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THE LOCAL WELL-POSEDNESS OF THE RELATIVISTIC VLASOV–MAXWELL–LANDAU SYSTEM WITH THE SPECULAR REFLECTION BOUNDARY CONDITION*

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Abstract. We prove the local-in-time well-posedness of the relativistic Vlasov–Maxwell–Landau system in a bounded domain Ω with the specular reflection condition. Our result covers the case when Ω is a nonconvex domain, e.g., solid torus. To the best of our knowledge, this is the first local well-posedness result for a nonlinear kinetic model with a self-consistent magnetic effect in a three-dimensional *bounded* domain.

Key words. relativistic Vlasov–Maxwell–Landau system, collisional plasma, specular reflection boundary condition, div-curl estimate, kinetic Fokker–Planck equation

MSC codes. 35Q83, 35Q84, 35Q61, 35K70, 35H10, 34A12

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1. Introduction.

Let $z = (t, x, p)$, where $t \in \mathbb{R}$, $x, p \in \mathbb{R}^3$ are the temporal, spatial, and momentum variables, respectively. For a spatial domain $\Omega \subset \mathbb{R}^3$, we denote the incoming/outgoing boundaries and the grazing sets, respectively, as follows:

$$\gamma_- = \{(x,p) : x \in \partial\Omega, n_x \cdot p < 0\}, \quad \gamma_+ = \{(x,p) : x \in \partial\Omega, n_x \cdot p > 0\},$$
$$\gamma_0 = \{(x,p) : n_x \cdot p = 0\},$$

where n_x is an outward unit normal vector at $x \in \partial\Omega$. Furthermore, we denote

$$p_0 = (1 + |p|^2)^{1/2}, \quad v(p) = \frac{p}{p_0}.$$

We study the relativistic Vlasov–Maxwell–Landau (RVML) system in a bounded domain:

$$\begin{aligned}
&\partial_t F^+ + v(p)\cdot\nabla_x F^+ + (\mathbf{E} + v(p)\times\mathbf{B})\cdot\nabla_p F^+ = \mathcal{C}(F^+,F^+) + \mathcal{C}(F^+,F^-),\\
&\partial_t F^- + v(p)\cdot\nabla_x F^- - (\mathbf{E} + v(p)\times\mathbf{B})\cdot\nabla_p F^- = \mathcal{C}(F^-,F^-) + \mathcal{C}(F^-,F^+),\\
&\partial_t \mathbf{E} - \nabla_x \times \mathbf{B} = -\int v(p)(F^+ - F^-)\,dp,\\
&\partial_t \mathbf{B} + \nabla_x \times \mathbf{E} = 0,\\
&\nabla_x \cdot \mathbf{E} = \int (F^+ - F^-)\,dp, \quad \nabla_x \cdot \mathbf{B} = 0,\\
&(\mathbf{E}\times n_x)|_{\partial\Omega} = 0, \quad (\mathbf{B}\cdot n_x)|_{\partial\Omega} = 0
\end{aligned} \tag{1.1}$$

with the initial conditions

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$$F^{\pm}(0,\cdot)=F_0^{\pm}(\cdot),\ \mathbf{E}(0,\cdot)=\mathbf{E}_0(\cdot),\ \mathbf{B}(0,\cdot)=\mathbf{B}_0(\cdot),$$

and the specular reflection boundary condition (SRBC)

$$F^{\pm}(t,x,p)=F^{\pm}(t,x,R_x p), \quad R_x p = p-2(p\cdot n_x)n_x.$$

Here F^+ and F^- are the density functions of ions and electrons, respectively, and $\mathcal{C}$ is the relativistic Landau collision operator given by

$$\mathcal{C}(f,g)(p)=\nabla_p\cdot\int_{\mathbb{R}^3}\Phi(P,Q)\big(\nabla_p f(p)g(q)-f(p)\nabla_q g(q)\big)\,dq, \tag{1.2}$$

where $\Phi(P,Q)$ is the Belyaev–Budker kernel defined in section 2 (see (2.7)). The RVML is a fundamental model of a hot dilute collisional plasma with magnetic and relativistic effects. Such a model is relevant for plasma fusion in tokamaks, where plasma particles may reach high velocities. For the formal derivation, see, for example, [27].

For the sake of simplicity, all the physical constants are set to 1 (cf. [32]) since the exact relationships among them do not play any role in our analysis. Our goal is to prove the local-in-time well-posedness of the RVML system near the relativistic Maxwellian

$$J(p)=e^{-p_0}, \tag{1.3}$$

which is called the Jüttner's solution. The global well-posedness of the RVML system was first established in [32] for the periodic boundary conditions, and later, this result was extended to the whole space in [36]. For the related studies of this model, see [28] and [35].

The presence of spatial boundaries is natural in kinetic models, and the study of boundary value problems is one of the foci of contemporary kinetic PDE theory. In this context, the investigation of hyperbolic kinetic models poses a formidable challenge due to the nonuniformly characteristic nature of the grazing set γ_0 associated with the free streaming operator $\partial_t+p\cdot\nabla_x$. Near the grazing set, the regularity of a solution is expected to deteriorate significantly, resulting in profound mathematical intricacies. The standard energy techniques, which typically rely on differentiating with respect to spatial and velocity variables, become inadequate in such a scenario.

Particularly noteworthy is the occurrence of singularities emanating from the grazing set in nonconvex domains [24], where hyperbolic kinetic PDEs are expected to yield solutions of, at best, bounded variation [19]. Furthermore, the introduction of magnetic effects can trigger singularities even in a half-space domain. An illustrative example is the one-dimensional relativistic Vlasov–Maxwell (RVM) system subject to the perfect conductor boundary conditions [16] (see also [15] for an example in a three-dimensional half-space). This specific case underscores the limited knowledge we possess, as only the global existence of a weak solution is currently known for the RVM system in a three-dimensional bounded domain [14].

In stark contrast, in convex domains and in the absence of magnetic effects, recent papers have demonstrated global well-posedness for several important hyperbolic plasma models such as the Vlasov–Poisson and Vlasov–Poisson–Boltzmann systems [21], [22], [5].

Conversely, when velocity diffusion is introduced, a higher degree of regularity near the grazing set is expected, owing to a hypoelliptic gain [31]. The nature of this regularity depends on the specific boundary conditions imposed on the outgoing boundary γ_-. Notably, a linear kinetic Fokker–Planck (KFP) equation with the inflow

(Dirichlet) boundary conditions is anticipated to exhibit at most Hölder regularity in both spatial and velocity variables [23].

However, in the presence of the SRBC, a unique avenue opens. Employing a flattening and extension strategy, one can extend the solution of the KFP equation to the entire space and invoke the S_p theory of KFP equations, akin to the Calderon–Zygmund theory for parabolic PDEs [20], [7], [8]. This approach yields Hölder regularity not only for the solution but also for its velocity gradient. Such an extension argument is unknown for other boundary conditions in kinetic theory.

In recent years, an L_2 to L_∞ framework has been developed for the Boltzmann equation in bounded domains (see [18] and [12], [13], [25] for further developments). The method is based on interpolating between the natural entropy or energy bound and the interplay between characteristics and velocity averaging in the collision [17]. However, this approach is less applicable to the Landau equation due to the absence of characteristic curves. We emphasize that a higher regularity of the velocity gradient is required to establish the uniqueness for the Landau equation due to the nonlinear diffusion term (see [25]). To handle the Landau and the Vlasov–Poisson–Landau equations with the SRBC, the authors of [20] and [7] combined the aforementioned mirror-extension method with the S_p estimate. Their results require merely C^2 regularity of domains and, hence, allow a solid tori domain, which resembles a tokamak.

Adapting the aforementioned framework to the RVML system poses a formidable challenge due to the anticipated low regularity of solutions to Maxwell's equations. The intricate nature of the relativistic Landau kernel, coupled with the presence of the relativistic transport term, introduces additional mathematical complexities. Our innovative approach involves deducing the regularity of solutions to Maxwell's equations by treating them as an elliptic system of the Hodge type. This inspired the development of a delicate iteration scheme, where we propagate temporal derivatives and employ a descent argument, leveraging div-curl estimates and a relativistic adaptation of the S_p estimates for KFP equations with the SRBC.

Our main result is, informally speaking, the following (see Theorem 3.10): if F_0-J, $\mathbf{E}_0$, $\mathbf{B}_0$ are of order ε in some sense, then the RVML system has a unique strong solution $[F,\mathbf{E}_f,\mathbf{B}_f]$ on $[0,T]$ for some $T>0$ such that $F^\pm-J,\mathbf{E}_f,\mathbf{B}_f$ are of order ε. Due to the delicate behavior of kinetic PDEs near the boundary, there have been few results on well-posedness for any kinetic models with a self-consistent magnetic effect in three-dimensional domains (see [6] for the result on RVM in a half-space). To the best of our knowledge, Theorem 3.10 provides the first well-posedness result for the system with the Vlasov–Maxwell structure in a three-dimensional *bounded* domain. In a separate paper [9], the first, second, and fourth authors established a global estimate and asymptotic stability for the RVML system near a global Jüttner's solution.

2. Notation and conventions. Before we state the main results, we introduce some notation. Throughout the paper, $T>0$ is a number.

- Geometric notation.

$$P=(p_0,p),\quad Q=(q_0,q),\quad P\cdot Q=p_0q_0-p\cdot q, \tag{2.1}$$

$$B_r(x_0)=\{x\in\mathbb{R}^3:|x-x_0|<r\},\quad \Omega_r(x_0)=\Omega\cap B_r(x_0), \tag{2.2}$$

$$\begin{aligned}&\mathbb{R}^3_\pm=\{x\in\mathbb{R}^3:\pm x_3>0\},\quad \mathbb{H}_\pm=\{(x,p)\in\mathbb{R}^3_\pm\times\mathbb{R}^3\},\\&\mathbb{H}^T_\pm=\{z\in(0,T)\times\mathbb{H}_\pm\},\quad \mathbb{R}^7_T=\{z\in(0,T)\times\mathbb{R}^6\},\\&\Sigma^T=(0,T)\times\Omega\times\mathbb{R}^3,\quad \Sigma^T_\pm=(0,T)\times\gamma_\pm.\end{aligned} \tag{2.3}$$

- Matrix notation.

$$
\begin{aligned}
&\mathbf{1}_3 = \operatorname{diag}(1,1,1), \quad \boldsymbol{R} = \operatorname{diag}(1,1,-1), \\
&\boldsymbol{\xi} = \operatorname{diag}(1,-1), \quad \boldsymbol{\xi}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{\xi}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned} \tag{2.4}
$$

- Relativistic kinetic transport operator.

$$
Y = \partial_t + v(p) \cdot \nabla_x.
$$

- Relativistic Belyaev–Budker kernel. We introduce

$$
\Lambda(P,Q) = (P \cdot Q)^2 \big((P \cdot Q)^2 - 1\big)^{-3/2}, \tag{2.5}
$$

$$
\begin{aligned}
S(P,Q) = &\big((P \cdot Q)^2 - 1\big)\mathbf{1}_3 - (p-q) \otimes (p-q) \\
&+ (P \cdot Q - 1)(p \otimes q + q \otimes p),
\end{aligned} \tag{2.6}
$$

$$
\Phi(P,Q) = \frac{\Lambda(P,Q)}{p_0 q_0} S(P,Q). \tag{2.7}
$$

- Function spaces. Let $G \subset \mathbb{R}^7$ be an open set.
  - $C(\overline{G})$ is the set of all bounded continuous functions on $\overline{G}$, and $C^k(\overline{G}), k \in \{1,2,\ldots\}$ is the subspace of $C(\overline{G})$ functions with partial derivatives up to order k belonging to $C(\overline{G})$.
  - $C_0^1(\overline{G})$ ($C_0^{0,1}(\overline{G})$) is the subset of $C^1(\overline{G})$ (Lipschitz) functions on G that vanish for large z. Similarly, one can define $C_0^k(\overline{G}), k \in \{2,3,\ldots\}$.
  - $C_0^\infty(G)$ is the set of all infinitely differentiable functions with the support contained in G.
  - *Anisotropic Hölder space.* For an open set $D \subset \mathbb{R}^6$ and $\alpha \in (0,1]$, by $C_{x,p}^{\alpha/3,\alpha}(D)$, we denote the set of all bounded functions $f = f(x,p)$ such that

$$
[f]_{C_{x,p}^{\alpha/3,\alpha}(D)} := \sup_{(x_i,p_i) \in \overline{D} : (x_1,p_1) \neq (x_2,p_2)} \frac{|f(x_1,p_1) - f(x_2,p_2)|}{(|x_1 - x_2|^{1/3} + |p_1 - p_2|)^\alpha} < \infty.
$$

    Furthermore, the norm is given by

$$
\|f\|_{C_{x,p}^{\alpha/3,\alpha}(D)} := \|f\|_{L_\infty(D)} + [f]_{C_{x,p}^{\alpha/3,\alpha}(D)}. \tag{2.8}
$$

  - *Weighted spaces on the kinetic boundary.* For a weight $\omega \geq 0$ on $\partial\Omega \times \mathbb{R}^3$, we set

$$
\|f\|^2_{L_2(\Sigma_\pm^T, \omega)} = \int_{\Sigma_\pm^T} f^2 \omega \, dS_x dp. \tag{2.9}
$$

  - *Traces.* Let $r \in [1,\infty)$ and $f \in L_r(\Sigma^T)$ be a function such that $Yf \in L_r(\Sigma^T)$. Then, the traces of f can be defined (see the details in Appendix D). In particular, there exist functions (f_T, f_0, f_+, f_-), which we call traces of f, such that a variant of Green's identity holds (see Proposition D.2).
  - *Weighted Lebesgue space.* For $\theta \in \mathbb{R}$ and $r \in [1,\infty]$, by $L_{r,\theta}(G)$ we denote the set of all Lebesgue measurable functions u such that

$$
\|u\|_{L_{r,\theta}(G)} := \|p_0^\theta u\|_{L_r(G)} < \infty.
$$

– *Weighted Sobolev spaces.* For $r \in [1, \infty]$, by $W^1_{r,\theta}(\mathbb{R}^3)$ we denote the Banach space of functions $u \in L_{r,\theta}(\mathbb{R}^3)$ such that the norm

$$\|u\|_{W^1_{r,\theta}(\mathbb{R}^3)} := \||u| + |\nabla_p u|\|_{L_{r,\theta}(\mathbb{R}^3)} < \infty.$$

For $\theta = 0$, we set $W^1_r(\mathbb{R}^3) := W^1_{r,0}(\mathbb{R}^3)$.

– *Dual Sobolev space.* Let $W^{-1}_{2,\theta}(\mathbb{R}^3)$ be the space of all distributions u such that

$$u = \partial_{p_i} \eta_i + \xi \tag{2.10}$$

for some $\xi, \eta_i \in L_{2,\theta}(\mathbb{R}^3), i = 1, 2, 3$. Furthermore, for $u \in W^{-1}_2(\mathbb{R}^3)$ and $f \in W^1_2(\mathbb{R}^3)$, by

$$\langle u, f\rangle = \int_{\mathbb{R}^3} (-\eta_i \cdot \partial_{p_i} f + \xi f)\, dp, \tag{2.11}$$

we denote the duality pairing between $W^{-1}_2(\mathbb{R}^3)$ and $W^1_2(\mathbb{R}^3)$, which is independent of the choice of η_i and ξ.

– *Nonrelativistic (Newtonian) kinetic Sobolev space.*

$$S^N_r(G) = \{f \in L_r(G) : (\partial_t + p \cdot \nabla_x) f, \nabla_p f, D^2_p f \in L_r(G)\},$$

and the norm is defined as follows:

$$\|f\|_{S^N_r(G)} = \||f| + |\nabla_p f| + |D^2_p f| + |(\partial_t + p \cdot \nabla_x) f|\|_{L_r(G)}. \tag{2.12}$$

– *Mixed-norm spaces.* For normed spaces X and Y, we write $u = u(x, y) \in XY$ if for each x, $u_x := u(x, \cdot) \in Y$, and

$$\|u\|_{XY} := \|\|u_x\|_Y\|_X < \infty.$$

– *Weighted unsteady relativistic kinetic Sobolev spaces.* Let $S_{r,\theta}(G) = \{f \in L_{r,\theta}(G) : Y f, \nabla_p f, D^2_p f \in L_{r,\theta}(G)\}$ be the Banach space with the norm

$$\|f\|_{S_{r,\theta}(G)} = \||f| + |\nabla_p f| + |D^2_p f| + |Y f|\|_{L_{r,\theta}(G)}. \tag{2.13}$$

In the case when $\theta = 0$, we set $S_r(G) = S_{r,0}(G)$.

– *Steady S_r spaces.* For $r \in [1, \infty]$, by $S_{r,\theta}(\Omega \times \mathbb{R}^3)$, we denote the set of all functions u on $\Omega \times \mathbb{R}^3$ such that

$$u, v(p) \cdot \nabla_x u, \nabla_p u, D^2_p u \in L_{r,\theta}(\Omega \times \mathbb{R}^3). \tag{2.14}$$

The norm is given by

$$\|u\|_{S_{r,\theta}(\Omega\times\mathbb{R}^3)} = \||u| + |v(p) \cdot \nabla_x u| + |\nabla_p u| + |D^2_p u|\|_{L_{r,\theta}(\Omega\times\mathbb{R}^3)}. \tag{2.15}$$

– *Vector fields.* We use boldface letters to denote vector fields. We write $\boldsymbol{u} \in X$, where X is some vector space if each component of $\boldsymbol{u}$ belongs to X.

• *Conventions.*
 – We assume the summation with respect to repeated indexes.
 – If functions f and g are defined on $D \subset \mathbb{R}^3$ and $\mathbb{R}^3$, respectively, and g vanishes outside D, then, for $x \notin D$, we set $(fg)(x) = 0$.
 – By $N = N(\cdots)$, we denote a constant depending only on the parameters inside the parentheses. The constants N might change from line to line. Sometimes, when it is clear what parameters N depends on, we omit them.

3. Main results. Let $f = (f^+, f^-)$ be perturbations of $F^{\pm}$ near the relativistic Maxwellian given by

$$F^{\pm} = J + J^{1/2} f^{\pm}$$

(see (1.3)). We denote

$$\begin{aligned} & Lf = -Af - Kf, \\ & A_{\pm} f = 2J^{-1/2}\mathcal{C}(J^{1/2} f^{\pm}, J), \quad Af = (A_+ f, A_- f), \\ & Kf = J^{-1/2}\mathcal{C}\big(J, J^{1/2}(f^+ + f^-)\big)\boldsymbol{\xi}_0, \\ & \Gamma_{\pm}(f,g) = J^{-1/2}\mathcal{C}\big(J^{1/2} f^{\pm}, J^{1/2}(g^+ + g^-)\big), \\ & \Gamma(f,g) = (\Gamma_+(f,g), \Gamma_-(f,g)), \end{aligned} \tag{3.1}$$

where $\mathcal{C}$ is defined by (1.2) and (2.5)–(2.7). Then, the triple $[f, \mathbf{E}, \mathbf{B}]$ satisfies the following system (see p. 276 in [32]):

$$\begin{aligned} Yf = & -\boldsymbol{\xi}(\mathbf{E} + v(p) \times \mathbf{B}) \cdot \nabla_p f + \frac{\boldsymbol{\xi}_1}{2}(v(p) \cdot \mathbf{E}) f + Af \\ & + \boldsymbol{\xi}(v(p) \cdot \mathbf{E}) J^{1/2} + Kf + \Gamma(f,f), \end{aligned} \tag{3.2}$$

$$f(0,\cdot) = f_0(\cdot), \ f(t,x,p) = f(t,x,R_x p), \ z \in \Sigma_-^T,$$

$$\partial_t \mathbf{E} - \nabla_x \times \mathbf{B} = -\boldsymbol{j}_f := -\int v(p) J^{1/2}(p) f(p) \cdot \boldsymbol{\xi}\, dp, \tag{3.3}$$

$$\partial_t \mathbf{B} + \nabla_x \times \mathbf{E} = 0, \tag{3.4}$$

$$\nabla_x \cdot \mathbf{E} = \rho_f := \int J^{1/2} f(p) \cdot \boldsymbol{\xi}\, dp, \quad \nabla_x \cdot \mathbf{B} = 0, \tag{3.5}$$

$$(\mathbf{E} \times n_x)_{|\partial\Omega} = 0, \ (\mathbf{B} \cdot n_x)_{|\partial\Omega} = 0, \ \mathbf{E}(0,\cdot) = \mathbf{E}_0(\cdot), \ \mathbf{B}(0,\cdot) = \mathbf{B}_0(\cdot), \tag{3.6}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_1$ are defined in (2.4). For the sake of convenience, we also call (3.2)–(3.6) the RVML system.

Before we state the definition of the strong solution to the RVML system, we introduce the notions of finite energy and strong solutions to the linear relativistic Landau equation

$$Yf - \nabla_p \cdot (\sigma_g \nabla_p f) + b \cdot \nabla_p f + (c + \lambda) f = \eta, \tag{3.7}$$

$$f(t,x,p) = f(t,x,R_x p), \, z \in \Sigma_-^T, \ f(0,\cdot) = f_0(\cdot), \tag{3.8}$$

where

$$\sigma_g(t,x,p) = \int_{\mathbb{R}^3} \Phi(P,Q)(2J + J^{1/2} g(t,x,q))\, dq. \tag{3.9}$$

Remark 3.1. The Landau equation (3.2) can be rewritten as (3.7) with $g = f^+ + f^-$ in (3.9), $\lambda = 0$, and b, c, η depending on f. See the details in the proof of Proposition 6.3 (cf. (6.17)).

Definition 3.1 (finite energy solution). *We say that*

$$f \in C([0,T]) L_2(\Omega \times \mathbb{R}^3) \cap L_2((0,T) \times \Omega) W_2^1(\mathbb{R}^3)$$

is a finite energy solution to (3.7)–(3.8) *if for any test function* ϕ *satisfying*

(3.10) $\phi \in L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3), Y\phi \in L_2(\Sigma^T),$

(3.11) $\phi \in C([0,T])L_2(\Omega\times\mathbb{R}^3),$

(3.12) $\phi(t,x,p) = \phi(t,x,R_x p),\ (t,x,p)\in\Sigma_-^T$ *(in the trace sense),*

and all $t\in[0,T]$*, one has*

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^3} (f\phi)(t,x,p) - f_0(x,p)\phi(0,x,p)\,dxdp - \int_{(0,t)\times\Omega\times\mathbb{R}^3} f(Y\phi)\,dz \\ + \int_{(0,t)\times\Omega\times\mathbb{R}^3}\Big((\nabla_p\phi)^T\sigma_g\nabla_p f + (b\cdot\nabla_p f)\phi + (c+\lambda)f\phi\Big)\,dz \\ = \int_{(0,t)\times\Omega}\langle\eta(\tau,x,\cdot),\phi(\tau,x,\cdot)\rangle\,dxd\tau,\end{aligned} \tag{3.13}$$

where $\langle\cdot,\cdot\rangle$ *is the duality pairing between* $W_2^{-1}(\mathbb{R}^3)$ *and* $W_2^1(\mathbb{R}^3)$ *(see* (2.11)*).*

Furthermore, let g*,* b*,* c*,* f*, and* η *be* t*-independent functions. Then, we say that* $f\in L_2(\Omega)W_2^1(\mathbb{R}^3)$ *is a finite energy solution to the steady equation*

$$\begin{aligned}& v(p)\cdot\nabla_x f - \nabla_p\cdot(\sigma_g\nabla_p f) + b\cdot\nabla_p f + (c+\lambda)f = \eta,\\ & f(x,p) = f(x,R_x p),\ z\in\gamma_-,\end{aligned} \tag{3.14}$$

if for any test function $\phi=\phi(x,p)$ *satisfying the conditions analogous to* (3.10)–(3.12)*, the "steady" counterpart of the identity* (3.13) *holds.*

Remark 3.2. By Lemma D.5, if ϕ satisfies the conditions (3.10), (3.12), and $\phi_0\in L_2(\Omega\times\mathbb{R}^3)$ in the trace sense (see Definition D.4), then $\phi\in C([0,T])L_2(\Omega\times\mathbb{R}^3)$.

Definition 3.2. *We say that* $f\in S_2(\Sigma^T)$ *is a strong solution to* (3.7)–(3.8) *if*

- *the identity* (3.7) *holds in the* $L_2(\Sigma^T)$ *sense,*
- *the initial condition and the SRBC in* (3.8) *hold a.e. for the trace functions* f_0 *and* $f_\pm$ *(see Definition* D.4*).*

Similarly, we define a strong solution to the steady counterpart of (3.7)–(3.8)*.*

Remark 3.3. By using the Green's identity in (D.4), one can show that any strong solution is also a finite energy solution. Conversely, if f is a finite energy solution such that $f\in S_2(\Sigma^T)$, then f is a strong solution.

Definition 3.3. *We say that the VML system* (3.2)–(3.6) *has a strong solution* $[f,\mathbf{E},\mathbf{B}]$ *on the time interval* $[0,T]$ *if*

– f *is a strong solution to the Landau equation* (3.2) *(see Definition* 3.2*),*
– $\mathbf{E},\mathbf{B}\in C^1\big([0,T],L_2(\Omega)\big)$*,*
– *for any* $t\in[0,T]$*,* $\mathbf{E}(t,\cdot),\mathbf{B}(t,\cdot)\in W_2^1(\Omega)$*, and* $(\mathbf{E}(t,\cdot)\times n_x)_{|\partial\Omega}\equiv 0$*,* $(\mathbf{B}(t,\cdot)\cdot n_x)_{|\partial\Omega}\equiv 0$*,*
– *the identities* (3.3)–(3.5) *hold in the* $L_2((0,T)\times\Omega)$ *sense.*

Assumption 3.4. The domain Ω satisfies the following variant of the div-curl estimate. For any $r\in(1,\infty)$ and any $\boldsymbol{u}\in L_r(\Omega)$ such that

– $\nabla_x\times\boldsymbol{u}\in L_r(\Omega)$, $\nabla_x\cdot\boldsymbol{u}\in L_r(\Omega)$,
– either $(\boldsymbol{u}\times n_x)_{|\partial\Omega}=0$ or $(\boldsymbol{u}\cdot n_x)_{|\partial\Omega}=0$,

one has $\boldsymbol{u}\in W_r^1(\Omega)$, and

$$\|\boldsymbol{u}\|_{W_r^1(\Omega)} \le N\||\nabla_x\times\boldsymbol{u}| + |\nabla_x\cdot\boldsymbol{u}| + |\boldsymbol{u}|\|_{L_r(\Omega)}, \tag{3.15}$$

where $N=N(r,\Omega)$.

Remark 3.5. Loosely speaking, if Ω can be transformed into a simply connected domain of class $C^{1,1}$ by removing a finite number of "cuts," then Ω satisfies Assumption 3.4. See Hypothesis 1.1 and Theorems 3.2–3.3 in [1]. We point out that a solid torus $B_1 \times S^1$ satisfies the aforementioned assumption since one needs to make a single cut to obtain a simply connected $C^{1,1}$ domain.

Remark 3.6. A reader might be familiar with a variant of (3.15) where the right-hand side (r.h.s.) does not contain the term $\|\boldsymbol{u}\|_{L_r(\Omega)}$ (see [34]). However, in the case when the boundary condition $(\boldsymbol{u} \cdot n_x)_{|\partial\Omega} = 0$ is imposed, such an estimate might be false if Ω is not simply connected. We refer the reader to a beautiful counterexample in section 9 of [4].

We will construct the solution to the VML system via a Picard type iteration argument. It turns out that to close such an argument, one needs to control the temporal derivatives up to order $m \geq 20$ of the particle density functions and the electromagnetic field (see (3.46)–(3.50)). To this end, one needs the initial data to satisfy certain regularity and compatibility conditions. Loosely speaking, those are the conditions on the temporal derivatives at $t = 0$. One can deduce the expression of such derivatives from the RVML system as follows. Given $[\partial_t^k f(0,x,p), \partial_t^k \mathbf{E}(0,x), \partial_t^k \mathbf{B}(0,x)]$, we formally apply the operator ∂_t^k to (3.2), (3.3)–(3.4), plug $t = 0$, and solve for $[\partial_t^{k+1} f(0,x,p), \partial_t^{k+1} \mathbf{E}(0,x), \partial_t^{k+1} \mathbf{B}(0,x)]$.

Definition 3.4. *We set* $[f_{0,0}, \mathbf{E}_{0,0}, \mathbf{B}_{0,0}] = [f_0, \mathbf{E}_0, \mathbf{B}_0]$. *Furthermore, given* $f_{0,j}(x,p)$, $\mathbf{E}_{0,j}(x)$, $\mathbf{B}_{0,j}(x), j = 0, \dots, k$, *we set*

(3.16)
$$f_{0,k+1} = -v(p) \cdot \nabla_x f_{0,k} + (A+K) f_{0,k} + \boldsymbol{\xi}_1 (v(p) \cdot \mathbf{E}_{0,k}) J^{1/2} + \sum_{j=0}^{k} \binom{k}{j} \times \Big(-\boldsymbol{\xi}(\mathbf{E}_{0,j} + v(p) \cdot \mathbf{B}_{0,j}) \cdot \nabla_p f_{0,k-j} + \frac{\boldsymbol{\xi}_1}{2} (v(p) \cdot \mathbf{E}_{0,j}) f_{0,k-j} + \Gamma(f_{0,j}, f_{0,k-j}) \Big),$$

(3.17)
$$\mathbf{E}_{0,k+1}(x) := \nabla_x \times \mathbf{B}_{0,k}(x) - \int_{\mathbb{R}^3} v(p) J^{1/2}(p) f_{0,k}(x,p) \cdot \boldsymbol{\xi} \, dp,$$

(3.18)
$$\mathbf{B}_{0,k+1} = -\nabla_x \times \mathbf{E}_{0,k}.$$

Assumption 3.7 (compatibility conditions). We assume

$$f_{0,k} \text{ is a finite energy solution to (3.16) with the SRBC, } k \leq m-1, \tag{3.19}$$

$$f_{0,k}(x,p) = f_{0,k}(x, R_x p),\ (x,p) \in \gamma_- \text{ (in the trace sense)},\ k \leq m-8, \tag{3.20}$$

$$(\mathbf{E}_{0,k} \times n_x)|_{\partial\Omega} \equiv 0,\ (\mathbf{B}_{0,k} \cdot n_x)_{\partial\Omega} \equiv 0,\ k \leq m-1, \tag{3.21}$$

$$\nabla \cdot \mathbf{B}_{0,k} \equiv 0, \quad \nabla \cdot \mathbf{E}_{0,k}(x) = \int J^{1/2}(p) f_{0,k}(x,p) \cdot \boldsymbol{\xi} \, dp,\ k \leq m-1, \tag{3.22}$$

where in (3.20), we implicitly assume that $f_{0,k}, \frac{p}{p_0} \cdot \nabla_x f_{0,k} \in L_2(\Omega \times \mathbb{R}^3)$, so that the trace is well defined.

Remark 3.8. Here, we show that for $k \geq 1$, (3.22) can be derived formally from (3.22) with $k = 0$. The first identity in (3.22) follows directly from (3.18). Due to (3.17), to prove the second one, it suffices to demonstrate that for $k = 0, \dots, m$,

$$\int_{\mathbb{R}^3} J^{1/2} f_{0,k+1} \cdot \boldsymbol{\xi}\, dp + \nabla_x \cdot \int_{\mathbb{R}^3} v(p) J^{1/2} f_{0,k} \cdot \boldsymbol{\xi}\, dp = 0. \tag{3.23}$$

To this end, we denote $F_{0,k} = J + J^{1/2} f_{0,k}$ and note that the function $F_{0,k+1}$ satisfies

(3.24)

$$\begin{aligned} F^+_{0,k+1} &= -v(p)\cdot\nabla_x F^+_{0,k} + \sum_{j=0}^{k} \binom{k}{j} \\ &\quad \times \left((\mathbf{E}_{0,j} + v(p)\times \mathbf{B}_{0,j})\cdot\nabla_p F^+_{0,k-j} + \mathcal{C}(F^+_{0,j}, F^+_{0,k-j}) + \mathcal{C}(F^+_{0,j}, F^-_{0,k-j}) \right) = 0, \end{aligned}$$

(3.25)

$$\begin{aligned} F^-_{0,k+1} &= -v(p)\cdot\nabla_x F^-_{0,k} + \sum_{j=0}^{k} \binom{k}{j} \\ &\quad \times \left(-(\mathbf{E}_{0,j} + v(p)\times \mathbf{B}_{0,j})\cdot\nabla_p F^-_{0,k-j} + \mathcal{C}(F^-_{0,j}, F^-_{0,k-j}) + \mathcal{C}(F^-_{0,j}, F^+_{0,k-j}) \right) = 0. \end{aligned}$$

The above identities can be derived by using the definition of A, K, Γ (see (3.1)) and the fact that $\mathcal{C}(J,J) = 0$. One can also deduce (3.24)–(3.25) by differentiating formally the first two equations in (1.1), plugging $t = 0$, and replacing $\partial_t^k F$ with $F_{0,k}$. Finally, subtracting (3.25) from (3.24), integrating over $p \in \mathbb{R}^3$, and using the definition of $\mathcal{C}$ (see (1.2)), we obtain (3.23).

Remark 3.9. One can show that Assumption 3.7 is satisfied if $f_0, \mathbf{E}_0$, and $\mathbf{B}_0$ are smooth compactly supported functions away from $\partial\Omega$, f_0 decays fast for large p, and (3.22) holds with $k = 0$.

We introduce the key functionals that will be controlled in the proof of the local existence. Let $\theta, \tau > 0$ be numbers, and let f and $[\mathbf{E}_f, \mathbf{B}_f]$ be sufficiently regular functions on Σ^τ and $(0,\tau)\times\Omega$, respectively.

Instant energy functionals. We introduce the baseline instant energy

$$\mathcal{E}_{||,f}(\tau) = \sum_{k=0}^{m} \left(\|\partial_t^k f(\tau,\cdot)\|^2_{L_2(\Omega\times\mathbb{R}^3)} + \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f](\tau,\cdot)\|^2_{L_2(\Omega)} \right), \tau > 0, \tag{3.26}$$

and the energy

$$\mathcal{E}_f(\tau) = \mathcal{E}_{||,f}(\tau) + \sum_{k=0}^{m-4} \|\partial_t^k f(\tau,\cdot)\|^2_{L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)}. \tag{3.27}$$

Higher regularity instant functional. Let $\Delta r \in (0, \frac{1}{42})$ and denote

$$r_1 = 2, \quad \frac{1}{r_i} = \frac{1}{r_{i-1}} - \left(\frac{1}{6} - \Delta r\right), i = 2,3,4, \tag{3.28}$$

$$r_2 \in (2,3), \quad r_3 \in (3,6), \quad r_4 > 14, \tag{3.29}$$

$$\begin{aligned} \mathcal{H}_f(\tau) &= \sum_{i=1}^{4} \sum_{k=0}^{m-4-i} \|\partial_t^k f(\tau,\cdot)\|^2_{S_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} \\ &\quad + \sum_{k=0}^{m-1} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f](\tau,\cdot)\|^2_{W_2^1(\Omega)} + \sum_{i=2}^{3} \sum_{k=0}^{m-4-i} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f](\tau,\cdot)\|^2_{W^1_{r_i}(\Omega)}. \end{aligned} \tag{3.30}$$

Total instant functional. The total instant functional is the sum of the total instant energy and the higher regularity functional:

$$\mathcal{I}_f(\tau) = \mathcal{E}_f(\tau) + \mathcal{H}_f(\tau), \tau > 0, \tag{3.31}$$

$$\mathcal{I}_f(0) \text{ is given by (3.31) with } \partial_t^k [f, \mathbf{E}_f, \mathbf{B}_f](\tau, \cdot) \text{ replaced with } [f_{0,k}, \mathbf{E}_{0,k}, \mathbf{B}_{0,k}]. \tag{3.32}$$

Dissipation functionals. The baseline dissipation is defined by

$$\mathcal{D}_{||,f}(\tau) = \sum_{k=0}^{m} \|\partial_t^k f(\tau, \cdot)\|^2_{L_2(\Omega) W_2^1(\mathbb{R}^3)} \tag{3.33}$$

and the total dissipation is

$$\mathcal{D}_f(\tau) = \mathcal{D}_{||,f}(\tau) + \sum_{k=0}^{m-4} \|\partial_t^k f(\tau, \cdot)\|^2_{L_2(\Omega) W^1_{2,\theta/2^k}(\mathbb{R}^3)}. \tag{3.34}$$

Total functional.

$$y_f(\tau) = \sup_{t \le \tau} \mathcal{I}_f(t) + \int_0^\tau \mathcal{D}_f(t)\, dt. \tag{3.35}$$

Here is the main result of the present paper.

THEOREM 3.10. *Let $m \ge 20$ be an integer, $r \in (14, \infty)$ be a number, and Ω be a $C^{1,1}$ bounded domain satisfying Assumption 3.4. Then, there exists a constant $\theta_0 = \theta_0(r_1, \dots, r_4) > 1$ such that for any $\theta \ge \theta_0$ there exist constants*

$$\begin{aligned} &M = M(r_1, \dots, r_4, m, \theta, \Omega) > 1, \\ &\varepsilon_0 = \varepsilon_0(\theta, r_1, \dots, r_4, m, \Omega) \in (0,1), \quad T = T(\theta, r_1, \dots, r_4, m, \Omega) \in (0,1) \end{aligned} \tag{3.36}$$

such that if $\mathcal{I}_f(0) < \infty$ (see (3.32)), and

–

$$\begin{aligned} \mathcal{E}_f(0) := &\sum_{k=0}^{m} \left(\|[\mathbf{E}_{0,k}, \mathbf{B}_{0,k}]\|^2_{L_2(\Omega)} + \|f_{0,k}\|^2_{L_2(\Omega \times \mathbb{R}^3)} \right) \\ &+ \sum_{k=0}^{m-4} \|f_{0,k}\|^2_{L_{2,\theta/2^k}(\Omega \times \mathbb{R}^3)} \le \varepsilon_0 / M \end{aligned} \tag{3.37}$$

(see (3.16)–(3.18)),

– *the compatibility conditions (3.19)–(3.22) in Assumption 3.7 hold,*

then the following assertions hold.

(i) *RVML system (3.2)–(3.6) has a strong solution $[f, \mathbf{E}_f, \mathbf{B}_f]$ (see Definition 3.4) on Σ^T such that*

$$y_f(T) < \varepsilon_0 \text{ (see (3.35))}. \tag{3.38}$$

(ii) *For $k \le m$, $\partial_t^k f$ is a finite energy solution (see Definition 3.1) to (3.2) differentiated k times in t with the initial data $\partial_t^k f(0, \cdot) \equiv f_{0,k}(\cdot)$ and SRBC.*

(iii) *For $k \le m - 1$, $\partial_t^k [\mathbf{E}_f, \mathbf{B}_f] \in C([0,T]) L_2(\Omega) \cap L_\infty((0,T)) W_2^1(\Omega)$ is a strong solution to Maxwell's equations (3.3)–(3.4) differentiated k times with the initial data $[\mathbf{E}_{0,k}, \mathbf{B}_{0,k}]$ and the perfect conductor BC, whereas $\partial_t^m [\mathbf{E}_f, \mathbf{B}_f] \in C([0,T]) L_2(\Omega)$ is a weak solution (see [11]).*

(iv) *The identities* $\nabla_x \cdot \partial_t^k \mathbf{E}_f = \partial_t^k \rho_f$, $\nabla_x \cdot \partial_t^k \mathbf{B}_f = 0$ *hold thanks to the compatibility conditions* (3.22) *and the continuity equation*

$$\partial_t(\partial_t^k \rho_f) + \nabla_x \cdot \partial_t^k \boldsymbol{j}_f = 0, k \le m$$

(see (3.3), (3.5)*).*

(v) *In addition, if* $[f_i, \mathbf{E}_{f_i}, \mathbf{B}_{f_i}], i = 1, 2$, *are strong solutions to the RVML system on* Σ^T *satisfying the bound* (3.38), *then we have* $f_1 = f_2$ *on* Σ^T *and* $\mathbf{E}_{f_1} = \mathbf{E}_{f_2}$, $\mathbf{B}_{f_1} = \mathbf{B}_{f_2}$ *on* $(0, T) \times \Omega$.

3.1. Iteration scheme. To prove the existence, we set up an iteration scheme (cf. [32]). Let $[f^0, \mathbf{E}^0, \mathbf{B}^0] = [f_0, \mathbf{E}_0, \mathbf{B}_0]$, and, given $[f^n, \mathbf{E}^n, \mathbf{B}^n]$, we set $[f^{n+1}, \mathbf{E}^{n+1}, \mathbf{B}^{n+1}]$ to be the strong solution to the following linear system (see Proposition 6.2):

$$\begin{aligned} &Y f^{n+1} + \boldsymbol{\xi}(\mathbf{E}^n + v(p) \times \mathbf{B}^n) \cdot \nabla_p f^{n+1} - \frac{\boldsymbol{\xi}}{2}(v(p) \cdot \mathbf{E}^n) f^{n+1} + L f^{n+1} \\ &\quad = \boldsymbol{\xi}_1 (v(p) \cdot \mathbf{E}^{n+1}) J^{1/2} + \Gamma(f^{n+1}, f^n), \end{aligned} \tag{3.39}$$

$$f^{n+1}(t, x, p) = f^{n+1}(t, x, R_x p), \ z \in \Sigma_-^T, \quad f^{n+1}(0, \cdot) \equiv f_{0,0}, \tag{3.40}$$

$$\partial_t \mathbf{E}^{n+1} - \nabla_x \times \mathbf{B}^{n+1} = -\int v(p) J^{1/2}(p) f^{n+1}(p) \cdot \boldsymbol{\xi} \, dp, \tag{3.41}$$

$$\partial_t \mathbf{B}^{n+1} + \nabla_x \times \mathbf{E}^{n+1} = 0, \tag{3.42}$$

$$\nabla_x \cdot \mathbf{E}^{n+1} = \int J^{1/2} f^{n+1}(p) \cdot \boldsymbol{\xi} \, dp, \quad \nabla_x \cdot \mathbf{B}^{n+1} = 0, \tag{3.43}$$

$$(\mathbf{E}^{n+1} \times n_x)_{|\partial\Omega} = 0, \quad (\mathbf{B}^{n+1} \cdot n_x)_{|\partial\Omega} = 0, \tag{3.44}$$

$$\mathbf{E}^{n+1}(0, \cdot) \equiv \mathbf{E}_{0,0}(\cdot), \quad \mathbf{B}^{n+1}(0, \cdot) \equiv \mathbf{B}_{0,0}(\cdot), \tag{3.45}$$

where $L = -A - K$ is the linearized Landau operator (see (3.1)).

Setting $f := f^{n+1}$, $g := f^n$, $\mathbf{E}_f := \mathbf{E}^{n+1}$, $\mathbf{E}_g := \mathbf{E}^n$, $\mathbf{B}_f = \mathbf{B}^{n+1}$, $\mathbf{B}_g := \mathbf{B}^n$ gives

$$\begin{aligned} &Y f + \boldsymbol{\xi}(\mathbf{E}_g + v(p) \times \mathbf{B}_g) \cdot \nabla_p f - \frac{\boldsymbol{\xi}}{2}(v(p) \cdot \mathbf{E}_g) f + L f \\ &\quad = \boldsymbol{\xi}_1 (v(p) \cdot \mathbf{E}_f) J^{1/2} + \Gamma(f, g), \\ &f(0, \cdot) = f_{0,0}, \ f(t, x, p) = f(t, x, R_x p), \ z \in \Sigma_-^T, \end{aligned} \tag{3.46}$$

$$\partial_t \mathbf{E}_f - \nabla_x \times \mathbf{B}_f = -\int v(p) J^{1/2}(p) f(p) \cdot \boldsymbol{\xi} \, dp, \tag{3.47}$$

$$\partial_t \mathbf{B}_f + \nabla_x \times \mathbf{E}_f = 0, \tag{3.48}$$

$$\nabla_x \cdot \mathbf{E}_f = \rho_f = \int J^{1/2}(p) f(p) \cdot \boldsymbol{\xi} \, dp, \quad \nabla_x \cdot \mathbf{B}_f = 0, \tag{3.49}$$

$$(\mathbf{E}_f \times n_x)_{|\partial\Omega} = 0, \ (\mathbf{B}_f \cdot n_x)_{|\partial\Omega} = 0, \ \mathbf{E}_f(0, \cdot) \equiv \mathbf{E}_0(\cdot), \ \mathbf{B}_f(0, \cdot) \equiv \mathbf{B}_0(\cdot). \tag{3.50}$$

The following proposition is the crux of the proof of the existence part in Theorem 3.10.

PROPOSITION 3.11 (propagation of smallness). *Invoke the assumptions of Theorem* 3.10 *and let* θ, M, ε_0, *and* T *be the constants as in* (3.36). *Let* $g = (g^+, g^-), \mathbf{E}_g, \mathbf{B}_g$ *be a triple such that for each* $k \in \{0, \ldots, m\}$,

$$\partial_t^k g \in C([0, T]) L_2(\Omega \times \mathbb{R}^3), \ \partial_t^k [\mathbf{E}_g, \mathbf{B}_g] \in C([0, T]) L_2(\Omega), \ k \le m, \tag{3.51}$$

$$g(t, x, p) = g(t, x, R_x p), \ (t, x, p) \in \Sigma_-^T, \tag{3.52}$$

$$\partial_t^k g(0, \cdot) = f_{0,k}(\cdot) \ (\textit{see}\ (3.16)), \tag{3.53}$$

$$\partial_t^k [\mathbf{E}_g, \mathbf{B}_g](0, \cdot) = [\mathbf{E}_{0,k} \mathbf{B}_{0,k}](\cdot) \ (\textit{see}\ (3.17)\text{–}(3.18)). \tag{3.54}$$

Then, if θ and M are sufficiently large and ε_0 is sufficiently small, and

$$y_g(T) < \varepsilon_0, \quad (3.37) \text{ holds}, \tag{3.55}$$

then the linear RVML system (3.46)–(3.50) *with the initial conditions* $[f_0, \mathbf{E}_0, \mathbf{B}_0]$ *has a unique strong solution* $[f, \mathbf{E}_f, \mathbf{B}_f]$ *(see Definition* 3.4*). Furthermore, we have*

$$y_f(T) < \varepsilon_0, \tag{3.56}$$

and in addition, the assertions analogous to (ii)–(v) *hold for* $\partial_t^k[f, \mathbf{E}_f, \mathbf{B}_f]$. *Moreover, the conditions* (3.51)–(3.54) *hold with* $[g, \mathbf{E}_g, \mathbf{B}_g]$ *replaced with* $[f, \mathbf{E}_f, \mathbf{B}_f]$.

4. Method of the proof and organization of the paper.

The goal of this section is to highlight the key difficulties and to delineate the main ideas in the proof of Theorems 3.10 and Proposition 3.11. For the sake of clarity, we will omit some technical details.

4.1. Unique solvability and the velocity Hessian estimates for the linear Landau equation.

First, we need to show that for each n, the iteration scheme (3.39)–(3.45) is well-posed. We will focus on the case when $n = 0$ and will only consider (3.46) with $g = f_0$. We want to show that it has a unique strong solution f (see Definition 3.2), under the assumption that f_0, $\mathbf{E}_0$, and $\mathbf{B}_0$ are sufficiently regular functions. In addition, our argument will enable us to deduce that f and $\nabla_p f$ are bounded functions, which is important for proving both the existence and the uniqueness parts of Theorem 3.10.

Uniqueness and S_2 regularity. The basic difficulty in establishing the uniqueness of the boundary value problems for the velocity diffusive kinetic equations lies in the fact that for the natural solution class

$$f \in L_2((0,T) \times \Omega) W_2^1(\mathbb{R}^3), \; Yf \in L_2((0,T) \times \Omega) W_2^{-1}(\mathbb{R}^3),$$

it is unknown if the traces are well defined and if the energy identity for the transport operator Y holds. On the other hand, if $f, Yf \in L_2(\Sigma^T)$, then the traces are well defined (see Appendix D), and if, additionally, f satisfies the SRBC, then a variant of the energy identity does hold (see Lemma D.5). To summarize, we first construct a solution to (3.46) in the natural energy class. We show the uniqueness by establishing the S_2 regularity, i.e.,

$$Yf, D_p^2 f \in L_2(\Sigma^T). \tag{4.1}$$

Mirror-extension argument and S_2 regularity. To prove (4.1), we use an extension argument, which first appeared in [20] and was later used in the studies of the Vlasov–Poisson–Landau [7] and linear Landau [8] equations. First, we localize in the spatial and momentum variables by deriving an equation for f multiplied by a suitable cutoff function (see (5.25)–(5.26)). By using a flattening and extension argument (see the proof of Lemma 5.10), near the boundary, we reduce (3.46) to a parabolic PDE on the whole space (5.46) with discontinuous drift coefficients $\mathcal{X}$ (see (5.37), (5.43)). We point out that such a drift term is absent when the boundary is flat. Thus, one needs to use a Calderon–Zygmund type result to obtain (4.1). However, in contrast to [20], [7], [8], our new equation (5.46) is quite different from the Newtonian KFP type equation (see (4.2)) since the coefficient in the transport term depends on both spatial and momentum variables. We are not aware of any Calderon–Zygmund type result for such an equation. To overcome this difficulty, we make a change of variables in the momentum variable, which enables us to reduce (5.46) to a Newtonian KFP equation on the whole space (see (5.65)). Finally, we use the S_2^N (see (2.12)) estimate of [10] to deduce (4.1). We remark that for other prominent boundary conditions in kinetic theory, e.g., inflow and diffuse boundary conditions, such an extension argument does not work.

Higher regularity. Near the spatial boundary, we work with the Newtonian KFP equation (5.65), which we derived from (3.46). By using the Sobolev embedding theorem for S_r^N spaces (see [30]) and the S_r^N regularity theory developed in [10], we conclude that

$$f, \nabla_p f \in L_\infty(\Sigma^T).$$

S_r^N theory on the whole space for a Newtonian KFP equation with rough coefficients. Here, we want to highlight one of the main ingredients of the present paper, that is, the Calderon–Zygmund theory for nonrelativistic KFP equations established in [10]. We explain the importance of this theory by considering the equation

$$\partial_t f + p \cdot \nabla_x f - \Delta_p f = \eta$$

in Σ^T with the initial condition $f_0 \equiv 0$ and the SRBC. Near the boundary, one can use a flattening and an extension argument as in [20], [7], [8] to derive the following equation for the "mirror extension" $\overline{f}$ on $(0,T) \times \mathbb{R}^3_y \times \mathbb{R}^3_w$ (see section 2.1 in [8]):

$$\partial_t \overline{f} + w \cdot \nabla_y \overline{f} - a^{ij}(y) \partial_{w_i w_j} \overline{f} - \nabla_w \cdot (X \overline{f}) = \overline{\eta} \text{ in } \mathbb{R}^7_T,$$

where X is the "geometric" term which is quadratic w, depends on the curvature of Ω, and is discontinuous across the hyperplane $\{y_3 = 0\} \times \mathbb{R}^3_w$. Before the work [10], the unique solvability in the class of strong solutions and the global L_r estimate of $D^2_w \overline{f}, (\partial_t + w \cdot \nabla_y) \overline{f}$ was unknown for the equation

$$\begin{aligned} &\partial_t u + w \cdot \nabla_y u - a^{ij}(t,y,w) \partial_{w_i w_j} u \\ &\quad + b^i \partial_{w_i} u + cu = \eta \text{ in } \mathbb{R}^7_T, \quad u(0,\cdot) \equiv 0, \end{aligned} \tag{4.2}$$

with $a \in L_\infty((0,T)) C^{\alpha/3,\alpha}_{y,w}(\mathbb{R}^{2d})$. In particular, in the papers [3] and [29], the authors imposed the uniform continuity assumption with respect to the following "kinetic distance":

$$d_{\text{kin}}\left((t,y,w),(t',y',w')\right) = \max\{|t-t'|^{1/2}, |y-y'-(t-t')w'|^{1/3}, |w-w'|\}.$$

It is easy to see that even in dimension 1, the function $a^{ij}(t,y,w) = 2 + \sin(y)$ is not uniformly continuous on $\mathbb{R}^7$ with respect to d_{kin}. In contrast, the theory developed in [10] covers (4.2) with more general leading coefficients a^{ij} including the ones satisfying the uniform continuity with respect to the metric $d((y,w),(y',w')) = |y-y'|^{1/3} + |w - w'|$ uniformly in time. As we mentioned above, to show the uniqueness and higher regularity of (3.46), we reduce it to the one of the form (4.2) (see (5.65)). It turns out that for such an equation,

$$a^{ij} \in L_\infty((0,T)) C^{\alpha/3,\alpha}_{y,w}(\mathbb{R}^6),$$

and, hence, the results of [10] are applicable.

4.2. Propagation of smallness. Here, we highlight the main difficulties in the proof of Proposition 3.11 and describe the key features of the argument.

Issue 1: *L_∞ bound of the electromagnetic field.* As usual, to control the "cubic" terms involving $[\mathbf{E}_f, \mathbf{B}_f]$ in the energy argument, we need an L_∞ bound $[\mathbf{E}_f, \mathbf{B}_f]$. However, in contrast to the Vlasov–Poisson–Landau equation, we do not expect an "instant" regularization of the electromagnetic field due to the hyperbolic nature of Maxwell's equations.

To overcome this issue, we rewrite the Maxwell system into two div-curl systems:

$$\begin{cases} \nabla_x \times \mathbf{E}_f = -\partial_t \mathbf{B}_f, \\ \nabla_x \cdot \mathbf{E}_f = \int J^{1/2}(p) f(p) \cdot \boldsymbol{\xi}_1 \, dp, \\ (\mathbf{E}_f \times n_x)|_{\partial\Omega} = 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \nabla_x \times \mathbf{B}_f = \partial_t \mathbf{E}_f + \int v(p) J^{1/2}(p) f(p) \cdot \boldsymbol{\xi}_1 \, dp, \\ \nabla_x \cdot \mathbf{B}_f = 0, \\ (\mathbf{B}_f \cdot n_x)|_{\partial\Omega} = 0. \end{cases} \tag{4.4}$$

If we have the bound of the $L_\infty((0,T))L_2(\Omega)$ norm of $\partial_t[\mathbf{E}_f, \mathbf{B}_f]$, then, by using the div-curl estimate (3.15) with $r = 2$, we can bound the $L_\infty((0,T))W_2^1(\Omega)$ norm of $[\mathbf{E}_f, \mathbf{B}_f]$. This yields an $L_\infty((0,T))L_6(\Omega)$ estimate of $[\mathbf{E}_f, \mathbf{B}_f]$ due to the Sobolev embedding $W_2^1(\Omega) \subset L_6(\Omega)$. To achieve this, we differentiate Maxwell's equations with respect to t and use the div-curl estimate. It is clear now that to close the iteration argument, we need to control certain norms for $\partial_t^k[\mathbf{E}_f, \mathbf{B}_f]$ and $\partial_t^k f$ for $k \le m$ for some m.

Issue 2: *existence of the temporal derivatives of* $[f, \mathbf{E}_f, \mathbf{B}_f]$. One needs to justify that the higher-order temporal derivatives $\partial_t^k[f, \mathbf{E}_f, \mathbf{B}_f], k \le m$, are sufficiently regular functions. Let us consider the case when $k = 1$. By differentiating formally the linear VML system (3.46)–(3.50), we write down the initial boundary value problem for $\partial_t[f, \mathbf{E}_f, \mathbf{B}_f]$ with the initial data $[f_{0,1}, \mathbf{E}_{0,1}, \mathbf{B}_{0,1}]$ defined in (3.16)–(3.18). We then use the well-posedness theory for the KFP equation in the finite energy solution class, developed in section 5 of the present paper, and the well-known results for Maxwell's equations (see [11]). See the details in Proposition 6.2 and Appendix G. To apply these theories, one needs to impose certain regularity and compatibility conditions on the "initial data" $[f_{0,1}, \mathbf{E}_{0,1}, \mathbf{B}_{0,1}]$ (see (3.19)–(3.22)).

The scheme. The basic structure of the argument is similar to that in [7]. The primary functional that needs to be controlled throughout the iteration argument is the energy norm, while the $L_\infty((0,T))S_r(\Omega \times \mathbb{R}^3)$ (see (2.14)–(2.15)) estimates are needed for establishing higher regularity bounds of the lower-order t-derivatives for the closure of the energy argument. Here, we explain the main steps of the argument and the motivation for designing the functional y_f (see (3.35)).

- First, we derive the energy bound of $\partial_t^k f, k \le m$, by using the estimates of the terms A, K, and $\Gamma(f, g)$ established in [32]. As usual, to close such estimates, one needs to control the L_∞ norms of $\partial_t^k[f, \mathbf{E}_f, \mathbf{B}_f], k \le m/2$.
- To gain the L_∞ regularity of the lower-order derivatives of the electromagnetic field via the W_r^1 div-curl estimate, one needs to descend from the top-order temporal derivatives to lower-order ones. As we descend, the electromagnetic field gains integrability.
- To prove the L_∞ estimates of $\partial_t^k f, k \le m/2$, we use the weighted $L_\infty^t S_r(\Omega \times \mathbb{R}^3)$ estimate. Due to the presence of the term $(v(p) \cdot \mathbf{E}_f)\sqrt{J}\boldsymbol{\xi}_1$ in the linear Landau equation (3.46) and the loss of t-derivatives in the higher regularity

estimate of $\partial_t^k[\mathbf{E}_f, \mathbf{B}_f]$, we combined the $L_\infty^t S_r$ estimate with a descend argument.

- The specific gap between the unweighted energy control (up to $k \le m$) and the weighted energy one (up to $k \le m-4$) is motivated by the global estimate for the RVML system established in the subsequent paper [9].

4.3. Organization of the paper. The rest of the paper is organized as follows. In section 5, we establish the results concerning the existence, uniqueness, and higher regularity of the strong solution to the linear Landau equations (3.7)–(3.8) and (3.14). Additionally, in this section, we present the well-posedness result for the KFP equation in the class of finite energy solutions. In sections 6 and 7, we prove Proposition 3.11 and Theorem 3.10, respectively. In Appendices A–G, we collect various auxiliary results.

5. Regularity theory of the linear relativistic Landau equation with the specular reflection boundary condition. The purpose of this section is to present the results on the unique solvability and certain estimates for the linear equation (3.7)–(3.8) and its "steady" counterpart (3.14). This section is organized as follows. First, in section 5.1, we present the results on the strong solutions (see Definition 3.2) to the unsteady linear Landau equations (3.7)–(3.8). Second, in section 5.2, we establish the "steady" counterparts of the aforementioned results. Finally, in section 5.3, we prove the well-posedness of the unsteady linear Landau equation in the class of finite energy solutions (see Definition 3.1).

5.1. Strong solutions to the unsteady Landau equations.

Assumption 5.1. There exist $\varkappa \in (0,1]$ and $K > 0$ such that

$$\|g\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{x,p}(\Omega\times\mathbb{R}^3)} \le K, \tag{5.1}$$

$$\|\nabla_p g\|_{L_\infty(\Sigma^T)} \le K. \tag{5.2}$$

Assumption 5.2. For a.e. $(t,x,p) \in \Sigma^T_-$,

$$g(t,x,p) = g(t,x,R_x p). \tag{5.3}$$

We will use the fact that if g is sufficiently small ($J + \sqrt{J}g$ is near Maxwellian), then the leading coefficients σ_g are uniformly nondegenerate.

LEMMA 5.3. *There exists $\varepsilon_\star > 0$ and $\delta_0 \in (0,1)$ such that if for some $T > 0$, one has*

$$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon_\star, \tag{5.4}$$

and (5.2) *holds, then*

$$\sigma_g(z)\xi_i\xi_j \ge \delta_0|\xi|^2,\ z \in \Sigma^T, \xi \in \mathbb{R}^3, \quad \|[\sigma_g, \nabla_p \sigma_g]\|_{L_\infty(\Sigma^T)} \le \delta_0^{-1}. \tag{5.5}$$

The constants $\varepsilon_\star > 0$ and $\delta_0 \in (0,1)$ are independent of T.

PROPOSITION 5.4 (unique solvability in weighted S_2 spaces). *Let*

- $\lambda \ge 0$, $\kappa \in (0,1)$, $\varkappa \in (0,1], K > 0$ *be numbers,*
- Ω *be a $C^{1,1}$ bounded domain,*
- $b = (b^1, b^2, b^3)^T$ *and c be bounded measurable functions on $\mathbb{R}^7_T$ such that for some $K > 0$,*

$$\|b\|_{L_\infty(\Sigma^T)} + \|c\|_{L_\infty(\Sigma^T)} \le K, \tag{5.6}$$

– *g satisfy* (5.1)–(5.3),
– *the condition* (5.4) *hold.*

Then, there exists $\theta = \theta(\kappa, \varkappa) > 0$ such that if

$$\eta \in L_{2,\theta}(\Sigma^T), \quad f_0 \in S_{2,\theta}(\Omega \times \mathbb{R}^3), \tag{5.7}$$

then the following assertions hold.

(i) *There exists a unique strong solution f to* (3.7)–(3.8) *(see Definition* 3.2*).*

(ii) *We have $f \in S_{2,\kappa\theta}(\Sigma^T)$, and, furthermore,*

$$\begin{aligned} &\|f\|_{L_2((0,T)\times\Omega)W^1_{2,\theta}(\mathbb{R}^3)} \\ &\quad \le N_1 \|\eta\|_{L_{2,\theta}(\Sigma^T)} + N_1 \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\|f\|_{S_{2,\kappa\theta}(\Sigma^T)} + \|f\|_{L_{14/5,\kappa\theta}(\Sigma^T)} + \|\nabla_p f\|_{L_{7/3,\kappa\theta}(\Sigma^T)} \\ &\quad \le N_2 \|\eta\|_{L_{2,\theta}(\Sigma^T)} + N_2(1+\lambda)\|f_0\|_{S_{2,\theta}(\Omega\times\mathbb{R}^3)} + N_2\|f\|_{L_{2,\theta}(\Sigma^T)}, \end{aligned} \tag{5.9}$$

where $N_1 = N_1(K, \delta_0, T)$, $N_2 = N_2(\delta_0, \varkappa, \kappa, K, \theta, \Omega) > 0$.

Remark 5.5. Invoke the assumptions of Proposition 5.4 and let f be the strong solution to (3.7)–(3.8). Then, f satisfies the mirror-extension property, which is defined (imprecisely) below. We will make this statement precise in the proof of the present remark (below).

Let $\xi_n, n \ge 1$, be a dyadic partition of unity in $\mathbb{R}^3$ and let $\chi_k, k = 1, \dots, m$ be a partition of unity in Ω. A strong solution f satisfies the mirror-extension property if, near the boundary, $f_{k,n} := f\chi_k\xi_n$ can be "extended" to a function $\tilde{\tilde{\mathcal{U}}}$ satisfying the identity

$$\begin{aligned} &\partial_t \tilde{\tilde{\mathcal{U}}} + v \cdot \nabla_y \tilde{\tilde{\mathcal{U}}} - \nabla_v \cdot (a(t,y,v)\nabla_v \tilde{\tilde{\mathcal{U}}}) \\ &+ b^i \partial_{v_i} \tilde{\tilde{\mathcal{U}}} + c\tilde{\tilde{\mathcal{U}}} = \psi \text{ in } \mathbb{R}^7_T \end{aligned}$$

for certain a, b, c, and ψ, which are "under control."

Proposition 5.6 (higher regularity of a strong solution). *Invoke the assumptions of Proposition* 5.4 *and let $r > 2$ be a number. Then, there exists a constant $\theta = \theta(\kappa, \varkappa, r) > 0$ such that if, additionally,*

$$\eta \in L_{2,\theta}(\Sigma^T) \cap L_{r,\theta}(\Sigma^T), \quad f_0 \in S_{2,\theta}(\Omega \times \mathbb{R}^3) \cap S_{r,\theta}(\Omega \times \mathbb{R}^3),$$

then for the strong solution to (3.7)–(3.8), *one has*

$$f \in S_{2,\kappa\theta}(\Sigma^T) \cap S_{r,\kappa\theta}(\Sigma^T), \tag{5.10}$$

$$\begin{aligned} \|f\|_{S_{r,\kappa\theta}(\Sigma^T)} \le N &\sum_{s\in\{2,r\}} \big(\|\eta\|_{L_{s,\theta}(\Sigma^T)} + (1+\lambda)\|f_0\|_{S_{s,\theta}(\Omega\times\mathbb{R}^3)}\big) \\ &+ N\|f\|_{L_{2,\theta}(\Sigma^T)}, \end{aligned} \tag{5.11}$$

where $N = N(\delta_0, \kappa, \varkappa, r, K, \theta, \Omega)$. Furthermore,

– *if $r \in (2,7)$, we have*

$$\|f\|_{L_{r_1,\kappa\theta}(\Sigma^T)} + \|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Sigma^T)} \le \text{r.h.s. of } (5.11), \tag{5.12}$$

where $r_1, r_2 > 1$ are numbers satisfying the relations

$$\frac{1}{r_1} = \frac{1}{r} - \frac{1}{7}, \quad \frac{1}{r_2} = \frac{1}{r} - \frac{1}{14}, \tag{5.13}$$

– *if* $r \in (7, 14)$,

$$\|f\|_{L_{\infty,\kappa\theta}(\Sigma^T)} + \|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Sigma^T)} \le \text{r.h.s. of } (5.11), \tag{5.14}$$

where r_2 *is defined in* (5.13),

– *if* $r > 14$, *then, for any* $\alpha \in (0, 1 - 14/r)$, *one has*

$$\begin{aligned}\sum_{s \in \{2,\infty\}} \|f\|_{L_\infty((0,T)\times\Omega) W^1_{s,\kappa\theta}(\mathbb{R}^3)} \\ + \|[f, \nabla_p f]\|_{L_\infty((0,T)) C^{\alpha/3,\alpha}_{x,p}(\Omega\times\mathbb{R}^3)} \le \text{r.h.s. of } (5.11).\end{aligned} \tag{5.15}$$

In all the estimates (5.12), (5.14), *and* (5.15), *one needs to take into account the dependence of* N *on* r_1, r_2, *and* α.

Remark 5.7. We point out that for the Newtonian KFP equation on the whole space, there is no loss of the weight in the momentum variable in the S^N_r (see (2.12)) estimate (see [10]). Furthermore, in [8], the present authors have established an S^N_r estimate with the loss of weight in the presence of a spatial boundary. In the relativistic case, we, loosely speaking, lose weight due to the presence of the spatial boundary and the relativistic transport term.

We will prove the assertions in the order we stated them.

Proof of Lemma 5.3. Denote

$$\sigma(p) = 2 \int_{\mathbb{R}^3} \Phi(P, Q) J(q)\, dq. \tag{5.16}$$

It is well known that σ is a bounded uniformly nondegenerate symmetric matrix-valued function (see Lemma B.2 (i)). The desired assertion follows from this and Lemma B.3. ∎

We will break down the proof of Proposition 5.4 into three significant steps. The initial two will be explained in Lemmas 5.8–5.10. Our argument goes as follows.

- First, we construct a variational solution, which we call the "finite energy weak solution" (see Lemma 5.8). It is a quadruple $(f, f^\star_+, f^\star_-, f^\star_T)$, where $f^\star_\pm$ and $f^\star_T$ are the functions that appear in the boundary terms in the integral formulation. We impose additional conditions $f_0 \in L_\infty(\Omega \times \mathbb{R}^3)$ and $\eta \in L_\infty(\Sigma^T), \nabla_p b \in L_\infty(\Sigma^T)$. The first two are needed to ensure that the boundary terms in the integral formulation are well defined for any test function $\phi \in C^{0,1}_0(\overline{\Sigma^T})$.
- By using a mirror-extension argument as in [20], [7], [8], we show that if θ is sufficiently large, then any "finite energy weak solution" is a strong solution (see Definition 3.2). To implement the mirror-extension argument in the integral formulation, one needs to work with general test functions that are Lipshitz up to the grazing set. This explains the necessity of the additional boundedness assumptions in the previous paragraph.
- We use a limiting argument to get rid of the boundedness conditions on f_0, η, and $\nabla_p b$. The resulting solution f satisfies $Yf \in L_2(\Sigma^T)$, so that the traces are well defined (see (D.2) in Lemma D.3), and they coincide with the functions $f^\star_\pm, f^\star_T, f_0$, and, in addition, the SRBC holds. We will also explain why the limiting procedure preserves the energy identity and the "mirror-extension property."

We first state a lemma concerning finite energy weak solutions to the general KFP equation

$$Yf - \nabla_p \cdot (a \nabla_p f) + b \cdot \nabla_p f + (c+\lambda) f = \eta, \tag{5.17}$$

$$f(t,x,p) = f(t,x,R_x p),\ z \in \Sigma_-^T,\ f(0,\cdot) = f_0(\cdot). \tag{5.18}$$

The nonrelativistic counterpart of this result was established in [8].

LEMMA 5.8. *Let*

– Ω *be a* $C^{1,1}$ *domain,*

– $a = a(z), z \in \mathbb{R}^7_T$ *be a bounded measurable function satisfying*

$$\delta |\xi|^2 \le a_{ij}(z) \xi_i \xi_j \le \delta^{-1} |\xi|^2 \quad \forall z \in \Sigma^T,\ \xi \in \mathbb{R}^3 \tag{5.19}$$

for some $\delta \in (0,1)$,

– b *and* c *satisfy* (5.6),

– $\nabla_p a, \nabla_p b \in L_\infty(\Sigma^T)$,

– $f_0 \in L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_\infty(\Omega \times \mathbb{R}^3)$,

– $\eta \in L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$.

Then, for any $\lambda \ge 0$, *there exists a quadruple* $(f,\ f_+^\star,\ f_-^\star,\ f_T^\star)$ *such that*

(i) $f, \nabla_p f \in L_{2,\theta}(\Sigma^T)$, $f_\pm^\star \in L_\infty(\Sigma_\pm^T)$, $f_T^\star \in L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_\infty(\Omega \times \mathbb{R}^3)$,

(ii) $f_-^\star(t,x,p) = f_+^\star(t,x,R_x p)$ *a.e. on* Σ_-^T,

(iii) *for any* $\phi \in C_0^{0,1}(\overline{\Sigma^T})$,

$$\begin{aligned}
&- \int_{\Sigma^T} (Y\phi) f \, dz \\
&+ \int_{\Omega \times \mathbb{R}^3} \big(f_T^\star(x,p) \phi(T,x,v) - f_0(x,p) \phi(0,x,p) \big) \, dx dp \\
&+ \int_{\Sigma_+^T} f_+^\star \phi \, |v(p) \cdot n_x| dS_x dp dt - \int_{\Sigma_-^T} f_-^\star \phi \, |v(p) \cdot n_x| dS_x dp dt \\
&+ \int_{\Sigma^T} (a \nabla_p f) \cdot \nabla_p \phi \, dz + \int_{\Sigma^T} (c+\lambda) f \phi \, dz + \int_{\Sigma^T} (b \cdot \nabla_p f) \phi \, dz = \int_{\Sigma^T} \eta \phi \, dz.
\end{aligned} \tag{5.20}$$

Furthermore, one has

$$\begin{aligned}
&\|f_T^\star\|_{L_{2,\theta}(\Omega \times \mathbb{R}^3)} + \|f\|_{L_2((0,T)\times\Omega) W^1_{2,\theta}(\mathbb{R}^3)} \\
&\quad \le N \|\eta\|_{L_{2,\theta}(\Sigma^T)} + N \|f_0\|_{L_{2,\theta}(\Omega \times \mathbb{R}^3)},
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
&\max\{\|f_T^\star\|_{L_\infty(\Omega \times \mathbb{R}^3)}, \|f_\pm^\star\|_{L_\infty(\Sigma_\pm^T)}, \|f\|_{L_\infty(\Sigma^T)}\} \\
&\quad \le \|\eta\|_{L_\infty(\Sigma^T)} + \|f_0\|_{L_\infty(\Omega \times \mathbb{R}^3)},
\end{aligned} \tag{5.22}$$

where $N = N(\delta, \theta, K, T)$.

We say that f *is a finite energy weak solution to* (5.17)–(5.18).

Remark 5.9. Here, we elaborate on various notions of weak solutions to the linear Landau equation (3.7)–(3.8) that we use in the present paper.

- *Finite energy solutions.* These are functions of class $C([0,T]) L_2(\Omega \times \mathbb{R}^3) \cap L_2((0,T) \times \Omega) W_2^1(\mathbb{R}^3)$ that satisfy the integral formulation of (3.7)–(3.8) with the test functions satisfying the SRBC. See Definition 3.1.
- *Finite energy weak solutions.* In the proof of the existence part in Proposition 5.4, we need to construct a "weak solution" in the class $f \in L_2((0,T) \times \Omega) W_2^1(\mathbb{R}^3)$ such that its integral formulation holds for test functions that

are Lipschitz up to the kinetic boundary $\partial\Omega \times \mathbb{R}^3$. This integral formulation is necessary for the mirror-extension trick, requiring well-defined "kinetic boundary" terms. Lemma 5.8 provides a weak solution meeting these requirements.

- *Very weak solutions.* We justify the uniqueness in the class of finite energy solutions via a duality argument, which works for a class of weaker solutions, which we call "very weak solutions." These are $L_2(\Sigma^T)$ functions satisfying an integral formulation with all derivatives "transferred" onto a test function satisfying the SRBC. See Definition 5.1.
- *Intermediate finite energy solutions.* In the proof of the existence of finite energy solutions (see the argument of Proposition 5.13), we first construct a solution in a slightly weaker class, where elements lack the temporal continuity in $L_2(\Omega \times \mathbb{R}^3)$ (see Definition 5.2).

We note that

$$\text{finite energy solution} \Longrightarrow \text{intermediate finite energy solution} \Longrightarrow \text{very weak solution},$$
$$\text{finite energy weak solution} \Longrightarrow \text{intermediate finite energy solution}.$$

The present authors also used the notion of the *finite energy weak solution* in the construction of a strong solution to a linear nonrelativistic Landau equation (see [8]).

Proof of Lemma 5.8. We repeat almost word-for-word the argument of Theorem 1.5 in [8] (see section 3 therein). Here, we delineate the argument. The main idea is to discretize the velocity diffusion to obtain a perturbed kinetic transport equation in a bounded domain for which the well-posedness problem is well understood (see, for example, [33] and [2]). We use the energy argument to derive uniform estimates with respect to the parameters of our approximation scheme. The key difficulty is to "preserve" the "boundary information" on $(0,T) \times \gamma_\pm$ in the weak* compactness argument. By designing a specific discretization of the velocity diffusion that respects the maximum principle, we are able to obtain L_∞ estimates of the solution and its traces that are uniform throughout the approximation scheme. ☐

We first prove the following lemma, which is Proposition 5.4 under more restrictive assumptions mentioned at the beginning of the section.

LEMMA 5.10. *Invoke the assumptions of Proposition* 5.4 *and assume, additionally,*

$$f_0 = 0, \quad \eta, \nabla_p b \in L_\infty(\Sigma^T). \tag{5.23}$$

Then, for sufficiently large $\theta = \theta(\kappa, \varkappa) > 0$, the following assertions hold.

(i) *Any finite energy weak solution f to* (5.17)–(5.18) *constructed in Lemma* 5.8 *must be a strong solution (see Definition* 3.2*).*

(ii) *The estimate* (5.9) *holds.*

Proof of Lemma 5.10. The proof is split into six steps. First, we localize in space and momentum variables and use a boundary flattening argument. Then, we use a mirror-extension argument (see Step 3) to "erase" the boundary conditions and reduce the equation to the one on the whole space. Then, we use a transformation that reduces the equation to a Newtonian KFP equation, and we apply the S_2^N (see (2.12)) estimate of [10].

Step 1: Localization. Let $\chi_k = \chi_k(x), k = 1, \dots, m$, be a standard partition of unity in Ω such that supp $\chi_1 \subset \Omega$, $0 \le \chi_k \le 1$, $k = 1, \dots, m$, and

$$|\nabla_x \chi_k| \le N/r_0, \quad \begin{cases} \chi_k = 1 & \text{in } B_{r_0/4}(\mathsf{x_k}), \\ \chi_k = 0 & \text{in } B^c_{r_0/2}(\mathsf{x_k}), \end{cases} \quad k = 2, \dots, m, \tag{5.24}$$

where $\mathsf{x_k} \in \partial\Omega, k = 2, \dots, m$.

Let $\xi_n = \xi_n(p), n \ge 1$, be sequence of functions such that

$$\begin{aligned} &\xi_0 \in C^\infty(B_1), \quad \xi_0 = 1 \text{ on } \{|p| \le 3/4\}, \\ &\xi_n \in C^\infty(\{2^{n-1} < |p| < 2^{n+3/2}\}), \quad \xi_n = 1 \text{ on } \{2^{n-1/2} \le |p| \le 2^{n+1}\}, \\ &|D^l_p \xi_n| \le N(l) 2^{-nl}, n, l \in \{1, 2, \dots\}. \end{aligned}$$

We will assume that $n \ge 2$ because the case $n = 1$ is handled in the same way.

We denote

$$f_{k,n}(z) := f(z)\chi_k(x)\xi_n(p)p_0^{\omega\theta} \tag{5.25}$$

and note that $f_{k,n}$ satisfies the identity

$$Y f_{k,n} - \nabla_p \cdot (\sigma_g \nabla_p f_{k,n}) + b \cdot \nabla_p f_{k,n} + (c + \lambda) f_{k,n} = \eta_{k,n} \tag{5.26}$$

in the sense of the integral identity (5.20), where

(5.27)
$$\begin{aligned} \eta_{k,n} = &(v(p) \cdot \nabla_x \chi_k) f p_0^{\omega\theta} \xi_n + \eta \chi_k p_0^{\omega\theta} \xi_n \\ &+ \chi_k \Big(- (\partial_{p_i} \sigma_g^{ij})(\partial_{p_j}(\xi_n p_0^{\omega\theta})) f - 2\sigma_g^{ij}(\partial_{p_i} f)(\partial_{p_j}(p_0^{\omega\theta}\xi_n)) - \sigma_g^{ij}\partial_{p_i p_j}(p_0^{\omega\theta}\xi_n) f \\ &\quad + \big(b \cdot \nabla_p(\xi_n p_0^{\omega\theta})\big) f \Big), \end{aligned}$$

where $\omega \in (0,1)$. We will focus on the near boundary case when $k \ge 2$. At the end of the proof, we discuss the case when $k = 1$. Our goal is to show that

$$\begin{aligned} &\| |Y f_{k,n}| + |D^2_p(f_{k,n})| \|_{L_2(\Sigma^T)} + \|f_{k,n}\|_{L_{14/5}(\Sigma^T)} + \|\nabla_p f_{k,n}\|_{L_{7/3}(\Sigma^T)} \\ &\quad \le N \|(|f| + |\nabla_p f| + |\eta|) 1_{2^{n-1} < |p| < 2^{n+3/2}}\|_{L_{2,\beta+\omega\theta}(\Sigma^T)} \end{aligned} \tag{5.28}$$

for some number $\beta = \beta(\varkappa) > 0$. When $n = 0$, the indicator function in (5.28) should be replaced with $1_{|p|<1}$.

If (5.28) is true, we take $\omega_1 \in (\omega, \frac{1+\omega}{2})$ and θ large, so that $\beta + \omega\theta < \omega_1\theta$, raise (5.28) to the power 2, and sum with respect to n and k. We obtain

$$\| |Y(f p_0^{\omega\theta})\|_{L_2(\Sigma^T)} + \|D^2_p(f p_0^{\omega\theta})| \|_{L_2(\Sigma^T)} \le N \| |f| + |\nabla_p f| + |\eta| \|_{L_{2,\omega_1\theta}(\Sigma^T)}. \tag{5.29}$$

Integrating by parts and using the Cauchy–Schwarz inequality, we get for $\varepsilon \in (0,1)$,

$$\begin{aligned} \int_{\Sigma^T} |\nabla_p f|^2 p_0^{2\omega_1\theta}\, dz \lesssim_{\theta,\omega_1} &\varepsilon \|\nabla_p f\|^2_{L_{2,\omega_1\theta}(\Sigma^T)} + \varepsilon^{-1}\|f\|^2_{L_{2,\omega_1\theta}(\Sigma^T)} \\ &+ \varepsilon \|D^2_p f\|^2_{L_{2,\omega\theta}(\Sigma^T)} + \varepsilon^{-1}\|f\|^2_{L_{2,(2\omega_1-\omega)\theta}(\Sigma^T)}. \end{aligned}$$

Due to our choice of ω_1, we have $2\omega_1 - \omega \le 1$. Hence, by choosing ε sufficiently small, we can drop the norm involving $\nabla_p f$ on the r.h.s. of (5.29) and replace ω_1 with 1 therein, and obtain the desired estimate of the weighted S_2 norm in (5.9). Similarly, we conclude the validity of the estimates of the second and the third terms of the l.h.s. of (5.9).

For the sake of convenience, we denote

$$U = f_{k,n} \quad H = \eta_{k,n}. \tag{5.30}$$

Without loss of generality, we may assume that $\omega = \frac{1}{2}$.

Step 2: Boundary flattening. We fix a point $\mathsf{x}_k \in \partial\Omega, k = 2, \dots, m$, and relabel it as x_0. There exists a function $\rho \in C_b^{1,1}(\mathbb{R}^2)$ such that

$$\begin{aligned} &\partial\Omega \cap B_{r_0}(x_0) \subset \{x : x_3 = \rho(x_1, x_2)\}, \\ &\Omega_{r_0}(x_0) := \Omega \cap B_{r_0}(x_0) \subset \{x : x_3 < \rho(x_1, x_2)\}. \end{aligned}$$

Let

$$\Psi : \Omega_{r_0}(x_0) \times \mathbb{R}^3 \to \mathbb{H}_- = \mathbb{R}^3_- \times \mathbb{R}^3, \quad (x, p) \to (y, w) \tag{5.31}$$

be the transformation given by

$$y = \psi(x), \quad w = (D\psi(x))p, \tag{5.32}$$

where ψ is the inverse of

$$\psi^{-1}(y) = \begin{pmatrix} y_1 \\ y_2 \\ \rho(y_1, y_2) \end{pmatrix} + y_3 \begin{pmatrix} -\rho_1^{(y_3)} \\ -\rho_2^{(y_3)} \\ 1 \end{pmatrix},$$

where $\rho_i = \partial_{x_i}\rho, i = 1, 2$, and ρ^ε is a standard mollification of ρ. It follows from the expression of the Jacobi matrix $(\frac{\partial x}{\partial y})$ (see (A.5)) that ψ is a local $C^{1,1}$ diffeomorphism.

A similar diffeomorphism was used to study the Newtonian KFP and the Landau equations in a bounded domain with the SRBC (see [20], [7], [8]). Ψ has two special features:

- it preserves the form of the Newtonian KFP equation in the sense explained in section 2.1 of [8];
- it preserves the SRBC, i.e.,

  $$\widehat{U}^\star_-(t, y_1, y_2, w) = \widehat{U}^\star_+(t, y_1, y_2, \boldsymbol{R}w), \text{ whenever } w_3 < 0, \tag{5.33}$$

  where

  $$\widehat{U}^\star_\pm(t, y_1, y_2, w) = U^\star_\pm(t, x(y_1, y_2, 0), p(y_1, y_2, 0, w)), \tag{5.34}$$

  and $U^\star_\pm$ were introduced in Lemma 5.8. The identity (5.33) follows from the fact that whenever $y_3 = 0$, one has

  $$\begin{aligned} (R_x p)(y, w) &= (p - 2(p \cdot n_x)n_x)(y, w) \\ &= \begin{pmatrix} w_1 + \rho_1 w_3 \\ w_2 + \rho_2 w_3 \\ \rho_1 w_1 + \rho_2 w_2 - w_3 \end{pmatrix} = \left(\frac{\partial x}{\partial y}\right)_{|y_3=0} \begin{pmatrix} w_1 \\ w_2 \\ -w_3 \end{pmatrix}, \end{aligned}$$

  where the Jacobi matrix is computed in (A.5).

The first property does not hold for the relativistic Fokker–Planck (see (5.39)). Nevertheless, this equation can still be reduced to a Newtonian KFP type equation (see Step 4 below).

Next, denote

$$\widehat{u}(t,y,w) = u(t,x(y),p(y,w)), \quad \mathsf{J}_\psi = \left|\det\left(\frac{\partial x}{\partial y}\right)\right|^2, \tag{5.35}$$

$$W = \frac{w}{\left(1+\left|\frac{\partial x}{\partial y}w\right|^2\right)^{1/2}}, \quad A = \left(\frac{\partial y}{\partial x}\right)\widehat{\sigma}_g\left(\frac{\partial y}{\partial x}\right)^T, \quad B = \left(\frac{\partial y}{\partial x}\right)\widehat{b}, \tag{5.36}$$

$$X = (X_1,X_2,X_3)^T = \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial p}{\partial y}\right)W = \left(\frac{\partial y}{\partial x}\right)\frac{\partial\left(\frac{\partial x}{\partial y}w\right)}{\partial y}W. \tag{5.37}$$

For a function Ξ on $\Omega_{r_0}(x_0)\times\mathbb{R}^3$, we denote

$$\widetilde{\Xi}(y,w) = \widehat{\Xi}(y,w)\mathsf{J}_\psi(y). \tag{5.38}$$

Changing variables in (5.20) (see section E.1), we conclude for any $\phi \in C_0^{0,1}([0,T]\times\overline{\Omega_{r_0}(x_0)}\times\mathbb{R}^3)$ (see the definition below the formula (2.7)), we have

(5.39)

$$\begin{aligned}
&-\int_{\mathbb{H}_-^T}(\partial_t\widehat{\phi}+W\cdot\nabla_y\widehat{\phi})\,\widetilde{U}\,dydwdt+\int_{\mathbb{H}_-}\widetilde{U}^*(T,y,w)\widehat{\phi}(T,y,w)\,dydw\\
&\quad+\int_{\mathbb{H}_-^T}\left((\nabla_w\widetilde{U})^TA\nabla_w\widehat{\phi}+\widetilde{U}X\cdot\nabla_w\widehat{\phi}+\left(B\cdot\nabla_w\widetilde{U}\right)\widehat{\phi}+(\widehat{c}+\lambda)\widetilde{U}\,\widehat{\phi}\right)dydwdt\\
&\quad+\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^3_+}|w_3|\widetilde{U}_+^\star\widehat{\phi}\,dy_1dy_2dwdt-\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^3_-}|w_3|\widetilde{U}_-^\star\widehat{\phi}\,dy_1dy_2dwdt\\
&=\int_{\mathbb{H}_-^T}\widehat{\phi}\,\widetilde{H}\,dydwdt,
\end{aligned}$$

where $\widetilde{U}_+^\star = \widehat{U}_+^\star\mathsf{J}_\psi|_{y_3=0}$.

Step 3: Mirror extension. For a function $\Xi=\Xi(x,p)$ on $\Omega_{r_0}(x_0)\times\mathbb{R}^3$, we denote

$$\overline{\Xi}(y,w) := \begin{cases}\widetilde{\Xi}(y,w), & (y,w)\in\mathbb{H}_-,\\ \widetilde{\Xi}(\boldsymbol{R}y,\boldsymbol{R}w), & (y,w)\in\mathbb{H}_+\end{cases} \tag{5.40}$$

(see (5.38)). We call $\overline{\Xi}$ the mirror extension of Ξ.

Next, let $G\subset\mathbb{R}^3$ be the even extension of $\psi(\Omega_{r_0}(x_0))\subset\overline{\mathbb{R}^3_-}$ across the plane $y_3=0$. We set

$$\mathcal{A}(t,y,w) = \begin{cases}A(t,y,w), & (t,y,w)\in(0,T)\times\overline{\psi(B_{r_0}(x_0))}\times\mathbb{R}^3,\\ \boldsymbol{R}\,A(t,\boldsymbol{R}y,\boldsymbol{R}w)\,\boldsymbol{R}, & (t,y,w)\in(0,T)\times(G\cap\mathbb{R}^3_+)\times\mathbb{R}^3,\end{cases} \tag{5.41}$$

$$\mathcal{B}(t,y,w) = \begin{cases}B(t,y,w), & (t,y,w)\in\overline{\mathbb{H}_-^T},\\ \boldsymbol{R}\,B(t,\boldsymbol{R}y,\boldsymbol{R}w), & (t,y,w)\in\mathbb{H}_+^T,\end{cases} \tag{5.42}$$

$$\mathcal{X}(y,w) = \begin{cases}X(y,w), & (y,w)\in\overline{\mathbb{H}_-},\\ \boldsymbol{R}\,X(\boldsymbol{R}y,\boldsymbol{R}w), & (y,w)\in\mathbb{H}_+,\end{cases} \tag{5.43}$$

$$\mathcal{W}(y,w) = \begin{cases}\dfrac{w}{\left(1+|V_1|^2\right)^{1/2}}, & (y,w)\in\overline{\psi(B_{r_0}(x_0))}\times\mathbb{R}^3,\\ \dfrac{w}{\left(1+|V_2|^2\right)^{1/2}}, & (y,w)\in(G\cap\mathbb{R}^3_+)\times\mathbb{R}^3,\end{cases} \tag{5.44}$$

where

$$V_1 = \left(\frac{\partial x}{\partial y}\right)(y)w, \quad V_2 = \left(\left(\frac{\partial x}{\partial y}\right)(\boldsymbol{R}y)\right)(\boldsymbol{R}w).$$

We also set $\mathcal{C}$ to be the even extension in y_3 and w_3 of $\widehat{c}$.

We now find an equation satisfied by $\overline{U}$. We fix a test function $\phi \in C^{0,1}([0,T) \times \overline{G} \times \mathbb{R}^3)$ vanishing for large z. Replacing $\widehat{\phi}$ with $\phi(t, \boldsymbol{R}y, \boldsymbol{R}w)$ in the identity (5.39) and changing variables $x \to \boldsymbol{R}x, w \to \boldsymbol{R}w$ give

(5.45)

$$\begin{aligned}
&- \int_{\mathbb{H}_+^T} (\partial_t \phi + \mathcal{W} \cdot \nabla_y \phi)\, \overline{U} dydwdt + \int_{\mathbb{H}_+} \phi(T, y, w) \overline{U}(y, w)\, dydw \\
&\quad + \int_{\mathbb{H}_+^T} \left((\nabla_w \overline{U})^T \mathcal{A} \nabla_w \phi + \overline{U} \mathcal{X} \cdot \nabla_w \phi + \left(\mathcal{B} \cdot \nabla_w \overline{U}\,\right) \phi + (\mathcal{C} + \lambda) \overline{U}\, \phi \right) dydwdt \\
&\quad + \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^3_-} |w_3| \overline{U}^{\star}_+ (t, y_1, y_2, \boldsymbol{R}w) \phi\, dy_1 dy_2 dwdt \\
&\quad - \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^3_+} |w_3| \overline{U}^{\star}_- (t, y_1, y_2, \boldsymbol{R}w) \phi\, dy_1 dy_2 dwdt \\
&= \int_{\mathbb{H}_+^T} \phi\, \overline{H}\, dydwdt.
\end{aligned}$$

Adding (5.45) to (5.39) with $\widehat{\phi}$ replaced with ϕ and using (5.33), we cancel the integrals over the incoming/outgoing boundaries and conclude that the mirror extension $\overline{U}$ satisfies the identity

$$\partial_t \overline{U} + \mathcal{W} \cdot \nabla_y \overline{U} - \nabla_w \cdot (\mathcal{A} \nabla_w \overline{U}) + \mathcal{B} \cdot \nabla_w \overline{U} - \nabla_w \cdot (\mathcal{X} \overline{U}) + (\mathcal{C} + \lambda) \overline{U} = \overline{H} \tag{5.46}$$

in the weak sense on $[0,T) \times \overline{G} \times \mathbb{R}^3$, i.e., for any $\phi \in C_0^{0,1}([0,T) \times \overline{G} \times \mathbb{R}^3)$,

(5.47)

$$\begin{aligned}
&- \int_{\mathbb{R}_T^7} (\partial_t \phi + \mathcal{W} \cdot \nabla_y \phi)\, \overline{U} dydwdt \\
&\quad + \int_{\mathbb{R}_T^7} \left((\nabla_w \overline{U})^T \mathcal{A} \nabla_w \phi + \overline{U} \mathcal{X} \cdot \nabla_w \phi + \left(\mathcal{B} \cdot \nabla_w \overline{U}\,\right) \phi + (\mathcal{C} + \lambda) \overline{U}\, \phi \right) dydwdt \\
&= \int_{\mathbb{R}_T^7} \phi\, \overline{H}\, dydwdt.
\end{aligned}$$

Step 4: Reducing (5.47) to a Newtonian KFP equation. Recall that $U(t, \cdot) = f_{k,n}(t, \cdot)$ is supported on $\Omega_{r_0/2}(x_0) \times \{2^{n-1} < |w| < 2^{n+3/2}\}$, and hence,

$$\widehat{U}(t, \cdot, \cdot) \text{ vanishes outside } B_{3r_0/4}(x_0) \times \{2^{n-3/2} < |w| < 2^{n+2}\} \tag{5.48}$$

for sufficiently small r_0. For any $y \in G$, we denote

$$\mathcal{W}_y(w) = \mathcal{W}(y, w) \tag{5.49}$$

(see (5.44)). By the assertion (ii) in Lemma A.2, for any $y \in G$, the mapping

$$\mathcal{W}_y : \{|w| < 2^{n+2}\} \to \mathbb{R}^3$$

is a diffeomorphism onto its image, and by (A.18)–(A.19), one has

$$\begin{aligned}(5.50)\qquad &\sup_{|w|<2^{n+2}} |D^j \mathcal{W}_y| \le N 2^{-jn}, j=0,1,2, \quad \sup_{\mathcal{W}_y(\{|w|<2^{n+2}\})} |D(\mathcal{W}_y)^{-1}| \le N 2^{3n},\\ &\sup_{\mathcal{W}_y(\{|w|<2^{n+2}\})} |D^2(\mathcal{W}_y)^{-1}| \le N 2^{5n},\end{aligned}$$

where $N = N(\Omega) > 0$. We also introduce the mapping

$$\Upsilon_n(y,w) = (y, \mathcal{W}(y,w)) : G \times \{|w| < 2^{n+2}\} \to \mathbb{R}^6. \tag{5.51}$$

Due to Lemma A.3 (i), Υ_n is a globally bi-Lipschitz map onto its image, so that, if we change variables

$$v = \mathcal{W}_y w \tag{5.52}$$

in (5.47), then the new integral identity (5.55) will hold on a set of Lipschitz test functions.

Next, for a function $\Xi = \Xi(y,w)$ on $G \times \{|w| < 2^{n+2}\}$, we set

$$\hat{\hat{\Xi}}(y,v) = \Xi(y, (\mathcal{W}_y)^{-1}(v)),\ (y,v) \in \Upsilon_n(G \times \{|w| < 2^{n+2}\}). \tag{5.53}$$

For the sake of convenience, we change the notation as follows:

$$\mathcal{U} := \overline{U}, \quad \mathcal{H} := \overline{H}. \tag{5.54}$$

We fix a test function

$$\phi \in C_0^{0,1}([0,T] \times \overline{G} \times \{|w| \le 2^{n+2}\})$$

and change variables

$$w = (\mathcal{W}_y)^{-1}(v)$$

in the identity (5.47). Due to the identity (E.4) in section E.2, we obtain

$$\begin{aligned}(5.55)\qquad & -\int_{\mathbb{R}^7_T} (\partial_t \hat{\hat{\phi}} + v \cdot \nabla_y \hat{\hat{\phi}}) \hat{\hat{\mathcal{U}}} \mathsf{J}_{\mathcal{W}} dy dv dt + \int_{\mathbb{R}^6} \hat{\hat{\phi}}(T,y,v) \hat{\hat{\mathcal{U}}}(T,y,v) \mathsf{J}_{\mathcal{W}} dy dv\\ &\quad + \int_{\mathbb{R}^7_T} \Big((\nabla_v \hat{\hat{\mathcal{U}}})^T \mathbb{A} \nabla_v \hat{\hat{\phi}} + \big((\mathbb{X} + \mathbb{G}) \cdot \nabla_v \hat{\hat{\phi}}\big) \hat{\hat{\mathcal{U}}}\\ &\quad + \big(\mathbb{B} \cdot \nabla_v \hat{\hat{\mathcal{U}}}\big) \hat{\hat{\phi}} + (\mathbb{C} + \lambda) \hat{\hat{\mathcal{U}}} \hat{\hat{\phi}} \Big) \mathsf{J}_{\mathcal{W}} dy dv dt\\ &= \int_{\mathbb{R}^7_T} \hat{\hat{\phi}} \hat{\hat{\mathcal{H}}} \mathsf{J}_{\mathcal{W}} dy dv dt,\end{aligned}$$

where

$$\mathsf{J}_{\mathcal{W}} = \left|\det \frac{\partial w}{\partial v}\right|, \tag{5.56}$$

$$\mathbb{A}(t,y,v) = \left(\frac{\partial v}{\partial w}\right) \hat{\hat{\mathcal{A}}}(t,y,v) \left(\frac{\partial v}{\partial w}\right)^T, \tag{5.57}$$

$$\mathbb{B}(t,y,v) = \left(\frac{\partial v}{\partial w}\right) \hat{\hat{\mathcal{B}}}(t,y,v), \quad \mathbb{C}(t,y,v) = (\widehat{c})(t,y,w(y,v)), \tag{5.58}$$

$$\mathbb{X}(t,y,v) = \left(\frac{\partial v}{\partial w}\right) \hat{\hat{\mathcal{X}}}(t,y,w(y,v)) 1_{y \in G, |w(y,v)| < 2^{n+2}}, \tag{5.59}$$

$$\mathbb{G}(t,y,v) = \left(\frac{\partial v}{\partial w}\right) \left(\frac{\partial w}{\partial y}\right) v\, 1_{y \in G, |w(y,v)| < 2^{n+2}}. \tag{5.60}$$

As we mentioned in the previous paragraph, thanks to Lemma A.3 (i), we may replace $\hat{\hat{\phi}}$ with any test function

$$\phi \in C_0^{0,1}\big([0,T] \times \Upsilon_n(\overline{G} \times \{|w| \le 2^{n+2}\})\big)$$

in the identity (5.55).

We now replace $\mathbb{A}$ with $\mathfrak{A}$ as follows so that $\mathfrak{A} = \mathbb{A}$ on the support of $\hat{\hat{\mathcal{U}}}$ contained in $\Upsilon_n(G \times \{|w| < 2^{n+2}\})$. Let $\zeta_n = \zeta_n(y,v)$ be a smooth cutoff function such that $0 \le \zeta_n \le 1$ and

$$\begin{aligned} &\zeta_n = 1 \text{ on } \Upsilon_n(G \times \{|w| < 2^{n+2}\}), \\ &|\nabla_x \zeta_n| + |\nabla_v \zeta_n| \le N(\Omega). \end{aligned} \tag{5.61}$$

Introduce

$$\mathfrak{A} = \mathbb{A}\zeta_n + (1-\zeta_n)\mathbf{1}_3, \tag{5.62}$$

$$\tilde{\tilde{U}} = \hat{\hat{U}} \mathsf{J}_{\mathcal{W}}. \tag{5.63}$$

We also extend $\mathbb{B}, \mathbb{X}$, and $\mathbb{C}$ by 0 outside $[0,T] \times \Upsilon(\overline{G} \times \{|w| \le 2^{n+2}\})$. It follows that for any $\phi \in C_0^{0,1}([0,T] \times \mathbb{R}^6)$ such that $\phi(T,\cdot) \equiv 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^7_T} \Big(- (\partial_t \phi + v \cdot \nabla_y \phi) \tilde{\tilde{\mathcal{U}}} + (\nabla_v \tilde{\tilde{\mathcal{U}}})^T \mathfrak{A} \nabla_v \phi + \lambda \tilde{\tilde{\mathcal{U}}} \phi \Big)\, dy dv dt \\ &\quad = \int_{\mathbb{R}^7_T} \big(- \hat{\hat{\mathcal{U}}}\, (\mathbb{X} + \mathbb{G}) \cdot \nabla_v \phi - \big(\mathbb{B} \cdot \nabla_v \hat{\hat{\mathcal{U}}}\big)\, \phi - \mathbb{C} \hat{\hat{\mathcal{U}}}\, \phi \big)\, \mathsf{J}_{\mathcal{W}} dy dv dt \\ &\qquad + \int_{\mathbb{R}^7_T} \phi\, \hat{\hat{\mathcal{H}}} \mathsf{J}_{\mathcal{W}} + (\nabla_v \mathsf{J}_{\mathcal{W}})^T \mathbb{A} (\nabla_v \phi) \hat{\hat{\mathcal{U}}}\, dy dv dt. \end{aligned} \tag{5.64}$$

In other words, the identity

$$\begin{aligned} &\partial_t \tilde{\tilde{\mathcal{U}}} + v \cdot \nabla_y \tilde{\tilde{\mathcal{U}}} - \nabla_v \cdot (\mathfrak{A} \nabla_v \tilde{\tilde{\mathcal{U}}}) + \lambda \tilde{\tilde{\mathcal{U}}} \\ &\quad = \Big(\nabla_v \cdot \big((\mathbb{X} + \mathbb{G}) \hat{\hat{\mathcal{U}}} \big) - \mathbb{B} \cdot \nabla_v \hat{\hat{\mathcal{U}}} - \mathbb{C} \hat{\hat{\mathcal{U}}} + \hat{\hat{\mathcal{H}}} \Big) \mathsf{J}_{\mathcal{W}} \\ &\qquad - \nabla_v \cdot \big(\mathbb{A} (\nabla_v \mathsf{J}_{\mathcal{W}}) \hat{\hat{\mathcal{U}}} \big) =: \text{r.h.s.} \end{aligned} \tag{5.65}$$

holds in the weak sense. For the reader's convenience, we briefly review the notation introduced above.

- $U = f_{k,n}$, $H = \eta_{k,n}$ (see (5.30), (5.27)),
- $\widehat{U}$ is U in coordinates (t,y,w),
- $\widetilde{U}$ is $\widehat{U}$ multiplied by the Jacobian determinant of the change of variables $(x,v) \to (y,w)$ (see (5.38)),
- $\mathcal{U} := \overline{U}$ is the mirror extension of $\widetilde{U}$ (see (5.40)),
- $\hat{\hat{\mathcal{U}}}$ is $\mathcal{U}$ in coordinates (t,y,v),
- $\tilde{\tilde{\mathcal{U}}}$ is $\hat{\hat{\mathcal{U}}}$ multiplied by the Jacobian determinant of the change of variables $w \to v$,
- $\sigma_g(z)$ (see (3.9)) is the matrix of the leading coefficients in the original equation,
- G is the even extension of $\psi(\Omega_{r_0}(x_0))$ across the plane $\{y_3 = 0\}$,
- A and B (see (5.36)) are the diffusion and drift coefficients on $(0,T) \times \mathbb{R}^3_- \times \mathbb{R}^3$ obtained after the change of variables $(x,v) \to (y,w)$,

– $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{X}$ (see (5.41)) are the drift, diffusion, and discount, and "geometric" coefficients "extended" across the boundary $\{y_3 = 0\} \times \mathbb{R}^3$,
– $\hat{\hat{\mathcal{A}}}$, $\hat{\hat{\mathcal{B}}}$, $\hat{\hat{\mathcal{C}}}$, $\hat{\hat{\mathcal{X}}}$ (see (5.53)) are the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{X}$ in the new coordinates t, y, v,
– $\mathbb{A}, \mathbb{B}, \mathbb{C}$, $\mathbb{X}$ are the drift, diffusion, discount, and "geometric" coefficients on $(0,T) \times \Upsilon_n(G \times \mathbb{R}^3)$ obtained after the change of variable $w \to v$,
– $\mathbb{G}$ (see (5.57)) is the second "geometric" coefficient due to the change of variables $w \to v$,
– $\mathfrak{A}$ (see (5.62)) is an extension of $\mathbb{A}$ to $\mathbb{R}^7_T$.

Step 5: S_2^N estimate in the t, y, v coordinates. We now apply Lemma F.6. We first check its conditions.

Estimates of the coefficients $\mathbb{A}$, $\mathfrak{A}$, $\mathbb{B}$, $\mathbb{X}$. In Lemma C.1, we show that the following bounds are valid:

$$N_0(\Omega) 2^{-6n} |\xi|^2 \le \mathfrak{A}(z) \xi_i \xi_j, \tag{5.66}$$

$$|\mathbb{A}| I_{(y,v) \in \Upsilon_n(G \times \{|p| < 2^{n+2}\})} + |\mathfrak{A}| \le N(\Omega, K), \tag{5.67}$$

$$\|\nabla_v \mathbb{A}\|_{L_\infty((0,T) \times \Upsilon_n(G \times \{|w| < 2^{n+2}\}))} + \|\nabla_v \mathfrak{A}\|_{L_\infty(\mathbb{R}^7_T)} \le N 2^n, \tag{5.68}$$

$$\|\mathfrak{A}\|_{L_\infty((0,T)) C^{\varkappa/3, \varkappa}_{y,v}(\mathbb{R}^6)} \le N(K, \Omega, \varkappa) 2^n, \tag{5.69}$$

$$\|\mathbb{B}\|_{L_\infty((0,T) \times \Upsilon_n(G \times \{|w| < 2^{n+2}\}))} \le N(\Omega, K) 2^{-n}, \tag{5.70}$$

$$\|\mathbb{X}\|_{L_\infty(\Upsilon_n(G \times \{|w| < 2^{n+2}\}))} \le N(\Omega), \tag{5.71}$$

$$\|\nabla_v \mathbb{X}\|_{L_\infty(\Upsilon_n(G \times \{|w| < 2^{n+2}\}))} \le N(\Omega) 2^{3n}, \tag{5.72}$$

$$\|\mathbb{G}\|_{L_\infty(\Upsilon_n(G \times \{|w| < 2^{n+2}\}))} \le N(\Omega), \tag{5.73}$$

$$\|\nabla_v \mathbb{G}\|_{L_\infty(\Upsilon_n(G \times \{|w| < 2^{n+2}\}))} \le N(\Omega) 2^{4n}. \tag{5.74}$$

L_2-integrability of the r.h.s. of (5.65). To show this, we need to first estimate $\mathsf{J}_{\mathcal{W}}$, $\hat{\hat{\mathcal{H}}}$, $\hat{\hat{\mathcal{U}}}$, $\nabla_v \hat{\hat{\mathcal{U}}}$.

First, we estimate $\mathsf{J}_{\mathcal{W}}$ (see (5.56)). By (5.50) and Lemma A.2 (ii) (see (A.19)) and (A.29) in Lemma A.3,

$$\begin{aligned} &N_1 \le |\mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}| \le N 2^{9n}, \\ &|\nabla_v \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}| \le N 2^{11n}, \\ &|\nabla_y \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}| \le N 2^{11n}, \\ &|D_v^2 \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}| \le N 2^{13n}, \end{aligned} \tag{5.75}$$

where $N_1 = N_1(\Omega)$, $N = N(\Omega)$.

Second, we bound $\hat{\hat{\mathcal{H}}}$, $\hat{\hat{\mathcal{U}}}$, $\nabla_v \hat{\hat{\mathcal{U}}}$. By (5.75),

$$\begin{aligned} \|\hat{\hat{\mathcal{H}}} \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}\|_{L_2(\mathbb{R}^7_T)} &\le N 2^{(9n)/2} \| |\hat{\hat{\mathcal{H}}}|^2 \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}\|^{1/2}_{L_1(\mathbb{R}^7_T)} \\ &= N 2^{(9n)/2} \|\overline{H}\|_{L_2((0,T) \times G \times \{|w| < 2^{n+2}\})} \le N 2^{(9n)/2} \|H\|_{L_2(\Sigma^T)}, \end{aligned} \tag{5.76}$$

where $N = N(\Omega)$. Similarly, by (5.75),

$$\|(|\hat{\hat{\mathcal{U}}}| + |\nabla_v \hat{\hat{\mathcal{U}}}|) \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}\|_{L_2(\mathbb{R}^7_T)} \le N(\Omega) 2^{(9n)/2} \| |U| + |\nabla_p U| \|_{L_2(\Sigma^T)}. \tag{5.77}$$

Next, combining (5.67)–(5.68), (5.70)–(5.74), and (5.75), we get

$$\|\text{r.h.s. of } (5.65)\|_{L_2(\mathbb{R}^7_T)} \le N 2^{\beta n} \|(|\hat{\hat{\mathcal{U}}}| + |\nabla_v \hat{\hat{\mathcal{U}}}| + |\hat{\hat{\mathcal{H}}}|) \mathsf{J}_{\mathcal{W}} 1_{y \in G, w(y,v) < 2^{n+2}}\|_{L_2(\mathbb{R}^7_T)}, \tag{5.78}$$

where $\beta > 0$ is some constant independent of n, ε, c, and θ, which might change from line to line. Furthermore, by (5.76)–(5.77), we get

$$\|\text{r.h.s. of (5.65)}\|_{L_2(\mathbb{R}^7_T)} \le N 2^{\beta n} \| |U| + |\nabla_p U| + |H| \|_{L_2(\Sigma^T)}. \tag{5.79}$$

S_2^N*-estimate in the* t, y, v *variables.* An application of Lemma F.6 with

$$\delta = N_0 2^{-6n} \,(\text{see}\,(5.66)), \quad K = N \,(\text{see}\,(5.69)), \tag{5.80}$$

gives $\tilde{\tilde{\mathcal{U}}} \in S_2^N(\mathbb{R}^7_T)$. Furthermore, by the same lemma (see (F.5)), one has

$$\|\tilde{\tilde{\mathcal{U}}}\|_{S_2^N(\mathbb{R}^7_T)} \le N 2^{\beta n} \big(\| |\text{r.h.s.}| + |\tilde{\tilde{\mathcal{U}}}| + |\nabla_v \tilde{\tilde{\mathcal{U}}}| \|_{L_2(\mathbb{R}^7_T)} \big). \tag{5.81}$$

By the fact that $\tilde{\tilde{\mathcal{U}}} = \hat{\hat{\mathcal{U}}} \mathsf{J}_{\mathcal{W}}$ (see (5.63)) and the bounds (5.75) and (5.77), we have

$$\begin{aligned} \| |\tilde{\tilde{\mathcal{U}}}| + |\nabla_v \tilde{\tilde{\mathcal{U}}}| \|_{L_2(\mathbb{R}^7_T)} &\le N 2^{\beta n} \| \mathsf{J}_{\mathcal{W}} (|\hat{\hat{\mathcal{U}}}|^2 + |\nabla_v \hat{\hat{\mathcal{U}}}|^2) 1_{y \in G, w(y,v) < 2^{n+2}} \|^{1/2}_{L_1(\mathbb{R}^7_T)} \\ &\le N 2^{\beta n} \| |U| + |\nabla_p U| \|_{L_2(\Sigma^T)}. \end{aligned} \tag{5.82}$$

Combining (5.81)–(5.82) with (5.79), we obtain

$$\| |(\partial_t + v \cdot \nabla_y) \tilde{\tilde{\mathcal{U}}}| + |D_v^2 \tilde{\tilde{\mathcal{U}}}| \|_{L_2(\mathbb{R}^7_T)} \le N 2^{\beta n} \| |U| + |\nabla_p U| + |H| \|_{L_2(\Sigma^T)}. \tag{5.83}$$

Step 6: Going back to the original variables t, x, p. *Estimate of* $D_p^2 U$. First, by the chain rule and change of variables,

$$\|D_p^2 U\|_{L_2((0,T) \times \Omega_{r_0}(x_0) \times \mathbb{R}^3)} \le N(\Omega) \| |\nabla_w \widehat{U}| + |D_w^2 \widehat{U}| \|_{L_2((0,T) \times \psi(\Omega_{r_0}(x_0)) \times \mathbb{R}^3)}. \tag{5.84}$$

Furthermore, recall

- $\mathcal{U} = \overline{U}$, where the latter is the even extension in the y_3, w_3 variables of the function $\widehat{U}$ (see (5.40));
- the definition of $\tilde{\tilde{\mathcal{U}}}$ in (5.63).

Then, by (5.50) and the estimates of $\mathsf{J}_{\mathcal{W}}$ and its derivatives (see (5.75)),

$$\begin{aligned} &\| |\nabla_w \widehat{U}| + |D_w^2 \widehat{U}| \|_{L_2((0,T) \times \psi(\Omega_{r_0}(x_0)) \times \mathbb{R}^3)} \\ &\quad \le N 2^{\beta n} \| (|\nabla_v \hat{\hat{\mathcal{U}}}|^2 + |D_v^2 \hat{\hat{\mathcal{U}}}|^2) \mathsf{J}_{\mathcal{W}} 1_{y \in G, |w(y,v)| < 2^{n+2}} \|^{1/2}_{L_1(\mathbb{R}^7_T)} \\ &\quad \le N 2^{\beta n} \| |\tilde{\tilde{\mathcal{U}}}| + |\nabla_v \tilde{\tilde{\mathcal{U}}}| + |D_v^2 \tilde{\tilde{\mathcal{U}}}| \|_{L_2(\mathbb{R}^7_T)}. \end{aligned} \tag{5.85}$$

Next, combining (5.83)–(5.85), we obtain

$$\|D_p^2 U\|_{L_2((0,T) \times \Omega_{r_0}(x_0) \times \mathbb{R}^3)} \le N 2^{\beta n} \| |U| + |\nabla_p U| + |H| \|_{L_2(\Sigma^T)}. \tag{5.86}$$

Since $U(t,x,\cdot), H(t,x,\cdot)$ vanish outside $\{2^{n-1} < |p| < 2^{n+3/2}\}$, we may replace the r.h.s. of (5.86) with

$$\| |U| + |\nabla_p U| + |H| \|_{L_{2,\beta}((0,T) \times \Omega \times \mathbb{R}^3)}. \tag{5.87}$$

Recall that

$$U = f \eta_k p_0^{\theta/2}, \quad H = \eta_{k,n}, \tag{5.88}$$

where $\eta_{k,n}$ is defined in (5.27). We conclude that the expression in (5.87) is less than

$$N\||f| + |\nabla_p f| + |\eta|\|_{L_{2,\theta/2+\beta}((0,T)\times\Omega\times\{2^{n-1}<|p|<2^{n+3/2}\})}.$$

Thus, the estimate (5.28) for $D_p^2 U$ is proved.

Estimate of the transport term. First, by the estimate (E.3) in Lemma E.1, we have

$$\|YU\|_{L_2(\Sigma^T)} \le N\|(\partial_t + W\cdot\nabla_y)\widehat{U}\|_{L_2(\mathbb{H}_-^T)} + N\||U| + |\nabla_p U|\|_{L_{2,1}(\Sigma^T)}. \tag{5.89}$$

Similarly, by the identity (E.4) in section E.2,

$$\|(\partial_t + W\cdot\nabla_y)\widehat{U}\|_{L_2(\mathbb{H}_-^T)} \le N\|\big(|\partial_t + v\cdot\nabla_y)\hat{\hat{\mathcal{U}}}|^2 + |\mathbb{G}|^2\,|\nabla_v \hat{\hat{\mathcal{U}}}|^2\big)\mathsf{J}_{\mathcal{W}}\|^{1/2}_{L_1(\mathbb{R}^7_T)}, \tag{5.90}$$

where $\mathbb{G}$ is defined in (5.60). Note that

$$(\partial_t + v\cdot\nabla_y)\tilde{\tilde{\mathcal{U}}} = \big((\partial_t + v\cdot\nabla_y)\hat{\hat{\mathcal{U}}}\big)\mathsf{J}_{\mathcal{W}} + (v\cdot\nabla_y \mathsf{J}_{\mathcal{W}})\hat{\hat{\mathcal{U}}}. \tag{5.91}$$

Then, by (5.90)–(5.91), the Jacobian estimate (5.75), and the estimate of $\mathbb{G}$ (5.73), and (5.77), we get

$$\|(\partial_t + W\cdot\nabla_y)\widehat{U}\|_{L_2(\mathbb{H}_-^T)} \le N\|(\partial_t + v\cdot\nabla_y)\hat{\hat{\mathcal{U}}}\|_{L_2(\mathbb{R}^7_T)} + N2^{\beta n}\||\widehat{U}| + |\nabla_p \widehat{U}|\|_{L_2(\mathbb{R}^7_T)}.$$

Combining (5.89) with (5.91) and (5.83) gives

$$\begin{aligned}\|YU\|_{L_2(\Sigma^T)} &\le N2^{\beta n}\||U| + |\nabla_p U| + |H|\|_{L_2(\Sigma^T)} + N\|\nabla_p U\|_{L_{2,1}(\Sigma^T)}\\ &\le N\|(|f| + |\nabla_p f| + |\eta|)\|_{L_{2,\theta/2+\beta}((0,T)\times\Omega\times\{2^{n-1/2}<|p|<2^{n+3/2}\})}\end{aligned}$$

provided that $\beta > 1$, which we may certainly assume. Thus, the estimate (5.28) holds for YU. Finally, note that by the embedding theorem for the $S_2^N(\mathbb{R}^7_T)$ space (see Theorem 2.1 [30]), the norms

$$\|\tilde{\tilde{\mathcal{U}}}\|_{L_{14/5}(\mathbb{R}^7_T)}, \|\nabla_v \tilde{\tilde{\mathcal{U}}}\|_{L_{7/3}(\mathbb{R}^7_T)}$$

are bounded by the r.h.s. of (5.81). Then, repeating the above argument, we prove the bound of the second and third terms on the r.h.s. of (5.28). ◻

Proof of Proposition 5.4. We first impose the additional assumptions (5.23), which will be removed at the end of the proof.

Existence. Let $b_n \in L_\infty((0,T)\times\Omega)W^1_\infty(\mathbb{R}^3), n \ge 1$, be a sequence of functions such that $b_n \to b$ a.e., and $\|b_n\|_{L_\infty(\Sigma^T)} \le N_1$ with N_1 independent of n. We set f_n to be a finite energy weak solution (see Lemma 5.8) to the equation

$$Yf_n - \nabla_p\cdot(\sigma_g\nabla_p f_n) + b_n\cdot\nabla_p f_n + (c+\lambda)f_n = \chi_n(\eta), f_n(0,\cdot) \equiv 0, \tag{5.92}$$

with the SRBC, where $\chi_n(t) = -n \vee t \wedge n$. By (5.9) in Proposition 5.4, we have

$$\|f_n\|_{S_{2,\kappa\theta}(\Sigma^T)} + \|f_n\|_{L_{14/5,\kappa\theta}(\Sigma^T)} + \|\nabla_p f_n\|_{L_{7/3,\kappa\theta}(\Sigma^T)} \le N\|\eta\|_{L_{2,\theta}(\Sigma^T)}. \tag{5.93}$$

By this estimate, there exists a function f such that $f_n \to f$ in the weak* topology of $S_{2,\kappa\theta}(\Sigma^T)$, so that the bound (5.9) is true for the limiting function f. We now show that f satisfies the initial condition and the SRBC. By Ukai's trace lemma (D.2) and (5.93),

$$\|f_n\|_{L_2(\Sigma_\pm^T, w|v\cdot n_x|)} \le N\|\eta\|_{L_{2,\theta}(\Sigma^T)}.$$

Then, $(f_n)_\pm \to f_\pm^\star$ in the weak* topology of $L_2(\Sigma_\pm^T, w|v\cdot n_x|)$, and $f_\pm^\star$ satisfy the SRBC. Furthermore, since $Yf_n \in L_2(\Sigma^T)$, we have Green's identity (D.1) with u replaced with f_n. Then, by using a limiting argument, we see that the integrals over $\Sigma_\pm^T$ converge to those with the integrand $f_\pm^\star$. Hence, the latter are the traces of f on $\Sigma_\pm^T$, and 0 is the trace on $\{t=0\}\times\Omega\times\mathbb{R}^3$. Testing (5.93) and passing to the limit in (5.92), we conclude that the identity (3.7) holds in the $L_2(\Sigma^T)$ sense, and, thus, f is a strong solution to (3.7)–(3.8) (see Definition 3.2). Finally, the "energy" estimate (5.8) is obtained via the same limiting argument.

To show the existence with the initial condition $f_0 \in S_{2,\theta}(\Omega\times\mathbb{R}^3)$ satisfying the SRBC, we reduce the problem to the case when $f_0\equiv 0$ by replacing $f(z)$ with $\mathsf{f}(z) = f(z) - \phi(t) f_0(x,p)$, where $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi(0)=1$. We note that f satisfies the identities

$$\partial_t \mathsf{f} + v(p)\cdot\nabla_x \mathsf{f} - \nabla_p\cdot(\sigma_g\nabla_p \mathsf{f}) + b\cdot\nabla_p \mathsf{f} + (c+\lambda)\mathsf{f} = \widetilde{\eta},$$
$$\mathsf{f}_-^*(t,x,p) = \mathsf{f}_+^*(t,x,R_x p), z\in\Sigma_-^T, \quad \mathsf{f}(0,\cdot)\equiv 0,$$

where

$$\widetilde{\eta} = \eta + \phi' f_0 + \phi\big(v(p)\cdot\nabla_x f_0) - \nabla_p\cdot(\sigma_g\nabla_p f_0) + b\cdot\nabla_p f_0 + (c+\lambda) f_0\big).$$

Since the L_∞ norms of $\sigma_g, \nabla_p\sigma_g, b, c$ are bounded by N (see (5.5) and (5.6)), we have

$$\|\widetilde{\eta}\|_{L_{2,\theta}(\Sigma^T)} \le N\|\eta\|_{L_{2,\theta}(\Sigma^T)} + N(1+\lambda)\|f_0\|_{S_{2,\theta}(\Sigma^T)}.$$

This concludes the proof of the existence part.

Uniqueness. Let f be a strong solution to (3.7) with $\eta\equiv 0, f_0\equiv 0$. Then, we may use a variant of the energy identity for functions satisfying the SRBC (see (D.4) in Lemma D.5) with $u=f$ and $\phi = fe^{-2\lambda' t}$. Integrating by parts in p and using the Cauchy–Schwarz inequality, we get

$$\int_{\Sigma^T}\left(\frac{\delta_0}{2}|\nabla_p f|^2 + (\lambda+\lambda'-N)|f|^2\right)dz \le 0, \tag{5.94}$$

where $N = N(K) > 0$ and δ_0 is the ellipticity constant of σ_g (see Lemma 5.3). Hence, taking $\lambda' > N$ gives $f\equiv 0$. The uniqueness is proved. ∎

Proof of Remark 5.5. Invoke all the notation in the proof of Lemma 5.10. We say that f satisfies the mirror-extension property if

$$\text{the identity (5.65) holds for } \tilde{\tilde{\mathcal{U}}} \text{ on } \mathbb{R}^7. \tag{5.95}$$

To show this, we regularize f by using an approximation scheme f_n defined as in the proof of Proposition 5.4 (see (5.92)). Then we construct $\tilde{\tilde{\mathcal{U}}}$ for such f_n and η_n. Since $f_n\to f$ in the weak* topology of $S_{2,\kappa\theta}(\Sigma^T)$, by passing to the limit in the integral formulation of (5.65) as $n\to\infty$, we conclude that (5.95) is true. ∎

Proof of Proposition 5.6. We inspect the proof of Proposition 5.4. We use a bootstrap method to show that $\tilde{\tilde{\mathcal{U}}}$ (see (5.63)) is of class $S_r^N(\mathbb{R}_T^7)$ and to estimate

$\|\tilde{\tilde{\mathcal{U}}}\|_{S_r^N(\mathbb{R}^7_T)}$. In particular, one needs to use an induction argument with "the base" at $S_2^N(\mathbb{R}^7_T)$. In the induction step, one uses the embedding theorem for S_r^N spaces (see Theorem 2.1 in [30]) combined with the S_r^N-estimate (F.6) in Lemma F.6 (ii). Refer to the proof of Theorem 1.7 in [8] on pp. 493–494. We point out that the embedding theorem in [30] is stated for $S_2^N(\mathbb{R}^7_T)$ with $T=\infty$. Nevertheless, it is easily seen that the case $T<\infty$ is treated by the same method, which involves using the explicit fundamental solution to $\partial_t + v\cdot\nabla_x - \Delta_v$ (cf. Lemma F.7). ∎

5.2. Strong solutions to steady linear Landau equations. In this section, we establish the results analogous to those in Propositions 5.4 and 5.6 for the steady KFP equation (3.14).

PROPOSITION 5.11 (steady S_r estimate in the presence of SRBC). *Invoke the assumptions of Proposition* 5.4. *In addition, assume that* g, b, c, *and* η *are independent of* t. *Let* $r\in[2,\infty)$. *Then, there exists a constant* $\theta=\theta(r,\varkappa,\kappa)>0$ *such that if, additionally,*

$$\eta\in L_{2,\theta}(\Omega\times\mathbb{R}^3)\cap L_{r,\theta}(\Omega\times\mathbb{R}^3), \tag{5.96}$$

then the following assertions hold.

(i) *There exists a unique strong solution* f *to* (3.14). *In addition,* $f\in L_2(\Omega)W^1_{2,\theta}(\mathbb{R}^3)$.

(ii) *For the strong solution* f *to* (3.14) *satisfying* $f\in L_2(\Omega)W^1_{2,\theta}(\mathbb{R}^3)$, *one has*

$$f\in S_{2,\kappa\theta}(\Omega\times\mathbb{R}^3)\cap S_{r,\kappa\theta}(\Omega\times\mathbb{R}^3), \tag{5.97}$$

and

$$\begin{aligned}&\|f\|_{S_{2,\kappa\theta}(\Omega\times\mathbb{R}^3)}+\|f\|_{S_{r,\kappa\theta}(\Omega\times\mathbb{R}^3)}\\&\quad\le N\Big(\|\eta\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}+\|\eta\|_{L_{r,\theta}(\Omega\times\mathbb{R}^3)}+\|f\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}\Big),\end{aligned} \tag{5.98}$$

where $N=N(\varkappa,\kappa,r,\delta_0,\theta,K,\Omega)$.

Furthermore, in the case when $r<6$, *we have*

$$\|f\|_{L_{r_1,\kappa\theta}(\Omega\times\mathbb{R}^3)}+\|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Omega\times\mathbb{R}^3)}\le r.h.s.\ \text{of } (5.98), \tag{5.99}$$

where $r_1,r_2>1$ *are the numbers satisfying the relations*

$$\frac{1}{r_1}>\frac{1}{r}-\frac{1}{6},\quad \frac{1}{r_2}>\frac{1}{r}-\frac{1}{12}. \tag{5.100}$$

In the case when $r\in(6,12)$,

$$\|f\|_{L_{\infty,\kappa\theta}(\Omega\times\mathbb{R}^3)}+\|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Omega\times\mathbb{R}^3)}\le r.h.s.\ \text{of } (5.98), \tag{5.101}$$

where r_2 *satisfies* (5.100). *Finally, in the case when* $r>12$,

$$\|[f,\nabla_p f]\|_{L_{\infty,\kappa\theta}(\Omega\times\mathbb{R}^3)}+\|[f,\nabla_p f]\|_{C^{\alpha/3,\alpha}_{x,p}(\Omega\times\mathbb{R}^3)}\le r.h.s.\ \text{of } (5.98), \tag{5.102}$$

where $\alpha\in(0,1-\frac{12}{r})$. *In* (5.99), (5.101), *and* (5.102), *one needs to take into account the dependence of* N *on the additional parameters such as* r_1,r_2, *and* α.

Proof of Proposition 5.11. We repeat the argument of Propositions 5.4–5.6 with the following modifications:

– One needs to use the steady counterparts of Theorem 2.6 and Corollary 2.8 in [10] (see Remark 2.11 therein).
– The estimates (5.99) and (5.101)–(5.102) are proved by using the embedding results in Lemma F.7. ∎

COROLLARY 5.12. *For any $\kappa \in (0,1)$, there exists $\theta = \theta(\kappa, r) > 0$ such that for any $f \in S_{r,\theta}(\Omega \times \mathbb{R}^3)$ satisfying the SRBC, the following assertions hold.*

(i) *If $r \in [2,7)$, we have*

$$\|f\|_{L_{r_1,\kappa\theta}(\Omega\times\mathbb{R}^3)} + \|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Omega\times\mathbb{R}^3)} \lesssim_{\theta,\kappa,r,r_1,r_2,\Omega} \|f\|_{S_{r,\theta}(\Omega\times\mathbb{R}^3)}, \tag{5.103}$$

where r_1 and r_2 are numbers satisfying (5.100).

(ii) *If $r \in (6,12)$,*

$$\|f\|_{L_{\infty,\kappa\theta}(\Omega\times\mathbb{R}^3)} + \|\nabla_p f\|_{L_{r_2,\kappa\theta}(\Omega\times\mathbb{R}^3)} \lesssim_{\theta,\kappa,r,r_2,\Omega} \|f\|_{S_{r,\theta}(\Omega\times\mathbb{R}^3)}, \tag{5.104}$$

where r_2 satisfies (5.100).

(iii) *If $r > 12$, then, for any $\alpha \in (0, 1 - 12/r)$, we have*

$$\|[f, \nabla_p f]\|_{C^{\alpha/3,\alpha}_{x,p}(\Omega\times\mathbb{R}^3)} \lesssim_{\theta,\kappa,r,\alpha,\Omega} \|f\|_{S_r(\Omega\times\mathbb{R}^3)}. \tag{5.105}$$

Proof of Corollary 5.12. Let

$$\eta := \frac{p}{p_0} \cdot \nabla_x f - \nabla_p \cdot (\sigma_g \nabla_p f)$$

and note that $\eta \in L_{r,\theta}(\Omega \times \mathbb{R}^3)$. Since f has the mirror-extension property (see Remark 5.5), the function $\tilde{\tilde{\mathcal{U}}}$ satisfies (cf. (5.65))

$$\begin{aligned}
&v \cdot \nabla_y \tilde{\tilde{\mathcal{U}}} - \nabla_v \cdot (\mathfrak{A} \nabla_v \tilde{\tilde{\mathcal{U}}}) \\
&= \big(\nabla_v \cdot ((\mathbb{X} + \mathbb{G})\hat{\hat{\mathcal{U}}}) + \hat{\hat{\mathcal{H}}}\big) \mathsf{J}_{\mathcal{W}} \\
&- \nabla_v \cdot \big(\mathbb{A}(\nabla_v \mathsf{J}_{\mathcal{W}})\hat{\hat{\mathcal{U}}}\big) - \big(v \cdot \nabla_y \mathsf{J}_{\mathcal{W}}\big)\, \hat{\hat{\mathcal{U}}}.
\end{aligned}$$

Then, applying the steady S_r^N estimate in Proposition 5.11 to the above equation and using the embedding theorem for the steady S_r^N spaces (see Lemma F.7), and going back to the original variables as in the proof of Lemma 5.10, we obtain the desired estimates (5.103)–(5.105). ∎

5.3. Finite energy solutions to unsteady KFP equations. The goal of this section is to establish the existence and uniqueness result for the unsteady linear Landau equation (3.7)–(3.8) in the class of finite energy solutions (Definition 3.1). In particular, we employ a duality argument to prove the uniqueness and utilize an approximation argument to establish existence. The well-posedness result is used to prove Lemma G.1 about differentiating finite energy solutions in t. This lemma plays a crucial role in demonstrating the temporal differentiability of the nonlinear RVML system (see assertion (ii) in Theorem 3.10 and (a)–(b) in Proposition 6.2).

PROPOSITION 5.13. *We invoke the assumptions of Proposition* 5.4 *and assume, additionally, that*

$$\|\nabla_p \cdot b\|_{L_\infty(\Sigma^T)} \le K. \tag{5.106}$$

Then, for any $\theta \geq 0$ *and*

$$f_0 \in L_{2,\theta}(\Sigma^T), \quad \eta \in L_2((0,T)\times\Omega)W^{-1}_{2,\theta}(\mathbb{R}^3),$$

there exists a unique finite energy solution to (3.7)–(3.8) *(see Definition* 3.1*). In addition, for any* $t \in (0,T)$, f *satisfies the energy identity*

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^3} \big(f^2(t,x,p) - f_0^2(x,p)\big)p_0^{2\theta}\,dxdp\\ &\quad+\int_{\Sigma^t}\big(\nabla_p(p_0^{2\theta} f)\big)^T\sigma_g\nabla_p f + \big((b\cdot\nabla_p f)f + (c+\lambda)f^2\big)p_0^{2\theta}\,dz = \int_{(0,t)\times\Omega}\langle \eta, f p_0^{2\theta}\rangle\,dxd\tau,\end{aligned} \tag{5.107}$$

where $\langle\cdot,\cdot\rangle$ *is defined in* (2.11)*. A similar result holds for the steady equation* (3.14)*.*

Before we prove Proposition 5.13, we first establish the uniqueness in the class of very weak solutions defined below.

Definition 5.1 (very weak solution). *We say that* f *is a very weak solution to* (3.7)–(3.8) *if*

$$f \in L_2(\Sigma^T),$$

and for any test function $\phi \in S_2(\Sigma^T)$ *satisfying SRBC and* $\phi(T,\cdot)\equiv 0$*, we have*

$$\begin{aligned}&-\int_{\Omega\times\mathbb{R}^3} f_0(x,p)\phi(0,x,p)\,dxdp\\ &\quad+\int_{\Sigma^T} f\Big(-Y\phi - \nabla_p\cdot(\sigma_g\nabla_p\phi) - f\nabla_p\cdot(b\phi) + (c+\lambda)\phi\Big)\,dz = \int_{(0,T)\times\Omega}\langle\eta,\phi\rangle\,dxd\tau.\end{aligned} \tag{5.108}$$

Remark 5.14. We note that due to Lemma D.5, any test function ϕ in Definition 5.1 belongs to $C([0,T])L_2(\Omega\times\mathbb{R}^3)$. Hence, any finite energy solution (see Definition 3.1) is a very weak solution provided that b is sufficiently regular. See also Remark 5.9 for a comparison with other notions of weak solutions used in this paper.

Lemma 5.15 (uniqueness of very weak solutions). *We invoke the assumptions of Proposition* 5.4 *and assume, additionally, that* $\nabla_p\cdot b\in L_\infty(\Sigma^T)$*. Then, the uniqueness holds for the problem* (3.7)–(3.8) *in the class of very weak solutions.*

Proof. Assume that $u^{(j)}, j=1,2,$ are very weak solutions to (3.7)–(3.8) and denote $u = u^{(1)} - u^{(2)}$. Then, for any function $\phi\in S_2(\Sigma^T)$ satisfying SRBC and the condition $\phi(T,\cdot)\equiv 0$, we have

$$\int_{\Sigma^T} u\,(-Y\phi - \nabla_p\cdot(\sigma_g\nabla_p\phi) - (\nabla_p\cdot b)\phi - b\cdot\nabla_p\phi + (c+\lambda)\phi)\,dz = 0.$$

Let $\zeta\in C_0^\infty(\mathbb{R}^3)$ be a nonnegative function such that $\zeta = 1$ on $B_1(0)$ and denote $\zeta_n(\cdot) = \zeta(\cdot/n), n>0$. We consider the equation

$$\begin{aligned}&-Y\phi_n - \nabla_p\cdot(\sigma_g\nabla_p\phi_n) - b\cdot\nabla_p\phi_n + (c+\lambda-\nabla_p\cdot b)\phi_n = u\zeta_n,\\ &\phi_n(T,\cdot)\equiv 0, \quad \phi_n(t,x,p) = \phi_n(t,x,R_x p),\, z\in\Sigma^T_-.\end{aligned}$$

Since $u\zeta_n \in L_{2,\theta}(\Sigma^T)$ for any $\theta > 0$, by Proposition 5.4 the above equation has a unique strong solution $\phi_n \in S_2(\Sigma^T)$. Then, we have

$$\int_{\Sigma^T} u^2 \zeta_n \, dz = 0,$$

and by nonnegativity of ζ, we have $u^2\zeta_n = 0$ a.e. Since n is arbitrary, we conclude $u \equiv 0$, as desired. ∎

Proof of Proposition 5.13. The uniqueness follows from Remark 5.14 and Lemma 5.15.

Existence. For the sake of clarity, we will only consider the case when $\theta = 0$, as the case when $\theta > 0$ is handled by the same argument. The proof is split into two steps.

We will need an auxiliary notion of finite energy solutions, which we call intermediate finite energy solutions.

Step 1: Construction of an intermediate finite energy solution.

DEFINITION 5.2. *We say that f is an intermediate finite energy solution if*

$$f \in L_\infty((0,T))L_2(\Omega \times \mathbb{R}^3) \cap L_2((0,T) \times \Omega)W_2^1(\mathbb{R}^3),$$

and for any test function ϕ satisfying the conditions (3.10)–(3.12), *and $\phi(T,x,p) \equiv 0$ (see Remark* 3.2*), one has*

$$\begin{aligned} (5.109) \qquad & -\int_{\Sigma^T} f(Y\phi)\, dz - \int_{\Omega\times\mathbb{R}^3} f_0(x,p)\phi(0,x,p)\, dxdp \\ & + \int_{\Sigma^T} \Big((\nabla_p \phi)^T \sigma_g \nabla_p f + (b\cdot\nabla_p f)\phi + (c+\lambda)\phi \Big)\, dz = \int_{\Sigma^T} \langle \eta, \phi\rangle \, dxd\tau \end{aligned}$$

(see (2.11)*).*

We refer to Remark 5.9 for a review of various definitions of weak solutions employed throughout this paper.

Proof by approximations and weak* compactness. Let $f_{0,n}, n \geq 1$, be a sequence of functions such that

$$\begin{aligned} (5.110) \qquad & f_{0,n} \in S_{2,2\theta}(\Omega \times \mathbb{R}^3), \quad f_{0,n} \text{ satisfies SRBC}, \\ & f_{0,n} \to f_0 \text{ in } L_2(\Omega\times\mathbb{R}^3), \end{aligned}$$

where θ is large. For example, one can choose $f_{0,n} \in C_0^\infty(\Omega \times \mathbb{R}^3)$ such that $f_{0,n} \to f_0$ in $L_2(\Omega\times\mathbb{R}^3)$, so that both conditions in (5.110) are satisfied.

Furthermore, let $\zeta, \xi \in C_0^\infty(\mathbb{R}^3)$ be functions such that $\int_{\mathbb{R}^3} \zeta\, dp = 1$ and $\xi = 1$ on $B_1(0)$. We set

$$\zeta_n(p) = n^{-3}\zeta(p/n), \quad \xi_n(p) = \xi(p/n).$$

For any function $h \in L_{1,loc}(\mathbb{R}^7_T)$, we denote

$$h_{(n)}(t,x,p) = (h *_p \zeta_n)(t,x,p).$$

Next, let $\eta_0, \boldsymbol{\eta}_1$ be any $L_2(\Sigma^T)$ functions such that

$$(5.111) \qquad \eta = \eta_0 + \nabla_p \cdot \boldsymbol{\eta}_1.$$

Since (5.110) is valid and one has

$$\xi_n \eta_0,\, \nabla_p \cdot (\xi_n \boldsymbol{\eta}_1)_{(n)} \in L_{2,\theta}(\Sigma^T) \quad \forall \theta > 0,$$

by Proposition 5.4 (i) there exists a unique strong solution $f_n \in S_2(\Sigma^T)$ to the equation

$$\begin{aligned} &Y f_n - \nabla_p \cdot (\sigma_g \nabla_p f_n) + b \cdot \nabla_p f_n + (c+\lambda) f_n = \xi_n \eta_0 + \nabla_p \cdot (\xi_n \boldsymbol{\eta}_1)_{(n)}, \\ &f_n(t,x,p) = f_n(t,x,R_x p),\, z \in \Sigma^T_-, \quad f_n(0,\cdot) = f_{0,n}(\cdot). \end{aligned} \tag{5.112}$$

By using the energy identity (D.4), integration by parts, and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} &\|f_n\|_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|f_n\|_{L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3)} \\ &\quad \le N\|f_n(0,\cdot)\|_{L_2(\Omega\times\mathbb{R}^3)} + N\||\eta_0| + |\boldsymbol{\eta}_1|\|_{L_2(\Sigma^T)}, \end{aligned} \tag{5.113}$$

where $N = N(\delta_0, K, T) > 0$.

By the weak* compactness argument, there exists a function f and a subsequence n' such that

$$\begin{aligned} &f \in L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3) \cap L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3), \\ &f_{n'} \to f \quad \text{in the weak* topology of } L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3), \\ &f_{n'} \to f \quad \text{in the weak topology of } L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3). \end{aligned}$$

Hence, by passing to the limit in the integral formulation (5.109) of (5.112), we conclude that f satisfies the integral formulation (5.109). Thus, f is an intermediate finite energy solution to (3.7)–(3.8). The uniqueness follows from Lemma 5.15. Taking liminf in (5.113) and then infimum over all $\eta_0, \boldsymbol{\eta}_1 \in L_2(\Sigma^T)$ satisfying (5.111), we obtain the estimate

$$\begin{aligned} &\|f\|_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|f\|_{L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3)} \\ &\quad \le N\|f(0,\cdot)\|_{L_2(\Omega\times\mathbb{R}^3)} + N\|\eta\|_{L_2((0,T)\times\Omega)W^{-1}_2(\mathbb{R}^3)}, \end{aligned} \tag{5.114}$$

where $N = N(\delta_0, K, T)$.

Step 2: Existence of a finite energy solution. We first show that

$$f_n \to f \text{ strongly in } L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3) \text{ and in } L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3). \tag{5.115}$$

We note that $w_n = f_n - f$ is an intermediate finite energy solution to

$$\begin{aligned} &Y w_n - \nabla_p \cdot (\sigma_g \nabla_p w_n) + b \cdot \nabla_p w_n + (c+\lambda) w_n \\ &\quad = (\xi_n - 1)\eta_0 + \nabla_p \cdot \big((\xi_n \boldsymbol{\eta}_1)_{(n)} - \boldsymbol{\eta}_1\big), \\ &w_n(t,x,p) = w_n(t,x,R_x p),\, z \in \Sigma^T_-, \quad w_n(0,\cdot) = f_{0,n}(\cdot) - f_0(\cdot). \end{aligned} \tag{5.116}$$

By the estimate (5.114) obtained in Step 1, we have

$$\begin{aligned} &\|w_n\|_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|w_n\|_{L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3)} \\ &\quad \le N\|f_{0,n} - f_0\|_{L_2(\Omega\times\mathbb{R}^3)} \\ &\qquad + N\||\eta_0 - \xi_n\eta_0| + |\boldsymbol{\eta}_1 - (\xi_n\boldsymbol{\eta}_1)_{(n)}|\|_{L_2(\Sigma^T)}. \end{aligned}$$

Passing to the limit, we prove (5.115). Since $f_n \in S_2(\Sigma^T)$, $f_{0,n} \in L_2(\Omega\times\mathbb{R}^3)$, and f_n satisfies SRBC, by Lemma D.5, we have $f_n \in C([0,T])L_2(\Omega\times\mathbb{R}^3)$. Then, due to the convergence (5.115), we conclude $f \in C([0,T])L_2(\Omega\times\mathbb{R}^3)$.

Finally, we prove the validity of the weak formulation (3.13). We fix arbitrary $t \in [0,T]$. By the energy identity (D.4) with t in place of T applied to (5.112), we get

$$\begin{aligned} -\int_{\Sigma^t} f_n Y \phi\, dz &+ \int_{\Omega\times\mathbb{R}^3} (f_n\phi)(t,x,p) - f_{0,n}(x,p)\phi(0,x,p)\, dxdp \\ &+ \int_{\Sigma^t} \Big((\nabla_p \phi)^T \sigma_g \nabla_p f_n + b\cdot\nabla_p f_n \phi + (c+\lambda)\phi\Big) dz \\ &= \int_{(0,t)\times\Omega} \eta_0 \xi_n \phi - (\nabla_p\phi)\cdot(\xi_n \boldsymbol{\eta}_1)_{(n)}\, dxd\tau. \end{aligned}$$

Passing to the limit as $n\to\infty$, we obtain the desired identity (3.13). Thus, f is the finite energy solution to (3.7)–(3.8), as desired.

6. Proof of Proposition 3.11.

The section is organized as follows. First, in section 6.1 we prove the desired estimate (3.56) given that the linear RVML system (3.46)–(3.50) is well-posed and the triple $[f,\mathbf{E}_f,\mathbf{B}_f]$ is sufficiently regular. See the details in Proposition 6.2. We justify the existence, uniqueness, and higher regularity in the proof of Proposition 6.2 in Appendix G. Denote

$$\sigma_{g^+ + g^-} = \underbrace{2\int_{\mathbb{R}^3} \Phi(P,Q)J(q)\, dq}_{=\sigma(p)} + \int_{\mathbb{R}^3} \Phi(P,Q)J^{1/2}(q) g(t,x,q)\cdot\boldsymbol{\xi}_0\, dq, \tag{6.1}$$

$$a^i_g(z) = -\int \Phi^{ij}(P,Q)J^{1/2}(q)\left(\frac{p_i}{2p_0} g(t,x,q) + \partial_{q_j} g(t,x,q)\right)\cdot\boldsymbol{\xi}_0\, dq, \tag{6.2}$$

$$\begin{aligned} C_g(z) = &-\frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0} + \partial_{p_i}\left(\sigma^{ij}\frac{p_j}{p_0}\right) \\ &-\int \left(\partial_{p_i} - \frac{p_i}{2p_0}\right)\Phi^{ij}(P,Q)J^{1/2}(q)\partial_{q_j} g(t,x,q)\cdot\boldsymbol{\xi}_0\, dq, \end{aligned} \tag{6.3}$$

$$\begin{aligned} Kg = -J^{-1/2}(p)\partial_{p_i}\Bigg(J(p)\int \Phi^{ij}(P,Q)J^{1/2}(q)\big(\partial_{q_j} g(t,x,q) \\ + \frac{q_j}{2q_0} g(t,x,q)\big)\cdot\boldsymbol{\xi}_0\, dq\Bigg)\boldsymbol{\xi}_0. \end{aligned} \tag{6.4}$$

The following lemma will be used many times in the paper.

LEMMA 6.1. *Under the assumptions of Proposition* 3.11, *we have*

$$\begin{aligned} &\|\partial_t^k g\|_{L_\infty((0,T)\times\Omega)W^1_{r,\theta/2^{k+9}}(\mathbb{R}^3)} \\ &\quad\le N_0 \sup_{\tau\le T}\sqrt{\mathcal{I}_g(\tau)} \le N_0\sqrt{\varepsilon_0},\ r\in\{2,\infty\}, k=0,1,\dots,m-8, \end{aligned} \tag{6.5}$$

$$\|\partial_t^k g\|_{L_\infty((0,T))C^{\alpha/3,\alpha}_{x,p}(\Omega\times\mathbb{R}^3)} \le N_0\sup_{\tau\le T}\sqrt{\mathcal{I}_g(\tau)} \le N_0\sqrt{\varepsilon_0},\ k=0,1,\dots,m-8, \tag{6.6}$$

$$\|\partial_t^k[\mathbf{E}_g,\mathbf{B}_g]\|_{L_\infty((0,T)\times\Omega)} \le N_0\sqrt{\varepsilon_0},\ k=0,1,\dots,m-7, \tag{6.7}$$

$$\|\partial_t^k a_g\|_{L_\infty(\Sigma^T)} \le N_0\sqrt{\varepsilon_0},\ k=0,1,\dots,m-8, \tag{6.8}$$

$$\||\sigma_{g^++g^-}| + |\nabla_p\sigma_{g^++g^-}| + |C_g|\|_{L_\infty(\Sigma^T)} \le N_0, \tag{6.9}$$

$$\||\partial_t^k\sigma_{g^++g^-}| + |\partial_t^k\nabla_p\sigma_{g^++g^-}| + |\partial_t^k C_g|\|_{L_\infty(\Sigma^T)} \le N_0\sqrt{\varepsilon_0},\ k=1,\dots,m-8, \tag{6.10}$$

where $\alpha\in(0,1-12/r_4)$. *Furthermore, for* $h=[\sigma_{g^++g^-},\nabla_p\sigma_{g^++g^-},C_g,a_g]$, *and* $i\in\{1,\dots,4\}$,

$$\|\partial_t^k h\|_{L_\infty((0,T))L_{r_i}(\Omega)L_\infty(\mathbb{R}^3)} \le N_0\sqrt{\varepsilon_0},\ k \le m-4-i, \tag{6.11}$$

where $r_i, i=1,\dots,4$, *are given by* (3.28), *and* $N_0 = N_0(r_1,\dots,r_4,\theta,\Omega,\alpha,m)$.

Proof. In this proof $N_0 = N_0(r_1,\dots,r_4,\theta,\Omega,\alpha,m)$ might change from line to line. We note that by the definition of $\mathcal{H}_g(T)$ in (3.30) and the assumption $y_g(T) < \varepsilon_0$ (see (3.55)), the fact that $r_4 > 12$, and the embedding theorem for steady S_p spaces (see (5.105)), we have for $k \le m-8$ and $r \in \{2,\infty\}$,

$$\begin{aligned}\|\partial_t^k g\|_{L_\infty((0,T)\times\Omega)W^1_{r,\theta/2^{k+9}}(\mathbb{R}^3)} &\lesssim_{r_4,\theta,k,\Omega} \|\partial_t^k g\|_{L_\infty((0,T))S_{r_4,\theta/2^{k+8}}(\Omega\times\mathbb{R}^3)} \\ &\le N_0\|\mathcal{H}_g\|^{1/2}_{L_\infty((0,T))} \le N_0\sqrt{\varepsilon_0}.\end{aligned} \tag{6.12}$$

By the same embedding result, we obtain (6.6). The estimate of the L_∞ norm $\partial_t^k[\mathbf{E}_g,\mathbf{B}_g]$ follows from the fact that $r_3 > 3$ (see (3.29)), the Sobolev embedding theorem, the definition of $\mathcal{H}_g(T)$ (see (3.30)), and the smallness assumption (3.55). Furthermore, using the identities (6.2)–(6.4), the estimate (B.7) with $r = \infty$ in Lemma B.3, and the bound (6.5) with $r=\infty$, we obtain the estimate of σ_g in (6.9)–(6.10). By (B.9)–(B.10), we obtain the estimate of a_g and C_g in (6.8)–(6.10).

Finally, using the bound (B.7) again, we get for fixed t, x,

$$\|\partial_t^k h(t,x,\cdot)\|_{L_\infty(\mathbb{R}^3)} \le N_0|\partial_t^k[\mathbf{E}_g,\mathbf{B}_g](t,x)| + N_0\|\partial_t^k[g,\nabla_p g](t,x,\cdot)\|_{L_{r_i}(\mathbb{R}^3)}.$$

Taking the $L_\infty((0,T))L_{r_i}(\Omega)$ norm and invoking the definition of $\mathcal{H}_g$ in (3.30), we get for $k \le m-4-i$ (cf. (6.12)),

$$\begin{aligned}&\text{l.h.s. of (6.11)}\\ &\quad\le N_0\|\partial_t^k[\mathbf{E}_g,\mathbf{B}_g]\|_{L_\infty((0,T))L_{r_i}(\Omega)} + N_0\|\partial_t^k[g,\nabla_p g]\|_{L_\infty((0,T))L_{r_i}(\Omega\times\mathbb{R}^3)} \le N_0\sqrt{\varepsilon_0}. \quad \square\end{aligned}$$

The next lemma asserts the well-posedness of the linear RVML system. We will prove it in Appendix G.

PROPOSITION 6.2. *Under the assumptions of Proposition* 3.11, *there exists a triple* $[f,\mathbf{E}_f,\mathbf{B}_f]$ *such that*

(a) $\partial_t^k f, k \le m-5$, *is a strong solution (see Definition* 3.2*) to the linear Landau equation* (3.46) *differentiated* k *times with respect to* t *with the initial condition* $\partial_t^k f = f_{0,k}$ *(see* (3.16)*)*,

(b) $\partial_t^k f, m-4 \le k \le m$, *is a finite energy solution to* (3.46) *(see Definition* 3.1*) differentiated* k *times with respect to* t *with the initial condition* $f_{0,k}$,

(c) $\partial_t^k[\mathbf{E}_f,\mathbf{B}_f], k \le m-1$, *is a strong solution to Maxwell's equations* (3.47)–(3.48) *differentiated* k *times with respect to* t *with the perfect conductor BC and the initial condition* $[\mathbf{E}_{0,k},\mathbf{B}_{0,k}]$ *(see* (3.17)–(3.18)*), whereas* $\partial_t^m f$ *is a weak solution to differentiated Maxwell's equations,*

(d) *for any* $k \le m$, *we have* $\partial_t^k[\nabla_x\cdot\mathbf{E}_f,\nabla_x\cdot\mathbf{B}_f] = \partial_t^k[\rho_f,0]$ *(cf.* (3.49)*)*,

(e)

$$\partial_t^k f(t,\cdot) \in L_\infty(\Omega)W^1_{\infty,\theta/2^{k+9}}(\mathbb{R}^3), k \le m-8, \text{ for any } t \in (0,T], \tag{6.13}$$

Furthermore, any two triples $[f^{(j)},\mathbf{E}_f^{(j)},\mathbf{B}_f^{(j)}], j=1,2$, *satisfying* (a)–(e) *must coincide.*

The next result shows that given the energy-dissipation control (see (3.27) and (3.34)), one can establish the higher-regularity control by estimating $\mathcal{H}_f(T)$ (see (6.14)).

PROPOSITION 6.3. *Assuming that Proposition* 6.2 *is valid, we have for* $\tau \le T$,

$$\begin{aligned}\mathcal{H}_f(\tau) = &\sum_{i=1}^{4}\sum_{k=0}^{m-4-i} \|\partial_t^k f(\tau,\cdot)\|^2_{S_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)}\\ &+\sum_{k=0}^{m-1}\|\partial_t^k[\mathbf{E}_f,\mathbf{B}_f](\tau,\cdot)\|^2_{W_2^1(\Omega)}+\sum_{i=2}^{3}\sum_{k=0}^{m-4-i}\|\partial_t^k[\mathbf{E}_f,\mathbf{B}_f](\tau,\cdot)\|^2_{W^1_{r_i}(\Omega)} \qquad (6.14)\\ &\le N\varepsilon_0\sup_{\tau\le T}\mathcal{I}_f(\tau)\\ &+N\sum_{k=0}^{m-1}\|\partial_t^k f\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)}+N\sum_{k=0}^{m}\|\partial_t^k[\mathbf{E}_f,\mathbf{B}_f]\|^2_{L_\infty((0,T))L_2(\Omega)}\\ &+N\sum_{k=0}^{m-4}\|\partial_t^k f\|^2_{L_\infty((0,T))L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)},\end{aligned}$$

where $N = N(r_1,\dots,r_4,\alpha,\Omega,\theta,m)$.

Proof. Here we estimate the functional $\mathcal{H}_f(\tau)$ (see (3.30)). For the sake of clarity, we assume, additionally,

$$\mathcal{H}_f(\tau) < \infty,\ \tau \le T, \qquad (6.15)$$

which is used to perform the descent argument (see section 4). This assumption will be removed at the end of this step.

First, we differentiate Maxwell's equations formally k times in the t variable and rewrite them as two systems of div-curl type as in (4.3)–(4.4). By the $W^1_{r_i}$ div-curl estimate (see (3.15)), we have

$$\begin{aligned}&\|\partial_t^k[\mathbf{E}_f,\mathbf{B}_f]\|_{L_\infty((0,T))W^1_{r_i}(\Omega)} \qquad (6.16)\\ &\quad\lesssim_\Omega \sum_{l=k}^{k+1}\|\partial_t^l[\mathbf{E}_f,\mathbf{B}_f]\|_{L_\infty((0,T))L_{r_i}(\Omega)}+\|\partial_t^k f\|_{L_\infty((0,T))L_{r_i}(\Omega\times\mathbb{R}^3)},\end{aligned}$$

where $k \le m-1$ if $i = 1$ ($r_1 = 2$), and $k \le m-4-i$ if $i \in \{2,3\}$. This gives the desired estimate of the second term on the l.h.s. of (6.14).

Next, differentiating (3.46) formally and using the expressions of A and $\Gamma(f,g)$ in (B.1) and (B.3), and those of $\sigma_{g^+ + g^-}$, a_g, and C_g (see (6.1)–(6.3)), we conclude that for each t, the function $u(t,\cdot) = \partial_t^k f(t,\cdot)$, $k \le m-5$, is a strong solution to the "steady" equation

$$\begin{aligned}&\frac{p}{p_0}\cdot\nabla_x u-\nabla_p\cdot(\sigma_{g^+ + g^-}\nabla_p u)+\boldsymbol{\xi}(\mathbf{E}_g+v(p)\times\mathbf{B}_g-a_g)\cdot\nabla_p u \qquad (6.17)\\ &\quad+\left(C_g-\frac{\boldsymbol{\xi}}{2}v(p)\cdot\mathbf{E}_g\right)u\\ &=-\partial_t^{k+1}f+\boldsymbol{\xi}_1(v(p)\cdot\partial_t^k\mathbf{E}_f)J^{1/2}+K(\partial_t^k f)+1_{k>0}\sum_{j=1}^{3}\sum_{k_1+k_2=k,k_1\ge 1}\eta^j_{k_1,k_2},\end{aligned}$$

$$u(t,x,p) = u(t,x,R_x p), (x,p)\in\gamma_-, \qquad (6.18)$$

$$\begin{aligned}&\eta^1_{k_1,k_2} = -\boldsymbol{\xi}\partial_t^{k_1}(\mathbf{E}_g+v(p)\times\mathbf{B}_g)\cdot\nabla_p(\partial_t^{k_2}f)+\frac{\boldsymbol{\xi}}{2}(v(p)\cdot\partial_t^{k_1}\mathbf{E}_g)\partial_t^{k_2}f,\\ &\eta^2_{k_1,k_2} = \left(\partial_{p_j}\partial_t^{k_1}\sigma^{ij}_{g^+ + g^-}-\partial_t^{k_1}a^i_g\right)(\partial_{p_i}\partial_t^{k_2}f)+(\partial_t^{k_1}C_g)\partial_t^{k_2}f,\\ &\eta^3_{k_1,k_2} = (\partial_t^{k_1}\sigma^{ij}_{g^+ + g^-})(\partial_{p_ip_j}\partial_t^{k_2}f).\end{aligned}$$

We apply Proposition 5.11 (see (5.98)) to $\partial_t^k f^{\pm}(t,\cdot)$ for each t with

$$b = \pm(\mathbf{E}_g + v(p)\times \mathbf{B}_g) - a_g, \quad c = C_g - \frac{\boldsymbol{\xi}}{2} v(p)\cdot \mathbf{E}_g.$$

We first check its assumptions (5.1)–(5.2), (5.3), (5.6).

We note that (5.1)–(5.2) in Assumption 5.1 hold with $K=1$ due to (6.5)–(6.6) in Lemma 6.1 provided that ε_0 is sufficiently small. Similarly, (5.6) with $K=1$ follow directly from (6.9) in Lemma 6.1. Finally, (5.3) is valid due to (3.52).

We fix $i=1,\dots,4$ and $0\le k\le m-4-i$. Then, by the estimates (5.98)–(5.99) and (5.101)–(5.102) with $\theta/2^{k+2i-1}$ in place of θ and $\kappa=1/2$ applied for each $t\in[0,T]$, we get

$$\begin{aligned}(6.19)\qquad &\|\partial_t^k f\|^2_{L_\infty((0,T))S_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} + 1_{i<4}\|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_{i+1},\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)}\\ &\le N\sum_{s\in\{2,r_i\}}\|\text{r.h.s. of (6.17)}\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)}\\ &\quad + N\|\partial_t^k f\|^2_{L_\infty((0,T))L_{2,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}(6.20)\qquad &\sum_{s\in\{2,r_i\}}\|\text{r.h.s. of (6.17)}\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)} \le \sum_{j=1}^{4}\mathcal{I}_{j,k},\\ &\mathcal{I}_{j,k} = 1_{k>0}\sum_{s\in\{2,r_i\}}\sum_{k_1+k_2=k,k_1\ge1}\|\eta^j_{k_1,k_2}\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)}, j=1,2,3,\\ &\mathcal{I}_{4,k} = \sum_{s\in\{2,r_i\}}\|\partial_t^{k+1} f\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)},\\ &\mathcal{I}_{5,k} = \|\partial_t^k \mathbf{E}_f\|^2_{L_\infty((0,T))L_{r_i}(\Omega)},\\ &\mathcal{I}_{6,k} = \sum_{s\in\{2,r_i\}}\|K(\partial_t^k f)\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)}.\end{aligned}$$

We chose weights $\theta/2^{k+2i}$ because for each $i=1,\dots,4$, and $k\le m-4-i$, we need to compensate for

(a) the "natural" weight loss in the steady S_p estimate with $\kappa=1/2$ (see (5.98)–(5.99) in Proposition 5.11),

(b) the presence of the term $\partial_t^{k+1} f\in L_{2,\theta/2^{k+1}}(\Sigma^T)$ on the r.h.s. of (6.17), which has a worse decay than $\partial_t^k f$.

Loosely speaking, due to (a)–(b), for each i, the loss factor in the weight parameter is $\frac14$, which leads to the factor 2^{-2i} in the "hierarchy of weights."

Estimates of $\mathcal{I}_{1,k}$ *and* $\mathcal{I}_{2,k}$. We will show that

$$\begin{aligned}(6.21)\qquad \mathcal{I}_{1,k}+\mathcal{I}_{2,k} &\le N\varepsilon_0\sum_{s\in\{2,r_i\}}\sum_{k_2=0}^{k-1}\|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_\infty((0,T))L_{s,\theta/2^{k_2+2i}}(\Omega\times\mathbb{R}^3)}\\ &\quad + N\varepsilon_0 1_{k\ge m-7}\sum_{k_2=0}^{2}\|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_\infty((0,T))L_{\infty,\theta/2^{k_2+9}}(\Omega\times\mathbb{R}^3)}.\end{aligned}$$

Recall that $k\le m-4-i$. We will consider the case when $k\ge m-7$ since the remaining case is easier to handle due to (6.5)–(6.10). Furthermore, splitting the sum into $k_1\le m-8$ and $k_1\ge m-7$ gives

(6.22)
$$
\begin{aligned}
\mathcal{I}_{1,k}+\mathcal{I}_{2,k} \le & \sum_{k_1+k_2=k: 1\le k_1\le m-8} \|\partial_t^{k_1} h\|^2_{L_\infty(\Sigma^T)} \\
& \times \sum_{s\in\{2,r_i\}} \|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)} \\
& + 1_{m-7\le k\le m-4-i} \sum_{k_1+k_2=k: k_1\ge m-7} \sum_{s\in\{2,r_i\}} \|(\partial_t^{k_1} h) p_0^{-2}\|^2_{L_\infty((0,T))L_s(\Omega\times\mathbb{R}^3)} \\
& \times \|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_{\infty,2+\theta/2^{k+2i-1}}(\Sigma^T)},
\end{aligned}
$$

where $h = [\mathbf{E}_g, \mathbf{B}_g, a_g, C_g, \sigma_{g^+ + g^-}, \nabla_p \sigma_{g^+ + g^-}]$. Due to (6.5)–(6.8) and (6.10) in Lemma 6.1,

$$
1_{1\le k_1\le m-8} \|\partial_t^{k_1} h\|^2_{L_\infty(\Sigma^T)} \le N\varepsilon_0, \tag{6.23}
$$

and by (6.11) in the same lemma, and by the fact that $\mathcal{H}_g(\tau)\le\varepsilon_0$, we get

$$
1_{k_1\le m-4-i} \sum_{s\in\{2,r_i\}} \|(\partial_t^{k_1} h) p_0^{-2}\|^2_{L_\infty((0,T))L_s(\Omega\times\mathbb{R}^3)} \le N\varepsilon_0.
$$

Next, since $k_1\ge 1$, we have $k_2\le k-1$, so that the second factor in the first term on the r.h.s. of (6.22) is bounded by

$$
\sum_{s\in\{2,r_i\}} \|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_\infty((0,T))L_{s,\theta/2^{k_2+2i}}(\Omega\times\mathbb{R}^3)},
$$

as desired. Furthermore, if $k_1\ge m-7$, one has $k_2\le 2$ (recall that $k\le m-5$), and hence, $k_2+9\le 11 < m-6\le k+2i-1$, which gives

$$
2+\theta/2^{k+2i-1} \le \theta/2^{k_2+9} \tag{6.24}
$$

for large θ. Then, for sufficiently large θ, the second factor in the second term on the r.h.s. of (6.22) is bounded by

$$
\|\partial_t^{k_2}[f,\nabla_p f]\|^2_{L_{\infty,\theta/2^{k_2+9}}(\Sigma^T)}.
$$

Thus, the inequality in (6.21) is true.

Estimate of $\mathcal{I}_{3,k}$. We will show that

$$
\begin{aligned}
\mathcal{I}_{3,k} \le & N\varepsilon_0 \sum_{s\in\{2,r_i\}} \sum_{k_2=0}^{k-1} \|\partial_t^{k_2} D_p^2 f\|^2_{L_\infty((0,T))L_{s,\theta/2^{k_2+2i}}(\Omega\times\mathbb{R}^3)} \\
& + N\varepsilon_0 1_{k\ge m-7} \sum_{s\in\{2,r_4\}} \sum_{k_2=0}^{2} \|\partial_t^{k_2} f\|^2_{L_\infty((0,T))S_{s,\theta/2^{k_2+8}}(\Omega\times\mathbb{R}^3)}.
\end{aligned} \tag{6.25}
$$

Inspecting the proof of (6.21) and using (6.23), we conclude

(6.26)
$$
\sum_{s\in\{2,r_i\}} \sum_{k_1+k_2=k, 1\le k_1\le m-8} \|\eta^3_{k_1,k_2}\|^2_{L_\infty((0,T))L_{s,\theta/2^{k_2+2i-1}}(\Omega\times\mathbb{R}^3)} \le \text{r.h.s. of (6.25)}.
$$

Hence, we may assume that $k_1\ge m-7$, $k_2\le 2$.

We denote

$$p_0^{\theta/2^{k+2i-1}} = p_0^{-2} p_0^{2+\theta/2^{k+2i-1}} =: w_1(p) w_2(p).$$

By using Hölder's inequality in the x, p variables with the exponents r_{i+1}/r_i and η_i/r_i, where $\eta_i := (r_i^{-1} - r_{i+1}^{-1})^{-1}$, we get

(6.27)

$$\begin{aligned}
\mathcal{I}_{3,k} &= \sup_{\tau \le T} \left(\int_{\Omega \times \mathbb{R}^3} |\partial_t^{k_1} \sigma_{g^+ + g^-}|^{r_i} (\tau, x, p) \, |D_p^2 \partial_t^{k_2} f(\tau, x, p)|^{r_i} \, p_0^{r_i(\theta/2^{k+2i-1})} \, dx dp \right)^{2/r_i} \\
&\le \sup_{\tau \le T} \mathcal{I}_{3,1,k}(\tau) \mathcal{I}_{3,2,k}(\tau),
\end{aligned}$$

$$\mathcal{I}_{3,1,k}(\tau) = \left(\int_{\Omega \times \mathbb{R}^3} |\partial_t^{k_1} \sigma_{g^+ + g^-}(\tau, x, p)|^{r_{i+1}} w_1^{r_{i+1}}(p) \, dx dp \right)^{2/r_{i+1}},$$

$$\mathcal{I}_{3,2,k}(\tau) = \left(\int_{\Omega \times \mathbb{R}^3} |D_p^2 \partial_t^{k_2} f(\tau, x, p)|^{\eta_i} \, w_2^{\eta_i}(p) \, dx dp \right)^{2/\eta_i}.$$

We estimate $\mathcal{I}_{3,1,k}$ first. Recalling the definition of r_{i+1} in (3.28) and using the embedding result in (5.103) in Corollary 5.12 with r_i in place of r, and invoking the definition of $\mathcal{H}_g(\tau)$ in (3.30), and the fact that $k_1 \le m - 4 - i$, we find

$$\|\partial_t^{k_1} g(\tau, \cdot)\|^2_{L_{r_{i+1}}(\Omega \times \mathbb{R}^3)} \le N \|\partial_t^{k_1} g(\tau, \cdot)\|^2_{S_{r_i, \theta/2^{k_1+2i}}(\Omega \times \mathbb{R}^3)} \le N \mathcal{H}_g(\tau) \le N \varepsilon_0.$$

Furthermore, differentiating the identity (6.1) and using the pointwise bound (B.7) in Lemma B.3, we find

(6.28)

$$\mathcal{I}_{3,1,k}(\tau) = \|p_0^{-2} \partial_t^{k_1} \sigma_{g^+ + g^-}(\tau, \cdot)\|^2_{L_{r_{i+1}}(\Omega \times \mathbb{R}^3)} \le N \|\partial_t^{k_1} g(\tau, \cdot)\|^2_{L_{r_{i+1}}(\Omega \times \mathbb{R}^3)} \le N \varepsilon_0.$$

We move to $\mathcal{I}_{3,2,k}$. We first note that since $k_2 \le 2$, (6.24) is valid, and hence, we may replace $w_2(p)$ with $p_0^{\theta/2^{k_2+8}}$. Furthermore, by the definition of $r_1, \dots r_4$ in (3.28) and the fact that $\Delta r < \frac{1}{42}$, we have

$$\frac{1}{\eta_i} = \frac{1}{r_i} - \frac{1}{r_{i+1}} = \frac{1}{6} - \Delta r \ge \frac{1}{6} - \frac{1}{42} \ge \frac{1}{7}.$$

Hence, by interpolating between L_2 and L_{r_4} ($r_4 > 14$), we obtain

$$\mathcal{I}_{3,2,k}(\tau) \le N \sum_{s \in \{2, r_4\}} \|\partial_t^{k_2} f(\tau, \cdot)\|^2_{S_{s, \theta/2^{k_2+8}}(\Omega \times \mathbb{R}^3)}. \tag{6.29}$$

Combining (6.26)–(6.29), we conclude that (6.25) holds.

Estimate of $\mathcal{I}_{5,k}$. In the case when $i = 1$ and $r_i = 2$, we keep $\mathcal{I}_{5,k}$ as is. In the remaining case $i > 1$, we first note that by Sobolev embedding and the fact that

$$1 - \frac{6}{r_{i-1}} > -\frac{6}{r_i},$$

which follows from (3.28), we have

$$\|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f](\tau, \cdot)\|_{L_{r_i}(\Omega)} \lesssim_\Omega \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f](\tau, \cdot)\|_{W^1_{r_{i-1}}(\Omega)}.$$

Hence, by (6.16) with i replaced with $i-1$, we obtain

$$\mathcal{I}_{5,k} = \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_{r_i}(\Omega)} \leq N \|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega\times\mathbb{R}^3)} + N \sum_{l=k}^{k+1} \|\partial_t^l [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega)}. \tag{6.30}$$

Estimate of $\mathcal{I}_{6,k}$. By (B.11) in Lemma B.5 and interpolation and Hölder's inequalities, for any $\varepsilon_1 \in (0,1)$ and sufficiently large θ, we have

$$\begin{aligned}\mathcal{I}_{6,k} &= \sum_{s\in\{2,r_i\}} \|K(\partial_t^k f)\|^2_{L_\infty((0,T))L_{s,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)} \\ &\leq N \sum_{s\in\{2,r_i\}} \|\partial_t^k f\|^2_{L_\infty((0,T))L_s(\Omega)W^1_s(\mathbb{R}^3)} \\ &\leq \varepsilon_1 \|D_p^2 \partial_t^k f\|^2_{L_\infty((0,T))L_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} + N\varepsilon_1^{-1} \|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_i,\theta/2^{k+2(i-1)}}(\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{6.31}$$

Finally, gathering all the estimates (6.20)–(6.21), (6.25), (6.30)–(6.31) gives

$$\begin{aligned}&\sum_{s\in\{2,r_i\}} \|\text{r.h.s. of (6.17)}\|^2_{L_{s,\theta/2^{k+2i-1}}(\Sigma^T)} \\ &\leq N\varepsilon_1^{-1} \sum_{l=k}^{k+1} \|\partial_t^l f\|^2_{L_\infty((0,T))L_{r_i,\theta/2^{l+2(i-1)}}(\Omega\times\mathbb{R}^3)} + \varepsilon_1 \|\partial_t^k D_p^2 f\|^2_{L_\infty((0,T))L_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} \\ &\quad + \|\partial_t^{k+1} f\|^2_{L_\infty((0,T))L_{2,\theta/2^{k+2i-1}}(\Omega\times\mathbb{R}^3)} + N 1_{i=1} \|\partial_t^k \mathbf{E}_f\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} \\ &\quad + N 1_{i>1} \left(\|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega\times\mathbb{R}^3)} + \sum_{l=k}^{k+1} \|\partial_t^l [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega)} \right) \\ &\quad + 1_{k>0} N\varepsilon_0 \sum_{s\in\{2,r_i\}} \sum_{k_2=0}^{k-1} \|\partial_t^{k_2} f\|^2_{L_\infty((0,T))S_{s,\theta/2^{k_2+2i}}(\Sigma^T)} \\ &\quad + N\varepsilon_0 1_{k\geq m-7} \sum_{k_2=0}^{2} \Bigg(\|\partial_t^{k_2} [f, \nabla_p f]\|^2_{L_{\infty,\theta/2^{k_2+9}}(\Sigma^T)} \\ &\quad + \sum_{s\in\{2,r_i\}} \|\partial_t^{k_2} f\|^2_{L_\infty((0,T))S_{s,\theta/2^{k_2+8}}(\Omega\times\mathbb{R}^3)} \Bigg).\end{aligned} \tag{6.32}$$

We note that the first term involving weighted $L_\infty^{t,x,p}$ norm on the r.h.s. is bounded by

$$\varepsilon_0 1_{k\geq m-7} \sum_{k_2=0}^{2} \|\partial_t^{k_2} [f, \nabla_p f]\|^2_{L_{\infty,\theta/2^{k_2+9}}(\Sigma^T)} \leq N\varepsilon_0 \sup_{\tau\leq T} \mathcal{I}_f(\tau)$$

due to the first inequality in (6.5).

Combining (6.16), (6.19), and (6.32), and summing up over $k \leq m-4-i$, and invoking the definition of $\mathcal{I}_f$ in (3.31), we get

$$\begin{aligned}&\sum_{k=0}^{m-4-i} \|\partial_t^k f\|^2_{L_\infty((0,T))S_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} + 1_{i<4} \sum_{k=0}^{m-4-i} \|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_{i+1},\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} \\ &\quad + 1_{i=1} \sum_{k=0}^{m-i} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))W^1_{r_i}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\quad + 1_{i>1} \sum_{k=0}^{m-4-i} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))W^1_{r_i}(\Omega)} \\
&\quad + 1_{i>1} \sum_{k=0}^{m-4-i} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_{r_{i+1}}(\Omega)} \\
&\le \varepsilon_1 \sum_{k=0}^{m-4-i} \|\partial_t^k f\|^2_{L_\infty((0,T))S_{r_i,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)} + N\varepsilon_0 \sup_{\tau\le T} \mathcal{I}_f(\tau) \\
&\quad + N\varepsilon_1^{-1} \sum_{k=0}^{m-3-i} \|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_i,\theta/2^{k+2(i-1)}}(\Omega\times\mathbb{R}^3)} \\
&\quad + N 1_{i>1} \sum_{k=0}^{m-4-i} \|\partial_t^k f\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega\times\mathbb{R}^3)} \\
&\quad + N 1_{i>1} \sum_{k=0}^{m-3-i} \|\partial_t^k [\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_{r_{i-1}}(\Omega)} \\
&\quad + N \sum_{k=0}^{m-3-i} \|\partial_t^k f\|^2_{L_\infty((0,T))L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)}.
\end{aligned}
$$

We point out that

- by choosing ε_1 sufficiently small, we may absorb the first term on the r.h.s. into the l.h.s.,
- the fourth term on the r.h.s. is bounded by the third one due to Hölder's inequality provided that θ is sufficiently large,
- if we replace i with $i-1$ in the second term on the l.h.s., we obtain the third one on the r.h.s.,
- if we replace i with $i-1$ in the fifth term on the l.h.s., the resulting term dominates the fifth term on the r.h.s.

Then, by using induction on i, we obtain the desired estimate (6.14) for the first and the third terms on the l.h.s. therein.

We note that the assumption (6.15) is actually not necessary, as one can use an induction argument by ascending from $k=0$ to $k=m-4-i$ and using the bounds (6.19) and (6.32). At each step of the induction argument, one needs to use the existence and uniqueness results

(a) for finite energy and strong solutions to the steady KFP equation (3.14) (see Propositions 5.11 and 5.13),
(b) for strong solutions to Maxwell's equation (see Chapter VII in [11]). □

6.1. Proof of the bound (3.56). In this proof, $N = N(r_1, \ldots, r_4, \alpha, \Omega, \theta, m)$. We will do some formal calculations below assuming that Proposition 6.2 is valid. We first prove the energy-dissipation estimate. Combining this inequality with $L^t_\infty S_p$ and $L^t_\infty W^1_2(\Omega)$ bounds in (6.14), we are able to close the estimate of $y_f(T)$.

Step 1: Energy-dissipation bound. Here, we estimate the total energy and dissipation, that is, $\mathcal{E}_f(\tau) + \int_0^\tau \mathcal{D}\, dt$ (see (3.27) and (3.34)). First, applying the standard energy identity for the weak solution to Maxwell's equations differentiated k times with respect to t and using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\frac{1}{2}\|\partial_t^k[\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_2(\Omega)} &\le \frac{1}{2}\|\mathbf{E}_{0,k}, \mathbf{B}_{0,k}\|^2_{L_2(\Omega)} + N\|\partial_t^k f\|^2_{L_2(\Sigma^T)} \\
&\quad + \|\partial_t^k \mathbf{E}_f\|^2_{L_2((0,T)\times\Omega)}.
\end{aligned}
\tag{6.33}
$$

Next, for the sake of convenience, we introduce

$$\theta_k = \begin{cases} \theta/2^k, \ k = 0, 1, \ldots, m-4, \\ 0, \ k = m-3, \ldots, m. \end{cases} \tag{6.34}$$

Differentiating the linear Landau equation (3.46) formally k times in the t variable and using a variant of the energy identity (5.107), we get for each $\tau > 0$,

$$\begin{aligned} &\frac{1}{2}\big(\|\partial_t^k f(\tau,\cdot)\|^2_{L_{2,\theta_k}(\Omega\times\mathbb{R}^3)} - \|f_{0,k}\|^2_{L_{2,\theta_k}(\Omega\times\mathbb{R}^3)}\big) \\ &\quad + \underbrace{\int_0^\tau \int_\Omega \langle L\partial_t^k f, \partial_t^k f \, p_0^{2\theta_k}\rangle \, dxdt}_{=I_1} \\ &\quad - \underbrace{\boldsymbol{\xi}_1 \int_{\Sigma^\tau} (v(p)\cdot\partial_t^k \mathbf{E}_f)(\partial_t^k f)\sqrt{J} p_0^{2\theta_k} dz}_{=I_2} \\ &= \underbrace{\int_0^\tau \int_\Omega \langle (\partial_t^k \Gamma(f,g)), (\partial_t^k f) \, p_0^{2\theta_k}\rangle \, dxdt}_{I_4} \\ &\quad + \underbrace{\frac{\boldsymbol{\xi}}{2}\sum_{k_1+k_2=k}\binom{k}{k_1}\int_{\Sigma^\tau} v(p)\cdot(\partial_t^{k_1}\mathbf{E}_g)(\partial_t^{k_2} f)(\partial_t^k f) p_0^{2\theta_k}\, dz}_{I_5} \\ &\quad - \underbrace{\boldsymbol{\xi}\sum_{k_1+k_2=k}\binom{k}{k_1}\int_{\Sigma^\tau}\big(\partial_t^{k_1}\mathbf{E}_g + v(p)\times(\partial_t^{k_1}\mathbf{B}_g)\big)\cdot(\nabla_p \partial_t^{k_2} f)(\partial_t^k f)\, p_0^{2\theta_k} dz}_{I_6}, \end{aligned} \tag{6.35}$$

where $f_{0,k}$ is defined in (3.16).

Estimate of "quadratic terms." Using the fact that $L = -A - K$ (see (3.1)) and combining the coercivity estimate of A in (B.13) in Lemma B.7 with the estimate of K in (B.14), we have

$$I_1 \geq \kappa \|\nabla_p \partial_t^k f\|^2_{L_{2,\theta_k}(\Sigma^\tau)} - N_1(\theta,k)\|\partial_t^k f\|^2_{L_2(\Sigma^\tau)}. \tag{6.36}$$

Next, by the Cauchy–Schwarz inequality, we get

$$I_2 \leq \|\partial_t^k \mathbf{E}_f\|^2_{L_2((0,\tau)\times\Omega)} + N\|\partial_t^k f\|^2_{L_2(\Sigma^\tau)}. \tag{6.37}$$

Estimate of "cubic terms." To estimate $I_4 - I_6$, we need to prove the following claim: for any nonnegative integers k_1, k_2 such that $k_1 + k_2 = k$, one has

$$\text{(i)} \left|\int_0^T \int_\Omega \langle (\partial_t^k \Gamma(f,g)), (\partial_t^k f)\, p_0^{2\theta_k}\rangle \, dxdt\right| \leq N\sqrt{\varepsilon_0} y_f(T), \tag{6.38}$$

$$\text{(ii)} \left|\int_{\Sigma^T} (|\partial_t^{k_1} f| + |\nabla_p \partial_t^{k_1} f|)|\partial_t^{k_2}[\mathbf{E}_g, \mathbf{B}_g]||\partial_t^k f|\, p_0^{2\theta_k} dz\right| \leq N\sqrt{\varepsilon_0}\, y_f(T). \tag{6.39}$$

Then, applying (i)–(ii), we get

$$I_4 + I_5 + I_6 \leq N\sqrt{\varepsilon_0} y_f(T). \tag{6.40}$$

First, we consider the case when $k \le m-4$, so that $\theta_k = \theta/2^k$. We start with (i). It suffices to consider the case when $m-7 \le k \le m-4$ since the remaining case is simpler thanks to (6.5). By the estimate (B.15) in Lemma B.8 with $r=2$ in and $L_\infty^{t,x} - L_2^{t,x} - L_2^{t,x}$ Hölder's inequality, the integral on the l.h.s. of (6.38) is bounded by

$$
\begin{aligned}
&N(\theta)\mathcal{J}_k^1\big(1_{k_1 \le m-8}\mathcal{J}_{k_2}^1\mathcal{J}_{k_1}^2 + 1_{m-7\le k_1, k_2 \le 3}\mathcal{J}_{k_2}^3\mathcal{J}_{k_1}^4\big),\\
&\mathcal{J}_l^1 = \|\partial_t^l f\|_{L_2((0,\tau)\times\Omega)W^1_{2,\theta/2^k}(\mathbb{R}^3)},\\
&\mathcal{J}_l^2 = \|\partial_t^l g\|_{L_\infty((0,\tau)\times\Omega)W^1_2(\mathbb{R}^3)},\\
&\mathcal{J}_l^3 = \|\partial_t^l f\|_{L_\infty((0,\tau)\times\Omega)W^1_{2,\theta/2^k}(\mathbb{R}^3)},\\
&\mathcal{J}_l^4 = \|\partial_t^l g\|_{L_2((0,\tau)\times\Omega)W^1_2(\mathbb{R}^3)}.
\end{aligned}
$$

By the definition of $\mathcal{D}_f$ in (3.34), for $l \le m-4$,

$$
\mathcal{J}_l^1 \le \left(\int_0^T \mathcal{D}_f(\tau)\,d\tau\right)^{1/2},
$$

and similarly, by the smallness assumption on $y_g(T)$ (see (3.55)), we get for $l \le m$

$$
\mathcal{J}_l^4 \le \left(\int_0^T \mathcal{D}_g(\tau)\,d\tau\right)^{1/2} \le \sqrt{\varepsilon_0}.
$$

Next, due to the bound (6.5) in Lemma 6.1, we have

$$
1_{k_1 \le m-8}\mathcal{J}_{k_1}^2 \le N\sqrt{\varepsilon_0}.
$$

Furthermore, observe that for $k_2 \le 3$ and $k \ge m-7$, one has $k_2 + 9 \le 12 < m-7 \le k$ (recall that $m \ge 20$), so that $1_{k_2\le 3, k\ge m-7}\,\theta/2^k < \theta/2^{k_2+9}$. By this and the first inequality in (6.5), we conclude

$$
1_{k_2 \le 3, k \ge m-7}\mathcal{J}_{k_2}^3 \le \|\partial_t^{k_2} f\|_{L_\infty((0,T)\times\Omega)W^1_{2,\theta/2^{k_2+9}}(\mathbb{R}^3)} \le N\|\mathcal{H}_f\|^{1/2}_{L_\infty((0,T))}.
$$

Combining the above estimates, we obtain (6.38).

The assertion (ii) is proved in a similar way. We note that in the case when $k \ge m-3$, we have $\theta_k = 0$, and the same argument gives the desired bounds (6.38)–(6.39).

Finally, gathering the estimates (6.35)–(6.37) and (6.40) and summing up over k, and invoking definitions of $\mathcal{E}_f$ and $\mathcal{D}_f$ in (3.27) and (3.34), respectively, we obtain

(6.41)

$$
\begin{aligned}
&\sup_{\tau\le T}\mathcal{E}_f(\tau) + \int_0^T \mathcal{D}_f(\tau)\,d\tau\\
&\quad = \sum_{k=0}^m \Big(\|\partial_t^k f\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|\partial_t^k f\|^2_{L_2((0,T)\times\Omega)W^1_2(\mathbb{R}^3)}\\
&\qquad\quad + \|\partial_t^k[\mathbf{E}_f, \mathbf{B}_f]\|^2_{L_\infty((0,T))L_2(\Omega)}\Big)\\
&\qquad + \sum_{k=0}^{m-4}\Big(\|\partial_t^k f\|^2_{L_\infty((0,T))L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)} + \|\partial_t^k f\|^2_{L_2((0,T)\times\Omega)W^1_{2,\theta/2^k}(\mathbb{R}^3)}\Big)\\
&\quad \le N\sum_{k=0}^m\Big(\|f_{0,k}\|^2_{L_2(\Omega\times\mathbb{R}^3)} + \|[\mathbf{E}_{0,k}, \mathbf{B}_{0,k}]\|^2_{L_2(\Omega)} + \|\partial_t^k f\|^2_{L_2(\Sigma^T)} + \|\partial_t^k \mathbf{E}_f\|^2_{L_2((0,T)\times\Omega)}\Big)\\
&\qquad + N\sum_{k=0}^{m-4}\|f_{0,k}\|^2_{L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)} + N\sqrt{\varepsilon_0}\,y_f(T).
\end{aligned}
$$

Step 2: Closing the estimate of y_f. Finally, combining (6.41) with (6.14), and using the smallness assumption on $[f_{0,k}, \mathbf{E}_{0,k}, \mathbf{B}_{0,k}]$ in (3.37), we obtain

(6.42)

$$\begin{aligned} y_f(T) &\le N\sqrt{\varepsilon_0}y_f(T) + N\sum_{k=0}^{m}\Big(\|f_{0,k}\|^2_{L_2(\Omega\times\mathbb{R}^3)} + \|[\mathbf{E}_{0,k}, \mathbf{B}_{0,k}]\|^2_{L_2(\Omega)}\Big) \\ &\quad + N\sum_{k=0}^{m-4}\|f_{0,k}\|^2_{L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)} + N\sum_{k=0}^{m}\Big(\|\partial_t^k f\|^2_{L_2(\Sigma^T)} + \|\partial_t^k \mathbf{E}_f\|^2_{L_2((0,T)\times\Omega)}\Big) \\ &\le N(\sqrt{\varepsilon_0}+T)y_f(T) + N\varepsilon_0/M. \end{aligned}$$

By choosing $\varepsilon_0 < (4N)^{-2}$, $T < (4N)^{-1}$, $M > 4N$, we obtain the desired estimate $y_f(T) < \varepsilon_0$.

7. Proof of Theorem 3.10. We first state an auxiliary result that is useful for establishing both the existence and the uniqueness of the solution to the RVML system. See also the proof of Lemma 8.2 in [25].

LEMMA 7.1. *Invoke the assumptions of Theorem* 3.10 *and let ε_0, θ, M, and T be the constants introduced in the statements of that theorem (see* (3.36)*). Furthermore, let $[g^{(j)}, \mathbf{E}_{g^{(j)}}, \mathbf{B}_{g^{(j)}}], j = 1,2,$ be functions satisfying* (3.51)–(3.55) *and let $[f^{(j)}, \mathbf{E}_{f^{(j)}}, \mathbf{B}_{f^{(j)}}], j = 1,2,$ be two strong solutions to the linear RVML system* (3.46)–(3.50) *with $[g, \mathbf{E}_g, \mathbf{B}_g]$ replaced with $[g^{(j)}, \mathbf{E}_{g^{(j)}}, \mathbf{B}_{g^{(j)}}], j = 1,2,$ such that $[f^{(j)}, \mathbf{E}_{f^{(j)}}, \mathbf{B}_{f^{(j)}}], j = 1,2,$ satisfy the conditions analogous to* (i)–(iv) *in Theorem* 3.10. *We also denote $f^{1,2} = f^{(1)} - f^{(2)}$, $\mathbf{E}_f^{1,2} = \mathbf{E}_{f^{(1)}} - \mathbf{E}_{f^{(2)}}$ and define $\mathbf{B}_f^{1,2}$, $g^{1,2}, \mathbf{E}_g^{1,2}, \mathbf{B}_g^{1,2}$ in the same way. Then, we have*

$$\begin{aligned} &\sum_{k=0}^{m-8}\Big(\|\partial_t^k f^{1,2}\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|\partial_t^k f^{1,2}\|^2_{L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3)} \\ &\quad + \|\partial_t^k[\mathbf{E}_f^{1,2}, \mathbf{B}_f^{1,2}]\|^2_{L_\infty((0,T))L_2(\Omega)}\Big) \le \frac{1}{2}\sum_{k=0}^{m-8}\Big(\big(\|\partial_t^k g^{1,2}\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} \\ &\quad + \|\partial_t^k g^{1,2}\|^2_{L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3)} + \|\partial_t^k[\mathbf{E}_g^{1,2}, \mathbf{B}_g^{1,2}]\|^2_{L_\infty((0,T))L_2(\Omega)}\big)\Big). \end{aligned} \tag{7.1}$$

Proof of Lemma 7.1. We inspect the argument we used to establish the energy-dissipation bound (6.41). In particular, we write down the equation satisfied by $\partial_t^k f^{1,2}$ and use a variant of the energy identity in (5.107). The "quadratic" terms in the energy identity are estimated in the same way as in (6.36)–(6.37). On the other hand, we need to slightly modify the estimates of the "cubic" terms. For the sake of clarity, we focus on the integral

$$\begin{aligned} I_4 &= \int_{(0,T)\times\Omega}\langle\partial_t^k\big(\Gamma(f^{(1)}, g^{(1)}) - \Gamma(f^{(2)}, g^{(2)})\big), \partial_t^k f^{1,2}\rangle\,dxd\tau \\ &= \underbrace{\int_{(0,T)\times\Omega}\langle\partial_t^k\big(\Gamma(f^{1,2}, g^{(1)})\big), \partial_t^k f^{1,2}\rangle\,dxd\tau}_{=I_{4,1}} + \underbrace{\int_{(0,T)\times\Omega}\langle\partial_t^k\big(\Gamma(f^{(2)}, g^{1,2})\big), \partial_t^k f^{1,2}\rangle\,dxd\tau}_{=I_{4,2}}. \end{aligned}$$

Inspecting the proof of (6.38), using the bounds $y_{g^{(1)}} < \varepsilon_0$, $y_{f^{(2)}} < \varepsilon_0$ combined with Lemma 6.1, and employing the Cauchy–Schwarz inequality, we conclude

$$I_{4,1} \le N\sqrt{\varepsilon_0}\sum_{k=0}^{m-8}\Big(\|\partial_t^k f^{1,2}\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|\partial_t^k f^{1,2}\|^2_{L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3)}\Big),$$

$$I_{4,2} \le N\sqrt{\varepsilon_0}\sum_{k=0}^{m-8}\left(\|\partial_t^k g^{1,2}\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} + \|\partial_t^k g^{1,2}\|^2_{L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3)}\right)$$
$$+\frac{1}{2}\|\partial_t^k f^{1,2}\|^2_{L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3)},$$

where $N = N(r_1,\dots,r_4,\theta,\Omega,\alpha,m)$. For the closure, one needs to take ε_0 and T sufficiently small. ∎

Proof of Theorem 3.10. We note that the uniqueness follows directly from the above lemma.

The existence is proved by passing to the limit in the iteration scheme (3.39)–(3.45). Since the argument is standard (see, for example, [25]), we will not present it here but point out major steps.

(1) By Proposition 6.2, the sequence $[f^n,\mathbf{E}^n,\mathbf{B}^n], n\ge 1$, is well defined, and by (3.56) in Proposition 3.11, one has $y_{f_n}(T)<\varepsilon_0$ for each n. By (7.1) in Lemma 7.1, the sequence $\partial_t^k f^n, k\le m-8$, is Cauchy in $L_\infty^t L_2^{x,p}\cap L_2^{t,x}W_2^1(\mathbb{R}^3)$, and $[\mathbf{E}^n,\mathbf{B}^n], n\ge 1$, is a Cauchy sequence in the space $L_\infty^t L_2^x$, and hence, $[f^n,\mathbf{E}^n,\mathbf{B}^n]$ converge to some $[f,\mathbf{E},\mathbf{B}]$. In addition, using (7.1) again and (3.56), we conclude that all the temporal derivatives up to order m also converge in the weak* topology of the same space.

(2) By using Green's identity (D.4), we write down the weak formulation of the system with a test function ϕ satisfying (3.10)–(3.12). Due to the uniform in n estimates in Proposition 3.11 and the fact that f_n converges to f, we may pass to the limit in the weak formulation. In particular, one needs to use Lemma 7.1 and (3.56) to pass to the limit in the integrals involving the Lorentz and collisional terms.

(3) Due to the convergence in (2), $\partial_t^k f\in L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)\cap L_2((0,T)\times\Omega)W_2^1(\mathbb{R}^3), k\le m$, is an intermediate finite energy solution (see Definition 5.2 in the proof of Proposition 5.13) to (3.2) formally differentiated k times in t with the SRBC and $\partial_t^k f(0,\cdot)\equiv f_{0,k}(\cdot)$. We point out that in the proof of the aforementioned proposition, we showed that any intermediate finite energy solution is a finite energy solution in the sense of Definition 3.1, and hence, $\partial_t^k f\in C([0,T])L_2(\Omega\times\mathbb{R}^3), k\le m$, as desired. Furthermore, since $\partial_t^k f\in S_2(\Sigma^T), k\le m-5$, by Remark 3.3, we conclude that $\partial_t^k f, k\le m-5$, is a strong solution to the k times differentiated Landau equation.

(4) By using a limiting argument, we conclude that $\partial_t^k[\mathbf{E}_f,\mathbf{B}_f]\in C([0,T])L_2(\Omega)\cap L_\infty((0,T))W_2^1(\Omega), k\le m-1$, is a strong solution to Maxwell's equations (3.3)–(3.4) with the perfect conductor BC, initial data $[\mathbf{E}_{0,k},\mathbf{B}_{0,k}]$, whereas $\partial_t^m[\mathbf{E}_f,\mathbf{B}_f]\in L_\infty((0,T))L_2(\Omega)$ is a weak solution. In addition, the identities in (3.5) formally differentiated k times in t are valid. The fact that $\partial_t^m[\mathbf{E}_f,\mathbf{B}_f]\in C([0,T])L_2(\Omega)$ can be proved by a mollification argument as in the proof of Theorem 4.1 in Chapter VII in [11]. ∎

Appendix A.

Lemma A.1. *Let* $\Psi:\Omega_{r_0}(x_0)\times\mathbb{R}^3\to\mathbb{H}_-$ *be a local diffeomorphism given by* (5.31)–(5.32). *Then, the following assertions hold.*

(i) *For*

$$C(y)=\left(\frac{\partial x}{\partial y}\right)^T\left(\frac{\partial x}{\partial y}\right),$$

one has

$$C^{i3}(y) = 0,\ i \in \{1,2\}, \quad \textit{if } y_3 = 0. \tag{A.1}$$

(ii) *For any* $y \in \psi(\Omega_{r_0}) \cap \{y_3 = 0\}$ *and any* w,

$$\left| \left(\frac{\partial x}{\partial y} \right) w \right| = \left| \left(\frac{\partial x}{\partial y} \right) \boldsymbol{R} w \right|. \tag{A.2}$$

(iii) *Let* u *be a function on* $\overline{\Omega_{r_0}(x_0)} \times \mathbb{R}^3$ *satisfying*

$$u(x,p) = u(x, R_x p), (x,p) \in \gamma_- \tag{A.3}$$

and denote

$$\begin{aligned}
&\mathsf{U}(x,p) = \int_{\mathbb{R}^3} \Phi(P,Q) u(x,q)\, dq, \quad \widehat{\mathsf{U}}(y,w) = \mathsf{U}(x(y), p(y,w)),\\
&\mathfrak{U}(y,w) = \left(\frac{\partial y}{\partial x} \right) \widehat{\mathsf{U}}(y,w) \left(\frac{\partial y}{\partial x} \right)^T.
\end{aligned}$$

Then, one has

$$\mathfrak{U}^{i3}(y,w) = -\mathfrak{U}^{i3}(y, \boldsymbol{R} w),\ i \in \{1,2\}, \quad \textit{if } y_3 = 0. \tag{A.4}$$

Proof. (i) We assume that $\rho^{(\varepsilon)}$ is a mollification of ρ with a standard mollifier ι and let $\rho^{(\varepsilon),j}$ be the mollification of ρ with the mollifier $y_j \iota, j = 1,2$. The assertion follows from the identities

$$\left(\frac{\partial x}{\partial y} \right) = \begin{pmatrix} 1 - y_3 \rho_{11}^{(y_3)} & -y_3 \rho_{12}^{(y_3)} & -\rho_1^{(y_3)} + y_3 (\rho_{11}^{(y_3),1} + \rho_{12}^{(y_3),2}) \\ -y_3 \rho_{12}^{(y_3)} & 1 - y_3 \rho_{22}^{(y_3)} & -\rho_2^{(y_3)} + y_3 (\rho_{12}^{(y_3),1} + \rho_{22}^{(y_3),2}) \\ \rho_1 & \rho_2 & 1 \end{pmatrix}, \tag{A.5}$$

$$\begin{aligned}
C(y_1,y_2,0) &= \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho_2 \\ -\rho_1 & -\rho_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1+\rho_1^2 & \rho_1\rho_2 & 0 \\ \rho_1\rho_2 & 1+\rho_2^2 & 0 \\ 0 & 0 & 1+\rho_1^2+\rho_2^2 \end{pmatrix},
\end{aligned}$$

where $\rho_{ij} = \partial_{y_i y_j} \rho$.

(ii) The desired identity follows from the equality

$$\left| \left(\frac{\partial x}{\partial y} \right) \boldsymbol{R} w \right|^2 = C^{ij}(y) (\boldsymbol{R} w)_i (\boldsymbol{R} w)_j$$

and (A.1).

(iii) We denote $q_0 = (1 + |q|^2)^{1/2}$,

$$\begin{aligned}
&w' = \left(\frac{\partial y}{\partial x} \right) q,\ \widehat{p}_0 = p_0(p(y,w)),\ \widehat{q}_0 = q_0(q(y,w')),\\
&\widehat{P}(y,w) = (\widehat{p}_0, p(y,w)),\ \widehat{Q}(y,w') = (\widehat{q}_0, q(y,w)).
\end{aligned}$$

Furthermore, changing variables $q = \left(\frac{\partial x}{\partial y}\right) w'$ gives

$$
\begin{aligned}
\widehat{\mathsf{U}}(y,w) &= \int_{\mathbb{R}^3} \Phi(\widehat{P},Q) u(x(y),q)\,dq \\
&= \underbrace{\left|\det\left(\frac{\partial x}{\partial y}\right)\right|}_{=\mathcal{J}} \int_{\mathbb{R}^3} \Phi(\widehat{P},\widehat{Q}) \widehat{u}(y,w')\,dw', \qquad \text{(A.6)}
\end{aligned}
$$

$$
\mathfrak{U}(y,w) = \mathcal{J} \int_{\mathbb{R}^3} \Xi(y,w,w') \widehat{u}(y,w')\,dw', \qquad \text{(A.7)}
$$

where $\widehat{u}(y,w') = u(x(y), q(y,w'))$ (cf. (5.35)), and

$$
\Xi(y,w,w') = \left(\frac{\partial y}{\partial x}\right) \Phi(\widehat{P},\widehat{Q}) \left(\frac{\partial y}{\partial x}\right)^T. \qquad \text{(A.8)}
$$

Furthermore, by the change of variables $w' \to \boldsymbol{R}w'$,

$$
\mathfrak{U}(y,\boldsymbol{R}w) = \mathcal{J} \int_{\mathbb{R}^3} \Xi(y,\boldsymbol{R}w,\boldsymbol{R}w') \widehat{u}(y,\boldsymbol{R}w')\,dw'. \qquad \text{(A.9)}
$$

Since u satisfies the SRBC (see (A.3)), we have

$$
\widehat{u}(y,\boldsymbol{R}w) = \widehat{u}(y,w) \quad \text{ if } y_3 = 0, \qquad \text{(A.10)}
$$

Thus, due to (A.6)–(A.10), to prove (A.4), it suffices to demonstrate that

$$
\Xi^{i3}(y,w,w') = -\Xi^{i3}(y,\boldsymbol{R}w,\boldsymbol{R}w'),\ i \in \{1,2\},\ \text{whenever } y_3 = 0. \qquad \text{(A.11)}
$$

Verification of (A.11). First, by the definition of Φ in (2.5)–(2.7),

$$
\Xi = \frac{\Lambda(\widehat{P},\widehat{Q})}{\widehat{p}_0 \widehat{q}_0} \left(\frac{\partial y}{\partial x}\right) S(\widehat{P},\widehat{Q}) \left(\frac{\partial y}{\partial x}\right)^T. \qquad \text{(A.12)}
$$

We will need the following identities:

$$
\widehat{p}_0 = \left(1 + \left|\left(\frac{\partial x}{\partial y}\right) w\right|^2\right)^{1/2}, \quad \widehat{q} = \left(1 + \left|\left(\frac{\partial x}{\partial y}\right) w'\right|^2\right)^{1/2}, \qquad \text{(A.13)}
$$

$$
\widehat{P} \cdot \widehat{Q} = \widehat{p}_0 \widehat{q}_0 - w^T \left(\frac{\partial x}{\partial y}\right)^T \left(\frac{\partial x}{\partial y}\right) w', \qquad \text{(A.14)}
$$

$$
p(y,w) \otimes q(y,w') = \left(\frac{\partial x}{\partial y}\right) w (w')^T \left(\frac{\partial x}{\partial y}\right)^T,
$$

$$
\left(\frac{\partial y}{\partial x}\right) p(y,w) \otimes q(y,w') \left(\frac{\partial y}{\partial x}\right)^T = w(w')^T,
$$

$$
\begin{aligned}
&\left(\frac{\partial y}{\partial x}\right) S(\widehat{P},\widehat{Q}) \left(\frac{\partial y}{\partial x}\right)^T \qquad \text{(A.15)} \\
&\quad = \left((\widehat{P} \cdot \widehat{Q})^2 - 1\right) \left(\frac{\partial y}{\partial x}\right) \left(\frac{\partial y}{\partial x}\right)^T \\
&\qquad - (w - w') \otimes (w - w') \\
&\qquad + (\widehat{P} \cdot \widehat{Q} - 1)(w \otimes w' + w' \otimes w).
\end{aligned}
$$

We first handle the factor $\Lambda(\widehat{P}, \widehat{Q})$ (see (2.5)). By (A.1) in assertion (i), we have

$$f(y,w,w') = f(y,\boldsymbol{R}w,\boldsymbol{R}w'), \text{ for } f = \widehat{p}_0, \widehat{q}_0, \widehat{P}\cdot\widehat{Q}, \quad \text{if } y_3 = 0, \tag{A.16}$$

and, hence, (A.16) is also true for $f = \Lambda(\widehat{P}, \widehat{Q})$.

We consider the remaining factor on the r.h.s. of (A.12). By (A.1) in the assertion (i), we conclude that (A.11) holds with Ξ replaced with the l.h.s. of (A.15). Thus, (A.11) holds, and the desired identity (A.4) is valid. ∎

LEMMA A.2. *Let M be a nondegenerate 3 by 3 matrix, and denote*

$$\mathfrak{W}(w) = \frac{w}{(1+|Mw|^2)^{1/2}} =: v. \tag{A.17}$$

Then, the following assertions hold.

(i)

$$\underbrace{|D^j \mathfrak{W}|}_{=D^j_w v} \le N(M)(1+|w|^2)^{-j/2}, j = 1,2. \tag{A.18}$$

(ii) *Let $m \ge 1$ be a number. Then, $\mathfrak{W}: \{|w| < m\} \to \mathbb{R}^3$ is a diffeomorphism onto its image, and*

$$\sup_{\mathfrak{W}(\{|w|<m\})} |\underbrace{D^j \mathfrak{W}^{-1}}_{=D^j_v w}| < N m^{2j+1}, j = 1,2,3, \tag{A.19}$$

$$\sup_{\mathfrak{W}(\{|w|<m\})} |\underbrace{D\big((D\mathfrak{W})\circ \mathfrak{W}^{-1}\big)}_{=D_v\big((D_w v)(w(v))\big)}| \le N m, \tag{A.20}$$

where $N = N(|M|)$.

Proof. (i) Let c_{ik} be the ik-th entry of the matrix $M^T M$. Then, by direct computations,

$$\begin{aligned} \frac{\partial v_i}{\partial w_j} &= \frac{\delta_{ij}}{(1+|Mw|^2)^{1/2}} - \frac{c_{jl} w_l w_i}{(1+|Mw|^2)^{3/2}}, \\ \frac{\partial^2 v_i}{\partial w_j \partial w_k} &= -\frac{\delta_{ij} c_{kl} w_l}{(1+|Mw|^2)^{3/2}} - \frac{c_{jk} w_i + \delta_{ki} c_{jl} w_l}{(1+|Mw|^2)^{3/2}} + \frac{3 c_{jl} c_{kl'} w_i w_l w_{l'}}{(1+|Mw|^2)^{5/2}}. \end{aligned} \tag{A.21}$$

Combining the above identities with the fact that

$$|Mw|^2 \ge N(M)|w|^2,$$

we prove the first assertion.

(ii) Multiplying both sides of (A.17) by M gives

$$|Mv|^2 = \frac{|Mw|^2}{1+|Mw|^2}, \quad 1 - |Mv|^2 = \frac{1}{1+|Mw|^2}, \tag{A.22}$$

$$|Mw|^2 = \frac{|Mv|^2}{1-|Mv|^2}, \quad w = \frac{v}{(1-|Mv|^2)^{1/2}}. \tag{A.23}$$

Note that $1 - |Mv|^2$ is bounded away from 0 on $\mathfrak{W}(\{|w| < m\})$, and hence $\mathfrak{W}: \{|w| < m\} \to \mathbb{R}^3$ is a diffeomorphism.

Next, differentiating the second identity in (A.23), we get

$$\begin{aligned} &\frac{\partial w_i}{\partial v_j} = \frac{\delta_{ij}}{(1-|Mv|^2)^{1/2}} + \frac{c_{jl} v_l v_i}{(1-|Mv|^2)^{3/2}}, \\ &\frac{\partial^2 w_i}{\partial v_j \partial v_k} = \frac{P_3(v)}{(1-|Mv|^2)^{5/2}}, \quad \frac{\partial^3 w_i}{\partial v_j \partial v_k \partial v_r} = \frac{P_5(v)}{(1-|Mv|^2)^{7/2}}, \end{aligned} \tag{A.24}$$

where $P_3(v)$ and $P_5(v)$ are certain polynomials of orders 3 and 5 with coefficients bounded by $N(|M|)$. Then, by (A.22) and (A.24), and the fact that

$$|v| = |\mathfrak{W}(w)| \le N(M), \tag{A.25}$$

for $v \in \mathfrak{W}(\{|w| < m\})$, we have

$$\left|\frac{\partial w_i}{\partial v_j}\right| \le N(1+|Mw|^2)^{3/2} \le N m^3,$$

where $N = N(M)$. Similarly, we prove the estimates of the second- and third-order derivatives.

Finally, to prove the bound (A.20), we note that by (A.21) and (A.22)–(A.23),

$$\frac{\partial v_i}{\partial w_j} = (1-|Mv|^2)^{1/2}(\delta_{ij} - c_{jl} v_i v_l).$$

Differentiating the above expression and using (A.25), we conclude

$$\left|D_v \frac{\partial v_i}{\partial w_j}\right| \le N(1-|Mv|^2)^{-1/2} \le N m,$$

so that (A.20) is true. $\square$

Lemma A.3. *Let $n \ge 0$, $G \subset \mathbb{R}^3$ be the even extension of $\psi(\Omega_{r_0}(x_0))$ across the plane $y_3 = 0$ (see Step 3 in the proof of Lemma 5.10). Let $\mathcal{W}$ and Υ_n be the mappings given by (5.44) and (5.51), respectively. Then, the following assertions hold.*

(i) *The mapping $\Upsilon_n : G \times \{|w| < 2^{n+2}\}$ is a bi-Lipschitz homeomorphism onto its image, and*

$$\Upsilon_n^{-1}(y,v) = (y, \mathsf{W}(y,v)), \tag{A.26}$$

where

$$\mathsf{W}(y,v) = \begin{cases} \frac{v}{\left(1-\left|\mathsf{M}(y)v\right|^2\right)^{1/2}}, \ (y,v) \in \Upsilon_n\big(\psi(\Omega_{r_0}(x_0)) \times \{|w| < 2^{n+2}\}\big), \\ \frac{v}{\left(1-\left|[\mathsf{M}(\boldsymbol{R}y)]\boldsymbol{R}v\right|^2\right)^{1/2}}, \ (y,v) \in \Upsilon_n\big(G \cap \mathbb{R}^3_+ \times \{|w| < 2^{n+2}\}\big), \end{cases}$$

and

$$\mathsf{M}(y) = \left(\frac{\partial x}{\partial y}\right)(y).$$

Furthermore, for the sake of convenience, we denote

$$v = \mathcal{W}(y,w), \quad w = \mathsf{W}(y,v).$$

(ii) *One has*

$$\text{(A.27)}\qquad \| \underbrace{\nabla_y \mathsf{W}}_{=\dfrac{\partial w}{\partial y}} \|_{L_\infty\left((0,T)\times \Upsilon_n(G\times\{|w|<2^{n+2}\})\right)} \le N 2^n,$$

$$\text{(A.28)}\qquad \| \underbrace{\nabla_y \nabla_w \mathcal{W}}_{=\dfrac{\partial^2 v}{\partial y \partial w}} \|_{L_\infty((0,T)\times G\times\{|w|<2^{n+2}\})} \le N 2^{-n},$$

$$\text{(A.29)}\qquad \|| \underbrace{\nabla_y \nabla_v \mathsf{W}}_{=\dfrac{\partial^2 w}{\partial y \partial v}} | + | \underbrace{\nabla_v \nabla_v \mathsf{W}}_{=\dfrac{\partial^2 w}{\partial v^2}} |\|_{L_\infty\left((0,T)\times \Upsilon_n(G\times\{|w|<2^{n+2}\})\right)} \le N 2^{5n},$$

where $N = N(\Omega)$.

Proof. (i) First, note that (A.26) follows from (A.23). We now show that Υ_n is bi-Lipschitz. Since Ω is a $C^{1,1}$ domain, we only need to show that the functions $\mathcal{W}, \mathsf{W}$ are continuous across the boundary $\{y_3 = 0\}\times\mathbb{R}^3$. To this end, it suffices to demonstrate that

$$\left| \left(\frac{\partial x}{\partial y}\right) w \right| = \left| \left(\frac{\partial x}{\partial y}\right) \boldsymbol{R} w \right| \quad \text{whenever } y_3 = 0.$$

The latter is true thanks to (A.2) in Lemma A.1 (ii).

(ii) Invoke the notation of Lemma A.2. Let $M(y)$ be either

$$\left(\frac{\partial x}{\partial y}\right)(y) \text{ or } \left(\left(\frac{\partial x}{\partial y}\right)(\boldsymbol{R} y)\right)\boldsymbol{R},$$

and $C(y) = (c_{ij}, i,j = 1,2,3) := M^T M$, and

$$v(y,w) = \frac{w}{(1+|Mw|^2)^{1/2}}.$$

First, we claim that the functions

$$\left(\frac{\partial v}{\partial w}\right), \left(\frac{\partial w}{\partial v}\right)$$

are continuous across the set $\{y_3 = 0\}\times\mathbb{R}^3$. This assertion follows from the explicit expressions of these functions (see (A.21) and (A.24)) and the identity (A.2). Hence, we only need to prove (A.28)–(A.29) away from $\{y_3 = 0\}\times\mathbb{R}^3$.

Next, by (A.23) and (A.22), whenever $y_3 \ne 0$, we have

$$\frac{\partial w_i}{\partial y_r} = \frac{(\partial_{y_r} c_{ll'}) v_l v_{l'} v_i}{(1-|Mv|^2)^{1/2}} = (\partial_{y_r} c_{ll'}) v_l v_{l'} v_i (1+|Mw|^2)^{1/2},$$

and this implies (A.27). Furthermore, by (A.21) and (A.24), away from $\{y_3 = 0\}$, the following identities hold:

$$\frac{\partial^2 v_i}{\partial w_j \partial y_r} = (\partial_{y_r} c_{ll'}) w_l w_{l'} \Big(-\frac{1}{2} \frac{\delta_{ij}}{(1+|Mw|^2)^{3/2}} + \frac{3}{2} \frac{c_{jl} w_l w_i}{(1+|Mw|^2)^{5/2}} \Big) - \frac{(\partial_{y_r} c_{jl}) w_l w_i}{(1+|Mw|^2)^{3/2}},$$

$$\frac{\partial^2 w_i}{\partial v_j \partial y_r} = (\partial_{y_r} c_{ll'} v_l v_{l'}) \Big(\frac{1}{2} \frac{\delta_{ij}}{(1-|Mv|^2)^{3/2}} - \frac{3}{2} \frac{c_{jk} v_k v_i}{(1-|Mv|^2)^{5/2}} \Big) + \frac{(\partial_{y_r} c_{jl}) v_l v_i}{(1-|Mv|^2)^{3/2}}.$$

The first identity implies (A.28). Furthermore, the second identity combined with (A.22), and (A.25) yield

$$\left| \frac{\partial^2 w_i}{\partial v_j \partial y_r} \right| \leq N(1+|Mw|^2)^{5/2} \leq N(\Omega) 2^{5n}.$$

The bound of $\nabla_v \nabla_v \mathsf{W}$ follows from (A.19) with $j=2$. The assertion (ii) is proved. ◻

Lemma A.4. *Let G be an bounded domain and $\psi : G \to \mathbb{R}^3$ be a diffeomorphism such that*

$$\|D\psi\|_{C(G)} \leq N_0, \quad \|D(\psi)^{-1}\|_{C(\psi(G))} \leq N_1.$$

Let a be a bounded matrix-valued function such that

$$\delta_1 |\xi|^2 \leq a^{ij}(v) \xi_i \xi_j \leq \delta_2 |\xi|^2 \quad \forall v \in G, \xi \in \mathbb{R}^3.$$

Denote

$$\tilde{a} = (D\psi)(a \circ \psi^{-1})(D\psi)^T.$$

Then, $\tilde{a}$ satisfies

$$\tilde{\delta}_1 |\xi|^2 \leq \tilde{a}^{ij}(w) \xi_i \xi_j \leq \tilde{\delta}_2 |\xi|^2, \, w \in \psi(G), \xi \in \mathbb{R}^3 \tag{A.30}$$

with

$$\tilde{\delta}_1 = c\, \delta_1 N_1^{-2}, \quad \tilde{\delta}_2 = c^{-1} \delta_2 N_0^2,$$

where $c \in (0,1)$.

Proof. To prove the lower bound, note that

$$\xi^T \tilde{a} \xi = ((D\psi)^T \xi)^T (a \circ \psi^{-1})((D\psi)^T \xi) \geq \delta_1 |(D\psi)^T \xi|^2 \geq c \delta_1 N_1^{-2}.$$

The upper bound follows from the same argument. ◻

Appendix B. Auxiliary results about the relativistic Landau equation near Jüttner's solution.

Lemma B.1 (Lemma 6 in [32]). *For sufficiently regular functions $f = (f^+, f^-)$, $g = (g^+, g^-)$, $h = (h^+, h^-)$ on $\mathbb{R}^3$, the following formulas hold:*

(B.1) $$Af = 2\nabla_p \cdot (\sigma \nabla_p f) - \frac{1}{2}(v(p))^T \sigma v(p) f + \nabla_p \cdot \big(\sigma v(p)\big) f,$$

(B.2) $$Kf = -J^{-1/2}(p)\partial_{p_i}\left(J(p)\int \Phi^{ij}(P,Q)J^{1/2}(q)\big(\partial_{q_j} f(q) + \frac{q_j}{2q_0} f(q)\big)\cdot \boldsymbol{\xi}_0\, dq\right)\boldsymbol{\xi}_0,$$

(B.3)
$$\begin{aligned}\Gamma_{\pm}(g,h) = &\left(\partial_{p_i} - \frac{p_i}{2p_0}\right)\partial_{p_j} g_{\pm}(p)\int \Phi^{ij}(P,Q)J^{1/2}(q)h(q)\cdot \boldsymbol{\xi}_0\, dq\\ &- \left(\partial_{p_i} - \frac{p_i}{2p_0}\right) g_{\pm}(p)\int \Phi^{ij}(P,Q)J^{1/2}(q)\partial_{q_j} h(q)\cdot \boldsymbol{\xi}_0\, dq,\end{aligned}$$

where σ is defined in (5.16).

LEMMA B.2 (Corollary 4.5 with $\alpha = -3$ in [26]). *Let σ be the function defined in* (5.16). *Then, the following assertions hold.*

(i) *There exist constants $N_1, N_2 > 0$ such that for any $\xi \in \mathbb{R}^3$*

(B.4) $$N_1|\xi|^2 \le \sigma^{ij}(p)\xi_i\xi_j \le N_2|\xi|^2.$$

(ii) *For any multi-index β,*

(B.5) $$|D_p^{\beta}\sigma(p)| \le N(\beta)p_0^{-|\beta|}.$$

LEMMA B.3. *Let $k \ge 0$ be an integer, $r \in (3/2, \infty]$, and $g \in W_r^k(\mathbb{R}^3)$. Then, for*

(B.6) $$I(p) = \int \Phi^{ij}(P,Q)J^{1/2}(q)g(q)\, dq,$$

we have

(B.7) $$\|D_p^k I\|_{L_\infty(\mathbb{R}^3)} \lesssim \|g\|_{W_r^k(\mathbb{R}^3)}.$$

Proof. By Theorem 3 in [32] (see p. 281 therein), for any multi-index $\beta = (\beta_1, \beta_2, \beta_3)$,

$$\begin{aligned}&D_p^{\beta}\int \Phi^{ij}(P,Q)J^{1/2}(q)g(q)\, dq\\ &\quad = \sum_{\beta^1+\beta^2\le\beta}\int \Theta_{\beta_1}(p,q)\Phi^{ij}(P,Q)J^{1/2}(q)\partial_{\beta_2} g(q)\phi^{\beta}_{\beta_1,\beta_2}(p,q)\, dq,\end{aligned}$$

where

$$\Theta_{\beta_1}(p,q) = \left(\partial_{p_1} + \frac{q_0}{p_0}\partial_{q_1}\right)^{\beta_1^1}\left(\partial_{p_2} + \frac{q_0}{p_0}\partial_{q_2}\right)^{\beta_2^1}\left(\partial_{p_3} + \frac{q_0}{p_0}\partial_{q_3}\right)^{\beta_3^1},$$

and $\phi^{\beta}_{\beta_1,\beta_2}$ is a smooth function satisfying the bound

$$|\phi^{\beta}_{\beta_1,\beta_2,\beta_3}(p,q)| \lesssim q_0^{|\beta|}p_0^{|\beta_1|-|\beta|}.$$

By using the above identity, the estimate

$$|\Theta_{\beta_1}(p,q)\Phi(P,Q)| \le N p_0^{-|\beta_1|}q_0^7(1 + |p-q|^{-1})$$

(see Lemma 2 on p. 277 in [32]), and Hölder's inequality with $r \in (3/2, \infty]$ and $r' = r/(r-1) \in [1,3)$, we obtain (B.7). ∎

The following lemma follows directly from Lemma 4 on p. 287 in [32].

LEMMA B.4. *For $r \in (3/2, \infty]$, $g \in W^1_r(\mathbb{R}^3)$, the following identity holds in the sense of distributions:*

$$\begin{aligned}\text{(B.8)}\qquad &\partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(q)\, dq \\ &= \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \frac{q_j}{2q_0} g(q)\, dq \\ &\quad - 4 \int \frac{P \cdot Q}{p_0 q_0} \Big((P\cdot Q)^2 - 1\Big)^{-1/2} J^{1/2}(q) g(q)\, dq - \kappa(p) J^{1/2}(p) g(p),\end{aligned}$$

where $\kappa(p) = 2^{7/2} \pi p_0 \int_0^\pi (1 + |p|^2 \sin^2\theta)^{-3/2} \sin(\theta)\, d\theta$.

LEMMA B.5. *Let $r \in (3/2, \infty]$, $g = (g^+, g^-) \in W^1_r(\mathbb{R}^3)$ and a_g, C_g, and Kg be given by* (6.2)–(6.4)*, respectively. Then, one has*

$$\text{(B.9)}\qquad \|a_g\|_{L_\infty(\mathbb{R}^3)} \le \|g\|_{W^1_r(\mathbb{R}^3)},$$

$$\text{(B.10)}\qquad \|C_g\|_{L_\infty(\mathbb{R}^3)} \le N + N \|g\|_{W^1_r(\mathbb{R}^3)},$$

$$\text{(B.11)}\qquad |Kg|(p) \le N J^{1/4}(p) \|g\|_{W^1_r(\mathbb{R}^3)},$$

where $N = N(r)$.

Proof. Estimate of a_g. The estimate follows from the definition of a_g (see (6.2)) and (B.7) with $k = 0, 1$ (see Lemma B.3).

Estimate of C_g. By the estimates of σ in (B.5), we only need to handle the integral term in (6.3), which we decompose as follows:

$$\begin{aligned}&\partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(q) \cdot \boldsymbol{\xi}_0\, dq \\ &\quad - \frac{p_i}{2p_0} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(q) \cdot \boldsymbol{\xi}_0\, dq =: C_{g,1} + C_{g,2}.\end{aligned}$$

Next, by the identity (B.8),

$$\begin{aligned}C_{g,1} &= \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \frac{q_j}{2q_0} g(q) \cdot \boldsymbol{\xi}_0\, dq \\ &\quad - 4 \int \frac{P\cdot Q}{p_0 q_0} \Big((P \cdot Q)^2 - 1\Big)^{-1/2} J^{1/2}(q) g(q) \cdot \boldsymbol{\xi}_0\, dq \\ &\quad - \kappa(p) J^{1/2}(p) g(p) \cdot \boldsymbol{\xi}_0 =: C_{g,1,1} + C_{g,1,2} + C_{g,1,3}.\end{aligned}$$

Applying the estimate (B.7) with $k = 0, 1$ to the terms $C_{g,1,1}$ and $C_{g,2}$, we get

$$|C_{g,1,1}| + |C_{g,2}| \le N \|g\|_{W^1_r(\mathbb{R}^3)}.$$

By a simple bound (see the formula (32) in [32])

$$P \cdot Q - 1 \ge N_1 \left(\frac{|p-q|^2}{q_0^2} 1_{|p-q| < (|p|+1)/2} + \frac{p_0}{q_0} 1_{|p-q| \ge (|p|+1)/2} \right)$$

and Hölder's inequality,

$$|C_{g,1,2}| \le N \|g\|_{L_r(\mathbb{R}^3)}.$$

Finally, we note that the last inequality also holds for $C_{g,1,3}$ since κ is a bounded function. Thus, (B.10) is valid.

Estimate of Kg. First, we split the integral in (B.2) as follows:

$$
\begin{aligned}
Kg = {} & (\partial_{p_i} p_0) J^{1/2}(p) \int \Phi^{ij}(P,Q) J^{1/2}(q) \Big(\partial_{q_j} g(q) + \frac{q_j}{2} g(q) \Big) \cdot \boldsymbol{\xi}_0 \, dq \, \boldsymbol{\xi}_0 \\
& - J^{1/2}(p) \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \frac{q_j}{2q_0} g(q) \cdot \boldsymbol{\xi}_0 \, dq \, \boldsymbol{\xi}_0 \\
& - J^{1/2}(p) \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(q) \cdot \boldsymbol{\xi}_0 \, dq \, \boldsymbol{\xi}_0 =: K_1 + K_2 + K_3.
\end{aligned}
$$

We observe that the following terms are similar:

- K_1 and $C_{g,2}$,
- K_2 and $C_{g,1,1}$,
- K_3 and $C_{g,1}$.

Hence, the estimate (B.11) is proved by repeating the above argument. ◻

LEMMA B.6. *Let g be a function satisfying Assumption 5.1 (see (5.1)–(5.2)). Then, for σ_g defined in (3.9), one has*

$$
\|\nabla_p \sigma_g\|_{L_\infty(\Sigma^T)} + \|\sigma_g\|_{L_\infty((0,T)) C^{\varkappa/3,\varkappa}_{x,p}(\Omega\times\mathbb{R}^3)} \le N(K). \tag{B.12}
$$

Proof. First, note that the estimate of $\nabla_p \sigma_g$ follows directly from (B.7) with $k=1$ and the assumption (5.2). Furthermore, for any $t \ge 0$ and $x_1, x_2 \in \Omega$, $p \in \mathbb{R}^3$,

$$
\sigma_g(t,x_1,p) - \sigma_g(t,x_2,p) = \int \Phi(P,Q) J^{1/2} \big(g(t,x_1,q) - g(t,x_2,q) \big) \, dq.
$$

Then, by (B.7) with $k=0$ and the assumption (5.1),

$$
\begin{aligned}
|\sigma_g(t,x_1,p) - \sigma_g(t,x_2,p)| & \le N \sup_{p\in\mathbb{R}^3} |g(t,x_1,p) - g(t,x_2,p)| \\
& \le N |x_1 - x_2|^{\varkappa/3}.
\end{aligned}
$$

Now the assertion follows from the above inequality, the L_∞ estimate of $|\nabla_p \sigma_g|$, and the interpolation inequality for Hölder spaces. ◻

LEMMA B.7 (cf. Lemma 7 of [32]). *For any $\theta \ge 0$, there exists $\kappa > 0$ such that for any $g = (g^+, g^-) \in W^1_{2,\theta}(\mathbb{R}^3), h = (h^+, h^-) \in L_{2,\theta}(\mathbb{R}^3)$,*

$$
-\langle Ag, g\, p_0^{2\theta} \rangle \ge \kappa \|\nabla_p g\|^2_{L_{2,\theta}(\mathbb{R}^3)} - N(\theta) \|g\|^2_{L_2(\mathbb{R}^3)}. \tag{B.13}
$$

Furthermore, for any $\varepsilon \in (0,1)$,

$$
\left| \int_{\mathbb{R}^3} (Kg) \cdot h\, p_0^{2\theta} dp \right| \le \varepsilon \|g\|^2_{W^1_2(\mathbb{R}^3)} + N(\theta) \varepsilon^{-1} \|h\|^2_{L_2(\mathbb{R}^3)}. \tag{B.14}
$$

Proof. In the case when $\theta = 0$, the estimate (B.13) is proved in Lemma 7 in [32]. The case $\theta > 0$ is handled by the same argument, and hence, we omit the proof. The bound (B.14) follows from (B.11) in Lemma B.5. ◻

LEMMA B.8. *For sufficiently regular functions $f_j = (f_j^+, f_j^-)$, $j = 1,2,3$, on $\mathbb{R}^3$ and any $r \in (3/2, \infty]$ and $\theta \ge 0$, we have*

$$
\begin{aligned}
\big| \langle \Gamma(f_1,f_2), f_3 p_0^{2\theta} \rangle \big| \lesssim_\theta {} & \|\nabla_p f_1\|_{L_{2,\theta}(\mathbb{R}^3)} \|f_2\|_{L_r(\mathbb{R}^3)} \|f_3\|_{W^1_{2,\theta}(\mathbb{R}^3)} \\
& + \|f_1\|_{L_{2,\theta}(\mathbb{R}^3)} \|\nabla_p f_2\|_{L_r(\mathbb{R}^3)} \|f_3\|_{W^1_{2,\theta}(\mathbb{R}^3)}.
\end{aligned} \tag{B.15}
$$

Proof. Invoke the explicit expression of $\Gamma(f_1, f_2)$ in (B.3). For the sake of simplicity, we assume that f_1 and f_2 are scalar functions, and we estimate a simplified integral given by (cf. (B.15))

$$I = \left\langle \left(\partial_{p_i} - \frac{p_i}{2p_0}\right) \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{p_j} f_1(p) f_2(q)\, dq, f_3 p_0^{2\theta} \right\rangle$$
$$- \left\langle \left(\partial_{p_i} - \frac{p_i}{2p_0}\right) \int \Phi^{ij}(P,Q) J^{1/2}(q) f_1(p) \partial_{q_j} f_2(q)\, dq, f_3 p_0^{2\theta} \right\rangle.$$

Integrating by parts in p gives

$$I = \left\langle \partial_{p_j} f_1 \int \Phi^{ij}(P,Q) J^{1/2}(q) f_2(q)\, dq, \left(-\partial_{p_i} - \frac{p_i}{2p_0}\right)(f_3 p_0^{2\theta}) \right\rangle$$
$$+ \left\langle f_1 \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} f_2(q)\, dq, \left(\partial_{p_i} + \frac{p_i}{2p_0}\right)(f_3 p_0^{2\theta}) \right\rangle =: I_1 + I_2.$$

Finally, applying the $L_2 - L_\infty - L_2$ Hölder's inequality to I_1 and I_2 and using the bound (B.7) with $k=0$, we obtain (B.15). ∎

Appendix C. Verification of estimates (5.66)–(5.74).

LEMMA C.1. *Estimates* (5.66)–(5.74) *are true.*

Proof. Ellipticity and boundedness of the leading coefficients.

(1) Bounds of $\mathcal{A}$ (see (5.41)). By Lemma 5.3 and (A.30), for sufficiently small r_0,

$$(\delta_0/4)|\xi|^2 \le A^{ij}(z)\xi_i\xi_j \le (4\delta_0^{-1})|\xi|^2 \quad \forall z \in \mathbb{R}^7_T, \xi \in \mathbb{R}^3,$$

where A is defined in (5.36), and hence, the same estimate also holds for $\mathcal{A}$.

(2) Bounds of $\mathfrak{A}$ (see (5.62)). First, we estimate $\mathbb{A}$ via Lemma A.4 with $\psi = \mathcal{W}_y$. By (A.18)–(A.19) with $m = 2^{n+2}$, the assumptions of Lemma A.4 hold with $N_0 = N_0' 2^{-n}$, $N_1 = N_1' 2^{3n}$, and $\delta_1 = \delta_0/4$. Then, by (A.30), we conclude that for any $\xi \in \mathbb{R}^3$ and $z \in (0,T) \times \Upsilon_n(G \times \{|p| < 2^{n+2}\})$,

$$N' 2^{-6n} \le \mathbb{A}^{ij}(z)\xi_i\xi_j \le N'', \tag{C.1}$$

and hence, the same bounds (with, perhaps, different constants N' and N'') are true for $\mathfrak{A}$.

Boundedness of $\nabla_v \mathfrak{A}$.

(1) Estimate of $\nabla_w \mathcal{A}$. By (B.12) in Lemma B.6 and the construction of the coefficients A (see (5.36)),

$$\|\nabla_w A\|_{L_\infty((0,T)\times\psi(\Omega_{r_0}(x_0))\times\mathbb{R}^3)} \le N(K,\Omega).$$

Then, by (5.41) we have

$$\|\nabla_w \mathcal{A}\|_{L_\infty((0,T)\times G\times\mathbb{R}^3)} \le N(K,\Omega). \tag{C.2}$$

(2) Estimate of $\mathfrak{A}$. First, we estimate $\hat{\hat{\mathcal{A}}}$, which is given (5.53). By (A.20) with $m = 2^{n+2}$,

$$\left\| \nabla_v \left(\frac{\partial v}{\partial w}\right)(y, w(y,v)) \right\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N 2^n. \tag{C.3}$$

By the chain rule and (C.2) and (A.19), one has

$$\|\nabla_v \hat{\hat{\mathcal{A}}}\|_{L_\infty((0,T)\times\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N2^{3n}.$$

Next, recall that $\mathbb{A}$ is defined in (5.57). Combining (5.50) (with $j=1$) and (C.3) with the last inequality, we get

$$\|\nabla_v \mathbb{A}\|_{L_\infty((0,T)\times\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N2^{n}.$$

Then, the definition of $\mathfrak{A}$ (see (5.62)) and the last inequality give the desired bound (5.68), that is,

$$\|\nabla_v \mathfrak{A}\|_{L_\infty((0,T)\times\mathbb{R}^6)} \le N2^{n}.$$

Hölder continuity of the leading coefficients. Here we verify (5.69).

(1) Estimate of $\mathcal{A}$ (see (5.41)). First, by the definition of A in (5.36) and (B.12) in Lemma B.6, we have

$$\|A\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{x,p}(\psi(\Omega_{r_0}(x_0))\times\mathbb{R}^3)} \le N(K,\Omega,\varkappa).$$

To show that $\mathcal{A}$ is Hölder continuous, that is,

$$\|\mathcal{A}\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{x,p}(G\times\mathbb{R}^3)} \le N(K,\Omega,\varkappa), \tag{C.4}$$

it suffices to check that

$$\mathcal{A} \quad \text{is continuous across } \{y_3=0\}\times\mathbb{R}^3. \tag{C.5}$$

Note that for any arbitrary 3 by 3 symmetric matrix $M=(m^{ij}, i,j=1,2,3)$,

$$\boldsymbol{R}M\boldsymbol{R} = \begin{pmatrix} m^{11} & m^{12} & -m^{13} \\ m^{12} & m^{22} & -m^{23} \\ -m^{13} & -m^{23} & m^{33} \end{pmatrix}.$$

Then, by the definition of $\mathcal{A}$ (see (5.41)), if the identity

$$A^{i3}(t,y_1,y_2,0,w) = -A^{i3}(t,y_1,y_2,0,\boldsymbol{R}w), i\in\{1,2\}, \tag{C.6}$$

is valid, then (C.5) is also true. The identity (C.6) follows from Lemma A.1 because Assumption 5.2 is valid as g satisfies the SRBC (see (5.3)).

(2) Estimate of $\mathfrak{A}$. First, we estimate $\hat{\hat{\mathcal{A}}}$ (see (5.53)). By (A.19) with $m=2^{n+2}$ and (A.27), we have

$$\begin{aligned} &\left\|\frac{\partial w}{\partial v}\right\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N(\Omega)2^{3n}, \\ &\left\|\frac{\partial w}{\partial y}\right\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N(\Omega)2^{n}. \end{aligned} \tag{C.7}$$

By the definition of $\hat{\hat{\mathcal{A}}}$ in (5.53) and (C.4), and (C.7), we have

$$\|\hat{\hat{\mathcal{A}}}\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{y,v}(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N(K,\Omega,\varkappa)2^{3n}. \tag{C.8}$$

Next, we estimate the Hölder norm of $\mathbb{A}$. First, we need to bound the Hölder norm of

$$\left(\frac{\partial v}{\partial w}\right)(y, w(y,v)).$$

By the chain rule, for any $(y,v) \in \Upsilon(G \times \{|w| < 2^{n+2}\})$,

$$\begin{aligned}\left|\nabla_y \left(\frac{\partial v}{\partial w}\right)(y, w(y,v))\right| &\le \left\|\nabla_y \frac{\partial v}{\partial w}\right\|_{L_\infty(G\times\{|w|<2^{n+2}\})} \\ &+ \left\|\nabla_w \frac{\partial v}{\partial w}\right\|_{L_\infty(G\times\{|w|<2^{n+2}\})} \left\|\frac{\partial w}{\partial y}\right\|_{L_\infty(\Upsilon_n(G\times\{|w|<2^{n+2}\}))}.\end{aligned} \tag{C.9}$$

Hence, by (A.28) in Lemma A.3 (ii), the first term on the l.h.s. of (C.9) is bounded by $N(\Omega)2^{-n}$. Furthermore, by (A.18) and (A.27), we conclude that the second term on the r.h.s. of (C.9) is also bounded by $N2^{-n}$, and hence,

$$\left\|\nabla_y \left(\frac{\partial v}{\partial w}\right)(y, w(y,v))\right\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N(\Omega)2^{-n}. \tag{C.10}$$

Furthermore, by the definition of $\mathbb{A}$ in (5.57), (C.8), and (C.3) and (C.10), we obtain

$$\|\mathbb{A}\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{y,v}(\Upsilon(\mathbb{R}^3\times\{|w|<2^{n+2}\}))} \le N(K,\Omega,\varkappa)2^n. \tag{C.11}$$

Finally, by the definition of $\mathfrak{A}$ (see (5.62)), the bound (C.11), and our choice of the cutoff function ζ_n (see (5.61)), we obtain

$$\|\mathfrak{A}\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{y,v}(\mathbb{R}^6)} \le N(K,\Omega,\varkappa)2^n$$

(see (5.69)).

Estimates of the lower-order terms. Invoke the definition $\mathbb{B}$ (see (5.36), (5.42), (5.53), (5.58)). By the assumption (5.6) and (5.36), and (5.42), we have

$$\|\mathcal{B}\|_{L_\infty((0,T)\times G\times\mathbb{R}^3)} \le N,$$

and then, by the first inequality in (5.50), we obtain (5.70).

Next, recall the definition of $\mathbb{X}$ (see (5.37), (5.43), (5.53), (5.59)). Note that by (5.37), for any $(y,w) \in \psi(\Omega_{3r_0/4}) \times \{|w| < 2^{n+2}\}$,

$$\begin{aligned}&|X(y,w)| + |\nabla_w X(y,w)| \\ &\le N(\Omega)(1+|w|^2)^{1/2} \le N2^n.\end{aligned} \tag{C.12}$$

Hence, by (5.43), the same bound is true for $\mathcal{X}$. Furthermore, by the definition of $\mathbb{X}$ (see (5.59)) and the first inequality in (5.50), we get

$$\|\mathbb{X}\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N(\Omega).$$

Next, recall the definition of $\hat{\hat{\mathcal{X}}}$ (see (5.53)). By the chain rule, (C.12), and (C.7) (cf. (C.8)), we get

$$\|\nabla_v \hat{\hat{\mathcal{X}}}\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N2^{4n}. \tag{C.13}$$

Finally, by the definition of $\mathbb{X}$ (see (5.59)), (C.13) and (C.3), and the first inequality in (5.50), we conclude that the bound (5.72) is valid, that is,

$$\|\nabla_v \mathbb{X}\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N 2^{3n}.$$

Finally, we estimate the second "geometric" coefficient $\mathbb{G}$ (see (5.60)). By (C.7) and the first inequality in (5.50),

$$\|\mathbb{G}\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N.$$

Furthermore, by differentiating (5.60) and using the estimates (C.3) and (C.7), and (A.29) combined with the first inequality in (5.50), we conclude

$$\|\nabla_v \mathbb{G}\|_{L_\infty(\Upsilon(G\times\{|w|<2^{n+2}\}))} \le N 2^{4n}. \qquad \square$$

Appendix D. Relativistic kinetic transport equation in a domain. All the assertions here are either contained in [2] or [33], or can be easily proved by adapting the arguments therein. We start by introducing the relativistic counterpart of the set of test functions in [2] (see Definition D.2).

DEFINITION D.1. *We say that $G \subset \Sigma^T \cup \Sigma^T_\pm$ is a good set if there is a positive lower bound of the length of the characteristic lines $(t+s, x+v(p)s, p)$ inside $\Sigma^T \cup \Sigma^T_\pm$ that intersect G.*

DEFINITION D.2. *Let Φ be the set of functions ϕ on Σ^T such that*
- *ϕ is continuously differentiable along the characteristic lines $(t+s, x+v(p)s, p)$,*
- *ϕ, $Y\phi$ are bounded functions on Σ^T,*
- *the support of ϕ is a bounded good set.*

Remark D.1. By following the argument of Lemma 2.1 in [14], one can show that

$$C^1_0\Big(\overline{\Sigma^T} \setminus \big(((0,T)\times\gamma_0) \cup (\{0\}\times\partial\Omega\times\mathbb{R}^3) \cup (\{T\}\times\partial\Omega\times\mathbb{R}^3)\big)\Big) \subset \Phi.$$

DEFINITION D.3. *For $r \in [1,\infty)$, we say that $\xi \in L_{r,\theta,\,loc}(\Sigma^T_\pm)$ if for any good set G, one has $\xi 1_G \in L_{r,\theta}(\Sigma^T_\pm)$.*

To define the traces of functions on Σ^T, we need the following assertion, which is similar to Proposition 1 in [2].

PROPOSITION D.2. *Let $r \in [1,\infty)$, $\theta \ge 0$ be numbers, and let $u \in L_{r,\theta}(\Sigma^T)$ be a function such that $Yu \in L_{r,\theta}(\Sigma^T)$. Then, there exist unique functions $u_\pm, u_T, u_0$ on $\Sigma^T_\pm$ and $\Omega\times\mathbb{R}^3$, respectively, such that*
- *$u_\pm \in L_{r,\theta,\,loc}(\Sigma^T)$, $u_T, u_0 \in L_{r,\theta}(\Omega\times\mathbb{R}^3)$,*
- *the following Green's identity holds for any $\phi\in\Phi$:*

$$\begin{aligned}
&\int_{\Sigma^T} (Yu)\phi + (Y\phi)u\,dz \\
&= \int_{\Omega\times\mathbb{R}^3} u_T(x,p)\phi(T,x,p)\,dxdp - \int_{\Omega\times\mathbb{R}^3} u_0(x,p)\phi(0,x,p)\,dxdp \\
&\quad + \int_{\Sigma^T_+} u_+\phi\,|v(p)\cdot n_x|\,dS_x dpdt - \int_{\Sigma^T_-} u_-\phi\,|v(p)\cdot n_x|\,dS_x dpdt.
\end{aligned} \tag{D.1}$$

DEFINITION D.4. *Such functions $u_\pm, u_T, u_0$ are called the traces of a function u.*

The next lemma shows that $u_\pm$ belongs to a certain weighted Lebesgue space (see [33]).

LEMMA D.3 (Ukai's trace lemma). *Let $r \geq 1$ and u be such that $u, Yu \in L_r(\Sigma^T)$. Then, we have $u_\pm \in L_r(\Sigma^T_\pm, w|v(p)\cdot n_x|)$, where $w(z) = \min\{1, l(z)\}$, and $l(z)$ is the length of the characteristic line $(t+s, x+v(p)s, p)$ inside $\Sigma^T \cup \Sigma^T_\pm$, and, in addition,*

$$\|u\|_{L_r(\Sigma^T_\pm, w|v(p)\cdot n_x|)} \leq N\|Yu\|_{L_r(\Sigma^T)} + N\|u\|_{L_r(\Sigma^T)}, \tag{D.2}$$

where $N = N(r,T)$, and the weighted Lebesgue space on the l.h.s. is defined in (2.9).

PROPOSITION D.4 (see Theorem 5.1.2 in [33]). *Let $r \in [1,\infty), \theta \geq 0$ be numbers and u and ϕ be the functions in the following class:*
- *$u, Yu \in L_{r,\theta}(\Sigma^T)$,*
- *either u_0 or u_T belongs to $L_{2,\theta}(\Omega\times\mathbb{R}^3)$,*
- *either u_+ or u_- belongs to $L_{2,\theta}(\Sigma^T_\pm, |v(p)\cdot n_x|)$.*

Then, we have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} (u_T\phi_T(x,p) - u_0\phi_0(x,p))\, p_0^{\theta r}\, dx dp \\
&+ \int_{\Sigma^T_+} u_+\phi_+ p_0^{\theta r}\, |v(p)\cdot n_x|\, dS_x dp dt - \int_{\Sigma^T_-} u_-\phi_- p_0^{\theta r}\, |v(p)\cdot n_x|\, dS_x dp dt \\
&= \int_{\Sigma^T} \big((Yf)\phi + (Y\phi)f\big) p_0^{\theta r}\, dz.
\end{aligned} \tag{D.3}$$

The following lemma shows that one can drop the "strong" integrability conditions on the traces u and ϕ on $\Sigma^T_\pm$ in Proposition D.4 if u and ϕ satisfy the SRBC.

LEMMA D.5. *We assume that*
- *$u, \phi, Yu, Y\phi \in L_{2,\theta}(\Sigma^T)$,*
- *either $u_0, \phi_0 \in L_{2,\theta}(\Omega\times\mathbb{R}^3)$ or the same holds for u_T, ϕ_T,*
- *u and ϕ satisfy the SRBC.*

Then, the following variant of the energy identity holds:

$$\int_{\Omega\times\mathbb{R}^3} \big((u\phi)(T,x,p) - (u\phi)(0,x,p)\big)\, p_0^{\theta r} dx dp = \int_{\Sigma^T} \big(u(Y\phi) + \phi(Yu)\big)\, p_0^{\theta r} dz. \tag{D.4}$$

In addition, $u, \phi \in C([0,T])L_2(\Omega\times\mathbb{R}^3)$.

Proof. We repeat the argument of Lemma 3.7 in [8]. The key idea is to cut off away from the grazing set so that the traces on $\Sigma^T_\pm$ of the regularized function ϕ_ε are of class $L_2(\Sigma^T, p_0^{2\theta}|v\cdot n_x|)$ and they satisfy the SRBC. Then, the Green's identity (D.3) is applicable. We list below a few minor modifications in the argument of the aforementioned lemma.
- We note that the integrals over $\Sigma^T_\pm$ cancel out thanks to the SRBC.
- One needs to modify integrals I_2 and I_3:

$$\begin{aligned}
I_2 = I_{2,1} + I_{2,2} &:= \int_{\mathbb{H}^T_-} (W\cdot\nabla_y \widehat{\phi})\xi_\varepsilon(y,w)\xi\left(\frac{t}{\varepsilon}\right)\xi\left(\frac{T-t}{\varepsilon}\right)\widehat{u}\, dy dw dt \\
&+ 2\int_{\mathbb{H}^T_-: \varepsilon^2 < y_3^2 + w_3^2 < 2\varepsilon^2} \widehat{u}\widehat{\phi}\xi\left(\frac{t}{\varepsilon}\right)\xi\left(\frac{T-t}{\varepsilon}\right)\frac{W_3 y_3}{\varepsilon^2}\,\xi'\left(\frac{y_3^2+w_3^2}{\varepsilon^2}\right) dy dw dt,
\end{aligned}$$

$$I_3 = I_{3,1} + I_{3,2} := -\int_{\mathbb{H}_-^T} (X \cdot \nabla_w \widehat{\phi}) \xi_\varepsilon(y,w) \xi\left(\frac{t}{\varepsilon}\right) \xi\left(\frac{T-t}{\varepsilon}\right) \widehat{u}\, dydwdt$$
$$- 2\int_{\mathbb{H}_-^T: \varepsilon^2 < y_3^2 + w_3^2 < 2\varepsilon^2} \widehat{u}\widehat{\phi} \xi\left(\frac{t}{\varepsilon}\right) \xi\left(\frac{T-t}{\varepsilon}\right) \frac{X_3 w_3}{\varepsilon^2}\, \xi'\left(\frac{y_3^2 + w_3^2}{\varepsilon^2}\right) dydwdt,$$

where W and X are defined by (E.1). Repeating the argument on p. 489 in [8], we conclude

$$\lim_{\varepsilon \to 0} (I_{2,2} + I_{3,2}) = 0.$$

The rest of the proof is the same as in Lemma 3.7 in [8]. ◻

Appendix E. Verification of the identities 5.39 and 5.55.

E.1. Identity 5.39. Let Ω be a $C^{1,1}$ bounded domain, and let

$$\psi : \Omega_{r_0}(x_0) \to \mathbb{R}^3$$

be a local $C^{1,1}$ diffeomorphism and $\Psi : (x,p) \to (y,w)$ be a mapping given by

$$y = \psi(x), \quad w = (D\psi)\, p.$$

For a function f vanishing outside $(0,T) \times \Omega_{r_0}(x_0) \cap \mathbb{R}^3$, we set

$$\widehat{f}(y,w) = f(\Psi^{-1}(y,w)) = f(x(y), p(y,w)).$$

We compute the transport term Y in the (t,y,w) variables. We repeat the calculations of Appendix A in [8] with minor modifications.

Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^7_T), \phi \in C_0^{0,1}(\mathbb{R}^7_T)$ be functions such that $f(\cdot,x,\cdot), \phi(\cdot,x,\cdot) = 0$ for $x \notin \Omega_{r_0}$. Using the chain rule gives

$$(\nabla_p \phi)(t, x(y), p(y,w)) = \left(\frac{\partial y}{\partial x}\right)^T \nabla_w \widehat{\phi}(t,y,w),$$
$$\big(\nabla_x \phi\big)(t, x(y), p(y,w))$$
$$= \left(\frac{\partial y}{\partial x}\right)^T \left[\nabla_y \widehat{\phi}(t,y,w) - \left(\frac{\partial p}{\partial y}\right)^T (\nabla_p \phi)(t, x(y), p(y,w))\right]$$
$$= \left(\frac{\partial y}{\partial x}\right)^T \nabla_y \widehat{\phi}(t,y,w) - \left(\frac{\partial y}{\partial x}\right)^T \left(\frac{\partial p}{\partial y}\right)^T \left(\frac{\partial y}{\partial x}\right)^T \nabla_w \widehat{\phi}(t,y,w).$$

Therefore,

$$\left(\frac{p^T}{\sqrt{1+|p|^2}}\right)(y,w) \big(\nabla_x \phi\big)(t, x(y), p(y,w))$$
$$= \frac{w^T}{\sqrt{1 + |\frac{\partial x}{\partial y} w|^2}} \nabla_y \widehat{\phi}(t,y,w) - \frac{w^T}{\sqrt{1 + |\frac{\partial x}{\partial y} w|^2}} \left(\frac{\partial p}{\partial y}\right)^T \left(\frac{\partial y}{\partial x}\right)^T \nabla_w \widehat{\phi}(t,y,w)$$
$$= W \cdot \nabla_y \widehat{\phi}(t,y,w) - X \cdot \nabla_w \widehat{\phi}(t,y,w),$$

where

$$W = \frac{w}{\sqrt{1 + |\frac{\partial x}{\partial y} w|^2}}, \; X = \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial p}{\partial y}\right) W = \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial (\frac{\partial x}{\partial y} w)}{\partial y}\right) W. \tag{E.1}$$

Thus, by the above computation, we have

$$\begin{aligned}(E.2)\qquad &\int_{\Sigma^T} (Y\phi) f\, dz\\ &= \int_{\mathbb{H}^T_-} (\partial_t \widehat{\phi} + W\cdot\nabla_y \widehat{\phi}) \widehat{f}\, \mathsf{J}\, dydwdt - \int_{\mathbb{H}^T_-} (X\cdot\nabla_w \widehat{\phi}) \widehat{f}\, \mathsf{J}\, dydwdt,\end{aligned}$$

where $\mathbb{H}^T_-$ is defined in (2.3), and

$$\mathsf{J} = \left(\det\left(\frac{\partial x}{\partial y}\right)\right)^2.$$

The following lemma is a consequence of (E.2).

LEMMA E.1. *Let Ψ be the local diffeomorphism given by (5.31)–(5.32), $r>1$ be a number, and $u\in L_r(\Sigma^T)$ be a function such that*
- $\nabla_p u \in L_{2,1}(\Sigma^T)$, $Yu\in L_2(\Sigma^T)$,
- $u(t,\cdot,p)=0$ *for* $x\notin\Omega_r(x_0)$.

Then, one has

$$\|Yu\|_{L_r(\Sigma^T)} \le N\|(\partial_t + W\cdot\nabla_y)\widehat{u}\|_{L_r(\mathbb{H}^T_-)} + N\|\nabla_p u\|_{L_{r,1}(\Sigma^T)}, \tag{E.3}$$

where $N=N(\Omega,r)$.

Proof. The estimate follows from (E.2) and the fact that for any $y\in\psi(\Omega_{r_0}(x_0))$ and $w\in\mathbb{R}^3$,

$$|W(y,w)|\le N(\Omega)\max\{|w|,1\}. \qquad \square$$

E.2. Identity 5.55. Invoke the notation of Step 4 in the proof of Lemma 5.10. Proceeding as in section E.1, for any test function ϕ, we have

$$\begin{aligned}(\nabla_w\phi)(t,y,w(y,v)) &= \left(\frac{\partial v}{\partial w}\right)^T \nabla_v \hat{\hat{\phi}}(t,y,v),\\ (\nabla_y\phi)(t,y,w(y,v)) &= \nabla_y\hat{\hat{\phi}}(t,y,v) - \left(\frac{\partial w}{\partial y}\right)^T \left(\frac{\partial v}{\partial w}\right)^T \nabla_v\hat{\hat{\phi}}(t,y,v).\end{aligned}$$

By this computation, we conclude

$$\int_{\mathbb{R}^7_T} (\mathcal{W}\cdot\nabla_y\phi)\mathcal{U}\, dydwdt = \int_{\mathbb{R}^7_T} (v\cdot\nabla_y\hat{\hat{\phi}})\hat{\hat{\mathcal{U}}}\, dydvdt - \int_{\mathbb{R}^7_T} (\mathbb{G}\cdot\nabla_v\hat{\hat{\phi}})\hat{\hat{\mathcal{U}}}\, dydvdt, \tag{E.4}$$

where

$$\mathbb{G} = \left(\frac{\partial v}{\partial w}\right)\left(\frac{\partial w}{\partial y}\right) v.$$

Appendix F. S_r theory for the KFP equation on the whole space.

Assumption F.1. $(\gamma_\star)$ There exists $R_0>0$ such that for any $z_0=(t_0,x_0,v_0)$ satisfying $t_0<T$ and $r\in(0,R_0]$,

$$\operatorname{osc}_{x,p}(a, Q_r(z_0)) \le \gamma_\star, \tag{F.1}$$

where

$$\text{(F.2)}\quad \begin{aligned}&\operatorname{osc}_{x,p}(a, Q_r(z_0))\\ &\quad = r^{-14} \int_{t_0-r^2}^{t_0} \int_{D_r(z_0,t)\times D_r(z_0,t)} |a(t,x_1,p_1) - a(t,x_2,p_2)|\, dx_1 dp_1 dx_2 dp_2\, dt,\end{aligned}$$

and

$$D_r(z_0,t) = \{(x,p) : |x - x_0 - (t-t_0)p_0|^{1/3} < r, |p - p_0| < r\}.$$

Remark F.2. Note that if $a \in L_\infty((-\infty,T))C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6)$ for some $\varkappa \in (0,1]$, then, for any $\gamma_\star \in (0,1)$, Assumption F.1 ($\gamma_\star$) holds with $R_0 = ([a]^{-1}_{L_\infty((-\infty,T))C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6)}\gamma_\star)^{1/\varkappa}$.

The following theorem is a simplified version of Theorem 2.4 of [10].

THEOREM F.3. *Let*
- $r > 1$, $K > 0$, $\lambda \ge 0$, $-\infty < S < T < \infty$ *be numbers,*
- a, b, c *satisfy the assumptions* (5.19) *and* (5.6) *with* $\mathbb{R}^7_T$ *in place of* Σ^T,
- $[a]_{L_\infty((S,T))C^{\varkappa/3,\varkappa}_{x,p}(\mathbb{R}^6)} \le L$ *for some* $\varkappa \in (0,1]$ *and* $L > 0$.

Then, for any $f \in L_r((S,T)\times\mathbb{R}^6)$, *the equation*

$$(\partial_t + p\cdot\nabla_x)u - a^{ij}\partial_{p_ip_j}u + b\cdot\nabla_p u + cu + \lambda u = f, \quad u(0,\cdot) = 0,$$

has a unique solution $u \in S^N_r((S,T)\times\mathbb{R}^6)$ *(see* (2.12)*). In addition,*

$$\begin{aligned}&\|u\| + \|\nabla_p u\| + \|D^2_p u\| + \|(\partial_t + p\cdot\nabla_x)u\|\\ &\quad + \|(-\Delta_x)^{1/3}u\| + \|\nabla_p(-\Delta_x)^{1/6}u\| \le N\|f\|,\end{aligned}$$

where $\|\cdot\| = \|\cdot\|_{L_r((S,T)\times\mathbb{R}^6)}$ *and* $N = N(\delta, \varkappa, r, K, T-S, L)$.

THEOREM F.4 (Corollary 2.6 of [10]). *Invoke the assumptions of Theorem* F.3 *and drop the Hölder continuity assumption on* a. *Then, there exist constants*

$$\kappa = \kappa(r) > 0, \quad \beta = \beta(r) > 0, \quad \gamma_\star = \delta^\kappa\widetilde{\gamma}_\star(r) > 0$$

such that if the condition (F.1) *in Assumption* F.1 ($\gamma_\star$) *holds, then for any* $u \in S^N_r((-\infty,T)\times\mathbb{R}^6)$ *and* $\lambda \ge 0$,

$$\text{(F.3)}\quad \begin{aligned}\|u\|_{S^N_r((-\infty,T)\times\mathbb{R}^6)} \le N\delta^{-\beta}(&\|(\partial_t + p\cdot\nabla_x)u - a^{ij}\partial_{p_ip_j}u + b\cdot\nabla_p u + cu\\ &+ \lambda u\|_{L_r((-\infty,T)\times\mathbb{R}^6)} + NR_0^{-2}\|u\|_{L_r((-\infty,T)\times\mathbb{R}^6)}),\end{aligned}$$

where $N = N(r,K)$, *and* $R_0 \in (0,1)$ *is the constant in Assumption* F.1 ($\gamma_\star$).

LEMMA F.5 (see Lemma D.6 in [8]). *Invoke the assumptions of Theorem* F.3 *and let*
- $T > 0$, $\lambda \ge 0$, $1 < q < r$ *be numbers,*
- $u \in S^N_q(\mathbb{R}^7_T)$ *be a function such that* $u(0,\cdot) \equiv 0$, *and*

$$h := (\partial_t + p\cdot\nabla_x)u - a^{ij}\partial_{p_ip_j}u + b\cdot\nabla_p u + (c+\lambda)u \in L_r(\mathbb{R}^7_T).$$

Then, $u \in S^N_r(\mathbb{R}^7_T)$.

LEMMA F.6. *Let*

– $\delta \in (0,1)$, $\varkappa \in (0,1]$, $K > 0$, $M > 0, \lambda \ge 0$ *be numbers,*
– $u \in L_2((0,T) \times \mathbb{R}^3_x) W^1_2(\mathbb{R}^3_p)$,
– $\delta |\xi|^2 < a^{ij}(z)\xi_i \xi_j < \delta^{-1}|\xi|^2$ *for any* $z \in \mathbb{R}^7_T$, $\xi \in \mathbb{R}^3$,
–

$$\|a\|_{L_\infty((0,T))C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6)}, \|\nabla_p a\|_{L_\infty(\mathbb{R}^7_T)} \le K\delta^{-1} \tag{F.4}$$

for some $K > 1$,
– $h \in L_2(\mathbb{R}^7_T)$,
– *for any* $\phi \in C^1_0([0,T] \times \mathbb{R}^6)$ *such that* $\phi(T,\cdot) \equiv 0$, *one has*

$$\int (-(\partial_t \phi + p \cdot \nabla_x \phi) u + (\nabla_p \phi)^T a \nabla_p u + \lambda u \phi - h\phi)\, dz = 0.$$

Then, the following assertions hold.

(i) *One has* $u \in S^N_2(\mathbb{R}^7_T)$ *and there exists* $\beta = \beta(\varkappa) > 0$ *such that*

$$\|u\|_{S^N_2(\mathbb{R}^7_T)} \le N\delta^{-\beta}\big(\|h\|_{L_2(\mathbb{R}^7_T)} + \||u| + |\nabla_p u|\|_{L_2(\mathbb{R}^7_T)}\big), \tag{F.5}$$

where $N = N(\varkappa, K) > 0$.

(ii) *If, additionally,* $h, \nabla_p u \in L_r(\mathbb{R}^7_T)$, *for some* $r \in (2,\infty)$, *then* $u \in S_r(\mathbb{R}^7_T)$, *and*

$$\|u\|_{S^N_r(\mathbb{R}^7_T)} \le N\delta^{-\beta}\big(\|h\|_{L_r(\mathbb{R}^7_T)} + \||u| + |\nabla_p u|\|_{L_r(\mathbb{R}^7_T)}\big), \tag{F.6}$$

where $\beta = \beta(r, \varkappa) > 0$, $N = N(r, \varkappa, K) > 0$.

Proof. In this proof, N is a constant independent of δ.

(i) *Step* 1*:* $u \in S^N_2(\mathbb{R}^7_T)$. For $t < 0$ we set u and h to be 0, and a to be $\mathbf{1}_3$. Then, for any smooth function ϕ with compact support in $(-\infty, T) \times \mathbb{R}^6$, we have

$$\int \Big(- (\partial_t \phi + p \cdot \nabla_x \phi) u + (\nabla_p \phi)^T a \nabla_p u - \lambda u \phi - h\phi \Big)\, dz = 0.$$

This implies that $(\partial_t + p \cdot \nabla_x) u \in L_2((-\infty,T) \times \mathbb{R}^3_x) W^{-1}_2(\mathbb{R}^3_p)$. Next, by Theorem F.3, the equation

$$\partial_t u_1 + p \cdot \nabla_x u_1 - a^{ij} \partial_{p_i p_j} u_1 + \lambda u_1 = h - \partial_{p_i} a^{ij} \partial_{p_j} u \tag{F.7}$$

has a unique solution $u_1 \in S^N_2((-\infty,T) \times \mathbb{R}^6)$ such that $u_1 1_{t \le 0} \equiv 0$. Then, for $U = u - u_1$, we have

$$U \in L_2((-\infty,T) \times \mathbb{R}^3_x) W^1_2(\mathbb{R}^3_p), \quad (\partial_t + p \cdot \nabla_x) U \in L_2((-\infty,T) \times \mathbb{R}^3_x) W^{-1}_2(\mathbb{R}^3_p),$$

and the identity

$$(\partial_t + p \cdot \nabla_x) U - \nabla_p \cdot (a \nabla_p U) + \lambda U = 0$$

holds in $L_2((-\infty,T) \times \mathbb{R}^3_x) W^{-1}_2(\mathbb{R}^3_p)$. Then, by "testing" the above identity with u in the sense of the duality pairing between $L_2((-\infty,T) \times \mathbb{R}^3) W^k_2(\mathbb{R}^3)$, $k = \pm 1$, and integrating by parts in p, we get for a.e. $s \in (-\infty, T)$,

$$\int_{\mathbb{R}^6} U^2(s,x,p)\, dx dp + \int_{(-\infty,s) \times \mathbb{R}^6} (\delta |\nabla_p U|^2 + \lambda U^2)\, dz = 0.$$

We conclude $U \equiv 0$, and hence, $u \in S_2^N((-\infty, T) \times \mathbb{R}^6)$.

Step 2: S_2 estimate. By the assumption (F.4) and Remark F.2, for any $\gamma_\star \in (0,1)$, the condition (F.1) in Assumption F.1 ($\gamma_\star$) holds with

$$R_0 = K^{-1/\varkappa} \delta^{1/\varkappa} \gamma_\star^{1/\varkappa}. \tag{F.8}$$

Furthermore, let

$$\beta > 0, \quad \kappa > 0, \quad \gamma_\star = \delta^\kappa \widetilde{\gamma}_\star(\varkappa) > 0$$

be the numbers in Theorem F.4 with $r = 2$. Then, by the estimate (F.3) applied to (F.7) and the assumption (F.4), we conclude that there exists $N = N(\varkappa)$ such that

$$\begin{aligned}
&\|u\|_{S_2^N(\mathbb{R}^7_T)} = \|u_1\|_{S_2^N(\mathbb{R}^7_T)} \\
&\le N\delta^{-\beta} \| |h| + |\partial_{p_i} a^{ij} \partial_{p_j} u| \|_{L_2(\mathbb{R}^7_T)} + NK^{2/\varkappa} \delta^{-2\kappa/\varkappa} \|u\|_{L_2(\mathbb{R}^7_T)} \\
&\le N\delta^{-\beta} \|h\|_{L_2(\mathbb{R}^7_T)} + NK\delta^{-\beta} \|\nabla_p u\|_{L_2(\mathbb{R}^7_T)} + NK^{2/\varkappa} \delta^{-2\kappa/\varkappa} \|u\|_{L_2(\mathbb{R}^7_T)}.
\end{aligned}$$

(ii) Applying Lemma F.5 to (F.7), we conclude that $u \in S_r^N(\mathbb{R}^7_T)$. The desired estimate (F.6) is obtained via Theorem F.4 in the same way as (F.5). ∎

LEMMA F.7 (embedding for the steady $S_p^N(\mathbb{R}^{2d})$ space). *Let $d \ge 1$, $p \in (1,\infty)$, and $u \in S_p^N(\mathbb{R}^{2d})$ (see (2.12)). Then, the following assertions hold.*

(i) *For any $p \in (1, 2d)$ and $q > p$ satisfying*

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{2d},$$

we have

$$\|u\|_{L_q(\mathbb{R}^{2d})} \lesssim_{d,p,q} \|u\|_{S_p^N(\mathbb{R}^{2d})}. \tag{F.9}$$

(ii) *For any $p \in (1, 4d)$ and $q > 1$ satisfying*

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{4d},$$

one has

$$\|\nabla_v u\|_{L_q(\mathbb{R}^{2d})} \lesssim_{d,p,q} \|u\|_{S_p^N(\mathbb{R}^{2d})}. \tag{F.10}$$

(iii) *For $p > 2d$,*

$$\|u\|_{L_\infty(\mathbb{R}^{2d})} \lesssim_{d,p} \|u\|_{S_p^N(\mathbb{R}^{2d})}. \tag{F.11}$$

Furthermore, if $p > 4d$ and $\alpha \in \left(0, 1 - \frac{4d}{p}\right)$,

$$\|[u, \nabla_v u]\|_{C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^{2d})} \lesssim_{d,p,\alpha} \|u\|_{S_p^N(\mathbb{R}^{2d})}. \tag{F.12}$$

Proof. (i)–(ii) We denote

$$f = v \cdot \nabla_x u - \Delta_v u + u.$$

Let $\Gamma(t,x,v;t',x',v')$ be the fundamental solution of the operator $(\partial_t + v \cdot \nabla_x) - \Delta_v$. It is well known that (see, for example, [30])

$$\Gamma(t,x,v;t',x',v') = (t-t')^{-2d} \mathfrak{p}\left(\frac{x - x' - (t-t')v'}{(t-t')^{3/2}}, \frac{v - v'}{(t-t')^{1/2}}\right),$$

where $\mathfrak{p}$ is a certain Gaussian function. Then, we have

$$u(x,v) = \int_0^\infty \int_{\mathbb{R}^{2d}} e^{-t}\,\Gamma(t,x,v;0,x',v') f(x',v')\,dx'dv'dt$$
$$= \int_0^\infty \int_{\mathbb{R}^{2d}} t^{-2d} e^{-t}\,\mathfrak{p}\Big(\frac{x-x'}{t^{3/2}}, \frac{v-v'}{t^{1/2}}\Big) \tilde f(x',v')\,dx'dv'dt,$$

where $\tilde f(x,v) = f(x+tv,v)$.

Next, let r be the number defined by the relation

$$\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}.$$

Then, by the Minkowski and Young inequalities,

$$\|u\|_{L_q(\mathbb{R}^{2d})} \le \|f\|_{L_p(\mathbb{R}^{2d})} \int_0^\infty e^{-t} t^{-2d} \|\mathfrak{p}(\frac{\cdot}{t^{3/2}}, \frac{\cdot}{t^{1/2}})\|_{L_r(\mathbb{R}^{2d})}\,dt.$$

Since $1 - 1/r < 1/(2d)$, the second factor on the r.h.s. is bounded by

$$N(d)\int_0^\infty e^{-t} t^{-2d(1-1/r)}\,dt < \infty,$$

and hence, the estimate (F.9) is valid. The second assertion (F.10) is proved in the same way.

(iii) A simple application of Hölder's inequality gives

$$|u(x,v)| \lesssim_d \|f\|_{L_p(\mathbb{R}^{2d})} \int_0^\infty e^{-t} t^{-2d/p}\,dt \lesssim_{d,p} \|f\|_{L_p(\mathbb{R}^{2d})}, \tag{F.13}$$

and hence, (F.11) is true. The proof of (F.12) follows from the identity

$$\nabla_v u(x,v) = \int_0^\infty \int_{\mathbb{R}^{2d}} e^{-t} t^{-2d-1/2}\,(\nabla_v \mathfrak{p})\Big(\frac{x-x'-tv'}{t^{3/2}}, \frac{v-v'}{t^{1/2}}\Big) f(x',v')\,dx'dv'dt$$

and the argument in (F.13). We omit the technical details. □

Appendix G. Proof of Proposition 6.2.

LEMMA G.1. *We invoke the assumptions of Proposition* 5.13 *and let f be the finite energy solution to* (3.7)–(3.8) $f \in C([0,T])L_{2,\theta}(\Omega\times\mathbb{R}^3) \cap L_2((0,T)\times\Omega)W^1_{2,\theta}(\mathbb{R}^3)$. *We assume, additionally, that for some* $0 \le \theta_1 \le \theta$,

$$\partial_t[g, \nabla_p g, b, \nabla_p\cdot b, c] \in L_\infty(\Sigma^T), \tag{G.1}$$
$$\partial_t \eta \in L_2((0,T)\times\Omega) W^{-1}_{2,\theta_1}(\mathbb{R}^3), \tag{G.2}$$

and, for

$$\begin{aligned} f_1(x,p) &:= -v(p)\cdot\nabla_x f_0(x,p) + \nabla_p\cdot(\sigma_{f_0}(x,p)\nabla_p f_0(x,p)) \\ &\quad - b(0,x,p)\cdot\nabla_p f_0(x,p) - c(0,x,p) f_0(x,p) + \eta(0,x,p) \end{aligned} \tag{G.3}$$

(understood in the sense of distributions), one has

$$f_1 \in L_{2,\theta_1}(\Omega\times\mathbb{R}^3),$$

where σ_{f_0} *is given by* (3.9) *with* g *replaced with* f_0. *In addition, we assume that* $f_0 \in L_2(\Omega)W_2^1(\mathbb{R}^3)$ *is a finite energy solution to the steady equation* (G.3) *with SRBC (see Definition* 3.1*), where* f_1 *is viewed as the r.h.s. Then,*

$$\partial_t f \in C([0,T])L_{2,\theta_1}(\Omega \times \mathbb{R}^3) \cap L_2((0,T) \times \Omega)W^1_{2,\theta_1}(\mathbb{R}^3),$$

and, furthermore, $u = \partial_t f$ *is a finite energy solution to*

$$\begin{aligned} &Yu - \nabla_p \cdot (\sigma_g \nabla_p u) + b \cdot \nabla_p u + cu = \eta_1, \ z \in \Sigma^T, \\ &u(t,x,p) = u(t,x,R_x p), (t,x,p) \in \Sigma^T_-, \quad u(0,x,p) = f_1(x,p), (x,p) \in \Omega \times \mathbb{R}^3, \end{aligned} \tag{G.4}$$

where

$$\eta_1 = \partial_t \eta - \big(-\nabla_p \cdot ((\partial_t \sigma_g)\nabla_p f) + (\partial_t b) \cdot \nabla_p f + (\partial_t c) f \big). \tag{G.5}$$

Proof of Lemma G.1. For the sake of clarity, we consider the case when $\theta, \theta_1 = 0$. The argument in the remaining case is the same as the one presented here. Let us first consider (G.4). By the definition of η_1 (see (G.5)), the assumptions of the present lemma, and the fact that $f \in L_2((0,T) \times \Omega)W_2^1(\mathbb{R}^3)$, we conclude

$$\eta_1 \in L_2((0,T) \times \Omega)W_2^{-1}(\mathbb{R}^3).$$

Then, by Proposition 5.13, the problem (G.4) has a unique finite energy solution (see Definition 3.1). Furthermore, we denote

$$\widetilde{f}(t,x,p) = \int_0^t u(s,x,p)\, ds + f_0(x,p).$$

To prove the lemma, it suffices to show that $\widetilde{f} \equiv f$.

Next, by using a simple identity

$$(\xi_1 \xi_2)(t) = (\xi_1\xi_2)(0) + \int_0^t [\xi_1'(s)\xi_2(s) + \xi_1(s)\xi_2'(s)]\, ds$$

with $\xi_1 = \sigma_g, b, c$ and $\xi_2 = \tilde{f}, \nabla_p \tilde{f}$, and (G.3)–(G.4), we formally conclude that $\tilde{f}$ is a finite energy solution to the equation

$$\begin{aligned} &Y\widetilde{f}(z) - \nabla_p \cdot (\sigma_g(z)\nabla_p \widetilde{f}(z)) + b(z) \cdot \nabla_p \widetilde{f}(z) + c(z)\widetilde{f}(z) - \eta(z) \\ &\quad = -\eta(0,x,p) + v(p) \cdot \nabla_x f_0(x,p) - \nabla_p \cdot (\sigma_{f_0}(x,p)\nabla_p f_0(x,p)) \\ &\qquad + b(0,x,p) \cdot \nabla_p f_0(x,p) + c(0,x,p) f_0(x,p) + u(0,x,p) \\ &\qquad + \int_0^t \Big(-\nabla_p \cdot \big((\partial_t \sigma_g(s,x,p))\nabla_p(\widetilde{f} - f)(s,x,p)\big) \\ &\qquad + (\partial_t b(s,x,p)) \cdot \nabla_p(\widetilde{f} - f)(s,x,p) + (\partial_t c(s,x,p))(\widetilde{f} - f)(s,x,p) \Big)\, ds \end{aligned}$$

with SRBC and the initial data $\widetilde{f}(0,\cdot) \equiv f_0(\cdot)$. We note that the sum of the nonintegral terms on the r.h.s. of the above identity equals 0 due to (G.3). Hence, the function $w = \widetilde{f} - f$ is a finite energy solution to

$$
\begin{aligned}
\text{(G.6)} \qquad & Yw(z) - \nabla_p \cdot (\sigma_g(z)\nabla_p w(z)) + b(z)\cdot\nabla_p w(z) + c(z)w(z) \\
&= \int_0^t \Big(- \nabla_p \cdot \big((\partial_t \sigma_g(s,x,p))\nabla_p w(s,x,p)\big) \\
&\quad + (\partial_t b(s,x,p))\cdot\nabla_p w(s,x,p) + (\partial_t c(s,x,p))w(s,x,p)\Big)\, ds
\end{aligned}
$$

with the SRBC and the initial data $w(0,\cdot)\equiv 0$. To make the above argument rigorous, one needs to work with the weak formulations of (3.13) and (G.4) and use the fact that f_0 is a finite energy solution to (G.3) with the SRBC.

Finally, by applying an "energy" type identity (5.107) to (G.6), using integration by parts in p, and the Cauchy–Schwarz inequality, we get

$$
\|w(t,\cdot)\|^2_{L_2(\Omega\times\mathbb{R}^3)} + \|\nabla_p w\|^2_{L_2(\Sigma^t)} \le N\, t(\|w\|^2_{L_2(\Sigma^t)} + \|\nabla_p w\|^2_{L_2(\Sigma^t)}), t\in[0,T],
$$

where N is independent of t. Taking $t\le(2N)^{-1}$ and using the Gronwall's inequality, we conclude that $w=0$ on Σ^{T_1} where $T_1=\min\{(2N)^{-1},T\}$. Similarly, we show that $w=0$ on Σ^t for $t\in[T_1,\min\{T_1+(2N)^{-1},T\}]$ and so on. Thus, $f\equiv\widetilde{f}$. ◻

Proof of Proposition 6.2. The uniqueness follows from the estimate (6.42) with vanishing "initial data" $f_{0,k},\mathbf{E}_{0,k},\mathbf{B}_{0,k}$.

To show the existence, we consider the iteration scheme $[h_{(n)},\mathbf{E}_{(n)},\mathbf{B}_{(n)}], n\ge 0$, such that $[h_{(0)},\mathbf{E}_{(0)},\mathbf{B}_{(0)}]=[f_{0,0},\mathbf{E}_{0,0},\mathbf{B}_{0,0}]$, and given $[h_1,\mathbf{E}_1,\mathbf{B}_1]=[h_{(n)},\mathbf{E}_{(n)},\mathbf{B}_{(n)}]$, the next iteration $[h_2,\mathbf{E}_2,\mathbf{B}_2]=[h_{(n+1)},\mathbf{E}_{(n+1)},\mathbf{B}_{(n+1)}]$ is defined as the strong solution to the system

$$
\begin{aligned}
\text{(G.7)} \qquad & Yh_2 + \boldsymbol{\xi}(\mathbf{E}_g + v(p)\times\mathbf{B}_g)\cdot\nabla_p h_2 - \frac{\boldsymbol{\xi}}{2}(v(p)\cdot\mathbf{E}_g)h_2 - Ah_2 \\
&= \boldsymbol{\xi}_1(v(p)\cdot\mathbf{E}_1)J^{1/2} + Kh_1 + \Gamma(h_2,g),\\
\text{(G.8)} \qquad & h_2(t,x,p) = h_2(t,x,R_x p),\ z\in\Sigma^T_-, \quad h_2(0,\cdot)\equiv f_{0,0},\\
\text{(G.9)} \qquad & \partial_t\mathbf{E}_2 - \nabla_x\times\mathbf{B}_2 = -\int v(p)J^{1/2}(p)h_1(p)\cdot\boldsymbol{\xi}\, dp,\\
\text{(G.10)} \qquad & \partial_t\mathbf{B}_2 + \nabla_x\times\mathbf{E}_2 = 0,\\
\text{(G.11)} \qquad & \nabla_x\cdot\mathbf{E}_2 = \int J^{1/2}h_1(p)\cdot\boldsymbol{\xi}\, dp, \quad \nabla_x\cdot\mathbf{B}_2 = 0,\\
\text{(G.12)} \qquad & (\mathbf{E}_2\times n_x)_{|\partial\Omega} = 0, \quad (\mathbf{B}_2\cdot n_x)_{|\partial\Omega} = 0,\\
\text{(G.13)} \qquad & \mathbf{E}_2(0,\cdot)\equiv\mathbf{E}_{0,0}(\cdot), \quad \mathbf{B}_2(0,\cdot)\equiv\mathbf{B}_{0,0}(\cdot).
\end{aligned}
$$

We assume that $[h_1,\mathbf{E}_1,\mathbf{B}_1]$ satisfies

$$
\begin{aligned}
\text{(G.14)} \qquad & \partial_t^k h_1 \in C([0,T])L_2(\Omega\times\mathbb{R}^3)\cap L_2((0,T))W^1_2(\Omega\times\mathbb{R}^3), k\le m,\\
\text{(G.15)} \qquad & \partial_t^k[\mathbf{E}_1,\mathbf{B}_1]\in C([0,T])L_2(\Omega), k\le m,\\
\text{(G.16)} \qquad & \partial_t^k h_1(0,\cdot) = f_{0,k}(\cdot)\,(\text{see}\,(3.16)),\ k\le m,\\
\text{(G.17)} \qquad & \partial_t^k[\mathbf{E}_1,\mathbf{B}_1](0,\cdot) = [\mathbf{E}_{0,k}\mathbf{B}_{0,k}](\cdot)\ (\text{see}\,(3.17)\text{–}(3.18)), k\le m,\\
\text{(G.18)} \qquad & \partial_t\rho_k + \nabla_x\cdot\mathbf{j}_k = 0\,(\text{in the sense of distributions}), k\le m,\\
& \text{where}\,\rho_k(t,x) = \int_{\mathbb{R}^3} J^{1/2}(p)\partial_t^k h_1(t,x,p)\cdot\boldsymbol{\xi}\, dp,\\
& \mathbf{j}_k(t,x) = \int_{\mathbb{R}^3} J^{1/2}(p)v(p)\partial_t^k h_1(t,x,p)\cdot\boldsymbol{\xi}\, dp,
\end{aligned}
$$

$$y^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1) \le N_1, \quad \sum_{k=0}^{m-1} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_1, \mathbf{B}_1]\|^2_{L_\infty((0,T))W_2^1(\Omega)} + \sum_{k=0}^{m-8} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_1, \mathbf{B}_1]\|^2_{L_\infty((0,T)\times\Omega)} \le N_1 N_2, \tag{G.19}$$

where $N_1, \lambda > 1$ are constants depending only on $r_1, \dots, r_4, \Omega, \theta, f_{0,k}, \mathbf{E}_{0,k}, \mathbf{B}_{0,k}, k \le m$, $N_2 = N_2(\Omega) > 1$, and

$$y^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1) = \sup_{\tau \le T} \mathcal{I}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau) + \int_0^T \mathcal{D}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau)\, d\tau, \tag{G.20}$$

$$\begin{aligned}\mathcal{D}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau) &= \sum_{k=0}^{m} \Big(\|e^{-\lambda\tau}(\sqrt{\lambda}|\partial_t^k f(\tau, \cdot)| + |\nabla_p \partial_t^k f(\tau, \cdot)|)\|^2_{L_2(\Omega\times\mathbb{R}^3)} \\ &\quad + \lambda \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_1, \mathbf{B}_1](\tau, \cdot)\|^2_{L_2(\Omega)} \Big) \\ &\quad + \sum_{k=0}^{m-4} \|e^{-\lambda\tau}(\sqrt{\lambda}|\partial_t^k f(\tau, \cdot)| + |\nabla_p \partial_t^k f(\tau, \cdot)|)\|^2_{L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)},\end{aligned} \tag{G.21}$$

$$\begin{aligned}\mathcal{E}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau) &= \sum_{k=0}^{m} \Big(\|e^{-\lambda\tau}\partial_t^k f(\tau, \cdot)\|^2_{L_2(\Omega\times\mathbb{R}^3)} + \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_1, \mathbf{B}_1](\tau, \cdot)\|^2_{L_2(\Omega)} \Big) \\ &\quad + \sum_{k=0}^{m-4} \|e^{-\lambda\tau}\partial_t^k f(\tau, \cdot)\|^2_{L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3)},\end{aligned} \tag{G.22}$$

$$\begin{aligned}\mathcal{I}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau) &= \mathcal{E}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau) \\ &\quad + \sum_{k=0}^{m-8} \Big(\|e^{-\lambda\tau}\partial_t^k h_1(\tau, \cdot)\|^2_{L_\infty(\Omega)W^1_{\infty,\theta/2^{k+9}}(\mathbb{R}^3)} + \sum_{s\in\{2, r_4\}} \|e^{-\lambda\tau} D_p^2 \partial_t^k h_1\|^2_{L_s(\Sigma^\tau)} \Big).\end{aligned} \tag{G.23}$$

We will show that the following assertions are true.

(i) $\partial_t^k[\mathbf{E}_2, \mathbf{B}_2], k \le m$, is a weak solution to Maxwell's equations (G.9)–(G.10) formally differentiated k times with respect to t with the perfect conductor BC and $\partial_t^k[\mathbf{E}_2, \mathbf{B}_2](0, \cdot) = [\mathbf{E}_{0,k}, \mathbf{B}_{0,k}], k \le m$. For $k \le m-1$, the same pair is a strong solution. In addition, the identities in (G.11) formally differentiated k times in t are valid.

(ii) $\partial_t^k h_2, k \le m$, is a finite energy solution to (G.7) differentiated formally k times with respect to t with the initial conditions $\partial_t^k h_2(0, \cdot) = f_{0,k}(\cdot)$ and with the SRBC.

(iii) the assumptions (G.14)–(G.18) hold with $[h_2, \mathbf{E}_2, \mathbf{B}_2]$ in place of $[h_1, \mathbf{E}_1, \mathbf{B}_1]$,

(iv)

$$y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) \le N_1, \quad \sum_{k=0}^{m-1} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T))W_2^1(\Omega)} + \sum_{k=0}^{m-8} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T)\times\Omega)} \le N_1 N_2. \tag{G.24}$$

The weight $e^{-\lambda\tau}$ enables us to close the $L_{\infty,\theta/2^{k+9}}(\Sigma^T)$ estimate via the unsteady S_p a priori estimates (see (5.9), (5.11)–(5.15) in Propositions 5.4 and 5.6) by leveraging the L_∞ and L_2 control in the estimates of the "free" terms $K(\partial_t^k h_1)$ and $\boldsymbol{\xi}_1(v(p)\cdot \partial_t^k \mathbf{E}_1)J^{1/2}$. See (G.31) and the paragraph below. Furthermore, the control of the last term in (G.23) is needed to estimate the L_{r_i} norm of the free term $K(\partial_t^k h_1)$, which appears in the unsteady S_p estimate.

If the assertions (i)–(iv) are true, then, by using a limiting argument (cf. the proof of Theorem 3.10 in section 7), we conclude that there exist $[f, \mathbf{E}_f, \mathbf{B}_f]$ such that the desired assertions (a)–(e) in Proposition 6.2 are valid.

Proof of (i)–(iv). We will prove assertions in the following order: (i), (iv), (ii), and (iii).

(i) We use the standard existence/uniqueness results for weak/strong solutions to Maxwell's equations with the perfect conductor boundary conditions (see, for example, Chapter VII, section 4, in [11]). In particular, the differentiated in t equations in (G.11) are satisfied due to the continuity equations for $\partial_t^k h_1, k \le m$ (see (G.18)) and the compatibility conditions (3.21)–(3.22) on the initial data $[\mathbf{E}_{0,k}, \mathbf{B}_{0,k}]$ combined with the fact that $\partial_t^k h_1(0,\cdot) \equiv f_{0,k}$ (see (G.16)). Thus, the assertion (i) is valid.

(iv) In this argument, $N = N(r_1, \dots, r_4, \theta, \Omega, m, \alpha)$. First, we prove the estimates (G.24), assuming that (ii)–(iii) are true. We modify the proof of the estimate (3.38) given in section 6.1.

L_∞ estimate of $\partial_t^k[\mathbf{E}_2, \mathbf{B}_2], k \le m-8$. We establish the second estimate in (G.24) and specify the constant N_2. In this argument, $N_2 = N_2(\Omega)$ is a constant which might change from line to line. By applying the W_2^1 div-curl estimate in (3.15) to Maxwell's equations differentiated k times in t and rewritten as div-curl systems (see (4.3)–(4.4)), and using the first bound in (G.19), we have

(G.25)
$$\begin{aligned}
&\sum_{k=0}^{m-1} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T))W_2^1(\Omega)} \\
&\quad\le N_2 \sum_{k=0}^{m} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T))L_2(\Omega)} + N_2 \sum_{k=0}^{m-1} \|e^{-\lambda\tau}\partial_t^k h_1\|^2_{L_\infty((0,T))L_2(\Omega\times\mathbb{R}^3)} \\
&\quad\le N_2 N_1
\end{aligned}$$

(recall $N_1 > 1$), which gives the bound of the first term in the second estimate in (G.24). The first term on the right-hand side of (G.25) is estimated in (G.26).

Next, using the W_6^1 div-curl estimate, the Sobolev embedding $W_2^1 \subset L_6$, and (G.25), and the $L_\infty^{t,x,p}$ bound of $\partial_t^k h_1, k \le m-8$, in (G.19), we conclude

$$\begin{aligned}
&\sum_{k=0}^{m-8} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T))W_6^1(\Omega)} \\
&\quad\le N_2 \sum_{k=0}^{m-7} \|e^{-\lambda\tau}\partial_t^k[\mathbf{E}_2, \mathbf{B}_2]\|^2_{L_\infty((0,T))L_6(\Omega)} + N_2 \sum_{k=0}^{m-8} \|e^{-\lambda\tau}\partial_t^k h_1\|^2_{L_\infty(\Sigma^T)} \le N_2 N_1.
\end{aligned}$$

Finally, thanks to the embedding $W_6^1 \subset L_\infty$, we obtain the desired estimate of the second term in the second estimate in (G.24).

Total energy estimate. First, we derive an estimate of the total instant energy and dissipation,

$$\sup_{\tau\le T}\mathcal{E}^{(\lambda)}(h_2,\mathbf{E}_2,\mathbf{B}_2,\tau)+\int_0^T\mathcal{D}^{(\lambda)}(h_2,\mathbf{E}_2,\mathbf{B}_2,\tau)\,d\tau \\ \le N\sqrt{\varepsilon_0}y^{(\lambda)}(h_2,\mathbf{E}_2,\mathbf{B}_2)+N\lambda^{-1}N_1N_2+N\mathcal{E}^{(\lambda)}(h_2,\mathbf{E}_2,\mathbf{B}_2,0), \tag{G.26}$$

where N_1 and N_2 are the constants in (G.19). We follow the argument of Step 1 in section 6.1 by making the following minor modifications:

- we add the weight $e^{-2\lambda t}$ to all the terms therein,
- we modify the integrals I_1 and I_2.

In particular, in I_1 (see (6.35)), one needs to replace Lf with $-Ah_2-Kh_1$. We then apply the estimate (B.13) in Lemma B.7:

$$\int_0^\tau\int_\Omega\langle -A(\partial_t^k h_2), p_0^{2\theta_k}\partial_t^k h_2\rangle e^{-2\lambda t}\,dxdt \\ \ge \kappa\|e^{-\lambda t}\nabla_p\partial_t^k h_2\|^2_{L_{2,\theta_k}(\Sigma^\tau)}-N\|e^{-\lambda t}\partial_t^k h_2\|^2_{L_2(\Sigma^\tau)}. \tag{G.27}$$

Furthermore, using the symmetry of the operator K and the bound (B.14), we get

$$-\int_{\Sigma^\tau}\big(K(\partial_t^k h_1)\big)(\partial_t^k h_2)e^{-2\lambda t}\,dz \\ \ge -(\kappa/2)\|e^{-\lambda t}\partial_t^k h_2\|^2_{L_2((0,\tau)\times\Omega)W_2^1(\mathbb{R}^3)}-N\|e^{-\lambda t}\partial_t^k h_1\|^2_{L_2(\Sigma^\tau)}. \tag{G.28}$$

We note that by the assumption (G.19) and the presence of the factor λ in the definition of $\mathcal{D}^{(\lambda)}(h_1,\mathbf{E}_1,\mathbf{B}_1,\tau)$ (see (G.21)), the last term on the r.h.s. of (G.28) can be replaced with $-NN_1\lambda^{-1}$. Furthermore, in the term I_2 in (6.35), we replace $\mathbf{E}_f$ with $\mathbf{E}_1$ and proceed as in (G.27).

Unsteady S_p estimate. Here, we estimate the remaining term in (G.23), which is the sum of squares of weighted $L_\infty^{t,x}W_\infty^1$ norms. This will be done via the *unsteady S_p* estimate. We first note that $u=e^{-\lambda t}\partial_t^k h_2, k\le m-8$, formally satisfies the identities

$$\begin{aligned} &Yu-\nabla_p\cdot(\sigma_{g^++g^-}\nabla_p u)+\boldsymbol{\xi}(\mathbf{E}_g+v(p)\times\mathbf{B}_g-a_g)\cdot\nabla_p u\\ &\qquad+\left(\lambda+C_g-\frac{\boldsymbol{\xi}}{2}v(p)\cdot\mathbf{E}_g\right)u\\ &\quad=e^{-\lambda t}\left(K(\partial_t^k h_1)+\boldsymbol{\xi}_1(v(p)\cdot\partial_t^k\mathbf{E}_1)J^{1/2}+1_{k>0}\sum_{j=1}^3\sum_{k_1+k_2=k,k_1\ge 1}\eta^j_{k_1,k_2}\right),\\ &u(t,x,p)=u(t,x,R_xp), z\in\Sigma_-^T,\quad u(0,\cdot)=f_{0,k}(\cdot)\,(\text{see (3.16)}),\\ &\eta^1_{k_1,k_2}=-\boldsymbol{\xi}\partial_t^{k_1}(\mathbf{E}_g+v(p)\times\mathbf{B}_g)\cdot\nabla_p(\partial_t^{k_2}h_2)+\frac{\boldsymbol{\xi}}{2}(v(p)\cdot\partial_t^{k_1}\mathbf{E}_g)\partial_t^{k_2}h_2,\\ &\eta^2_{k_1,k_2}=\big(\partial_{p_j}\partial_t^{k_1}\sigma^{ij}_{g^++g^-}-\partial_t^{k_1}a_g^i\big)(\partial_{p_i}\partial_t^{k_2}h_2)+(\partial_t^{k_1}C_g)\partial_t^{k_2}h_2,\\ &\eta^3_{k_1,k_2}=(\partial_t^{k_1}\sigma^{ij}_{g^++g^-})(\partial_{p_ip_j}\partial_t^{k_2}h_2). \end{aligned} \tag{G.29}$$

For $i\le 4$ and $k\le m-8$, we apply the unsteady S_p estimates in (5.9), (5.11), and (5.15) with $\theta/2^{k+2i}$ in place of θ and $\kappa=\frac{1}{2}$, and we get

$$\begin{aligned} &\|e^{-\lambda t}\partial_t^k h_2\|^2_{S_{r_i,\theta/2^{k+2i+1}}(\Sigma^T)}+1_{i=4}\|e^{-\lambda t}\partial_t^k h_2\|^2_{L_\infty((0,T)\times\Omega)W^1_{\infty,\theta/2^{k+2i+1}}(\mathbb{R}^3)}\\ &\quad\le N\sum_{s\in\{2,r_i\}}\left(\|e^{-\lambda t}(\text{r.h.s of (G.29)})\|^2_{L_{s,\theta/2^{k+2i}}(\Sigma^T)}+\lambda^2\|f_{0,k}\|^2_{S_{s,\theta/2^{k+2i}}(\Omega\times\mathbb{R}^3)}\right)\\ &\qquad+N\|e^{-\lambda t}\partial_t^k h_2\|^2_{L_{2,\theta/2^{k+2i}}(\Sigma^T)}. \end{aligned} \tag{G.30}$$

We follow the argument in the proof of (6.14) in Proposition 6.3 with minor modifications:

- The loss of decay in the p variable is different from that in the $L^t_\infty S_p$ estimate in (6.14) since the term $\partial_t^{k+1} f$ is on the l.h.s. in the present argument (cf (b) below formula (6.20)).
- The main difference is the estimate of the "free" terms $e^{-\lambda t} K \partial_t^k h_1$ and $v(p) \cdot \partial_t^k \mathbf{E}_1 J^{1/2}$, as the rest of the terms on the r.h.s. of (G.29) are handled in the same way as in the proof of (6.14) (cf. (6.21) and (6.25)).

Let us consider the first two terms on the r.h.s. of (G.29). By interpolating between L_2 and L_∞, exploiting the presence of the factor λ in front of the L_2 norm of $\mathbf{E}_1$ in $\mathcal{D}^{(\lambda)}(h_1, \mathbf{E}_1, \mathbf{B}_1, \tau)$, and using the $L^{t,x}_\infty$ bound of $\mathbf{E}_1$ in (G.19), we get

$$\begin{aligned}&\|e^{-\lambda t} v(p) \cdot \partial_t^k \mathbf{E}_1 J^{1/2}\|^2_{L_{r_i}((0,T)\times\Omega)} \\ &\quad \le \|e^{-\lambda t}\partial_t^k \mathbf{E}_1\|^{4/r_i}_{L_2((0,T)\times\Omega)} \|e^{-\lambda t}\partial_t^k \mathbf{E}_1\|^{2-4/r_i}_{L_\infty((0,T)\times\Omega)} \le N_2^{1-2/r_i} N_1 \lambda^{-2/r_i}.\end{aligned} \tag{G.31}$$

Next, by (B.11) in Lemma B.5 and the interpolation inequality, we have (cf. (6.31))

$$\begin{aligned}&\|e^{-\lambda t} K(\partial_t^k h_1)\|^2_{L_{r_i,\theta/2^{k+2i}}(\Sigma^T)} \le N \|e^{-\lambda t}\partial_t^k h_1\|^2_{L_{r_i}((0,T)\times\Omega) W^1_{r_i}(\mathbb{R}^3)} \\ &\quad \le \lambda^{-1/r_i} \|e^{-\lambda t} D_p^2 \partial_t^k h_1\|^2_{L_{r_i}(\Sigma^T)} + N\lambda^{1/r_i}\|e^{-\lambda t}\partial_t^k h_1\|^2_{L_{r_i}(\Sigma^T)}.\end{aligned}$$

Furthermore, since $k \le m-8$, by interpolating between L_2 and L_∞ and using the bounds of h_1 in (G.19) (cf. (G.31)), the last term is bounded by

$$N N_1 \lambda^{1/r_i} \lambda^{-2/r_i} = N N_1 \lambda^{-1/r_i}.$$

By the above argument, Hölder's inequality, and the fact that the last term on the r.h.s. in (G.23) is bounded by N_1 (see (G.19)), we get

$$\begin{aligned}&\sum_{s\in\{2,r_i\}} \|e^{-\lambda t} K(\partial_t^k h_1)\|^2_{L_{s,\theta/2^{k+2i}}(\Sigma^T)} \\ &\quad \le 2\lambda^{-1/r_i} \sum_{s\in\{2,r_i\}} \|e^{-\lambda t} D_p^2\partial_t^k h_1\|^2_{L_s(\Sigma^T)} + N N_1 \lambda^{-1/r_i} \le N N_1 \lambda^{-1/r_i}.\end{aligned} \tag{G.32}$$

Thus, combining (G.30) with the estimates of the "free terms" (G.31)–(G.32) and with the bounds of nonlinear terms (cf. (6.21)–(6.25)), we obtain

$$\begin{aligned}&\sum_{k=0}^{m-8} \Big(\|e^{-\lambda t}\partial_t^k h_2\|^2_{L_\infty((0,T)\times\Omega) W^1_{\infty,\theta/2^{k+9}}(\mathbb{R}^3)} + \sum_{i=1}^4 \|e^{-\lambda t}\partial_t^k h_2\|^2_{S_{r_i,\theta/2^{k+2i+1}}(\Sigma^T)}\Big) \\ &\quad \le N\varepsilon_0 y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) + N N_1 N_2 \lambda^{-1/r_4} + N\lambda^2 \mathcal{S}_f(0),\end{aligned} \tag{G.33}$$

where

$$\mathcal{S}_f(0) := \sum_{k=0}^{m-8} \sum_{s\in\{2,r_4\}} \|f_{0,k}\|^2_{S_{s,\theta/2^{k+8}}(\Omega\times\mathbb{R}^3)}.$$

Finally, gathering (G.26) and (G.33) gives

$$\begin{aligned} y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) &\le N\sqrt{\varepsilon_0}\, y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) + N\lambda^{-1/r_4} N_1 \\ &\quad + N\mathcal{E}_f(0) + N\lambda^2 \mathcal{S}_f(0),\end{aligned}$$

where $\mathcal{E}_f(0)$ is defined in (3.37). Choosing $\varepsilon_0 < (2N)^{-2}$ gives

$$y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) \le N(N_1 \lambda^{-1/r_4} + \mathcal{E}_f(0) + \lambda^2 \mathcal{S}_f(0)).$$

Furthermore, choosing $\lambda > (4N)^{r_4}$ gives $N_1 N \lambda^{-1/r_4} < N_1/4$. Finally, choosing $N_1 > (4/3)N\big(\mathcal{E}_f(0) + \lambda^2 \mathcal{S}_f(0)\big)$, we obtain

$$y^{(\lambda)}(h_2, \mathbf{E}_2, \mathbf{B}_2) \le N_1,$$

as desired.

(ii) First, we note that by the estimates of the free terms (G.31)–(G.32), the assumption on $f_{0,0}$ in the statement of Theorem 3.10, and Propositions 5.4–5.6, the problem (G.29) with $k=0$ has a unique strong solution h_2, and, in addition,

$$\begin{aligned} h_2 \in{} & C([0,T])L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_2((0,T)\times\Omega)W^1_{2,\theta}(\mathbb{R}^3) \\ & \cap S_{r_i,\theta/2^{2i+1}}(\Sigma^T) \cap L_\infty((0,T)\times\Omega)W^1_{\infty,\theta/2^9}(\mathbb{R}^3), i=1,\dots,4. \end{aligned} \tag{G.34}$$

Next, we use an induction argument.

Claim 1. We assume that for some $k_0 \in \{1,\dots,m-8\}$, and all $k \le k_0 - 1$, one has

$$\partial_t^k h_2 \in C([0,T])L_{2,\theta/2^k}(\Omega\times\mathbb{R}^3) \cap L_2((0,T)\times\Omega)W^1_{2,\theta/2^k}(\mathbb{R}^3), \tag{G.35}$$

$$\begin{aligned} \partial_t^k h_2 \in{} & S_{r_i,\theta/2^{k+2i+1}}(\Sigma^T) \\ & \cap L_\infty((0,T)\times\Omega)W^1_{\infty,\theta/2^{k+9}}(\mathbb{R}^3),\ i=1,\dots,4, \end{aligned} \tag{G.36}$$

$$u = e^{-\lambda t}\partial_t^k f \text{ is a strong solution to (G.29).} \tag{G.37}$$

Then, we claim that (G.35)–(G.37) hold for all $k \le k_0$.

Claim 2. Invoke the definition of θ_k in (6.34). We assume that for some $k_0 \in \{m-7,\dots,m\}$ and all $k \le k_0 - 1$, one has

$$\partial_t^k h_2 \in C([0,T])L_{2,\theta_k}(\Omega\times\mathbb{R}^3) \cap L_2((0,T)\times\Omega)W^1_{2,\theta_k}(\mathbb{R}^3), \tag{G.38}$$

$$u = e^{-\lambda t}\partial_t^k f \text{ is a finite energy solution to (G.29).} \tag{G.39}$$

Then, (G.38)–(G.39) hold for all $k \le k_0$.

Proof of Claim 1. To justify the differentiation with respect to t and (G.35) with k_0 in place of k, we use Lemma G.1 with $\partial_t^{k_0-1} h_2$ in place of f, and f_0 and f_1 replaced with $f_{0,k_0-1} \in L_{2,\theta/2^{k_0-1}}(\Omega\times\mathbb{R}^3)$ and $f_{0,k_0} \in L_{2,\theta/2^{k_0}}(\Omega\times\mathbb{R}^3)$, respectively, and

$$b = \pm(\mathbf{E}_g + v(p)\times\mathbf{B}_g) - a_g, c = \left(C_g \mp \frac{1}{2}v(p)\cdot\mathbf{E}_g\right),$$

$$\eta = \text{r.h.s. of (G.29) with } k \text{ replaced with } k_0 - 1. \tag{G.40}$$

We check the conditions of Lemma G.1. First, it follows from the argument of (G.26) that

$$\eta \in L_2((0,T)\times\Omega)W^{-1}_{2,\theta/2^{k_0-1}}(\mathbb{R}^3), \quad \partial_t \eta \in L_2((0,T)\times\Omega)W^{-1}_{2,\theta/2^{k_0}}(\mathbb{R}^3). \tag{G.41}$$

Finally, we check the condition $\nabla_p \cdot b \in L_\infty(\Sigma^T)$ with $b = a_g$, where a_g is defined in (6.2). We note that

$$\partial_{p_i} a_g^i(t,x,p) = -\partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \frac{p_i}{2p_0} g(t,x,q) \cdot (1,1)\, dq$$
$$+ \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(t,x,q) \cdot (1,1)\, dq = I_1 + I_2.$$

By the estimate (B.7) with $k=1$,

$$\|I_1\|_{L_\infty(\Sigma^T)} \le N \|g\|_{L_\infty((0,T)\times\Omega) W_2^1(\mathbb{R}^3)}. \tag{G.42}$$

Next, to handle I_2, we will use the identity (B.8):

$$\begin{aligned}
&\partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \partial_{q_j} g(q)\, dq \\
&= \partial_{p_i} \int \Phi^{ij}(P,Q) J^{1/2}(q) \frac{q_j}{2q_0} g(q)\, dq \\
&\quad - 4 \int \frac{P\cdot Q}{p_0 q_0} \Big((P\cdot Q)^2 - 1 \Big)^{-1/2} J^{1/2}(q) g(q)\, dq - \kappa(p) J^{1/2}(p) g(p),
\end{aligned} \tag{G.43}$$

where $\kappa(p) = 2^{7/2}\pi p_0 \int_0^\pi (1+|p|^2\sin^2\theta)^{-3/2}\sin(\theta)\, d\theta$. By (B.7) with $k=1$, the first term on the r.h.s. of (G.43) is bounded by the r.h.s. of (G.42). The remaining terms are handled similarly. Thus, $\|\nabla_p \cdot a_g\|_{L_\infty((0,T)\times\Omega\times\mathbb{R}^3)}$ is bounded by the r.h.s. of (G.42). Hence, by Lemma G.1, $\partial_t^{k_0} h_2$ is a finite energy solution to (G.29), and (G.35) holds with k replaced with k_0, as claimed.

Next, to deduce that $\partial_t^{k_0} h_2$ is a strong solution that satisfies the desired S_{r_i} regularity in (G.36), we use Propositions 5.4–5.6 combined with the argument of (G.33). Thus, Claim 1 is proved.

Proof of Claim 2. We repeat the proof of Claim 1 with one minor modification. We note that to apply Lemma G.1, we need (G.41) to hold, where η is defined in (G.40). This estimate was established in the proof of the energy bound (G.26). See the argument of (6.41). In particular, to handle the cubic terms (cf. (6.38)–(6.39)), we need to control certain weighted $L_\infty^{t,x,p}$ norms of $\partial_t^k [h_2, \nabla_p h_2], k \le m/2$, which was done in Claim 1 (see (G.36)).

(iii) Since (ii) is valid, we only need to verify the continuity equation (G.18). To this end, we note that the functions $H_2 = J + J^{1/2} h_2$ and $[\mathbf{E}_1, \mathbf{B}_1]$, satisfy the identities

$$\begin{aligned}
&Y H_2^+ + (\mathbf{E}_g + v(p)\times \mathbf{B}_g)\cdot \nabla_p H_2^+ \\
&\quad = \mathcal{C}(H_2^+, G^+ + G^-) + \mathcal{C}\big(J, J^{1/2}(h_2^+ + h_2^- - g^+ - g^-)\big) \\
&\quad\quad - (\mathbf{E}_1 - \mathbf{E}_g)\cdot \nabla_p J,
\end{aligned} \tag{G.44}$$

$$\begin{aligned}
&Y H_2^- - (\mathbf{E}_g + v(p)\times \mathbf{B}_g)\cdot \nabla_p H_2^- \\
&\quad = \mathcal{C}(H_2^-, G^- + G^+) + \mathcal{C}\big(J, J^{1/2}(h_2^+ + h_2^- - g^+ - g^-)\big) \\
&\quad\quad + (\mathbf{E}_1 - \mathbf{E}_g)\cdot \nabla_p J.
\end{aligned} \tag{G.45}$$

Differentiating formally the above identities k times in t and integrating over $p\in\mathbb{R}^3$, we obtain the continuity equation (G.18). ∎

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