Cramér-Rao Bounds and Resolution Benefits of Sparse Arrays in Measurement-Dependent SNR Regimes

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Abstract—**This paper derives new non-asymptotic characterization of the Cramér-Rao Bound (CRB) of any sparse array as a function of the angular separation between two far-field narrowband sources in certain regimes characterized by a low Signal-to-Noise Ratio (SNR). The primary contribution is the derivation of matching upper and lower bounds on the CRB in a certain measurement-dependent SNR (MD-SNR) regime, where one can zoom into progressively lower SNR as the number of sensors increases. This tight characterization helps to establish that sparse arrays such as nested and coprime arrays provably exhibit lower CRB compared to Uniform Linear Arrays (ULAs) in the specified SNR regime.**

Index Terms—**Cramér-Rao bound, measurement-dependent SNR, non-asymptotic guarantees, sparse arrays, super-resolution.**

I. Introduction

Achieving high resolution is crucial in many imaging problems including optical imaging, microscopy, radar, astronomy, and medical imaging. Sparse arrays such as nested and coprime arrays are capable of achieving superior resolution over a uniform array with the same number of physical sensors, due to their large aperture and lack of spatial ambiguity [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. Therefore, sparse array geometries have gained significant attention in recent times, especially for emerging applications in autonomous sensing, millimeter-wave communication, and joint sensing and communication [11], [12], [13], [14], [15].

While there exists a significant body of work on theoretical analysis of sparse arrays and coarray-based algorithms [10], [16], [17], existing results do not offer sharp *non-asymptotic comparisons* between sparse arrays and Uniform Linear Arrays (ULAs) for resolving closely-spaced sources *in certain low SNR regimes*. Existing analytical results on the performance of coarray-based methods can be broadly classified into two categories. The first category focuses on the performance of specific algorithms, such as (coarray) MUSIC and (coarray) ESPRIT [16], [18], [19]. In [19], a non-asymptotic performance analysis of the Coarray ESPRIT algorithm was presented, where the authors characterized the number of temporal snapshots required to bound the matching distance error by a specified parameter. Papers in the second category study fundamental performance limits, such as those established by the Cramér-Rao Bound (CRB). In CRB computations, two main types of array signal models are typically considered: the conditional model (CM) and the unconditional (or stochastic) model (UM) [20]. In [16], [21], [22], the authors demonstrated that the CM cannot be used to compute the CRB when the number of sources, K, exceeds the number of sensors, M, as the associated Fisher Information Matrix (FIM) becomes singular in such cases. In contrast, for UM, if the source signals are statistically uncorrelated, the FIM remains nonsingular even in underdetermined scenarios. In a follow-up study [23], the authors analyzed the behavior of the CRB for coprime and nested arrays when there are fewer sources than sensors under certain asymptotic assumptions. They demonstrated that if the number of sensors goes to infinity ($M \to \infty$) and if the SNR is significantly larger than the number of sensors (SNR $\gg M$), the CRB decreases at a rate of $O(M^5)$. This result suggests that coprime and nested arrays can achieve better asymptotic (in M) performance than ULAs in relatively high SNR regimes.

While the above findings offer a valuable understanding of the asymptotic behavior of the CRB for sparse arrays as a function of SNR and the number of sensors in high SNR scenarios, their simplified expressions do not shed any light into the behavior of CRB in non-asymptotic settings when M is not very large, or the SNR is low. Specifically, there is a lack of analysis regarding the behavior of the CRB as a function of the angular separation between two closely spaced sources when the SNR is allowed to change (inversely) with the number of sensors, M. This emulates a "Measurement-dependent SNR" or MD-SNR regime which can be relevant for practical applications such as massive MIMO communication with a total power constraint over all antennas. In such cases, the per-antenna SNR can decrease with the number of sensors M, i.e., SNR $\propto 1/M$, in order to maintain a constant total SNR [24], [25]. This parameterization enables investigating the robustness of sparse arrays relative to ULAs in "low" SNR regimes, where the definition of "low" depends on M (i.e. a SNR which is low for small M may not be low for large M).

Our Contributions: In a previous work [26], we explored the super-resolution advantages of sparse arrays over ULAs by focusing on such a MD-SNR regime, where both SNR and

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M vary jointly. We derived certain bounds on the CRB in this regime and obtained partial results for comparing nested array and ULA. Building on this foundation, this paper seeks to significantly generalize the findings in [26] and addresses a more fundamental question: *Is it possible to derive matching lower and upper bounds on the CRB as a function of M for any sparse array geometries under a total power constraint?* Such matching bounds will enable exact non-asymptotic comparisons between the CRBs of any sparse array and ULA. This paper provides an affirmative answer to this question. We rigorously establish that certain sparse arrays can provably offer lower CRB than ULA in the MD-SNR regimes, and the gap between their performance increases with the number of sensors. This result is particularly significant as existing research on CRB of source separation has not adequately shed light on such a low SNR regime. Our analysis is non-asymptotic and provides an exact rate for any M.

II. Problem Formulation and Review of Past Work

A. Measurement Model

Consider L i.i.d temporal snapshots received at a linear (sparse) array of M antennas, from $K = 2$ narrowband far-field sources (with wavelength λ) whose direction- of-arrivals (DOAs) are given by $\boldsymbol{\theta} = \{\theta_0, \theta_0 + \Delta\}$

$$\mathbf{y}_l = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{s}_l + \mathbf{n}_l, \quad 1 \leqslant l \leqslant L$$

Here, $\mathbf{A}(\boldsymbol{\theta}) \in \mathbb{C}^{M\times 2}$ is the array manifold matrix given by

$$\mathbf{A}(\boldsymbol{\theta})_{m,1} = e^{j\pi d_m \sin\theta_0}, \mathbf{A}(\boldsymbol{\theta})_{m,2} = e^{j\pi d_m \sin(\theta_0+\Delta)},$$

with $\frac{\lambda d_m}{2}$ denoting the location of the m_{th} antenna for $d_m \in \mathcal{D} = \{d_1, d_2, \ldots, d_M\}$. The (time-varying) source amplitudes are given by the vector $\mathbf{s}_l \in \mathbb{C}^2$. We make the following statistical assumptions which are used to derive the stochastic or unconditional Cramér-Rao bound for sparse arrays [16], [21], [22], [23], [27]

- (A1) The L snapshots of source amplitudes $\boldsymbol{s}_l$ are i.i.d random vectors distributed as $\boldsymbol{s}_l \sim \mathcal{CN}(\mathbf{0}, p\mathbf{I}_2)$, where p is the power of the sources (assumed to be equal in this paper).
- (A2) The noise $\mathbf{n}_l$ is assumed to be a spatially and temporally white Gaussian process, uncorrelated from the source signals, i.e., $\mathbf{n}_t \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I})$ and $\mathbb{E}(\mathbf{n}_l\mathbf{s}_l^H) = 0$.

We introduce the following notations

$$\begin{cases} w = \pi\sin(\theta_0) \\ \delta = \pi\left(\sin(\theta_0+\Delta) - \sin(\theta_0)\right) \end{cases} \tag{1}$$

where δ represents the separation in spatial frequencies between the sources. Under (A1–A2), the measurements $\{\mathbf{y}_l\}_{l=1}^{L}$ are i.i.d random vectors distributed as

$$\mathbf{y}_l \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{R}(\delta)\right), \quad \mathbf{R}(\delta) = p\mathbf{A}(\boldsymbol{\theta})\mathbf{A}^H(\boldsymbol{\theta}) + \sigma_n^2\mathbf{I} \tag{2}$$

Since our goal in this paper is to understand the resolution benefits of sparse arrays, we focus on the CRB of the separation parameter δ. In order to derive exact *non-asymptotic* and tractable CRB expressions which can help us gain a clear understanding of the behavior of CRB(δ), we make the following additional assumption

- (A3) The parameters p, σ_n, θ_0 are assumed to be known, and the separation δ is the only unknown parameter of interest.

As we will show, indeed it is possible to derive novel and insightful conclusions about CRB(δ) even if we assume p, σ_n, θ_0 to be known. Moreover, our results can serve as a benchmark for comparing CRB expressions (and how they may increase) when p, σ_n, θ_0 are unknown.

B. Review CRB of Nested and ULA in Measurement-Dependent SNR Regimes

In [26], we aimed to understand the super-resolution benefits of nested arrays over ULA by zooming into certain "low SNR regimes", by analytically comparing their CRB under assumptions (A1)-(A3). This was achieved by letting p/σ_n^2 (SNR per antenna) vary with the number of sensors M:

$$\frac{p}{\sigma_n^2} \times M^{\alpha} = c \qquad \text{(MD-SNR)}, \tag{3}$$

where c and α are given positive constants. The setting $\alpha = 1$ denotes the practical scenario (such as massive MIMO systems), where the total power across all antennas is constant [24], [25]. Notice that with this parameterization, as M increases, we are able to zoom into progressively smaller SNR regimes. In particular, under the MD-SNR regime, SNR $= c/M^{\alpha}$, for settings where c is a constant, the SNR remains below 0 dB. This formulation therefore enables us to analyze system performance in very low SNR settings, controlled by M.

We considered the following specific arrays

- ULA: $\mathcal{D}_{\text{ULA}} = \{1, 2, \ldots, M\}$
- (Generalized) Nested array:

  $$\mathcal{D}_{\text{Nest}} = \{1, 2, \ldots, N_1, N_1{+}1, 2(N_1{+}1), \ldots, N_2(N_1{+}1)\}$$

  where $N_1 = \mu_{\text{Nest}} M$ for some $0 < \mu_{\text{Nest}} < 1$ and $N_1 + N_2 = M$.

For $c = 1$, we demonstrated that for sufficiently large M, there exist constants c_1 and c_2 such that

$$\frac{\text{CRB}_{\text{Nest}}(\delta)}{\text{CRB}_{ULA}(\delta)} \leqslant \begin{cases} c_1 M^{-2\alpha}, & 0 < \alpha < 1 \\ c_2 M^{-2}, & \alpha \geqslant 1 \end{cases}$$

We aim to generalize this result by addressing the following question: *Is it possible to obtain "matching" lower and upper bounds on CRB (δ) as a function of M, for an arbitrary sparse array in the aforementioned low SNR regime?* Such matching bounds (if available) will enable us to compare any two (arbitrary) sparse geometries and accurately characterize their performance gaps. In the following section, we derive new results that answer the above question in the affirmative.

III. Main Result: Exact Scaling of CRB in Low SNR Regimes for Sparse Arrays

Under assumptions (A1-A3), the unconditional Fisher Information $\text{F}(\delta)$ is given by [28], [29]

$$\text{F}(\delta) = L\left[\text{vec}^H\left(\frac{\partial\mathbf{R}(\delta)}{\partial\delta}\right)\mathbf{W}(\delta)\text{vec}\left(\frac{\partial\mathbf{R}(\delta)}{\partial\delta}\right)\right], \tag{4}$$

where $\mathbf{W}(\delta) = \left(\mathbf{R}(\delta)^{-1}\right)^T \otimes \mathbf{R}(\delta)^{-1}$. Denoting $\mathbf{v}_{\mathcal{D}} = \text{vec}(\partial\mathbf{R}/\partial\delta)$, the CRB of δ for a given array $\mathcal{D}$ is given by [21], [26]

$$\mathrm{CRB}_{\mathcal{D}}(\delta)=\frac{1}{F(\delta)}=\frac{1}{L\mathbf{v}_{\mathcal{D}}^{H}\mathbf{W}(\delta)\mathbf{v}_{\mathcal{D}}} \tag{5}$$

We also consider a coprime array with elements given by $\mathcal{D}_{\mathrm{CP}}$:

$$\mathcal{D}_{\mathrm{CP}}=\{N_1,2N_1,\ldots,2N_2N_1\}\bigcup\{N_2,2N_2,\ldots,N_1N_2\}$$

where $N_2=\mu_{\mathrm{cp}}M$ for some $0<\mu_{\mathrm{cp}}<1$ and $N_1+2N_2-1=M$, (N_1,N_2) are coprime integers. The following theorem provides an exact characterization of $\mathrm{CRB}(\delta)$ in the MD-SNR regime.

Theorem 1: Consider an (arbitrary) array geometry $\mathcal{D}$ with M antennas, and two narrowband far-field sources with angular separation δ. Under assumptions (A1-A3), the unconditional CRB for estimating δ scales in the following manner in the MD-SNR regime defined in (3) for every constant $\mathbf{c}>0$, and $\alpha\geqslant 1$,

$$\mathrm{CRB}_{\mathcal{D}}(\delta)=\Theta\left(L^{-1}\sigma_n^4\|\mathbf{v}_{\mathcal{D}}\|_2^{-2}\right) \tag{6}$$

Specifically, the CRBs for ULA, nested and coprime arrays satisfy

$$\begin{aligned}\mathrm{CRB}_{\mathrm{ULA}}&=\Theta\left(L^{-1}M^{2(\alpha-1)}(M^2-1)^{-1}\right)\\ \mathrm{CRB}_{\mathrm{Nest}}&=\Theta\left(L^{-1}M^{2\alpha}G_{\mathrm{Nest}}(\mu_{\mathrm{Nest}},M)^{-1}\right)\\ \mathrm{CRB}_{\mathrm{CP}}&=\Theta\left(L^{-1}M^{2\alpha}G_{\mathrm{CP}}(\mu_{\mathrm{CP}},M)^{-1}\right)\end{aligned} \tag{7}$$

where

$$\begin{aligned}&G_{\mathrm{Nest}}(\mu_{\mathrm{Nest}},M)\\ &=\gamma_1M^6+\gamma_2M^5+\gamma_3M^4+\gamma_4M^3-M^2/6\\ &\begin{cases}\gamma_1=-\mu_{\mathrm{Nest}}^6/2+4\mu_{\mathrm{Nest}}^5/3-\mu_{\mathrm{Nest}}^4+\mu_{\mathrm{Nest}}^2/6\\ \gamma_2=-\mu_{\mathrm{Nest}}^5+8\mu_{\mathrm{Nest}}^4/3-2\mu_{\mathrm{Nest}}^3+\mu_{\mathrm{Nest}}/3\\ \gamma_3=-\mu_{\mathrm{Nest}}^4/2+5\mu_{\mathrm{Nest}}^3/3-7\mu_{\mathrm{Nest}}^2/6+1/6\\ \gamma_4=\mu_{\mathrm{Nest}}^2/3-\mu_{\mathrm{Nest}}/3\end{cases}\end{aligned} \tag{8}$$

and

$$\begin{aligned}&G_{\mathrm{CP}}(\mu_{\mathrm{CP}},M)\\ &=\alpha_1M^6+\alpha_2M^5+\alpha_3M^4+\alpha_4M^3+\alpha_5M^2\\ &\begin{cases}\alpha_1=-8\mu_{\mathrm{CP}}^6+16\mu_{\mathrm{CP}}^5-\frac{28}{3}\mu_{\mathrm{CP}}^4+\frac{4}{3}\mu_{\mathrm{CP}}^3+\frac{1}{6}\mu_{\mathrm{CP}}^2\\ \alpha_2=-8\mu_{\mathrm{CP}}^5+\frac{32}{3}\mu_{\mathrm{CP}}^4-\frac{22}{3}\mu_{\mathrm{CP}}^3+2\mu_{\mathrm{CP}}^2\\ \alpha_3=6\mu_{\mathrm{CP}}^4-\frac{8}{3}\mu_{\mathrm{CP}}^3-\frac{5}{6}\mu_{\mathrm{CP}}^2+\frac{2}{3}\mu_{\mathrm{CP}}\\ \alpha_4=4\mu_{\mathrm{CP}}^3-\frac{14}{3}\mu_{\mathrm{CP}}^2+\frac{4}{3}\mu_{\mathrm{CP}}\\ \alpha_5=-2\mu_{\mathrm{CP}}^2+\frac{2}{3}\mu_{\mathrm{CP}}\end{cases}\end{aligned} \tag{9}$$

Proof: We consider the following lower and upper bounds for the CRB,

$$\frac{1}{\|\mathbf{v}_{\mathcal{D}}\|_2^2\sigma_{\max}(\mathbf{W}(\delta))}\leqslant\frac{1}{\mathbf{v}_{\mathcal{D}}^{H}\mathbf{W}(\delta)\mathbf{v}_{\mathcal{D}}}\leqslant\frac{1}{\|\mathbf{v}_{\mathcal{D}}\|_2^2\sigma_{\min}(\mathbf{W}(\delta))} \tag{10}$$

Here $\sigma_{\max}(\mathbf{W}(\delta))$ and $\sigma_{\min}(\mathbf{W}(\delta))$ are the largest and smallest eigenvalue of $\mathbf{W}(\delta)$ respectively.[1] Let $\sigma_M\leqslant\sigma_{M-1}\leqslant\cdots\leqslant\sigma_1$ denote the eigenvalues of the covariance matrix $\mathbf{R}(\delta)$. It is straightforward to show that

$$\begin{aligned}\sigma_M&=\sigma_{M-1}=\cdots=\sigma_3=\sigma_n^2,\\ \sigma_1&=\sigma_n^2+p\widetilde{\sigma}_1,\sigma_2=\sigma_n^2+p\widetilde{\sigma}_2\end{aligned} \tag{11}$$

where

$$\widetilde{\sigma}_2=M-\phi_{\mathcal{D}}(\delta,M),\widetilde{\sigma}_1=M+\phi_{\mathcal{D}}(\delta,M)$$

$$\phi_{\mathcal{D}}(\delta,M)=\left|\sum_{i=1}^{M}e^{jd_i\delta}\right|,\quad d_i\in\mathcal{D}$$

Due to the properties of the Kronecker product we have,

$$\sigma_{\max}(\mathbf{W}(\delta))=\sigma_{\max}^2(\mathbf{R}(\delta)^{-1}),\ \sigma_{\min}(\mathbf{W}(\delta))=\sigma_{\min}^2(\mathbf{R}(\delta)^{-1})$$

Thus, we have

$$\begin{aligned}\sigma_{\max}(\mathbf{W}(\delta))&=\sigma_n^{-4}\\ \sigma_{\min}(\mathbf{W}(\delta))&=(\sigma_n^2+p\widetilde{\sigma}_1)^{-2}\end{aligned} \tag{12}$$

Note that for any arbitrary array geometry, $\mathcal{D}$, we have

$$0\leqslant\phi_{\mathcal{D}}(\delta,M)\leqslant\sum_{i=1}^{M}\left|e^{jd_i\delta}\right|=M,\quad\delta\neq0$$

Thus $M\leqslant\widetilde{\sigma}_1\leqslant 2M$. By replacing eigenvalues from (12) in (10), we get the following bounds on $\mathrm{CRB}_{\mathcal{D}}(\delta)$

$$L^{-1}\|\mathbf{v}_{\mathcal{D}}\|_2^{-2}\sigma_n^4\leqslant\mathrm{CRB}_{\mathcal{D}}(\delta)\leqslant \tag{13}$$

$$(1+2pM/\sigma_n^2)^2L^{-1}\sigma_n^4\|\mathbf{v}_{\mathcal{D}}\|_2^{-2} \tag{14}$$

In the MD-SNR regimes, $pM^{\alpha}/\sigma_n^2=c$ is assumed to be constant, which implies $pM/\sigma_n^2\leqslant c$, for $\alpha\geqslant1$. Thus,

$$L^{-1}\sigma_n^4\|\mathbf{v}_{\mathcal{D}}\|_2^{-2}\leqslant\mathrm{CRB}_{\mathcal{D}}(\delta)\leqslant CL^{-1}\sigma_n^4\|\mathbf{v}_{\mathcal{D}}\|_2^{-2} \tag{15}$$

where $C=(1+2c)^2$ is a constant. Hence

$$\mathrm{CRB}_{D}(\delta)=\Theta\left(L^{-1}\sigma_n^4\|\mathbf{v}_{\mathcal{D}}\|_2^{-2}\right) \tag{16}$$

Now, we derive $\|\mathbf{v}_{\mathcal{D}}\|_2$ for ULA, nested, and coprime array. For a ULA, it is straightforward to show that [26],

$$\|\mathbf{v}_{\mathrm{ULA}}\|_2^2=2p^2\sum_{i=1}^{M}i^2(M-i)=\frac{p^2}{6}M^2(M^2-1) \tag{17}$$

For the nested array, we have

$$\|\mathbf{v}_{\mathrm{Nest}}\|_2^2=2p^2\left[(N_1+1)^2\sum_{i=1}^{N_2-2}i^2(N_2-i-1)+\sum_{i=1}^{N_2(N_1+1)-1}i^2+\sum_{i=1}^{N_1}i^2(N_1-i)\right]$$

[1] Note that $\mathbf{W}(\delta)$ is full-rank.

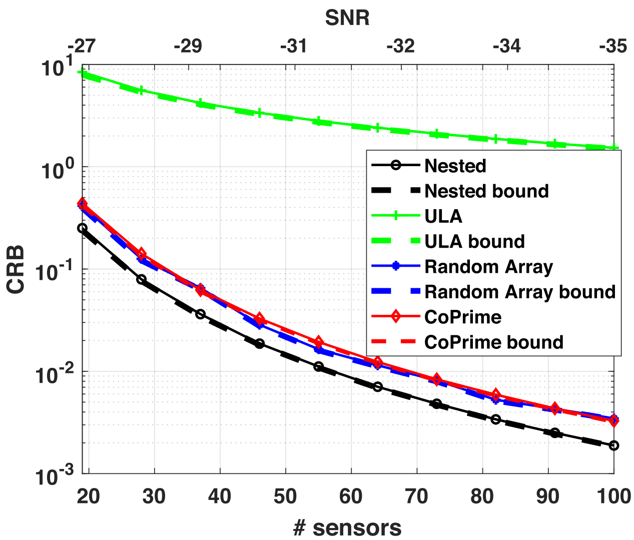

Fig. 1. CRB(δ) as a function of Number of sensors for ULA and nested, coprime array, Random array, overlaid with bounds on CRB(δ) derived in Theorem 1 for each array geometry. $M = \{19, 28, 37, \ldots, 100\}, \delta = \pi/10, w = \pi/3, \mu_{\text{Nest}} = 0.5, \mu_{cp} = 0.34$, Location of M sensors in the random array are drawn from a uniform distribution: $\mathcal{U}\left[1, \max\left(\max\left(\mathcal{D}_{\text{CP}}\right), \max\left(\mathcal{D}_{\text{Nest}}\right)\right)\right]$.

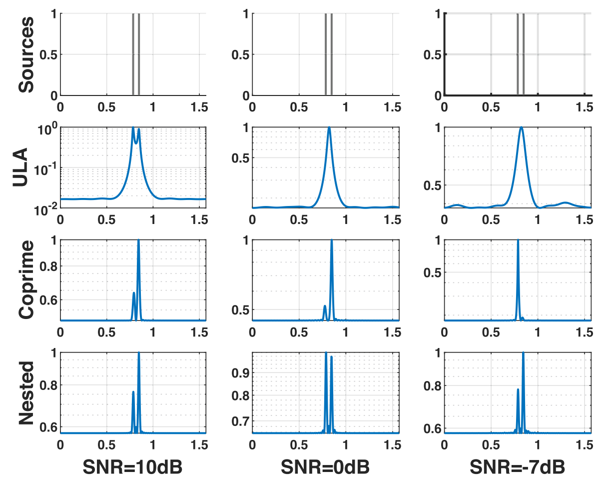

Fig. 2. DOA estimation using MUSIC algorithm on a ULA (second-row), a coprime array (third-row), and nested array (forth-row). SNR varies column-wise with values $\{10\text{ dB}, 0\text{ dB}, -7\text{ dB}\}$. $\delta = \pi/50, M = 32, L = 10, \mu_{\text{CP}} = 0.28, \mu_{\text{Nest}} = 0.5$. In all plots, x-axis shows source powers and y-axis shows source locations $\{w, w + \delta\}$.

By replacing $N_1 = \mu_{\text{Nest}} M$, $N_2 = (1 - \mu_{\text{Nest}})M$, in the above expression can be written as follows,

$$\|\mathbf{v}_{\text{Nest}}\|_2^2 = p^2 G_{\text{Nest}}(\mu_{\text{Nest}}, M) \tag{18}$$

where G_{Nest} is defined in (8). Similarly, for the coprime array, we have

$$\|\mathbf{v}_{\text{CP}}\|_2^2 = p^2 G_{\text{CP}}(\mu_{\text{CP}}, M) \tag{19}$$

where G_{CP} is defined in (9). By using (17)–(19), (3) in (6) we obtain (7). ∎

Remark In the MD-SNR regime, when $\alpha = 1$, CRB_{ULA} decays as M^{-2}, whereas CRB_{Nest} and CRB_{CP} decay as M^{-4}, since both $G_{\text{CP}}, G_{\text{Nest}}$ grows as M^6. Thus the ratio of CRBs of ULA and nested array is given by

$$\frac{\text{CRB}_{\text{Nest}}}{\text{CRB}_{\text{ULA}}} = \Theta\left(\frac{M^2(M^2 - 1)}{G_{\text{Nest}}(\mu_{\text{Nest}}, M)}\right) = \Theta(M^{-2})$$

In [26], we only derived an upper bound on this ratio. The above result shows that the upper bound is indeed tight in the MD-SNR regime. Our results provide an exact non-asymptotic characterization of CRB as a function of M and establish that sparse arrays can offer enhanced super-resolution error compared to a uniform array as we zoom into low SNR regimes characterized by MD-SNR.

IV. Numerical Experiments

We empirically validate our theoretical bounds by comparing CRB(δ) for the three aforementioned array geometries in the MD-SNR regime. In addition, we show the tightness of the derived bound in Theorem 1 for different sensor array geometries.

Fig. 1 shows the CRB(δ) of a ULA, nested array, coprime array, and Random array (See Fig. 1 for details) overlaid with the theoretical scaling given by (7) in the MD-SNR regime with $\alpha = 1.5, c = 0.2$. In this regime, we vary the number of sensors from $M = 19$ to $M = 100$, thus the SNR decreases from -26.5 dB to -35 dB. As can be seen, the bound is tight for every array pattern. Moreover, the nested array consistently demonstrates the lowest CRB across all values of M, and the gap between the CRB of the ULA and those of the sparse arrays, such as the nested and coprime arrays, widens as the number of sensors increases. This trend can be attributed to the significantly larger total aperture as well as geometry of nested and coprime arrays compared to that of ULA. As M increases, the disparity in the total aperture size between the sparse arrays and the ULA becomes larger, further enhancing the advantage of the sparse arrays.

We also compare the performance of MUSIC algorithm for DOA estimation applied on ULA, nested, and coprime arrays at different values of the SNR. In Fig. 2, we plot the MUSIC spectrum for $K = 2$ closely spaced sources with $\delta = \pi/50$ using a total of $M = 32$ physical sensors, $\mu_{\text{CP}} = 0.28, \mu_{\text{Nest}} = 0.5$, and $L = 10$ snapshots, for different SNR regimes: $\{10\text{ dB}, 0\text{ dB}, -7\text{ dB}\}$. In high SNR regimes, all three arrays can resolve the two sources. However, the performance of ULA degrades earlier than nested and coprime arrays as SNR decreases.

V. Conclusion

We derived an exact characterization for the CRB of source separation in a measurement-dependent SNR regime that allows us to progressively zoom into low SNR regimes as a function of the number of sensors. Our analysis demonstrates that in the MD-SNR regime, sparse arrays can provably achieve lower CRB than ULAs, with the performance gap expanding as the number of sensors increases. Moreover, the derived error bound offers valuable insights into the behavior of sparse arrays in resource-starved scenarios (such as low SNR regimes) that are relevant for many emerging applications.

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