|M'| \ge k \ge 2$).

Let $S \subseteq V(G)$ be some arbitrary set of size at most k. We claim that the following property holds: If there exists an obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, then there exists an obstruction $\mathbb{O}'$ for G such that $V(\mathbb{O}') \cap S = \emptyset$ and $V(\mathbb{O}') \cap (V(\widehat{C}) \setminus B) = \emptyset$. Clearly, if there does not exist any obstruction $\mathbb{O}$ for G that is not covered by $\mathcal{W}$ and such that $V(\mathbb{O}) \cap S = \emptyset$, then our proof is complete. Hence, we next suppose that such an obstruction exists, and we let $\mathbb{O}'$ be such an obstruction that minimizes $|V(\mathbb{O}') \cap (V(\widehat{C}) \setminus B)|$. We claim that for this obstruction $\mathbb{O}'$, it holds that $V(\mathbb{O}') \cap (V(\widehat{C}) \setminus B) = \emptyset$, which would complete the proof. Suppose, by way of contradiction, that this claim is false. Then, as $V(C^\star) \subseteq B$, there exists $C \in \widehat{\mathcal{C}} \setminus \mathcal{C}^\star$ and $v \in V(C)$ such that $v \in V(\mathbb{O}')$. By Lemma 4.12, $|V(\mathbb{O}') \cap V(C)| = 1$ and $\mathbb{O}'$ is an AW where v is a terminal.

Let us first suppose that v is not the shallow terminal of $\mathbb{O}'$. Then, by Lemma 4.14, there exist $(k+1)$ vertices $u \in A'$ such that $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ is an obstruction. However, as $|S| \le k$, this means that there exists $u \in A' \setminus S$ such that $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ is an obstruction. As $A' \subseteq B$ and $G[(V(\mathbb{O}') \setminus \{v\}) \cup \{u\}]$ has fewer vertices from $V(\widehat{C}) \setminus B$ than $\mathbb{O}'$, we have reached a contradiction to the choice of $\mathbb{O}'$.

As the choice of v was arbitrary, we derive that $V(\mathbb{O}') \cap (V(\widehat{C}) \setminus B)$ contains exactly one vertex, which we denote by w, that is the shallow terminal of $\mathbb{O}'$. In this case, by Corollary 4.19, there exist $(k+1)$ vertices $u \in \widehat{A}$ such that $G[(V(\mathbb{O}') \setminus \{w\}) \cup \{u\}]$ is an obstruction. However, as $|S| \le k$, this means that there exists $u \in \widehat{A} \setminus S$ such that $G[(V(\mathbb{O}') \setminus \{w\}) \cup \{u\}]$ is an obstruction. As $\widehat{A} \subseteq B$ and $G[(V(\mathbb{O}') \setminus \{w\}) \cup \{u\}]$ has no vertices from $V(\widehat{C}) \setminus B$, we have again reached a contradiction to the choice of $\mathbb{O}'$. This completes the proof. □

4.1 Bounded Intersection Two Families Lemma

At the heart of our marking scheme to handle shallow terminals is in fact the special case of Lemma 1.1 where $c = 2$. Indeed, viewing this case in a more abstract manner, let us give a rough description of the relation between it and the statement of Lemma 1.1. For all $c_1, c_2 \in M'$, we have sets $A_1, A_2, \ldots, A_t$ and $B_1, B_2, \ldots, B_t$, which are defined as follows. First, the universe is the set of all vertices and pairs of vertices in N. Second, let W denote a set of vertices $w \in V(\widehat{C}) \setminus (A^\star \cup A')$ such that (i) w is adjacent to c_1 and c_2, and (ii) w has at least one induced path in $G[N]$, say P_w, which contains no vertex adjacent to w, so that the two following properties hold:

- For all distinct $w, w' \in W$, w is adjacent to at least one vertex on $P_{w'}$.
- For every induced path P in $G[N]$ that has no vertex adjacent to some vertex in $V(\widehat{C}) \setminus (A^\star \cup A')$, there also exists a vertex in W that is not adjacent to any vertex on P.

These properties mean, in a sense, that W is a minimal set that "covers" all induced paths in $G[N]$ that can potentially create AWs together with c_1 and c_2 as centers. Then, $t = |W|$, and denote $W = \{w_1, w_2 \ldots, w_t\}$. For every vertex $w_i \in W$, we create the new set A_i, which contains all the neighbors of w_i in N, and the new set B_i, which is equal to $V(P_{w_i})$. Clearly, for all $i \in [t]$, $A_i \cap B_i = \emptyset$, and due to Observation 4.15, for all distinct $i, j \in [t]$, $|A_i \cap B_j| \in \{1, 2\}$.

Let us now turn to the proof of Lemma 1.1. For convenience, let us restate it.

Lemma 1.1 (Bounded Intersection Two Families Lemma). *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be families over a universe U such that (i) for every $i \leq m$, $A_i \cap B_i = \emptyset$, and (ii) for every $j \neq i$, $|A_i \cap B_j| \in \{1, \ldots, c\}$. Then, $m \leq \sum_{t=0}^{c} \binom{|U|}{t}$.*

Proof. Let $|U| = n$, and let $d = \sum_{t=0}^{c} \binom{n}{t}$. Let D be the set of all subsets of U of size at most c (including the empty set). Note that we have $|D| = d$. Fix a bijection between D and $\{1, 2, \ldots, d\}$. We construct an incidence vector $\mathbf{v}_i$ for each set A_i, where $\mathbf{v}_i$ is indexed by the subsets of U of size up to c. More precisely, we have a vector $\mathbf{v}_i \in \{0,1\}^d$, where $\mathbf{v}_i[X] = 1$ if and only if $X \subseteq A_i$. Let us note that $\mathbf{v}_i[\emptyset] = 1$ for all $1 \leq i \leq m$. We consider these vectors as elements of the vector space $\mathbb{F}_2^d$. Similarly, we construct vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ for each set $B_1, B_2, \ldots, B_m$. We first claim that for every $i \in [m]$, we have $\mathbf{v}_i \cdot \mathbf{u}_i = 1$. This follows from the fact that $A_i \cap B_i = \emptyset$. We next claim that for each $i, j \in [m]$, where $i \neq j$, we have $\mathbf{v}_i \cdot \mathbf{u}_j = 0$. This follows from the following observation. Let $C_{ij} = A_i \cap B_j$. Then, as $|C_{ij}| \in [c]$, we have that $2^{C_{ij}} \subseteq D$, where $2^{C_{ij}}$ denotes the collection of all subsets of C_{ij}. Now, observe that $\mathbf{v}_i[X]\mathbf{u}_j[X] = 1$ if and only if $X \subseteq C_{ij}$. As $|2^{C_{ij}}|$ is an even number (greater than or equal to 2), it follows that $\mathbf{v}_i \cdot \mathbf{u}_j = 0$ over the field $\mathbb{F}_2$.

Now, suppose that $m > d$. Then, the collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is not linearly independent in $\mathbb{F}_2^d$. Hence, there is a vector, say $\mathbf{v}_m$, such that $\mathbf{v}_m = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_{m-1}\mathbf{v}_{m-1}$, where $\alpha_j \in \mathbb{F}_2$ for each $j \in [m-1]$. We claim that there is a vector $\mathbf{v}_i$ such that $\mathbf{v}_i \cdot \mathbf{u}_m = 1$ for some $i \in [m-1]$. This follows from the following equation.

$$\mathbf{v}_m \cdot \mathbf{u}_m = \left(\sum_{j=1}^{m-1} \alpha_j \mathbf{v}_j\right) \cdot \mathbf{u}_m$$

$$\implies 1 = \sum_{j=1}^{m-1} \alpha_j (\mathbf{v}_j \cdot \mathbf{u}_m)$$

However, this is a contradiction. Hence, $m \leq d$. This concludes the proof of this lemma. □

5 BOUNDING THE MAXIMUM SIZE OF A CLIQUE OF NON-MODULE COMPONENTS

Let $\eta = 2^{10} \cdot 4(k+5)\binom{|M|}{10}$. Recall that C is the set of connected components of $G - M$, $\mathcal{D}$ is the set of connected components in C that are modules, and $\overline{\mathcal{D}} = C \setminus \mathcal{D}$. Let $(\mathbb{P}, \beta)$ be a clique path of $G[V(\overline{\mathcal{D}})]$, $V(\mathbb{P}) = \{x_1, x_2, \ldots, x_t\}$, and for each $i \in [t]$ we let $B_i = \beta(x_i)$. Furthermore, let

$\beta(\mathbb{P}) = \cup_{i=1}^{t}\beta(x_i)$. Let B_i be a bag such that $|B_i| > \eta$. Toward bounding the size of B_i, we mark some of the vertices in B_i and delete all the unmarked vertices in B_i from G. In fact, in a step, we *only delete* one unmarked vertex and then repeat the whole kernelization algorithm on the reduced instance. In the following, we describe the precise marking procedure.

Marking Scheme. To define our marking scheme, we first introduce some notations. We define two functions, namely, $\mathsf{id}^i_{\mathsf{lft}}, \mathsf{id}^i_{\mathsf{rht}} : B_i \to [t]$. Intuitively, these functions denote how far or close a vertex appears in the bags that are to the left and right of B_i, respectively. For a vertex $v \in B_i$, $\mathsf{id}^i_{\mathsf{lft}}(v)$ is the smallest integer $x \in [t]$ such that $v \in B_x$, and $\mathsf{id}^i_{\mathsf{rht}}(v)$ is the largest integer $y \in [t]$ such that $v \in B_y$. Note that for each $v \in B_i$, we have $\mathsf{id}^i_{\mathsf{lft}}(v) \leq i \leq \mathsf{id}^i_{\mathsf{rht}}(v)$. A frame $\mathbb{F} = (X, Y)$ in G is a pair of vertex subsets such that $X \subseteq M$ of size at most 10 and $Y \subseteq X$. A vertex $v \in V(G)$ is said to *fit* a frame $\mathbb{F} = (X, Y)$ if $N_G(v) \cap X = Y$. We now move to the construction of the set $H_i \subseteq B_i$, of marked vertices. For each frame $\mathbb{F}$ in G, we create four sets $L^{\mathbb{F},i}_{\mathsf{far}}, L^{\mathbb{F},i}_{\mathsf{cls}}, R^{\mathbb{F},i}_{\mathsf{far}}, R^{\mathbb{F},i}_{\mathsf{cls}} \subseteq B_i$ of marked vertices each of size at most $k+5$ (and add these vertices to H_i) as follows:

- We create the set $L^{\mathbb{F},i}_{\mathsf{far}}$ as follows. Let W be the set of unmarked vertices in B_i, which fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $L^{\mathbb{F},i}_{\mathsf{far}}$. Else, let $W_{\mathsf{low}} \subseteq W$ be the set of $k+5$ vertices with lowest $\mathsf{id}^i_{\mathsf{lft}}$ values among the vertices in W. Add W_{low} to $L^{\mathbb{F},i}_{\mathsf{far}}$.
- We create the set $L^{\mathbb{F},i}_{\mathsf{cls}}$ as follows. Let W be the set of unmarked vertices in B_i, which fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $L^{\mathbb{F},i}_{\mathsf{cls}}$. Else, let $W_{\mathsf{high}} \subseteq W$ be the set of $k+5$ vertices with highest $\mathsf{id}^i_{\mathsf{lft}}$ values among the vertices in W. Add W_{high} to $L^{\mathbb{F},i}_{\mathsf{cls}}$.
- We create the set $R^{\mathbb{F},i}_{\mathsf{far}}$ as follows. Let W be the set of unmarked vertices in B_i, which fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $R^{\mathbb{F},i}_{\mathsf{far}}$. Else, let $W_{\mathsf{high}} \subseteq W$ be the set of $k+5$ vertices with highest $\mathsf{id}^i_{\mathsf{rht}}$ values among the vertices in W. Add W_{high} to $R^{\mathbb{F},i}_{\mathsf{far}}$.
- We create the set $R^{\mathbb{F},i}_{\mathsf{cls}}$ as follows. Let W be the set of unmarked vertices in B_i, which fit the frame $\mathbb{F}$. If $|W| \leq k+5$, then add all the vertices in W to $R^{\mathbb{F},i}_{\mathsf{cls}}$. Else, let $W_{\mathsf{low}} \subseteq W$ be the set of $k+5$ vertices with lowest $\mathsf{id}^i_{\mathsf{rht}}$ values among the vertices in W. Add W_{low} to $R^{\mathbb{F},i}_{\mathsf{cls}}$.

Notice that $|H_i| \leq 2^{10} \cdot 4(k+5)\binom{|M|}{10} = \eta$. Before proceeding further, we observe (Observations 5.1 and 5.2) certain useful properties regarding a frame $\mathbb{F}$ to which $v \in B_i \setminus H_i$ fits and the vertices in $L^{\mathbb{F},i}_{\mathsf{far}}, R^{\mathbb{F},i}_{\mathsf{far}}, L^{\mathbb{F},i}_{\mathsf{cls}}$, and $R^{\mathbb{F},i}_{\mathsf{cls}}$.

Observation 5.1. *For a frame $\mathbb{F} = (X, Y)$ to which $v \in B_i \setminus H_i$ fits and a vertex $w \in N_G(v)$, the following holds:*

- *If $w \in Y$, then $L^{\mathbb{F},i}_{\mathsf{far}} \cup R^{\mathbb{F},i}_{\mathsf{far}} \subseteq N_G(w)$.*
- *If $w \in V(G) \setminus M$, then at least one of $L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$ or $R^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$ holds.*

Proof. In the first case, it follows from the definition that $L^{\mathbb{F},i}_{\mathsf{far}} \cup R^{\mathbb{F},i}_{\mathsf{far}} \subseteq N_G(w)$. Now we prove the second part of the observation. First, consider the case when both v and w belong to B_i. In this case, the second claim holds because B_i is a clique, $L^{\mathbb{F},i}_{\mathsf{far}} \subseteq B_i$ and $R^{\mathbb{F},i}_{\mathsf{far}} \subseteq B_i$. So let us assume that $w \notin B_i$. However, $w \in N_G(v)$ and hence both v and w lie in the same bag, say B_j, on the clique path $\mathbb{P}$. Since the bags in which w is present occur consecutively on $\mathbb{P}$, we have that all these bags either appear left of B_i or right of B_i. Let us consider the case when all the bags containing w appear left of B_i. The other case when all the bags containing w appear right of B_i is symmetric. We will show that $L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\} \subseteq N_G(w)$. Toward this, we will show that for every $x \in L^{\mathbb{F},i}_{\mathsf{far}} \setminus \{w\}$, there exists a bag that contains both x and w. For a vertex z, let s_z denote the leftmost bag on $\mathbb{P}$ in which z appears and e_z denote the rightmost bag on $\mathbb{P}$ in which z appears. Recall that v is an unmarked vertex in B_i

and thus $s_x \leq s_v \leq i \leq e_x$. Furthermore, we know that $s_x \leq j < i$. This implies that x also belongs to B_j. Hence, we have shown that $L^{\mathbb{F},i}_{\text{far}} \setminus \{w\} \subseteq N_G(w)$. This concludes the proof. □

Observation 5.2. *For a frame $\mathbb{F} = (X, Y)$ to which $v \in B_i \setminus H_i$ fits and a vertex $w \notin N_G(v)$, the following holds:*

- *If $w \in X \setminus Y$, then $(L^{\mathbb{F},i}_{\text{cls}} \cup R^{\mathbb{F},i}_{\text{cls}}) \cap N_G(w) = \emptyset$.*
- *If $w \in V(G) \setminus M$, then at least one of $L^{\mathbb{F},i}_{\text{cls}} \cap N_G(w) = \emptyset$ or $R^{\mathbb{F},i}_{\text{cls}} \cap N_G(w) = \emptyset$ holds.*

Proof. In the first case, it follows from the definition that $(L^{\mathbb{F},i}_{\text{cls}} \cup R^{\mathbb{F},i}_{\text{cls}}) \cap N_G(w) = \emptyset$. In the second case, if $w \notin V(\overline{\mathcal{D}})$, then the claim trivially holds. Otherwise, v and w lie in the clique path $\mathbb{P}$. Since $w \notin N_G(v)$, there is no bag that contains both v and w, and $v \in B_i$. On the one hand, w appears only in the bags (strictly) to the left of B_i, in which case v being an unmarked vertex implies that $L^{\mathbb{F},i}_{\text{cls}} \cap N_G(w) = \emptyset$. On the other hand, if w appears only in the bags (strictly) to the right of B_i, we have $R^{\mathbb{F},i}_{\text{cls}} \cap N_G(w) = \emptyset$. □

Next, we give a reduction rule that deletes unmarked vertices from B_i in G.

Reduction Rule 5.1. Let v be a vertex in $B_i \setminus H_i$. Delete v from G—that is, the resulting instance is $(G - \{v\}, k)$.

Lemma 5.3. *Reduction Rule 5.1 is safe.*

Before moving to the proof of Lemma 5.3, we note that using it we immediately obtain the following lemma.

Lemma 5.4. *If Reduction Rule 5.1 is not applicable, then for each $j \in [t]$, we have $|B_j| \leq \eta$.*

Proof. Follows from the safeness of Reduction Rule 5.1 (Lemma 5.3) and the fact that $|H_j| \leq \eta$, for each $j \in [t]$. □

In the remainder of this section, we focus on the proof of Lemma 5.3. Let v be a vertex in $B_i \setminus H_i$ and $G' = G - \{v\}$. We will show that (G, k) is a Yes-instance of IVD if and only if (G', k) is a Yes-instance of IVD. In the forward direction, let S be a solution to IVD in (G, k). As $G - S$ is an interval graph and so are all its induced subgraphs, we therefore have that $S \setminus \{v\}$ is a solution to IVD in (G', k).

In the reverse direction, let S be a solution to IVD in (G', k). We will show that $G - S$ is an interval graph. Suppose not, then there must be an obstruction in $G - S$. Note that all the obstructions in $G - S$ are guaranteed to contain v, as otherwise the obstruction is also present in $G' - S$, which contradicts that S is a solution to IVD in (G', k). This implies that $S \cup \{v\}$ is a $(k + 1)$-solution for G. Recall that $\mathcal{W}$ is $(k + 1)$-necessary, and therefore $S \cup \{v\}$ hits $\mathcal{W}$. Since $v \notin M$ and $\mathcal{W} \subseteq 2^M$, we derive that S hits $\mathcal{W}$. But then any obstruction in $G - S$ is not covered by $\mathcal{W}$ since $v \notin M$. This together with the fact that M is a 9-redundant solution implies that for any obstruction $\mathbb{O}'$ in $G - S$, we have $|V(\mathbb{O}') \cap M| \geq 10$. Moreover, such an obstruction can either be a cycle, a †-AW, or a ‡-AW on at least 10 vertices. Among all obstructions in $G - S$ (containing v), we will proof the correctness of the lemma by carefully choosing an (available) obstruction, and in each case arriving at some contradiction. In the following, we describe the choice of the obstruction $\mathbb{O}$ in $G - S$:

(1) If $G - S$ has an induced cycle Q (containing v) of length at least 10, then $\mathbb{O}$ is set to Q.
(2) Otherwise, $\mathbb{O}$ is an obstruction in $G - S$ (containing v) of minimum possible size, and over all such minimum sized obstructions, $\mathbb{O}$ maximizes the number of vertices from B_i.

We will consider cases depending on which type of obstruction $\mathbb{O}$ is, as well as the role that v plays in $\mathbb{O}$. In the case when $\mathbb{O}$ is an induced cycle, our goal will be to obtain an obstruction not containing v in $G - S$. In all other cases, we either will obtain an obstruction not containing v, or a smaller sized obstruction, or an obstruction that has the same number of vertices as $\mathbb{O}$ but has more vertices from B_i than $\mathbb{O}$ has from B_i. In each such case, this will contradict the choice of $\mathbb{O}$.

Next, we consider the cases depending on whether $\mathbb{O}$ is a cycle, a †-AW, or a ‡-AW. We remark that whenever we are dealing with a particular case, we will assume that the cases stated prior to it are not applicable.

$\mathbb{O}$ Is a Cycle

Let us first note that $|V(\mathbb{O}) \cap B_i| \leq 2$ as B_i is a clique. Let x, y be the neighbors of v in $\mathbb{O}$, and note that they lie in $M \cup \beta(\mathbb{P})$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $|V(\mathbb{O}) \cap M| \geq 10$. Let $\hat{M} = M \cap V(\mathbb{O})$, $M' \subseteq \hat{M}$ of size 3 such that $\hat{M} \cap \{x, y\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap \{x, y\})$. Next, consider the sets $L_{\text{far}} = L^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$ and $R_{\text{far}} = R^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$. Since $|S| \leq k$, $v \notin H_i$, and B_i is a clique, therefore $L_{\text{far}}, R_{\text{far}} \neq \emptyset$. Let $z \in M' \setminus \{x, y\}$, which exists since $|M'| = 3$. Now suppose that there is $v^* \in L_{\text{far}} \cup R_{\text{far}}$ such that $(v^*, x), (v^*, y) \in E(G)$, then we claim that we can obtain a cycle on at least four vertices not containing v in $G - S$. Since v^* fits $\mathbb{F}$, therefore $(v^*, z) \notin E(G)$. Consider the paths P_{xz} and P_{yz} from x to z and y to z in $\mathbb{O} - \{v\}$, respectively. Furthermore, let x^* and y^* be the last vertices in P_{xz} and P_{yz} which are adjacent to v^*. Note that x^* and y^* exists since $(x, v^*), (y, v^*) \in E(G)$. But then the path from x^* to y^* in $\mathbb{O} - \{v\}$ along with v^* forms an induced cycle on at least four vertices in $G - S$ that does not contain v.

Next, we assume that any vertex in $L_{\text{far}} \cup R_{\text{far}}$ is adjacent to at most one of x, y. From Observation 5.1 (together with $(x, y) \notin E(G)$), it follows that either $L_{\text{far}} \subseteq N_G(x)$ and $R_{\text{far}} \subseteq N_G(y)$, or $R_{\text{far}} \subseteq N_G(x)$ and $L_{\text{far}} \subseteq N_G(y)$, must hold. Suppose that $L_{\text{far}} \subseteq N_G(x)$ and $R_{\text{far}} \subseteq N_G(y)$ (the other case is symmetric). Consider vertices $u^* \in L_{\text{far}}$ and $v^* \in R_{\text{far}}$. Note that $(u^*, x), (v^*, y), (u^*, v^*) \in E(G)$ and $(u^*, y), (v^*, x), (u^*, z), (v^*, z) \notin E(G)$. Consider the paths P_{xz} and P_{yz} from x to z and y to z in $\mathbb{O} - \{v\}$, respectively. Let x^* be the last vertex in the path P_{xz} such that $N_G(x^*) \cap \{u^*, v^*\} \neq \emptyset$. Similarly, let y^* be the last vertex in the path P_{yz} such that $N_G(y^*) \cap \{u^*, v^*\} \neq \emptyset$. Let P_{x^*z} and P_{zy^*} be the paths from x^* to z and z to y^* in $\mathbb{O} - \{v\}$, respectively. Notice that $G[V(P_{x^*z}) \cup V(P_{zy^*}) \cup \{u^*, v^*\}]$ contains an induced cycle (not containing v) on at least four vertices.

$\mathbb{O}$ Is a †-AW

Let $\mathbb{O}$ comprise of the base path $\text{base}(\mathbb{O}) = (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and center c (as in the definition in Section 2). Furthermore, let $P(\mathbb{O}) = (t_\ell, b_1, b_2, \ldots, b_z, t_r)$, and let $b_0 = t_\ell$, and $b_{z+1} = t_r$. Let $\hat{M} = M \cap V(\mathbb{O})$. Recall that $\mathbb{O}$ is not covered by $\mathcal{W}$, and thus $|\hat{M}| \geq 10$. Let M' be a subset of $\hat{M}$ of size 8 such that $\hat{M} \cap \{c, t, t_\ell, t_r, b_1, b_2, b_{z-1}, b_z\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap N_G(v))$. Next, we define the following sets, whose vertices will be used to either construct an obstruction not containing v, or an obstruction containing v but with (strictly) smaller size, or an obstruction with same number of vertices as $\mathbb{O}$ but containing strictly more vertices from B_i than $\mathbb{O}$ contains from B_i. Let $L_{\text{far}} = L^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$, $L_{\text{cls}} = L^{\mathbb{F},i}_{\text{cls}} \setminus (S \cup V(\mathbb{O}))$, $R_{\text{far}} = R^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$, and $R_{\text{cls}} = R^{\mathbb{F},i}_{\text{cls}} \setminus (S \cup V(\mathbb{O}))$. Notice that $|V(\mathbb{O}) \cap B_i| \leq 3$, since no †-AW contains a clique of size 4 and $G[B_i]$ is a clique. This together with the fact that $v \notin H_i$ and $|S| \leq k$ implies that $L_{\text{far}}, L_{\text{cls}}, R_{\text{far}}, R_{\text{cls}} \neq \emptyset$. Next, we consider cases depending on the role that v plays in the obstruction $\mathbb{O}$.

Suppose v Is the Shallow Terminal. In this case, $(v, c) \in E(G)$, and therefore from Observation 5.1, one of $L_{\text{far}} \subseteq N_G(c)$ or $R_{\text{far}} \subseteq N_G(c)$ must hold. Consider the case when $L_{\text{far}} \subseteq N_G(c)$ (the other

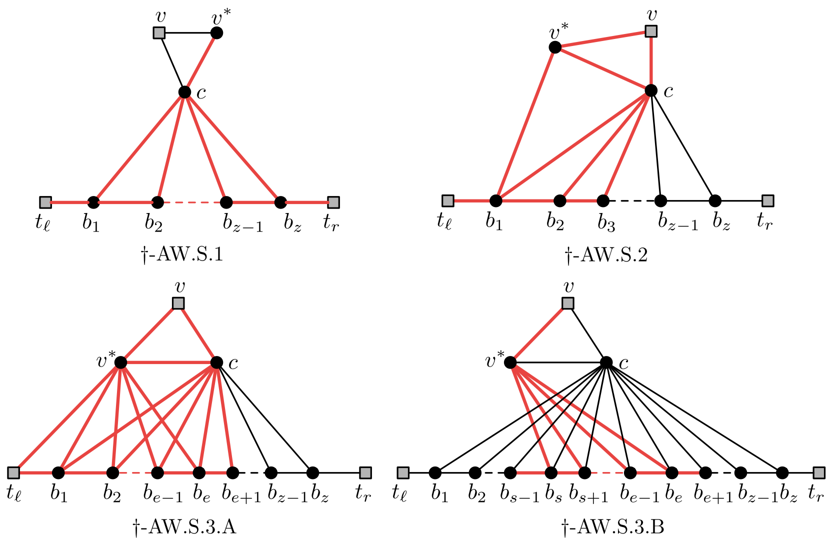

Fig. 2. Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = t$.

case is symmetric), and let v^* be a vertex in L_{far}. Next, we consider the following cases based on the neighborhood of v^* in $\mathbb{O}$ (Figure 2).

Case †-AW.S.1. $\boldsymbol{N_G(v^*) \cap V(P(\mathbb{O}))| = 0}$. In this case, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{v^*\}]$ is a †-AW in $G' - S$.

Case †-AW.S.2. If $\boldsymbol{|N_G(v^*) \cap V(P(\mathbb{O}))| = 1}$. If $(v^*, t_\ell) \in E(G)$, then $G[\{v^*, c, t_\ell, b_1\}]$ is an induced cycle on four vertices not containing v in $G - S$. Analogous argument can be given when $(v^*, t_r) \in E(G)$. Therefore, we assume that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_i\}$, where $i \in [z]$. If $i \in [z] \setminus \{1, z\}$, then $G[\{v^*, v, b_i, b_{i-1}, b_{i-2}, b_{i+1}, b_{i+2}\}]$ is a long claw in $G - S$. This cannot happen, as any obstruction in $G - S$ is of size at least 10. If none of the preceding cases are applicable, then $N_G(v^*) \cap V(P(\mathbb{O})) \in \{\{b_1\}, \{b_z\}\}$. Suppose that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_1\}$ (the other case is symmetric), then $G[\{c, v, v^*, b_1, b_2, b_3, t_\ell\}]$ is a whipping top in $G - S$.

Case †-AW.S.3. $\boldsymbol{|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2}$. If neighbors of v^* are not consecutive in the path $P(\mathbb{O})$, then we can obtain an induced cycle on at least four vertices in $G[\{v^*\} \cup V(P(\mathbb{O}))]$, and therefore we assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive. By the construction of $\mathbb{F}$ and v^*, we know that there are at least seven vertices in $P(\mathbb{O})$ that are non-adjacent to v^* (recall that we are in the case when v is the shallow terminal). This also implies that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Without loss of generality, we assume that $(v^*, t_r) \notin E(G)$. Next, we consider the following cases based on whether or not $(v^*, t_\ell) \in E(G)$:

(A) $\boldsymbol{(v^*, t_\ell) \in E(G)}$. In this case, there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. Let $V' = \{v, v^*, c, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW with $|V'| < |V(\mathbb{O})|$, a contradiction to the choice of $\mathbb{O}$.

(B) $\boldsymbol{(v^*, t_\ell) \notin E(G)}$. Let b_s and b_e be the first and the last vertices in $P(\mathbb{O})$ that are adjacent to v^*, respectively. Notice that $s \neq e$ (since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$), and $\{b_s, b_{s+1}, \ldots b_e, b_{e+1}\} \subset \{b_1, b_2, \ldots b_z\}$ (strict subset). Let $V' = \{v, v^*\} \cup \{b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e+1}\}$. Observe that $|V'| < |V(\mathbb{O})|$ and $G[V']$ is a †-AW.

Suppose v Is the Center. In this case, $(t_\ell, v), (t_r, v) \notin E(G)$. Since $v \notin H_i$ and each vertex in $L_{\text{cls}} \cup R_{\text{cls}}$ fits the frame $\mathbb{F}$, from Observation 5.2 one of the following holds: (1) $N_G(t_\ell) \cap L_{\text{cls}} = \emptyset$

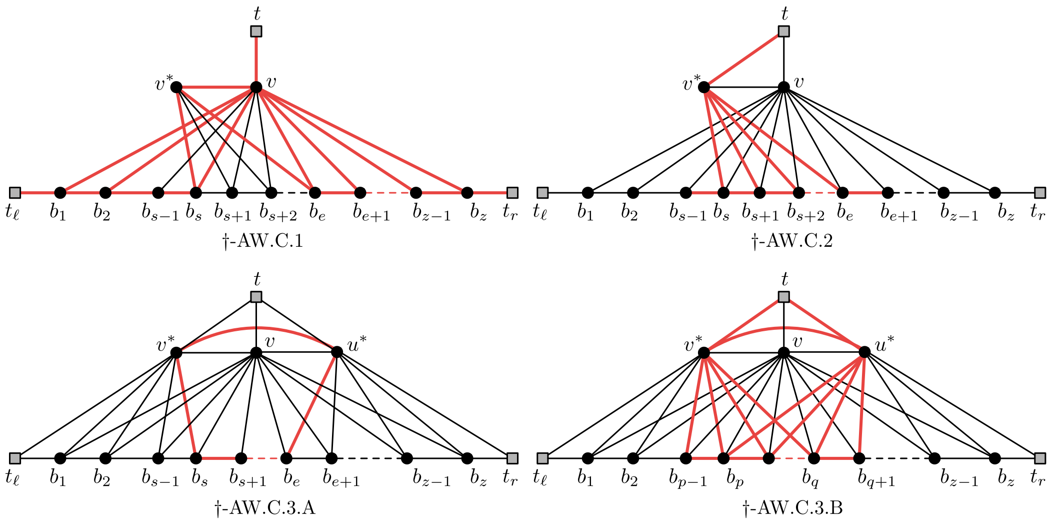

Fig. 3. Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = c$.

and $N_G(t_r) \cap R_{\text{cls}} = \emptyset$, (2) $N_G(t_r) \cap L_{\text{cls}} = \emptyset$ and $N_G(t_\ell) \cap R_{\text{cls}} = \emptyset$, (3) $N_G(t_\ell) \cap L_{\text{cls}} = \emptyset$ and $N_G(t_r) \cap L_{\text{cls}} = \emptyset$, or (4) $N_G(t_\ell) \cap R_{\text{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\text{cls}} = \emptyset$. Consider a vertex $v^* \in L_{\text{cls}} \cup R_{\text{cls}}$, and let b_s and b_e be the first and the last vertices in the path $P(\mathbb{O})$ that are adjacent to v^*, respectively. The existence and distinctness of b_s, b_e follow from the fact that $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 5$, which in turn is implied from the choice of M' and v^* fitting the frame $\mathbb{F}$. The neighbors of v^* in $P(\mathbb{O})$ must be consecutive, as otherwise we can obtain an induced cycle of length at least 4, which does not contain v. We further consider subcases based on whether or not the following two criterions are satisfied (Figure 3):

(1) $t \in N_G(v^*)$;
(2) $N_G(v^*) \cap \{t_\ell, t_r\} = \emptyset$.

Case †-AW.C.1. $\boldsymbol{t \notin N_G(v^*)}$. If $\{t_\ell, t_r\} \subseteq N_G(v^*)$, then $G[\{v^*, t_\ell, b_1, v, b_z, t_r, t\}]$ is a whipping top. Here, we rely on the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive and b_1 and b_z are not adjacent as $\mathbb{O}$ has at least 11 vertices. From the preceding, we can assume that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Let $V' = (V(\mathbb{O}) \setminus \{b_{s+1}, b_{s+2}, \ldots, b_{e-1}\}) \cup \{v^*\}$. Notice that $|V'| < |V(\mathbb{O})|$ since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 5$ and neighbors of v^* are consecutive. Moreover, $G[V']$ is an (induced) †-AW or a net, which is of strictly smaller size than $\mathbb{O}$, contradicting the choice of $\mathbb{O}$. Here, we crucially rely on the fact that $|N_G(v^*) \cap \{t_\ell, t_r\}| \leq 1$.

Case †-AW.C.2. $\boldsymbol{t \in N_G(v^*)}$ and $\boldsymbol{N_G(v^*) \cap \{t_\ell, t_r\} = \emptyset}$. In this case, $G[\{v^*, t, b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e+1}\}]$ forms an (induced) †-AW in $G - S$ that does not contain v.

If Cases †-AW.C.1 and †-AW.C.2 are not applicable, then for each $u \in L_{\text{cls}} \cup R_{\text{cls}}$ we have $t \in N_G(u)$ and $N_G(u) \cap \{t_\ell, t_r\} \neq \emptyset$. Furthermore, $v \notin H_i$, $(t_\ell, v), (t_r, v) \notin E(G)$, and each vertex in $L_{\text{cls}} \cup R_{\text{cls}}$ fits the frame $\mathbb{F}$. Therefore, one of the following must hold: (1) $N_G(t_\ell) \cap L_{\text{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\text{cls}} = \emptyset$ or (2) $N_G(t_r) \cap L_{\text{cls}} = \emptyset$ and $N_G(t_\ell) \cap R_{\text{cls}} = \emptyset$. Thus, for each $u \in L_{\text{cls}} \cup R_{\text{cls}}$, we have $|N_G(u) \cap \{t_\ell, t_r\}| = 1$. We assume that $N_G(t_\ell) \cap L_{\text{cls}} = \emptyset$ and $N_G(t_r) \cap R_{\text{cls}} = \emptyset$ (the other case is symmetric). Next, we consider a vertex $u^* \in L_{\text{cls}}$ and a vertex $v^* \in R_{\text{cls}}$. Notice that (by the preceding discussion) $t \in N_G(u^*) \cap N_G(v^*)$, $t_\ell \notin N_G(u^*)$, $t_r \in N_G(u^*)$, $t_r \notin N_G(v^*)$, and $t_\ell \in N_G(v^*)$. Additionally, since $u^*, v^* \in B_i$, we have $(u^*, v^*) \in E(G)$. We now consider the remaining case.

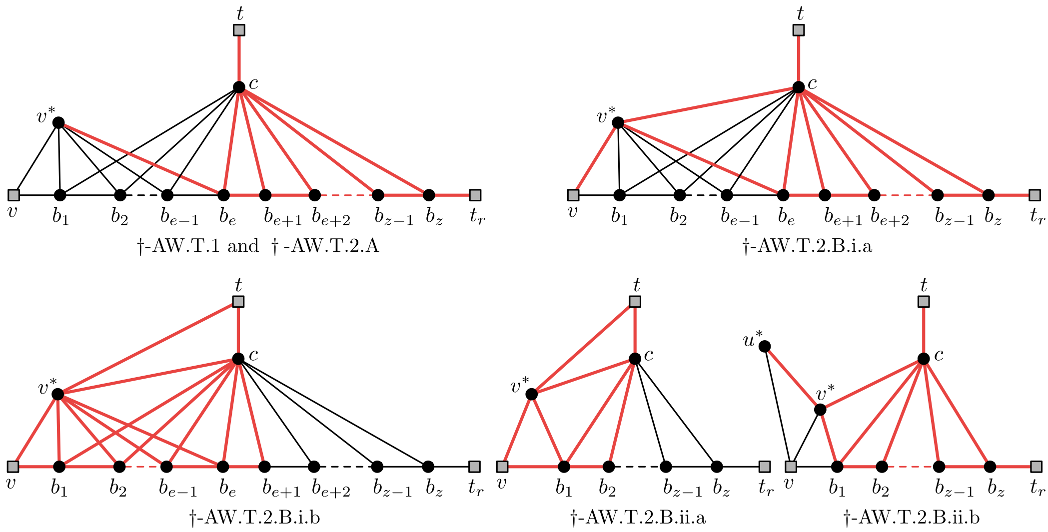

Fig. 4. Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = t_\ell$.

Case †-AW.C.3. $\boldsymbol{t \in N_G(u^*) \cap N_G(v^*)}$, $\boldsymbol{N_G(u^*) \cap \{t_\ell, t_r\} = \{t_r\}}$, and $\boldsymbol{N_G(v^*) \cap \{t_\ell, t_r\} = \{t_\ell\}}$. We consider the following subcases:

(A) $\boldsymbol{u^*}$ and $\boldsymbol{v^*}$ have no common neighbor in $\boldsymbol{P(\mathbb{O})}$. Then $G[\{u^*, v^*\} \cup V(P(\mathbb{O}))]$ contains an (induced) cycle on at least four vertices.

(B) Otherwise, $\boldsymbol{u^*}$ and $\boldsymbol{v^*}$ have at least one common neighbor in $\boldsymbol{P(\mathbb{O})}$. Let b_p and b_q be the first and the last common neighbors of u^* and v^* in $P(\mathbb{O})$, respectively. Notice that $b_{p-1} \in N_G(v^*)$ and $b_{p-1} \notin N_G(u^*)$. This follows from the fact that $t_\ell, b_q \in N_G(v^*)$, neighbors of v^* are consecutive vertices in $P(\mathbb{O})$, $t_\ell \notin N_G(u^*)$, and p is the first common neighbor of u^* and v^* in $P(\mathbb{O})$. Similarly, we can argue that $b_{q+1} \in N_G(u^*)$ and $b_{q+1} \notin N_G(v^*)$. Consider the set $V' = \{t, v^*, u^*\} \cup \{b_{p-1}, b_p, \dots, b_q, b_{q+1}\}$. Notice that $G[V']$ is a ‡-AW or a tent that does not contain v.

Suppose v Is One of the Non-Shallow Terminals. We consider the case when $v = t_\ell$. By a symmetric argument, we can handle the case when $v = t_r$. If $c \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{cls}} \cup R_{\mathsf{cls}}$ we have $(u, c) \notin E(G)$, as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $c \in \beta(\mathbb{P})$, and then from Observation 5.2, at least one of $L_{\mathsf{cls}} \cap N_G(c) = \emptyset$ or $R_{\mathsf{cls}} \cap N_G(c) = \emptyset$ holds. Let $X_{\mathsf{cls}} \in \{L_{\mathsf{cls}}, R_{\mathsf{cls}}\}$ be a set such that $X_{\mathsf{cls}} \cap N_G(c) = \emptyset$. Similarly, if $b_1 \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_1) \in E(G)$, as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_1 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_1)$ or $R_{\mathsf{far}} \subseteq N_G(b_1)$ holds (see Observation 5.1). Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_1)$. Next, we consider cases based on whether or not $b_1 \in B_i$ (Figure 4).

Case †-AW.T.1. $\boldsymbol{b_1 \in B_i}$. Consider a vertex $v^* \in X_{\mathsf{cls}}$. Note that $(v^*, b_1) \in E(G)$ since $b_1 \in B_i$, and $(v^*, c) \notin E(G)$ by the choice of v^*. Additionally, $(v^*, t) \notin E(G)$, and otherwise $G[\{t, c, b_1, v^*\}]$ is cycle on four vertices in $G - S$. Recall that v^* fits the frame $\mathbb{F}$ (and $(b_1, v^*) \in E(G)$), and therefore there exists b_e such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$, where $e \in [z-1]$ (possibly $e = 1$). This together with the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive (otherwise, we obtain an induced cycle on at least four vertices not containing v) implies that $(v^*, t_r) \notin E(G)$. But then $G[\{t, c, v^*\} \cup \{b_e, b_{e+1}, \dots, b_z, t_r\}]$ is a †-AW (or a net) that does not contain v.

Case †-AW.T.2. $\boldsymbol{b_1 \notin B_i}$. Consider a vertex $v^* \in Y_{\mathsf{far}} \cup \{u \in X_{\mathsf{cls}} \mid (u, b_1) \in E(G)\}$, and the following cases based on its neighborhood in $\mathbb{O}$:

(A) $\boldsymbol{(v^*, c) \notin E(G)}$. In this case, $(v^*, t) \notin E(G)$, and otherwise $G[\{v^*, t, c, b_1\}]$ is an induced cycle on four vertices. Recall that v^* fits the frame $\mathbb{F}$, and therefore there are at least five vertices in $P(\mathbb{O})$ that are non-adjacent to v^*. This together with the fact that $(b_1, v^*) \in E(G)$ implies that there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. But then $G[V']$ is a †-AW (or a net) not containing v in $G - S$, where $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$.

(B) $\boldsymbol{(v^*, c) \in E(G)}$. We further consider the following cases:
(i) There exists $\boldsymbol{e \in [z] \setminus \{1\}}$ such that $\boldsymbol{b_e \in N_G(v^*)}$ and $\boldsymbol{b_{e+1} \notin N_G(v^*)}$. By the choice of M' and the fact that v^* fits $\mathbb{F}$, we have $e \leq z-2$. Consider the following cases based on whether or not $(t, v^*) \in E(G)$:
  (a) $\boldsymbol{(t, v^*) \notin E(G)}$. Let $V' = \{t, c, v^*, v, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$. Furthermore, either $|V'| < |V(\mathbb{O})|$ or $|V'| = |V(\mathbb{O})|$ and $|V' \cap B_i| > |V(\mathbb{O}) \cap B_i|$. Here, we rely on the fact that $b_1 \notin B_i$. In either case, we obtain a contradiction to the choice of $\mathbb{O}$.
  (b) $\boldsymbol{(t, v^*) \in E(G)}$. Let $V' = \{t, c, v^*, v\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ and $|V'| < |V(\mathbb{O})|$, which contradicts the choice of $\mathbb{O}$.
(ii) Otherwise, if (i) does not hold, then the only neighbors of v^* in $P(\mathbb{O})$ are b_1 and v. Consider the following cases based on whether or not $(t, v^*) \in E(G)$:
  (a) $\boldsymbol{(t, v^*) \in E(G)}$. In this case, $G[\{v, v^*, t, c, b_1, b_2\}]$ is a tent.
  (b) $\boldsymbol{(t, v^*) \notin E(G)}$. We consider a vertex in $u^* \in X_{\mathsf{cls}}$ to obtain the desired obstruction. We can assume that $(b_1, u^*) \notin E(G)$ as $X_{\mathsf{cls}} \cap N_G(c) = \emptyset$ and Case †-AW.T.2.A is not applicable. Furthermore, $(b_j, u^*) \notin E(G)$, for each $j \in [z] \setminus \{1\}$, and otherwise $G[\{v, u^*\} \cup \{b_1, b_2, \ldots b_j\}]$ will contain an induced cycle on at least four vertices. Let $V' = (V(\mathbb{O}) \setminus \{v\}) \cup \{v^*, u^*\}$. Observe that $G[V']$ is a †-AW that does not contain v.

Suppose v Is Either b_1 or b_z. Suppose $v = b_1$ (the other case is symmetric). If $t_\ell \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, t_\ell) \in E(G)$, as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $t_\ell \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(t_\ell)$ or $R_{\mathsf{far}} \subseteq N_G(t_\ell)$ holds (see Observation 5.1). Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$. Similarly, if $b_2 \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_2) \in E(G)$, as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_2 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_2)$ or $R_{\mathsf{far}} \subseteq N_G(b_2)$ holds. Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_2)$. Next, we consider cases depending on the neighborhood of vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ in $\mathbb{O}$ (Figure 5).

Case †-AW.B.1. There is a vertex $\boldsymbol{v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}}$ such that $\boldsymbol{\{t_\ell, b_2\} \subseteq N_G(v^*)}$. There exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This follows from the fact that $(v^*, b_2) \in E(G)$ and v^* fits the frame $\mathbb{F}$. Next, we consider the subcases based on whether or not $(v^*, c), (v^*, t) \in E(G)$:

(A) $\boldsymbol{(v^*, c) \in E(G), (v^*, t) \notin E(G).}$ Let $V' = \{t, c, v^*, t_\ell, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW that does not contain v.

(B) $\boldsymbol{(v^*, c) \in E(G), (v^*, t) \in E(G).}$ Let $V' = \{t, c, v^*, v, t_\ell\} \cup \{b_2, b_3, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW that has strictly fewer vertices than $\mathbb{O}$.

(C) $\boldsymbol{(v^*, c) \notin E(G).}$ Notice that in this case $(v^*, t) \notin E(G)$, and otherwise $G[\{v^*, t, c, b_2\}]$ is an induced cycle on four vertices. Let $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is an induced †-AW that does not contain v.

Case †-AW.B.2. Suppose that for every $\boldsymbol{u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}}$, we have $\boldsymbol{(u, c) \in E(G)}$. Since Case †-AW.B.1 is not applicable, we can assume that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \not\subseteq N_G(u)$. By the construction of X_{far} and Y_{far}, we know that for each $u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ we have $\{t_\ell, b_2\} \cap$

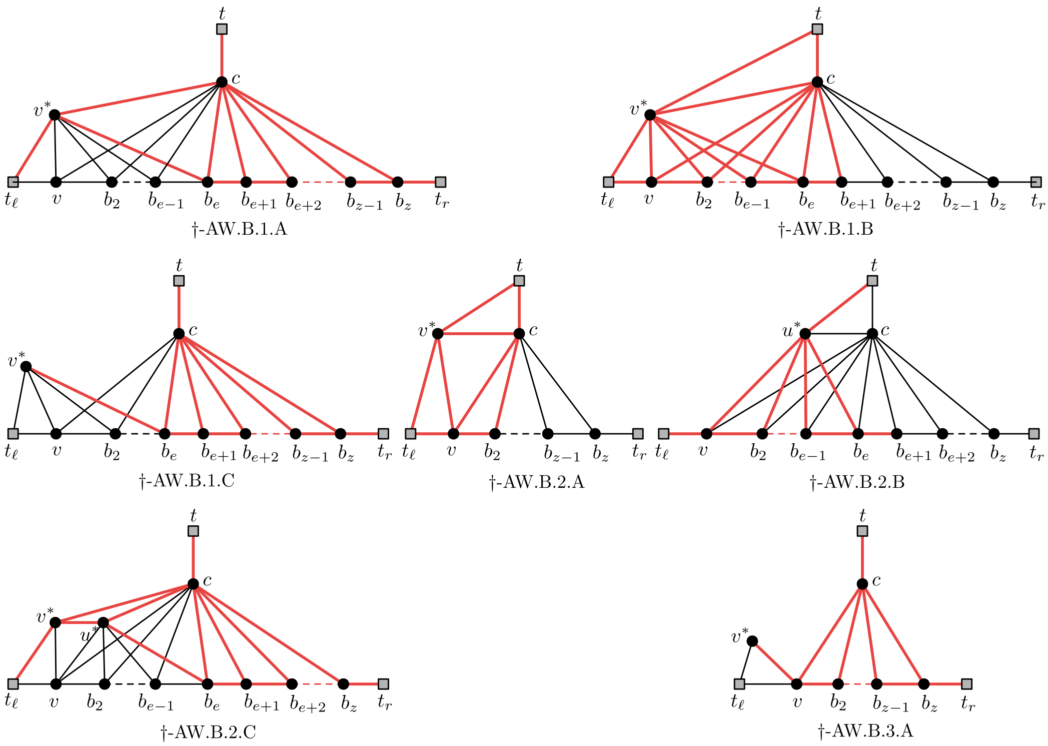

Fig. 5. Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = b_1$.

$N_G(u) \neq \emptyset$, and $X_{\text{far}}, Y_{\text{far}} \neq \emptyset$. Consider a vertex $v^* \in X_{\text{far}}$ and a vertex $u^* \in Y_{\text{far}}$. We have that $(v^*, c), (u^*, c), (v^*, t_\ell), (u^*, b_2) \in E(G)$ and $(v^*, b_2), (u^*, t_\ell) \notin E(G)$. Next, we consider cases based on whether or not t adjacent to v^* and u^*:

(A) $\boldsymbol{(t, v^*) \in E(G)}$**.** Recall that $b_2 \notin N_G(v^*)$ and $t_\ell, t, c \in N_G(v^*)$. But then $G[\{c, v, v^*, b_2, t_\ell, t\}]$ is a tent in $G - S$.

(B) $\boldsymbol{(t, u^*) \in E(G)}$**.** There exists $e \in [z-2]$ such that $b_e \in N_G(u^*)$ and $b_{e+1} \notin N_G(u^*)$. This follows from the fact that $(u^*, b_2) \in E(G)$ and u^* fits the frame $\mathbb{F}$. Let $V' = \{b_2, b_3, \ldots, b_e, b_{e+1}\} \cup \{t, u^*, t_\ell, v\}$. Then $G[V']$ is a †-AW in $G - S$ which has strictly fewer vertices than $\mathbb{O}$.

(C) $\boldsymbol{(t, v^*), (t, u^*) \notin E(G)}$**.** We start by arguing that v^* cannot be adjacent to b_j, where $j \in [z] \setminus \{1\}$. For $j = 2$, it follows from the choice of v^*. Next, consider the smallest $j > 2$ such that $(v^*, b_j) \in E(G)$. Then, $G[\{v, v^*\} \cup \{b_2, b_3, \ldots, b_j\}]$ is an induced cycle on at least four vertices, which is a contradiction, as we assume previously stated cases are not applicable. Therefore, we assume that the only neighbor of v^* in $P(\mathbb{O})$ are v and t_ℓ. Next, we argue about neighbors of u^* in $P(\mathbb{O})$. There exists $e \in [z-2]$ such that $b_e \in N_G(u^*)$ and $b_{e+1} \notin N_G(u^*)$. This follows from the fact that $(u^*, b_2) \in E(G)$ and u^* fits the frame $\mathbb{F}$. Let $V' = \{t, c, t_\ell, t_r, v^*, u^*\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$ that does not contain v.

Case †-AW.B.3. Suppose that there is $\boldsymbol{u \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(u, c) \notin E(G)}$, and for all $\boldsymbol{u \in X_{\text{far}} \cup Y_{\text{far}}}$ we have $\boldsymbol{\{t_\ell, b_2\} \not\subseteq N_G(u)}$. Consider vertices $v^* \in X_{\text{far}}$ and $u^* \in Y_{\text{far}}$, and the following subcases.:

(A) $\boldsymbol{(v^*, c) \notin E(G)}$**.** This implies that $(v^*, t) \notin E(G)$, and otherwise $G[v^*, c, t, v]$ is a cycle on four vertices. As Case †-AW.B.1 is not applicable, for each $u \in Y_{\text{far}}$ we have $(u, b_2) \in E(G)$

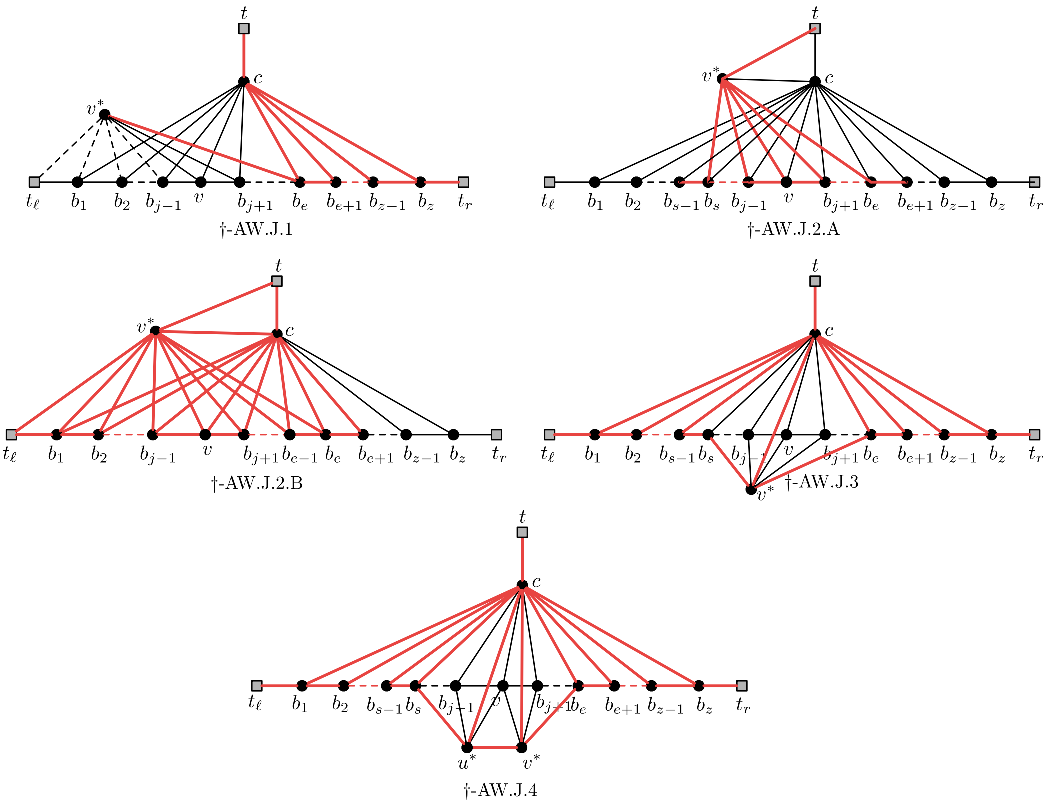

Fig. 6. Construction of an obstruction when $\mathbb{O}$ is †-AW and $v = b_j$, where $j \in [z-1] \setminus \{1\}$.

and $(u, t_\ell) \notin E(G)$. Note that since v is unmarked, therefore $Y_{\text{far}} \neq \emptyset$. From the preceding discussions, we obtain that $t_\ell \notin B_i$. Observe that v^* cannot be adjacent to any b_j, where $j \geq 2$, since the neighbors of v^* in $P(\mathbb{O})$ must be consecutive, $(v^*, t_\ell) \in E(G)$, and $(v^*, b_2) \notin E(G)$. But then $G[(V(\mathbb{O}) \setminus \{t_\ell\}) \cup \{v^*\}]$ is a †-AW with the same number of vertices as $\mathbb{O}$ but with more vertices from B_i.

(B) $\boldsymbol{(u^*, c) \notin E(G)}$. Since Case †-AW.B.3.A is not applicable, we can assume that $(v^*, c) \in E(G)$. Observe that $G[\{c, v^*, u^*, b_2\}]$ is a cycle on four vertices. Here, we rely on the fact that $(v^*, b_2) \notin E(G)$.

Suppose That $\boldsymbol{v}$ *Is a Base Vertex* $\boldsymbol{b_j}$*, Where* $\boldsymbol{j \in [z] \setminus \{1, z\}}$. Let $X_{\text{far}} \in \{L_{\text{far}}, R_{\text{far}}\}$ be a set such that $X_{\text{far}} \subseteq N_G(b_{j-1})$ and $Y_{\text{far}} \in \{L_{\text{far}}, R_{\text{far}}\}$ be a set such that $Y_{\text{far}} \subseteq N_G(b_{j+1})$. We note that existence of X_{far} and Y_{far} is guaranteed from Observation 5.1. Next, we consider cases based on the neighborhood of vertices in X_{far} and Y_{far} in $\mathbb{O}$ (Figure 6).

Case †-AW.J.1. There is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, c) \notin E(G)}$. Note that as $(v^*, c) \notin E(G)$, we have $(v^*, t) \notin E(G)$, and otherwise $G[\{v, v^*, c, t\}]$ would be an induced cycle on four vertices. All the neighbors of v^* on $P(\mathbb{O})$ must be consecutive. This together with the choice of $\mathbb{F}$ and v^* implies that one of (a) $\{t_\ell, b_1\} \cap N_G(v^*) = \emptyset$ or (b) $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ (the other case is symmetric). Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $t_r, b_z \notin N_G(v^*)$ and $N_G(v^*) \cap \{v, b_{j-1}, b_{j+1}\} \neq \emptyset$. We note that e could possibly be equal to j. Let $V' = \{t, c, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that

$|V'| < |V(\mathbb{O})|$ since $j \in [z] \setminus \{1, z\}$. Moreover, $G[V']$ is a †-AW in $G - S$, which contradicts the choice of $\mathbb{O}$.

Note that if Case †-AW.J.1 is not applicable, then for each $u \in X_{\text{far}} \cup Y_{\text{far}}$ we have $(u, c) \in E(G)$. Next, we consider cases based on whether or not the following conditions are satisfied for a vertex $u \in X_{\text{far}} \cup Y_{\text{far}}$:

(1) $(u, t) \in E(G)$;
(2) $\{b_{j-1}, b_{j+1}\} \subseteq N_G(u)$.

Case †-AW.J.2. If there is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, t) \in E(G)}$. We start by recalling the following. Since M is a 9-redundant solution and $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $|M \cap V(\mathbb{O})| \geq 10$, which implies that $|V(\mathbb{O})| \geq 10$. By the choice of $\mathbb{F}$ and the fact that $2 \leq j \leq z - 1$ (where $v = b_j$), we have at least four vertices in $V(P(\mathbb{O}))$ that are non-adjacent to v^*. Moreover, by our assumption that there is no obstruction that is an induced cycle on at least four vertices, we have that all the neighbors of v^* in $P(\mathbb{O})$ must be consecutive. From the preceding discussions, we can conclude that at least one of $\{b_1, b_2, t_\ell\} \cap N_G(v^*) = \emptyset$ or $\{b_{z-1}, b_z, t_r\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{b_{z-1}, b_z, t_r\} \cap N_G(v^*) = \emptyset$ holds (the other case is symmetric). We further consider the following subcases based on whether or not $t_\ell \in N_G(v^*)$:

(A) $\boldsymbol{t_\ell \notin N_G(v^*)}$. Let $s \in [j]$ such that b_s is the first vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $(t_\ell, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Additionally, let $e \in [z-2]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*), (b_{z-1}, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Notice that $s \neq e$, since by the construction of the sets X_{far} and Y_{far} we have that v^* is incident to v and at least one of the vertices in $\{b_{j-1}, b_{j+1}\}$. Let $V' = \{t, v^*\} \cup \{b_{s-1}, b_s, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, c, b_z \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

(B) $\boldsymbol{t_\ell \in N_G(v^*)}$. Let $e \in [z-2]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*), (b_{z-1}, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Let $V' = \{t, v^*, c, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, b_z \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

Case †-AW.J.3. There is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, t) \notin E(G)}$ and $\boldsymbol{\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)}$. Notice that all the neighbors of v^* on $P(\mathbb{O})$ must be consecutive, and there are at least four vertices on $P(\mathbb{O})$ that are non-adjacent to v^*. This follows from the facts that M is a 9-redundant solution, $\mathbb{O}$ is not covered by $\mathcal{W}$, $G - S$ has no obstructions that are induced cycles, and the choices of $\mathbb{F}$ and v^*. From the preceding discussions, we can conclude that one of $\{t_\ell, b_1\} \cap N_G(v^*) = \emptyset$ or $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ must hold. Suppose that $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ (other case is symmetric). Let $e \in [z - 1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $t_r, b_z \notin N_G(v^*)$ and $\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)$. Additionally, let $s \in [z - 1] \cup \{0\}$ be the lowest integer such that $(v^*, b_s) \in E(G)$ (b_s could possibly be same as b_{j-1} or $b_0 = t_\ell$). Let $V' = \{t, c, v^*, t_\ell, t_r\} \cup \{b_1, b_2, \ldots, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is an induced †-AW in $G - S$, which does not contain v. Here, we rely on the fact that Case †-AW.J.1 is not applicable, due to which we have $(v^*, c) \in E(G)$.

Case †-AW.J.4. For all $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$, we have $\boldsymbol{(v^*, t) \notin E(G)}$ and $\boldsymbol{\{b_{j-1}, b_{j+1}\} \not\subseteq N_G(v^*)}$. The non-applicability of Case †-AW.J.1, †-AW.J.2, and †-AW.J.3 (together with the constructions of X_{far} and Y_{far}) imply that for each $u \in X_{\text{far}} \cup Y_{\text{far}}$ we have $(u, c) \in E(G)$, $(u, t) \notin E(G)$, and $|N_G(u) \cap \{b_{j-1}, b_{j+1}\}| = 1$. Next, consider a vertex $u^* \in X_{\text{far}}$ and $v^* \in Y_{\text{far}}$. Let $s \in [j-1] \cup \{0\}$ such that b_s is the first vertex in $P(\mathbb{O})$ adjacent to u^*, which exists since $(u^*, b_{j-1}) \in E(G)$. Additionally, let $e \in [z+1]$ such that b_e is the last vertex in $P(\mathbb{O})$ adjacent to v^*, which exists since $(v^*, b_{j+1}) \in E(G)$. Recall that

$(u^*, b_{j-1}), (v^*, b_{j+1}) \in E(G)$ and $(u^*, b_{j+1}), (v^*, b_{j-1}) \notin E(G)$. Moreover, the neighbors of u^* and the neighbors of v^* in $P(\mathbb{O})$ must be consecutive vertices in $P(\mathbb{O})$, respectively. From the preceding discussions, we can conclude that $s \neq e$. Now, we let $V' = \{t, c, v^*, u^*\} \cup \{t_\ell, b_1, b_2, b_{s-1}, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z, t_r\}$. Observe that $G[V']$ is a †-AW (or a net) in $G - S$ that does not contain v.

$\mathbb{O}$ Is a ‡-AW

Let $\mathbb{O}$ comprise of the base path base($\mathbb{O}$) $= (b_1, b_2, \ldots, b_z)$, non-shallow terminals t_ℓ and t_r, shallow terminal t, and centers c_1 and c_2 (as in the definition in Section 2). Furthermore, let $P(\mathbb{O}) = (t_\ell, b_1, b_2, \ldots, b_z, t_r)$, $b_0 = t_\ell$, and $b_{z+1} = t_r$. Let $\hat{M} = M \cap V(\mathbb{O})$, M' be a subset of $\hat{M}$ of size 9 such that $\hat{M} \cap \{c_1, c_2, t, t_\ell, t_r, b_1, b_2, b_{z-1}, b_z\} \subseteq M'$, and $\mathbb{F} = (M', M' \cap N_G(v))$. Next, we define the sets, the vertices from which will be used to either construct an obstruction not containing v, an obstruction containing v but with (strictly) smaller size, or an obstruction with same number of vertices as $\mathbb{O}$ but containing more vertices from B_i. Let $L_{\text{far}} = L^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$, $L_{\text{cls}} = L^{\mathbb{F},i}_{\text{cls}} \setminus (S \cup V(\mathbb{O}))$, $R_{\text{far}} = R^{\mathbb{F},i}_{\text{far}} \setminus (S \cup V(\mathbb{O}))$, and $R_{\text{cls}} = R^{\mathbb{F},i}_{\text{cls}} \setminus (S \cup V(\mathbb{O}))$. Notice that $|V(\mathbb{O}) \cap B_i| \leq 4$, since no obstruction contains a clique of size 5 and $G[B_i]$ is a clique. This together with the fact that $v \notin H_i$ and $|S| \leq k$ implies that $L_{\text{far}}, L_{\text{cls}}, R_{\text{far}}, R_{\text{cls}} \neq \emptyset$. Next, we consider cases depending on the role that v plays in $\mathbb{O}$.

Suppose That v Is the Shallow Terminal. For a vertex $u \in L_{\text{far}} \cup R_{\text{far}}$, we have $\{c_1, c_2\} \cap N_G(u) \neq \emptyset$. This follows from Observation 5.1 and the fact that $(v, c_1), (v, c_2) \in E(G)$. Next, consider the following cases depending on the neighborhood of vertices in $L_{\text{far}} \cup R_{\text{far}}$ in $\mathbb{O}$.

Case ‡-AW.S.1. There is $\boldsymbol{v^* \in L_{\text{far}} \cup R_{\text{far}}}$ such that $\boldsymbol{c_1, c_2 \in N_G(v^*)}$. We further consider subcases based on other neighbors (if any) of v^* in $\mathbb{O}$ (Figure 7):

(A) $\boldsymbol{|N_G(v^*) \cap V(P(\mathbb{O}))| = 0.}$ In this case, $G[(V(\mathbb{O}) \setminus \{v\}) \cup \{v^*\}]$ is a ‡-AW in $G - S$.
(B) $\boldsymbol{|N_G(v^*) \cap V(P(\mathbb{O}))| = 1.}$ If $(v^*, t_\ell) \in E(G)$, then $G[\{v^*, c_2, t_\ell, b_1\}]$ is an induced cycle on four vertices. Analogous argument can be given when $(v^*, t_r) \in E(G)$. Therefore, we assume that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_i\}$, where $i \in [z]$. If $i \in [z] \setminus \{1, z\}$, then $G[\{v^*, v, b_i, b_{i-1}, b_{i-2}, b_{i+1}, b_{i+2}\}]$ is a long claw in $G - S$. If none of the preceding cases are applicable, then $N_G(v^*) \cap V(P(\mathbb{O}))$ is either $\{b_1\}$ or $\{b_z\}$. Suppose that $N_G(v^*) \cap V(P(\mathbb{O})) = \{b_1\}$ (the other case is symmetric), then $G[\{c_2, v, v^*, b_1, b_2, b_3, t_\ell\}]$ is a whipping top in $G - S$.
(C) $\boldsymbol{|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2.}$ If neighbors of v^* are not consecutive in the path $P(\mathbb{O})$, then we can obtain an induced cycle on at least four vertices in $G[\{v^*\} \cup V(P(\mathbb{O}))]$, and therefore we assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive. By the construction of $\mathbb{F}$ and v^*, we know that there are at least seven vertices in $P(\mathbb{O})$ that are non-adjacent to v^*. From the preceding discussions, we can conclude that $|\{t_\ell, t_r\} \cap N_G(v^*)| \leq 1$. Assume that $(v^*, t_r) \notin E(G)$ (the other case is symmetric). Next, we consider the following cases based on whether or not $(v^*, t_\ell) \in E(G)$:
(i) $\boldsymbol{(v^*, t_\ell) \in E(G).}$ In this case, there exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. Let $V' = \{v, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW with $|V'| < |V(\mathbb{O})|$.
(ii) $\boldsymbol{(v^*, t_\ell) \notin E(G).}$ Let b_s and b_e be the first and the last vertex in $P(\mathbb{O})$ that are adjacent to v^*, respectively. Notice that $s \neq e$ (since $|N_G(v^*) \cap V(P(\mathbb{O}))| \geq 2$), and $\{b_s, b_{s+1}, \ldots b_e, b_{e+1}\} \subset \{b_1, b_2, \ldots b_z\}$. Let $V' = \{v, v^*\} \cup \{b_{s-1}, b_s, b_{s+1}, \ldots, b_e, b_{e+1}\}$. Observe that $|V'| < |V(\mathbb{O})|$, and $G[V']$ is a †-AW.

Case ‡-AW.S.2. For all $\boldsymbol{u \in L_{\text{far}} \cup R_{\text{far}}}$, we have $\boldsymbol{|\{c_1, c_2\} \cap N_G(v^*)| = 1}$. From Observation 5.1, we know that for each $c' \in \{c_1, c_2\}$, we have that one of $L_{\text{far}} \subseteq N_G(c')$ or $R_{\text{far}} \subseteq N_G(c')$ holds.

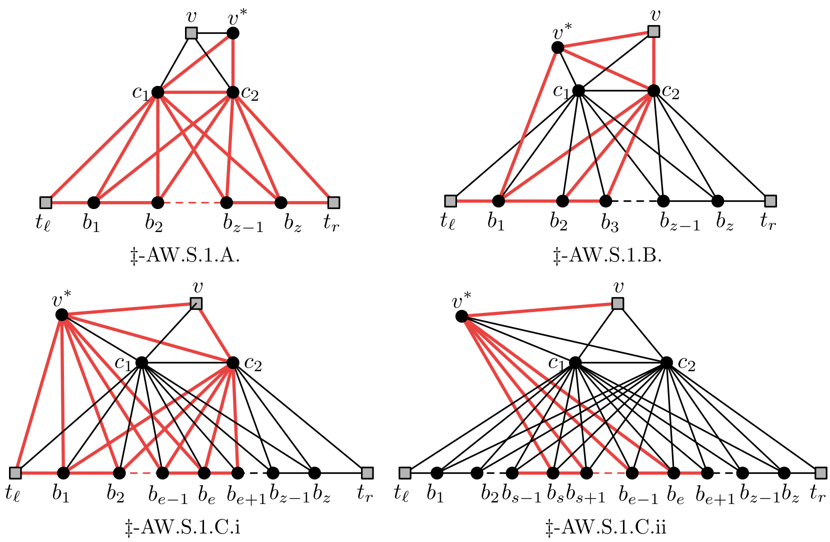

Fig. 7. Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = t$.

Moreover, from our assumption that for each $u \in L_{\text{far}} \cup R_{\text{far}}$ we have $|\{c_1, c_2\} \cap N_G(v^*)| = 1$, it cannot be the case that $L_{\text{far}} \subseteq N_G(c_1)$ and $L_{\text{far}} \subseteq N_G(c_2)$. Similarly, it cannot be the case that $R_{\text{far}} \subseteq N_G(c_1)$ and $R_{\text{far}} \subseteq N_G(c_2)$. From the preceding discussions, we can conclude that one of $L_{\text{far}} \subseteq N_G(c_1)$ and $R_{\text{far}} \subseteq N_G(c_2)$, or $R_{\text{far}} \subseteq N_G(c_1)$ and $L_{\text{far}} \subseteq N_G(c_2)$, holds. Suppose $L_{\text{far}} \subseteq N_G(c_1)$ and $R_{\text{far}} \subseteq N_G(c_2)$ (the other case is symmetric). Next, consider a vertex $u^* \in L_{\text{far}}$ and a vertex $v^* \in R_{\text{far}}$. By our assumption and non-applicability of Case ‡-AW.S.1, we have $(u^*, c_1), (v^*, c_2) \in E(G)$ and $(u^*, c_2), (v^*, c_1) \notin E(G)$. Moreover, $u^*, v^* \in B_i$, and therefore $(u^*, v^*) \in E(G)$. But then $G[\{u^*, v^*, c_1, c_2\}]$ is an induced cycle on four vertices.

Suppose $\boldsymbol{v}$ *Is One of the Centers.* Suppose $v = c_1$ (the other case is symmetric). From Observation 5.2, we know that at least one of $N_G(t_r) \cap L_{\text{cls}} = \emptyset$ or $N_G(t_r) \cap R_{\text{cls}} = \emptyset$ holds. Let $X_{\text{cls}} \in \{L_{\text{cls}}, R_{\text{cls}}\}$ be a set such that $N_G(t_r) \cap X_{\text{cls}} = \emptyset$. Consider a vertex $v^* \in X_{\text{cls}}$, and let b_s and b_e be the first and last vertex in the path $P(\mathbb{O})$ that are adjacent to v^*, respectively. Since M is a 9-redundant solution and $\mathbb{O}$ is not covered by $\mathcal{W}$, we have that $|M \cap V(\mathbb{O})| \geq 10$. This together with the choice of $\mathbb{F}$ and v^*, and the fact that $V(\text{base}(\mathbb{O})) \subseteq N_G(v)$, implies that b_s and b_e exist and are distinct. Moreover, from the preceding we can also conclude that $|N_G(v^*) \cap V(\text{base}(\mathbb{O}))| \geq 5$. We also note that $e \leq z$ since $(v^*, t_r) \notin E(G)$. The neighbors of v^* in $P(\mathbb{O})$ must be consecutive, and otherwise we can obtain an induced cycle of length at least 4 that does not contain v. We further consider subcases based on whether or not $t, c_2 \in N_G(v^*)$ (Figure 8).

Case ‡-AW.C.1. $\boldsymbol{t, c_2 \notin N_G(v^*)}$. Let $V' = \{v^*, v, c_2, t, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $|V'| < |V(\mathbb{O})|$ since $|N_G(v^*) \cap V(\text{base}(\mathbb{O}))| \geq 5$ and neighbors of v^* are consecutive. Moreover, $G[V']$ is a ‡-AW or a tent, which is of strictly smaller size than $\mathbb{O}$, contradicting the choice of $\mathbb{O}$. Here, we crucially rely on the fact that $t_r \notin N_G(v^*)$.

Case ‡-AW.C.2. $\boldsymbol{t \notin N_G(v^*)}$ and $\boldsymbol{c_2 \in N_G(v^*)}$. Let $V' = (V(\mathbb{O}) \setminus \{b_{s+1}, b_{s+2}, \ldots, b_{e-2}, b_{e-1}\}) \cup \{v^*\}$. Notice that $|V'| < |V(\mathbb{O})|$ (since $|N_G(v^*) \cap V(\text{base}(\mathbb{O}))| \geq 5$) and $G[V']$ is a ‡-AW.

Case ‡-AW.C.3. $\boldsymbol{t \in N_G(v^*)}$ and $\boldsymbol{c_2 \notin N_G(v^*)}$. Recall that $N_G(v^*) \cap \{b_1, b_2, \ldots, b_z\} \neq \emptyset$. Consider a vertex $b_j \in N_G(v^*) \cap \{b_1, b_2, \ldots, b_z\}$. The graph $G[\{v^*, t, c_2, b_j\}]$ is an induced cycle on fourvertices.

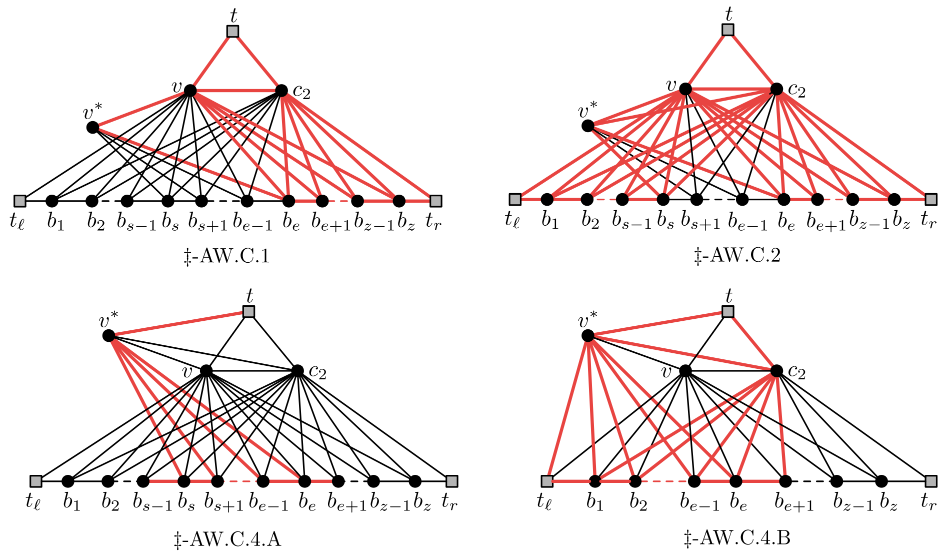

Fig. 8. Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = c_1$.

Case ‡-AW.C.4. $\boldsymbol{t \in N_G(v^*)}$ **and** $\boldsymbol{c_2 \in N_G(v^*)}$. We further consider the following subcases based on whether or not $(t_\ell, v^*) \in E(G)$:

(A) $\boldsymbol{(t_\ell, v^*) \notin E(G)}$. Let $V' = \{t, v^*\} \cup \{b_{s-1}, b_s, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$ that does not contain v.

(B) $\boldsymbol{(t_\ell, v^*) \in E(G)}$. Let $V' = \{t, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ that does not contain v.

Suppose $\boldsymbol{v}$ *Is One of the Non-Shallow Terminals.* We consider the case when $v = t_\ell$. By a symmetric argument, we can handle the case when $v = t_r$. If $c_2 \notin \beta(\mathbb{P})$, then for each $u \in L_{\text{cls}} \cup R_{\text{cls}}$ we have $(u, c_2) \notin E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $c_2 \in \beta(\mathbb{P})$, and then using Observation 5.2 we obtain that at least one of $L_{\text{cls}} \cap N_G(c_2) = \emptyset$ or $R_{\text{cls}} \cap N_G(c_2) = \emptyset$ holds. Let $X_{\text{cls}} \in \{L_{\text{cls}}, R_{\text{cls}}\}$ be a set such that $X_{\text{cls}} \cap N_G(c_2) = \emptyset$. Similarly, if $b_1 \notin \beta(\mathbb{P})$, then for each $u \in L_{\text{far}} \cup R_{\text{far}}$ we have $(u, b_1) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_1 \in \beta(\mathbb{P})$, and then using Observation 5.1 we obtain that at least one of $L_{\text{far}} \subseteq N_G(b_1)$ or $R_{\text{far}} \subseteq N_G(b_1)$ holds. Let $Y_{\text{far}} \in \{L_{\text{far}}, R_{\text{far}}\}$ be a set such that $Y_{\text{far}} \subseteq N_G(b_1)$. Next, we consider cases based on whether or not $b_1 \in B_i$ (Figure 9).

Case ‡-AW.T.1. $\boldsymbol{b_1 \in B_i}$. Consider a vertex $v^* \in X_{\text{cls}}$. Note that $(b_1, v^*) \in E(G)$ since $b_1 \in B_i$, and $(v^*, c_2) \notin E(G)$, by the choice of v^*. Additionally, $(v^*, t) \notin E(G)$, and otherwise $G[\{t, c_2, b_1, v^*\}]$ is an induced cycle on four vertices in $G - S$. Recall that v^* fits the frame $\mathbb{F}$ (and $(b_1, v^*) \in E(G)$), and therefore there exists $e \in [z - 2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This together with the fact that neighbors of v^* in $P(\mathbb{O})$ are consecutive (otherwise, we obtain an induced cycle on at least 4 vertices not containing v) implies that $(v^*, t_r) \notin E(G)$. Next, we consider cases based on whether or not $(v^*, c_1) \in E(G)$:

(A) $\boldsymbol{(v^*, c_1) \in E(G)}$. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ not containing v.

(B) $\boldsymbol{(v^*, c_1) \notin E(G)}$. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$ not containing v.

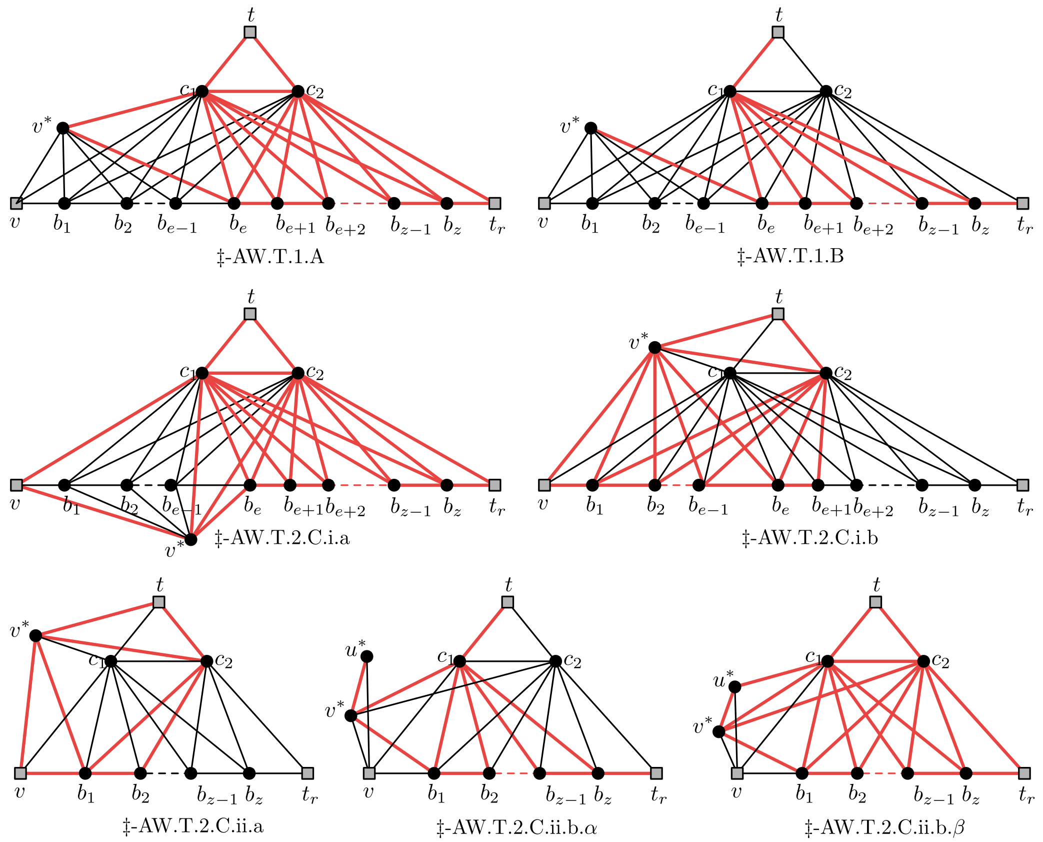

Fig. 9. Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = t_\ell$.

Case ‡-AW.T.2. $\boldsymbol{b_1 \notin B_i}$. Consider a vertex $v^* \in Y_{\text{far}} \cup \{u \in X_{\text{cls}} \mid (u, b_1) \in E(G)\}$ and the following cases based on its neighborhood in $\mathbb{O}$:

(A) $\boldsymbol{(v^*, c_2) \notin E(G)}$**.** Notice that this case is the same as Case ‡-AW.T.1, and therefore we can obtain an obstruction in a similar way.

(B) $\boldsymbol{(v^*, c_1) \notin E(G)}$**.** Observe that $(v^*, t) \notin E(G)$, and otherwise $G[\{v^*, b_1, c_1, t\}]$ is an induced cycleon four vertices in $G - S$. Now, we can obtain an obstruction as in Case ‡-AW.T.1.B.

(C) $\boldsymbol{(v^*, c_1), (v^*, c_2) \in E(G)}$**.** We further consider the following cases based on the neighborhood of v^* in $P(\mathbb{O})$:

(i) There exists $\boldsymbol{e \in [z] \setminus \{1\}}$ such that $\boldsymbol{(v^*, b_e) \in N_G(v^*)}$ and $\boldsymbol{(v^*, b_{e+1}) \notin N_G(v^*)}$. Observe that by the choices of $\mathbb{F}$ and v^*, we have $e < z - 1$. Consider the following cases based on whether or not $(t, v^*) \in E(G)$:

(a) $\boldsymbol{(t, v^*) \notin E(G)}$**.** Let $V' = \{t, c_1, c_2, v^*, v, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$. Furthermore, either $|V'| < |V(\mathbb{O})|$ or $|V'| = |V(\mathbb{O})|$ and $|V' \cap B_i| > |V(\mathbb{O}) \cap B_i|$. Here, we rely on the fact that $b_1 \notin B_i$. In either case, we obtain a contradiction to the choice of $\mathbb{O}$.

(b) $\boldsymbol{(t, v^*) \in E(G)}$**.** Let $V' = \{t, v^*, c_2, v\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ and $|V'| < |V(\mathbb{O})|$.

(ii) If (i) does not hold, then the only neighbors of $\boldsymbol{v^*}$ in $\boldsymbol{P(\mathbb{O})}$ are $\boldsymbol{b_1}$ and $\boldsymbol{v}$. Consider the following cases based on whether or not $(t, v^*) \in E(G)$:

(a) $\boldsymbol{(t, v^*) \in E(G)}$**.** In this case, $G[\{v, v^*, t, c_2, b_1, b_2\}]$ is a tent.
(b) $\boldsymbol{(t, v^*) \notin E(G)}$**.** We consider a vertex $u^* \in X_{\mathsf{cls}}$ to obtain the desired obstruction. Recall that from the construction of X_{cls}, we have $(u^*, c_2) \notin E(G)$. Moreover, by the premise of Case ‡-AW.T.2.C, we have $(v^*, c_2) \in E(G)$. From the preceding discussions, we can conclude that $(u^*, t) \notin E(G)$, as otherwise $G[\{u^*, v^*, c_2, t\}]$ is an induced cycle on four vertices. We assume that $(u^*, b_1) \notin E(G)$, and otherwise u^* would satisfy the premise of Case ‡-AW.T.2.A and we can obtain an obstruction using it. Additionally, $(u^*, b_j) \notin E(G)$, for each $j \in [z] \setminus \{1\}$, and otherwise $G[\{v, u^*\} \cup \{b_1, b_2, \ldots b_j\}]$ will contain an induced cycle on at least four vertices, which is an obstruction containing v with strictly less number of vertices than $\mathbb{O}$. Next, we consider the following cases depending on whether or not $(u^*, c_1) \in E(G)$:
$\boldsymbol{(\alpha)}$ $\boldsymbol{(u^*, c_1) \notin E(G)}$**.** Let $V' = \{t, c_1, u^*, v^*, t_r\} \cup \{b_1, b_2, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$, which does not contain v.
$\boldsymbol{(\beta)}$ $\boldsymbol{(u^*, c_1) \in E(G)}$**.** Let $V' = \{t, c_1, c_2, u^*, v^*, t_r\} \cup \{b_1, b_2, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$, which does not contain v.

Suppose $\boldsymbol{v}$ *Is* $\boldsymbol{b_1}$ *or* $\boldsymbol{b_z}$. Suppose $v = b_1$ (the other case is symmetric). If $t_\ell \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, t_\ell) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $t_\ell \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(t_\ell)$ or $R_{\mathsf{far}} \subseteq N_G(t_\ell)$ holds (see Observation 5.1). Let $X_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$. Similarly, if $b_2 \notin \beta(\mathbb{P})$, then for each $u \in L_{\mathsf{far}} \cup R_{\mathsf{far}}$ we have $(u, b_2) \in E(G)$ as it fits the frame $\mathbb{F}$ and $N_G(u) \setminus (M \cup \beta(\mathbb{P})) = N_G(v) \setminus (M \cup \beta(\mathbb{P})) = \emptyset$. Otherwise, $b_2 \in \beta(\mathbb{P})$, and then at least one of $L_{\mathsf{far}} \subseteq N_G(b_2)$ or $R_{\mathsf{far}} \subseteq N_G(b_2)$ holds (see Observation 5.1). Let $Y_{\mathsf{far}} \in \{L_{\mathsf{far}}, R_{\mathsf{far}}\}$ be a set such that $Y_{\mathsf{far}} \subseteq N_G(b_2)$. Next, we consider cases depending on the neighborhood of vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$ in $\mathbb{O}$ (Figure 10).

Case ‡-AW.B.1. There is $\boldsymbol{v^* \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}}$ such that $\boldsymbol{\{t_\ell, b_2\} \subseteq N_G(v^*)}$. There exists $e \in [z-2]$ such that $b_e \in N_G(v^*)$ and $b_{e+1} \notin N_G(v^*)$. This follows from the choices of $\mathbb{F}$ and v^*, and the facts that $(v^*, b_2) \in E(G)$ and v^* fits $\mathbb{F}$. We assume that the neighbors of v^* in $P(\mathbb{O})$ are consecutive, as otherwise we can obtain an obstruction that is an induced cycle on at least four vertices. Next, we consider the subcases based on whether or not $(v^*, c_1), (v^*, c_2), (v^*, t) \in E(G)$:

(A) $\boldsymbol{(v^*, c_2) \in E(G), (v^*, t) \in E(G)}$**.** Let $V' = \{t, c_2, v^*, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a ‡-AW such that $|V'| < |V(\mathbb{O})|$.
If Case ‡-AW.B.1.A is not applicable, then $(v^*, c_2) \notin E(G)$ or $(v^*, t) \notin E(G)$ must hold.
(B) $\boldsymbol{(v^*, t) \notin E(G)}$**.** We consider the following cases:
(i) $\boldsymbol{(v^*, c_1) \notin E(G)}$**.** Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots b_z\}$. Observe that $G[V']$ is a †-AW in $G - S$ not containing v.
(ii) $\boldsymbol{(v^*, c_1) \in E(G)}$**.** Let $V' = \{t, c_1, c_2, v^*, t_r, t_\ell\} \cup \{b_e, b_{e+1} \ldots b_z\}$. Observe that $G[V']$ contains a ‡-AW not containing v, which is present in $G - S$. We note that such an obstruction can be found both when $(v^*, c_2) \in E(G)$ and when $(v^*, c_2) \notin E(G)$.
(C) $\boldsymbol{(v^*, c_2) \notin E(G)}$**.** Since Case ‡-AW.B.1.B is not applicable, we can assume that $(v^*, t) \in E(G)$. But then $G[\{v^*, b_2, c_2, t\}]$ is a cycle on four vertices.

Case ‡-AW.B.2. For all $\boldsymbol{u \in X_{\mathsf{far}} \cup Y_{\mathsf{far}}}$, we have $\boldsymbol{\{t_\ell, b_2\} \not\subseteq N_G(u)}$. Furthermore, by the construction of X_{far} and Y_{far}, we know that $X_{\mathsf{far}} \subseteq N_G(t_\ell)$, $Y_{\mathsf{far}} \subseteq N_G(b_2)$, and $X_{\mathsf{far}}, Y_{\mathsf{far}} \neq \emptyset$. Hence, for any pair of vertices $u^* \in X_{\mathsf{far}}$ and $v^* \in Y_{\mathsf{far}}$, we have that $(u^*, t_\ell), (v^*, b_2) \in E(G)$ and $(u^*, b_2), (v^*, t_\ell) \notin E(G)$ (since Case ‡-AW.B.1 is not applicable). Next, we consider cases based on whether or not t and c_2 are adjacent to vertices in $X_{\mathsf{far}} \cup Y_{\mathsf{far}}$:

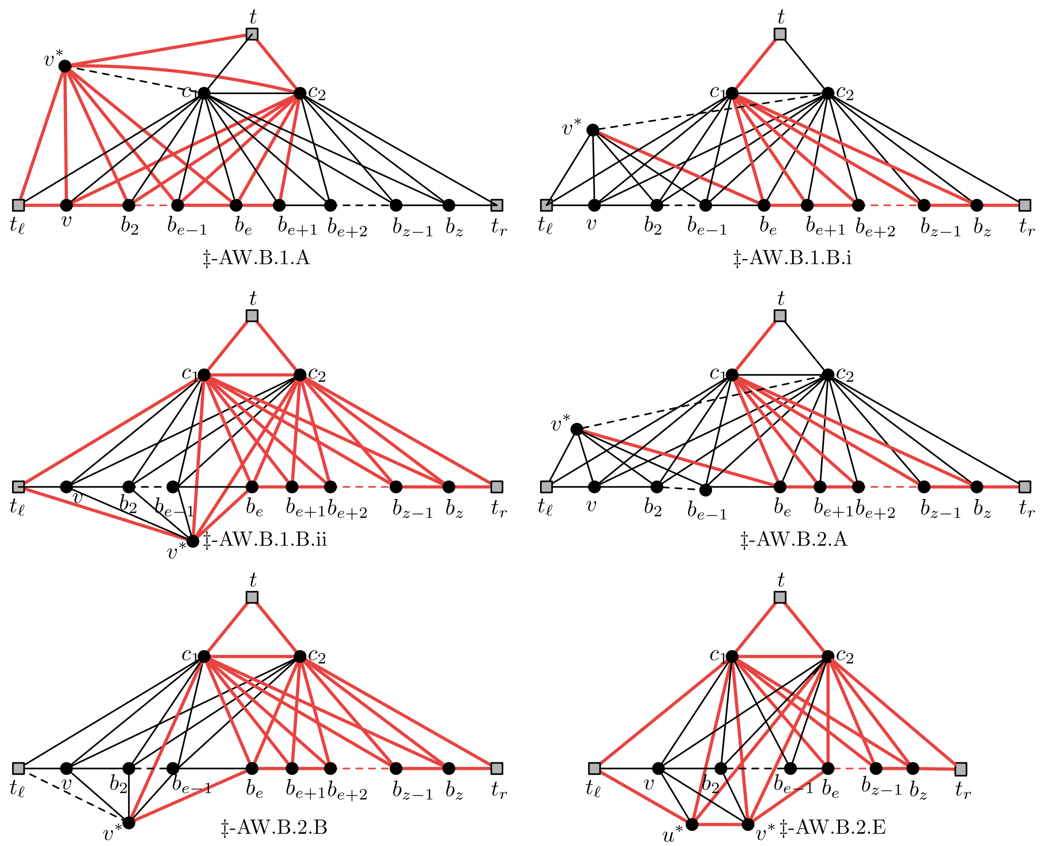

Fig. 10. Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = b_1$.

(A) Consider the case when there is $\boldsymbol{v^* \in X_{\mathrm{far}} \cup Y_{\mathrm{far}}}$ such that $\boldsymbol{(v^*, c_1) \notin E(G)}$. In this case, $(v^*, t) \notin E(G)$, and otherwise we obtain an induced cycle $G[\{v^*, v, c_1, t\}]$ on four vertices. Let $e \in [z-2]$ such that b_e is the last vertex in base($\mathbb{O}$) that is adjacent to v^*. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $G[V']$ is a †-AW that excludes v.

Hereafter, we assume that for each $u \in X_{\mathrm{far}} \cup Y_{\mathrm{far}}$, we have $(u, c_1) \in E(G)$.

(B) Consider the case when there is $\boldsymbol{v^* \in X_{\mathrm{far}} \cup Y_{\mathrm{far}}}$ such that $\boldsymbol{(v^*, c_2) \notin E(G)}$. In this case, $(v^*, t) \notin E(G)$, and otherwise $G[v^*, t, c_2, v]$ is a cycle on four vertices. Let $e \in [z-2]$ such that b_e is the last vertex in base($\mathbb{O}$) that is adjacent to v^*. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Notice that $G[V']$ is a ‡-AW that has either fewer vertices than $\mathbb{O}$ or has the same number of vertices as $\mathbb{O}$ but has more vertices from B_i (than $\mathbb{O}$ has from B_i). Here, we rely on the fact that $t_\ell \notin B_i$, which is ensured by the fact that $Y_{\mathrm{far}} \neq \emptyset$ and $Y_{\mathrm{far}} \cap N_G(t_\ell) = \emptyset$.

Hereafter, we will assume that for each $u \in X_{\mathrm{far}} \cup Y_{\mathrm{far}}$ we have $c_1, c_2 \in N_G(u)$.

(C) If there is $\boldsymbol{u^* \in X_{\mathrm{far}}}$ such that $\boldsymbol{(u^*, t) \in E(G)}$. Recall that $(u^*, t_\ell) \in E(G)$ and $(u^*, b_2) \notin E(G)$. In this case, $G[\{t, u^*, c_2, t_\ell, v, b_2\}]$ is a tent.

(D) If there is $\boldsymbol{v^* \in Y_{\mathrm{far}}}$ such that $\boldsymbol{(v^*, t) \in E(G)}$. Recall that,$(v^*, b_2) \in E(G)$ and $(v^*, t_\ell) \notin E(G)$. Let $e \in [z-2]$ such that b_e is the last vertex in base($\mathbb{O}$) that is adjacent to v^*. Note that $e \geq 2$ as $v^* \in Y_{\mathrm{far}} \subseteq N_G(b_2)$. Let $V' = \{t, v^*, t_\ell, b_{e+1}\} \cup \{v, b_3, \ldots, b_e\}$. Observe that $G[V']$ is a †-AW in $G - S$ with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude c_1, c_2 and include v^*.

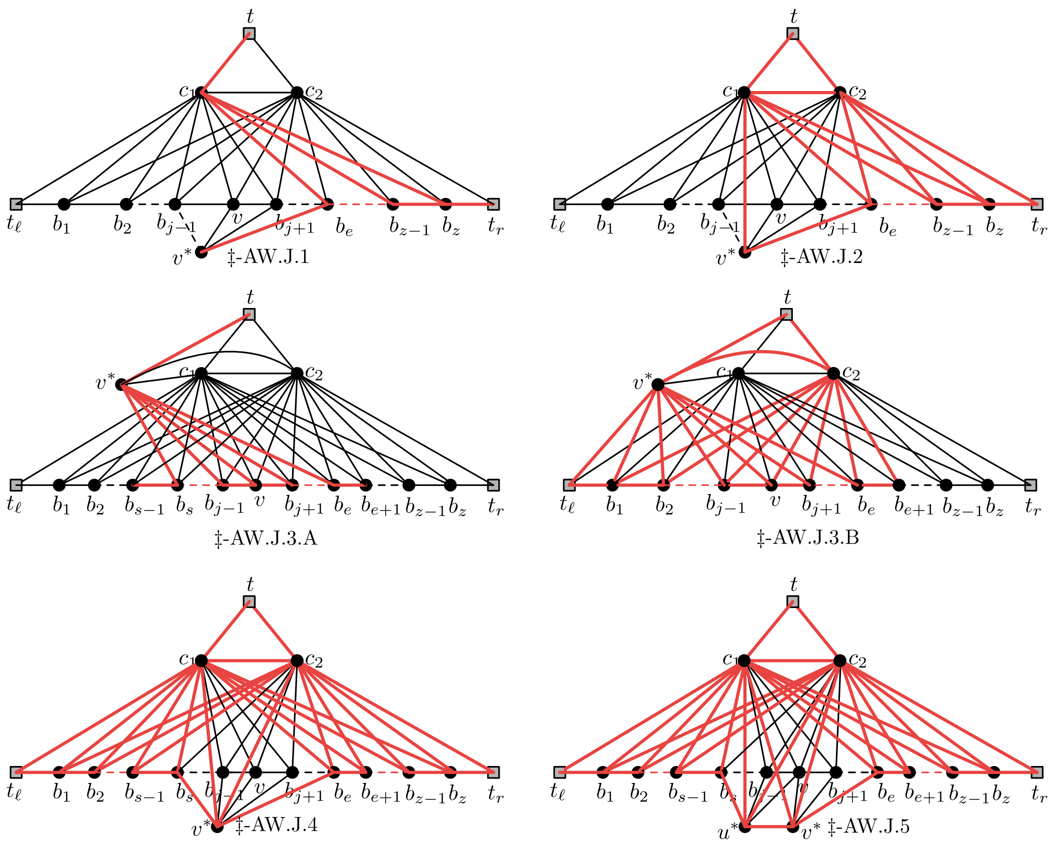

Fig. 11. Construction of an obstruction when $\mathbb{O}$ is ‡-AW and $v = b_j$, where $j \in [z-1] \setminus \{1\}$.

(E) Consider a vertex $\boldsymbol{u^* \in X_{\text{far}}}$ and a vertex $\boldsymbol{v^* \in Y_{\text{far}}}$. Since all the previous cases are not applicable, therefore $(u^*, c_1), (u^*, c_2), (v^*, c_1), (v^*, c_2) \in E(G)$, and $(u^*, t), (v^*, t) \notin E(G)$. Recall that neighbors of u^*, v^* in $P(\mathbb{O})$ are consecutive. Furthermore, $(v^*, t_\ell) \notin E(G)$ and there is no b_j adjacent to u^*, where $j \geq 2$. Let $e \in [z-2]$ such that b_e is the last neighbor of v^* in $P(\mathbb{O})$. Now, let $V' = \{t_\ell, u^*, v^*, c_1, c_2, t\} \cup \{b_e, b_{e+1}, \ldots, b_z, t_r\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ that does not contain v.

Suppose $\boldsymbol{v = b_j}$*, Where* $\boldsymbol{j \in [z] \setminus \{1, z\}}$. Let $X_{\text{far}} \in \{L_{\text{far}}, R_{\text{far}}\}$ be a set such that $X_{\text{far}} \subseteq N_G(b_{j-1})$ and $Y_{\text{far}} \in \{L_{\text{far}}, R_{\text{far}}\}$ be a set such that $Y_{\text{far}} \subseteq N_G(b_{j+1})$. The existence of X_{far} and Y_{far} is guaranteed from Observation 5.1. Recall that $|M'| = 9$. Thus, $|V(P(\mathbb{O})) \cap M'| \geq 6$, and therefore v must have at least four non-neighbors in $V(P(\mathbb{O})) \cap M'$. From the preceding, we can conclude that one of $|(\{t_\ell\} \cup \{b_1, b_2, \ldots, b_{j-2}\}) \cap (M' \setminus N_G(v))| \geq 2$ or $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \ldots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$ holds. Assume that $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \ldots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$ holds (the other case is symmetric). For each $u \in X_{\text{far}} \cup Y_{\text{far}}$, the neighbors of u in $P(\mathbb{O})$ must be consecutive, and otherwise we can obtain an induced cycle on at least four vertices. From the preceding discussions, together with the facts that $(u, v) \in E(G)$ and u fits $\mathbb{F}$, we can conclude that $\{t_r, b_z\} \cap N_G(u) = \emptyset$. Here, we rely on our assumption that $|(\{t_r\} \cup \{b_{j+2}, b_{j+3}, \ldots, b_z\}) \cap (M' \setminus N_G(v))| \geq 2$. We consider cases based on the neighborhood of vertices in $X_{\text{far}} \cup Y_{\text{far}}$ in $\mathbb{O}$ (Figure 11).

Case ‡-AW.J.1. If there is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, c_1) \notin E(G)}$. Note that if $(v^*, c_1) \notin E(G)$, then $(v^*, t) \notin E(G)$, and otherwise $G[\{v, v^*, c_1, t\}]$ is a cycle on four vertices. Additionally, the neighbors of v^* in $P(\mathbb{O})$ must be consecutive, and otherwise we can obtain an induced cycle on at least four vertices. Since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v, v^*) \in E(G)$, there exists $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that adjacent to v^*. Let $V' = \{t, c_1, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a †-AW with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude c_2, t_ℓ, b_1 and include v^*.

Case ‡-AW.J.2. If there is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, c_2) \notin E(G)}$. Since Case ‡-AW.J.1 is not applicable, we can assume that $(v^*, c_1) \in E(G)$. Note that if $(v^*, c_2) \notin E(G)$, then $(v^*, t) \notin E(G)$, and otherwise $G[\{v, v^*, c_2, t\}]$ is a cycle on four vertices. Additionally, the neighbors of v^* in $P(\mathbb{O})$ must be consecutive. Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v, v^*) \in E(G)$. Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW (or a net) with strictly fewer vertices than $\mathbb{O}$, as we (at least) exclude t_ℓ, b_1 and include v^*.

Note that if Cases ‡-AW.J.1 and ‡-AW.J.2 are not applicable, then for each $u \in X_{\text{far}} \cup Y_{\text{far}}$ we have $(u, c_1), (u, c_2) \in E(G)$. Moreover, by our assumption, we have $N_G(u) \cap \{t_r, b_z\} = \emptyset$. The cases we consider next are based on whether or not the following conditions are satisfied for a vertex $u \in X_{\text{far}} \cup Y_{\text{far}}$:

(1) $(u, t) \in E(G)$;
(2) $\{b_{j-1}, b_{j+1}\} \subseteq N_G(u)$.

Case ‡-AW.J.3. If there is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, t) \in E(G)}$. We further consider the following subcases based on whether or not $t_\ell \in N_G(v^*)$:

(A) $\boldsymbol{t_\ell \notin N_G(v^*)}$**.** Let $s \in [j]$ such that b_s is the first vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $(t_\ell, v^*) \notin E(G)$ and $(v^*, v) \in E(G)$. Additionally, let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Notice that $s \neq e$, since by the construction of the sets X_{far} and Y_{far} we have that v^* is incident to v and at least one of the vertices in $\{b_{j-1}, b_{j+1}\}$. Let $V' = \{t, v^*\} \cup \{b_{s-1}, b_s, \ldots, b_e, b_{e+1}\}$. Observe that $G[V']$ is a †-AW in $G - S$ with $|V'| < |V(\mathbb{O})|$. Here, we rely on the fact that $e \leq z - 1$.
(B) $\boldsymbol{t_\ell \in N_G(v^*)}$**.** Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Let $V' = \{t, v^*, c_2, t_\ell\} \cup \{b_1, b_2, \ldots, b_e, b_{e+1}\}$ is a ‡-AW in $G - S$. Moreover, $|V'| < |V(\mathbb{O})|$ since $t_r, c_1 \notin V'$ and $V' \subseteq V(\mathbb{O}) \cup \{v^*\}$.

Case ‡-AW.J.4. If there is $\boldsymbol{v^* \in X_{\text{far}} \cup Y_{\text{far}}}$ such that $\boldsymbol{(v^*, t) \notin E(G)}$ and $\boldsymbol{\{b_{j-1}, b_{j+1}\} \subseteq N_G(v^*)}$. Notice that all the neighbors of v^* on $P(\mathbb{O})$ must be consecutive. Let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $\{t_r, b_z\} \cap N_G(v^*) = \emptyset$ and $(v^*, v) \in E(G)$. Additionally, let $s \in [z-1] \cup \{0\}$ be the lowest integer such that $(v^*, b_s) \in E(G)$ (b_s could possibly be same as b_{j-1} or $b_0 = t_\ell$). Let $V' = \{t, c_1, c_2, v^*, t_r\} \cup \{b_1, b_2, \ldots, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ that does not contain v.

Case ‡-AW.J.5. For all $\boldsymbol{u \in X_{\text{far}} \cup Y_{\text{far}}}$, we have $\boldsymbol{c_1, c_2 \in N_G(u)}$, $\boldsymbol{(u, t) \notin E(G)}$, and $\boldsymbol{\{b_{j-1}, b_{j+1}\} \nsubseteq N_G(u)}$. Additionally, we have $X_{\text{far}} \subseteq N_G(b_{j-1})$ and $Y_{\text{far}} \subseteq N_G(b_{j+1})$. Next, consider a vertex $u^* \in X_{\text{far}}$ and a vertex $v^* \in Y_{\text{far}}$. Let $s \in [j-1] \cup \{0\}$ such that b_s is the first vertex in $P(\mathbb{O})$ that is adjacent to u^*, which exists since $(u^*, b_{j-1}) \in E(G)$. Additionally, let $e \in [z-1]$ such that b_e is the last vertex in $P(\mathbb{O})$ that is adjacent to v^*, which exists since $(t_r, v^*), (b_z, v^*) \notin E(G)$ and $(v^*, b_{j+1}) \in E(G)$. Notice that $s \neq e$. Let $V' = \{t, c_1, c_2, v^*, u^*\} \cup \{t_\ell, b_1, b_2, b_{s-1}, b_s\} \cup \{b_e, b_{e+1}, \ldots, b_z, t_r\}$. Observe that $G[V']$ is a ‡-AW in $G - S$ that does not contain v.

We have exhaustively considered all the cases and obtained a desired type of obstruction for each of the cases. This concludes the proof of Lemma 5.3.

6 BOUNDING THE LENGTH OF A CLIQUE PATH

Let us first recall the various sets we are dealing with, with respect to the instance (G, k) of IVD:

- A $(k+2)$-necessary family $\mathcal{W} \subseteq 2^M$ along with a solution M that is 9-redundant with respect to $\mathcal{W}$ (see Lemma 3.2).
- Every set in $\mathcal{W}$ has size at least 2 (see Reduction Rule 3.1).
- C is the set of connected components of $G - M$, $\mathcal{D}$ is the set of connected components in C that are modules, and $\overline{\mathcal{D}} = C \setminus \mathcal{D}$. We know that $|V(\mathcal{D})| \leq k^{O(1)}$ (see Lemma 4.3 Reduction Rule 4.2, and Observation 4.5) and $|\overline{\mathcal{D}}| \leq k^{O(1)}$ (see Observation 4.2).
- Each component in $\mathcal{D}$ has at most $k + 1$ vertices (see Reduction Rule 4.3). The preceding together with Lemma 5.4 implies that every maximal clique (and hence every clique) in $G - M$ has size bounded by η, where $\eta = 2^{10} \cdot 4(k + 5)\binom{|M|}{10}$.

Let us now turn to the problem of bounding the sizes of non-module components. Observe that to bound this, it is sufficient to "bound the length of the clique path" of a non-module component. This together with the fact that each maximal clique is bounded will lead to the desired result. Our approach mirrors that of other works [3, 36] but also requires additional structural observations corresponding to interval graphs and its obstructions [8, 28]. Each non-module component is a clique path in $G - M$.

Let $\mathbb{K} = (K, \beta)$ be a clique path of a non-module component C, where K is the path $(x_1, x_2, \ldots, x_t)$, and for each $i \in [t]$ we let $B_i = \beta(x_i)$. (In the remainder of this section, we will be working with this fixed clique path $\mathbb{K}$ and the component C.) We will refer to the sets B_i, $1 \leq i \leq t$, as the *bags* in $\mathbb{K}$. We will assume that for any two distinct bags B, B' in $\mathbb{K}$, neither $B \subseteq B'$ nor $B' \subseteq B$. Any bag B_i in the clique path $\mathbb{K}$ has at most $\eta = 2^{10} \cdot 4(k + 5)\binom{|M|}{10}$ vertices (because every maximal clique in $G - M$ has size bounded by η, by Lemma 5.4). We let $\beta(\mathbb{K}) = \cup_{i=1}^{t} \beta(x_i)$. Furthermore, for a subpath K' of K, by $\mathbb{K}' = (K', \beta')$ we denote the sub-clique path induced by K'. In other words, for $x \in V(K')$, $\beta'(x) = \beta(x)$. Moreover, by $\beta(\mathbb{K}')$, we denote the set $\cup_{x \in V(K')} \beta(x)$. Note that there is a vertex in M that has a neighbor as well as a non-neighbor in C.

In this section, we consider the problem of reducing the number of bags in $\mathbb{K}$. Toward our goal, we will devise a collection of "marking schemes" that mark some polynomially (in k) many bags in $\mathbb{K}$ such that the obstructions are "well behaved" in the region between any two consecutive marked bags. In particular, our marking schemes ensure that if any obstruction intersects an unmarked region of the clique path, then the intersection is an *induced path*. Then, we design reduction rules that "preserve" a minimum separator of the unmarked region. More precisely, we identify an *irrelevant vertex* or an *irrelevant edge*, then delete it or contract it in the graph. The correctness of these reduction rules follows from the structural properties ensured by the marking schemes.

Let us now define some notations that will be required in this section. Note that these notations apply to $\mathbb{K} = (K, \beta)$ as well as any sub-clique path of it. We fix an ordering (from left to right) of the bags of $\mathbb{K}$, which is given by the path K of the clique path $\mathbb{K}$. We will maintain a set of bags $\mathcal{B}_{\mathsf{Marked}}$ in $\mathbb{K}$, which we will call *marked bags*. Initially, $\mathcal{B}_{\mathsf{Marked}} = \emptyset$, and we will add some carefully chosen bags in $\mathbb{K}$ to it as we proceed:

(1) For two bags B_i and B_j in $\mathbb{K}$, $1 \leq i \leq j \leq t$, by $\mathbb{K}[B_i, B_j] = (K', \beta')$ we denote the sub-clique path of $\mathbb{K}$ between B_i and B_j (including B_i and B_j).
(2) For a sub-clique path $\mathbb{K}'$ of $\mathbb{K}$, $\mathsf{B}_{\mathsf{left}}(\mathbb{K}')$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}')$ denote the leftmost bag and the rightmost bag of $\mathbb{K}'$, respectively. Observe that $\mathbb{K}' = \mathbb{K}[\mathsf{B}_{\mathsf{left}}(\mathbb{K}'), \mathsf{B}_{\mathsf{right}}(\mathbb{K}')]$. All other bags of $\mathbb{K}'$, except $\mathsf{B}_{\mathsf{left}}(\mathbb{K}')$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}')$, are called *interior bags* of $\mathbb{K}'$.

(3) For a sub-clique path $\mathbb{K}'$ of $\mathbb{K}$, let $\mathsf{C}(\mathbb{K}') = \mathsf{B}_{\mathsf{left}}(\mathbb{K}') \cap \mathsf{B}_{\mathsf{right}}(\mathbb{K}')$ and $\mathsf{I}(\mathbb{K}') = \beta(\mathbb{K}') \setminus (\mathsf{B}_{\mathsf{left}}(\mathbb{K}') \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}'))$. Here, 'C' stands for "common" and 'I' stands for "internal."
(4) We say that a vertex $v \in \beta(\mathbb{K})$ is a *marked vertex* if there is a marked bag that contains it, and otherwise it is an *unmarked vertex*.
(5) Consider a collection of bags $\mathcal{B}^\star$. We say that two distinct bags $B, B' \in \mathcal{B}^\star$ are *consecutive* if $\mathbb{K}[B, B']$ contains no other bags in $\mathcal{B}^\star$ except for B and B'.
(6) We say that two distinct bags B, B' in $\mathbb{K}$ are *adjacent* if there is no other bag that lies between them—that is, $\mathbb{K}[B, B']$ has only two bags, namely B and B'.
(7) For a bag B in $\mathbb{K}$, B^{-1} and B^{+1} denote the bags (if they exist) adjacent to B on its left and right, respectively.

6.1 Partition into Manageable Clique Paths

In this section, we partition the clique path $\mathbb{K}$ into a collection of so-called "manageable clique paths," which are well structured with respect to the set M. We will construct our first set of marked bags, denoted by $\mathcal{B}_{\mathsf{Marked}}(I)$, based on the edges between the vertices in $\beta(\mathbb{K})$ and M. Let us initialize $\mathcal{B}_{\mathsf{Marked}}(I)$ as the set containing the first and the last bags of $\mathbb{K}$. We begin by stating a property of interval graphs, which will be useful later.

Observation 6.1. *Let H be an interval graph and let H' be the graph obtained by one of the following operations:*

(a) *For $v \in V(H)$, $H' = H - \{v\}$.*
(b) *For $(u, v) \in E(H)$, $H' = H/(u, v)$.*

Then, H' is an interval graph. Furthermore, the size of any clique in H' is upper bounded by the size of a maximum clique in H.

The preceding observation follows from the definition of interval graphs and their interval representation [28]. In particular, statement (b) follows from the observation that an interval representation of $H/(u, v)$ can be obtained by taking an interval representation of H and "merging" the intervals of u and v.

In the following, we will define (auxiliary) graphs that will be helpful in obtaining some useful bags in $\mathbb{K}$. To this end, consider a vertex $m \in M$. Let H_m be the bipartite graph with vertex bipartition $N_G(m) \cap \beta(\mathbb{K})$ and $\beta(\mathbb{K}) \setminus N_G(m)$, where $u \in N_G(m) \cap \beta(\mathbb{K})$ and $v \in \beta(\mathbb{K}) \setminus N_G(m)$ are adjacent in H_m if and only if $(u, v) \in E(G)$. Next, we prove the following lemma about the graph H_m. (Recall that η is an upper bound on the size of any clique in $G - M$.)

Lemma 6.2. *For $m \in M$, let Y_m be a maximum matching in H_m. Then, $|Y_m| \leq 2\eta$.*

Proof. Suppose, toward a contradiction, that $|Y_m| > 2\eta$. Let T be the graph obtained from $G[\beta(\mathbb{K})]$ by contracting all the edges in Y_m. Additionally, for each edge (u, v) in Y_m, let w_{uv} be the vertex resulting from its contraction. Recall that $G - M$ is an interval graph of maximum clique size at most η, which together with Observation 6.1 implies that both $G[\beta(\mathbb{K})]$ and T are also interval graphs, and that the maximum size of a clique in these graphs is upper bounded by η. Next, let $\widetilde{T}$ be the graph $T[\{w_{uv} \mid (u, v) \in Y_m\}]$. We note that the definition of $\widetilde{T}$ relies on the fact that Y_m is a matching in H_m, and thus it has $|Y_m| > 2\eta$ many vertices. From the construction of $\widetilde{T}$ and Observation 6.1, it follows that $\widetilde{T}$ is also an interval graph and that the size of any clique in $\widetilde{T}$ is bounded by η. Interval graphs are perfect graphs, and on a perfect graph H we know that $\omega(H)\alpha(H) \geq |V(H)|$, where $\omega(H)$ and $\alpha(H)$ denote the size of a maximum clique and a maximum independent set in H, respectively [47] (or Theorem 3.3 [28]). This implies that there is an independent set in $\widetilde{T}$ of size at least $|Y_m|/\eta > 2$. Consider an independent set of size

3 in $\widetilde{T}$ and the corresponding edges of the matching Y_m. It follows that these three edges and the vertex m form a long claw $\mathbb{O}$ in G, which is an obstruction of size 7. Since Reduction Rule 3.1 is not applicable, each set in $\mathcal{W}$ is of size at least 2. Moreover, $|V(\mathbb{O}) \cap M| = 1$. Therefore, $\mathbb{O}$ is not covered by $\mathcal{W}$. But then, since M is a 9-redundant solution, each obstruction in G that is not covered by $\mathcal{W}$ must contain at least 10 vertices from M. But this is a contradiction. Thus, we deduce that $|Y_m| > 2\eta$ cannot hold. □

For each $m \in M$, we compute a maximum matching Y_m in the graph H_m. Then, for each edge in Y_m, we pick a bag in $\mathbb{K}$ that contains this edge and add it to $\mathcal{B}_{\mathsf{Marked}}(I)$. Let us observe that we have added at most $2\eta|M|$ bags to $\mathcal{B}_{\mathsf{Marked}}(I)$. Before proceeding further, we will add some more bags to $\mathcal{B}_{\mathsf{Marked}}(I)$ that give us some additional structural properties. To this end, we state the following observation, which will be useful in designing the following marking scheme for bags in $\mathbb{K}$.

Observation 6.3. *Let $m_1, m_2 \in M$ be (distinct) vertices such that $\{m_1, m_2\} \notin \mathcal{W}$ and $(m_1, m_2) \notin E(G)$. Then, $(N_G(m_1) \cap N_G(m_2)) \setminus M$ induces a clique in G.*

Proof. This observation is the special case of Lemma 4.6 with $M' = M, u = m_1, v = m_2$, and $u, v \in M$. □

Next, consider (distinct) $m_1, m_2 \in M$ such that $\{m_1, m_2\} \notin \mathcal{W}$, $(m_1, m_2) \notin E(G)$, and $(N_G(m_1) \cap N_G(m_2)) \setminus M \neq \emptyset$. Let $B(m_1, m_2)$ be a bag in $\mathbb{K}$ such that $(N_G(m_1) \cap N_G(m_2)) \cap \beta(\mathbb{K}) \subseteq B(m_1, m_2)$. We note that the existence of $B(m_1, m_2)$ is guaranteed from Observation 6.3. We add $B(m_1, m_2)$ to the set $\mathcal{B}_{\mathsf{Marked}}(I)$. We are now ready to state our first bag-marking scheme.

Marking Scheme I. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(I)$ to $\mathcal{B}_{\mathsf{Marked}}$.

Note that $|\mathcal{B}_{\mathsf{Marked}}(I)|$ is at most $2\eta|M| + |M|^2 + 2$. This bound is obtained because(i) $\mathcal{B}_{\mathsf{Marked}}(I)$ contains the first and last bags of $\mathbb{K}$, (ii) at most 2η bags in $\mathbb{K}$ were added corresponding to the matching Y_m for each $m \in M$ (and H_m), and (iii) for (distinct) $m_1, m_2 \in M$, such that $\{m_1, m_2\} \notin \mathcal{W}$ and $(m_1, m_2) \notin E(G)$, we added a bag to $\mathcal{B}_{\mathsf{Marked}}(I)$. Thus, using Marking Scheme I, we have marked at most $\boxed{2\eta|M| + |M|^2 + 2 < 4\eta|M|}$ bags in $\mathbb{K}$. Here, we used the fact that $\eta \geq |M|$.

Next, we state an observation regarding vertices that are not present in any bag in $\mathcal{B}_{\mathsf{Marked}}(I)$, which will be useful later. We note that this observation is quite similar to Observation 4.11 of Section 4.

Observation 6.4. *Consider a vertex $v \in \beta(\mathbb{K})$ such that there is no bag in $\mathcal{B}_{\mathsf{Marked}}(I)$ that contains v. For (distinct) vertices $u, w \in N_G(v) \cap M$, at least one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds.*

Proof. Consider $v \in \beta(\mathbb{K})$ such that there is no bag in $\mathcal{B}_{\mathsf{Marked}}(I)$ that contains v, and (distinct) vertices $u, w \in N_G(v) \cap M$. Suppose, by way of contradiction, that $\{u, w\} \notin \mathcal{W}$ and $(u, v) \notin E(G)$. This together with Observation 6.3 implies that $(N_G(u) \cap N_G(w)) \setminus M$ induces a clique in G. From the preceding and Marking Scheme I, it follows that there is a bag $B(u, w)$ in $\mathcal{B}_{\mathsf{Marked}}(I)$ such that $(N_G(u) \cap N_G(w)) \setminus M \subseteq B(u, w)$. However, $v \in (N_G(u) \cap N_G(w)) \setminus M$, and hence $v \in B(u, w)$. This contradicts that v is not contained in any bag in $\mathcal{B}_{\mathsf{Marked}}(I)$. □

Let $B_\ell, B_r \in \mathcal{B}_{\mathsf{Marked}}(I)$ be two consecutive marked bags in $\mathbb{K}$. We define the graph $G[B_\ell, B_r]$ to be the graph induced on the vertices appearing in the sub-clique path $\mathbb{K}[B_\ell, B_r]$ excluding the vertices in B_ℓ and B_r. In other words, $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$, where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$. Note that although $G[\beta(\mathbb{K}[B_\ell, B_r])]$ is a connected subgraph of G, $G[B_\ell, B_r]$ need not be a connected graph. We refer to a connected component of $G[B_\ell, B_r]$ as an *obtruded component of* $\mathbb{K}[B_\ell, B_r]$. We extend this definition to say that an induced subgraph H of $G[\beta(\mathbb{K})]$ is an *obtruded* *component of* $\mathbb{K}$ if there are consecutive marked bags $B_\ell, B_r \in \mathcal{B}_{\text{Marked}}(I)$ such that H is an obtruded component of $\mathbb{K}[B_\ell, B_r]$. We remark the following regarding vertices of $\mathbb{K}$ outside $\cup_{B \in \mathcal{B}_{\text{Marked}}(I)} B$.

Observation 6.5. *For each $v \in \beta(\mathbb{K}) \setminus (\cup_{B \in \mathcal{B}_{\text{Marked}}(I)} B)$, there is an obtruded component H of $\mathbb{K}$ such that $v \in V(H)$.*

In the following, we prove a property regarding the obtruded components of $\mathbb{K}$.

Lemma 6.6. *Let H be an obtruded component of $\mathbb{K}$. For each $m \in M$, either we have $V(H) \subseteq N_G(m)$ or we have $V(H) \cap N_G(m) = \emptyset$.*

Proof. Suppose, toward a contradiction, that there exists $m \in M$ that has both a neighbor and a non-neighbor from the set $V(H)$ in G. Because H is connected, this implies that there is an edge $e \in E(H)$ such that one endpoint of e lies in $N_G(m) \cap \beta(\mathbb{K})$ and the other endpoint of e lies in $\beta(\mathbb{K}) \setminus N_G(m)$ (i.e., $e \in E(H_m)$). Furthermore, by construction, both these endpoints are different from all the vertices belonging to the edges of the matching Y_m in H_m. Therefore, $Y_m \cup \{e\}$ is also a matching in H_m. However, this is a contradiction, as Y_m is a maximum matching in H_m. This concludes the proof. □

Let us fix a pair of consecutive marked bags $B_\ell, B_r \in \mathcal{B}_{\text{Marked}}(I)$ and consider the obtruded components of $\mathbb{K}[B_\ell, B_r]$. Note that Lemma 6.6 can be interpreted as follows. Any obtruded component of $\mathbb{K}[B_\ell, B_r]$ is a "module with respect to M." The following lemma shows that all but at most 4η of these obtruded components are actually modules in the graph G.

Lemma 6.7. *All but at most 4η of the obtruded components of $\mathbb{K}[B_\ell, B_r]$ are modules in G.*

Proof. Let H be an obtruded component of $\mathbb{K}[B_\ell, B_r]$. For any vertex $v \in B_\ell \cup B_r$, there are at most two obtruded components in $\mathbb{K}[B_\ell, B_r]$ with the property that v has both a neighbor and a non-neighbor in the component. Indeed, if this were not the case, then we would have obtained a long claw in $G[\beta(\mathbb{K})] - M$, which is a contradiction. Notice that there are at most 2η vertices in $B_\ell \cup B_r$. Hence, it follows that all but at most 4η obtruded components of $\mathbb{K}[B_\ell, B_r]$ have the following property: Each vertex $v \in B_\ell \cup B_r$ is adjacent either to all vertices of this obtruded component or to none of them. Finally, observe that the neighborhood of a vertex in an obtruded component H, excluding the neighbors that belong to H itself, is a subset of $M \cup B_\ell \cup B_r$. Hence, it follows from the preceding arguments and Lemma 6.6 that all but at most 4η obtruded components of $\mathbb{K}[B_\ell, B_r]$ are modules in G. □

Let us note another useful property of the obtruded components.

Lemma 6.8. *Let H be an obtruded component of $\mathbb{K}[B_\ell, B_r]$. Then, there is a sub-clique path $\mathbb{K}_H^{\text{obs}}$ of $\mathbb{K}[B_\ell, B_r]$ such that $V(H) \subseteq \beta(\mathbb{K}_H^{\text{obs}}) \subseteq V(H) \cup B_\ell \cup B_r$.*

Proof. Since H is a connected graph and $\mathbb{K}$ is a path decomposition, it follows from the definition of a path decomposition that the set of bags of $\mathbb{K}$ that have non-empty intersection with $V(H)$ forms a sub-clique path $\mathbb{K}_H^{\text{obs}}$ of $\mathbb{K}$. Furthermore, as H is a connected component of $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$, where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$, it follows that $V(H) = \beta(\mathbb{K}_H^{\text{obs}}) \setminus (B_\ell \cup B_r)$. Therefore, $\mathbb{K}_H^{\text{obs}}$ is a sub-clique path of $\mathbb{K}[B_\ell, B_r]$ and $V(H) \subseteq \beta(\mathbb{K}_H^{\text{obs}}) \subseteq V(H) \cup B_\ell \cup B_r$. □

The obtruded components of $\mathbb{K}[B_\ell, B_r]$ can be divided into two groups, those that are modules in G and the rest. We will first consider the problem of reducing the module obtruded components.

6.1.1 Handling Obtruded Modules of $\mathbb{K}$. In this section, our goal will be to upper bound the total number of vertices across all bags B that have that following property: B has non-empty intersection with at least one obtruded component of $\mathbb{K}$ that is a module in G. First, we will only reduce the total number of vertices in the obtruded components of $\mathbb{K}$ that are modules in G. To achieve this, we will employ Lemma 4.3 (see Section 4). To this end, consider a pair of consecutive marked bags B_ℓ, B_r in $\mathcal{B}_{\mathsf{Marked}}(I)$. Let $\widehat{C}$ be the set of obtruded components of $\mathbb{K}[B_\ell, B_r]$ that are modules in G. Note that by the construction, $\widehat{C}$ is the set of connected components in $G[B_\ell, B_r] = G[V[B_\ell, B_r]]$ (where $V[B_\ell, B_r] = \beta(\mathbb{K}[B_\ell, B_r]) \setminus (B_\ell \cup B_r)$) that are modules. Thus, from the definition of a path decomposition, it follows that $\widehat{C}$ is a subcollection of the collection of all the connected components in $G - (M \cup B_\ell \cup B_r)$ that are modules. Moreover, note that $|M \cup B_\ell \cup B_r| \leq |M| + 2\eta$.

Now we apply Lemma 4.3 for $\widehat{M} = B_\ell \cup B_r$, and obtain a subset Z of $V(\widehat{C})$ of size at most $8(k+1)^3(|M| + 2\eta)^{10}$ such that the following holds:

> If $S \subseteq V(G)$ of size at most k and $\mathbb{O}$ is an obstruction in $G - S$ that is not covered by $\mathcal{W}$, then there is another obstruction $\mathbb{O}'$ in $G - S$ such that $\mathbb{O}' \cap (V(\widehat{C}) \setminus Z) = \emptyset$.

This gives the following reduction rule.

Reduction Rule 6.1. Suppose there is $v \in V(\widehat{C}) \setminus Z$. Then, delete v from the graph G. In other words, the resulting instance is $(G - \{v\}, k)$.

Lemma 6.9. *Reduction Rule 6.1 is safe.*

Proof. Let $v \in V(\widehat{C}) \setminus Z$, and $G' = G - \{v\}$. We will show that (G, k) is a Yes-instance of IVD if and only if (G', k) is. In the forward direction, let S be a solution to (G, k). As $G' - S$ is an induced subgraph of $G - S$, Observation 6.1 implies that S is a solution to (G', k).

In the reverse direction, let S' be a solution to (G', k). We claim that S' is a solution to (G, k). Let $S_v = S' \cup \{v\}$ and observe that it is a solution of size $k + 1$ in G. Toward a contradiction, suppose that this claim is false. Then, there is an obstruction $\mathbb{O}$ in $G - S'$. Notice that $\mathbb{O}$ is not covered by $\mathcal{W}$—indeed, if $\mathbb{O}$ were covered by $\mathcal{W}$, then because $S_v \cap M = S' \cap M$ and $\mathcal{W} \subseteq 2^M$ is a $(k+2)$-necessary family, it would have followed that $V(\mathbb{O}) \cap S' \neq \emptyset$. Thus, Lemma 4.3 implies that there is an obstruction $\mathbb{O}'$ in $G - S'$ that is disjoint from $V(\widehat{C}) \setminus Z$. The obstruction $\mathbb{O}'$ does not contain the vertex v, hence it is also an obstruction in $(G - \{v\}) - S' = G - S_v$. Since we have reached a contradiction, the proof is complete. ◻

If Reduction Rule 6.1 is not applicable, then we can assume that the (total) number of vertices in $V(\widehat{C})$ is bounded by $8(k+1)^3(|M| + 2\eta)^{10}$. In the following lemma, we bound the number of bags in $\mathbb{K}$ that have non-empty intersection with $V(\widehat{C})$.

Lemma 6.10. *The number of bags in* $\mathbb{K}$ *having non-empty intersection with* $V(\widehat{C})$ *is bounded by* $48(k+1)^3(|M| + 2\eta)^{10}$.

Proof. Let us first note that any bag in $\mathbb{K}$ that contains at least one vertex of $V(\widehat{C})$ is a subset of $V(\widehat{C}) \cup B_\ell \cup B_r$ and is also a bag in $\mathbb{K}[B_\ell, B_r]$. To prove the desired claim, we create a special set of bags $\mathcal{S}$, as follows. First, add B_ℓ, B_r to $\mathcal{S}$. Recall that B_ℓ appears before B_r in the ordering of the bags given by $\mathbb{K}$. For each $x \in B_\ell$, let B^x be the first bag in $\mathbb{K}[B_\ell, B_r]$ that does *not* contain x, where if such a bag does not exist we then set $B^x = B_r$. Similarly, for each $y \in B_r$, let $\widehat{B}^y$ be the first bag in $\mathbb{K}[B_\ell, B_r]$ that contains y, which exists since $y \in B_r$. We add all the bags in $\{B^x \mid x \in B_\ell\} \cup \{\widehat{B}^y \mid y \in B_r\}$ to $\mathcal{S}$. Next, for each $v \in V(\widehat{C})$, let F^v and L^v be the first bag and last bag in $\mathbb{K}[B_\ell, B_r]$ containing v, respectively. We further add each bag in $\{F^v \mid v \in V(\widehat{C})\} \cup \{L^v \mid v \in V(\widehat{C})\}$ to $\mathcal{S}$.

Recall that $|V(\widehat{C})| \leq 8(k+1)^3(|M|+2\eta)^{10}$ and $\eta \geq |M| \geq k \geq 1$, and thus we can obtain that $|\mathcal{S}| \leq |B_\ell| + |B_r| + 2|V(\widehat{C})| + 2 \leq 3 \cdot 8(k+1)^3(|M|+2\eta)^{10} = 24(k+1)^3(|M|+2\eta)^{10}$. Consider any two bags B_1, B_2 in $\mathcal{S}$ (where B_1 appears before B_2 in the ordering given by $\mathbb{K}$) such that there is no bag from $\mathcal{S}$ in $\mathbb{K}[B_1, B_2]$ other than B_1 and B_2. We call the sub-clique path $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ (which might be empty) a *restricted region* of the sub-clique path $\mathbb{K}[B_\ell, B_r]$. In the following, we will argue that all the bags belonging to the same restricted region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{C})$. We make this argument with respect to B_1 and B_2. To this end, consider the collection of bags $\mathcal{S}' = \{X \in \mathbb{K}[B_1, B_2] \mid X \notin \{B_1, B_2\}\}$. We will argue that for any $X, Y \in \mathcal{S}'$, we have $X \cap (B_\ell \cup B_r \cup V(\widehat{C})) = Y \cap (B_\ell \cup B_r \cup V(\widehat{C}))$. Toward this, consider some $X, Y \in \mathcal{S}'$ such that X appears before Y in the ordering given by $\mathbb{K}$. We consider two cases as follows, and in each of the cases we rely on the property that in a clique path, the set of bags containing a fixed vertex forms a sub-clique path:

- There is $v \in (X \setminus Y) \cap (B_\ell \cup B_r \cup V(\widehat{C}))$. Note that $v \notin B_r$, as otherwise it belongs to $X \cap B_r$ but not to Y, which violates the sub-clique path property of a clique path. Consider the subcase where $v \in B_\ell$. This implies that v belongs to each bag in $\mathbb{K}[B_\ell, X]$. But as $v \notin Y$, the bag $B^v \in \mathcal{S}$ must belong to $\mathbb{K}[X, Y]$. This contradicts the fact that $\mathbb{K}[X, Y]$ does not contain any bag from $\mathcal{S}$. Next, consider the subcase where $v \in V(\widehat{C})$. Again, as $v \in X$ and $v \notin Y$, we have that the bag L^v must belong to $\mathbb{K}[X, Y]$, which is a contradiction.
- There is $v \in (Y \setminus X) \cap (B_\ell \cup B_r \cup V(\widehat{C}))$. Note that $v \notin B_\ell$, as otherwise it belongs to $B_\ell \cap Y$ but not to X, which violates the sub-clique path property of a clique path. Consider the subcase where $v \in B_r$. This implies that v belongs to each bag in $\mathbb{K}[Y, B_r]$. But as $v \notin X$, the bag $\widehat{B}^v \in \mathcal{S}$ must belong to $\mathbb{K}[X, Y]$. This contradicts the fact that $\mathbb{K}[X, Y]$ does not contain any bag from $\mathcal{S}$. Next, consider the subcase where $v \in V(\widehat{C})$. Again, as $v \notin X$ and $v \in Y$, we have that the bag F^v must belong to $\mathbb{K}[X, Y]$, which is a contradiction.

From the preceding, we conclude that bags in the same restricted region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{C})$. In what follows, we will show why this statement implies that in any restricted region there can be at most one bag that has non-empty intersection with $V(\widehat{C})$. Before showing that the claim is true, let us argue that having this claim concludes the proof. Indeed, since $|\mathcal{S}| \leq 24(k+1)^3(|M|+2\eta)^{10}$ and $B_\ell, B_r \in \mathcal{S}$, there are at most $24(k+1)^3(|M|+2\eta)^{10}$ restricted regions that can have non-empty intersection with $V(\widehat{C})$. Each one of these regions has only one bag that has non-empty intersection with $V(\widehat{C})$. Adding up the bags in $\mathcal{S}$ itself, we conclude that there are at most $48(k+1)^3(|M|+2\eta)^{10}$ bags in $\mathbb{K}$ that contain a vertex from $V(\widehat{C})$.

We now turn to show that in any restricted region, there can be at most one bag that has non-empty intersection with $V(\widehat{C})$. For this purpose, consider some restricted region $\mathbb{K}[B_1^{+1}, B_2^{-1}]$. Then, all bags in this region contain the same set of vertices from $B_\ell \cup B_r \cup V(\widehat{C})$. Suppose that this region contains some vertex $v \in V(\widehat{C})$. By the definition of $\widehat{C}$, there exists an obtruded component H of $\mathbb{K}[B_\ell, B_r]$ that contains v. Because v belongs to every bag in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ and by Lemma 6.8, it follows that H contains all vertices across all bags in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ apart from those in $B_\ell \cup B_r$. Thus, all vertices across all bags in $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ belong to $B_\ell \cup B_r \cup V(\widehat{C})$. Because distinct bags on a clique path correspond to distinct sets of vertices, this means that $\mathbb{K}[B_1^{+1}, B_2^{-1}]$ can only contain a single bag that has a non-empty intersection with $V(\widehat{C})$. This concludes the proof. □

Recall that there are at most $4\eta|M|$ pairs of consecutive marked bags in $\mathcal{B}_{\mathsf{Marked}}(I)$. Applying Reduction Rule 6.1 for every such pair, we obtain the following. There are at most $48(k+1)^3(|M|+2\eta)^{10} \cdot 4\eta|M|$ bags of $\mathbb{K}$ that contain vertices from obtruded modules. Let $C(\mathbb{K})$ denote the set of vertices that appear in obtruded modules. Let $\mathcal{B}_{\mathsf{Marked}}(II)$ denote the collection of all bags in $\mathbb{K}$ that contain a vertex in $C(\mathbb{K})$.

Marking Scheme II. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(II)$ to $\mathcal{B}_{\mathsf{Marked}}$.

From Lemma 6.10, we obtain that we have marked at most $\boxed{48(k+1)^3(|M|+2\eta)^{10} \cdot 4\eta|M|}$ bags of $\mathbb{K}$ using Marking Scheme II.

6.1.2 Obtaining Manageable Clique Paths. In this section, we will focus on the obtruded components of $\mathbb{K}$ that are not modules in G. To this end, we mark some more bags in $\mathbb{K}$ so that the regions between unmarked bags have additional structural properties. We will refer to the sub-clique paths obtained by this process as *manageable clique paths*. In the following, we start by defining some notation that will be helpful in describing this marking scheme.

Let B_ℓ, B_r be two consecutive bags in $\mathcal{B}_{\mathsf{Marked}}(I)$, where B_ℓ appears before B_r in the ordering given by $\mathbb{K}$. Next, consider a non-module obtruded component H of $\mathbb{K}[B_\ell, B_r]$ (and note that it contains an unmarked vertex), and let $\mathbb{K}_H^{\mathsf{obt}}$ be the sub-clique path of $\mathbb{K}[B_\ell, B_r]$ provided by Lemma 6.8. Let $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_H^{\mathsf{obt}})$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_H^{\mathsf{obt}})$ be the first and last bags of $\mathbb{K}_H^{\mathsf{obt}}$, respectively. Before moving on to our next marking scheme, we construct two sets of bags, $\mathcal{L}_1(H)$ and $\mathcal{L}_2(H)$. Initially, we have $\mathcal{L}_1(H) = \{\mathsf{B}_{\mathsf{left}}(\mathbb{K}_H^{\mathsf{obt}}), \mathsf{B}_{\mathsf{right}}(\mathbb{K}_H^{\mathsf{obt}})\}$. We note that the construction of $\mathcal{L}_1(H)$ is quite similar to the construction of $\mathcal{S}$ used in the proof of Lemma 6.10. For each $u \in B_\ell$, let $B^u(H)$ be the first bag in $\mathbb{K}_H^{\mathsf{obt}}$ that does not contain u, where if such a bag does not exist we then set $B^u(H) = \mathsf{B}_{\mathsf{right}}(\mathbb{K}_H^{\mathsf{obt}})$. Additionally, for each $v \in B_r \setminus B_\ell$, let $\widehat{B}^v(H)$ be the first bag in $\mathbb{K}_H^{\mathsf{obt}}$ that contains v, where if such a bag does not existwe then set $\widehat{B}^v(H) = \mathsf{B}_{\mathsf{right}}(\mathbb{K}_H^{\mathsf{obt}})$. We add all the bags in $\{B^u(H) \mid u \in B_\ell\} \cup \{\widehat{B}^v(H) \mid v \in B_r \setminus B_\ell\}$ to $\mathcal{L}_1(H)$. We initialize $\mathcal{L}_2(H) = \mathcal{L}_1(H)$. For each bag $B \in \mathcal{L}_1(H)$ in $\mathbb{K}_H^{\mathsf{obt}}$, we add to $\mathcal{L}_2(H)$ the bags adjacent to B, namely B^{-1} and B^{+1} (if they exist) in $\mathbb{K}_H^{\mathsf{obt}}$. Note that the number of bags in $\mathcal{L}_2(H)$ is bounded by 10η.[4]

For consecutive marked bags $B_\ell, B_r \in \mathcal{B}_{\mathsf{Marked}}(I)$ in $\mathbb{K}$, let $\mathcal{H}(B_\ell, B_r)$ be the set of non-module obtruded components of $\mathbb{K}[B_\ell, B_r]$. Furthermore, let $\mathcal{L}(B_\ell, B_r)$ be the union of the sets $\mathcal{L}_2(H)$ taken over all $H \in \mathcal{H}(B_\ell, B_r)$. From Lemma 6.7, we know that there are at most 4η obtruded components of $\mathbb{K}[B_\ell, B_r]$ that are not modules. Thus, the number of bags in $\mathcal{L}(B_\ell, B_r)$ is bounded by $40\eta^2$. Finally, let $\mathcal{B}_{\mathsf{Marked}}(III)$ be the union of the sets of bags $\mathcal{L}(B_\ell, B_r)$ taken over all B_ℓ and B_r that are consecutive marked bags in $\mathcal{B}_{\mathsf{Marked}}(I)$. Recall that $|\mathcal{B}_{\mathsf{Marked}}(I)|$ is bounded by $4\eta|M|$. Thus, the number of bags in $\mathcal{B}_{\mathsf{Marked}}(III)$ is bounded by $160\eta^3|M|$. We are now ready to state our third marking scheme.

Marking Scheme III. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(III)$ to $\mathcal{B}_{\mathsf{Marked}}$.

Note that we marked at most $\boxed{160\eta^3|M|}$ bags using the preceding marking scheme. We now further partition $\mathbb{K}$ using the bags marked in the preceding scheme.

In the following, we will give some useful properties regarding the region between consecutive marked bags in $\mathcal{B}_{\mathsf{Marked}}(III)$. To this end, let $B_\ell, B_r \in \mathcal{B}_{\mathsf{Marked}}(I)$ be consecutive marked bags in $\mathbb{K}$, where we consider marked bags only in $\mathcal{B}_{\mathsf{Marked}}(I)$. We assume that B_ℓ appears before B_r in the ordering given by $\mathbb{K}$. Consider an obtruded non-module component H of $\mathbb{K}[B_\ell, B_r]$, and let $\mathbb{K}_H^{\mathsf{obt}}$ be the sub-clique path provided by Lemma 6.8. Note that from the lemma, bags marked in $\mathcal{B}_{\mathsf{Marked}}(II)$ do not occur in $\mathbb{K}_H^{\mathsf{obt}}$. In the following, we write $\mathbb{K}_X, \mathbb{K}_Y, \ldots$ and so forth to denote

[4]The number 10 in 10η is a slightly larger constant than what can actually be achieved, and we use this constant only to simplify calculations.

various sub-clique paths of $\mathbb{K}$. Here $X, Y, \ldots$ are used as indices to identify these clique paths, unless we state otherwise.

Definition 6.11 (Manageable Clique Path). Let $\mathbb{K}_X$ be a sub-clique path of $\mathbb{K}$ such that it contains at least one other bag apart from $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$. The sub-clique path $\mathbb{K}_X$ is called a *manageable clique path* if all interior bags of $\mathbb{K}_X$ do not lie in $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \cup \mathcal{B}_{\mathsf{Marked}}(III)$. Furthermore, if $\mathbb{K}_X$ is a sub-clique path of $\mathbb{K}[B_\ell, B_r]$, where B_ℓ, B_r are consecutive marked bags in $\mathcal{B}_{\mathsf{Marked}}(I)$, then $\mathbb{K}_X$ is called a (B_ℓ, B_r)-*manageable clique path*.

We note that in the preceding definition, there is a non-module obtruded component H of $\mathbb{K}[B_\ell, B_r]$ such that $\beta(\mathbb{K}_X) \subseteq \beta(\mathbb{K}_H^{\mathsf{obt}}) \subseteq V(H) \cup B_\ell \cup B_r$, where $\mathbb{K}_H^{\mathsf{obt}}$ is the sub-clique path provided by Lemma 6.8 (also see Observation 6.5). Observe that a manageable clique path $\mathbb{K}_X$ is not a clique in G, since it contains at least three distinct bags of the clique path $\mathbb{K}$. Further observe that for the manageable clique path $\mathbb{K}_X$, the bags $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$ are not necessarily marked bags. However, this is true for any *maximal* manageable clique path $\mathbb{K}_X$ (i.e., those manageable clique paths that are not contained in another manageable clique path). Then observe that the endpoint bags of $\mathbb{K}_X$ must lie in $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(III)$, since any manageable clique path is contained in a non-module obtruded component of $\mathbb{K}[B_\ell, B_r]$ for some consecutive pair of bags $B_\ell, B_r \in \mathcal{B}_{\mathsf{Marked}}(I)$, and these end bags are not in $\mathcal{B}_{\mathsf{Marked}}(II)$ by the definition of $\mathcal{B}_{\mathsf{Marked}}(III)$ (recall that from Lemma 6.8, we have $\beta(\mathbb{K}_H^{\mathsf{obt}}) \subseteq V(H) \cup B_\ell \cup B_r$). This gives us the following observation (from Marking Schemes I and III).

Observation 6.12. *The number of maximal manageable clique paths in $\mathbb{K}$ is upper bounded by $160\eta^3|M|$.*

Next, we derive the following property using the notations we introduced earlier. Consider a manageable clique path $\mathbb{K}_X$ that is a sub-clique path of the clique path $\mathbb{K}_H^{\mathsf{obt}}$, where H is a non-module obtruded component of $\mathbb{K}[B_\ell, B_r]$. (Note that $\mathbb{K}_X$ is a (B_ℓ, B_r)-manageable clique path.)

Lemma 6.13. *For any two bags B, B' in a (B_ℓ, B_r)-manageable clique path $\mathbb{K}_X$, we have $B \cap (B_\ell \cup B_r) = B' \cap (B_\ell \cup B_r)$.*

Proof. Let us consider a maximal (B_ℓ, B_r)-manageable clique path $\widehat{\mathbb{K}_X}$ that contains $\mathbb{K}_X$ as a sub-clique path. Furthermore, let H be a non-module obtruded component of $\mathbb{K}[B_\ell, B_r]$ such that $\beta(\widehat{\mathbb{K}_X}) \subseteq \beta(\mathbb{K}_H^{\mathsf{obt}})$, where $\mathbb{K}_H^{\mathsf{obt}}$ is the sub-clique path provided by Lemma 6.8.

We will prove the lemma for $\widehat{\mathbb{K}_X}$, thereby implying the lemma for $\mathbb{K}_X$. Recall that by the construction of $\mathcal{L}_1(H)$ and $\mathcal{L}_2(H)$, $\widehat{\mathbb{K}_X}$ contains no bag from $\mathcal{L}_1(H)$. Consider two bags S, T in $\widehat{\mathbb{K}_X}$ such that S appears before T in the ordering given by $\mathbb{K}$. We consider the following cases, and in each of the cases we rely on the property that in a clique path, the set of bags containing a fixed vertex forms a sub-clique path:

- There is $v \in (S \setminus T) \cap (B_\ell \cup B_r)$. Note that $v \notin B_r$, as otherwise it belongs to $S \cap B_r$ but not to T, which violates the sub-clique path property of a clique path. From the preceding, we can conclude that $v \in B_\ell$. This implies that v belongs to each bag in $\mathbb{K}[B_\ell, S]$. But as $v \notin T$, the bag $B^v(H) \in \mathcal{L}_1(H)$ must belong to $\mathbb{K}[S, T]$. This contradicts the fact that $\widehat{\mathbb{K}_X}$ does not contain any bag from $\mathcal{L}_1(H)$.
- There is $v \in (T \setminus S) \cap (B_\ell \cup B_r)$. Note that $v \notin B_\ell$, as otherwise it belongs to $T \cap B_\ell$ but not to S, which violates the sub-clique path property of a clique path. From the preceding, we can conclude that $v \in B_r$. This implies that v belongs to each bag in $\mathbb{K}[T, B_r]$. But as $v \notin S$, the bag $\widehat{B}^v(H) \in \mathcal{L}_1(H)$ must belong to $\mathbb{K}[S, T]$. This contradicts the fact that $\widehat{\mathbb{K}_X}$ does not contain any bag from $\mathcal{L}_1(H)$.

This concludes the proof. □

We will conclude this section by deriving a few more properties of manageable clique paths, which will be useful later. Consider a (B_ℓ, B_r)-manageable clique path $\mathbb{K}_X$, and recall that $\mathsf{C}(\mathbb{K}_X) = \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cap \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$.

Observation 6.14. *For $m \in M$, either $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X) \subseteq N_G(m)$ or $(\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)) \cap N_G(m) = \emptyset$. Furthermore, for $v \in \beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ and distinct $u, w \in N_G(v) \cap M$, at least one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds.*

Proof. Consider $m \in M$. Note that all the vertices in $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ must belong a single obtruded component. Thus, the first part of the observation follows from Lemma 6.6, and the second part of the observation follows from Observation 6.4. □

For a manageable clique path $\mathbb{K}_X$, let us define $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = M \cap N(\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X))$, and $\mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_X) = M \setminus \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Note that $N(\mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_X)) \cap \beta(\mathbb{K}_X) \subseteq \mathsf{C}(\mathbb{K}_X)$, and thus, in the notation in the previous sentence, the word ‘Priv’ stands for the possible “private” neighbors (in M) of vertices in $\mathsf{C}(\mathbb{K}_X)$. The following observation will be helpful in ruling out the case when there is a vertex $v \in \mathsf{C}(\mathbb{K}_X)$ and a vertex $m \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ such that $(v, m) \notin E(G)$.

Observation 6.15. *Consider $v \in \mathsf{C}(\mathbb{K}_X)$ and $m \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ such that $(v, m) \notin E(G)$. Then, $G[\beta(\mathbb{K}_X)]$ is a clique in G.*

Proof. Notice that $\mathsf{C}(\mathbb{K}_X) \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ is a clique in G, and thus $G[\mathsf{C}(\mathbb{K}_X)]$ is a clique. Additionally, every vertex in $\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is adjacent to every vertex in $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ in the graph G. Therefore, if there is a pair of non-adjacent vertices $u, w \in \beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$, then $\mathbb{O} = G[\{u, v, w, m\}]$ is an induced cycle on 4 vertices. Since Reduction Rule 3.1 is not applicable, each set in $\mathcal{W}$ has size at least 2, and hence $\mathbb{O}$ is not covered by $\mathcal{W}$. But then any obstruction that is not covered by $\mathcal{W}$ must intersect M in at least 10 vertices. Hence, we arrive at a contradiction. □

Observation 6.16. *For a manageable clique path $\mathbb{K}_X$, each of the following holds:*

(1) *For any $v \in \beta(\mathbb{K}_X)$ and $m \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, we have $(v, m) \in E(G)$.*
(2) *For each $u \in \mathsf{C}(\mathbb{K}_X)$ and $v \in \beta(\mathbb{K}_X)$, where $u \neq v$, we have $(u, v) \in E(G)$.*
(3) *For distinct $m_1, m_2 \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, at least one of $\{m_1, m_2\} \in \mathcal{W}$ or $(m_1, m_2) \in E(G)$ holds.*

Proof. The first item follows from Observation 6.14 and 6.15 because $G[\beta(\mathbb{K}_X)]$ cannot be a clique. Since $\mathsf{C}(\mathbb{K}_X)$ is a clique that is contained in every bag of $\mathbb{K}_X$ in G, the second item of the observation follows. Last, the third item follows from Observation 6.14 and the definition of $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. □

Observation 6.17. *Let $\mathbb{K}_X$ be a manageable clique path, and let $\mathbb{K}'$ be any sub-clique path of $\mathbb{K}_X$, such that $G[\beta(\mathbb{K}')]$ is not a clique. Then, $\mathsf{C}(\mathbb{K}_X) \subseteq \mathsf{C}(\mathbb{K}')$ and $\mathsf{M}_{\mathsf{All}}(\mathbb{K}') = \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ (Figure 12).*

Proof. By the definition of a path decomposition, any vertex that belongs to both $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$ must also belong to every bag in between these two bags, and particularly to both $\mathsf{B}_{\mathsf{left}}(\mathbb{K}')$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}')$. Thus, it follows that $\mathsf{C}(\mathbb{K}_X) \subseteq \mathsf{C}(\mathbb{K}')$. This containment directly implies that $\mathsf{M}_{\mathsf{All}}(\mathbb{K}') \subseteq \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. However, we need to show that these two sets are in fact equal. To this end, consider a vertex $m \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. By Observation 6.14, we have that $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X) \subseteq N_G(m)$, and therefore $\beta(\mathbb{K}') \setminus \mathsf{C}(\mathbb{K}') \subseteq N_G(m)$. Thus, unless $\beta(\mathbb{K}') \setminus \mathsf{C}(\mathbb{K}')$ is empty, the last containment implies that $m \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}')$. However, $\beta(\mathbb{K}') \setminus \mathsf{C}(\mathbb{K}')$ cannot be empty, since then $\mathbb{K}'$ would have induced a clique. □

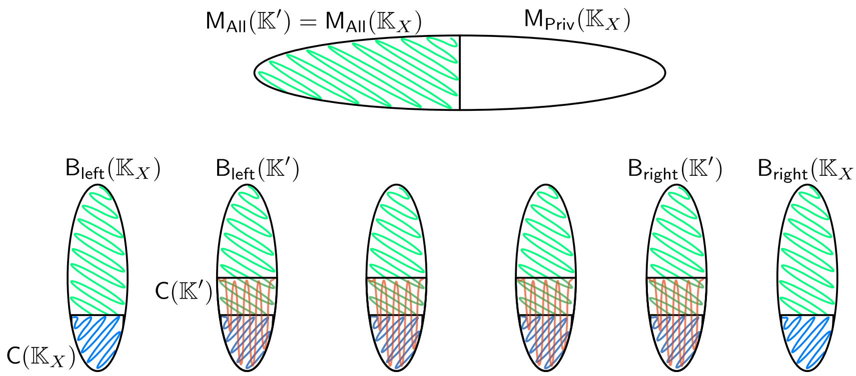

Fig. 12. A manageable clique path $\mathbb{K}_X$, a sub-clique path $\mathbb{K}'$, and an illustration of various sets in Observation 6.17.

6.2 Handling Manageable Clique Paths

We start by recalling that the number of maximal manageable clique paths is bounded by $160\eta^3|M|$. For the sake of simplicity, intuitively speaking, our next marking scheme will mark bags, which will help us ensure that after this marking scheme we are able to apply Observation 6.17. To this end, we let $\mathcal{B}_{\mathsf{Marked}}(IV)$ be the set of bags in $\mathbb{K}$, which contains, for every maximal manageable clique path that exactly has three bags, the middle bag of it. Notice that for each maximal manageable clique path $\mathbb{K}_X$, for $A = \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)\setminus(\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X))^{+1}$ and $A' = \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)\setminus(\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X))^{-1}$, if $G[\beta(\mathbb{K}_X)\setminus(A\cup A')]$ is a clique, then all the bags in $\mathbb{K}_X$ must belong to the set $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \cup \mathcal{B}_{\mathsf{Marked}}(III) \cup \mathcal{B}_{\mathsf{Marked}}(IV)$. (Recall our assumption that in $\mathbb{K}$, there are no two distinct bags where one is a subset of the other.) For simplicity in our arguments later, we mark all the bags in $\mathcal{B}_{\mathsf{Marked}}(IV)$—that is, we have the following marking scheme.

Marking Scheme IV. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(IV)$ to $\mathcal{B}_{\mathsf{Marked}}$.

We note that by the preceding marking scheme, we have marked at most $\boxed{160\eta^3|M|}$ many bags.

In the following, consider a (not necessarily maximal) manageable clique path $\mathbb{K}_X$. Recall that $\mathsf{C}(\mathbb{K}_X) = \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cap \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$ and $\mathsf{I}(\mathbb{K}_X) = \beta(\mathbb{K}_X) \setminus (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X))$. Observe that no vertex in $\mathsf{I}(\mathbb{K}_X)$ belongs to any marked bag (among all bags marked so far). Further recall that $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = M \cap N(\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X))$, and $\mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_X) = M \setminus \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.

We will devise a sequence of marking schemes that mark a polynomial in k number of bags in $\mathbb{K}_X$ such that the obstructions are "well behaved" with respect to the marked bags, where, loosely speaking, well behavedness will be captured by the obstruction being a path in each of the manageable clique paths. Intuitively speaking, this will allow us to focus mainly on AWs, as (large) cycles already have such a property. To this end, we have the following definition related to an obstruction.

Definition 6.18 (Manageable Obstruction). For a manageable clique path $\mathbb{K}_X$, an obstruction $\mathbb{O}$ is called $\mathbb{K}_X$-*manageable* if either $\mathbb{O}$ is an induced cycle on at least four vertices or it is an AW where no terminal of $\mathbb{O}$ belongs to $\mathsf{I}(\mathbb{K}_X)$. Furthermore, we say that $\mathbb{O}$ is a *manageable obstruction* if it is $\mathbb{K}_X$-manageable for every manageable clique path $\mathbb{K}_X$.

Observation 6.19. *If $\mathbb{K}_X$ is a manageable clique path that is a sub-clique path of another manageable clique path $\mathbb{K}_{\widetilde{X}}$ and $\mathbb{O}$ is a $\mathbb{K}_{\widetilde{X}}$-manageable obstruction, then $\mathbb{O}$ is also a $\mathbb{K}_X$-manageable obstruction.*

Recall that an induced cycle $\mathbb{O}$ on at least four vertices is a manageable obstruction (see Definition 6.18), and this allows us to mainly focus on AWs. Intuitively speaking, our goal is to mark a polynomial in k number of bags in each maximal manageable clique path $\mathbb{K}_X$ so that, for any set S of $k+2$ or fewer vertices, if there is an AW $\mathbb{O}$ in $G-S$, then (i) either $\mathbb{O}$ is already a $\mathbb{K}_X$-manageable obstruction, or (ii) there is an AW $\mathbb{O}'$ in $G-S$ such that all its vertices appear in marked bags (including the bags that we mark in our upcoming marking scheme). We present Lemma 6.20 to characterize the intersection between a manageable clique path $\mathbb{K}_X$ and an induced path P in G.

Lemma 6.20. *Let $\mathbb{K}_X$ be a manageable clique path. Let $P = (v_1, v_2, \ldots, v_t)$ be an induced path in G such that all of the following conditions are satisfied:*

(1) $(V(P) \setminus \{v_1, v_t\}) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$,[5]
(2) $V(P) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, *and*
(3) $V(P) \cap (V(G) \setminus \mathsf{I}(\mathbb{K}_X)) \neq \emptyset$.

Then, $V(P) \cap \mathsf{C}(\mathbb{K}_X) = \emptyset$. Furthermore, if $v_1, v_t \notin \mathsf{I}(\mathbb{K}_X)$, then the following properties hold:

- $P_X = P[V(P) \cap \beta(\mathbb{K}_X)]$ *is an induced path in G between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$.*
- $P_X - (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X))$ *is an induced path in $G[\mathsf{I}(\mathbb{K}_X)]$.*

Proof. Consider a vertex $v \in (V(P) \cap \mathsf{I}(\mathbb{K}_X)) \setminus \{v_1, v_t\}$, and let v_{-1} and v_{+1} be its two neighbors in P. Recall that $N(\mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_X)) \cap \beta(\mathbb{K}_X) \subseteq \mathsf{C}(\mathbb{K}_X)$, and hence $N_G(v) \cap \mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_X) = \emptyset$. We thus observe, because $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ and $V(P) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, it follows that the vertices v_{-1} and v_{+1} must belong to $\beta(\mathbb{K}_X)$. Furthermore, $\mathsf{C}(\mathbb{K}_X)$ is a clique, and for any $w \in \mathsf{C}(\mathbb{K}_X)$ we have $N_G(v) \subseteq N_G(w)$ (see Observation 6.16). Therefore, $V(P) \cap \mathsf{C}(\mathbb{K}_X) = \emptyset$. Indeed, if it were not the case, then we obtain a chord in the induced path P between a vertex $w \in V(P) \cap \mathsf{C}(\mathbb{K}_X)$ and (at least) one of v_{-1} or v_{+1} due to the containment $\{v_{-1}, v_{+1}\} \subseteq N_G(v) \subseteq N_G(w)$. This shows that $V(P) \cap \mathsf{C}(\mathbb{K}_X) = \emptyset$ (i.e., it concludes the proof of first part of the lemma).

Now, we turn to prove the second part of the lemma, and thus we assume that $v_1, v_t \notin \mathsf{I}(\mathbb{K}_X)$. Toward this, consider the set $V(P) \cap \beta(\mathbb{K}_X)$, and let $v_s \in \mathsf{I}(\mathbb{K}_X)$ be the vertex with the smallest index (i.e., subscript) in P that belongs to the set $\mathsf{I}(\mathbb{K}_X)$. The existence of such a vertex v_s follows from the assumption that $(V(P) \setminus \{v_1, v_t\}) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$. Moreover, note that $s \in \{2, \ldots, t-1\}$ due to the assumption that $v_1, v_t \notin \mathsf{I}(\mathbb{K}_X)$. Let v_e (possibly the same as v_s) be the vertex with the largest index in P that belongs to $\mathsf{I}(\mathbb{K}_X)$ such that for every $i \in \{s, s+1, \ldots, e\}$, $v_i \in \mathsf{I}(\mathbb{K}_X)$. As before, we have that $e \in \{2, \ldots, t-1\}$.

Next, we consider the vertices v_{s-1} and v_{e+1} along with the induced subpath $P' = P[\{v_{s-1}, v_s, \ldots, v_{e+1}\}]$. From the construction of v_{s-1} and v_{e+1}, the premise that $V(P) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, and the first part of the lemma, it follows that $v_{s-1}, v_{e+1} \notin \mathsf{I}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cup \mathsf{C}(\mathbb{K}_X)$. Moreover, $(v_{s-1}, v_s), (v_e, v_{e+1}) \in E(G)$, and for $v^* \in \{v_s, v_e\}$, we have $N_G(v^*) \subseteq \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Therefore, $v_{s-1}, v_{e+1} \in (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)) \setminus \mathsf{C}(\mathbb{K}_X)$. Without loss of generality, we assume that $v_{s-1} \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$. Then, $v_{e+1} \notin \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$, since otherwise we have the chord (v_{s-1}, v_{e+1}) in P. This implies that $v_{e+1} \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$. Therefore, $P' = P[\{v_{s-1}, v_s, \ldots, v_{e+1}\}]$ is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$.

Notice that v_{s-1-i}, for any $i \geq 2$, cannot belong to $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$, since otherwise there will be a chord in P (between v_{s-1-i} and v_{s-1}). We note that v_{s-2} could possibly belong to $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ but not to $\mathsf{C}(\mathbb{K}_X)$. Symmetrically, we derive that v_{e+1+i}, for any $i \geq 2$, cannot belong to $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$, whereas v_{e+2} could possibly belong to $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ but not to $\mathsf{C}(\mathbb{K}_X)$. Let $s^* \in \{s-1, s-2\}$ be the smallest index such that $v_{s^*} \in V(P) \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X))$, and let $e^* \in \{e+1, e+2\}$

[5]This implies that P has at least three vertices.

be the largest index such that $v_{e^*} \in V(P) \cap (\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X))$. From this, we conclude that $P^* = P[\{v_{s^*}, v_{s^*+1}, \ldots, v_{e^*}\}]$ is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$.

Thus, to complete the proof of the lemma, it remains to show that $v_i \notin \mathsf{I}(\mathbb{K}_X)$ for all $i \in \{1, 2, \ldots, s-2\} \cup \{e+2, e+3, \ldots, t\}$. Suppose not, then there is an integer $i^* \in [s-2] \cup \{e+2, e+3, \ldots, t\}$ such that $v_{i^*} \in \mathsf{I}(\mathbb{K}_X)$. Since $v_{i^*} \in \mathsf{I}(\mathbb{K}_X)$, it must hold that v_{i^*} belong to a bag, say B^* in $\mathbb{K}_X$, which is different from $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$. Recall that P' is a subpath of P from $v_{s-1} \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ to $v_{e+1} \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$. Therefore, P' intersects every bag in the manageable clique path $\mathbb{K}_X$. In particular, it contains a vertex different from v_{i^*}, say v', from B^*. But then $(v', v_{i^*}) \in E(G)$ is a chord in the induced path P, which is a contradiction. This concludes the proof of the lemma. □

Observation 6.21. *Let $v \in \beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$. Then, v is not a center vertex of any AW in G that is not covered by $\mathcal{W}$.*

Proof. Let $\mathbb{O}$ be an AW in G that is not covered by $\mathcal{W}$, and suppose that $v \in \beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ is a center vertex of $\mathbb{O}$. Then, v must be adjacent (in G) to all the vertices of base($\mathbb{O}$). As M is a 9-redundant solution, there are at least five vertices of M in base($\mathbb{O}$), and therefore there are vertices $m_1, m_2 \in M$ such that $(m_1, m_2) \notin E(G)$ and $(m_1, v), (m_2, v) \in E(G)$. Moreover, from Observation 6.14, for (distinct) $u, w \in N_G(v) \cap M$ one of $\{u, w\} \in \mathcal{W}$ or $(u, w) \in E(G)$ holds. But $(m_1, m_2) \notin E(G)$, and therefore $\{m_1, m_2\} \in \mathcal{W}$ must hold. This contradicts the fact that $\mathbb{O}$ is not covered by $\mathcal{W}$. □

Toward Our Case Distinction. Let us now consider the interaction between manageable clique paths and the obstructions in the graph that are not covered by $\mathcal{W}$. Let $\mathbb{O}$ be any AW (not covered by $\mathcal{W}$) in G. Recall that $P(\mathbb{O})$ denotes the extended base of $\mathbb{O}$ (including terminal vertices, t_ℓ and t_r). In what follows, we consider two cases based on the intersection between the vertex set of $\mathbb{O}$ and $\mathsf{I}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Before this, for the sake of clarity and summarization, let us recall the following facts:

(1) The obstruction $\mathbb{O}$ is an AW in G that is not covered by $\mathcal{W}$.
(2) The sets $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$ are cliques in G, and $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ separates $\mathsf{I}(\mathbb{K}_X)$ from the rest of the graph.
(3) Every vertex of $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is adjacent to all vertices in $\beta(\mathbb{K}_X)$ in G (by Observation 6.16).
(4) The vertices of $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$, and particularly $\mathsf{I}(\mathbb{K}_X)$, cannot be the center vertices of any AW in G that is not covered by $\mathcal{W}$ (by Observation 6.21). Therefore, every vertex of $\mathsf{I}(\mathbb{K}_X)$ is either a base vertex or a terminal of the AW $\mathbb{O}$.

6.2.1 $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$. The goal of this section will be to show that any AW $\mathbb{O}$ in G that is not covered by $\mathcal{W}$, and satisfies $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, is in fact already a $\mathbb{K}_X$-manageable obstruction. To this end, we let $\mathbb{O}$ be an AW in G. Furthermore, we remind that c_1 and c_2 are the centers of $\mathbb{O}$ (in case $\mathbb{O}$ is a †-AW, we have $c = c_1 = c_2$), t_ℓ, t_r are the non-shallow terminals, t is the shallow terminal, base($\mathbb{O}$) is the base, and $P(\mathbb{O})$ is the extended base.

In the following, we obtain some useful properties of $\mathbb{O}$ that satisfies the premise of this section—that is, $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$. This will be done in a sequence of four statements, after which we will be able to obtain the desired result. We first observe that the center(s) must belong to $\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.

Observation 6.22. *If $\mathbb{O}$ is an AW not covered by $\mathcal{W}$ and $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$, then $c_1, c_2 \in \mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.*

Proof. Consider $v \in V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X)$. Because $v \in \mathsf{I}(\mathbb{K}_X)$, we have $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, hence no vertex outside $\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ can be a center (as c_1, c_2 must belong to $N_G(v)$). Moreover, recall that by Observation 6.21, no vertex in $\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ can be a center vertex of an AW in G (that is uncovered by $\mathcal{W}$). Therefore, we have that $c_1, c_2 \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cup \mathsf{C}(\mathbb{K}_X)$. □

Second, we observe that the non-shallow terminals do not belong to $\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ (which already brings us close to the goal of this section), the base does not traverse $\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, and the shallow terminal does not belong to $\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.

Observation 6.23. *If $\mathbb{O}$ is an AW not covered by $\mathcal{W}$, $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, then $t_\ell, t_r \notin \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Furthermore, $V(\text{base}(\mathbb{O})) \cap (\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)) = \emptyset$ and $t \notin \mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.*

Proof. From Observation 6.22, $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ implies that $c_1, c_2 \in \mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. From Observation 6.16, we have that any vertex of $\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is adjacent to every vertex in $\beta(\mathbb{K}_X)$ in G. As $c_1 \in \mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is not adjacent to t_r, we obtain that $t_r \notin \beta(\mathbb{K}_X)$. Toward a contradiction, consider the case where $t_r \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, we have $\{c_1, t_r\} \notin \mathcal{W}$. But then from Observation 6.16, we obtain that $(c_1, t_r) \in E(G)$. This contradicts that $\mathbb{O}$ is an AW in G. From the preceding, we obtain that $t_r \notin \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. An analogous argument can be given to show that $t_\ell \notin \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. This proves the first part of the observation.

Next, toward a contradiction, suppose that there exists $w \in V(\text{base}(\mathbb{O})) \cap (\mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X))$. By the assumption that $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, we have $w \notin \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Hence, $w \in \mathsf{C}(\mathbb{K}_X)$, which means (by Observation 6.16) that w is adjacent to every vertex in $\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Let $v \in V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X)$ (which exists by the assumption that $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$) and u be the neighbor of v in $P(\mathbb{O})$ that is different than w. Recall that $N_G(v) \subseteq \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, and therefore $u \in \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. However, this implies that $P(\mathbb{O})[\{v, u, w\}]$ is a cycle on three vertices, contradicting that $P(\mathbb{O})$ is an induced path.

Finally, if $t \in \mathsf{C}(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, then $(t, v) \in E(G)$ ($\mathbb{O}$ is not covered by $\mathcal{W}$), which is a contradiction. This completes the proof. □

Third, we consider induced subgraph $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)]$ of $P(\mathbb{O})$. Due to Lemma 6.20, the following lemma is almost immediate.

Lemma 6.24. *If $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, then $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)]$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_X)$. And further, P_X is a subpath of $\text{base}(\mathbb{O})$.*

Proof. We note that $P(\mathbb{O})$ is an induced path in G and $\mathbb{O}$ is not covered by $\mathcal{W}$. We further note that the following conditions are satisfied:

(1) $(V(P(\mathbb{O})) \setminus \{v_1, v_t\}) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$, where $v_1 = t_\ell$ and $v_t = t_r$. This follows from our assumption that $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$.
(2) $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$, as this is one of our assumptions.
(3) $V(P(\mathbb{O})) \cap (V(G) \setminus \mathsf{I}(\mathbb{K}_X)) \neq \emptyset$ and $t_\ell, t_r \notin \mathsf{I}(\mathbb{K}_X)$. This follows from the fact that $t_\ell, t_r \notin \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$, which is obtained from Observation 6.23

Thus, using Lemma 6.20, we obtain that $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)]$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$, and P_X is a subpath of $\text{base}(\mathbb{O})$. □

Using Lemma 6.24, we obtain the following observation.

Observation 6.25. *If* $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$ *and* $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$*, then* $t \notin \beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.

Proof. From Observation 6.23, we can obtain that $t \notin \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Now, toward a contradiction, suppose that $t \in \beta(\mathbb{K}_X)$. Using Lemma 6.24, we obtain that $P_X = P(\mathbb{O})[\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)]$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$, and P_X is a subpath of base($\mathbb{O}$). But then P_X intersects every bag in $\mathbb{K}_X$ and t must lie in one of the bags in $\mathbb{K}_X$. From this, we conclude that there is $v \in V(P(\mathbb{O}))$ such that $(t, v) \in E(G)$, which again contradicts that $\mathbb{O}$ is an AW in G. □

The next lemma, whose proof was the goal of this section, follows directly from the preceding results and the definition of $\mathbb{K}_X$-manageable obstructions. Indeed, Observation 6.23 states that the non-shallow terminals cannot belong to $\beta(\mathbb{K}_X)$, and Observation 6.25 states that the shallow terminal cannot belong to $\beta(\mathbb{K}_X)$.

Lemma 6.26. *Let* $\mathbb{K}_X$ *be a manageable clique path. Let* $\mathbb{O}$ *be an AW in* G *such that* $\mathbb{O}$ *is not covered by* $\mathcal{W}$, $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) \neq \emptyset$*, and* $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \emptyset$*. Then,* $\mathbb{O}$ *is a* $\mathbb{K}_X$*-manageable obstruction that satisfies the following:*

(i) *The center vertices* c_1, c_2 *of* $\mathbb{O}$ *lie in* $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cup \mathsf{C}(\mathbb{K}_X)$.
(ii) *The terminals* t_ℓ, t_r, t *lie outside* $\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$.
(iii) *And* $P = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X))]$ *is an induced path between a vertex in* $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ *and a vertex in* $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_X) \setminus \mathsf{C}(\mathbb{K}_X)$ *that contains a vertex of* $\mathsf{I}(\mathbb{K}_X)$*. Furthermore,* P *is a subpath of* base($\mathbb{O}$).

6.2.2 $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$ *or* $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \neq \emptyset$. Irrespective of whether $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$ or $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \neq \emptyset$, let us first observe that since $\mathbb{O}$ is an AW, for any clique A in G, we have $|V(A) \cap V(\mathbb{O})| \leq 4$. This implies that $|V(\mathbb{O}) \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X))| \leq 8$. Moreover, since $\mathbb{O}$ is not covered by $\mathcal{W}$, for distinct $m, m' \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cap V(\mathbb{O})$, we have $(m, m') \in E(G)$ (see Observation 6.16). Thus, $|V(\mathbb{O}) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)| \leq 4$. From this, we obtain the following inequality:

$$|V(\mathbb{O}) \cap (\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X))| \leq 12.$$

Let c_1, c_2 be the center vertices of $\mathbb{O}$ (in the case of a †-AW, we have $c = c_1 = c_2$). Then, depending on whether $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$ or $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \neq \emptyset$, we note the following:

- First, suppose that $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$. In this subcase, from Observation 6.21, we have $V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_X) \subseteq \{t_\ell, t_r, t\}$ (possibly, $V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$).
- Second, suppose that there is a vertex $m \in V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Recall that every vertex in $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is adjacent to all the vertices in $\mathsf{I}(\mathbb{K}_X)$. Thus, in this subcase, $|V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_X)| \leq 2$, and otherwise $m \in V(P(\mathbb{O}))$ will be adjacent to three vertices of $V(\mathbb{O}) \setminus \{c_1, c_2\}$ (see Observations 6.16 and 6.21).

In summary, $V(\mathbb{O}) \cap (\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X))$ contains at most 15 vertices: up to 12 of these vertices are in $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{left}}(\mathbb{K}_X) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_X)$, and up to 3 of these vertices are in $\mathsf{I}(\mathbb{K}_X)$. We will use these bounds to derive our next marking scheme. In particular, since we deal with an obstruction whose intersection with $\beta(\mathbb{K}_X) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$ is upper bounded by a fixed constant, the relevance of the tool of representative families (defined in Section 2) is presented as a possibility—intuitively, we would like to capture enough vertices to represent every possibility of how the (up to) 3 vertices from $\mathsf{I}(\mathbb{K}_X)$ can "behave" within the small intersection. Toward that end, we proceed as follows.

Computation of Representative Families. We first restrict our attention to only a *maximal* manageable clique path $\widehat{\mathbb{K}_X}$. Consider a tuple $\mathcal{R} = (R, R_B, R_I)$, where R is a graph on the vertex set

$R_B \cup R_I$ (these are new dummy vertices), $|R_B| \leq 12$ and $|R_I| \leq 3$. Furthermore, consider a set $Z \subseteq \mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X})$ of $|R_B|$ vertices, a bijective function $f : Z \to R_B$, and an integer $d \in [3]$. For every such tuple (R, Z, f, d), we will perform a computation of a representative family as follows. Here, the family to be represented is $\mathcal{A}_{R,Z,f,d}$, the family of all d-sized subsets $Y \subseteq \mathsf{I}(\widehat{\mathbb{K}_X})$ such that the following condition is satisfied:

> There exists an isomorphism φ between $G[Z \cup Y]$ and R whose restriction to Z is equal to f—that is, for all $z \in Z$, we have $\varphi(z) = f(z)$.

Intuitively, we consider every "frame" that consists of the following: (i) the identity and topology of the (up to) 12 vertices in $\mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X})$ that lie in the intersection—this includes the specification of what are the identities of these vertices (given by Z) and what are the edges among them in G (given by $R[R_B]$), and (ii) the topology of the (up to) 3 vertices in $\mathsf{I}(\widehat{\mathbb{K}_X})$ that lie in the intersection (given by $R[R_I]$) and the edges between them and the previously mentioned 12 vertices (given by R). However, this information is not sufficient, and we require to also have explicit restriction of which vertex in Z is mapped to which vertex in R, and this is provided to us by the function f.

Next, consider the matroid $\mathcal{M} = (U, \mathcal{I})$, where $U = V(G)$ and $\mathcal{I} = \{U' \subseteq U \mid |U'| \leq d + k + 2\}$. Notice that $\mathcal{M}$ is a uniform matroid with universe size at most $|V(G)|$, and therefore it is representable over a field of size $|V(G)| + 1$ (see [13]). Furthermore, for such a field, the field operations can be done in time polynomial in $|V(G)|$ (even with very simple implementations). Thus, using Proposition 2.2, we find a $(k + 2)$-representative family $\widehat{\mathcal{A}_{R,Z,f,d}} \subseteq^{k+2}_{rep} \mathcal{A}_{R,Z,f,d}$ in polynomial time.

Marking Based on the Representative Families. We now construct a set $\mathbb{K}(\text{Rep}, \widehat{\mathbb{K}_X})$ of bags in $\widehat{\mathbb{K}_X}$ as follows. For every tuple (R, Z, f, d) defined earlier for the (maximal) manageable clique path $\widehat{\mathbb{K}_X}$, and for every vertex v that belongs to at least one set in $\widehat{\mathcal{A}_{R,Z,f,d}}$, we choose (arbitrarily) a bag in $\widehat{\mathbb{K}_X}$ that contains v and add this bag to the set $\mathbb{K}(\text{Rep}, \widehat{\mathbb{K}_X})$. Finally, we let $\mathcal{B}_{\mathsf{Marked}}(V)$ be the union of the bags in $\mathbb{K}(\text{Rep}, \widehat{\mathbb{K}_X})$ across every maximal manageable clique path $\widehat{\mathbb{K}_X}$.

Marking Scheme V. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(V)$ to $\mathcal{B}_{\mathsf{Marked}}$.

Toward bounding the number of bags we marked using the preceding marking scheme, consider a maximal manageable clique path $\widehat{\mathbb{K}_X}$ with end bags $\mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X})$, $\mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X})$. We observe that there are at most $O(1)$ choices for the graph R and its partition into R_B and R_I. Furthermore, there are at most $\binom{|\mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X})|}{\leq 12}$ choices for Z and at most $O(1)$ choices for f given the choice of Z. Thus, by Proposition 2.2, there are at most $O(k^3)$ sets in $\widehat{\mathcal{A}_{R,Z,f,d}}$ and each set contains at most $d \leq 3$ vertices. Hence, overall, we marked at most $O((2\eta + |M|)^{12} k^3)$ bags in the maximal manageable clique path $\widehat{\mathbb{K}_X}$. As there are at most $O(\eta^3 |M|)$ manageable clique paths in $\mathbb{K}$, Marking Scheme V marks at most $\boxed{O(\eta^{15}|M|k^3)}$ bags.

In the following, we prove a property regarding bags marked by Marking Scheme V.

Lemma 6.27. *Let S be a set of size at most $k + 2$ that intersects every set in $\mathcal{W}$, $\mathbb{K}_X$ be a manageable clique path, and $\widehat{\mathbb{K}_X}$ be the maximal manageable clique path such that $\mathbb{K}_X$ is a sub-clique path of $\widehat{\mathbb{K}_X}$. Additionally, let $\mathbb{O}$ be an AW in $G - S$ that is not covered by $\mathcal{W}$ such that $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X) = \emptyset$ or $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) \neq \emptyset$. Then, there is also an AW $\mathbb{O}'$ in $G - S$ that is not covered by $\mathcal{W}$ such that (i) $\mathbb{O}' - \mathsf{I}(\widehat{\mathbb{K}_X}) = \mathbb{O} - \mathsf{I}(\widehat{\mathbb{K}_X})$ and (ii) each vertex in $V(\mathbb{O}') \cap \mathsf{I}(\widehat{\mathbb{K}_X})$ appears in some marked bag from $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \ldots \cup \mathcal{B}_{\mathsf{Marked}}(V)$.*

Proof. Note that by Observation 6.17, $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_X) = \mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X})$. By the premise of the lemma, we have that $V(\text{base}(\mathbb{O})) \cap \mathsf{I}(\widehat{K_X}) = \emptyset$ or $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \neq \emptyset$.

Consider the graph $R = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\widehat{\mathbb{K}_X}) \cup \mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}))]$ (where we forget the "labeling" of the vertices, i.e., the graph R is supposed to be on $|V(R)|$ dummy vertices). Let $Z' = V(\mathbb{O}) \cap (\beta(\widehat{\mathbb{K}_X}) \cup \mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}))$. Furthermore, let $Z = Z' \cap (\mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X}))$ and $Y = Z' \setminus Z$. Observe that $Y \subseteq \mathsf{I}(\widehat{\mathbb{K}_X})$. Note that if $Y = \emptyset$, then trivially, $\mathbb{O}' = \mathbb{O}$ is a $\mathbb{K}_X$-manageable obstruction (see Definition 6.18). Thus, hereafter we assume that $Y \neq \emptyset$.

From the earlier discussion in this section, it follows that $|V(R)| \leq 15$, $|Z| \leq 12$, and $1 \leq |Y| \leq 3$. Let $d = |Y|$. Moreover, f is the function that maps every vertex in Z to the vertex in R that was originally labeled by Z.

Notice that $Y \in \mathcal{A}_{R,Z,f,d}$. Thus, from Proposition 2.2, there is a set $Y' \in \widehat{\mathcal{A}_{R,Z,f,d}}$ such that the following condition holds:

> There is an isomorphism φ between $G[Z \cup Y']$ and R whose restriction to Z is equal to f.

Since both Y and Y' are subsets of $\mathsf{I}(\widehat{\mathbb{K}_X})$, their neighbors in G belong to $\mathsf{M}_{\mathsf{All}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{left}}(\widehat{\mathbb{K}_X}) \cup \mathsf{B}_{\mathsf{right}}(\widehat{\mathbb{K}_X}) \cup \mathsf{I}(\widehat{\mathbb{K}_X})$. Let $\mathbb{O}' = G[(V(\mathbb{O}) \setminus Y) \cup Y']$. Note that both $N(Y) \cap V(\mathbb{O}) \subseteq Z$ and $N(Y') \cap V(\mathbb{O}') \subseteq Z$. Together with the preceding condition, we thus obtain that $\mathbb{O}'$ is isomorphic to $\mathbb{O}$. Hence, $\mathbb{O}'$ is an AW in $G - S$ with the property that all of the vertices of $\mathbb{O}'$ from $\widehat{\mathbb{K}_X}$ appear in the marked bags from $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \ldots \cup \mathcal{B}_{\mathsf{Marked}}(V)$. □

6.3 Nice Clique Paths and Nice Obstructions

We now consider a pair of consecutive marked bags in $\mathbb{K}$ that were marked by Marking Schemes I through V. In particular, for each maximal manageable clique path $\mathbb{K}_X$, we marked a collection of bags in $\mathbb{K}_X$ via Marking Scheme V, which (further) partitions $\mathbb{K}_X$ into sub-clique paths, which will be called *nice clique paths*.

Definition 6.28 (Nice Clique Path). Let $\mathbb{K}_Y$ be a sub-clique path of $\mathbb{K}$ such that it contains at least one bag apart from $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$. Then, $\mathbb{K}_Y$ is called a *nice clique path* if all interior bags of $\mathbb{K}_Y$ are unmarked in $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \ldots \cup \mathcal{B}_{\mathsf{Marked}}(V)$.

Note that any nice clique path has at least three bags and it is contained in a manageable clique path, and therefore it is also a manageable clique path. We also note that the end bags of a nice clique path $\mathbb{K}_Y$ need not be marked, and this is only true for *maximal* nice clique paths. In the following, we define the notion of *nice obstructions*.

Definition 6.29 (Nice Obstruction). Let $\mathbb{K}_Y$ be a nice clique path and $\mathbb{O}$ be an obstruction. Furthermore, let $J = V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))$. The obstruction $\mathbb{O}$ is called a $\mathbb{K}_Y$*-nice obstruction* (or $\mathbb{K}_Y$*-nice*) if one of the following holds:

(1) $J \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$, or
(2) $G[J]$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$.

Moreover, $\mathbb{O}$ is a *nice obstruction* if it is $\mathbb{K}_Y$-nice for every nice clique path $\mathbb{K}_Y$.

The following observation is easily obtained from the preceding definition and the fact that *a nice clique path is also a manageable clique path.*

Observation 6.30. *If $\mathbb{K}_Y$ is a nice clique path that is a sub-clique path of another nice clique path $\mathbb{K}_Z$, and $\mathbb{O}$ is a $\mathbb{K}_Z$-nice obstruction, then $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction. Hence, an obstruction $\mathbb{O}$ is a nice obstruction if it is $\mathbb{K}_Y$-nice for every maximal nice clique path $\mathbb{K}_Y$.*

Proof. Let $J_Z = V(\mathbb{O}) \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))$ and $J_Y = V(\mathbb{O}) \cap \beta(\mathbb{K}_Y)$. If $J_Z \subseteq \mathsf{B}_{\text{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Z)$, then as $(\mathsf{B}_{\text{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Z)) \cap \beta(\mathbb{K}_Y) \subseteq \mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y)$, we have $J_Y \subseteq \mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y)$. Hence, $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction.

Otherwise, J_Z is an induced path from a vertex in $\mathsf{B}_{\text{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ to a vertex in $\mathsf{B}_{\text{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$. If $J_Y \subseteq (\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y))$, then $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction. Otherwise, $J_Y \setminus (\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y)) \neq \emptyset$, and hence $J_Y \cap \mathsf{I}(\mathbb{K}_Y) \neq \emptyset$. Consider a vertex $v \in J_Y \cap \mathsf{I}(\mathbb{K}_Y)$. Note that as J_Z is an induced path from a vertex in $\mathsf{B}_{\text{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ to a vertex in $\mathsf{B}_{\text{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$, $(\mathsf{B}_{\text{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$, and $(\mathsf{B}_{\text{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$, therefore v is an internal vertex of the path J_Z. We will show that $J_Y \cap \mathsf{C}(\mathbb{K}_Y) = \emptyset$. To see the preceding statement, toward a contradiction, we consider a vertex $u \in J_Y \cap \mathsf{C}(\mathbb{K}_Y)$. This together with Observation 6.16 implies that $N(v) \subseteq \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\text{All}}(\mathbb{K}_Y) \subseteq N[u]$. This contradicts the fact that J_Z is an induced path (as v is an internal vertex of J_Z). Hence, $J_Y \cap \mathsf{C}(\mathbb{K}_Y) = \emptyset$, and therefore from Lemma 6.20 (invoking it with $\mathbb{K}_Y$ and J_Z), J_Y is an induced path from a vertex in $\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex $\mathsf{B}_{\text{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Hence, $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction. □

The following lemma shows that an induced cycle on at least four vertices, which is not covered by $\mathcal{W}$, is always a nice obstruction. We recall that by definition, a chordless cycle on four vertices is a manageable obstruction.

Lemma 6.31. *Let $\mathbb{O}$ be a chordless cycle on at least four vertices that is not covered by $\mathcal{W}$. Then, $\mathbb{O}$ is a nice obstruction.*

Proof. Let us consider a maximal nice clique path $\mathbb{K}_Y$, and suppose that $J = V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)) \not\subseteq \mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y)$. Consider a vertex $v \in J \setminus (\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y))$. Since $\mathsf{I}(\mathbb{K}_Y) = \beta(\mathbb{K}_Y) \setminus (\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y))$, we have that $v \in J \cap \mathsf{I}(\mathbb{K}_Y)$. As $\mathbb{O}$ is not covered by $\mathcal{W}$, there is a pair of (distinct) vertices $m_1, m_2 \in M \cap V(\mathbb{O})$ such that the path segment P between m_1 and m_2 in $\mathbb{O}$ contains the vertex v and $V(\mathbb{O}) \setminus V(P) \neq \emptyset$. Here, we rely on the fact that $\mathbb{O}$ is not covered by $\mathcal{W}$, and therefore $|M \cap V(\mathbb{O})| \geq 10$, which implies that $|V(\mathbb{O})| \geq 10$. Let P^* be the subpath of P from $m_1^* \in M$ to $m_2^* \in M$ containing v such that $|V(P^*) \cap M| = 2$. Note that P^* exists and could possibly be the same as P. As $V(\mathbb{O}) \setminus V(P) \neq \emptyset$, $(m_1^*, m_2^*) \notin E(G)$. Next, we argue that $m_1^*, m_2^* \notin \mathsf{M}_{\text{All}}(\mathbb{K}_Y)$. Consider the case when both $m_1^*, m_2^* \in \mathsf{M}_{\text{All}}(\mathbb{K}_Y)$. Since $\mathbb{O}$ is not covered by $\mathcal{W}$, from Observation 6.16, we have $(m_1^*, m_2^*) \in E(G)$, which is a contradiction. Next, suppose that $m_1^* \in \mathsf{M}_{\text{All}}(\mathbb{K}_Y)$ and $m_2^* \in \mathsf{M}_{\text{Priv}}(\mathbb{K}_Y)$ (the other case is symmetric). In this case, we have that $(v, m_2^*) \notin E(G)$. Observe that v has no neighbor outside $\beta(\mathbb{K}_Y) \cup \mathsf{M}_{\text{All}}(\mathbb{K}_Y)$ and m_1 is adjacent to all vertices in $\beta(\mathbb{K}_Y) \cup (\mathsf{M}_{\text{All}}(\mathbb{K}_Y) \cap V(\mathbb{O}))$ (Observation 6.16). Now let u be the neighbor of v in the subpath of P^* from v to m_2^*. Observe that $u \in \beta(\mathbb{K}_Y)$, and therefore we obtain a chord (m_1^*, u) in P^*, which is a contradiction. Therefore, $m_1^*, m_2^* \notin \mathsf{M}_{\text{All}}(\mathbb{K}_Y)$, and thus we have that $V(P^*) \cap \mathsf{M}_{\text{All}}(\mathbb{K}_Y) = \emptyset$. Observe that P^* satisfies the premise of Lemma 6.20, as the endpoints of P^* lie outside $\beta(\mathbb{K}_Y)$, and it contains an internal vertex from $\mathsf{I}(\mathbb{K}_Y)$, and $V(P^*) \cap \mathsf{M}_{\text{All}}(\mathbb{K}_Y) = \emptyset$. Therefore, $P^*[V(P^*) \cap \beta(\mathbb{K}_Y)]$ is an induced path from a vertex in $\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex in $\mathsf{B}_{\text{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ such that $P^* - (\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\text{right}}(\mathbb{K}_Y))$ is an induced path contained in $\mathsf{I}(\mathbb{K}_Y)$. Note that the endpoints of P^* in $\mathbb{O}$ belong to $\mathsf{M}_{\text{Priv}}(\mathbb{K}_Y)$. The preceding together with the fact that $P^*[V(P^*) \cap \beta(\mathbb{K}_Y)]$ is an induced path from a vertex in $\mathsf{B}_{\text{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex in $\mathsf{B}_{\text{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ implies that J cannot contain a vertex that does not belong to $V(P^*)$ (as otherwise, we can obtain a chord in $\mathbb{O}$). Thus, we conclude that $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction. Finally, as this argument holds for every nice clique path, the lemma follows. □

Next, for each obstruction (not covered by $\mathcal{W}$), we argue about existence of a nice obstruction.

Lemma 6.32. *Let $S \subseteq V(G)$ be a set of size at most $k+2$ that intersects each set in $\mathcal{W}$. If $\mathbb{O}$ is an obstruction in $G - S$, then there is a nice obstruction $\mathbb{O}'$ in $G - S$.*

Proof. Since S intersects each set in $\mathcal{W}$ and $\mathbb{O}$ is an obstruction in $G - S$, therefore $\mathbb{O}$ is not covered by $\mathcal{W}$. Thus, $\mathbb{O}$ contains at least 10 vertices from M. If $\mathbb{O}$ is a chordless cycle, then by Lemma 6.31 it is a nice obstruction. Now, we assume that $\mathbb{O}$ is an AW, and suppose that it is not a nice obstruction. Let $\mathbb{O}'$ be an obstruction in $G-S$ such that for each manageable path $\mathbb{K}_X$ we have that (i) either $\mathbb{O}'$ is a $\mathbb{K}_X$-manageable obstruction or (ii) each vertex in $V(\mathbb{O}') \cap V(\mathbb{K}_X)$ appears in some marked bag from $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \ldots \cup \mathcal{B}_{\mathsf{Marked}}(V)$. Note that $\mathbb{O}'$ can be obtained by iterative application of Lemma 6.26 or Lemma 6.27 for every maximal manageable clique path $\mathbb{K}_X$, depending on the sets $V(\mathrm{base}(\mathbb{O})) \cap \mathsf{I}(\mathbb{K}_X)$ and $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_X)$. Note that each application of these lemmas modifies the obstruction $\mathbb{O}'$ only within the corresponding maximal manageable clique path. Moreover, these lemmas also ensure that $\mathbb{O}'$ is not covered by $\mathcal{W}$, since $\mathbb{O}$ was not covered by $\mathcal{W}$. Thus, we have that $\mathbb{O}'$ is not covered by $\mathcal{W}$.

We claim that $\mathbb{O}'$ is a nice obstruction in $G - S$. From Observation 6.30, it is enough to argue that $\mathbb{O}'$ is a $\mathbb{K}_Y$-nice obstruction for every maximal nice clique paths. For the rest of the proof, fix a maximal nice clique path $\mathbb{K}_Y$ and let $\mathbb{K}_X$ be the maximal manageable clique path of which $\mathbb{K}_Y$ is a sub-clique path. Recall that $\mathbb{O}'$ is either a $\mathbb{K}_X$-manageable obstruction or each vertex in $V(\mathbb{O}') \cap V(\mathbb{K}_X)$ appears in some marked bag from $\mathcal{B}_{\mathsf{Marked}}(I) \cup \mathcal{B}_{\mathsf{Marked}}(II) \ldots \cup \mathcal{B}_{\mathsf{Marked}}(V)$. As $V(\mathbb{K}_Y) \subseteq V(\mathbb{K}_X)$, in the latter case, $V(\mathbb{O}') \cap V(\mathbb{K}_Y) \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$ must hold, and thus we can conclude that $\mathbb{O}'$ is a $\mathbb{K}_Y$-nice obstruction. We now focus on the case when $\mathbb{O}'$ is a $\mathbb{K}_X$-manageable obstruction. As a nice clique path is also a manageable clique path, from Observation 6.19 we can obtain that $\mathbb{O}'$ is a $\mathbb{K}_Y$-manageable obstruction that is not covered by $\mathcal{W}$. We must now show that $V(\mathbb{O}') \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))$ is either a subset of $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$ or it is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. In other words, we show that $\mathbb{O}'$ is a $\mathbb{K}_Y$-nice obstruction. Next, we consider the following cases:

(1) Consider the case when $V(\mathrm{base}(\mathbb{O}')) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$. Since $\mathbb{O}'$ is a $\mathbb{K}_Y$-manageable obstruction, the terminals of $\mathbb{O}'$ must lie in the marked bags or in M, and hence they cannot belong to vertices in $\mathsf{I}(\mathbb{K}_Y) = \beta(\mathbb{K}_Y) \setminus (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$. Since $\mathbb{O}'$ is not covered by $\mathcal{W}$, from Observation 6.21 we can obtain that the center vertices of $\mathbb{O}'$ do not belong to $\mathsf{I}(\mathbb{K}_Y)$. From the preceding discussions, together with the assumption that $V(\mathrm{base}(\mathbb{O}')) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$, we conclude that $V(\mathbb{O}') \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)) \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$. Therefore, $\mathbb{O}'$ is $\mathbb{K}_Y$-nice.

(2) Consider the case when $V(P(\mathbb{O}')) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) \neq \emptyset$. Recall that $\mathbb{O}'$ is $\mathbb{K}_Y$-manageable. Thus, the terminals of $\mathbb{O}'$ must lie in the marked bags or in M, and hence they cannot belong to vertices in $\mathsf{I}(\mathbb{K}_Y)$. By using an argument similar to the one used for the previous case, we can deduce that the centers cannot belong to $\mathsf{I}(\mathbb{K}_Y)$. Finally, if there is a vertex $v \in \mathsf{I}(\mathbb{K}_Y) \cap \mathrm{base}(\mathbb{O}')$, consider its two (non-adjacent) neighbors x, y in $P(\mathbb{O}')$. Notice that since $N_G(v) \subseteq \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$, we have $x, y \in \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Since $\mathbb{O}'$ is not covered by $\mathcal{W}$, using Observation 6.16 we can deduce that at most one of x, y can belong to $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. If $x \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$ and $y \notin \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$, which means that $y \in \beta(\mathbb{K}_Y)$, then using Observation 6.16, we have that $(x, y) \in E(G)$. We can give similar arguments for the case when $x \notin \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$ and $y \in \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Thus, we now assume that $x, y \in \beta(\mathbb{K}_Y)$. Consider a vertex $u \in V(P(\mathbb{O}')) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$, which exists by our assumption in this case. Then u, v are both adjacent to x, y (see Observation 6.16), contradicting that $P(\mathbb{O}')$ is an induced path. Thus, we conclude that $V(\mathbb{O}') \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)) \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$. Therefore, $\mathbb{O}'$ is $\mathbb{K}_Y$-nice.

(3) Otherwise, we have $\mathrm{base}(V(\mathbb{O}')) \cap \mathsf{I}(\mathbb{K}_Y) \neq \emptyset$ and $V(P(\mathbb{O}')) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$. Recall that $\mathbb{O}'$ is $\mathbb{K}_Y$-manageable. Then, Lemma 6.26 implies the following. $P = G[V(\mathbb{O}') \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Further, P is a subpath of $\mathrm{base}(\mathbb{O}')$. Furthermore, the centers c_1, c_2 of $\mathbb{O}'$ lie in $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) \cup \mathsf{C}(\mathbb{K}_Y)$, whereas the terminals $t_\ell, t_r, t \notin \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Hence,

$G[V(\mathbb{O}') \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Therefore, $\mathbb{O}'$ is $\mathbb{K}_Y$-nice.

Hence, $\mathbb{O}'$ is $\mathbb{K}_Y$-nice, and further we can conclude that $\mathbb{O}'$ is a nice obstruction in $G - S$. □

Corollary 6.33. *Let $\mathbb{K}_Y$ be a nice clique path, and let $\mathbb{O}$ be an AW such that it is a nice obstruction not covered by $\mathcal{W}$. Further, let $G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ be an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Then, the following holds:*

(i) *$V(\mathbb{O}) \cap (\mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)) = \{c_1, c_2\}$, the centers of $\mathbb{O}$.*
(ii) *The terminals $t_\ell, t_r, t \notin \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$.*
(iii) *And $G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ is a subpath of $\mathrm{base}(\mathbb{O})$, and $G[V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y)]$ is an induced path in $G[\mathsf{I}(\mathbb{K}_Y)]$.*

Proof. Let $J = V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))$. Note that J contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. We can obtain that $\mathrm{base}(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y) \neq \emptyset$ as follows. If $\mathrm{base}(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$, then using the arguments similar to the arguments of Case 1 in the proof of Lemma 6.32, we can obtain that $V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y) = \emptyset$ (and thus, reaching a contradiction). Similarly, using the arguments of Case 2 in the proof of Lemma 6.32, we can obtain that $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$.

Consider the first property, and we have $\{c_1, c_2\} \subseteq V(\mathbb{O}) \cap (\mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y))$ (from Lemma 6.26). We argue that indeed these sets are equal. Suppose not. Note that no terminal vertex of $\mathbb{O}$ lies in $\mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$ (from Lemma 6.26), hence any vertex $w \in (V(\mathbb{O}) \cap (\mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y))) \setminus \{c_1, c_2\}$ must be from $base(\mathbb{O})$. Note that $G[J]$ is a subpath of $\mathrm{base}(\mathbb{O})$ (from Lemma 6.26) and it contains at least three vertices (at least one from each of $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$, $\mathsf{I}(\mathbb{K}_Y)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$), and any vertex of $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) \cup \mathsf{C}(\mathbb{K}_Y)$ must be adjacent to all vertices of $\beta(\mathbb{K}_Y)$ (by Observation 6.16). From the preceding, we obtain that $w \in V(\mathrm{base}(\mathbb{O}))$ is adjacent to at least three vertices, which contradicts that $\mathrm{base}(\mathbb{O})$ is an induced path.

Note that the second property follows directly from Lemma 6.26. Now consider the third property, and we have that $G[J]$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Further, $G[J]$ is a subpath of $\mathrm{base}(\mathbb{O})$. We only need to argue that $G[V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y)]$ is an induced path in $G[\mathsf{I}(\mathbb{K}_Y)]$. Observe that $G[V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y)]$ is an induced subgraph of the path $G[J]$. Suppose that it is not connected. Then there must be vertices u, a_1, a_2, a_3, v in the path $G[J]$ that occur in this sequence (not necessarily as a subpath) such that $u, v, a_2 \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$ and $a_1, a_3 \in \mathsf{I}(\mathbb{K}_Y)$. But then, say $u, a_2 \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)$ (the other case is symmetric). This is a contradiction, since $G[J]$ is an induced path, and (u, a_2) is an edge in the clique $G[\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)]$. Therefore, $G[V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y)]$ must be a connected induced subgraph of $G[J]$, and hence it is a induced path in $G[\mathsf{I}(\mathbb{K}_Y)]$. □

We will require a strengthening of the preceding corollary that allows us to “replace” the path $P = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ in $\mathbb{O}$ with another path P' between the endpoint bags of $\mathbb{K}_Y$ and obtain a new obstruction.

Let $\mathbb{O}$ be a nice obstruction in G that is not covered by $\mathcal{W}$. Consider a nice clique path $\mathbb{K}_Y$, and let $P = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$. From Definition 6.29, either $V(P) \subseteq \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$, or P is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex in $\mathsf{I}(\mathbb{K}_Y)$. Consider the latter case (i.e., when P is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex in $\mathsf{I}(\mathbb{K}_Y)$), and let u and v be the endpoints of P in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$, respectively. Note that as P contains an internal vertex (from $\mathsf{I}(\mathbb{K}_Y)$), $(u, v) \notin E(G)$. Let P' be any other induced path between u and v in $G[\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)]$ such that $V(P') \subseteq V(P) \cup \mathsf{I}(\mathbb{K}_Y)$. In the following lemma, we show how we can obtain another nice obstruction using P'.

Lemma 6.34. *There is a nice obstruction $\mathbb{O}'$ that is not covered by $\mathcal{W}$ such that $\mathbb{O}'$ is an induced subgraph of $G[(V(\mathbb{O}) \setminus V(P)) \cup V(P')]$.*

Proof. Let $\mathbb{O}' = G[(V(\mathbb{O}) \setminus V(P)) \cup V(P')]$. We now have two cases. First consider the case when $\mathbb{O}$ is a chordless cycle. Let us argue that $\mathbb{O}'$ is a nice chordless cycle that is not covered by $\mathcal{W}$. Let u^* and v^* be the neighbors of u and v, respectively, in $\mathbb{O}$ that lie outside $V(P)$. Since $\mathbb{O}$ is a $\mathbb{K}_Y$-nice obstruction and $V(P) = \beta(\mathbb{K}_Y) \cap V(\mathbb{O})$, it follows that $\mathbb{O}[u^*, v^*] = G[V(\mathbb{O}) \setminus V(P)]$ is a path between u^* and v^* such that all vertices of this path lie outside $\beta(\mathbb{K}_Y)$. We claim that $\mathbb{O}[u^*, v^*] \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$. As P contains at least 3 vertices, any vertex in $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$ is adjacent to all vertices of P (by Observation 6.16.), this claim follows. Then, it is clear that $\mathbb{O}' = G[V(\mathbb{O}[u^*, v^*]) \cup V(P')]$ is a cycle in G. Next, observe that $V(\mathbb{O}') \setminus \mathsf{I}(\mathbb{K}_Y) \subseteq V(\mathbb{O}) \setminus \mathsf{I}(\mathbb{K}_Y)$, since $V(P') \subseteq V(P) \cup \mathsf{I}(\mathbb{K}_Y)$. Further, $N_G(w) \cap V(\mathbb{O}[u^*, v^*]) = \emptyset$ for every vertex $w \in V(\mathbb{O}') \cap \mathsf{I}(\mathbb{K}_Y)$, since $N_G(w) \subseteq \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Therefore, there are no edges in G between $V(\mathbb{O}') \cap \mathsf{I}(\mathbb{K}_Y)$ and $V(\mathbb{O}[u^*, v^*])$. Finally, as $\mathbb{O}$ is not covered by $\mathcal{W}$ and M is a 9-redundant solution, there must be at least 10 vertices of $\mathbb{O}$ in M, which implies that $\mathbb{O}'$ contain at least 10 vertices. Hence, $\mathbb{O}'$ contains a chordless cycle in G on at least 10 vertices. We note that $\mathbb{O}'$ is not covered by $\mathcal{W}$, as $V(\mathbb{O}) \cap M = V(\mathbb{O}') \cap M$ (and $|V(\mathbb{O}) \cap M| \geq 10$). Finally, from Lemma 6.31, it follows that $\mathbb{O}'$ is a nice obstruction.

Now we consider the case when $\mathbb{O}$ is an AW. Recall that $V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Furthermore, we have the following properties from Corollary 6.33:

(i) $V(\mathbb{O}) \cap (\mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)) = \{c_1, c_2\}$, where c_1, c_2 are the centers of $\mathbb{O}$.
(ii) The terminals $t_\ell, t_r, t \notin \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$.
(iii) The path $P\ (= G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))])$ is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Furthermore, P is a subpath of base$(\mathbb{O})$, and $G[V(\mathbb{O}) \cap \mathsf{I}(\mathbb{K}_Y)] = P[V(P) \cap \mathsf{I}(\mathbb{K}_Y)]$ is an induced path in $G[\mathsf{I}(\mathbb{K}_Y)]$.

Let us define $Q = G[(V(P(\mathbb{O})) \setminus V(P)) \cup V(P')]$. Then, we construct $\mathbb{O}'$ by replacing P with P' in $\mathbb{O}$. Let us argue that $\mathbb{O}'$ is also an AW. Let u^* and v^* be the neighbors of u and v, respectively, in $P(\mathbb{O})$ that lie outside $V(P)$. (We note that u^* and v^* exist, as P is a subpath of base$(\mathbb{O})$ and vertices in $V(\text{base}(\mathbb{O}))$ are internal vertices of $P(\mathbb{O})$.) Let P_{u^*} and P_{v^*} be the subpaths of $P(\mathbb{O})$ from t_ℓ to u^* and from v^* to t_r, respectively. Note that Q is a path from t_ℓ to t_r such that $V(Q) \setminus \beta(\mathbb{K}_Y) = V(P_{u^*}) \cup V(P_{v^*}) = V(P(\mathbb{O})) \setminus \beta(\mathbb{K}_Y)$. Moreover, as $V(P') \subseteq V(P) \cup \mathsf{I}(\mathbb{K}_Y)$ (together with Observation 6.14 and 6.16), it follows that Q is an induced path from t_ℓ to t_r. Similarly, we can argue that there is no edge between any vertex of Q and the shallow terminal t of $\mathbb{O}$. Moreover, each vertex of $V(Q) \cap \mathsf{I}(\mathbb{K}_Y)$ is adjacent to c_1 and c_2. Finally, recall that there are at least five vertices in base$(\mathbb{O}) \setminus \beta(\mathbb{K}_Y)$ that lie in M, as it is a 9-redundant solution and $\mathbb{O}$ is not covered bv $\mathcal{W}$. Hence, Q contains at least five internal vertices. Hence, $\mathbb{O}'$ is an AW. Furthermore, by construction, $\mathbb{O}'$ is a nice obstruction that is not covered by $\mathcal{W}$.

In each of the cases, by construction, $\mathbb{O}'$ is a nice obstruction that is not covered by $\mathcal{W}$. Moreover, $\mathbb{O} - \beta(\mathbb{K}_Y) = \mathbb{O}' - \beta(\mathbb{K}_Y)$. □

Consider a maximal nice clique path $\mathbb{K}_Y$ with endpoint bags $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$. Before moving on to our next marking scheme, we construct two sets of bags, $\mathcal{T}_1(\mathbb{K}_Y)$ and $\mathcal{T}_2(\mathbb{K}_Y)$. Initially, we have $\mathcal{T}_1(\mathbb{K}_Y) = \{\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y), \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)\}$. For each $u \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)$, let $\widehat{B_u(\mathbb{K}_Y)}$ be the last bag in $\mathbb{K}_Y$ that contains u. Additionally, for each $v \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)$, let $\widehat{B_v(\mathbb{K}_Y)}$ be the first bag in $\mathbb{K}_Y$ that contains v. We add all the bags in $\{\widehat{B_u(\mathbb{K}_Y)} \mid u \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)\} \cup \{\widehat{B_v(\mathbb{K}_Y)} \mid v \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus$

$\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y)\}$ to $\mathcal{T}_1(\mathbb{K}_Y)$. We initialize $\mathcal{T}_2(\mathbb{K}_Y) = \mathcal{T}_1(\mathbb{K}_Y)$. Furthermore, for each bag $B \in \mathcal{T}_1(\mathbb{K}_Y)$ in $\mathbb{K}_Y$, we add to $\mathcal{T}_2(\mathbb{K}_Y)$ the bags adjacent to B, namely B^{-1} and B^{+1} (if they exist) in $\mathbb{K}_Y$. Note that the number of bags in $\mathcal{T}_2(\mathbb{K}_Y)$ is bounded by $O(\eta)$. Finally, we let $\mathcal{B}_{\mathsf{Marked}}(VI)$ be the union of the sets $\mathcal{T}_2(\mathbb{K}_Y)$ taken over all maximal nice clique paths $\mathbb{K}_Y$.

Marking Scheme VI. Add all the bags in $\mathcal{B}_{\mathsf{Marked}}(VI)$ to $\mathcal{B}_{\mathsf{Marked}}$.

We marked at most $O(\eta)$ bags for each nice clique path. Recall that we have at most $O(\eta^3|M|)$ manageable clique paths, and for each manageable clique path we marked at most $O(\eta^{15}|M|k^3)$ bags in $\mathbb{K}$ using Marking Schemes IV and V, which partitioned the manageable clique path into nice clique paths. Hence, in Marking Scheme VI, we marked at most $\boxed{O(\eta^{16}|M|k^3)}$ bags in $\mathbb{K}$.

Definition 6.35 (Simple Clique Paths). Let B_i, B_j be a pair of consecutive marked bags (considering all marking schemes of the section up to now) in a nice clique path $\mathbb{K}_Y$. Then, $\mathbb{K}_Z = \mathbb{K}[B_i, B_j]$ is called a *simple clique path.*

Note that any simple clique path is also a nice clique path. Next, we state an observation regarding a simple clique path $\mathbb{K}_Z$. We note that this observation is similar to Lemma 6.13 presented in Section 6.1.2.

Observation 6.36. *Consider a pair B_i, B_j of consecutive marked bags in a maximal nice clique path $\mathbb{K}_Y$ such that $\mathbb{K}_Z = \mathbb{K}[B_i, B_j]$ contains at least three bags. Then, for any $B, B' \in \mathbb{K}_Z$, we have $B \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)) = B' \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$. Therefore, $\beta(\mathbb{K}_Z) \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)) \subseteq \mathsf{C}(\mathbb{K}_Z)$.*

Proof. As $\mathbb{K}_Z = \mathbb{K}[B_i, B_j]$ contains at least three bags where B_i and B_j are consecutive marked bags, both B_i and B_j must belong to $\mathcal{T}_2(\mathbb{K}_Y) \setminus \mathcal{T}_1(\mathbb{K}_Y)$. Thus, $\mathbb{K}_Z$ has no bags from $\mathcal{T}_1(\mathbb{K}_Y)$. Without loss of generality, assume that B appears before B' in $\mathbb{K}_Y$. If there is $u \in B \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$ such that $u \notin B' \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$, then there is a bag strictly before B' and on/after B that belongs to $\mathcal{T}_1(\mathbb{K}_Y)$. This contradicts that $\mathbb{K}_Z$ contains no bags from $\mathcal{T}_1(\mathbb{K}_Y)$. Similarly, when there is $u \in B' \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$ such that $u \notin B \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y))$, we can obtain a contradiction to the fact that $\mathbb{K}_Z$ contains no bags from $\mathcal{T}_1(\mathbb{K}_Y)$. This concludes the proof. □

In the next observation, we recall a property of interval graphs that will be useful later.

Observation 6.37 (See [8]). *For a connected interval graph, any minimal separator of it is an intersection of adjacent bags in its clique path.*

Let us now consider a simple clique path $\mathbb{K}_Z$ contained in a nice clique path $\mathbb{K}_Y$. In the following, by a *separator* in $\mathbb{K}_Z$, we mean a separator of $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ in the graph $G[\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)]$. From Observation 6.37, any minimal separator in $\mathbb{K}_Z$ lies in the intersection of two adjacent bags in $\mathbb{K}_Z$, after excluding the vertices in $\mathsf{C}(\mathbb{K}_Z)$.

Lemma 6.38. *Let $\mathbb{K}_Z$ be a simple clique path with at least three bags that is contained in the maximal nice clique path $\mathbb{K}_Y$. Furthermore, let S be a minimal solution of size at most $k+2$ in G that contains a vertex in $\mathsf{I}(\mathbb{K}_Z)$, and let $S_Z = (S \cap \beta(\mathbb{K}_Z)) \setminus \mathsf{C}(\mathbb{K}_Z)$. Then, S_Z is a separator in $\mathbb{K}_Z$. Furthermore, S_Z is a minimal separator in $\mathbb{K}_Z$. For any other separator S^*_Z in $\mathbb{K}_Z$ such that $S_Z \setminus \mathsf{I}(\mathbb{K}_Z) = S^*_Z \setminus \mathsf{I}(\mathbb{K}_Z)$ and $S^* = (S \setminus S_Z) \cup S^*_Z$ has size at most $k+2$, the set S^* is also a solution.*

Proof. Consider a vertex $w \in S \cap \mathsf{I}(\mathbb{K}_Z)$, and note that this vertex lies in S_Z. Then, consider an obstruction $\mathbb{O}^*$ such that $S \cap V(\mathbb{O}^*) = \{w\}$. Since S is a minimal solution, such an obstruction must exist. Moreover, as S is a solution of size at most $k+2$, it must cover $\mathcal{W}$. As all vertices of $\mathcal{W}$ lie in M and $w \in S \setminus M$, the set $S_w = S \setminus \{w\}$ also covers $\mathcal{W}$. Now, consider the obstruction $\mathbb{O}^*$ in $G - S_w$.

Note that $\mathbb{O}^*$ is not covered by $\mathcal{W}$, as otherwise S_w covers $\mathcal{W}$, and thus it intersects $\mathbb{O}^*$, which is a contradiction to the choice of $\mathbb{O}^*$. Hence, $\mathbb{O}^*$ is an obstruction that is not covered by $\mathcal{W}$ that is present in $G - S_w$. Then, by Lemma 6.32, there is a nice obstruction $\mathbb{O}$ in $G - S_w$ that is not covered by $\mathcal{W}$. Since $\mathbb{O}$ is not present in $G - S$, we must have that $S \cap V(\mathbb{O}) = \{w\}$. Let us further note that $w \in \mathsf{I}(\mathbb{K}_Y)$, since $\mathsf{I}(\mathbb{K}_Z) \subseteq \mathsf{I}(\mathbb{K}_Y)$ by the premise of the lemma.

Let $P = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$, and note that P contains w. Then, by the definition of a nice obstruction (see Definition 6.29), P is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Y)$. Let $P_Z = P[V(P) \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))]$. And again, because $w \in V(P)$ (and note that $\mathbb{K}_Z$ is also a nice clique path), P_Z is an induced path from a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ to a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ that contains a vertex of $\mathsf{I}(\mathbb{K}_Z)$.

Let u and v be the end vertices of the path P in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$, respectively. Let us note that the induced path P contains an internal vertex $w \in \mathsf{I}(\mathbb{K}_Y)$, and therefore $(u, v) \notin E(G)$. Let u_z and v_z be the endpoints of P_Z in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$, respectively. Then, as before, as P_Z is an induced path from u_z to v_z containing $w \in \mathsf{I}(\mathbb{K}_Z)$, $(u_z, v_z) \notin E(G)$. Now we argue (using Observation 6.36) that $u_z, v_z \notin \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$. Toward a contradiction, assume that $u_z \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$, then $u_z \in \mathsf{C}(\mathbb{K}_Y)$, which implies $u_z \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)$, which then implies that $(u_z, v_z) \in E(G)$. Indeed, $V(P_Z) \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)) = \emptyset$, since $V(P_Z) \cap \mathsf{C}(\mathbb{K}_Z) = \emptyset$. Finally, note that $u, v, u_z, v_z \notin S$, as these vertices belong to the obstruction $\mathbb{O}$ in $G - S_w$, where $w \in S \cap \mathsf{I}(\mathbb{K}_z)$.

Now suppose that S_Z is not a separator in $\mathbb{K}_Z$. Then, there is a path P'_Z in $G[\beta(\mathbb{K}_Z)] - (S \cup \mathsf{C}(\mathbb{K}_Z))$ between u_z and v_z. We note that such a path exists, because if there is a path from some $\widehat{u} \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ and $\widehat{v} \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ (which exists as S_Z is not a separator), then we can obtain a path from u_z to v_z, as $(u_z, \widehat{u}), (v_z, \widehat{v}) \in E(G)$. We note that $w \notin V(P'_Z)$, as $w \in S_Z$. Additionally, $V(P'_Z) \cap \mathsf{C}(\mathbb{K}_Z) = \emptyset$, which means $V(P'_Z) \cap (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)) = \emptyset$. Then, consider an induced path P' from u to v in $G[(V(P) \setminus V(P_Z)) \cup V(P'_Z)]$. Observe that $V(P') \cap S = \emptyset$, by construction, and $V(P') \subseteq V(P) \cup \mathsf{I}(\mathbb{K}_Y)$. Then, by Lemma 6.34, there is a nice obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subseteq (V(\mathbb{O}) \setminus V(P)) \cup V(P')$. By choice of $\mathbb{O}$ and P', we have $V(\mathbb{O}') \cap S = \emptyset$. But this is a contradiction. Hence, S_Z must be a separator in $\mathbb{K}_Z$.

Let us now argue that when S_Z contains a vertex in $\mathsf{I}(\mathbb{K}_Z)$, then it is a minimal separator in $\mathbb{K}_Z$. As we have argued that S_Z is a separator in $\mathbb{K}_Z$, there are two adjacent bags B, B' in $\mathbb{K}_Z$ such that $(B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z) \subseteq S_Z$ (see Observation 6.37). We claim that $S_Z = (B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z)$—that is, S_Z is a minimal separator in $\mathbb{K}_Z$. Our arguments are similar to the one in the previous paragraph. Toward a contraction, assume that $S_Z \neq (B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z)$. (Recall that $(B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z) \subseteq S_Z$.) Let $S'_Z = (B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z)$, and note that $S_Z \setminus S'_Z \subseteq \mathsf{I}(\mathbb{K}_Z)$. Consider an arbitrary vertex $w \in S_Z \setminus S'_Z$, and let $S_w = S \setminus \{w\}$. Note that S_w covers $\mathcal{W}$. Then, as S is a minimal solution, there is an obstruction $\mathbb{O}^*$ such that $V(\mathbb{O}^*) \cap S = \{w\}$. From Lemma 6.32, there is a nice obstruction $\mathbb{O}$ that is not covered by $\mathcal{W}$ such that $V(\mathbb{O}) \cap S = \{w\}$. Let $P_Z = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))]$. (Note that P_Z contains w.) By the definition of a nice obstruction, P_Z is an induced path from a vertex $u_z \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ to a vertex $v_z \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ that contains the vertex $w \in \mathsf{I}(\mathbb{K}_Z)$. But then the path P_Z exists in $G - S_w$, whereas any path from u_z to v_z in $G[\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)]$ must intersect S'_Z and $S'_Z \subseteq S_w$. This is a contradiction. Hence, $S_Z = S'_Z$—that is, S_Z is a minimal separator in $\mathbb{K}_Z$.

Let us now argue that for any other separator S^*_Z in $\mathbb{K}_Z$ such that $S_Z \setminus \mathsf{I}(\mathbb{K}_Z) = S^*_Z \setminus \mathsf{I}(\mathbb{K}_Z)$ and $S^* = (S \setminus S_Z) \cup S^*_Z$ has size at most $k + 2$, the set S^* is also a solution. Suppose not. Note that S^* covers $\mathcal{W}$ since $S^* \cap M = S \cap M$. Now consider an obstruction $\mathbb{O}'$ in $G - S^*$, and note that it is not covered by $\mathcal{W}$. Therefore, by Lemma 6.32, there is a nice obstruction $\mathbb{O}$ in $G - S^*$ that is not covered by $\mathcal{W}$. Let $w \in (S_Z \setminus S^*_Z) \cap V(\mathbb{O})$. (Note that $w \in \mathsf{I}(\mathbb{K}_Z)$ by the choice of S^*_Z, and it exists as S is a solution and $\mathbb{O}$ is an obstruction in $G - S^*$.) Let $P_Z = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))]$, and note that P_Z contains w. Then, by the definition of a nice obstruction, P_Z is an induced path from a vertex $u_z \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ to a vertex $v_z \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ that contains a vertex $w \in \mathsf{I}(\mathbb{K}_Z)$. But any such path must intersect S^*_Z and therefore S^*, which contradicts the assumption that $S^* \cap V(\mathbb{O}) = \emptyset$. This concludes the proof. □

Let us now identify and mark a collection of minimal separators in $\mathbb{K}_Z$ such that if there is a solution, then there is a solution contained in the marked bags. To this end, we first obtain some "useful" subsets. Consider a simple clique path $\mathbb{K}_Z$. Let $\mathcal{S}(\mathbb{K}_Z)$ denote the collection of subsets $T \subseteq (\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z)) \setminus \mathsf{C}(\mathbb{K}_Z)$ for which $|T| \leq k$ and there is a pair of adjacent bags B_T, B'_T in $\mathbb{K}_Z$ such that (i) $T = (B_T \cap B'_T) \setminus (\mathsf{C}(\mathbb{K}_Z) \cup \mathsf{I}(\mathbb{K}_Z))$ and (ii) $(B_T \cap B'_T) \cap \mathsf{I}(\mathbb{K}_Z) \leq k$. We can bound $|\mathcal{S}(\mathbb{K}_Z)|$ as follows.

Lemma 6.39. $|\mathcal{S}(\mathbb{K}_Z)| \leq 2k + 1$.

Proof. Let us index the bags of $\mathbb{K}_Z$ by natural numbers starting from 1. Let p be the smallest index for which there is $T_p \in \mathcal{S}(\mathbb{K}_Z)$ such that $T_p = (B_p \cap B_{p+1}) \setminus (\mathsf{C}(\mathbb{K}_Z) \cup \mathsf{I}(\mathbb{K}_Z))$. Similarly, let q be the largest index for which there is $T_q \in \mathcal{S}(\mathbb{K}_Z)$ such that $T_q = (B_q \cap B_{q+1}) \setminus (\mathsf{C}(\mathbb{K}_Z) \cup \mathsf{I}(\mathbb{K}_Z))$. Note that for any $T \in \mathcal{S}(\mathbb{K}_Z)$, we have $T \subseteq T_p \cup T_q$. Furthermore, we can order the sets in $\mathcal{S}(\mathbb{K}_Z)$, denoted by '$<$', such that for any $T < T' \in \mathcal{S}(\mathbb{K}_Z)$ we have $T \cap \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \supseteq T' \cap \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z)$ and $T \cap \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \subseteq T' \cap \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z)$. Moreover, as T and T' are distinct subsets of $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus (\mathsf{C}(\mathbb{K}_Z) \cup \mathsf{I}(\mathbb{K}_Z))$, one of those inclusions must be strict. Finally, observe that $T_p \cup T_q$ contains at most $2k$ vertices of $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$. Therefore, $|\mathcal{S}(\mathbb{K}_Z)| \leq 2k + 1$. □

We will construct a collection $\mathcal{B}_{\mathsf{Marked}}(VII)$ as follows. For each simple clique path $\mathbb{K}_Z$ and for each $T \in \mathcal{S}(\mathbb{K}_Z)$, we select a pair of adjacent bags B_T, B'_T in $\mathbb{K}_Z$ such that $B_T \cap B'_T$ is of minimum cardinality and contains T, and add them to $\mathcal{B}_{\mathsf{Marked}}(VII)$. Note that $B_T \cap B'_T$ is a minimal separator in $\mathbb{K}_Z$.

Marking Scheme VII. Add all bags in $\mathcal{B}_{\mathsf{Marked}}(VII)$ to $\mathcal{B}_{\mathsf{Marked}}$.

We note that using the preceding marking scheme, we mark at most $\boxed{\mathcal{O}(\eta^{16}|M|k^4)}$ bags in $\mathbb{K}$, which follows from the number of bags marked by Marking Scheme VI. We have the following lemma, which states that the collection of marked bags in $\mathcal{B}_{\mathsf{Marked}}$ contains a solution if one exists.

Lemma 6.40. *Let S be a minimal solution of cardinality at most k. Then, there is another minimal solution S' of size at most $|S|$ such that all vertices of S' lie in marked bags.*

Proof. Consider any simple clique path $\mathbb{K}_Z$. Suppose that S contains an unmarked vertex in $\mathbb{K}_Z$. Then, S contains a vertex in $\mathsf{I}(\mathbb{K}_Z)$. Then, by Lemma 6.38, $S_Z = S \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))$ is a minimal separator in $\mathbb{K}_Z$. Therefore, there is a pair of consecutive bags B, B' in $\mathbb{K}_Z$ such that $S_Z = (B \cap B') \setminus \mathsf{C}(\mathbb{K}_Z)$. Let $T_Z = S_Z \setminus \mathsf{I}(\mathbb{K}_Z)$. Then, note that $T_Z \in \mathcal{S}(\mathbb{K}_Z)$ since (i) $|T_Z| \leq |S_Z| \leq k$, (ii) $T_Z = (B \cap B') \setminus (\mathsf{C}(\mathbb{K}_Z) \cup \mathsf{I}(\mathbb{K}_Z))$, and (iii) $(B \cap B') \cap \mathsf{I}(\mathbb{K}_Z) \leq |S_Z| \leq k$. Now, corresponding to T_Z, we have marked a pair of adjacent bags B_1, B'_1 in $\mathcal{B}_{\mathsf{Marked}}(VII)$ such that $S'_Z = (B_1 \cap B'_1) \setminus \mathsf{C}(\mathbb{K}_Z)$ is a minimal separator in $\mathbb{K}_Z$ containing T_Z. Note that $|S'_Z| \leq |S_Z|$ and $S_Z \setminus \mathsf{I}(\mathbb{K}_Z) = S'_Z \setminus \mathsf{I}(\mathbb{K}_Z)$. Then, again by Lemma 6.38, $S' = (S \setminus S_Z) \cup S'_Z$ is a solution and $|S'| \leq |S|$. This concludes the proof of this lemma. □

Now we consider the problem of reducing the set of unmarked vertices in $\mathbb{K}$.

Lemma 6.41. *Let v be an unmarked vertex in a simple clique path $\mathbb{K}_Z$ such that v is contained in only one bag. Then (G, k) is a* Yes-instance *of* IVD *if and only if $(G - \{v\}, k)$ is a* Yes-instance *of* IVD.

Proof. In the forward direction, let S be a solution in G of size at most k. Clearly, S is a solution in $G - \{v\}$ as well. Now, we consider the reverse direction. Let S be a solution of size at most

k in $G - \{v\}$, and suppose that it is not a solution in G. Observe that $S \cup \{v\}$ is a solution in G of cardinality at most $k + 1$, and therefore it hits each set in $\mathcal{W}$. Furthermore, as $v \notin M$, S hits every set in $\mathcal{W}$. Now consider an obstruction $\mathbb{O}$ in $G - S$, and clearly it includes v. It follows that the obstruction $\mathbb{O}$ is not covered by $\mathcal{W}$, and $V(\mathbb{O}) \cap M$ contains at least 10 vertices. Let us consider $\mathbb{O}$ in the graph G along with the set S. Observe that $N(v) \subseteq B \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Z)$, where B is the (unique) bag in $\mathbb{K}_Z$ containing v. Every pair of vertices in $(B \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Z)) \setminus S$ is either an edge in $G - S$ or a pair in $\mathcal{W}$ (using Observation 6.16). Therefore, v does not have a pair of non-adjacent neighbors in $\mathbb{O}$. Hence, $\mathbb{O}$ is not a chordless cycle, and so it is an AW. Now, by Lemma 6.32, there is a nice obstruction $\mathbb{O}'$ in $G - S$. (Note that $\mathbb{O}'$ must contain v, as S is a solution in $G - \{v\}$.) Note that all terminals of $\mathbb{O}'$ lie in marked bags. As v is an unmarked vertex, by Observation 6.21 and Corollary 6.33, v lies in base($\mathbb{O}$) and therefore $N(v)$ must contain a pair of non-adjacent vertices, which is a contradiction. But then v is not part of the obstruction $\mathbb{O}'$. This implies that that $\mathbb{O}'$ is an obstruction in $G - (S \cup \{v\})$, which is also a contradiction. Hence, S must also be a solution in G. This concludes the proof of this lemma. □

The vertices that satisfy the premise of the preceding lemma are called *irrelevant vertices*. The preceding lemma gives the following reduction rule.

Reduction Rule 6.2. Let $\mathbb{K}_Z$ be a simple clique path. Then, pick an unmarked vertex in $\mathbb{K}_Z$ that is contained in only one bag, and delete it from the graph G. The resulting instance is $(G - \{v\}, k)$.

If the preceding reduction rule is not applicable, then there are no unmarked vertices in any nice clique path $\mathbb{K}_Y$ that are contained in only one bag. Then, observe that for any unmarked bag B in $\mathbb{K}_Y$, we have $B = (B \cap B^{-1}) \cup (B \cap B^{+1})$. Let us now consider the remaining of the unmarked vertices in $\mathbb{K}$.

Lemma 6.42. *Let $\mathbb{K}_Z$ be a simple clique path that contains an unmarked vertex. Then, there is an edge (u, v) such that at least one of its endpoints is an unmarked vertex, and there is only one bag in $\mathbb{K}_Z$ that contains this edge.*

Proof. Let us walk in $\mathbb{K}_Z$ starting from $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z)$, and let B be the first bag in $\mathbb{K}_Z$ that contains an unmarked vertex. Let us partition the bag B into three parts as follows, $A_2 = B^{-1} \cap B^{+1} \subseteq B$, $A_1 = (B \cap B^{-1}) \setminus A_2$, and $A_3 = B \cap B^{+1} \setminus A_2$. Note that $B \cap B^{-1} = A_1 \cup A_2$, and $B \cap B^{+1} = A_2 \cup A_3$. Note that $A_1 \neq \emptyset$, and otherwise $B = A_2 \cup A_3 \subseteq B^{+1}$, which is a contradiction as B is a maximal clique in the clique path $\mathbb{K}_Z$, and hence $B \nsubseteq B^{+1}$. Similarly, we can argue that $A_3 \neq \emptyset$. Now, consider an unmarked vertex $u \in B$ and observe that $u \in A_3$, by choice of B. Next, we choose a vertex $v \in A_1$, and clearly it is distinct from u. Furthermore, as $v \notin B^{+1}$ and $u \notin B^{-1}$, we have that the edge (u, v) is present only in B. □

In the following, we select an edge $e = (u, v)$ given by Lemma 6.42 that lies in a simple clique path $\mathbb{K}_Z$. We call such an edge an *irrelevant edge*.

Observation 6.43. *Let (u, v) be an irrelevant edge in a simple clique path $\mathbb{K}_Z$ such that u is an unmarked vertex. Then, $u \in \mathsf{I}(\mathbb{K}_Z)$ and $v \notin \mathsf{C}(\mathbb{K}_Z)$.*

Proof. Since u is unmarked, $u \notin \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z)$ since those bags are marked. Therefore, $u \in \mathsf{I}(\mathbb{K}_Z)$. And suppose that $v \in \mathsf{C}(\mathbb{K}_Z)$. Then, as the vertex u lies in at least two consecutive bags B and B' (since Reduction Rule 6.2 is not applicable), the edge (u, v) is present in both B and B'. But this contradicts the definition of an irrelevant edge. □

Lemma 6.44. *Let (u, v) be an irrelevant edge in a simple clique path $\mathbb{K}_Z$. Then, there is no minimal separator in $\mathbb{K}_Z$ that contains both u and v.*

Proof. Recall that $\mathbb{K}_Z \setminus \mathsf{C}(\mathbb{K}_Z)$ is a clique path with endpoint bags $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$ and $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$. Therefore, every minimal separator of these endpoint bags is the intersection of a pair of adjacent bags in $\mathbb{K}_Z \setminus \mathsf{C}(\mathbb{K}_Z)$. If both u and v were in a minimal separator, then the irrelevant edge (u, v) appears in at least two bags, which is a contradiction. Therefore, there is no minimal separator that contains both u and v. □

Observation 6.45. *A minimal solution of size at most $k + 2$ in G contains at most one of u and v, where (u, v) is an irrelevant edge in the simple clique path $\mathbb{K}_Z$.*

Proof. Suppose not. Let S be a minimal solution in G that contains both of u and v, and suppose that v is unmarked. Then, as S contains the vertex v that lies in $\mathsf{I}(\mathbb{K}_Z)$, $S_Z = S \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z))$ is a minimal separator in $\mathbb{K}_Z$ (see Observation 6.43). Now, by our assumption, S_Z contains both u and v, whereas by Lemma 6.44 S_Z contains at most one of them. This is a contradiction. □

Lemma 6.46. *Let $e = (u, v)$ be an irrelevant edge in $\mathbb{K}_Z$, where u is an unmarked vertex. Then, (G, k) is a* Yes*-instance of* IVD *if and only if $(G/e, k)$ is a* Yes*-instance of* IVD.

Proof. To begin with, let us note that as (u, v) is an irrelevant edge in $\mathbb{K}_Z$ and u is an unmarked vertex, $u \in \mathsf{I}(\mathbb{K}_Z)$ and $v \in \beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$, by Observation 6.43. Let $\mathbb{K}_Y$ be a maximal nice clique path that contains the simple clique path $\mathbb{K}_Z$. In other words, $\mathbb{K}_Z$ was obtained from $\mathbb{K}_Y$ by Marking Scheme VI. As $u, v \in \beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Z)$, and $(\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \cup \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y)) \cap \beta(\mathbb{K}_Z) \subseteq \mathsf{C}(\mathbb{K}_Z)$, we have $u, v \in \mathsf{I}(\mathbb{K}_Y)$ (see Observation 6.36).

Let z^* denote the vertex obtained by contracting the irrelevant edge $e = (u, v)$. Let S be a solution of size at most k in G. Let $S' = (S \setminus \{u, v\}) \cup \{z^*\}$ whenever $S \cap \{u, v\} \neq \emptyset$, and $S' = S$ otherwise. In the first case, observe that $G/e - S'$ is isomorphic to $G - (S \cup \{u, v\})$. And in the second case, $G/e - S'$ is isomorphic to $(G - S)/e$. As interval graphs are closed under edge contractions and vertex deletions (Observation 6.1), we have that S' is a solution in G/e of size at most k.

Now, suppose that S' is a solution of size at most k in G/e. We have two cases depending on whether or not $z^* \in S'$. First consider the case when $z^* \in S'$. Then, $S = (S' \setminus \{z^*\}) \cup \{u, v\}$ is a solution of size $k + 1$ in G, as $G - S$ is isomorphic to $G/e - S'$. As S is a solution of size at most $k + 1$, from Observation 6.45, it is not a minimal solution. Hence, there is $S^* \subsetneq S$ that is solution of cardinality at most k.

Now consider the case when $z^* \notin S'$. In this case, let $S = S' \cup \{u, v\}$, and observe that it has size at most $k + 2$. As $G - S$ is isomorphic to $G/e - (S' \cup \{z^*\})$, we have that S is a solution in G. As $\mathcal{W}$ is $(k+2)$-necessary, S hits each set in $\mathcal{W}$, which then implies that S' hits each set in $\mathcal{W}$ (since $u, v \notin M$). We claim that S' is a solution of size k in G. Suppose not, and let there be an obstruction $\mathbb{O}'$ in $G - S'$. As S' hits $\mathcal{W}$, we have that $\mathbb{O}'$ is not covered by $\mathcal{W}$. Now, from Lemma 6.32, there is a nice obstruction $\mathbb{O}$ in $G - S'$ that is not covered by $\mathcal{W}$. Then, $V(\mathbb{O}) \cap M$ contains at least 10 vertices, since M is a 9-redundant solution.

First, suppose that $V(\mathbb{O}) \cap \{u, v\} = \emptyset$. Then, clearly $\mathbb{O}$ is present in G/e (since $G - \{u, v\} = G/e - \{z^*\}$), and furthermore it is disjoint from S'. But this is a contradiction, since S' is a solution in G/e. Next, suppose that $V(\mathbb{O}) \cap \{u, v\}$ is exactly one of u or v. We claim that $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains an obstruction. As $N_G(u) \cup N_G(v) \subseteq \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$, they have no neighbors in $V(\mathbb{O}) \setminus (\beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y))$ (see Observation 6.16). Now, as $P = G[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y))]$ contains a vertex from $\mathsf{I}(\mathbb{K}_Y)$ and $\mathbb{O}$ is a nice obstruction (see Definition 6.29), P must be an induced path between a vertex $a_y \in \mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex $b_y \in \mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$. Let us note that P must contain at least 3 vertices, and hence $(a_y, b_y) \notin E(G)$. Also observe that in G/e, $N_{G/e}(z^*) \subseteq N_G(u) \cup N_G(v) \subseteq \beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Now we have the two following cases depending on $\mathbb{O}$:

- Consider the case when $\mathbb{O}$ is a chordless cycle. As $\mathbb{O}$ is a nice obstruction, we have $|V(\mathbb{O}) \cap M| \geq 10$. And as P contains at least 3 vertices, $V(\mathbb{O}) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$, as any such vertex will have edges to all vertices of P (using Observation 6.16). And since $u, v \in \mathsf{I}(\mathbb{K}_Y)$, we have $(N_G(u) \cup N_G(v)) \cap (V(\mathbb{O}) \cap \mathsf{M}_{\mathsf{Priv}}(\mathbb{K}_Y)) = \emptyset$. Recall that as $\mathbb{O}$ is not covered by $\mathcal{W}$, $\mathbb{O}$ contains at least 10 vertices from M (and $V(\mathbb{O}) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$). Thus, we can conclude that $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains a chordless cycle.

- Next, we consider the case when $\mathbb{O}$ is an AW. As $\mathbb{O}$ is a nice obstruction that contains a vertex from $\mathsf{I}(\mathbb{K}_Y)$, by Corollary 6.33 we have the following: (i) $P \subseteq P(\mathbb{O})$ and $P \cap \mathsf{I}(\mathbb{K}_Y) \subseteq \text{base}(\mathbb{O})$; (ii) $V(P(\mathbb{O})) \cap \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y) = \emptyset$, as $P(\mathbb{O})$ is an induced path, P contains at least 3 vertices and any vertex in $\mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$ is adjacent to every vertex in $\mathbb{K}_Z$ (using Observation 6.16); (iii) all terminals t_ℓ, t_r, t of $\mathbb{O}$ lie outside $\beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$; and (iv) $\{c_1, c_2\} \subseteq \mathsf{C}(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)$. Hence, $V(\mathbb{O}) \cap (\beta(\mathbb{K}_Y) \cup \mathsf{M}_{\mathsf{All}}(\mathbb{K}_Y)) = V(P) \cup \{c_1, c_2\}$. Therefore, $(N_G(u) \cup N_G(v)) \cap V(\mathbb{O}) \subseteq V(P) \cup \{c_1, c_2\}$.

  Note that $\{u, v\} \cap V(P) \subset V(\text{base}(\mathbb{O}))$, as $u, v \in I(\mathbb{K}_Y)$. And since $|\text{base}(\mathbb{O}) \cap M| \geq 5$, we have that P is a strict subset of $P(\mathbb{O})$. Therefore, $(N_G(u) \cup N_G(v)) \cap (V(\mathbb{O}) \setminus \{c_1, c_2\})$ is a strict subset of $V(P(\mathbb{O}))$ and $u, v \in N_G(c_1) \cap N_G(c_2)$. Hence, $G/e[(V(P(\mathbb{O})) \setminus \{u, v\}) \cup \{z^*\}]$ contains an induced path P^* from t_ℓ to t_r with at least 6 internal vertices including z^* in G/e. Observe that the internal vertices of P^* are adjacent to centers c_1, c_2 and not adjacent to the shallow terminal t of $\mathbb{O}$. Now it follows that $\{t_\ell, t_r, t\}$ form an asteroidal triple in $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$. Hence, $G/e[(V(\mathbb{O}) \setminus \{u, v\}) \cup \{z^*\}]$ contains an AW. Further observe that this obstruction lies in $G/e - S'$, which is a contradiction.

Now we consider the case that both u, v are present in $\mathbb{O}$. Recall that $\mathbb{O}$ is not covered by $\mathcal{W}$, and therefore it contains at least 10 vertices of the 9-redundant solution M. We claim that $\mathbb{O}/e$ is an obstruction in G/e. Indeed, if $\mathbb{O}$ is a chordless cycle, then as it contains at least 10 vertices in M, it follows that $\mathbb{O}/e$ is also a chordless cycle on at least 9 vertices. Otherwise, $\mathbb{O}$ is a nice AW. Now, recall that u is an unmarked vertex in $\mathsf{I}(\mathbb{K}_Z) \subseteq \mathsf{I}(\mathbb{K}_Y)$. As before, let $P = \mathbb{O}[V(\mathbb{O}) \cap (\beta(\mathbb{K}_Z) \setminus \mathsf{C}(\mathbb{K}_Y))]$ and observe that $P[V(P) \cap \mathsf{I}(\mathbb{K}_Y)] \neq \emptyset$. Therefore, by Corollary 6.33, we have that P is an induced path between a vertex in $\mathsf{B}_{\mathsf{left}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$ and a vertex in $\mathsf{B}_{\mathsf{right}}(\mathbb{K}_Y) \setminus \mathsf{C}(\mathbb{K}_Y)$, which is a subpath of base($\mathbb{O}$). Observe that $u, v \in V(P)$, and therefore they are in base($\mathbb{O}$). Finally, recall that base($\mathbb{O}$) contains at least 5 internal vertices of M. Therefore, $P(\mathbb{O})/e$ is an induced path between t_ℓ and t_r with at least 6 internal vertices including z^* in G/e. Hence, it follows that $\mathbb{O}/e$ is an AW of the same type as $\mathbb{O}$, and further it is present in G/e. Finally, observe that $\mathbb{O}/e$ is an obstruction in G/e that is disjoint from S'. This is a contradiction.

Having obtained a contradiction in all cases, we must conclude that S' is a solution in G, and recall that it has size at most k. This concludes the proof of this lemma. □

The preceding lemma (Lemma 6.46) gives us the following reduction rule.

Reduction Rule 6.3. Let (u, v) be an irrelevant edge in the simple clique path $\mathbb{K}_Z$, where u is an unmarked vertex. Then, contract the edge (u, v) in the graph G. The resulting instance is $(G/e, k)$.

When Reduction Rule 6.3 is not applicable, then there are no unmarked vertices in any simple clique path. Then, we conclude that all vertices in the clique path $\mathbb{K}$ are marked. Finally, we apply the preceding marking schemes and reduction rules for every clique path in $G - M$, and conclude that all the vertices in $G - M$ are marked. We now proceed to bound the number of vertices in the graph.

7 BOUNDING THE NUMBER OF VERTICES

Let (G, k) be an instance of IVD on which none of the reduction rules apply. In the following, we bound the number of vertices in G. Recall that we start by computing a 9-redundant solution M, whose size is bounded by $\mathcal{O}(k^{10})$ (see Lemma 3.2). Next, we consider the connected components of $G - M$. First, we bound the total number of vertices in the module components of $G - M$ by

$O(k^3|M|^{10}) = O(k^{103})$ (see Observation 4.5). Then, we bound the total number of vertices in the non-module components of $G - M$ by a collection marking rules (and the non-applicability of a number of reduction rules). From Observation 4.2, we obtain that the number of non-module components in $G-M$ is bounded by $O(k|M|) = O(k^{11})$. We note that each non-module component is a clique path. Then, we consider a clique path $\mathbb{K}$ of a non-module connected component in $G-M$ and bound the size of the maximum clique in it by $\eta = O(k|M|^{10}) = O(k^{101})$ (see Lemma 5.4). Next, we focus on bounding the number of bags in a clique path $\mathbb{K}$ that is a non-module component in $G - M$. In the following, for a fixed non-module clique path $\mathbb{K}$, we summarize the number of bags we marked using each of our bag-marking schemes in Section 6:

(1) Using Marking Scheme I, we mark at most $O(\eta|M|)$ bags in $\mathbb{K}$.
(2) Using Marking Scheme II, we mark at most $O(k^3\eta^{11}|M|)$ bags in $\mathbb{K}$.
(3) Using Marking Scheme III, we mark at most $O(\eta^3|M|)$ bags in $\mathbb{K}$.
(4) Using Marking Scheme IV, we mark at most $O(\eta^3|M|)$ bags in $\mathbb{K}$.
(5) Using Marking Scheme V, we mark at most $O(k^3\eta^{15}|M|)$ bags in $\mathbb{K}$.
(6) Using Marking Scheme VI, we mark at most $O(k^3\eta^{16}|M|)$ bags in $\mathbb{K}$.
(7) Using Marking Scheme VII, we mark at most $O(k^4\eta^{16}|M|)$ bags in $\mathbb{K}$.

From the preceding, we obtain that the number of marked bags for each (non-module) clique path is upper bounded by $O(k^4\eta^{16}|M|) = O(k^{1630})$. Further, since none of the reduction rules is applicable, there is no vertex in G that belongs to an unmarked bag of a non-module component. There are at most $O(k^{11})$ non-module components in $G - M$, and a bag in a clique path of a non-module component has size at most η. Thus, the total number of vertices in G is bounded by $O(k^{1630} \cdot k^{11} \cdot k^{101}) = O(k^{1742})$.

8 CONCLUSION

In this article, we proved that the IVD problem admits a polynomial kernel. We remark that the degree in the polynomial that bounds the kernel size can be improved to be about a 100 at the cost of significantly more involved arguments. In particular, this can be done by considering a solution M of lower redundancy and far more involved case analysis for bounding the clique size and clique paths of $G - M$ in Sections 5 and 6. However, obtaining a kernel of size around $O(k^{10})$ will require new ideas. We leave this as an interesting open problem. We also believe that our techniques and methods, especially the two families lemma (Lemma 1.1), will be useful in other algorithmic applications.

REFERENCES

[1] Akanksha Agrawal, Sudeshna Kolay, Daniel Lokshtanov, and Saket Saurabh. 2016. A faster FPT algorithm and a smaller kernel for block graph vertex deletion. In *LATIN 2016: Theoretical Informatics*. Lecture Notes in Computer Science, Vol. 9644. Springer, 1–13.
[2] Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. 2017. Feedback vertex set inspired kernel for chordal vertex deletion. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'17)*. SIAM, 1383–1398.
[3] Akanksha Agrawal, Daniel Lokshtanov, Pranabendu Misra, Saket Saurabh, and Meirav Zehavi. 2019. Feedback vertex set inspired kernel for chordal vertex deletion. *ACM Transactions on Algorithms* 15, 1 (2019), Article 11, 28 pages.
[4] Akanksha Agrawal, Daniel Lokshtanov, Amer E. Mouawad, and Saket Saurabh. 2018. Simultaneous feedback vertex set: A parameterized perspective. *ACM Transactions on Computation Theory* 10, 4 (2018), Article 18, 25 pages.
[5] Amotz Bar-Noy, Reuven Bar-Yehuda, Ari Freund, Joseph Naor, and Baruch Schieber. 2001. A unified approach to approximating resource allocation and scheduling. *Journal of the ACM* 48, 5 (2001), 1069–1090.
[6] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. 2009. On problems without polynomial kernels. *Journal of Computer and System Sciences* 75, 8 (2009), 423–434.
[7] Béla Bollobás. 1965. On generalized graphs. *Acta Mathematica Hungarica* 16, 3-4 (1965), 447–452.

[8] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. 1999. *Graph Classes: A Survey*. SIAM, Philadelphia, PA.
[9] Yixin Cao. 2016. Linear recognition of almost interval graphs. In *Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'16)*. SIAM, 1096–1115.
[10] Yixin Cao. 2017. Unit interval editing is fixed-parameter tractable. *Information and Computation* 253 (2017), 109–126.
[11] Yixin Cao and Dániel Marx. 2015. Interval deletion is fixed-parameter tractable. *ACM Transactions on Algorithms* 11, 3 (2015), Article 21, 35 pages.
[12] Yixin Cao and Dániel Marx. 2016. Chordal editing is fixed-parameter tractable. *Algorithmica* 75, 1 (2016), 118–137.
[13] Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. 2015. *Parameterized Algorithms*. Springer-Verlag, Cham, Switzerland.
[14] Marek Cygan, Łukasz Kowalik, and Marcin Pilipczuk. 2013. Open problems from Workshop on Kernels.
[15] Holger Dell and Dániel Marx. 2012. Kernelization of packing problems. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'12)*. SIAM, 68–81.
[16] Holger Dell and Dieter van Melkebeek. 2014. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. *Journal of the ACM* 61, 4 (2014), Article 23, 27 pages.
[17] Reinhard Diestel. 2010. *Graph Theory* (4th ed.). Springer-Verlag, Heidelberg, Germany.
[18] Rodney G. Downey and Michael R. Fellows. 2013. *Fundamentals of Parameterized Complexity*. Springer, London.
[19] Andrew Drucker. 2015. New limits to classical and quantum instance compression. *SIAM Journal on Computing* 44, 5 (2015), 1443–1479.
[20] Jörg Flum and Martin Grohe. 2006. *Parameterized Complexity Theory*. Springer-Verlag, Berlin, Germany.
[21] Fedor V. Fomin, Bart M. P. Jansen, and Michal Pilipczuk. 2014. Preprocessing subgraph and minor problems: When does a small vertex cover help? *Journal of Computer and System Sciences* 80, 2 (2014), 468–495.
[22] Fedor V. Fomin, Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. 2012. Planar f-deletion: Approximation, kernelization and optimal FPT algorithms. In *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS'12)*. IEEE, Los Alamitos, CA, 470–479.
[23] Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, and Saket Saurabh. 2016. Efficient computation of representative families with applications in parameterized and exact algorithms. *Journal of the ACM* 63, 4 (2016), Article 29, 60 pages.
[24] Fedor V. Fomin and Saket Saurabh. 2014. Kernelization methods for fixed-parameter tractability. In *Tractability: Practical Approaches to Hard Problems*, L. Bordeaux, Y. Hamadi, and P. Kohli (Eds.). Cambridge University Press, Cambridge, UK, 260–282.
[25] Lance Fortnow and Rahul Santhanam. 2011. Infeasibility of instance compression and succinct PCPs for NP. *Journal of Computer and System Sciences* 77, 1 (2011), 91–106.
[26] Toshihiro Fujito. 1998. A unified approximation algorithm for node-deletion problems. *Discrete Applied Mathematics* 86, 2-3 (1998), 213–231.
[27] Delbert Fulkerson and Oliver Gross. 1965. Incidence matrices and interval graphs. *Pacific Journal of Mathematics* 15, 3 (1965), 835–855.
[28] Martin Charles Golumbic. 1980. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, NY.
[29] Jiong Guo and Rolf Niedermeier. 2007. Invitation to data reduction and problem kernelization. *ACM SIGACT News* 38, 1 (2007), 31–45.
[30] Danny Hermelin, Stefan Kratsch, Karolina Soltys, Magnus Wahlström, and Xi Wu. 2015. A completeness theory for polynomial (turing) kernelization. *Algorithmica* 71, 3 (2015), 702–730.
[31] Danny Hermelin and Xi Wu. 2012. Weak compositions and their applications to polynomial lower bounds for kernelization. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'12)*. SIAM, 104–113.
[32] Bart M. P. Jansen. 2013. *The Power of Data Reduction: Kernels for Fundamental Graph Problems*. Ph.D. Dissertation. Utrecht University.
[33] Bart M. P. Jansen, Daniel Lokshtanov, and Saket Saurabh. 2014. A near-optimal planarization algorithm. In *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'14)*. SIAM, 1802–1811.
[34] Bart M. P. Jansen and Marcin Pilipczuk. 2017. Approximation and kernelization for chordal vertex deletion. In *Proceedings of the T28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'17)*. SIAM, 1399–1418.
[35] Bart M. P. Jansen and Marcin Pilipczuk. 2017. Approximation and kernelization for chordal vertex deletion. In *Proceedings of the 28th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'17*. SIAM, 1399–1418.
[36] Bart M. P. Jansen and Marcin Pilipczuk. 2018. Approximation and kernelization for chordal vertex deletion. *SIAM Journal on Discrete Mathematics* 32, 3 (2018), 2258–2301.
[37] Yuping Ke, Yixin Cao, Xiating Ouyang, Wenjun Li, and Jianxin Wang. 2018. Unit interval vertex deletion: Fewer vertices are relevant. *Journal of Computer and System Sciences* 95 (2018), 109–121.
[38] David Kendall. 1969. Incidence matrices, interval graphs and seriation in archeology. *Pacific Journal of Mathematics* 28, 3 (1969), 565–570.

[39] Eun Jung Kim and O-Joung Kwon. 2017. A polynomial kernel for block graph deletion. *Algorithmica* 79, 1 (2017), 251–270.

[40] Eun Jung Kim, Alexander Langer, Christophe Paul, Felix Reidl, Peter Rossmanith, Ignasi Sau, and Somnath Sikdar. 2016. Linear kernels and single-exponential algorithms via protrusion decompositions. *ACM Transactions on Algorithms* 12, 2 (2016), Article 21, 41 pages.

[41] Stefan Kratsch. 2014. Recent developments in kernelization: A survey.

[42] Stefan Kratsch and Magnus Wahlström. 2012. Representative sets and irrelevant vertices: New tools for kernelization. In *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS'12)*. IEEE, Los Alamitos, CA, 450–459.

[43] David C. Lay. 2006. *Linear Algebra and Its Applications*. Pearson/Addison-Wesley.

[44] C. Lekkeikerker and J. Boland. 1962. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae* 51, 1 (1962), 45–64.

[45] John M. Lewis and Mihalis Yannakakis. 1980. The node-deletion problem for hereditary properties is NP-complete. *Journal of Computer and System Sciences* 20, 2 (1980), 219–230.

[46] Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. 2012. Kernelization—Preprocessing with a guarantee. In *The Multivariate Algorithmic Revolution and Beyond: Essays Dedicated to Michael R. Fellows on the Occasion of His 60th Birthday*. Springer, Heidelberg, Germany, 129–161.

[47] László Lovász. 1972. A characterization of perfect graphs. *Journal of Combinatorial Theory, Series B* 13, 2 (1972), 95–98.

[48] Carsten Lund and Mihalis Yannakakis. 1994. On the hardness of approximating minimization problems. *Journal of the ACM* 41, 5 (1994), 960–981.

[49] Dániel Marx. 2009. A parameterized view on matroid optimization problems. *Theoretical Computer Science* 410, 44 (2009), 4471–4479.

[50] Dániel Marx. 2010. Chordal deletion is fixed-parameter tractable. *Algorithmica* 57, 4 (2010), 747–768.

[51] Rolf Niedermeier. 2006. *Invitation to Fixed-Parameter Algorithms*. Vol. 31. Oxford University Press, Oxford, UK.

[52] James G. Oxley. 2006. *Matroid Theory*. Vol. 3. Oxford University Press, New York, NY.

[53] Mihalis Yannakakis. 1978. Node- and edge-deletion NP-complete problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC'78)*. ACM, New York, NY, 253–264.

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