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Erdős–Pósa property of obstructions to interval graphs

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Abstract

A class of graphs $\mathcal{F}$ admits the Erdős–Pósa property if for any graph G, either G has k vertex-disjoint "copies" of the graphs in $\mathcal{F}$, or there is a set $S \subseteq V(G)$ of $f(k)$ vertices that intersects all copies of the graphs in $\mathcal{F}$. For any graph class $\mathcal{G}$, it is natural to ask whether the family of obstructions to $\mathcal{G}$ has the Erdős–Pósa property. In this paper, we prove that the family of obstructions to interval graphs—namely, the family of chordless cycles and asteroidal witnesses (AWs)—admits the Erdős–Pósa property. In turn, this yields an algorithm to decide whether a given graph G has k vertex-disjoint AWs and chordless cycles, or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all AWs and chordless cycles.

KEYWORDS
asteroidal witness, Erdős–Pósa property, interval graphs

1 | INTRODUCTION

Packing and covering problems are ubiquitous in both graph theory and computer science. The duality between packing and covering problems lies at the heart of not only fundamental combinatorial proofs, but also well-known algorithmic methods such as the primal-dual method for approximation and win/win-approach for parameterized analysis. The very essence of this duality is encompassed by a well-known property called the Erdős–Pósa property. This property, being both simple and powerful, has been extensively studied for over five decades (see below). In the context of any graph class $\mathcal{G}$, the most natural question that arises in this regard is as follows—do obstructions to $\mathcal{G}$ have the Erdős–Pósa property?

Having this view in mind, we focus on the class of interval graphs. Arguably, this is the most basic class of graphs that can be viewed as geometric inputs—indeed, an interval graph is the intersection graph of a family of intervals on real lines. In general, interval graphs are among the most well-studied classes of graphs in the literature. In particular, the usage of interval graphs as models is relevant to a wide variety of applications, ranging from resource allocation in operations research and scheduling theory to assembling contiguous subsequences in DNA mapping. From an algorithmic point of view, the structural properties of interval graphs are also intensively studied as they allow to design polynomial-time algorithms for well-known problems in computer science, such as Independent Set and Hamiltonian Path, that are NP-hard on general graphs. Our main contribution is the first proof that obstructions to interval graphs admit the Erdős–Pósa property.

Before we turn to consider our contribution in more detail, we present a gentle introduction to the rich realm of studies of Erdős–Pósa properties. For this purpose, we first define packing and covering problems. Let $\preccurlyeq$ be a containment relation (of a graph into another graph), and let $\mathcal{F}$ be a family of graphs. For example, we can define the containment relationship $\preccurlyeq$ as follows: for graphs G and H, $H \preccurlyeq G$ if and only if H is an induced subgraph/subgraph/minor/topological minor of G. In this setting, $(\mathcal{F}, \preccurlyeq)$-Packing is the problem whose input consists of a graph G and an integer k, and the objective is to decide if G has k vertex-disjoint subsets, $S_1, S_2, \ldots, S_k \subseteq V(G)$, where for each $i \in [k]$, there exists $F \in \mathcal{F}$ such that $F \preccurlyeq G[S_i]$. For example, if $\mathcal{F} = \{F\}$ and the relation refers to induced subgraphs, then we simply ask whether G has k vertex-disjoint "exact copies" of F. The $(\mathcal{F}, \preccurlyeq)$-Covering problem has the same input, but its objective is to decide if there is a set $S \subseteq V(G)$ of size at most k such that there does not exist $F \in \mathcal{F}$ that satisfies $F \preccurlyeq G - S$. Some well-known examples of packing problems (and their corresponding covering problems) are Maximum Matching (Vertex Cover), Vertex-Disjoint s-t Paths (s-t-Separator), Cycle Packing (Feedback Vertex Set), P_3-Packing (Cluster Vertex Deletion), and Triangle Packing (Triangle Free Deletion). Kőnig's and Menger's theorems are cornerstones of Graph Theory in general, and of the study of packing and covering problems in particular, which have also found a wide variety of applications in computer science. For example, Menger's theorem is particularly relevant to survivable network design (see, e.g., [5, 49]) and combinatorial optimization (see, e.g., [20, 46]). Formally, Kőnig's theorem states that in bipartite graphs, the maximum size of matching *equals* the minimum size of a vertex cover [14, 32]. Menger's theorem also exhibits equality—it states that for a given graph G and a pair of vertices s and t, either G has k vertex-disjoint paths between s and t or there is a set $S \subseteq V(G) \setminus \{s, t\}$ of size k such that $G - S$ has no path between s and t [14, 36]. Both theorems relate a packing problem to a covering problem,[1] by exhibiting equality between the size of a maximum packing and the size of a minimum covering. However, most natural packing and covering problems are not known to exhibit such an equality; in fact, frequently such an equality is *proven* not to exist. By simply relaxing the notion of equality, we enter the rich realm of the Erdős–Pósa properties.

[1] For example, Kőnig's theorem addresses the class $\mathcal{F} = \{F\}$ such that F is the graph on a single edge, where $\preccurlyeq$ refers to induced subgraphs/subgraphs.

1.1 | The Erdős–Pósa property

A celebrated theorem by Erdős and Pósa [15] states that for any graph G, either there is a set of k vertex-disjoint cycles in G, or there is a set $S \subseteq V(G)$ of $f(k) = \mathcal{O}(k \log k)$ vertices that intersects (covers) all cycles of G.[2] Notably, Erdős and Pósa [15] also showed that there exists a constant c and infinitely many pairs (G, k) such that G has neither k vertex-disjoint cycles nor a set $S \subseteq V(G)$ of $ck \log k$ vertices that covers all cycles of G. That is, not only equality cannot be expected, but also any function $f(k) = o(k \log k)$. We remark that later, Simonovits [48] provided concrete examples which realize the lower bound. The result of Erdős and Pósa [15] initiated a flurry of extensive study of the so called "Erdős–Pósa property" for various families of graphs as well as containment relationships. Formally, a family of graphs $\mathcal{F}$ and a containment relation $\preccurlyeq$ are said to admit the Erdős–Pósa property if there exists a function $f(\cdot)$ such that given a graph G and an integer k, either there are k vertex-disjoint subsets $S_1, \ldots S_k \subseteq V(G)$ so that for each $i \in [k]$, there is $F \in \mathcal{F}$ satisfying $F \preccurlyeq G[S_i]$, or there is a set $S \subseteq V(G)$ of size at most $f(k)$ so that there is no $F \in \mathcal{F}$ satisfying $F \preccurlyeq G - S$. Here, the first question that comes to mind is—do all natural families of graphs $\mathcal{F}$ and containment relationships $\preccurlyeq$ exhibit the Erdős–Pósa property?

The answer to this question is negative. For example, consider a fixed graph H, and let $\mathcal{F}(H)$ be the family of graphs that contain H as a minor. Robertson and Seymour [44] showed that $\mathcal{F}(H)$ with the containment relation referring to subgraphs admits the Erdős–Pósa property if and only if H is a planar graph. This result generalizes the result in [15]. However, the function $f(\cdot)$ given by [44] is exponential—can it be made polynomial? A few years ago, the bound was improved to $\mathcal{O}(k \log^c k)$ by Chekuri and Chuzhoy [12] following a more general approach that is applicable to other families as well. A well-known example of a different flavor concerns odd cycles. Specifically, for $\mathcal{F}$ being the family of odd-length cycles, Dejter and Neumann-Lara [13] showed that $\mathcal{F}$ (for subgraphs and induced subgraphs) does not admit the Erdős–Pósa property.

Since the emergence of the result of Erdős and Pósa [15], a multitude of studies on the Erdős–Pósa property have appeared in the literature for several combinatorial objects beyond graphs. This includes extensions to digraphs [21, 23, 35, 42, 47], rooted graphs [9, 26, 28, 38], labeled graphs [25], signed graphs [3, 24], hypergraphs [1, 6, 7], matroids [17], helly-type theorems [22], H-minors [45], H-immersions [18, 34], and H-butterfly directed minors [2] (also see [43]). This list is not comprehensive but rather illustrative. We refer to surveys such as [40] for more information. Even for subfamilies of cycles alone, there is a vast literature devoted to the Erdős–Pósa property. Studies of the Erdős–Pósa property for subfamilies of cycles include, for example, long cycles (subgraphs) [4, 37], directed cycles (subgraphs and induced subgraphs) [21, 42], holes[3] (induced subgraphs) [31] and cycles intersecting a prescribed vertex set [29, 38]. Not all subfamiles of cycles admit the Erdős–Pósa property. For example, recall the result stated earlier regarding the family of odd cycles [13]. For this subfamily of cycles alone, there has been a sequence of research about finding classes of graphs for which the family of odd cycles (subgraphs and induced subgraphs) admits the Erdős–Pósa property. This includes planar graphs [16], graphs with certain connectivity constraints [27, 30, 39, 51], and more [41]. Not only the subfamily of odd cycles does not admit the Erdős–Pósa property, but also subfamilies such as the family of all holes of length at least 5 [31].

[2]In the terminology of packing and covering, we address the class $\mathcal{F}$ of all cycles, where $\preccurlyeq$ refers to induced subgraphs/subgraphs.

[3]Throughout this paper, we use the term hole to refer to a chordless cycle of length at least 4.

A large number of the results above can be viewed as the question of packing or covering obstructions to a class of graphs. In some of these papers, this view is explicitly stated as the motivation behind the conducted studies. For example, the classic result by Erdős and Pósa [15] regards the question of packing and covering obstructions to forests. The results concerning odd cycles address obstructions to bipartite graphs. The setting of the work about packing and covering holes, as presented by [31], addresses obstructions to chordal graphs. Furthermore, Kőnig's theorem relates to obstructions to edgeless graphs, and the work by Robertson and Seymour [44] relates to obstructions to subfamilies of minor free graphs. We remark that other results can also be interpreted in this manner. Given that the class of interval graphs is among the most basic, well-studied families of graphs, we find it important to study the Erdős–Pósa [15] property with respect to it. Let $\mathcal{F}$ be the family of holes and asteroidal witnesses (AWs); see Section 2. It is well known that the class of interval graphs is precisely the class of graphs that exclude every graph in $\mathcal{F}$ as an induced subgraph [8, 19]. Given this clean characterization, the following question naturally arises:

> Does the family of holes and AWs—that is, obstructions to interval graphs—admit the Erdős–Pósa property?

1.2 | Our contribution

We provide an affirmative answer to the question above. Moreover, the dependency of the size of the covering set on k in our result is only $\mathcal{O}(k^2 \log k)$.[4] Specifically, we obtain the following theorem, where obstructions refer to AWs and holes.

Theorem 1. *Let G be a graph, and let $k \in \mathbb{N}$. At least one of the following conditions holds: (i) G has k vertex-disjoint obstructions; (ii) there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph.*

In other words, we show that there exists a constant c such that, for any graph G and $k \in \mathbb{N}$, either G has k vertex-disjoint obstructions, or there exists a subset of $V(G)$ of size at most $c \cdot k^2 \log k$ that intersects all obstructions of G. As a consequence of our main theorem, we also derive an algorithm to decide whether an input graph G has k vertex-disjoint obstructions (to interval graphs), or there exists a set of $\mathcal{O}(k^2 \log k)$ vertices in G that hits all such obstructions. It remains an interesting open question to "shave" the logarithmic factor to achieve a bound of $\mathcal{O}(k^2)$; we remark that this logarithmic factor appears due to our use of the work by [31] in a black box manner.

We conclude the introduction with a high-level (informal) overview of our proof. We begin by easily "getting rid" of all holes due to the work by [31], as well as all small AWs. Now, the heart of our proof consists of two main components. First, we exhibit the Erdős–Pósa property of the family of AWs on graphs that have a clique caterpillar (i.e., a tree decomposition i.e., a caterpillar, where every bag is a clique). Second, we show how this result can be utilized to

[4] In fact, all of our arguments achieve the dependency $\mathcal{O}(k^2)$, but we gain an extra $\log k$ factor due to an invocation of the above-mentioned result by Kwon and Kim [31]. Shaving off the $\log k$ factor in the result by [31] will automatically also shave it off from our result.

derive our main theorem by analyzing conflict-free sets with respect to a modular tree decomposition of the graph. Let us now elaborate on each component.

To analyze the case of a clique caterpillar, we present a procedure that at each iteration, finds an AW $\mathbb{O}$ with specific properties, inserts a set S of $\mathcal{O}(k)$ new vertices into a set $S^\star$ initialized to be empty, and removes the vertices in S from the graph (only for the sake of the execution of the procedure). Specifically, the set S consists of the terminals, centers and a few base vertices of $\mathbb{O}$, as well as all of the vertices of a "small" separator between the non-shallow terminals of $\mathbb{O}$ that we push as much as possible to the right of the caterpillar. The procedure terminates once the graph becomes an interval graph. Hence, it is clear that if at most $\mathcal{O}(k)$ iterations take place, then $S^\star$ is a set of size $\mathcal{O}(k^2)$ that intersects all AWs, which implies that our job is done. Otherwise, we require an intricate analysis to establish the existence of k vertex-disjoint AWs. This analysis consists of showing the following two items:

1. Consider the AWs found by the above procedure, and order them in a sequence based on the iteration where they where found (from small to large). Based this sequence, we construct another sequence of AWs of the same length (of possibly *different* AWs) where each AW has the following property: the subpath of its base that lies after the separator is vertex-disjoint from any AW positioned after it in the sequence.
2. From the new sequence obtained in the first item, we extract a *subsequence* of AWs where each AW has a stronger property: the entire subpath of its base is vertex-disjoint from any AW positioned after it in the sequence.

Towards the proof of the second item, we first show the following. Consider any sequence "resembling" the one found by our procedure, and any pair of AWs $\mathbb{O}$ and $\mathbb{O}'$ in that sequence such that $\mathbb{O}'$ appears before $\mathbb{O}$. Then, only the leftmost terminal and base vertex of $\mathbb{O}$ can belong to the part of the base path of $\mathbb{O}'$ that lies before the separator associated with $\mathbb{O}'$, and even that is only possible under certain conditions. This result then allows us to further argue about the relation between every *three* AWs in the sequence with respect to the "left sides of separators." Having established this relation, the argument about a complete sequence is derived.

Towards the proof of the first item, we first show that for any AW $\mathbb{O}$ in the sequence, we can find a path between a vertex in the separator associated with $\mathbb{O}$ and the right terminal of $\mathbb{O}$ that avoids all AWs that appear after $\mathbb{O}$ in the sequence. Then, by relying on structural results by Cao and Marx [11], we argue that this path can be used to replace part of $\mathbb{O}$ so that the result is yet another AW.

Let us now turn to our analysis of the general case—specifically, we explain how it is reduced to instances of the case of a clique caterpillar. We define "problematic" nodes in the modular tree decomposition of the input graph as the nodes associated with subgraphs that contain at least one AW that is not present in any of the subgraphs associated with their children. This definition also immediately gives rise to an association between nodes and AWs, so that each AW is associated with exactly one node. We observe that maximal modules of problematic nodes are vertex-disjoint, and that each problematic node has "many" children. It is also easily shown that the set of all problematic nodes can be partitioned into two sets that have no "conflict"—that is, on the unique path between every two nodes of one set, there exists a node of the other set. The point in analyzing each conflict-free set P separately is that for each problematic node in such a set, we prove that there exist at least k vertices in the subgraph associated with that node that do not belong to any subgraph associated with its problematic descendants from P. In particular, this allows us to examine each problematic node *individually*, and associate an instance of the clique caterpillar case with it (the construction of the caterpillar decomposition itself partially follows from structural results by Cao and Marx [11]). Specifically, we are able to collect the sets of AWs found in each instance, and argue that (after some modification) all of these AWs across all the sets are in fact vertex-disjoint. This result then allows us to handle the "packing perspective" of the proof. We remark that although we can create $\mathcal{O}(k)$ instances of the clique caterpillar case, and each individual instance can create a gap of $\mathcal{O}(k^2)$, we eventually get a gap of only $\mathcal{O}(k^2)$ rather than $\mathcal{O}(k^3)$ as we argue that the sum of the contributions to the gap of all individual instances is $\mathcal{O}(k^2)$.

2 | PRELIMINARIES

For $n \in \mathbb{N}$, we use $[n]$ as a shorthand for $\{1, 2, \ldots, n\}$. Given a function $f : A \to B$ and a subset $A' \subset A$, we use $f|_{A'}$ to denote the restriction of f to A'.

2.1 | Basic graph theory

Let us remind some terminology required for this paper. We refer to the book of Diestel [14] for a comprehensive introduction. Given a graph G, we denote its vertex set and its edge set by $V(G)$ and $E(G)$, respectively. Given a set $\mathcal{C}$ of subgraphs of G, denote $V(\mathcal{C}) = \bigcup_{C \in \mathcal{C}} V(C)$. The disjoint union of (vertex-disjoint) graphs $H_1, H_2, \ldots, H_t$ is the graph of the vertex set $V(H_1) \cup V(H_2) \cup \cdots \cup V(H_t)$ and edge set $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_t)$. Moreover, when graph G is clear from context, denote $n = |V(G)|$ and $m = |E(G)|$. Given a subset $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. Moreover, a graph H is an *induced subgraph* of G if there exists $U \subseteq V(G)$ such that $G[U] = H$. For a set of vertices $X \subseteq V(G)$, $G - X$ denotes the induced subgraph $G[V(G) \setminus X]$, that is, the graph obtained by deleting the vertices in X from G. We say that X is a *clique* if for all distinct vertices $u, v \in X$, we have that $\{u, v\} \in E(G)$, and that X is an independent set if for all distinct vertices $u, v \in X$, we have that $\{u, v\} \notin E(G)$. Given a vertex $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v in G. We say that v is *simplicial* if $N_G(v)$ is a clique.

A *path* $P = x_1 - x_2 - \cdots - x_\ell$ in G is a subgraph of G, where $V(P) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(P) = \{\{x_i, x_{i+1}\} | i \in [\ell - 1]\} \subseteq E(G)$, where $\ell \in [n]$. A *walk* $W = x_1 - x_2 - \cdots - x_\ell$ in G is a sequence of (not necessarily distinct) vertices of G such that for all $i \in [\ell - 1]$, $\{x_i, x_{i+1}\} \in E(G)$. A *cycle* $C = x_1 - x_2 - \cdots - x_\ell - x_1$ in G is a subgraph of G where $V(C) = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(G)$ and $E(C) = \{\{x_i, x_{i+1}\} | i \in [\ell - 1]\} \cup \{\{x_1, x_\ell\}\} \subseteq E(G)$. We say that $\{u, v\} \in E(G)$ is a *chord* of P if $u, v \in V(P)$ but $\{u, v\} \notin E(P)$. Similarly, we say that $\{u, v\} \in E(G)$ is a *chord* of C if $u, v \in V(C)$ but $\{u, v\} \notin E(C)$. A path P or cycle C is said to be *induced* (or, alternatively, *chordless*) if it has no chords. We use the term *hole* to refer to a chordless cycle on at least four vertices. A *caterpillar* is a tree T for which there exists a subpath P of T, called a *central path*, such that the removal of the vertices of P from T results in an edgeless graph. (We remark that a caterpillar can have multiple central paths). Given a rooted tree T and a vertex $v \in V(T)$, we use $T|_v$ to denote the subtree of T rooted at v. Moreover, child(v) denotes the set of children of v in T, and desc(v) denotes the set of descendants of v in T (we do not treat a node as a descendant of itself). A *chordal graph* is a graph that has no *hole*.

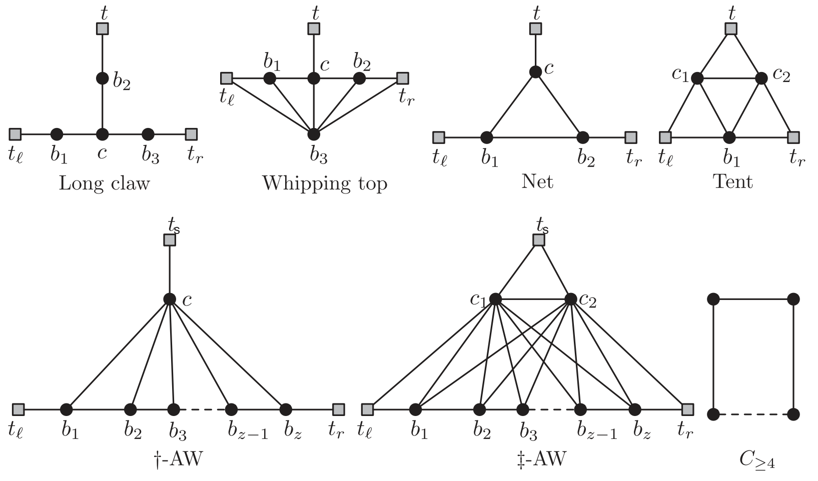

FIGURE 1 The set of obstructions for an interval graph

2.2 | Interval graphs

An *interval graph* is a graph G that does not contain any of the following graphs, called *obstructions*, as an induced subgraph (see Figure 1).

- Long Claw. A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_3\}, \{t, b_2\}, \{c, b_1\}, \{c, b_2\}, \{c, b_3\}\}$.
- Whipping Top. A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t, c, b_1, b_2, b_3\}$ and $E(\mathbb{O}) = \{\{t_\ell, b_1\}, \{t_r, b_2\}, \{c, t\}, \{c, b_1\}, \{c, b_2\}, \{b_3, t_\ell\}, \{b_3, b_1\}, \{b_3, c\}, \{b_3, b_2\}, \{b_3, t_r\}\}$.
- †-AW. A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t_s, c\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t_s, c\}, \{t_\ell, b_1\}, \{t_r, b_z\}\} \cup \{\{c, b_i\} | i \in [z]\} \cup \{\{b_i, b_{i+1}\} | i \in [z-1]\}$, and $z \geq 2$. A †-AW where $z = 2$ is called a *net*.
- ‡-AW. A graph $\mathbb{O}$ such that $V(\mathbb{O}) = \{t_\ell, t_r, t_s, c_1, c_2\} \cup \{b_1, b_2, \ldots, b_z\}$, where $t_\ell = b_0$ and $t_r = b_{z+1}$, $E(\mathbb{O}) = \{\{t_s, c_1\}, \{t_s, c_2\}, \{c_1, c_2\}, \{t_\ell, b_1\}, \{t_r, b_z\}, \{t_\ell, c_1\}, \{t_r, c_2\}\} \cup \{\{c, b_i\} | i \in [z]\} \cup \{\{b_i, b_{i+1}\} | i \in [z-1]\}$, and $z \geq 1$. A ‡-AW where $z = 1$ is called a *tent*.
- Hole. A chordless cycle on at least four vertices.

We remark that interval graphs have other equivalent definitions in the literature, and that the one we present above is based on [33]. Notice that each of the obstructions $\mathbb{O}$ above is inclusion-wise minimal, that is, there does not exist an obstruction $\mathbb{O}'$ such that $V(\mathbb{O}') \subsetneq V(\mathbb{O})$.

An *asteroidal triple* (AT) in a graph G is a triple u, v, w of vertices such that for each pair of these vertices, there is a path in G that does not contain any vertex from the closed neighborhood of the third vertex of the triple. An *AW* is inclusion-wise minimal induced subgraph containing an AT. Observe that long claws, whipping tops, †-AWs and ‡-AWs are all AWs, but we shall reserve this name for †-AWs and ‡-AWs. In each of the first four obstructions, the vertices t_ℓ, t_r, and t_s are called *terminals*, the vertices c, c_1, and c_2 are called *centers*, and the other vertices are called *base vertices*. To simplify notation, when we consider a †-AW, we use c_1 and c_2 to refer to c (this allows us to refer to a †-AW and a ‡-AW in a unified manner). Furthermore, the vertex t_s is called the *shallow terminal*. In the case of an AW $\mathbb{O}$, the induced path on the set of base vertices is called the *base* of the AW, and it is denoted by base($\mathbb{O}$). Moreover, we say that the induced path on the set of base vertices, t_ℓ and t_r is the *extended base* of the AW, and it is denoted by $P(\mathbb{O})$. Given a graph G, a vertex v is *shallow in G* if G has at least one AW where v is the shallow terminal.

2.3 | Tree decomposition

A *tree decomposition* of a graph G is a pair (T, β) where T is a tree, and $\beta : V(T) \to 2^{V(G)}$ is a function that satisfies the following properties:

(i) $\bigcup_{x \in V(T)} \beta(x) = V(G)$,
(ii) for any edge $\{u, v\} \in E(G)$ there is a node $x \in V(T)$ such that $u, v \in \beta(x)$, and
(iii) for any $v \in V(G)$, the collection of nodes $T_v = \{x \in V(T) | v \in \beta(x)\}$ induces a subtree of T.

For $v \in V(T)$, we call $\beta(v)$ the *bag* of v. In case T is a path, then (T, β) is also called a *path decomposition*, and in case T is a caterpillar then (T, β) is also called a *caterpillar decomposition*. We refer to the vertices in $V(T)$ as *nodes*. A *clique path* (*clique caterpillar*) of a graph G is a path decomposition (resp. *caterpillar decomposition*) of G where every bag is a distinct maximal clique. We remark that not every graph admits a clique caterpillar.

2.4 | Modules

Let G be a graph. A subset $M \subseteq V(G)$ is a *module* if for all $u, w \in M$ and $v \in V(G) \setminus M$, either both u and w are adjacent to v or both u and w are not adjacent to v. A module is *nontrivial* if neither $V(M) = \emptyset$ nor $V(M) = V(G)$.

The following simple proposition asserts that a "large" obstruction cannot intersect a module in more than one vertex unless it is contained in that module.

Proposition 2.1 (Proposition 4.4 [11]). *Let M be a module in G and $\mathbb{O}$ be an obstruction. If $|V(\mathbb{O})| > 4$, then either $V(\mathbb{O}) \subseteq V(M)$ or $|V(\mathbb{O}) \cap V(M)| \leq 1$.*

Intuitively, a *modular tree decomposition* of a graph $G = (V, E)$ is a linear-size representation of all its modules [50]. Formally, it consists of a rooted tree T, a function $f : V(T) \to 2^{V(G)} \setminus \{\emptyset\}$ and a function $g : V(T) \to \{0, 1\}$, which satisfy the following properties:

1. M is a nonempty module of G if and only if there is a node $v \in V(T)$ for which, either $M = f(v)$, or both $g(v) = 1$ and there is a subset U of the set of children of v such that $M = \bigcup_{u \in U} f(u)$.
2. Every $v, u \in V(T)$ that have the same parent in T satisfy $f(v) \cap f(u) = \emptyset$. Further, for a node $v \in V(T)$ with child nodes $v_1, v_2, \ldots, v_\ell \in V(T)$, we have $f(v) = \cup_{i=1}^{\ell} f(v_i)$.
3. If $r \in V(T)$ is the root node of T, then $f(r) = V(G)$. And if $v \in V(T)$ is a leaf node, then $|f(v)| = 1$.
4. $|V(T)| \leq 2n - 1$.

Furthermore, no node in T has exactly one child.

We remark that g above is meant to indicate (by assigning 1) every vertex that has the property that, for *every* subset of its children, the union of the sets assigned by f to the children in the subset yields a module. In particular, it follows from the first condition that if a vertex does not have this property, it should be assigned 0. Further, if a vertex is assigned 0, then it does not have any subset of children such that the union of the sets assigned to them by f yields a module.

Every graph G admits a modular tree decomposition. In fact, such a decomposition can be constructed in $O(n^2)$ time and $O(n)$ space:

Proposition 2.2 (Tedder et al. [50]). *Given a graph* G, *a modular tree decomposition exists and it can be constructed in* $\mathcal{O}(n^2)$ *time and* $\mathcal{O}(n)$ *space.*

3 | HITTING CHORDLESS CYCLES AND SMALL OBSTRUCTIONS

We start by stating the following proposition, which already handles holes.

Proposition 3.1 (Kim and Kwon [31]). *Let* G *be a graph, and let* $k \in \mathbb{N}$. *At least one of the following conditions holds:* (i) G *has* k *vertex-disjoint holes;* (ii) *there exists a subset* $D \subseteq V(G)$ *of size* $\mathcal{O}(k^2 \log k)$ *such that* $G - D$ *is a chordal graph.*

We proceed with the following simple lemma to deal with small ATs.

Lemma 3.1. *Let* G *be a graph, and let* $k \in \mathbb{N}$. *At least one of the following conditions holds:* (i) G *has* k *vertex-disjoint obstructions on at most* $\max\{2k, 10\}$ *vertices;* (ii) *there exists a subset* $D \subseteq V(G)$ *of size* $\mathcal{O}(k^2)$ *such that* $G - D$ *has no obstruction on at most* $\max\{2k, 10\}$ *vertices.*

Proof. If the first condition holds, then we are done. Thus, suppose that it does not hold, which means that G has no k vertex-disjoint obstructions on at most $\max\{2k, 10\}$ vertices. Let $\mathcal{O}$ be a set of maximum size of vertex-disjoint obstructions in G on at most $\max\{2k, 10\}$ vertices. Then, $|\mathcal{O}| \leq k - 1$, which implies that $|V(\mathcal{O})| \leq (k-1) \cdot \max\{2k, 10\} = \mathcal{O}(k^2)$. By the maximality of $\mathcal{O}$, we have that $G - V(\mathcal{O})$ has no obstruction on at most $\max\{2k, 10\}$ vertices. Thus, the second condition holds. □

As a corollary of Proposition 3.1 and Lemma 3.1, we have the following.

Corollary 3.1. *Let* G *be a graph, and let* $k \in \mathbb{N}$. *At least one of the following conditions holds:* (i) G *has* k *vertex-disjoint obstructions;* (ii) *there exists a subset* $D \subseteq V(G)$ *of size* $\mathcal{O}(k^2 \log k)$ *such that* $G - D$ *is a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices.*

4 | THE CASE OF A CLIQUE CATERPILLAR

This section analyzes the Erdős–Pósa Property of AWs on graphs with a clique caterpillar. Let us begin with a definition.

Definition 4.1. Let G be a graph. A clique caterpillar (T, β) of G is *nice* if every shallow vertex belongs to the bag of only one node of T and that node is a leaf.

The objective of this section is to prove the following lemma.

Lemma 4.1. *Let $k \in \mathbb{N}$, and let G be a graph with a nice clique caterpillar (T, β), such that G is chordal and has no obstruction on at most 10 vertices.*[5] *Then, at least one of the following conditions holds: (i) G has k vertex-disjoint AWs; (ii) there exists a subset $D \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such that $G - D$ is an interval graph.*

To simplify statements in this section, let us fix $k \in \mathbb{N}$ and a chordal graph G with a nice clique caterpillar (T, β), which has no obstruction on at most ten vertices. Thus, whenever we discuss an obstruction in G, that obstruction is necessarily an AW on more than ten vertices. Moreover, let us fix a central path of T, and call it P. We denote $P = p_1 - p_2 - \cdots - p_d$ for $d = |V(P)|$. We think of P as a path oriented from p_1 to p_d. For a vertex $v \in V(G)$, we let first(v) be the first node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define first(v) =nil), and we let last(v) be the last node p on P such that $v \in \beta(p)$ (if such a vertex does not exist, define last(v) =nil). The notation $p_i < p_j$ means that $i < j$ (similarly, we define $\leq$). Note that as nonterminal vertices of an AW have nonadjacent neighbors, we have the following observation.

Observation 4.1. Let $\mathbb{O}$ be an AW in G. For every nonterminal vertex v of $\mathbb{O}$, there exists $p \in V(P)$ such that $v \in \beta(p)$.

Observation 4.1 implies that the notation presented next is well-defined. In what follows, when we consider an AW $\mathbb{O}$, we index the base vertices $b_1^{\mathbb{O}}, b_2^{\mathbb{O}}, \ldots, b_{\eta^{\mathbb{O}}}^{\mathbb{O}}$ such that first$(b_1^{\mathbb{O}}) \leq$ first$(b_{\eta^{\mathbb{O}}}^{\mathbb{O}})$. When $\mathbb{O}$ is clear from context, we simplify the notation, also in the context of terminal and center vertices.[6] Note that $\eta \geq 5$, as G does not have AWs on at most 10 vertices (we use this observation implicitly throughout our arguments, for example, to assume that $b_1, b_2, b_{\eta-2}, b_{\eta-1}$, and b_η are distinct vertices). We remark that clearly, for all $i \in \{2, 3, \ldots, \eta - 1\}$, first$(b_i) \leq$ last$(b_{i-1}) <$ first(b_{i+1}) (this is also stated as Proposition 8.4 in [11]).

Our analysis relies on a notion of a special type of obstruction, defined by Cao and Marx [11], to exploit the "almost linear nature" of a caterpillar. To this end, we have the following notation. Given an AW $\mathbb{O}$, $\widehat{N}(\mathbb{O})$ denotes the set of vertices $v \in V(G)$ such that v is adjacent to every vertex in base$(\mathbb{O})$. We also need to give three definitions.

[5]We remark that the existence of the clique caterpillar already implies that G is chordal [8, 19].

[6]For example, if we consider an AW denoted by $\mathbb{O}, \mathbb{O}'$ and $\mathbb{O}^i$, then we use b_1 (b_η), b_1' ($b_{\eta'}'$), b_1^i ($b_{\eta^i}^i$) to refer to the first (last) base vertex of $\mathbb{O}, \mathbb{O}'$ and $\mathbb{O}^i$, respectively.

Definition 4.2 (Definition 8.5 in [11]). An AW $\mathbb{O}$ in G is *minimal* if there does not exist an AW $\mathbb{O}'$ such that $\text{last}(b_1) \leq \text{last}(b_1') \leq \text{first}(b'_{\eta'}) \leq \text{first}(b_\eta)$, and $\text{last}(b_1) < \text{last}(b_1')$ or $\text{first}(b'_{\eta'}) < \text{first}(b_\eta)$.

We stress that in the definition above we do not compare the base paths of $\mathbb{O}$ and $\mathbb{O}'$ in the sense that one should be a subpath of the other—we make the comparison based only on two particular vertices on each of these base paths and the nodes in P that correspond to them.

Definition 4.3 (Definition 8.7 in [11]). An AW $\mathbb{O}$ in G is *short* if $P(\mathbb{O})$ is the shortest path between t_ℓ and t_r in $G[\beta(p_i) \cup \beta(p_{i+1}) \cup \cdots \cup \beta(p_j) \cup \{t_\ell, t_r\}] - \widehat{N}(\mathbb{O})$, where $p_i = \text{last}(b_1)$ and $p_j = \text{first}(b_\eta)$.

Definition 4.4 (Based on Lemma 8.9 in [11]). An AW $\mathbb{O}$ in G is *first* if there does not exist an AW $\mathbb{O}'$ such that $\text{first}(b'_{\eta'}) < \text{first}(b_\eta)$.

We say that an AW is *good* if it is first, minimal and short. The following proposition asserts that a good AW exists. In this context, recall that we implicitly assume that G is not an arbitrary graph, but in particular it is a graph that has a nice clique caterpillar.

Proposition 4.1 (Lemma 8.8 and Proof of Theorem 2.4 (Page 31) [11]). *If G is not an interval graph, then it has a good AW.*

Before we proceed with our analysis, we state one more proposition by [11] that is used later.

Proposition 4.2 (Claim 5 [11]). *Let $\mathbb{O}$ be a good AW. For any vertex $v \in (\beta(p_1) \cup \beta(p_2) \cup \cdots \beta(p_i)) \setminus \widehat{N}(\mathbb{O})$, where $p_i = \text{first}(b_{\eta-2})$, it holds that v is not adjacent to any vertex that is shallow in G.*

4.1 | Procedure SeparateProcedure

Now, we present a procedure called SeparateProcedure. Initialize $G^1 = G$ and $i = 1$. Now, as long as G^i is not an interval graph, we execute the following instructions:

1. Let $\mathbb{O}^i$ be a good AW in G^i, whose existence is guaranteed by Proposition 4.1.
2. Denote $p_j = \text{first}(b^i_{\eta^i-2})$ and $p_q = \text{last}(b^i_1)$. For all $\delta \in [d]$, denote $\beta^i(p_\delta) = \beta(p_\delta) \cap V(G^i)$. Let $\gamma^i = \gamma$ be the index in $\{q, q+1, \ldots, j-1\}$ such that
   - there does not exist an index $\delta \in \{\gamma+1, \gamma+2, \ldots, j-1\}$ such that $|(\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$, and
   - $|(\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1})) \setminus \widehat{N}(\mathbb{O}^i)| < 8k$.

If such an index γ does not exist, define $\gamma = \text{nil}$. Intuitively, γ is the largest index of a “small” separator in $G^i \setminus \widehat{N}(\mathbb{O})$ between b^i_1 and $b^i_{\eta-2}$.

3. Denote $S^i = (\beta^i(p_\gamma) \cap \beta^i(p_{\gamma+1}))\setminus\widehat{N}(\mathbb{O}^i)$ if $\gamma \neq \text{nil}$, and $S^i = \emptyset$ otherwise.
4. Define $G^{i+1} = G^i - ((V(\mathbb{O}^i)\setminus V(\text{base}(\mathbb{O}^i))) \cup \{b_1^i, b_2^i, b_3^i, b_{\eta^i-3}^i, b_{\eta^i-2}^i, b_{\eta^i-1}^i, b_{\eta^i}^i\} \cup S^i)$.
5. Increment i by 1.

Let $i^\star$ denote the last index i considered by SeparateProcedure. In particular, $G^{i^\star}$ is an interval graph. Let us denote $S^\star = V(G)\setminus V(G^{i^\star})$. Then, $G - S^\star$ is an interval graph. Furthermore, note that $|S^\star| = \mathcal{O}(i^\star \cdot k)$.

To analyze this procedure, we first have the following immediate observation.

Observation 4.2. If $i^\star \leq 2k$, then $S^\star \subseteq V(G)$ is a set of size $\mathcal{O}(k^2)$ such that $G - S^\star$ is an interval graph.

Thus, to prove Lemma 4.1, it is sufficient to prove the following claim, which will be the focus of the rest of this section.

Lemma 4.2. *If $i^\star > 2k$, then G has k vertex-disjoint obstructions.*

In what follows, we suppose that $i^\star > 2k$. To prove this lemma, we first need to introduce the following definitions.

Definition 4.5. Let $i \in [2k]$. We say that an AW $\mathbb{O}$ in G is *i-relevant* if it is an AW in G^i, $t_s = t_s^i$, $t_\ell = t_\ell^i$, $t_r = t_r^i$, $c_1 = c_1^i$, $c_2 = c_2^i$, $b_1 = b_1^i$, $b_2 = b_2^i$, $b_{\eta-2} = b_{\eta^i-2}^i$, $b_{\eta-1} = b_{\eta^i-1}^i$ and $b_\eta = b_{\eta^i}^i$.[7] If in addition $b_3 = b_3^i$ and $b_{\eta-3} = b_{\eta-3}^i$, then we say that $\mathbb{O}$ is *highly i-relevant*.

Due to Step 4 of SeparateProcedure, the following observation follows directly from Definition 4.5.

Observation 4.3. Let $i, i' \in [2k]$ where $i > i'$, and let $\mathbb{O}'$ be an i'-relevant AW. Then, G^i does not contain any vertex from $(V(\mathbb{O}')\setminus V(\text{base}(\mathbb{O}')) \cup \{b_1', b_2', b_3', b_{\eta'-3}', b_{\eta'-2}', b_{\eta'-1}^{i'}, b_{\eta'}'\} \cup S^{i'}$.

Definition 4.6. We say that a tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ is *relevant* if for all $i \in [2k]$, $\widehat{\mathbb{O}}^i$ is i-relevant.

We further need the following notation. For every $i \in [2k]$, $\text{before}(i) = \beta(p_1) \cup \beta(p_2) \cup \cdots \cup \beta(p_{\gamma^i})$ if $\gamma^i \neq \text{nil}$ and $\text{before}(i) = \emptyset$ otherwise. The heart of the proof of Lemma 4.2 is given by two statements (Lemmas 4.4 and 4.6 in this section). Towards the first one, let us first prove the following claim.

Lemma 4.3. *For all $i, i' \in [2k]$, where $i > i'$, i-relevant AW $\mathbb{O}$ and i'-relevant AW $\mathbb{O}'$, it holds that*

1. $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i') \subseteq \{t_\ell, b_1\}$,

[7]That is, $\mathbb{O}$ and the AW $\mathbb{O}^i$ considered in the ith iteration of SeparateProcedure have the same terminals, centers and two first and three last base vertices.

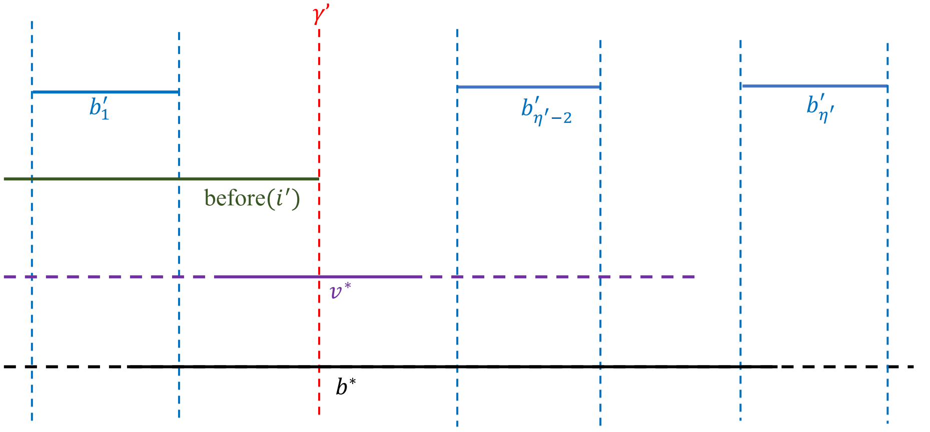

FIGURE 2 Illustration for the proof of Lemma 4.3

2. $|V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')| \leq 1$, *and*
3. *if* $b_1 \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')$ *then* $t_\ell \notin \bigcup_{i\in[d]}\beta(p_i)$.

Proof. Arbitrarily select $i, i' \in [2k]$ such that $i > i'$, an i-relevant AW $\mathbb{O}$ and an i'-relevant AW $\mathbb{O}'$. For simplicity, denote $\gamma' = \gamma^{i'}$ (recall that $\gamma^{i'}$ is set in SeparateProcedure). Suppose that there exists $v^\star \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')$, else we are done. (Note that this implies that $\gamma' \neq \text{nil}$, and in particular $S^{i'} \neq \emptyset$.) Recall that $\text{before}(i') = \beta(p_1) \cup \beta(p_2) \cup \cdots \cup \beta(p_{\gamma'})$ and $p_{\gamma'} < \text{first}(b'_{\eta'-2})$. Now, we have that $\text{first}(b_{\eta'-2}) \leq \text{last}(b_{\eta'-2}) < \text{first}(b_{\eta'})$. Because $\mathbb{O}$ is i-relevant and $\mathbb{O}'$ is i'-relevant where $i' < i$, we have that $\text{first}(b'_{\eta'}) \leq \text{first}(b_\eta)$. Further, since $v^\star \in V(\mathbb{O})$, which is a subset of $V(G^i)$, Observation 4.3 implies that $v^\star$ is neither a center nor a terminal of $\mathbb{O}'$, and it is neither $b'_{\eta'-1}$ nor $b'_{\eta'}$. Thus, $v^\star$ and $b'_{\eta'}$ are not adjacent (specifically, $\text{last}(v^\star) < \text{first}(b'_{\eta'})$), and therefore $v^\star \notin \widehat{N}(\mathbb{O}')$. Since $v^\star \in \text{before}(i')$, we have that $\text{first}(v^\star) \leq p_{\gamma'}$. Since $v^\star \notin S^{i'}$ (as $S^{i'} \cap V(\mathbb{O}) = \emptyset$), we have that $\text{last}(v^\star) \leq p_{\gamma'}$; see Figure 2. From this, we first derive that $\text{last}(v^\star) < \text{first}(b_\eta)$. This means that $v^\star$ is not a center of $\mathbb{O}$. Moreover, since (T, β) is nice, it is not shallow. Therefore, $v^\star$ belongs to the extended base of $\mathbb{O}$, which we denoted by $P(\mathbb{O})$. Since $\text{last}(v^\star) \leq p_{\gamma'} < \text{first}(b_\eta)$ and $P(\mathbb{O})$ is an induced path, we have that $P(\mathbb{O})$ must contain at least one vertex from $\beta(p_{\gamma'}) \cap \beta(p_{\gamma'+1})$ with one neighbor (on $P(\mathbb{O})$) from $\beta(p_1) \cup \cdots \cup \beta(p_{\gamma'})$ and the other neighbor (on $P(\mathbb{O})$) from $\beta(p_{\gamma'+1}) \cup \cdots \cup \beta(p_d)$. Since $\mathbb{O}$ is i-relevant and $i > i'$, this vertex cannot belong to $S^{i'}$. This means that $\mathbb{O}$ contains as a base vertex $b^\star$ that is adjacent to all the vertices of $\text{base}(\mathbb{O}')$.

Since $\mathbb{O}$ is i-relevant and $i > i'$, and because $v^\star \in V(\mathbb{O}) \cap V(\mathbb{O}')$, Observation 4.3 implies that $v^\star$ must be a base vertex of $\mathbb{O}'$, and it can be neither b'_1 nor b'_2. In particular, we derive that $b^\star$ is a neighbor of $v^\star$. Recall that we have argued that $v^\star$ and $b^\star$ belong to $P(\mathbb{O})$. As $P(\mathbb{O})$ is an induced path, $b^\star$ cannot be adjacent to the other neighbor of $v^\star$ on $P(\mathbb{O})$ (if one exists). Let us suppose that such a neighbor exists, and denote it by $n^\star$. We claim that $n^\star \notin \beta(p_1) \cup \cdots \cup \beta(p_d)$. To show this, suppose by way of contradiction that this claim is false. Because $b^\star$ is adjacent to all the vertices of $\text{base}(\mathbb{O}')$ but not to $n^\star$ that is supposed to belong to $\beta(p_1) \cup \cdots \cup \beta(p_d)$, this means that either $\text{last}(n^\star) < \text{last}(b'_1)$ or $\text{first}(b'_{\eta'}) < \text{first}(n^\star)$. However, as we have already argued that $\text{last}(v^\star) < \text{first}(b'_{\eta'})$, while

$n^\star$ and $v^\star$ are neighbors, it is impossible that $\text{first}(b'_{\eta'}) < \text{first}(n^\star)$, so $\text{last}(n^\star) < \text{last}(b'_1)$. Moreover, as $v^\star$ can be neither b'_1 nor b'_2 but it belongs to $\text{base}(\mathbb{O}')$, we have that $\text{last}(b'_1) < \text{first}(v^\star)$, which means that $\text{last}(n^\star) < \text{first}(v^\star)$. However, this implies that $n^\star$ and $v^\star$ are not adjacent, which is a contradiction. Thus, $n^\star \notin \beta(p_1) \cup \cdots \cup \beta(p_d)$.

Overall, we have shown that $v^\star \in V(P(\mathbb{O}))$, $v^\star$ is adjacent to $b^\star$ on $P(\mathbb{O})$, and that either $v^\star$ has no other neighbor on $P(\mathbb{O})$ or its other neighbor does not belong to $\bigcup_{i\in[d]}\beta(p_i)$. By Observation 4.1, this means that either $v^\star = t_\ell$ or both $v^\star = b_1$ and $t_\ell \notin \bigcup_{i\in[d]}\beta(p_i)$. To conclude the proof, it remains to show that $|V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')| \le 1$. Suppose, by way of contradiction, that this claim is false, and let $u^\star \in V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')$ such that $u^\star \neq v^\star$. Then, as the choice of $v^\star$ was arbitrary, we again derive that $u^\star = t_\ell$ or both $u^\star = b_1$ and $t_\ell \notin \bigcup_{i\in[d]}\beta(p_i)$. Then, without loss of generality, suppose that $v^\star = t_\ell$ and $u^\star = b_1$. However, this means that $v^\star \notin \bigcup_{i\in[d]}\beta(p_i)$. This is a contradiction because $v^\star \in \text{before}(i')$. □

We now present the first statement that lies at the heart of the proof.

Lemma 4.4. *Let $i, i', \hat{i} \in [2k]$ be such that $i > i' > \hat{i}$, let $\mathbb{O}$ be an i-relevant AW, let $\mathbb{O}'$ be an i'-relevant AW, and let $\widehat{\mathbb{O}}$ be an $\hat{i}$-relevant AW. For at least one index $j \in \{i', \hat{i}\}$, the following condition holds: $V(\mathbb{O}) \cap V(\mathbb{O}^j) \cap \text{before}(j) = \varnothing$.*

Proof. Arbitrarily select $i, i', \hat{i} \in [2k]$ such that $i > i' > \hat{i}$, an i-relevant AW $\mathbb{O}$, an i'-relevant AW $\mathbb{O}'$ and an $\hat{i}$-relevant AW $\widehat{\mathbb{O}}$. Suppose, by way of contradiction, that $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i') \neq \varnothing$ and $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}) \neq \varnothing$ By Conditions 1, 2, and 3 in Lemma 4.3 with respect to i', we have that either (a) $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i') = \{t_\ell\}$, or (b) $V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i') = \{b_1\}$ and $t_\ell \notin (\bigcup_{i\in[d]}\beta(p_i))$. Furthermore, by Conditions 1, 2, and 3 in Lemma 4.3 with respect to $\hat{i}$, we have that either (c) $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}) = \{t_\ell\}$, or (d) $V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}) = \{b_1\}$ and $t_\ell \notin (\bigcup_{i\in[d]}\beta(p_i))$.

Observe that (a) and (d) cannot happen simultaneously, as well as (b) and (c) cannot happen simultaneously. This means that either $t_\ell \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}))$ or $b_1 \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}))$. Let x denote the vertex in $\{t_\ell, b_1\}$ such that $x \in (V(\mathbb{O}) \cap V(\mathbb{O}') \cap \text{before}(i')) \cap (V(\mathbb{O}) \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i}))$. In particular, $x \in V(\mathbb{O}') \cap V(\widehat{\mathbb{O}}) \cap \text{before}(\hat{i})$. By Condition 1 in Lemma 4.3, we have that $x \in \{t'_\ell, b'_1\}$. This means that $\{t'_\ell, b'_1\} \cap V(\mathbb{O}) \neq \varnothing$. However, this is a contradiction since $\mathbb{O}$ is i-relevant and $\mathbb{O}'$ is i'-relevant where $i' < i$. This completes the proof. □

As a corollary to this lemma, we have the following.

Corollary 4.1. *Let $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ be a relevant tuple. There exist k indices, $i_1 < i_2 < \cdots < i_k$, so that for every two indices $x, y \in \{i_1, i_2, \ldots, i_k\}$ where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \text{before}(x) = \varnothing$.*

Proof. For all $i \in [2k]$, Lemma 4.4 implies that there exists at most one index $j \in [i-1]$ such that $V(\mathbb{O}^i) \cap V(\mathbb{O}^j) \cap \text{before}(j) = \emptyset$. If such an index exists, denote it by j_i, and else define $j_i = 1$. Let us initialize $I = [2k]$ and $J = \emptyset$. For $i = 2k, 2k-1, \ldots, 1$ (in this order): If $i \in I$, then insert i into J and remove i and j_i from I. Notice that at the end, the indices in J satisfy that for every two indices $x, y \in J$, where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \text{before}(x) = \emptyset$. Moreover, at each iteration, we disallow at most one new index in $[2k]$ from being inserted in J while we also insert one index into J. Thus, $|J|$ is at least half of $2k$, which completes the proof. $\square$

Towards the statement of the second lemma that lies at the heart of our proof, let us first state an immediate observation and one additional lemma.

Observation 4.4. The vertex set of an AW in G can contain at most four vertices of a clique in G.

Lemma 4.5. *Let* $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ *be a relevant tuple. For all* $i \in [2k-1]$, *there exists a path in* $G^i - \widehat{N}(\mathbb{O}^i)$ *from a vertex in* $S^i \cup \{b_1^i\}$ *to* $b_{\eta-2}^i$ *that does not contain any of the vertices of the AWs* $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.

Proof. Arbitrarily select $i \in [2k-1]$. Denote $U = \bigcup_{j=i+1}^{2k} V(\widehat{\mathbb{O}}^j)$. Moreover, denote $T = S^i$ if $\gamma^i \neq \text{nil}$ and $T = \{b_1^i\}$ otherwise. Let y be some vertex in T. To prove the lemma, it suffices to show that there exists a path in G^i from y to $b_{\eta-2}^i$ that does not contain any vertex from U. In addition, if $\gamma^i \neq \text{nil}$ then denote $\mu = \gamma^i$, and otherwise let μ denote $j-1$ where j is the index such that $p_j = \text{last}(b_1^i)$. Moreover, let α denote the index j such that $p_j = \text{first}(b_{\eta^i-2}^i)$. By the definition of μ, we know that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $|(\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)| \geq 8k$. Notice that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})$ is a clique. Therefore, by Observation 4.4, we have that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, $|((\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)) \setminus U| \geq 8k - 4(2k-i) \geq 1$.

We have thus shown that for all $\delta \in \{\mu+1, \mu+2, \ldots, \alpha-1\}$, there exists at least one vertex, which we denote by x_δ, that belongs to $((\beta^i(p_\delta) \cap \beta^i(p_{\delta+1})) \setminus \widehat{N}(\mathbb{O}^i)) \setminus U$. Observe that $y - x_\delta - x_{\delta+1} - \cdots - x_{\alpha-1} - b_{\eta^i-2}^i$ is a walk in $G^i - \widehat{N}(\mathbb{O}^i)$ from y to $b_{\eta-2}^i$ that does not contain any vertex from U. Clearly, if $G[\{y, x_\delta, x_{\delta+1}, \ldots, x_{\alpha-1}, b_{\eta^i-2}^i\}]$ contains a walk from y to $b_{\eta^i-2}^i$, then it also contains a path from y to $b_{\eta^i-2}^i$. This completes the proof. $\square$

We are now ready to prove the second statement central to the proof of Lemma 4.2.

Lemma 4.6. *Let* $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ *be a relevant tuple. For all* $i \in [2k-1]$ *such that* $\mathbb{O}^i$ *is a highly i-relevant good AW in* G^i, *there exists an i-relevant AW* $\mathbb{O}'$ *such that the following condition holds: the base path of* $\mathbb{O}'$ *has a subpath* Q *from a vertex in* $S^i \cup \{b_1^i\}$ *to* b_η^i *that does not contain any of the vertices of the AWs* $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.

Proof. Arbitrarily select $i \in [2k-1]$. Due to Observation 4.3, if (a) $\{b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}\} \cap S^i \neq \emptyset$ or (b) $|V(\mathrm{base}(\mathbb{O}^i))| \leq 6$, the claim is true: We can simply choose $\mathbb{O}'$ to be $\mathbb{O}^i$, and the path Q as the subpath of the base of $\mathbb{O}^i$ from $\{b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}\} \cap S^i$ (in case (a)) or from b^i_1 (in case (b)) to b^i_η. Hence, we next assume this is not the case. Denote $U = \bigcup_{j=i+1}^{2k} V(\mathbb{O}^j)$. For simplicity of notation, denote $\mathbb{O} = \mathbb{O}^i$. Note that $\{t_s, t_\ell, t_r, c_1, c_2, b_1, b_2, b_3, b_{\eta-3}, b_{\eta-2}, b_{\eta-1}, b_\eta\} \cap U = \emptyset$. Now, notice that to prove the lemma, it suffices to show that there exists an induced path from b_1 to b_η in $G^i - U$ (a) whose second vertex is b_2 and two before last vertices are $b_{\eta-2}$ and $b_{\eta-1}$, (b) whose internal vertices are all adjacent to c_1 and c_2 and are all neither equal nor adjacent to t_ℓ, t_r and t_s, and (c) has a subpath $Q^\star$ from a vertex in $S^i \cup \{b^i_1\}$ to b^i_η with no vertex from U. Indeed, together with the centers and terminals of $\mathbb{O}$, we will thus get the desired AW $\mathbb{O}'$.

Let W be an arbitrary induced path from b_3 to $b_{\eta-3}$ in $G^i - \widehat{N}(\mathbb{O})$ (as $|V(\mathrm{base}(\mathbb{O}))| \geq 7$, $3 < \eta - 3$). Since W is induced, every vertex on W must belong to $(\beta^i(p_\alpha) \cup \beta^i(p_{\alpha+1}) \cup \cdots \beta^i(p_\gamma)) \setminus \widehat{N}(\mathbb{O})$, where $\alpha = \mathrm{last}(b_3)$ and $p_\gamma = \mathrm{first}(b_{\eta-3})$. Because $\mathbb{O}$ is a good AW in G^i, by Proposition 4.2, for any vertex $v \in (\beta^i(p_1) \cup \beta^i(p_2) \cup \cdots \beta^i(p_\gamma)) \setminus \widehat{N}(\mathbb{O})$, it holds that v is not adjacent to any vertex that is shallow in G^i and in particular not to t_s. Thus, no vertex on W is adjacent to t_s. Moreover, since c_1 and c_2 are adjacent to all vertices on $\mathrm{base}(\mathbb{O})$, they are adjacent to all vertices in $\beta^i(p_\alpha) \cup \beta^i(p_{\alpha+1}) \cup \cdots \beta^i(p_\gamma)$, and in particular to all vertices on W.

Let w be any internal vertex on W. On the one hand, because W is an induced path, $\mathrm{last}(b_3) < \mathrm{last}(w)$. This means that if w was adjacent to any vertex in $\{t_\ell, b_1\}$, then we could have replaced $b_2 - b_3$ by w in $P(\mathbb{O})$ and obtain a walk (which contains a path) shorter than $P(\mathbb{O})$ between t_ℓ and t_r in $G[V(P(\mathbb{O})) \cup \beta(p_\alpha) \cup \beta(p_{\alpha+1}) \cup \cdots \cup \beta(p_\gamma)] - \widehat{N}(\mathbb{O})$, which is a contradiction as $\mathbb{O}$ is good and hence it is in particular short. Thus, w is not adjacent to any vertex in $\{t_\ell, b_1\}$. On the other hand, because W is an induced path, $\mathrm{first}(w) < \mathrm{first}(b_{\eta-3})$. Symmetrically to the former case, we deduce that w is not adjacent to any vertex in $\{b_{\eta-1}, b_\eta, t_r\}$.

Up until now, our arguments imply that to prove the lemma, it is sufficient to show that there exists an induced path from b_3 to $b_{\eta-3}$ in $G^i - \widehat{N}(\mathbb{O})$ having a subpath $Q^\star$ from a vertex in $S^i \cup \{b^i_3\}$ to $b_{\eta-3}$ with no vertex from U. Indeed, by appending $b_1 - b_2$ to the beginning of such a path and $b_{\eta-2} - b_{\eta-1} - b_\eta$ to the end of such a path, while removing b_3 ($b_{\eta-3}$) if the first (last) internal vertex of the path is adjacent to b_2 ($b_{\eta-2}$), we derive an induced path satisfying properties (a), (b) and (c) as required.

By Lemma 4.5, there exists a path Q' in $G^i - (\widehat{N}(\mathbb{O}) \cup U)$ from a vertex $v^\star$ in $S^i \cup \{b_1\}$ to $b_{\eta-2}$. Notice that $b_{\eta-3} \in \beta(\mathrm{first}(b_{\eta-2}))$ and that for every vertex $v \in S^i \cup \{b_1\}$, $\mathrm{first}(v) < \mathrm{first}(b_{\eta-2})$. This means that Q' has at least one vertex adjacent to $b_{\eta-3}$. Moreover, if $v^\star = b_1$, then because $\mathrm{last}(b_1) < \mathrm{first}(b_3) \leq \mathrm{first}(b_{\eta-3})$, Q' has at least one vertex adjacent to b_3. Also recall that $\{b^i_{\eta-3}, b^i_{\eta-2}, b^i_{\eta-1}\} \cap S^i \neq \emptyset$. We thus derive that there exists a path, and hence also an induced one, in $G^i - (\widehat{N}(\mathbb{O}) \cup U)$ from a vertex $v^\star \in S^i \cup \{b_3\}$, such that either $v^\star = b_3$ or $\mathrm{last}(b_3) < \mathrm{last}(v^\star)$, to $b_{\eta-3}$. Let us denote such a path by Q. Note that if $v^\star \in \{b^i_3\}$, then the proof is already complete.

Suppose now that $v^\star \in S^i$ and $\mathrm{last}(b_3) < \mathrm{last}(v^\star)$. Let b_j be the first vertex on $\mathrm{base}(\mathbb{O})$ that belongs to S^i (note that such a vertex must exist). Then, $G' = G^i[\{b_3, b_4, ..., b_j\} \cup V(Q)]$

has an induced path from b_3 to $b_{\eta-3}$ having a subpath from a vertex in S^i (which may or may not be $v^\star$) to $b_{\eta-3}$. Indeed, we derive such a path by taking any shortest path from b_3 to $b_{\eta-3}$ in G'—such a path necessarily contains a subpath from some vertex $u \in S^i$ (where u does not belong to U as $U \cap S^i = \emptyset$) to $\beta_{\eta-3}$ whose internal vertices all belong to $V(Q)$. This completes the proof. □

As a corollary to Lemma 4.6, we have the following.

Corollary 4.2. *There exists a relevant tuple* $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ *such that for all* $i \in [2k]$, *the following condition holds: the base path of* $\widehat{\mathbb{O}}^i$ *has a subpath* Q *from a vertex in* $S^i \cup \{b_1^i\}$ *to* b_η^i *that does not contain any of the vertices of the AWs* $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$.

Proof. To apply induction, we claim that for all $j \in [2k]$, we have a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k] \setminus [j-1]$, the following conditions hold: (i) the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the AWs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$, and (ii) for all $i \in [j-1]$, $\widehat{\mathbb{O}}^i$ is a highly i-relevant good AW in G^i. The proof is by induction on j, and it is clear that if it is correct for $j = 1$, then we will derive the lemma. In the base case, where $j = 2k$, the claim is true since $(\mathbb{O}^1, \mathbb{O}^2, \ldots, \mathbb{O}^{2k})$, as computed by SeparateProcedure, is a relevant tuple where every AW is highly relevant and good as required (the disjointness condition holds trivially).

Now, suppose that $j < 2k$. By the inductive hypothesis, there exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k] \setminus [j]$, the following conditions hold: (i) the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the AWs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$, and (ii) for all $i \in [j]$, $\widehat{\mathbb{O}}^i$ is a highly i-relevant good AW in G^i. Apply Lemma 4.6 with $i = j$ to obtain a j-relevant AW $\mathbb{O}'$ such that the following condition holds: the base path of $\mathbb{O}'$ has a subpath Q from a vertex in $S^i \cup \{b_1^j\}$ to $b_{\eta^j}^j$ that does not contain any of the vertices of the AWs $\widehat{\mathbb{O}}^{j+1}, \widehat{\mathbb{O}}^{j+2}, \ldots, \widehat{\mathbb{O}}^{2k}$. Thus, we replace $\widehat{\mathbb{O}}^j$ by $\mathbb{O}'$ in $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, , \ldots, \widehat{\mathbb{O}}^{2k})$ to obtain a tuple as required to prove the claim. □

We are now ready to prove Lemma 4.2. As noted earlier, together with Observation 4.1, this proof also concludes the proof of Lemma 4.1.

Proof of Lemma 4.2. By Corollary 4.2, there exists a relevant tuple $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^{2k})$ such that for all $i \in [2k]$, the following condition holds: the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex in $S^i \cup \{b_1^i\}$ to b_η^i that does not contain any of the vertices of the AWs $\widehat{\mathbb{O}}^{i+1}, \widehat{\mathbb{O}}^{i+2}, \ldots, \widehat{\mathbb{O}}^{2k}$. By Corollary 4.1, there exist k indices, $i_1 < i_2 < \cdots < i_k$, such that for every two indices $x, y \in \{i_1, i_2, \ldots, i_k\}$, where $x < y$, $V(\widehat{\mathbb{O}}^y) \cap V(\widehat{\mathbb{O}}^x) \cap \text{before}(x) = \emptyset$. Without loss of generality, suppose that $i_1 = 1, i_2 = 2, \ldots, i_k = k$ (the arguments to follow hold for any $i_1 < i_2 < \cdots < i_k$).

We claim that $\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^k$ are vertex-disjoint, which would complete the proof. To prove this claim, we arbitrarily choose $i, j \in [k]$ such that $i < j$. First note that as $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are i-relevant and j-relevant, we have that the terminals and centers of $\widehat{\mathbb{O}}^i$ do not belong to $\widehat{\mathbb{O}}^j$. Moreover, the base path of $\widehat{\mathbb{O}}^i$ has a subpath Q from a vertex $v^\star$ in $S^i \cup \{b_1^i\}$ to $b_{\eta^i}^i$ that has no vertex of $\widehat{\mathbb{O}}^j$. Let W denote the subpath of the base path of $\widehat{\mathbb{O}}^i$ from b_1^i to $v^\star$. Hence, to conclude that $\widehat{\mathbb{O}}^i$ and $\widehat{\mathbb{O}}^j$ are vertex-disjoint, it remains to show that no vertex of W belongs to $\widehat{\mathbb{O}}^j$. Notice that $V(W) \subseteq V(\widehat{\mathbb{O}}^i) \cap \text{before}(i)$. By our choice of $(\widehat{\mathbb{O}}^1, \widehat{\mathbb{O}}^2, \ldots, \widehat{\mathbb{O}}^k)$, it holds that $V(\widehat{\mathbb{O}}^i) \cap \text{before}(i)$ does not have any vertex of $\widehat{\mathbb{O}}^j$. Thus, the proof is complete. □

5 | DECOMPOSITION OF MODULES

Let us begin with the following simple observation, on which we rely implicitly in our arguments, and which follows immediately from the definition of a modular tree decomposition. For simplicity, we use the abbreviations $f|_v = f|_{V(T|_v)}$ and $g|_v = g|_{V(T|_v)}$.

Observation 5.1. Let G be a graph with a modular tree decomposition (T, f, g), and let $v \in V(T)$. Then, $(T|_v, f|_v, g|_v)$ is a modular tree decomposition of $G[f(v)]$.

We proceed by introducing the definition of a *problematic set* and a *problematic node*:

Definition 5.1. Let G be a graph with a modular tree decomposition (T, f, g). The *set of problematic obstructions* of a node $v \in V(T)$, denoted by $\text{prob}_G(v)$, is the set of all obstructions $\mathbb{O}$ in $G[f(v)]$ such that for every child u of v in T, $\mathbb{O}$ is not an obstruction in $G[f(u)]$, that is, $V(\mathbb{O}) \setminus f(u) \neq \emptyset$. When G is clear from context, it is omitted.

Definition 5.2. Let G be a graph with a modular tree decomposition (T, f, g). A node $v \in V(T)$ is *problematic* if $\text{prob}(v) \neq \emptyset$. The set of problematic nodes is denoted by $\text{prob}_G(T)$. When G is clear from context, it is omitted.

We have the following simple observation, which follows directly from the definition of a problematic set of obstructions.

Observation 5.2. Let G be a graph with a modular tree decomposition (T, f, g). The sets $\text{prob}(v), v \in \text{prob}(T)$, define a partition of the set of obstructions of G. That is, for all $u, v \in V(T), \text{prob}(u) \cap \text{prob}(v) = \emptyset$, and the set of obstructions of G is precisely $\bigcup_{v \in \text{prob}(T)} \text{prob}(v)$.

We proceed to argue that nodes assigned 1 by g are nonproblematic.

Lemma 5.1. *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ such that $g(v) = 1$. Then, v is not a problematic node.*

Proof. Suppose, by way of contradiction, that the lemma is false. Note that the leaves ℓ of a modular tree decomposition satisfy $|f(\ell)| = 1$. So, v is not a leaf. Let $v_1, v_2, ..., v_t$, $t \geq 1$, be the children of v. Then, $G[f(v)]$ has an obstruction $\mathbb{O}$ on more than 10 vertices such that for all $i \in [t]$, $\mathbb{O}$ is not an obstruction in $G[f(v_i)]$. Let $i \in [t]$ be the smallest index such that $V(\mathbb{O}) \subseteq f(v_1) \cup f(v_2) \cup \cdots \cup f(v_i)$. Since $G[f(v_1)]$ does not contain $\mathbb{O}$, $i \geq 2$. Moreover, by the choice of i and since $G[f(v_i)]$ does not contain $\mathbb{O}$, $V(\mathbb{O}) \cap f(v_i) \neq \emptyset$ and $V(\mathbb{O}) \setminus f(v_i) \neq \emptyset$. Since $f(v_1) \cup f(v_2) \cup \cdots \cup f(v_{i-1})$ and $f(v_i)$ are both modules (because $g(v) = 1$), by Proposition 2.1, $|V(\mathbb{O}) \cap (f(v_1) \cup f(v_2) \cup \cdots \cup f(v_{i-1}))| = 1$ and $|V(\mathbb{O}) \cap f(v_i)| = 1$. However, this means that $|V(\mathbb{O})| = 2$, which is a contradiction as no obstruction consists of only two vertices. □

We further observe that a problematic node should have "many" children.

Lemma 5.2. *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node. Then, v has at least* $\max\{2k, 10\} + 1$ *children in T.*

Proof. Since $v \in V(T)$ is a problematic node, $G[f(v)]$ has an obstruction $\mathbb{O}$ such that for every child u of v in T, $\mathbb{O}$ is not an obstruction in $G[f(u)]$. Moreover, since G has no obstruction on at most $\max\{2k, 10\}$ vertices, we have that $|V(\mathbb{O})| \geq \max\{2k, 10\} + 1$. By Proposition 2.1, $|V(\mathbb{O}) \cap f(u)| \leq 1$ for every child u of v. Since $f(v) = \bigcup_{u \in \mathrm{child}(v)} f(u)$, this means that v has at least $\max\{2k, 10\} + 1$ children in T. □

To proceed, we need the following definition and notation.

Definition 5.3. Let G be a graph with a modular tree decomposition (T, f, g). A subset $P \subseteq \mathrm{prob}(T)$ has a conflict if there exist $v, u \in P$ such that u is a descendant of v in T and on the (unique) path between u and v in T no vertex belongs to $\mathrm{prob}(T) \setminus P$.

Definition 5.4. Let G be a graph with a modular tree decomposition (T, f, g). For a node $v \in V(T)$, $\mathrm{pack}_G(v)$ is the maximum number of vertex-disjoint obstructions in $\mathrm{prob}(v)$. When G is clear from context, it is omitted.

Note that a problematic node is precisely a node such that $\mathrm{pack}(v) \geq 1$. We proceed with our analysis of problematic nodes using the following two lemmata.

Lemma 5.3. *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices, and which does not have k vertex-disjoint obstructions. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathrm{prob}(T)$ with no conflicts. Then, for each $v \in P$ and each child u of v in T such that u has a problematic descendant, there exist at least k vertices in $f(u)$ that do not belong to $\bigcup_{w \in \mathrm{desc}(u) \cap P} f(w)$.*

Proof. We arbitrarily select $v \in P$ and a child u of v in T such that u has a problematic descendant. Note that $u \notin P$, otherwise, as $v \in P$ is the parent of u, we have a conflict in P. Since P has no conflicts and u has a problematic descendant, there exists a problematic node $x \notin P$ that is either u or a descendant of u such that the path between u and x in T

contains no vertices in P. Indeed, for any descendant x' of u such that $x' \in P$, as P has no conflicts, the path between v and x' must contain some problematic vertex $x'' \notin P$. We first claim that among the children of x, there are less than k children that are each problematic or has a problematic descendant. To see this, note that for each child y of x that is problematic or has a problematic descendant, the subgraph $G[f(y)]$ has an obstruction, and that for distinct children y, y' of x, it holds that $f(y) \cap f(y') = \emptyset$. Thus, since G does not have k vertex-disjoint obstructions, the claim follows.

By Lemma 5.2, we further have that x has at least $2k + 1$ children. To conclude the proof, note that for every child y of x that is not problematic and has no problematic descendant, we have at least one vertex in $f(y) \subseteq f(u)$ that does not belong to $\bigcup_{w\in \mathrm{desc}(u)\cap P} f(w)$. As we have shown that x has less than k children that are problematic or have problematic descendants, and x has at least $2k + 1$ children, we know that x has at least k non-problematic children with no problematic descendant. Thus, we derive that there exist at least k vertices in $f(u)$ that do not belong to $\bigcup_{w\in \mathrm{desc}(u)\cap P} f(w)$. □

Lemma 5.4. *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G. Let $P \subseteq \mathrm{prob}(T)$ with no conflicts. Then, G has* $\min\{k, \sum_{v\in P} \mathrm{pack}(v)\}$ *vertex-disjoint obstructions.*

Proof. Suppose that G does not have k vertex-disjoint obstructions, else the proof is complete. For the proof, we strengthen the statement of the lemma as follows. We claim that G has $\sum_{v\in P} \mathrm{pack}(v)$ vertex-disjoint obstructions that belong to $\bigcup_{v\in P} \mathrm{prob}(v)$. We prove the claim by induction on $|V(T)|$. In the basis, where $|V(T)| \leq 1$, the claim is trivially true.

Now, suppose that $|V(T)| \geq 2$, and let r denote the root of T. Let $v_1, v_2, \ldots, v_t$ be the children of r in T. For all $i \in [t]$, let us denote $G_i = G[f(v_i)]$, $T_i = T|_{v_i}, f_i = f|_{V(T_i)}$ and $g_i = g|_{V(T_i)}$. Note that (T_i, f_i, g_i) is a modular tree decomposition of G_i. Moreover, denote $P_i = P \cap V(T_i)$. By the inductive hypothesis, G_i has $\sum_{v\in P_i} \mathrm{pack}(v)$ vertex-disjoint obstructions that belong to $\bigcup_{v\in P_i} \mathrm{prob}(v)$, and let us denote a set of such vertex-disjoint obstructions by $\mathcal{O}_i$. As the subtrees $T_i, i \in [t]$, are vertex-disjoint, we have that $\mathcal{O} = \bigcup_{i\in[t]} \mathcal{O}_i$ is a set of $\sum_{v\in(P\setminus\{r\})} \mathrm{pack}(v)$ vertex-disjoint obstructions. If $r \notin P$, the proof is complete, and therefore we next suppose that $r \in P$.

Let $\mathcal{O}_r$ denote a set of $\mathrm{pack}(r)$ vertex-disjoint obstructions in $\mathrm{prob}(r)$, such that among all such sets, it minimizes $|V(\mathcal{O}_r) \cap V(\mathcal{O})|$. We claim that $V(\mathcal{O}_r) \cap V(\mathcal{O}) = \emptyset$, which would complete the proof, as then $\mathcal{O}_r \cup \mathcal{O}$ would be a set of $\sum_{v\in P} \mathrm{pack}(v)$ vertex-disjoint obstructions. Suppose, by way of contradiction, that this claim is false. Since then $V(\mathcal{O}_r) \cap V(\mathcal{O}) \neq \emptyset$, there exists $\mathbb{O} \in \mathcal{O}_r$ that contains at least one vertex in $f(u)$ for a descendant $u \in P$ of r in T. Suppose, without loss of generality, that v_1 is the child of r such that u is a descendant of v_1. Then, as G has no obstruction on at most 10 vertices, we have that for all $\mathbb{O}' \in \mathcal{O}_r$, $|V(\mathbb{O}') \cap f(v_1)| \leq 1$. In particular, $|V(\mathbb{O}) \cap f(v_1)| = 1$ and since $|\mathcal{O}_r| \leq k - 1$ (because G does not have k vertex-disjoint obstructions), $|V(\mathcal{O}_r) \cap f(v_1)| \leq k - 1$. By Lemma 5.3, there exist at least k vertices in $f(v_1)$ that do not belong to $\bigcup_{w\in \mathrm{desc}(v_1)\cap P} f(w)$. Thus, there exists a vertex $x \in f(v_1)$ that does not belong to any obstruction in $\mathcal{O}_r \cup \mathcal{O}$. Because $f(v_1)$ is a module and $|V(\mathbb{O}) \cap f(v_1)| = 1$, by replacing the vertex in $V(\mathbb{O}) \cap f(v_1)$ by x in $\mathbb{O}$, we obtain another obstruction $\widehat{\mathbb{O}} \in \mathrm{prob}(r)$. However, then $(\mathcal{O}_r\setminus\{\mathbb{O}\}) \cup \{\widehat{\mathbb{O}}\}$ is a set of $\mathrm{pack}(r)$ vertex-disjoint obstructions in prob(r) that uses less vertices from $V(\mathcal{O})$ than $\mathcal{O}_r$. This contradicts the choice of $\mathcal{O}_r$, and hence the proof is complete. □

We also show that prob(T) can be divided into two sets with no conflicts.

Lemma 5.5. *Let G be a graph with a modular tree decomposition (T, f, g). There exists a partition (P_1, P_2) of* prob(T) *such that neither P_1 has a conflict nor P_2 has a conflict.*

Proof. To prove this claim, consider the tree T' constructed from T and prob(T), where $V(T') = \text{prob}(T)$ and $E(T') = \{(u, v)|$ the path from u to v in T contains no other vertex of ,prob$(T)\}$. Then consider a 2-coloring of T', and let P_1 and P_2 be the two color classes. Observe that P_1 and P_2 are conflict-free. Indeed, P_1 and P_2 are independent sets in T', and for any two vertices $u, v \in P_1$, the path between them in T must contain a vertex of prob(T) which lies in P_2; a similar statement holds for P_2. We output (P_1, P_2) as the required partition of prob(T). □

Specific classes of interval graphs, called *prereduced graphs* and *nice interval graphs*, were defined by Cao and Marx as follows.

Definition 5.5 (Cao and Marx [11]). A graph G is *reduced* if it satisfies the following properties:

1. Every nontrivial module of G is a clique.
2. G does not have any obstruction on at most 10 vertices.

Definition 5.6 (Cao and Marx [11]). A graph G is *nice* if it satisfies the following properties:

1. G is chordal.
2. G does not have any obstruction on at most 10 vertices.
3. Every vertex in G that is a shallow terminal of at least one obstruction is simplicial.

These definitions were in particular used to derive the following results.

Proposition 5.1 (Theorem 2.1 [11]). *Let G be a reduced graph. Every vertex in G that is a shallow terminal of at least one obstruction is simplicial.*

Proposition 5.2 (Proposition 8.3 [11]). *Any nice graph has a nice clique caterpillar (T, β).*

As a corollary of these two propositions, we have the following.

Corollary 5.1. *Any chordal reduced graph has a nice clique caterpillar (T, β).*

Let us derive a consequence of Corollary 5.1 with respect to a modular tree decomposition.

Lemma 5.6. *Let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$ be a problematic node such that for every child u of v in T, $f(u)$ is a clique. Then, $G[f(v)]$ has a nice clique caterpillar.*

Proof. By Corollary 5.1, it is sufficient to show that every nontrivial module of G is a clique. By Observation 5.1, Lemma 5.1 and since v is problematic, we have that $g(v) = 0$. Thus, by Observation 5.1 and the definition of a modular tree decomposition, every nontrivial module of G is a subset of $f(u)$ for a child u of v in T. Since for every child u of v in T, $f(u)$ is a clique, the proof is complete. □

Towards the proof of the main result of this section, we need one additional notation.

Definition 5.7. Let G be a graph with a modular tree decomposition (T, f, g), and let $v \in V(T)$. Then, $\text{clique}(G, v)$ denotes the graph obtained from $G[f(v)]$ by turning each $f(u)$, $u \in \text{child}(v)$, into a clique. That is, $V(\text{clique}(G, v)) = f(v)$ and $E(\text{clique}(G, v)) = E(G[f(v)]) \cup (\bigcup_{u \in \text{child}(v)} \{\{x, y\} : x, y \in f(u)\})$.

We first use this definition to prove the following result.

Lemma 5.7. *Let G be a graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$. Then, the set of obstructions in* $\text{clique}(G, v)$ *is precisely* $\text{prob}_G(v)$.

Proof. In one direction, let $\mathbb{O}$ be an obstruction in $\text{prob}_G(v)$. To show that $\mathbb{O}$ is an obstruction in $\text{clique}(G, v)$, it is sufficient to show that $\mathbb{O}$ does not contain two vertices between whom an edge was added. However, as G has no obstruction on at most 10 vertices, we have that $|V(\mathbb{O})| > 10$. Then, as $\mathbb{O}$ is not present in $G[f(u)]$ for any child u of v in T, by Proposition 2.1, its has at most one vertex in $f(u)$ for any child u of v in T. This concludes this direction.

In the other direction, let $\mathbb{O}$ be an obstruction in $\text{clique}(G, v)$. Note that the vertex set of $\text{clique}(G, v)[f(u)]$, $u \in \text{child}(v)$, is a clique. Thus, $\mathbb{O}$ is not present in $G[f(u)]$ for any child u of v in T. Thus, to show that $\mathbb{O}$ is an obstruction in $\text{prob}_G(v)$, it is sufficient to show that $\mathbb{O}$ is an obstruction in G. In turn, this means that it is sufficient to show that $\mathbb{O}$ does not contain two vertices between whom an edge was added. Suppose, by way of contradiction, that it does. Then, there exist $u \in \text{child}(v)$ and $x, y \in f(u)$ such that $x, y \in V(\mathbb{O})$. Note that $f(u)$ is also a module in $\text{clique}(G, v)$. Thus, Proposition 2.1 implies that $|V(\mathbb{O})| \leq 4$, which means that $\mathbb{O}$ is a hole. However, this is not possible as x, y are adjacent to each other and have the same neighborhood in $\text{clique}(G, v)$. This concludes the proof. □

The following lemma lies at the heart of the main result of this section.

Lemma 5.8. *Let $k \in \mathbb{N}$, and let G be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Let (T, f, g) be a modular tree decomposition of G, and let $v \in V(T)$.*

Then, at least one of the following conditions holds: (i) $\text{pack}(v) \geq k$*; (ii) there exists a subset* $D \subseteq V(G)$ *of size* $\mathcal{O}(k^2)$ *that intersects the vertex set of every obstruction in* $\text{prob}(v)$.

Proof. Suppose that $\text{pack}(v) < k$, else the proof is complete. By Lemma 5.7, the set of obstructions in $\text{clique}(G, v)$ is precisely $\text{prob}_G(v)$. Therefore, $\text{clique}(G, v)$ has less than k vertex-disjoint obstructions. By Lemma 5.6, $\text{clique}(G, v)$ has a nice clique caterpillar. Thus, by Lemma 4.1, there exists a subset $D \subseteq f(v)$ of size $\mathcal{O}(k^2)$ such that $\text{clique}(G, v) - D$ is an interval graph. That is, D intersects the vertex set of every obstruction in $\text{clique}(G, v)$, which means that D intersects the vertex set of every obstruction in $\text{prob}_G(v)$. □

We are now ready to prove the main result of this section.

Lemma 5.9. *Let* $k \in \mathbb{N}$*, and let* G *be a chordal graph that has no obstruction on at most* $\max\{2k, 10\}$ *vertices. Then, at least one of the following conditions holds: (i)* G *has* k *vertex-disjoint obstructions; (ii) there exists a subset* $D \subseteq V(G)$ *of size* $\mathcal{O}(k^2)$ *such that* $G - D$ *is an interval graph.*

Proof. Suppose that G does not have k vertex-disjoint obstructions, else we are done. By Lemma 5.5, there exists a partition (P_1, P_2) of $\text{prob}(T)$ such that neither P_1 has a conflict nor P_2 has a conflict. By Lemma 5.4, for each $i \in [2]$, G has $\sum_{v \in P_i} \text{pack}(v)$ vertex-disjoint obstructions. Thus, since G does not have k vertex-disjoint obstructions, for each $i \in [2]$, $\sum_{v \in P_i} \text{pack}(v) < k$. This means that $\sum_{v \in \text{prob}(T)} \text{pack}(v) < 2k$.

By Lemma 5.8, for all $v \in \text{prob}(T)$, there exists a subset $D_v \subseteq V(G)$ of size $\mathcal{O}((\text{pack}(v) + 1)^2)$[8] that intersects the vertex set of every obstruction in $\text{prob}(v)$. Denote $D = \bigcup_{v \in \text{prob}(T)} D_v$. Then, $|D| = \mathcal{O}(\sum_{v \in \text{prob}(T)} (\text{pack}(v) + 1)^2)$. By Observation 5.2, we have that $G - D$ is an interval graph. Thus, to conclude the proof, it remains to show that $\sum_{v \in \text{prob}(T)} (\text{pack}(v) + 1)^2 = \mathcal{O}(k^2)$. Since for all $v \in \text{prob}(T)$, $\text{pack}(v) \geq 1$, it is sufficient to show that $\sum_{v \in \text{prob}(T)} (\text{pack}(v))^2 = \mathcal{O}(k^2)$. Recall that $\sum_{v \in \text{prob}(T)} \text{pack}(v) < 2k$. Thus, $\sum_{v \in \text{prob}(T)} (\text{pack}(v))^2 \leq (\sum_{v \in \text{prob}(T)} \text{pack}(v)) \cdot (\sum_{v \in \text{prob}(T)} \text{pack}(v)) < 2k \cdot 2k = \mathcal{O}(k^2)$. This completes the proof. □

6 | PUTTING IT ALL TOGETHER

Finally, we are ready to prove our main theorem.

Proof of Theorem 1. By Corollary 3.1, at least one of the following conditions hold: (i) G has k vertex-disjoint obstructions; (ii) there exists a subset $D' \subseteq V(G)$ of size $\mathcal{O}(k^2 \log k)$ such that $G - D'$ is a chordal graph that has no obstruction on at most $\max\{2k, 10\}$ vertices. In the first case, our proof is complete, and thus we next suppose that the second case applies. Then, by Lemma 5.9, at least one of the following conditions hold: (i) $G - D'$ has k vertex-disjoint obstructions; (ii) there exists a subset $\widehat{D} \subseteq V(G)$ of size $\mathcal{O}(k^2)$ such

[8]By Lemma 5.8, since we cannot have a packing of $\ell = \text{pack}(v) + 1$ obstructions, there is a hitting set D of size $\mathcal{O}(\ell^2) = \mathcal{O}((\text{pack}(v) + 1)^2)$.

that $(G - D') - \widehat{D}$ is an interval graph. In the first case, our proof is complete. In the second case, we have that $D = D' \cup \widehat{D}$ is a set of size $\mathcal{O}(k^2 \log k)$ such that $G - D$ is an interval graph, which again completes the proof. □

Before we turn to prove a corollary of our main theorem, we need one more proposition.

Proposition 6.1 (Cao [10]). *There exists an $\mathcal{O}(nm)$-time algorithm that, given a graph G, outputs an integer d' such that the following conditions hold:*

1. *there exists a subset $D' \subseteq V(G)$ of size at most d' such that $G - D'$ is an interval graph;*
2. *$d' \leq 8d$ for the integer d that is the minimum size of a subset $D \subseteq V(G)$ such that $G - D$ is an interval graph.*

As a consequence of Theorem 1, we derive the following corollary.

Corollary 6.1. *There exist a constant $c \in \mathbb{N}$ and an $\mathcal{O}(nm)$-time algorithm that, given a graph G and an integer $k \in \mathbb{N}$, correctly concludes which one of the following conditions holds:*[9]

1. *G has k vertex-disjoint obstructions;*
2. *there exists a subset $D \subseteq V(G)$ of size at most $ck^2 \log k$ such that $G - D$ is an interval graph.*

Proof. Let c' be the constant in the $\mathcal{O}$ notation in Theorem 1, and denote $c = 8c'$. Our algorithm is as follows. Given a graph G and an integer $k \in \mathbb{N}$, it calls the algorithm in Proposition 6.1 to obtain an integer d'. If $d' > ck^2 \log k$, then it concludes that the first condition holds, and otherwise it concludes that the second condition holds. Clearly, the algorithm runs in time $\mathcal{O}(nm)$.

If the algorithm concluded that the second condition holds, then by the correctness of the algorithm in Proposition 6.1, this conclusion is correct. Otherwise, by Proposition 6.1, there does not exist a subset $D \subseteq V(G)$ of size at most $c'k^2 \log k$ such that $G - D$ is an interval graph. By Theorem 1, this means that G has k vertex-disjoint obstructions. Thus, the conclusion of the algorithm is again correct. □

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[9] In particular, at least one condition holds, and if both conditions hold, then the algorithm can choose either of the two conditions.

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