

q -Rational and q -Real Binomial Coefficients

John Machacek^a, Nicholas Ovenhouse^b

^aUniversity of Oregon, Eugene, OR, USA

^bYale University, New Haven, CT, USA

Abstract

We consider q -binomial coefficients built from the q -rational and q -real numbers defined by Morier-Genoud and Ovsienko in terms of continued fractions. We establish versions of both the q -Pascal identity and the q -binomial theorem in this setting. These results are then used to find more identities satisfied by the q -analogues of Morier-Genoud and Ovsienko, including a Chu–Vandermonde identity and q -Gamma function identities.

Keywords: q -analog, q -binomial coefficient, q -identities

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1. Introduction

The classical “ q -integer” is the polynomial $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Since this is also equal to $\frac{1-q^n}{1-q}$, it is also common to define the q -analogue of a non-integer number α by the expression $\frac{1-q^\alpha}{1-q}$. Recently, Morier-Genoud and Ovsienko have defined a different q -analogue of rational [10] and real numbers [12] by using q -deformations of their continued fraction expressions. For $\alpha \in \mathbb{R}$, we will use $[\alpha]_q$ to denote the

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q -analogue from [10, 12]. In the case of $\alpha \in \mathbb{Q}$, this definition produces a rational function of q with integer coefficients, unlike the more common q -number $\frac{1-q^\alpha}{1-q}$. More generally, for $\alpha \in \mathbb{R}$ the q -analogue is a Laurent series in q with integer coefficients.

The main goal of this paper is to demonstrate that many definitions involving the q -numbers $\frac{1-q^\alpha}{1-q}$ have alternative versions which use the Morier-Genoud–Ovsienko q -analogs $[\alpha]_q$, and that many of the results and identities involving $\frac{1-q^\alpha}{1-q}$ are still true for the versions which use $[\alpha]_q$. Our main example will be to generalize the q -binomial coefficients $\binom{n}{k}_q$ to the case when n is not an integer, and to see that many of the expected properties and identities hold in this new case.

The binomial coefficients can be written as $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, and this of course makes sense even when n is not an integer. Likewise, we can define q -analogues

$$\binom{\alpha}{k}_q := \frac{[\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q}{[k]_q!}$$

which will be Laurent series in q (and in fact a rational function if $\alpha \in \mathbb{Q}$). Again, we stress that this sort of definition has been made before (see e.g. [5]) using the expression $\frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$, but the latter is not a rational function nor even a Laurent series in q . This latter version of q -binomial coefficient satisfies many nice properties. They satisfy q -Pascal identities and q -analogues of the binomial theorem. We will show that our $\binom{\alpha}{k}_q$, using the q -numbers $[\alpha]_q$, satisfy all of the same nice identities.

The main idea which permits the generalizations mentioned above is to replace the expression q^α with some expression which is a Laurent series in q (or a rational function for $\alpha \in \mathbb{Q}$), but which retains many of the nice properties which make $\frac{1-q^\alpha}{1-q}$ a good q -analogue of the number α . Note that when n is an integer, then $[n+1]_q - [n]_q = q^n$. We suggest that for real numbers α , the function $\{\alpha\}_q := [\alpha+1]_q - [\alpha]_q$ is the appropriate substitute for q^α . The first observation which indicates that this is a good idea is the identity $[\alpha]_q = \frac{1-\{\alpha\}_q}{1-q}$. This observation, though very simple, seems to have many nice and interesting consequences, allowing one to use $\{\alpha\}_q$ in place of q^α in many existing q -analogues. As mentioned above, we will focus in this paper on the q -binomial coefficients $\binom{\alpha}{k}_q$, and see that many identities involving them hold in this new setting after replacing q^α by $\{\alpha\}_q$.

2. q -Rational and q -Real Numbers

We will recall the definition of q -deformed rational numbers from [10].

Definition 2.1. If $\alpha > 1$ has continued fraction expansion

$$\alpha = [a_1, a_2, \dots, a_{2m}] = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2m}}}}$$

then we define the q -analogue of α as the rational function $[\alpha]_q$ given by the q -deformed continued fraction:

$$[\alpha]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_{q^{-1}} + \frac{q^{a_2}}{[a_3]_q + \frac{q^{a_3}}{[a_4]_{q^{-1}} + \frac{q^{a_4}}{\ddots + \frac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}$$

Example 2.2. The continued fraction for $\alpha = \frac{52}{23}$ is $[2, 3, 1, 5]$, so its q -analogue is

$$\left[\frac{52}{23} \right]_q = [2]_q + \frac{q^2}{[3]_{q^{-1}} + \frac{q}{[5]_{q^{-1}}}} = \frac{1 + 3q + 5q^2 + 7q^3 + 8q^4 + 8q^5 + 7q^6 + 6q^7 + 4q^8 + 2q^9 + q^{10}}{1 + 2q + 3q^2 + 4q^3 + 4q^4 + 3q^5 + 3q^6 + 2q^7 + q^8}$$

In [12] the authors obtain a formal series q -analog $[\alpha]_q$ for any positive real number α by first taking a sequence of rational numbers $(\alpha_j)_{j \geq 0}$ converging to α . Next the formal power series $[\alpha]_q$ is defined to be the limit of $([\alpha_j]_q)_{j \geq 0}$ (which is shown to be well-defined).

Theorem 2.3 ([12], Theorem 1). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and let $\alpha_1, \alpha_2, \alpha_3, \dots$ be a sequence of rationals with $\lim_{n \rightarrow \infty} \alpha_n = \alpha$. Then the limit $\lim_{n \rightarrow \infty} [\alpha_n]_q$ exists as a formal Laurent series with integer coefficients (i.e. each coefficient eventually stabilizes).*

Definition 2.4. For irrational $\alpha \in \mathbb{R}$, the q -analogue $[\alpha]_q \in \mathbb{Z}((q))$ is defined as the Laurent series guaranteed by Theorem 2.3. That is, for any sequence of rationals $\alpha_n \rightarrow \alpha$, if $[\alpha_n]_q = \sum_k \varkappa_k^{(n)} q^k$, then define

$$[\alpha]_q := \sum_k \varkappa_k q^k, \quad \text{where } \varkappa_k := \lim_{n \rightarrow \infty} \varkappa_k^{(n)}$$

Example 2.5. The continued fraction expansion of $\pi/2$ begins with $[1, 1, 1, 3, 31, 1, 145, \dots]$. We can therefore approximate it by the sequence of rationals

$$\frac{3}{2}, \frac{11}{7}, \frac{344}{219}, \frac{355}{226}, \dots$$

The series expansions of these q -rationals are given by

$$\begin{aligned} \left[\frac{3}{2} \right]_q &= 1 + q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + q^8 - q^9 + \dots \\ \left[\frac{11}{7} \right]_q &= 1 + q^2 - q^3 + q^4 - 2q^6 + 4q^7 - 5q^8 + 4q^9 - 7q^{11} + \dots \\ \left[\frac{344}{219} \right]_q &= 1 + q^2 - q^3 + q^4 - q^5 + 2q^6 - 3q^7 + 3q^8 - 4q^9 + 6q^{10} - 7q^{11} + \dots \end{aligned}$$

The series $\left[\frac{344}{219} \right]_q$ already agrees with $\left[\frac{\pi}{2} \right]_q$ to much higher order than what is shown above.

Proposition 2.6 ([10, 12]). *For a real number $\alpha > 1$, we have*

$$[\alpha + 1]_q = 1 + q[\alpha]_q$$

Proposition 2.6 follows easily from Definition 2.1, and can be used to define the q -analog for $\alpha \leq 1$. That is, we simply define $[\alpha - 1]_q := \frac{1}{q}([\alpha]_q - 1)$. We then obtain the following corollary by iteration, which shows that q -rational numbers need not always be power series in q , but can in fact be Laurent series.

Corollary 2.7. *For a real number $\alpha \in \mathbb{R}$, and a positive integer n , we have*

$$[\alpha + n]_q = [n]_q + q^n [\alpha]_q$$

$$[\alpha - n]_q = \frac{1}{q^n} ([\alpha]_q - [n]_q)$$

For integers, the q -analogue $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$ is equal to $\frac{1-q^n}{1-q}$, and for this reason it is common to take the q -analogue of an arbitrary real number α to be the expression $\frac{1-q^\alpha}{1-q}$. The q -rational numbers from Definition 2.1 are quite different from this expression, but they have many interesting properties and connections to other areas of mathematics, some of which we will briefly mention now.

The numerators and denominators of q -rationals have interesting combinatorial interpretations as certain generating functions. Let $[a_1, \dots, a_{2m}] = \frac{r}{s}$, with q -version given by the rational function $[\frac{r}{s}]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$. There is a certain poset F (called a *fence*), where the numbers $a_1, \dots, a_{2m}$ determine the number of up and down steps in the Hasse diagram. Then the polynomial $\mathcal{R}(q)$ is the rank generating function of the lattice $J(F)$, of order ideals of F . That is,

$$\mathcal{R}(q) = \sum_{I \in J(F)} q^{|I|}$$

A version of this statement appears in [10] (Theorem 4). These polynomials were also studied from this fence poset perspective in [9]. A different, yet equivalent, perspective is the following. There is a *border strip* skew Young diagram λ/μ , whose shape is determined by the numbers $a_1, \dots, a_{2m}$. It is obtained by rotating the Hasse diagram of the fence poset F by 45° counter-clockwise, and then replacing each vertex with a box. Then there is a simple bijection between order ideals of F and north-east lattice paths on λ/μ , where the order ideal is the set of boxes underneath the lattice path. If we let $L(\lambda/\mu)$ be the set of north-east lattice paths on the shape λ/μ , and for a path p , let $|p|$ be the number of boxes under the path, the statement above can be re-phrased as

$$\mathcal{R}(q) = \sum_{p \in L(\lambda/\mu)} q^{|p|}$$

This interpretation, in terms of lattice paths and skew Young diagrams, was used in [13] to give another combinatorial meaning of the polynomials $\mathcal{R}(q)$ as counting the sizes of certain varieties over the finite field with q elements. In particular, $q^{|\mu|} \mathcal{R}(q)$ is the number of $\mathbb{F}_q$ -points in a union of Schubert cells in some Grassmannian, where the union is over the Schubert cells indexed by partitions ν with $\mu \leq \nu \leq \lambda$.

The q -rationals also have a significant connection to cluster algebras. Cluster algebras of “type A” can be realized as the homogeneous coordinate rings of Grassmannians $\text{Gr}_2(n)$, and also as the ring of functions on Penner’s *decorated Teichmüller space* of an ideal polygon in the hyperbolic plane. In both cases, the cluster combinatorics are determined by triangulations of a polygon. Any triangulation of a polygon, and a choice of a diagonal not in that triangulation, determines a continued fraction $[a_1, \dots, a_{2m}]$. It was explained in [10] (appendix B.2) that $\mathcal{R}(q)$ is a specialization of the F -polynomial of the corresponding cluster variable.

A nice exposition of some of the combinatorial formulas mentioned above, and the connection with cluster algebras, is given in [3]. The q -rationals are also related to Jones polynomials of certain knots. This is explained in [10] (appendix A). This is related to work of Lee and Schiffler [8], who showed the Jones polynomials of certain knots have a cluster algebra interpretation.

Some instances where q -reals are used include the deformation of modular groups in [6] and also of Conway–Coxeter friezes in [11]. The q -real numbers are (Laurent) series, and their convergence properties have been studied in [7]. Also in [6], it was shown that the q -deformations of quadratic irrationals have a particularly nice closed-form algebraic expression.

3. A Substitute for q^α

It is our hope that we might convince the reader not only that q -rationals and q -reals are interesting for the reasons mentioned in the previous section, but also that they satisfy many nice identities analogous to those satisfied by the more commonly used expressions $\frac{1-q^\alpha}{1-q}$. Many identities involving these q -analogues will inevitably involve the expression q^α , which is certainly not a rational function. In this section, we define a Laurent series denoted $\{\alpha\}_q$ (which is a rational function for $\alpha \in \mathbb{Q}$) which we suggest is a good replacement for q^α . For the remainder of the paper, we will demonstrate many examples of identities satisfied by the q -rational and q -real numbers which resemble well-known identities, but with q^α replaced by $\{\alpha\}_q$.

We remind the reader that we will always work in the ring of formal power series $\mathbb{Z}[[q]]$ or Laurent series $\mathbb{Z}((q))$. In particular, all limits will be understood in the sense of power series (i.e. each coefficient eventually stabilizes).

Note the simple fact that when n is an integer, $q^n = [n+1]_q - [n]_q$. We propose that this forward difference is the “correct” substitute for q^α when α is not an integer.

Definition 3.1. For $\alpha \in \mathbb{R}$, define $\{\alpha\}_q := [\alpha+1]_q - [\alpha]_q$.

Example 3.2. Here are some examples of $\{\alpha\}_q$ for rational values of α :

$$\begin{aligned} \left\{\frac{1}{2}\right\}_q &= \frac{1+q^2}{1+q}, & \left\{\frac{5}{3}\right\}_q &= q \frac{1+q^2+q^3}{1+q+q^2}, & \left\{\frac{25}{7}\right\}_q &= q^3 \frac{1+q+2q^2+q^3+q^4+q^5}{1+2q+2q^2+q^3+q^4}, \\ & & \left\{-\frac{12}{5}\right\}_q &= q^{-3} \frac{1+q+q^2+q^3+q^4}{1+2q+q^2+q^3} \end{aligned}$$

Note that in all these examples, there is a factor of $q^{\lfloor \alpha \rfloor}$. This is always the case, and this follows from Theorem 4.6 and Proposition 3.6(c) below.

Example 3.3. For any positive integer n , we have

$$\left\{\frac{1}{n}\right\}_q = \frac{1+q+q^2+\cdots+q^{n-2}+q^n}{1+q+q^2+\cdots+q^{n-1}} = 1 + (q-1) \frac{q^{n-1}}{[n]_q}$$

Expanded as a Laurent series, we have $\left\{\frac{1}{n}\right\}_q = 1 - q^{n-1} + 2q^n - \cdots$. In particular, we have $\lim_{n \rightarrow \infty} \left\{\frac{1}{n}\right\}_q = 1$.

Example 3.4. For an irrational example, we consider $\alpha = \sqrt{2} + 1 \approx 2.41421$. The q -analogue $[\sqrt{2}+1]_q$ was computed in [12] to be

$$[\sqrt{2}+1]_q = 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} - 55q^{13} + \cdots$$

Using Proposition 3.6(a) below, we can use this expression to compute $\{\sqrt{2}+1\}_q$:

$$\left\{\sqrt{2}+1\right\}_q = q^2 - q^4 + q^5 + 2q^6 - 3q^7 - 3q^8 + 9q^9 + 2q^{10} - 25q^{11} + 11q^{12} + 62q^{13} + \cdots$$

Note how, just as in Example 3.2, this can be written as $q^{\lfloor \alpha \rfloor}$ times a power series with constant term 1.

Remark 3.5. Although we focus in this paper on Morier-Genoud and Ovsienko's q -real numbers from [10, 12], the definition of $\{\alpha\}_q$ makes sense for any q -analogue, since it is defined simply by subtraction. As we show below in Proposition 3.6, all essential and important properties of $\{\alpha\}_q$ follow from the shift property $[\alpha + 1]_q = 1 + q[\alpha]_q$. So in principle, most results presented here would also hold for any other q -analogues which satisfy this shift property.

The following summarizes some properties of $\{\alpha\}_q$. We invite the reader to keep in mind the analogy of $\{\alpha\}_q$ with q^α , and to note that parts (c) and (d) correspond to basic rules of exponents $q^{\alpha+n} = q^\alpha q^n$ and $q^{-\alpha} = (q^{-1})^\alpha$. Furthermore, part (b) shows that the q -rationals $[\alpha]_q$ also have a form which resembles $\frac{1-q^\alpha}{1-q}$.

Proposition 3.6. For $\alpha \in \mathbb{R}$ and an integer n ,

$$\begin{aligned} (a) \quad & \{\alpha\}_q = 1 + (q-1)[\alpha]_q & (d) \quad & \text{If } \alpha \in \mathbb{Q}, \text{ then } \{-\alpha\}_q = \{\alpha\}_{q^{-1}} \\ (b) \quad & \frac{1 - \{\alpha\}_q}{1 - q} = [\alpha]_q & (e) \quad & \{\alpha\}_q = \frac{[\alpha + n]_q - [\alpha]_q}{[n]_q} \\ (c) \quad & \{\alpha + n\}_q = q^n \{\alpha\}_q \end{aligned}$$

Proof. Part (a) follows immediately from Proposition 2.6. Part (b) follows from (a) by algebraic manipulation.

For part (c), note that by a combination of part (a) and Proposition 2.6, we have

$$\begin{aligned} \{\alpha + 1\}_q &= 1 + (q-1)[\alpha + 1]_q & (\text{part (a)}) \\ &= 1 + (q-1)(1 + q[\alpha]_q) & (\text{Prop 2.6}) \\ &= q(1 + (q-1)[\alpha]_q) \\ &= q\{\alpha\}_q & (\text{part (a)}) \end{aligned}$$

The general case for $\{\alpha + n\}_q$ then follows by induction.

For part (d), we use the fact that $[-\alpha]_q = -q^{-1}[\alpha]_{q^{-1}}$ (Proposition 2.8 from [6]). Then we have

$$\begin{aligned} \{-\alpha\}_q &= [-(\alpha-1)]_q - [-\alpha]_q \\ &= q^{-1}([\alpha]_{q^{-1}} - [\alpha-1]_{q^{-1}}) & ([6], \text{Prop 2.8}) \\ &= q^{-1}([\alpha]_{q^{-1}} - q([\alpha]_{q^{-1}} - 1)) & (\text{Corollary 2.7}) \\ &= 1 + (q^{-1} - 1)[\alpha]_{q^{-1}} \\ &= \{\alpha\}_{q^{-1}} & (\text{part (a)}) \end{aligned}$$

Part (e) is a simple calculation using Corollary 2.7 and the fact that $[n]_q = \frac{1-q^n}{1-q}$. $\square$

Remark 3.7. By Corollary 2.7, we have $[\alpha + n]_q = [n]_q + q^n[\alpha]_q$, and part (e) of Proposition 3.6 is saying we also have $[\alpha + n]_q = [\alpha]_q + \{\alpha\}_q[n]_q$, so there is some symmetry in the roles of α and n . However, $[\alpha + \beta]_q \neq [\alpha]_q + \{\alpha\}_q[\beta]_q$ in general. It is only true if either α or β is an integer. For example, when $\alpha = \beta = 1/2$, we have $[1]_q = 1$, but on the other hand, $[\frac{1}{2}]_q + \{\frac{1}{2}\}_q[\frac{1}{2}]_q = q^{\frac{2+q+q^2}{1+2q+q^2}}$.

Remark 3.8. In part (c) of Proposition 3.6, it is important that n is an integer. The more general version of this statement is *not* true. That is, $\{\alpha + \beta\}_q \neq \{\alpha\}_q\{\beta\}_q$ in general. For example, when $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, we have $\{2\}_q = q^2$, but $\{\frac{1}{2}\}_q\{\frac{3}{2}\}_q = q^{\frac{1+2q^2+q^4}{1+2q+q^2}}$.

Remark 3.9. The reason that part (d) of Proposition 3.6 is stated only for rational α is the following. The proof of part (d) uses Proposition 2.8 from [6], which states that $\left[\frac{1}{x}\right]_q = \frac{1}{[x]_{q^{-1}}}$. However, as those authors point out immediately following the result, this equality no longer makes sense for irrational x , because $\left[\frac{1}{x}\right]_q$ is a series in q , while $\frac{1}{[x]_{q^{-1}}}$ is a series in q^{-1} . When $x \in \mathbb{Q}$, both can be thought of as rational functions in q , and thus it makes sense to compare them.

4. q -Rational and q -Real Binomial Coefficients

The q -integers $[n]_q = 1 + q + \dots + q^{n-1}$ are the building blocks of many other q -analogues. Perhaps the most well-known are the q -factorial $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q$ and the q -binomial coefficients $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Another very useful notation in the theory of q -analogues is the q -Pochhammer symbol

$$(x; q)_n := (1-x)(1-qx)(1-q^2x) \dots (1-q^{n-1}x)$$

$$(x; q)_\infty := \lim_{n \rightarrow \infty} (x; q)_n = \prod_{k=0}^{\infty} (1 - q^k x)$$

The q -factorial and q -binomial coefficients can both be expressed in terms of the q -Pochhammer symbol by

$$[n]_q! = \frac{(q; q)_n}{(1-q)^n} \quad \text{and} \quad \binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

We generalize the q -binomial coefficient to the case when $n = \alpha \in \mathbb{R}$ is not an integer by the following formula.

Definition 4.1. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$, we define the q -binomial coefficient $\binom{\alpha}{k}_q$ as follows. For special base cases, define $\binom{0}{0}_q := 1$, $\binom{\alpha}{0}_q = 1$ for $\alpha \neq 0$, and $\binom{0}{k}_q = 0$ for $k \neq 0$. Otherwise, define

$$\binom{\alpha}{k}_q := \frac{[\alpha]_q [\alpha-1]_q \dots [\alpha-k+1]_q}{[k]_q!}$$

where the factors in the numerator are q -rational or q -real numbers. Note that $\binom{\alpha}{k}_q$ is a Laurent series (with integer coefficients), which happens to be a rational function when $\alpha \in \mathbb{Q}$.

Example 4.2. $\binom{5/3}{3}_q = \frac{[5/3]_q [2/3]_q [-1/3]_q}{[3]_q [2]_q [1]_q} = \frac{-(1+q+2q^2+q^3)}{1+4q+10q^2+16q^3+19q^4+16q^5+10q^6+4q^7+q^8}$

These generalized q -binomial coefficients satisfy the usual q -Pascal identities, using $\{\alpha\}_q$ in place of q^α where appropriate.

Proposition 4.3. For a real number $\alpha \in \mathbb{R}$ and integer $k \geq 0$, we have

$$(a) \quad \binom{\alpha}{k}_q = q^k \binom{\alpha-1}{k}_q + \binom{\alpha-1}{k-1}_q$$

$$(b) \binom{\alpha}{k}_q = \binom{\alpha-1}{k}_q + \{\alpha-k\}_q \binom{\alpha-1}{k-1}_q$$

Proof. Thanks to Corollary 2.7, Proposition 3.6(e), and Remark 3.7, the proof is essentially the same as for the classical q -binomial coefficients (i.e. the case when α is an integer).

(a) Look at the right-hand side:

$$\begin{aligned} q^k \binom{\alpha-1}{k}_q + \binom{\alpha-1}{k-1}_q &= \frac{q^k [\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k]_q}{[k]_q!} + \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k+1]_q}{[k-1]_q!} \\ &= \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k+1]_q}{[k]_q!} (q^k [\alpha-k]_q + [k]_q) \end{aligned}$$

By Corollary 2.7, the expression in parentheses is equal to $[\alpha]_q$, and this gives the result.

(b) Again, look at the right-hand side and combine:

$$\begin{aligned} \binom{\alpha-1}{k}_q + \{\alpha-k\}_q \binom{\alpha-1}{k-1}_q &= \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k]_q}{[k]_q!} + \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k+1]_q \{\alpha-k\}_q}{[k-1]_q!} \\ &= \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k+1]_q}{[k]_q!} ([\alpha-k]_q + \{\alpha-k\}_q [k]_q) \end{aligned}$$

By Remark 3.7, the expression in parentheses is equal to $[\alpha]_q$. $\square$

Now we give several other formulas for $\binom{\alpha}{k}_q$, some of which will be useful later.

Proposition 4.4. *The q -binomial coefficients have the following alternate formulations: for $\alpha \in \mathbb{R}$,*

$$\begin{aligned} (a) \quad \binom{\alpha}{k}_q &= \prod_{i=0}^{k-1} \frac{[\alpha]_q - [i]_q}{[k]_q - [i]_q} & (d) \quad \binom{\alpha}{k}_q &= \frac{(\{\alpha\}_q; q^{-1})_k}{(q; q)_k} \\ (b) \quad \binom{\alpha}{k}_q &= \frac{1}{q^{\binom{k}{2}} [k]_q!} \prod_{i=0}^{k-1} ([\alpha]_q - [i]_q) & (e) \quad \binom{\alpha+k-1}{k}_q &= \frac{(\{\alpha\}_q; q)_k}{(q; q)_k} \\ (c) \quad \binom{\alpha}{k}_q &= \frac{(-\{\alpha\}_q)^k}{q^{\binom{k}{2}}} \cdot \frac{(\{\alpha\}_q^{-1}; q)_k}{(q; q)_k} \end{aligned}$$

Proof. Parts (a) and (b) follow directly from Corollary 2.7.

To verify (d), use Proposition 3.6, parts (b) and (c), to write

$$\binom{\alpha}{k}_q = \frac{(1 - \{\alpha\}_q)(1 - \{\alpha-1\}_q) \cdots (1 - \{\alpha-k+1\}_q)}{(q; q)_k} \quad (3.6(b))$$

$$= \frac{(1 - \{\alpha\}_q)(1 - q^{-1}\{\alpha\}_q) \cdots (1 - q^{-(k-1)}\{\alpha\}_q)}{(q; q)_k} \quad (3.6(c))$$

The numerator is equal to $(\{\alpha\}_q; q^{-1})_k$, and part (d) follows. Part (c) is easily obtained from (d) by some algebraic manipulations. Part (e) follows from a similar calculation as part (d), using Proposition 3.6(c). $\square$

Remark 4.5. Parts (c), (d), and (e) of Proposition 4.4 support the claim that $\{\alpha\}_q$ is a good substitute for q^α , as they resemble the corresponding formulas when $\alpha = n$ is an integer:

$$\binom{n}{k}_q = (-1)^k q^{kn - \binom{k}{2}} \frac{(q^{-n}; q)_k}{(q; q)_k} = \frac{(q^n; q^{-1})_k}{(q; q)_k} = \frac{(q^{n-k}; q)_k}{(q; q)_k}$$

For a (formal) Laurent series $f(q) = \sum_k c_k q^k$ we let $\text{ord}(f)$ denote the *order* of f which is the minimal value of k such that $c_k \neq 0$. The order is a valuation on the ring of Laurent series. We now give a rephrasing of a theorem from [12].

Theorem 4.6 ([12], Theorem 2 and [6], Proposition 2.4). *For $\alpha \in \mathbb{R}$, the order of the Laurent series $[\alpha]_q$ is given by*

$$\text{ord}([\alpha]_q) = \begin{cases} 0 & \text{if } \alpha \geq 1 \\ \infty & \text{if } \alpha = 0 \\ \lfloor \alpha \rfloor & \text{if } \alpha < 0 \\ \lfloor \frac{1}{\alpha} \rfloor & \text{if } 0 < \alpha < 1 \end{cases}$$

It will be useful to know the order of the Laurent series given by q -binomial coefficients to be able to ensure certain combinations and evaluations are well-defined. To reduce the number of cases, the following lemma does not consider ordinary integer q -binomial coefficients $\binom{n}{k}_q$ for a nonnegative integer n since these are always either a polynomial with nonzero constant term or identically zero (this latter case happens when $k > n$).

Lemma 4.7. *Consider $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Let $\beta = \alpha - \lfloor \alpha \rfloor$ and set $b = \text{ord}([\beta]_q)$. Assume that $N \in \mathbb{Z}$ with $N \leq \alpha < N + 1$ so that $\lfloor \alpha \rfloor = N$, then*

$$\text{ord}\left(\binom{\alpha}{k}_q\right) = \begin{cases} 0 & k \leq N \\ b - \binom{k-N}{2} & k > N \text{ and } N \geq 0 \\ Nk - \binom{k}{2} & N < 0. \end{cases}$$

Proof. Since $[k]_q!$ is a polynomial with constant term equal to 1 it follows that $\frac{1}{[k]_q!}$ is a power series with constant term equal to 1. Thus the order of $\binom{\alpha}{k}_q$ is completely determined by the order of the falling factorial $[\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q$ in the numerator. First consider the case that $N < 0$, then $\text{ord}([\alpha-j]_q) = N-j$ and

$$\text{ord}([\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q) = \sum_{j=0}^{k-1} (N-j) = Nk - \binom{k}{2}.$$

If $k \leq N$, then $N \geq 0$ and the order will be 0 as each term in the falling factorial will have order equal to 0 by Theorem 4.6. Last consider the case $N \geq 0$ with $k > N$, then

$$\begin{aligned} \text{ord}([\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q) &= \text{ord}([\alpha-N]_q) + \text{ord}([\alpha-N-1]_q \cdots [\alpha-k+1]_q) \\ &= \text{ord}([\beta]_q) + \text{ord}([\alpha-N-1]_q \cdots [\alpha-k+1]_q) \\ &= b - \sum_{j=1}^{k-N-1} j \\ &= b - \binom{k-N}{2} \end{aligned}$$

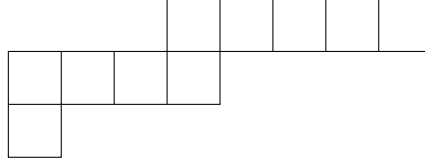


Figure 1: The snake graph G_α for $\alpha = \frac{52}{23}$.

and the lemma is proven. $\square$

5. Combinatorial Interpretation

In Section 2, we mentioned some combinatorial interpretations of q -rational numbers appearing in the literature. Using these, it is easy to give some combinatorial interpretations of the q -rational binomial coefficients. In this section, we will describe this combinatorial model, and give some examples.

Let $\alpha \in \mathbb{Q}$ with continued fraction $\alpha = [a_1, a_2, \dots, a_{2m}]$. Define a planar graph G_α built out of squares, where each square is attached to the previous either on the right or above. One can describe such a graph by a word in the alphabet $\{R, U\}$ by starting with an initial square, and then attaching one square either “right” or “up” according to the letters in the word, so that the number of squares in the graph is one more than the length of the word. Define G_α by the word $U^{a_1-1}R^{a_2}U^{a_3}R^{a_4} \dots U^{a_{2m-1}}R^{a_{2m}-1}$. That is, the number of consecutive “up” and “right” steps are given by the continued fraction coefficients (with the first and last being off-by-one). These graphs G_α are often called “snake graphs” in the literature (e.g. in [14] and [2]).

Example 5.1. Recall Example 2.2, where we considered $\frac{52}{23} = [2, 3, 1, 5]$. The corresponding graph has “up” and “right” steps in the sequence UR^3UR^4 . The graph $G_{52/23}$ is shown in figure 1.

As mentioned in Section 2, if $[\alpha]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ (with $\alpha > 1$), then the numerator $\mathcal{R}(q)$ is the generating function for an area statistic on north-east lattice paths in G_α . More specifically, if $\mathcal{L}(G_\alpha)$ is the set of north-east lattice paths in G_α from the bottom-left to the top-right corner, with $|p|$ denoting the area (i.e. the number of boxes) underneath the path p , then $\mathcal{R}(q) = \sum_{p \in \mathcal{L}(G_\alpha)} q^{|p|}$. This is an equivalent re-statement of Theorem 4 from [10], which was stated in the present form (in terms of the snake graph language) in [3] and [13]. Furthermore, the denominator $\mathcal{S}(q)$ has the same interpretation, but for the smaller snake graph obtained from G_α by removing the initial vertical column of boxes.

Definition 5.2. Let $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{Q}$ with $\alpha > k$, and snake graph G_α . Define $\mathcal{L}^{(k)}(G_\alpha)$ to be the set of k -tuples $(p_1, \dots, p_k)$ of north-east lattice paths in G_α , where each p_i is required to begin with at least $i - 1$ consecutive “up” steps. For $p = (p_1, \dots, p_k) \in \mathcal{L}^{(k)}(G_\alpha)$, let $|p| = \sum_{i=1}^k |p_i|$ denote the sum of the areas under the paths.

Theorem 5.3. Suppose $\alpha \in \mathbb{Q}$ with $\alpha > 1$, and let $0 \leq k < \alpha$ be an integer. If $[\alpha]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$, then

$$\binom{\alpha}{k}_q = \frac{\sum_{p \in \mathcal{L}^{(k)}(G_\alpha)} q^{|p| - \binom{k}{2}}}{[k]_q! \mathcal{S}(q)^k}$$

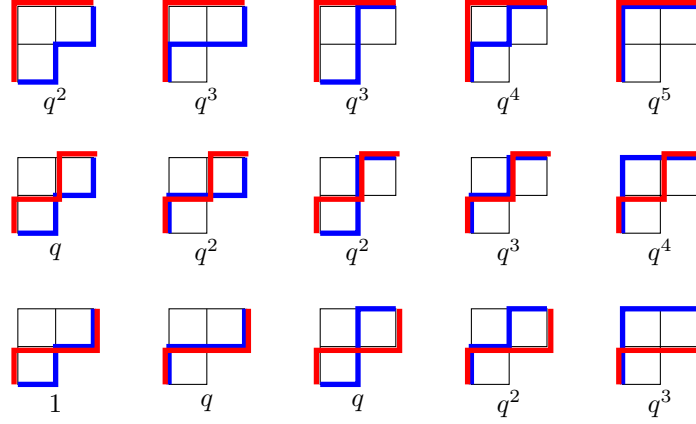


Figure 2: An illustration of $\mathcal{L}^{(k)}(G_\alpha)$ for $\alpha = \frac{5}{2}$ and $k = 2$. Below each picture is the monomial $q^{|p|-1}$, contributing to the sum from Theorem 5.3.

Proof. Let $[\alpha]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ as above. Note that for $i < \alpha$, $\alpha - i$ will have continued fraction $[a_1 - i, a_2, \dots, a_{2m}]$. Therefore the truncated snake graphs obtained by removing the first column of boxes will be the same for all $\alpha - i$ (for $0 \leq i < k$), and hence all the $[\alpha - i]_q$ will have the same denominator $\mathcal{S}(q)$. Let $\mathcal{R}_i(q)$ be their numerators, so that $[\alpha - i]_q = \frac{\mathcal{R}_i(q)}{\mathcal{S}(q)}$. Then the q -binomial coefficient will be given by

$$\binom{\alpha}{k}_q = \frac{\mathcal{R}(q)\mathcal{R}_1(q)\mathcal{R}_2(q)\cdots\mathcal{R}_{k-1}(q)}{\mathcal{S}(q)^k[k]_q!}$$

When $i < \alpha$, the graph $G_{\alpha-i}$ is obtained from G by simply removing the first i squares. There is a bijection between $\mathcal{L}(G_{\alpha-i})$ and the set of lattice paths in G_α which begin with at least $i - 1$ up steps (by identifying $G_{\alpha-i}$ with a subgraph of G_α). This bijection is not weight-preserving, but rather the weights differ by a factor q^{i-1} since the corresponding path in G_α goes over $i - 1$ extra boxes compared to its counterpart in $G_{\alpha-i}$. Adding these contributions for all $i = 1, 2, \dots, k$ gives the extra factor of $q^{-\binom{k}{2}}$. $\square$

Example 5.4. The continued fraction for $\frac{5}{2}$ is $[2, 2]$. The graph G_α therefore has 3 squares, with up-right sequence UR . Let $\mathcal{L}^{(2)}(G_{5/2})$ be the set of ordered pairs of paths (p_1, p_2) in $G_{5/2}$, where p_2 starts with an up step. The q -binomial coefficient $\binom{5/2}{2}_q$ is given by $\frac{1+3q+4q^2+4q^3+2q^4+q^5}{(1+q)^3}$. The numerator is the sum over $\mathcal{L}^{(2)}(G_{5/2})$ of $q^{|p_1|+|p_2|-1}$. The elements of $\mathcal{L}^{(2)}(G_{5/2})$ are shown in Figure 2, along with the corresponding monomials. In the figure, p_1 is in blue, and p_2 is in red.

6. The q -Binomial Theorem

Recall Newton's generalized binomial theorem: for any $\alpha \in \mathbb{R}$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{and} \quad \frac{1}{(1-x)^\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} x^k$$

When n is a positive integer, there are well-known q -versions:

$$\prod_{k=0}^{n-1} (1 + q^k x) = \sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k}_q x^k \quad \text{and} \quad \prod_{k=0}^{n-1} \frac{1}{1 - q^k x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q x^k$$

These can both be obtained as special cases of the general “ q -binomial theorem” (see e.g. [4, (1.3.2)]):

$$\frac{(ax; q)_{\infty}}{(x; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k$$

In particular, substituting $a = q^n$ gives one version, and substituting $x \mapsto -q^n x$ and $a = q^{-n}$ gives the other. Keeping with our basic philosophy, we will see that using $\{\alpha\}_q$ in place of q^n gives the appropriate q -analogue using $\binom{\alpha}{k}_q$.

Theorem 6.1. *For $\alpha \in \mathbb{R}$, we have*

$$(a) \quad \frac{(-x; q)_{\infty}}{(-\{\alpha\}_q x; q)_{\infty}} = \prod_{k=0}^{\infty} \frac{1 + q^k x}{1 + \{\alpha + k\}_q x} = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{\alpha}{k}_q x^k$$

$$(b) \quad \frac{(\{\alpha\}_q x; q)_{\infty}}{(x; q)_{\infty}} = \prod_{k=0}^{\infty} \frac{1 - \{\alpha + k\}_q x}{1 - q^k x} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k}_q x^k$$

Proof. Substitute $a = \{\alpha\}_q$ into the general q -binomial theorem to get

$$\frac{(\{\alpha\}_q x; q)_{\infty}}{(x; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{(\{\alpha\}_q; q)_k}{(q; q)_k} x^k$$

By Proposition 4.4(e), we have $\frac{(\{\alpha\}_q; q)_k}{(q; q)_k} = \binom{\alpha + k - 1}{k}_q$. This gives part (b).

Similarly, for part (a), we can substitute $-\{\alpha\}_q x$ for x , and $a = \{\alpha\}_q^{-1}$ to get

$$\frac{(-x; q)_{\infty}}{(-\{\alpha\}_q x; q)_{\infty}} = \sum_{k=0}^{\infty} (-1)^k \{\alpha\}_q^k \frac{(\{\alpha\}_q^{-1}; q)_k}{(q; q)_k} x^k$$

By Proposition 4.4(c), we have $(-1)^k \{\alpha\}_q^k \frac{(\{\alpha\}_q^{-1}; q)_k}{(q; q)_k} = q^{\binom{k}{2}} \binom{\alpha}{k}_q$. □

We now define q -analogs of $(1 + x)^{\alpha}$ and $\frac{1}{(1 - x)^{\alpha}}$ for which it will be convenient to give names to these series. So, we will call them B_{α} and b_{α} respectively. These series will be Laurent series in q with integer coefficients.

Definition 6.2. Let $\alpha \in \mathbb{R}$, and define $B_{\alpha}(q, x)$ and $b_{\alpha}(q, x)$ as the following elements of $\mathbb{Z}((q))[[x]]$:

$$B_{\alpha}(q, x) := \sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{\alpha}{k}_q x^k = \frac{(-x; q)_{\infty}}{(-\{\alpha\}_q x; q)_{\infty}}$$

$$b_{\alpha}(q, x) := \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k}_q x^k = \frac{(\{\alpha\}_q x; q)_{\infty}}{(x; q)_{\infty}}$$

The following result is a q -analogue of the simple statement that $(1 + x)^{\alpha} = (1 + x)(1 + x)^{\alpha-1}$.

Proposition 6.3. For $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$,

- (a) $B_{\alpha+1}(q, x) = (1+x) \cdot B_{\alpha}(q, qx) = (1 + \{\alpha\}_q x) \cdot B_{\alpha}(q, x)$
- (b) $b_{\alpha+1}(q, x) = \frac{1}{1-x} \cdot b_{\alpha}(q, qx) = \frac{1}{1 - \{\alpha\}_q x} \cdot b_{\alpha}(q, x)$
- (c) $B_{\alpha+n}(q, x) = B_n(q, x) \cdot B_{\alpha}(q, q^n x) = B_n(q, \{\alpha\}_q x) \cdot B_{\alpha}(q, x)$
- (d) $b_{\alpha+n}(q, x) = b_n(q, x) \cdot b_{\alpha}(q, q^n x) = b_n(q, \{\alpha\}_q x) \cdot b_{\alpha}(q, x)$

Proof. Parts (c) and (d) obviously follow from (a) and (b) by induction. Parts (a) and (b) follow from the product formula in Theorem 6.1. Parts (a) and (b) can also be derived by manipulating the power series using the q -Pascal identity (Proposition 4.3).

$$\begin{aligned}
B_{\alpha}(q, x) &= \sum_{k \geq 0} q^{\binom{k}{2}} \binom{\alpha}{k}_q x^k \\
&= \sum_{k \geq 0} q^{\binom{k}{2}} q^k \binom{\alpha-1}{k}_q x^k + \sum_{k \geq 0} q^{\binom{k}{2}} \binom{\alpha-1}{k-1}_q x^k \\
&= \sum_{k \geq 0} q^{\binom{k}{2}} q^k \binom{\alpha-1}{k}_q x^k + \sum_{k \geq 0} q^{\binom{k}{2}} q^k \binom{\alpha-1}{k}_q x^{k+1} \\
&= (1+x) \sum_{k \geq 0} q^{\binom{k}{2}} \binom{\alpha-1}{k}_q (qx)^k \\
&= (1+x) B_{\alpha-1}(q, qx)
\end{aligned}$$

For the second equality in part (a), use Proposition 4.3 (b). The calculations for b_{α} are similar. $\square$

Example 6.4. The power series $b_{1/2}(q, x)$ is a q -analogue of the function $\frac{1}{\sqrt{1-x}}$. By definition, we have

$$b_{1/2}(q, x) = \sum_{k=0}^{\infty} \binom{1/2 + k - 1}{k}_q x^k$$

Using the fact that $\{\frac{1}{2}\}_q = \frac{1+q^2}{1+q}$, we get from Proposition 4.4(e), and some algebraic manipulation, that

$$\binom{1/2 + k - 1}{k}_q = \frac{\left(\frac{1+q^2}{1+q}; q\right)_k}{(q; q)_k} = \prod_{i=0}^{k-1} \frac{1+q-q^i-q^{i+2}}{1+q-q^{i+1}-q^{i+2}}$$

The first few terms are given by

$$\begin{aligned}
b_{1/2}(q, x) &= 1 + \frac{q}{1+q}x + \frac{q(1+q+q^2)}{(1+q)^3}x^2 + \frac{q(1+2q+q^3+q^4)}{(1+q)^4}x^3 \\
&\quad + \frac{q(1+4q+7q^2+8q^3+7q^4+5q^5+2q^6+q^7)}{1+6q+16q^2+26q^3+30q^4+26q^5+16q^6+6q^7+q^8}x^4 + \dots
\end{aligned}$$

7. q -Calculus

Definition 7.1. For a function $f(x)$, its “ q -derivative” is

$$D_q f(x) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(qx) - f(x)}{(q-1)x}$$

The function $B_\alpha(q, x)$ is the q -analogue of $(1+x)^\alpha$, which has derivative $\frac{d}{dx} B_\alpha(1, x) = \alpha(1+x)^{\alpha-1} = \alpha B_{\alpha-1}(1, x)$. Similarly, $b_\alpha(1, x) = \frac{1}{(1-x)^\alpha}$ has derivative $\frac{d}{dx} b_\alpha(1, x) = \alpha b_{\alpha+1}(1, x)$. The following are q -analogues of these statements:

Proposition 7.2. *The q -derivatives of $B_\alpha(q, x)$ and $b_\alpha(q, x)$ are given as follows.*

- (a) $D_q B_\alpha(q, x) = [\alpha]_q B_{\alpha-1}(q, qx)$
- (b) $D_q b_\alpha(q, x) = [\alpha]_q b_{\alpha+1}(q, x)$

Proof. We only need the identity $D_q x^n = [n]_q x^{n-1}$, and we can differentiate the power series term-by-term:

$$\begin{aligned} D_q B_\alpha(q, x) &= D_q \sum_{n \geq 0} q^{\binom{n}{2}} \binom{\alpha}{n}_q x^n \\ &= \sum_{n \geq 0} q^{\binom{n}{2}} \binom{\alpha}{n}_q [n]_q x^{n-1} \\ &= \sum_{n \geq 0} q^{\binom{n+1}{2}} \binom{\alpha}{n+1}_q [n+1]_q x^n \end{aligned}$$

At this point, note that $\binom{n+1}{2} = \binom{n}{2} + n$, so we can write $q^{\binom{n+1}{2}} = q^{\binom{n}{2}} \cdot q^n$, and it is easy to check that

$$\binom{\alpha}{n+1}_q [n+1]_q = [\alpha]_q \cdot \binom{\alpha-1}{n}_q$$

Using these two observations, we have that

$$D_q B_\alpha(q, x) = [\alpha]_q \sum_{n \geq 0} q^{\binom{n}{2}} \binom{\alpha-1}{n}_q q^n x^n = [\alpha]_q B_{\alpha-1}(q, qx)$$

The computation for b_α is similar, but does not give an extra factor of q^n :

$$\begin{aligned} D_q b_\alpha(q, x) &= D_q \sum_{n \geq 0} \binom{\alpha+n-1}{n}_q x^n \\ &= \sum_{n \geq 0} \binom{\alpha+n-1}{n}_q [n]_q x^{n-1} \\ &= \sum_{n \geq 0} \binom{\alpha+n}{n+1}_q [n+1]_q x^n \\ &= [\alpha]_q \sum_{n \geq 0} \binom{\alpha+n}{n}_q x^n \\ &= [\alpha]_q b_{\alpha+1}(q, x) \end{aligned}$$

□

Remark 7.3. Combining these formulas with the identities from Theorem 6.3, we obtain the q -differential equations

$$D_q B_\alpha(q, x) = \frac{[\alpha]_q}{1+x} B_\alpha(q, x) \quad \text{and} \quad D_q b_\alpha(q, x) = \frac{[\alpha]_q}{1-x} b_\alpha(q, qx)$$

Remark 7.4. Theorem 6.1 can also be proved using these q -derivative expressions. Starting with the differential equation above, we can obtain a functional equation satisfied by $B_\alpha(q, x)$:

$$\begin{aligned} \frac{[\alpha]_q}{1+x} B_\alpha(q, x) &= \frac{B_\alpha(q, qx) - B_\alpha(q, x)}{(q-1)x} \\ [\alpha]_q (q-1)x B_\alpha(q, x) &= (1+x) (B_\alpha(q, qx) - B_\alpha(q, x)) \\ (1 + (1 + [\alpha]_q (q-1))x) B_\alpha(q, x) &= (1+x) B_\alpha(q, qx) \\ B_\alpha(q, x) &= \frac{1+x}{1 + \{\alpha\}_q x} B_\alpha(q, qx) \end{aligned}$$

In the last step, we have used that $\{\alpha\}_q = 1 + (q-1)[\alpha]_q$ (Proposition 3.6(a)). Now that we have this functional equation, we can use it again on the right-hand side to replace $B_\alpha(q, qx)$ with $\frac{1+qx}{1+\{\alpha+1\}_q x} B_\alpha(q, q^2x)$. Repeatedly applying the functional equation gives the infinite product formula from Theorem 6.1.

The calculation for $b_\alpha(q, x)$ is similar.

8. Some More Identities

Let us now demonstrate some more identities which hold in our setting. We first give another q -analog of the Pascal identity for q -deformed rational and real numbers. The identity in the following proposition for q -integers appeared in [16, Theorem 1].

Proposition 8.1. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\binom{\alpha-1}{k}_q + \binom{\alpha-1}{k-1}_q = \left(\frac{2 - q^k - \{\alpha-k\}_q}{1 - \{\alpha\}_q} \right) \binom{\alpha}{k}_q$$

Proof. We begin computing

$$\begin{aligned} \binom{\alpha-1}{k}_q + \binom{\alpha-1}{k-1}_q &= \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k]_q}{[k]_q!} + \frac{[\alpha-1]_q [\alpha-2]_q \cdots [\alpha-k+1]_q}{[k-1]_q!} \\ &= \left(\frac{[\alpha-k]_q}{[\alpha]_q} + \frac{[k]_q}{[\alpha]_q} \right) \frac{[\alpha]_q [\alpha-1]_q \cdots [\alpha-k+1]_q}{[k]_q!} \\ &= \left(\frac{(1 - \{\alpha-k\}_q) + (1 - q^k)}{1 - \{\alpha\}_q} \right) \binom{\alpha}{k}_q \end{aligned}$$

and the result follows noting we have made use of Proposition 3.6 part (b). $\square$

We also have a Chu–Vandermonde identity in the setting of q -deformed rational and real numbers.

Corollary 8.2 (q -Chu–Vandermonde). For $\alpha \in \mathbb{R}$ and $k, n \in \mathbb{Z}$,

$$\binom{\alpha+n}{k}_q = \sum_{j=0}^k q^{j(n-k+j)} \binom{n}{k-j}_q \binom{\alpha}{j}_q$$

Proof. We start with Theorem 6.3 part (c) which states that $B_{\alpha+n}(q, x) = B_n(q, x) \cdot B_\alpha(q, q^n x)$. As a summation this means

$$\sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{\alpha+n}{k}_q x^k = \left(\sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k}_q x^k \right) \left(\sum_{k=0}^{\infty} q^{\binom{k}{2}+nk} \binom{\alpha}{k}_q x^k \right)$$

from which we will equate the coefficient of x^k on each side. Thus we obtain

$$\begin{aligned} q^{\binom{k}{2}} \binom{\alpha+n}{k}_q &= \sum_{j=0}^k \left(q^{\binom{k-j}{2}} \binom{n}{k-j}_q \right) \left(q^{\binom{j}{2}+nj} \binom{\alpha}{j}_q \right) \\ &= \sum_{j=0}^k q^{\binom{k-j}{2}+j(n-k+j)} \binom{n}{k-j}_q \binom{\alpha}{j}_q \end{aligned}$$

and the corollary immediately follows. $\square$

Let us now give a lemma that is a consequence of the q -Chu–Vandermonde identity.

Lemma 8.3. *For $\alpha \in \mathbb{R}$ and $\ell, m, n \in \mathbb{Z}$,*

$$\sum_{j \geq 0} q^{\ell(j-n+\ell)+j(m-n+j)} \binom{\ell}{n-j}_q \binom{\alpha}{m+j}_q = q^{(m-\ell)(n-\ell)} \binom{\alpha+\ell}{m+n}_q$$

Proof. We begin by applying Corollary 8.2 to $\binom{\alpha+\ell}{m+n}_q$ and simplifying to obtain

$$\begin{aligned} \binom{\alpha+\ell}{m+n}_q &= \sum_k q^{k(\ell-(m+n)+k)} \binom{\alpha}{k}_q \binom{\ell}{m+n-k}_q \\ &= \sum_j q^{(m+j)(\ell-n+j)} \binom{\alpha}{m+j}_q \binom{\ell}{n-j}_q \end{aligned}$$

which completes the proof after comparing exponents. Indeed, it is the case that

$$(m-\ell)(n-\ell) + (m+j)(\ell-n+j) = \ell(j-n+\ell) + j(m-n+j)$$

which completes the proof. $\square$

The next proposition in the case of integers is found in Riordan's book [15, Section 1.4 Equation (10)], and a q -integer version was given in [1, Section 3.9].

Proposition 8.4. *For $\alpha \in \mathbb{R}$ and $m, n \in \mathbb{Z}$,*

$$\binom{\alpha}{m}_q \binom{\alpha}{n}_q = \sum_{\ell \geq 0} q^{(m-\ell)(n-\ell)} \binom{n}{\ell}_q \binom{m}{\ell}_q \binom{\alpha+\ell}{m+n}_q$$

Proof. We begin computing making use of the q -Chu–Vandermonde identity in Corollary 8.2 as well as other simplifications.

$$\begin{aligned}
\binom{\alpha}{m}_q \binom{\alpha}{n}_q &= \binom{\alpha}{m}_q \binom{(\alpha - m) + m}{n}_q \\
&= \sum_{j \geq 0} q^{j(m-n+j)} \binom{\alpha}{m}_q \binom{m}{n-j}_q \binom{\alpha - m}{j}_q \quad (\text{Corollary 8.2}) \\
&= \sum_{j \geq 0} q^{j(m-n+j)} \binom{m+j}{n}_q \binom{n}{j}_q \binom{\alpha}{m+j}_q \\
&= \sum_{j \geq 0} q^{j(m-n+j)} \binom{n}{j}_q \binom{\alpha}{m+j}_q \sum_{\ell \geq 0} q^{\ell(j-n+\ell)} \binom{j}{n-\ell}_q \binom{m}{\ell}_q \quad (\text{Corollary 8.2}) \\
&= \sum_{\ell \geq 0} \sum_{j \geq 0} q^{j(m-n+j)+\ell(j-n+\ell)} \binom{m}{\ell}_q \binom{n}{j}_q \binom{j}{n-\ell}_q \binom{\alpha}{m+j}_q \\
&= \sum_{\ell \geq 0} \sum_{j \geq 0} q^{j(m-n+j)+\ell(j-n+\ell)} \binom{m}{\ell}_q \binom{n}{\ell}_q \binom{\ell}{n-j}_q \binom{\alpha}{m+j}_q \\
&= \sum_{\ell \geq 0} \binom{m}{\ell}_q \binom{n}{\ell}_q \sum_{j \geq 0} q^{j(m-n+j)+\ell(j-n+\ell)} \binom{\ell}{n-j}_q \binom{\alpha}{m+j}_q
\end{aligned}$$

The proof is then completed by using Lemma 8.3. Unlabeled steps above are routine algebraic manipulations, and many of these steps are simply rearrangement of the factors in the falling factorials in the q -binomials. $\square$

It is well-known that for the classical q -binomial coefficients, $\lim_{n \rightarrow \infty} \binom{n}{k}_q = \frac{1}{(q; q)_k}$. We have the following generalization of this fact:

Proposition 8.5. *Let $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$ be any sequence of real numbers for which $\lim_{n \rightarrow \infty} \alpha_n = \infty$. Then*

$$\lim_{n \rightarrow \infty} \binom{\alpha_n}{k}_q = \frac{1}{(q; q)_k}$$

Proof. If $k = 0$, then we immediately find that both sides are equal to 1. So, we may assume that $k \geq 1$. By Proposition 4.4(e), we have

$$\binom{\alpha}{k}_q = \frac{(\{\alpha - k + 1\}_q; q)_k}{(q; q)_k} = \frac{(q^{1-k}\{\alpha\}_q; q)_k}{(q; q)_k}$$

The numerator is the product

$$(1 - q^{1-k}\{\alpha\}_q)(1 - q^{2-k}\{\alpha\}_q) \cdots (1 - \{\alpha\}_q)$$

If $N = \lfloor \alpha \rfloor$ is the integer part of α , then Proposition 3.6 says that $\{\alpha\}_q = q^N \{\alpha - N\}_q$. Since we are considering $\lim_{n \rightarrow \infty} \alpha_n$ and k is constant, we may assume that $N \gg k$. Therefore the power series for $\{\alpha\}_q$ has no terms of degree less than N . The product given above is therefore of the form

$$1 + \left(\text{terms of degree at least } N - k + 1 \right)$$

Since the sequence α_n increases without bound, we can eventually find n large enough so that $\alpha_n > N$, and so $\lim_{n \rightarrow \infty} (q^{1-k} \{\alpha_n\}_q; q)_k = 1$. We therefore have

$$\lim_{n \rightarrow \infty} \binom{\alpha_n}{k}_q = \lim_{n \rightarrow \infty} \frac{(q^{1-k} \{\alpha_n\}_q; q)_k}{(q; q)_k} = \frac{1}{(q; q)_k}$$

□

9. A New q -Analogue of the Γ Function

Note that the q -Pochhammer symbol is equal to $(x; q)_k = \frac{(x; q)_\infty}{(q^k x; q)_\infty}$. It is therefore natural (e.g. see the appendix of [4]) to define a version when $k = \alpha$ is not an integer by the formula $\frac{(x; q)_\infty}{(q^\alpha x; q)_\infty}$. Our basic philosophy of replacing q^α with the function $\{\alpha\}_q$ leads us to the following definition.

Definition 9.1. For $\alpha \in \mathbb{R}$, define the generalized q -Pochhammer symbol by

$$(x; q)_\alpha := \frac{(x; q)_\infty}{(\{\alpha\}_q x; q)_\infty} = \frac{1}{b_\alpha(q, x)} = B_\alpha(q, -x)$$

Remark 9.2. Using this new notation, Theorem 6.1 can be re-stated as follows:

$$(x; q)_\alpha = \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \binom{\alpha}{k}_q x^k$$

$$\frac{1}{(x; q)_\alpha} = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k}_q x^k$$

For a positive integer n , the q -factorial is related to the Pochhammer symbol by $[n]_q! = \frac{(q; q)_n}{(1-q)^n}$. Accordingly, the q -Gamma function is usually taken to be $\frac{(q; q)_\infty}{(q^x; q)_\infty (1-q)^{x-1}}$. This leads us to a definition of a new q -Gamma function after the following lemma.

Lemma 9.3. The evaluation $B_{\alpha-1}(q, -q)$ gives a well-defined series in $\mathbb{Z}((q))$ if and only if $\alpha \geq 1$.

Proof. A term in the expansion of $B_{\alpha-1}(q, -q)$ looks like $(-1)^k q^{\binom{k}{2} + k} \binom{\alpha-1}{k}_q$. If $N \leq \alpha - 1 < N + 1$ for $N \in \mathbb{Z}$ and $k > \alpha - 1$, then by Lemma 4.7 when $N \geq 0$ we have

$$\text{ord} \left((-1)^k q^{\binom{k}{2} + k} \binom{\alpha-1}{k}_q \right) = \binom{k}{2} + k + b - \binom{k-N}{2}$$

where $b \geq 1$ is as in the lemma. In this case the order is a strictly increasing function in k , implying each degree has only finitely many terms contributing. So, $B_{\alpha-1}(q, -q)$ is well-defined as a formal power series when $\alpha \geq 1$.

However, in the case that $N < 0$

$$\text{ord} \left((-1)^k q^{\binom{k}{2}+k} \binom{\alpha-1}{k}_q \right) = \binom{k}{2} + k + Nk - \binom{k}{2} = (N+1)k$$

which is identically 0 when $N = -1$. For $N < -1$ we find the order approaches $-\infty$ as k increases. Thus $B_{\alpha-1}(q, -q)$ is not defined even as a formal Laurent series for $\alpha < 1$. $\square$

Definition 9.4. Define a function $\Gamma_q: \mathbb{R} \setminus \mathbb{Z}_{\leq 0} \rightarrow \mathbb{R}((q))$ as follows: For $\alpha \geq 1$ we set

$$\Gamma_q(\alpha) := \frac{(q; q)_{\alpha-1}}{(1-q)^{\alpha-1}} = \frac{1}{(1-q)^{\alpha-1} b_{\alpha-1}(q, q)} = \frac{B_{\alpha-1}(q, -q)}{(1-q)^{\alpha-1}}$$

which is well-defined by Lemma 9.3. In this expression, the numerator $(q; q)_{\alpha-1}$ is the one from Definition 9.1, and the denominator $\frac{1}{(1-q)^{\alpha-1}}$ is taken to be the formal power series $\sum_n \binom{\alpha+n-2}{n} q^n$, where the binomial coefficients $\binom{\alpha+n-2}{n} \in \mathbb{R}$ are the ordinary ones (not the q -analogues). For $\alpha < 1$ we define

$$\Gamma_q(\alpha) := \frac{\Gamma_q(\alpha+1)}{[\alpha]_q}$$

which is well-defined since it is obtained through division by a Laurent series.

Note that in general $\Gamma_q(\alpha)$ is a Laurent series with real (not integer) coefficients. For example,

$$\Gamma_q\left(\frac{3}{2}\right) = 1 + \frac{1}{2}q - \frac{5}{8}q^2 - \frac{3}{16}q^3 + \frac{115}{128}q^4 - \frac{401}{256}q^5 + \frac{2383}{1024}q^6 - \frac{8139}{2048}q^7 + \dots$$

has rational coefficients. Here the noninteger coefficients come from the binomial coefficients $\binom{\frac{3}{2}}{k}$ in the factor of $(1-q)^{\frac{3}{2}}$ in the denominator. So, $\Gamma_q(\alpha)$ will have integer, rational, or real coefficients depending on if α is an integer, rational number, or real number because of the presence of $\binom{\alpha}{k}$.

For integers, the usual Γ function satisfies $\Gamma(n+1) = n!$. To avoid confusion with our q -Gamma function we have given in Definition 9.4, we will denote the classical q -Gamma function by $G_q(x)$. For the classical q -Gamma function, we have $G_q(n+1) = [n]_q!$, and more generally the well known property that $G_q(x+1) = \frac{1-q^x}{1-q} G_q(x)$. We required an analogous shift property to hold for $\alpha < 1$, but we actually have the corresponding property for our new q -Gamma function for all input values.

Proposition 9.5. For $\alpha \in \mathbb{R}$ with $\alpha \geq 1$, we have

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$$

Proof. Use Proposition 6.3 to write $B_{\alpha}(q, -q)$ as $(1 - \{\alpha\}_q) B_{\alpha-1}(q, -q)$, and factor $\frac{1}{(1-q)^{\alpha}}$ as $\frac{1}{1-q} \cdot \frac{1}{(1-q)^{\alpha-1}}$. Then we have that $\Gamma_q(\alpha+1) = \frac{1-\{\alpha\}_q}{1-q} \Gamma_q(\alpha)$, and by Proposition 3.6, the factor on the left is equal to $[\alpha]_q$. $\square$

For the classical (integer) q -binomial coefficients, they can be written in terms of q -factorials and q -Pochhammer symbols as

$$\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

Using the definitions given above, we have similar expressions for the q -rational and q -real binomial coefficients.

Proposition 9.6. For $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$, and positive integer k , we have

$$\binom{\alpha}{k}_q = \frac{(q; q)_\alpha}{(q; q)_k (q; q)_{\alpha-k}} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(k+1) \Gamma_q(\alpha-k+1)}$$

Proof. Assume $\alpha \geq 1$. The two expressions on the right-hand side are obviously equal by the definition of $\Gamma_q(\alpha)$. To see the first equality, notice that

$$\frac{(q; q)_\alpha}{(q; q)_{\alpha-k}} = \frac{(\{\alpha-k+1\}_q; q)_\infty}{(\{\alpha+1\}_q; q)_\infty} = (\{\alpha\}_q, q^{-1})_k$$

Dividing by $(q; q)_k$ gives $\binom{\alpha}{k}_q$ by Proposition 4.4(d).

In the case that $\alpha < 1$ we need to use our shift property. First notice that

$$\binom{\frac{[\alpha-k]_q}{[\alpha]_q}}{k}_q \binom{\alpha}{k}_q = \binom{\alpha-1}{k}_q$$

for $\alpha \neq 0$. Hence, if $\binom{\alpha}{k}_q$ has the form specified by the proposition then we see that $\binom{\alpha-1}{k}_q$ does as well by using Proposition 9.5. $\square$

For the classical Γ function, we have Euler's reflection formula, $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$. It turns out that for the q -version, this expression is a power series which always has integer coefficients.

Proposition 9.7. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. Then $\Gamma_q(\alpha)\Gamma_q(1-\alpha) \in \mathbb{Z}((q))$. In other words, it is a Laurent series with integer coefficients.

Proof. There are two cases to consider: when $0 < \alpha < 1$ and when $\alpha > 1$.

Consider first the case that $\alpha > 1$, and let $N = \lfloor \alpha \rfloor$ be the integer part of α . Then $-N < 1-\alpha < -(N-1)$. Then by definition we have

$$\Gamma_q(\alpha) = B_{\alpha-1}(q, -q)b_{\alpha-1}(1, q) \quad \text{and} \quad \Gamma_q(1-\alpha) = \frac{B_{N+1-\alpha}(q, -q)b_{N+1-\alpha}(1, q)}{[1-\alpha]_q[2-\alpha]_q \cdots [N+1-\alpha]_q}$$

Observe that the only possibility for non-integer coefficients comes from the factors of $b_\alpha(1, q) = \frac{1}{(1-q)^\alpha}$, whose coefficients are the scalar non-integer binomial coefficients of the form $\binom{\alpha+k-1}{k}$. When we multiply these together in the above expression, we get a factor of

$$b_{\alpha-1}(1, q)b_{N+1-\alpha}(1, q) = \frac{1}{(1-q)^{\alpha-1}(1-q)^{N+1-\alpha}} = \frac{1}{(1-q)^N} = \sum_{k=0}^{\infty} \binom{N+k-1}{k} q^k$$

In particular, this has integer coefficients. The other factors of B_α and $\frac{1}{[1-\alpha]_q}$, etc, all have integer coefficients.

Secondly, there is the case where $0 < \alpha < 1$. In this case both α and $1-\alpha$ are less than 1, so we have

$$\Gamma_q(\alpha) = \frac{\Gamma_q(\alpha+1)}{[\alpha]_q} = \frac{B_\alpha(q, -q)b_\alpha(1, q)}{[\alpha]_q} \quad \text{and} \quad \Gamma_q(1-\alpha) = \frac{\Gamma_q(2-\alpha)}{[1-\alpha]_q} = \frac{B_{1-\alpha}(q, -q)b_{1-\alpha}(1, q)}{[1-\alpha]_q}$$

As above in the other case, when we multiply these we get a factor of $b_\alpha(1, q)b_{1-\alpha}(1, q) = \frac{1}{1-q}$, which has integer coefficients. The remaining factors all have integer coefficients as well. $\square$

Example 9.8. Above, we gave the first several terms of $\Gamma_q\left(\frac{3}{2}\right)$. We can use this to compute

$$\Gamma_q\left(\frac{1}{2}\right) = \frac{\Gamma_q\left(\frac{3}{2}\right)}{\left[\frac{1}{2}\right]_q} = q^{-1} + \frac{3}{2} - \frac{1}{8}q - \frac{13}{16}q^2 + \frac{91}{128}q^3 - \frac{171}{256}q^4 + \frac{779}{1024}q^5 - \frac{3373}{2048}q^6 + \dots$$

As an example of Proposition 9.7, we can square this expression to obtain

$$\Gamma_q\left(\frac{1}{2}\right)^2 = \Gamma_q\left(\frac{1}{2}\right)\Gamma_q\left(1 - \frac{1}{2}\right) = q^{-2} + 3q^{-1} + 2 - 2q - q^2 + q^3 - 2q^5 + \dots$$

Euler's reflection formula says that $\Gamma\left(\frac{1}{2}\right)^2 = \pi$, so this Laurent series with integer coefficients is, in some sense, a q -analogue of the number π .

There is another situation where we can obtain Laurent series with integer coefficients, given in the following proposition.

Proposition 9.9. *Let $\frac{a}{b} \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then $\Gamma_q\left(\frac{a}{b}\right)^b$ is a Laurent series with integer coefficients.*

Proof. The idea is similar to the proof of Proposition 9.7, but even simpler. The only occurrences of non-integer coefficients come from the power series expansion of the factor $\frac{1}{(1-q)^{\frac{a-b}{b}}}$. Raising this to the power b gives $\frac{1}{(1-q)^{a-b}}$, which has integer coefficients. $\square$

Example 9.10. When $\alpha = \frac{2}{3}$, we have

$$\Gamma_q\left(\frac{2}{3}\right) = q^{-1} - \frac{2}{3} - \frac{1}{9}q - \frac{166}{81}q^2 + \frac{803}{243}q^3 - \frac{1553}{729}q^4 + \dots$$

Raising this to the third power gives

$$\Gamma_q\left(\frac{2}{3}\right)^3 = q^{-3} - 2q^{-2} + q^{-1} - 6 + 18q - 21q^2 + 27q^3 - 69q^4 + \dots$$

10. Further Questions

We conclude with some lingering questions which were not addressed in the present paper, and which we feel would be interesting directions for further study.

Question 10.1. *What other q -analogues have a version which uses q -rational or q -real numbers?*

In this article we have given some examples where the common q -analogue of a number α , given by $\frac{1-q^\alpha}{1-q}$, can be replaced by Morier-Genoud and Ovsienko's q -rationals and q -reals, with the latter version retaining many of the desirable properties of the former. The main observation was that $\{\alpha\}_q = [\alpha + 1]_q - [\alpha]_q$ was a good substitute for the expression q^α in these examples. The success of this basic idea and philosophy in the examples given here suggest that maybe this idea can be pushed even further. It would be interesting to find even more well-known q -analogues for which replacing q^α by $\{\alpha\}_q$ gives a new q -analogue (and one which is still interesting and meaningful!).

Question 10.2. *What can be said about the representation of q -rational binomial coefficients as a ratio of two relatively prime polynomials?*

Theorem 5.3 gives a combinatorial interpretation for some representation of the q -rational binomial coefficients as a rational function. However, the formula in this theorem may not be in lowest terms. For example, after applying Theorem 5.3 to the q -rational binomial coefficient $\binom{5/3}{3}_q$ a factor $(1+q)$ must be canceled to obtain the expression in Example 4.2.

Question 10.3. *Do the coefficients of the series representation of q -real binomial coefficients have some combinatorial interpretation?*

By definition, $\binom{\alpha}{k}_q$ is a Laurent series with integer coefficients. One might wonder if these integers have some combinatorial meaning. This question seems difficult however, since the meaning of the integer coefficients of the q -reals themselves is still somewhat mysterious.

Question 10.4. *Is there a more explicit q -analogue of the expression $\frac{\pi}{\sin(\pi\alpha)}$ appearing in Euler's reflection formula, which uses the q -rational (or q -real) number $[\alpha]_q$?*

Euler's reflection formula for the classical Γ function says $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$, and in Proposition 9.7, we showed that $\Gamma_q(\alpha)\Gamma_q(1-\alpha)$ is a series with integer coefficients (despite the fact that $\Gamma_q(\alpha)$ need not have integer coefficients). In what sense are these series a “good” q -analogue of the expression $\frac{\pi}{\sin(\pi\alpha)}$? Is there some q -deformation of the elements of this expression (the number π and the sine function) which make this explicit?

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