Mathematische Annalen (2024) 390:5005–5051
https://doi.org/10.1007/s00208-024-02876-y

Mathematische Annalen

Asymptotics of the solution to the perfect conductivity problem with *p*-Laplacian

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Received: 24 November 2023 / Revised: 13 March 2024 / Accepted: 7 April 2024 /
Published online: 30 April 2024

Abstract
We study the perfect conductivity problem with closely spaced perfect conductors embedded in a homogeneous matrix where the current-electric field relation is the power law $J = \sigma |E|^{p-2} E$. The gradient of solutions may be arbitrarily large as ε, the distance between inclusions, approaches to 0. To characterize this singular behavior of the gradient in the narrow region between two inclusions, we capture the leading order term of the gradient. This is the first gradient asymptotics result on the nonlinear perfect conductivity problem.

Mathematics Subject Classification 35B40 · 35J92 · 35Q74 · 74E30 · 74G70

1 Introduction and main results

Our study is instigated by the damage analysis in the fiber composite materials [6]. Particularly, when fibers are closely packed and in high-contrast to the background matrix in terms of material properties, the electric field could be amplified by the composite micro-structure. In this article, we investigate the specific scenario in which the fiber inclusions are perfect conductors, and the background matrix follows the current-electric field relation described by the power law:

Hongjie Dong was partially supported by the NSF under agreement DMS-2055244. Zhuolun Yang was partially supported by Simons Foundation Institute Grant Award ID 507536 and the AMS-Simons Travel Grant. Hanye Zhu was partially supported by the NSF under agreement DMS-2055244.

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$$J = \sigma|E|^{p-2}E, \quad p > 1, \tag{1.1}$$

where J, E, and σ represent current, electric field, and conductivity, respectively. This power law has physical relevance across various materials, including dielectrics, plastic moulding, plasticity phenomena, viscous flows in glaciology, electro-rheological and thermo-rheological fluids. We refer to [5, 12, 24, 27, 31, 40, 41] and the references therein.

Before stating our results, let us describe the mathematical setup: let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, and let $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ be two C^2 open sets with $\mathrm{diam}(\mathcal{D}_i^0) > c > 0$ and $\mathrm{dist}(\mathcal{D}_1^0 \cup \mathcal{D}_2^0, \partial\Omega) > c > 0$, touching at the origin with the inner normal direction of $\partial\mathcal{D}_1^0$ being the positive x_n-axis. We write the variable x as (x', x_n), where $x' \in \mathbb{R}^{n-1}$. For $\varepsilon > 0$, translating $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ by $\varepsilon/2$ along the x_n-axis, we obtain

$$\mathcal{D}_1^\varepsilon := \mathcal{D}_1^0 + (0', \varepsilon/2) \quad \text{and} \quad \mathcal{D}_2^\varepsilon := \mathcal{D}_2^0 - (0', \varepsilon/2).$$

We denote $\widetilde{\Omega}^\varepsilon := \Omega \backslash \overline{(\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon)}$.

The perfect conductivity problem incorporating the power law (1.1) can be modeled by the following p-Laplace equation with $p > 1$:

$$\begin{cases} -\mathrm{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ u_\varepsilon = U_i^\varepsilon & \text{on } \overline{\mathcal{D}_i^\varepsilon},\ i = 1, 2, \\ \displaystyle\int_{\partial\mathcal{D}_i^\varepsilon} |Du_\varepsilon|^{p-2}Du_\varepsilon \cdot \nu = 0, & i = 1, 2, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\varphi \in C^2(\partial\Omega)$ is given, $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outer normal vector on $\partial\mathcal{D}_1^\varepsilon \cup \partial\mathcal{D}_2^\varepsilon$ (pointing away from $\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon$), and U_1^ε, U_2^ε are two constants determined by $(1.2)_3$. Here and throughout the paper, we adopt the notation

$$Du_\varepsilon \cdot \nu(x) := \lim_{t \to 0_+} \frac{u_\varepsilon(x + t\,\nu(x)) - u_\varepsilon(x)}{t} \quad \text{and} \quad Du_\varepsilon(x) := [Du_\varepsilon \cdot \nu(x)]\nu(x)$$

for $x \in \partial\mathcal{D}_i^\varepsilon$, $i = 1, 2$.

The solution $u_\varepsilon \in W^{1,p}(\Omega)$ can be viewed as the unique function which has the minimal energy in appropriate function space: $I_p[u] = \min_{v \in \mathscr{A}^\varepsilon} I_p[v]$, where

$$\begin{aligned} I_p[v] &:= \int_\Omega |Dv|^p, \quad v \in \mathscr{A}^\varepsilon, \\ \mathscr{A}^\varepsilon &:= \{v \in W^{1,p}(\Omega) : Dv \equiv 0 \ \text{in } \mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon, v = \varphi \ \text{on } \partial\Omega\}. \end{aligned} \tag{1.3}$$

We refer the reader to the Appendix of [8] for the derivation of (1.2) and its equivalence with (1.3). Although this derivation specifically addresses the case when $p = 2$, the argument can be readily applied to $p > 1$ with slight modifications.

The perfect conductivity problem (1.2) with $p = 2$ has undergone thorough studies. It was proved by Ammari et al. in [3, 4] that, when $\mathcal{D}_1$ and $\mathcal{D}_2$ are disks of comparable radii in $\mathbb{R}^2$, the blow-up rate of the gradient of the solution is $\varepsilon^{-1/2}$ as ε goes to zero; Yun in [44, 45] generalized the above mentioned result for two strictly convex inclusions in $\mathbb{R}^2$. These gradient estimates in dimension $n = 2$ were localized and extended to higher dimensions by Bao, Li, and Yin in [8]:

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}^\varepsilon)} \leq \begin{cases} C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 2, \\ C|\varepsilon \ln \varepsilon|^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 3, \\ C\varepsilon^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n \geq 4. \end{cases}$$

These bounds were shown to be optimal in the paper and they are independent of the shape of inclusions, as long as the inclusions are relatively strictly convex. Moreover, numerous studies have been conducted into characterizing the asymptotic behavior of Du_ε, which are significant in practical applications. For further works on the linear perfect conductivity problem, see e.g. [1, 2, 10, 11, 13, 17, 21, 25, 28–30, 32–34, 39] and the references therein.

The study on the nonlinear perfect conductivity problem (1.2) is less comprehensive. The only results were given by Gorb and Novikov [26] and Ciraolo and Sciammetta [15]. They proved that for $n \geq 2$,

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}^\varepsilon)} \leq \begin{cases} C\varepsilon^{-\frac{n-1}{2(p-1)}} & \text{when } p > \dfrac{n+1}{2}, \\ C\varepsilon^{-1}|\ln \varepsilon|^{\frac{1}{1-p}} & \text{when } p = \dfrac{n+1}{2}, \\ C\varepsilon^{-1} & \text{when } 1 < p < \dfrac{n+1}{2}. \end{cases}$$

These bounds were shown to be optimal in their respective papers. In this paper, we give a more precise characterization of the gradient by capturing its leading order term in the asymptotics expansion.

It is noteworthy that, for the linear case in dimension two, solutions to the perfect conductivity problem and the insulated conductivity problem, representing the two extremes of conductivity, are harmonic conjugate to each other as shown in [4]. Therefore, the behavior of their gradients is essentially identical due to the Cauchy–Riemann equation. The authors of this paper in [20] studied the insulated conductivity problem with p-Laplacian, and identified the optimal blow-up exponent in dimension two. It turns out the gradient behaves significantly different from that of the solution to (1.2) in dimension two. This showcases an intriguing feature of the nonlinear conductivity problem. For more results on the linear insulated conductivity problem, we refer to [9, 18, 19, 35, 36, 43, 46].

To study the asymptotic behavior of u_ε, the solution to (1.2), it is important to study the limiting problem (1.3) with $\varepsilon = 0$. We will show that the minimizing problem (1.3) with $\varepsilon = 0$ is equivalent to

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2}Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_0 & \text{on } \overline{\mathcal{D}_1^0} \cup \overline{\mathcal{D}_2^0}, \\ \displaystyle\int_{\partial\mathcal{D}_1^0 \cup \partial\mathcal{D}_2^0} |Du_0|^{p-2}Du_0 \cdot \nu = 0, & \\ u_0 = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

for a constant U_0 when $p \geq (n+1)/2$, and is equivalent to

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2}Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_i & \text{on } \overline{\mathcal{D}_i^0}\backslash\{0\},\ i = 1, 2, \\ \displaystyle\int_{\partial\mathcal{D}_i^0} |Du_0|^{p-2}Du_0 \cdot \nu = 0, & i = 1, 2, \\ u_0 = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

for constants U_1 and U_2 when $p < (n+1)/2$. We would like to clarify a misunderstanding in the papers [14, 15]. In [14, 15], the authors implicitly claimed that the minimizing problem (1.3) with $\varepsilon = 0$ is universally equivalent to (1.4), which is not the case. We will justify this in Theorem 2.4. We emphasize that while the minimizer u_0 of (1.3) with $\varepsilon = 0$ always takes the same value in $\overline{\mathcal{D}_1^0}$ and $\overline{\mathcal{D}_2^0}$ when $p \geq (n+1)/2$, it may take different values when $1 < p < (n+1)/2$. On the other hand, the flux along $\partial\mathcal{D}_1^0$, denoted by

$$\mathcal{F} := \int_{\partial\mathcal{D}_1^0} |Du_0|^{p-2}Du_0 \cdot \nu, \tag{1.6}$$

might not be zero when $p \geq (n+1)/2$, but it must be zero when $p < (n+1)/2$.

By the regularity of $\mathcal{D}_1^\varepsilon$ and $\mathcal{D}_2^\varepsilon$, we can assume that near the origin, the part of $\partial\mathcal{D}_1^\varepsilon$ and $\partial\mathcal{D}_2^\varepsilon$, denoted by Γ_+^ε and Γ_-^ε, are respectively the graphs of two C^2 functions in terms of x'. That is,

$$\Gamma_+^\varepsilon = \left\{x_n = \frac{\varepsilon}{2} + h_1(x'),\ |x'| < 1\right\}, \quad \Gamma_-^\varepsilon = \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'),\ |x'| < 1\right\},$$

where h_1 and h_2 are relatively convex C^2 functions satisfying

$$h_1(0') = h_2(0') = 0, \quad D_{x'}h_1(0') = D_{x'}h_2(0') = 0, \tag{1.7}$$

$$c_1|x'|^2 \leq h_1(x') - h_2(x') \quad \text{for } 0 < |x'| < 1, \tag{1.8}$$

and

$$\|h_1\|_{C^2} \leq c_2, \quad \|h_2\|_{C^2} \leq c_2, \tag{1.9}$$

with some positive constants c_1, c_2. For $x_0 \in \widetilde{\Omega}^\varepsilon$, $0 < r < 1 - |x_0'|$, we denote

$$\Omega^\varepsilon_{x_0,r} := \left\{(x', x_n) \in \widetilde{\Omega}^\varepsilon \;\Big|\; -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'),\; |x' - x_0'| < r\right\},$$

and $\Omega^\varepsilon_r := \Omega^\varepsilon_{0,r}$. We also denote

$$\Gamma^\varepsilon_{+,r} := \Gamma^\varepsilon_+ \cap \overline{\Omega}^\varepsilon_r, \quad \Gamma^\varepsilon_{-,r} := \Gamma^\varepsilon_- \cap \overline{\Omega}^\varepsilon_r.$$

We use $B_r(x_0)$ to denote the open ball of radius r centered at x_0 and we set

$$B_r := B_r(0), \quad \Omega^\varepsilon_r(x_0) := \widetilde{\Omega}^\varepsilon \cap B_r(x_0).$$

Throughout this paper, we denote

$$\underline{\delta}(x) := \varepsilon + |x'|^2 \tag{1.10}$$

and

$$\delta(x) := \varepsilon + h_1(x') - h_2(x'). \tag{1.11}$$

By (1.7)–(1.9), it can be easily seen that

$$\min\{c_1, 1\}\underline{\delta}(x) \le \delta(x) \le \max\{c_2, 1\}\underline{\delta}(x), \quad \text{for} \quad x \in \Omega_1.$$

We denote

$$\Theta(\varepsilon) := \begin{cases} \varepsilon^{\frac{2p-n-1}{2(p-1)}}, & p > \frac{n+1}{2}, \\ |\ln \varepsilon|^{-\frac{1}{p-1}}, & p = \frac{n+1}{2}, \\ 1, & 1 < p < \frac{n+1}{2}, \end{cases} \tag{1.12}$$

and

$$K = \begin{cases} \dfrac{\det\left(D^2_{x'}(h_1 - h_2)(0')\right)^{\frac{1}{2}}\Gamma(p-1)}{(2\pi)^{\frac{n-1}{2}}\Gamma\left(p - \frac{n+1}{2}\right)}, & \text{when} \;\; p > \dfrac{n+1}{2}, \\ \dfrac{\det\left(D^2_{x'}(h_1 - h_2)(0')\right)^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}{(2\pi)^{\frac{n-1}{2}}}, & \text{when} \;\; p = \dfrac{n+1}{2}, \end{cases} \tag{1.13}$$

where $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}$ is the Gamma function defined for $z > 0$.

Our main result is the following asymptotic expansion of $Du_\varepsilon(x)$ for sufficiently small ε and x'.

Theorem 1.1 *Let h_1, h_2 be C^2 functions satisfying (1.7)–(1.9), $p > 1$, $n \geq 2$, $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (1.2), u_0 be the minimizer of (1.3) with $\varepsilon = 0$, $\mathcal{F}$ be given in (1.6), U_1, U_2 be the constants in $(1.5)_2$, $\delta(x)$ be defined in (1.11), $\Theta(\varepsilon)$ be given in (1.12), and K be defined in (1.13). Then there exist constants $\beta \in (0, 1)$ depending only on n and p, and $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that the following holds:*

(i) *If $p \geq (n+1)/2$, for $\varepsilon \in (0, 1)$ and $x \in \Omega^\varepsilon_{1/4}$, we have*

$$Du_\varepsilon(x) = \big(0', \delta(x)^{-1}\Theta(\varepsilon)(\mathrm{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)} + f_0(\varepsilon))\big) + \mathbf{f}_1(x, \varepsilon), \tag{1.14}$$

where $f_0 : \mathbb{R} \to \mathbb{R}$ is a function of ε and $\mathbf{f}_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function of x and ε, such that

$$\begin{aligned} \lim_{\varepsilon \to 0} f_0(\varepsilon) &= 0, \\ |\mathbf{f}_1(x, \varepsilon)| &\leq C_1\Big(\delta(x)^{\beta/2-1}\Theta(\varepsilon)(|K\mathcal{F}|^{1/(p-1)} + |f_0(\varepsilon)|) \\ &\qquad + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}}\Big). \end{aligned} \tag{1.15}$$

(ii) *If $1 < p < (n+1)/2$, for $\varepsilon \in (0, 1)$ and $x \in \Omega^\varepsilon_{1/4}$, we have*

$$Du_\varepsilon(x) = \big(0', \delta(x)^{-1}(U_1 - U_2 + g_0(\varepsilon))\big) + \mathbf{g}_1(x, \varepsilon),$$

where $g_0 : \mathbb{R} \to \mathbb{R}$ is a function of ε and $\mathbf{g}_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function of x and ε, such that

$$\begin{aligned} \lim_{\varepsilon \to 0} g_0(\varepsilon) &= 0, \\ |\mathbf{g}_1(x, \varepsilon)| &\leq C_1\Big(\delta(x)^{\beta/2-1}(|U_1 - U_2| + |g_0(\varepsilon)|) + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}}\Big). \end{aligned} \tag{1.16}$$

As a consequence of the asymptotic expansion in Theorem 1.1, we provide a pointwise upper bound of Du_ε.

Remark 1.2 Under the hypotheses of Theorem 1.1, there exist constants $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that for sufficiently small $\varepsilon > 0$, and any $x \in \Omega^\varepsilon_{1/4}$, we have

$$|Du_\varepsilon(x)| \leq C_1 \|\varphi\|_{L^\infty(\partial\Omega)} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + |x'|^2} + e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}}\Big). \tag{1.17}$$

In fact, when $1 < p < (n+1)/2$, (1.17) follows directly from Proposition 2.2. When $p \geq (n+1)/2$, since $u_0 = U_0$ on $\overline{\mathcal{D}^0_1} \cup \overline{\mathcal{D}^0_2}$, we have $|\mathcal{F}| \leq C\|\varphi\|^{p-1}_{L^\infty(\partial\Omega)}$ and thus

(1.17) follows from (1.14). Indeed, one can see the boundedness of Du_0 on $\Gamma^0_{+,1/2}$ by Lemma 2.1, and on $\partial\mathcal{D}^0_1\setminus\Gamma^0_{+,1/2}$ by classical gradient estimates (see e.g. [37]).

Another direct consequence of Theorem 1.1 is the following pointwise positive lower bound of $D_n u_\varepsilon$ near the origin, provided the coefficient of the leading order term in the asymptotic expansion is positive.

Remark 1.3 Under the hypotheses of Theorem 1.1, if either

$$p \geq \frac{n+1}{2} \quad \text{and} \quad \mathcal{F} > 0 \tag{1.18}$$

or

$$1 < p < \frac{n+1}{2} \quad \text{and} \quad U_1 > U_2 \tag{1.19}$$

holds, then there exist constants $\kappa_1, \kappa_2 \in (0, 1/4)$, $\gamma > 0$ depending only on n, p, c_1, c_2, $\|\varphi\|_{L^\infty(\partial\Omega)}$, $\mathcal{F}$ (when $p \geq (n+1)/2$), and $U_1 - U_2$ (when $p < (n+1)/2$), such that for sufficiently small $\varepsilon > 0$, and any $x \in \Omega^\varepsilon_{1/4}$ satisfying

$$\begin{cases} |x'| \leq |\ln\varepsilon|^{-\gamma}, & \text{when} \quad p > \dfrac{n+1}{2}, \\ |x'| \leq \kappa_1(\ln|\ln\varepsilon|)^{-1}, & \text{when} \quad p = \dfrac{n+1}{2}, \\ |x'| \leq \kappa_2, & \text{when} \quad 1 < p < \dfrac{n+1}{2}, \end{cases}$$

we have

$$\begin{cases} D_n u_\varepsilon(x) \geq \dfrac{1}{2}\delta(x)^{-1}\Theta(\varepsilon)(K\mathcal{F})^{1/(p-1)}, & \text{when} \quad p \geq \dfrac{n+1}{2}, \\ D_n u_\varepsilon(x) \geq \dfrac{1}{2}\delta(x)^{-1}(U_1 - U_2), & \text{when} \quad 1 < p < \dfrac{n+1}{2}. \end{cases} \tag{1.20}$$

Indeed, when $p \geq (n+1)/2$, (1.20) follows directly from Theorem 1.1 by setting

$$\begin{cases} |f_0(\varepsilon)| \leq \dfrac{1}{6}(K\mathcal{F})^{1/(p-1)}, \\ C_1\delta(x)^{\beta/2} \leq \dfrac{1}{7}, \\ C_1\|\varphi\|_{L^\infty(\partial\Omega)}e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}} \leq \dfrac{1}{6}\delta(x)^{-1}\Theta(\varepsilon)(K\mathcal{F})^{1/(p-1)}, \end{cases}$$

and the case when $p \in (1, (n+1)/2)$ follows similarly.

Next, we provide a concrete example whose coefficient of the leading order term in the asymptotic expansion is positive.

Proposition 1.4 *Let $\Omega = B_5 \subset \mathbb{R}^n$, $\mathcal{D}_1 = B_2(0', 2)$, $\mathcal{D}_2 = B_2(0', -2)$, $\varphi = x_n$, and u_0 be the minimizer of* (1.3) *with $\varepsilon = 0$. Then either* (1.18) *or* (1.19) *is satisfied.*

Our proof of the main result Theorem 1.1 relies on the following C^β bound of the gradient, which may be of independent interest.

Proposition 1.5 *Let h_1, h_2 be C^2 functions satisfying* (1.7)–(1.9)*, $p > 1$, $n \geq 2$, $\varepsilon \in (0, 1)$, and $u_\varepsilon \in W^{1,p}(\Omega)$ be a solution of* (1.2)*. Then there exist constants $\beta \in (0, 1)$ depending only on n and p, and $C > 0$ depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/4}$ and $\underline{\delta}(x) = \varepsilon + |x'|^2$, it holds that*

$$[Du_\varepsilon]_{C^\beta(\Omega^\varepsilon_{x,\sqrt{\underline{\delta}(x)}/4})} \leq C\underline{\delta}(x)^{-\beta/2}\|Du_\varepsilon\|_{L^\infty(\Omega^\varepsilon_{x,\sqrt{\underline{\delta}(x)}/2})}. \tag{1.21}$$

Remark 1.6 It can be seen from the proof in Sect. 3 that Proposition 1.5 holds as long as $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ is a solution of

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ u_\varepsilon = U_1^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_\varepsilon = U_2^\varepsilon & \text{on } \Gamma_-^\varepsilon, \end{cases}$$

for some arbitrary constants U_1^ε and U_2^ε. Moreover, the same estimates also hold for any solution $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ of

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_\pm^\varepsilon, \end{cases}$$

which might be useful for obtaining sharper blow-up estimates for the insulated conductivity problem with p-Laplacian (see [20]).

We briefly describe the steps of proving Theorem 1.1. First, we establish a pointwise upper bound of the gradient in terms of $U_1^\varepsilon - U_2^\varepsilon$ for arbitrary given U_1^ε and U_2^ε (Proposition 2.2). Then we use mean oscillation estimates to prove a $C^{1,\beta}$ estimate (Proposition 1.5). Note that Proposition 1.5 implies a power gain of order $\delta^{\beta/2}$ for the oscillation of the gradient in the x_n direction. Because of this power gain and Proposition 2.2, we then derive an asymptotic expansion of Du_ε in terms of $U_1^\varepsilon - U_2^\varepsilon$ (Proposition 4.1). When $p \geq (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converge to 0 as $\varepsilon \to 0$. In this case, we use the flux conditions to derive the convergence rate for $U_1^\varepsilon - U_2^\varepsilon$ (Theorem 4.4). When $p < (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converges to $U_1 - U_2$ (Theorem 2.5). Finally, Theorem 1.1 follows from putting all the ingredients above together.

We would like to point out that weaker versions of Proposition 4.1 were proved in [15, 26]. They derived an asymptotic expansion of Du_ε only on the upper and lower boundaries $\Gamma_\pm^\varepsilon$ by constructing suitable barrier functions and using comparison principle. However, in our opinion, it appears that there is a gap in their proofs. It is nontrivial to see that the normal derivative is bounded in the case when $U_1^\varepsilon - U_2^\varepsilon$ is small as lack of control of the oscillation of the solution in the x' direction (see [15, p. 6174] and [26, p. 740]). This gap can be filled by Proposition 2.2 of this paper. We would like to remark that our argument in Proposition 4.1 is more robust in the sense that the proof does not rely on the fundamental solution of the p-Laplace equation or the maximum principle. In fact, the only place this paper involves the maximum principle is Proposition 2.2. See also Remark 2.3. If one can give an alternative proof of (2.7) in Proposition 2.2 without using the maximum principle, then our results can be extended to nonlinear systems of p-Laplace type.

The rest of the paper is organized as follows: In Sect. 2, we provide some preliminary estimates and results. In Sect. 3, we use mean oscillation estimates to prove Proposition 1.5. A convergence rate of $U_1^\varepsilon - U_2^\varepsilon$ when $p \geq (n+1)/2$ is provided in Sect. 4. Finally, the proofs of Theorem 1.1 and Proposition 1.4 are given in Sect. 5.

2 Preliminaries

In this section, we provide some preliminary results.

Lemma 2.1 *Let h_1, h_2 be C^2 functions satisfying (1.7)–(1.9), $p > 1$, $n \geq 2$, $\varepsilon \in [0, 1)$, and $v \in W^{1,p}(\Omega_1^\varepsilon)$ be a solution of*

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = 0 & \text{in } \Omega_1^\varepsilon, \\ v = 0 & \text{on } \Gamma_\pm^\varepsilon. \end{cases} \tag{2.1}$$

Then there exist constants $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that

$$|v(x)| + |Dv(x)| \leq C_1 e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}} \|v\|_{L^p(\Omega_1^\varepsilon \setminus \Omega_{1/2}^\varepsilon)} \quad \text{for } x \in \overline{\Omega_{1/2}^\varepsilon}. \tag{2.2}$$

Proof The proof of this lemma essentially follows that of [7, Theorem 1.1], with some modification. For simplicity, we omit the superscript ε in the proof. Without loss of generality, we may assume $\varepsilon \in [0, 1/256)$ and $|x'| < 1/16$ since otherwise (2.2) follows from classical estimates for the p-Laplace equation (see e.g. [37]). For any $0 < t < s < 1$, let $\eta = \eta(x')$ be a cutoff function such that $\eta = 1$ in Ω_t, $\eta = 0$ in $\Omega_1 \setminus \Omega_s$, and $|D\eta| \leq C(s-t)^{-1}$. Multiplying $v\eta^p$ on both sides of (2.1) and integrating by parts, we have

$$\int_{\Omega_1} |Dv|^p \eta^p + p|Dv|^{p-2} Dv \cdot D\eta v \eta^{p-1} = 0.$$

By Young's inequality,

$$\int_{\Omega_1} |Dv|^p \eta^p \leq p \int_{\Omega_1} |Dv|^{p-1} \eta^{p-1} |v||D\eta| \leq \frac{1}{2} \int_{\Omega_1} |Dv|^p \eta^p + C \int_{\Omega_1} |v|^p |D\eta|^p.$$

Therefore,

$$\int_{\Omega_t} |Dv|^p \le \frac{C}{(s-t)^p} \int_{\Omega_s \setminus \Omega_t} |v|^p.$$

Since $v = 0$ on Γ_-, by the Poincaré inequality in the x_n direction, we have

$$\int_{\Omega_s \setminus \Omega_t} |v|^p \le C(\varepsilon + s^2)^p \int_{\Omega_s \setminus \Omega_t} |Du|^p.$$

Therefore,

$$\int_{\Omega_t} |Dv|^p \le C^* \left(\frac{\varepsilon + s^2}{s-t}\right)^p \int_{\Omega_s \setminus \Omega_t} |Dv|^p. \tag{2.3}$$

Let $t_0 = r \in (\sqrt{\varepsilon}, 1/2)$ and $t_j = (1 - jr)r$ for $j \in \mathbb{N}$ such that $j \le 1/r$. Taking $s = t_j, t = t_{j+1}$ in (2.3), we have

$$\int_{\Omega_{t_{j+1}}} |Dv|^p \le 2^p C^* \int_{\Omega_{t_j} \setminus \Omega_{t_{j+1}}} |Dv|^p.$$

Adding both sides by $2^p C^* \int_{\Omega_{t_{j+1}}} |Dv|^p$ and dividing both sides by $1 + 2^p C^*$, we have

$$\int_{\Omega_{t_{j+1}}} |Dv|^p \le \frac{2^p C^*}{1 + 2^p C^*} \int_{\Omega_{t_j}} |Dv|^p.$$

Let $k = \lfloor \frac{1}{2r} \rfloor$ and iterate the above inequality k times. We have

$$\int_{\Omega_{r/2}} |Dv|^p \le \left(\frac{2^p C^*}{1 + 2^p C^*}\right)^k \int_{\Omega_r} |Dv|^p \le C\mu^{\frac{1}{r}} \int_{\Omega_1 \setminus \Omega_{1/2}} |v|^p, \tag{2.4}$$

where $\mu \in (0, 1)$ and C are constants depending only on n, p, c_1, and c_2. Now we take $r = 4(\sqrt{\varepsilon} + |x'|)$ and (2.2) follows from classical estimates for the p-Laplace equation in $\Omega_{x',\varepsilon+|x'|^2}$ and (2.4). □

Next, we derive a pointwise upper bound of the gradient in terms of $U_1^\varepsilon - U_2^\varepsilon$.

Proposition 2.2 *Let h_1, h_2 be C^2 functions satisfying (1.7)–(1.9), $p > 1$, $n \ge 2$, $\varepsilon \in [0, 1)$, U_1^ε, U_2^ε be arbitrary constants, and $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ be a solution of*

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2} Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ u_\varepsilon = U_1^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_\varepsilon = U_2^\varepsilon & \text{on } \Gamma_-^\varepsilon. \end{cases}$$

Then there exist constants $C_1, C_2 > 0$ *depending only on* n, p, c_1, *and* c_2, *such that for* $x \in \Omega^{\varepsilon}_{1/4}$, *it holds that*

$$|Du_{\varepsilon}(x)| \le C_1\Big(\frac{|U_1^{\varepsilon} - U_2^{\varepsilon}|}{\varepsilon + |x'|^2} + \|u_{\varepsilon}\|_{L^{\infty}(\Omega_1^{\varepsilon}\setminus\Omega_{1/2}^{\varepsilon})} e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}}\Big). \tag{2.5}$$

Moreover, if $\varepsilon \in (0, 1)$, $u_{\varepsilon} \in W^{1,p}(\Omega)$ *is the solution to* (1.2) *and* U_1^{ε}, *and* U_2^{ε} *are the same constants in* (1.2)*, we have*

$$\inf_{\partial\Omega} \varphi \le U_1^{\varepsilon},\ U_2^{\varepsilon} \le \sup_{\partial\Omega} \varphi, \tag{2.6}$$

and for $x \in \Omega^{\varepsilon}_{1/4}$,

$$|Du_{\varepsilon}(x)| \le C_1\Big(\frac{|U_1^{\varepsilon} - U_2^{\varepsilon}|}{\varepsilon + |x'|^2} + \|\varphi\|_{L^{\infty}(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}}\Big). \tag{2.7}$$

Proof We first give the proof of (2.5). Take a point $x_0 \in \Omega^{\varepsilon}_{1/4}$. In order to estimate the gradient at x_0, we first estimate the oscillation of u_{ε} in $\Omega^{\varepsilon}_{x_0,\underline{\delta}(x_0)/8}$, where $\underline{\delta}(x_0) = \varepsilon + |x_0'|^2$. Without loss of generality, we may assume that $U_1^{\varepsilon} \ge U_2^{\varepsilon}$. Let v be the solution to

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = 0 & \text{in } \Omega_1^{\varepsilon}, \\ v = 0 & \text{on } \Gamma_{\pm}^{\varepsilon}, \\ v = u_{\varepsilon} - U_1^{\varepsilon} & \text{on } \partial\Omega_1^{\varepsilon} \cap \{x \in \mathbb{R}^n : |x'| = 1\}. \end{cases}$$

By Lemma 2.1,

$$|v(x)| \le C_1 e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}} \|u_{\varepsilon}\|_{L^{\infty}(\Omega_1^{\varepsilon}\setminus\Omega_{1/2}^{\varepsilon})} \quad \text{for } x \in \Omega^{\varepsilon}_{1/2}.$$

Since $v \ge u_{\varepsilon} - U_1^{\varepsilon}$ on $\partial\Omega_1^{\varepsilon}$, by the comparison principle, we have

$$u_{\varepsilon}(x) - U_1^{\varepsilon} \le v(x) \quad \text{in } \Omega_1^{\varepsilon}.$$

Similarly, let w be the solution to

$$\begin{cases} -\operatorname{div}(|Dw|^{p-2}Dw) = 0 & \text{in } \Omega_1^{\varepsilon}, \\ w = 0 & \text{on } \Gamma_{\pm}^{\varepsilon}, \\ w = u_{\varepsilon} - U_2^{\varepsilon} & \text{on } \partial\Omega_1^{\varepsilon} \cap \{x \in \mathbb{R}^n : |x'| = 1\}. \end{cases}$$

We have

$$|w(x)| \le C_1 e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}} \|u_{\varepsilon}\|_{L^{\infty}(\Omega_1^{\varepsilon}\setminus\Omega_{1/2}^{\varepsilon})} \quad \text{for } x \in \Omega^{\varepsilon}_{1/2},$$

and

$$u_\varepsilon(x) - U_2^\varepsilon \geq w(x) \quad \text{in } \Omega_1^\varepsilon.$$

Therefore,

$$\operatorname*{osc}_{\Omega^\varepsilon_{x_0,\underline{\delta}(x_0)/8}} u_\varepsilon \leq |U_1^\varepsilon - U_2^\varepsilon| + C_1\|u_\varepsilon\|_{L^\infty(\Omega_1^\varepsilon\setminus\Omega_{1/2}^\varepsilon)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x_0'|}}.$$

Then the gradient estimate (2.5) follows from classical boundary and interior estimates for the p-Laplace equation (see e.g. [37]).

Next we prove (2.6). Indeed, if $U_i^\varepsilon = \max\{U_1^\varepsilon, U_2^\varepsilon\} > \sup_{\partial\Omega}\varphi$, by the maximum principle and the Hopf lemma (see [42, Theorem 5]), $Du_\varepsilon \cdot \nu > 0$ on $\partial\mathcal{D}_i^\varepsilon$, which violates $(1.2)_3$.

Finally, (2.7) follows directly from (2.5), (2.6), and the maximum principle. □

Remark 2.3 For systems of p-Laplace type, instead of (2.6), one can still show that

$$|U_i^\varepsilon| \leq C\|\varphi\|_{C^1(\partial\Omega)}, \quad i = 1, 2$$

holds for some ε-independent constant C, by using classical boundary and interior gradient estimates away from the neck region $\Omega_{1/2}^\varepsilon$. However, it is not clear to us whether (2.7) (or a weaker version of it) is still true.

In the following, we justify the equivalence between the minimizing problem (1.3) with $\varepsilon = 0$ and the equations (1.4)–(1.5).

Theorem 2.4 *u_0 is the minimizer of* (1.3) *with $\varepsilon = 0$ if and only if $u_0 \in W^{1,p}(\Omega)$ satisfies* (1.4) *when $p \geq (n+1)/2$ and satisfies* (1.5) *when $p < (n+1)/2$.*

Proof First, we prove that (1.4) has at most one solution $u \in W^{1,p}(\Omega)$. The same conclusion applies to (1.5). Let $u_1, u_2 \in W^{1,p}(\Omega)$ be two solutions of (1.4). Multiplying the equation by $u_1 - u_2$ and integrating by parts, we have for $j = 1, 2$,

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}^0} |Du_j|^{p-2}Du_j \cdot D(u_1-u_2)\,dx - \int_{\partial\Omega} |Du_j|^{p-2}Du_j\cdot\nu(u_1-u_2)\,dS \\ &\quad - \sum_{i=1}^2 \int_{\partial D_i^0} |Du_j|^{p-2}Du_j\cdot\nu(u_1-u_2)\,dS \\ &= \int_{\widetilde{\Omega}^0} |Du_j|^{p-2}Du_j\cdot D(u_1-u_2)\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}^0} \Big(|Du_1|^{p-2}Du_1 - |Du_2|^{p-2}Du_2\Big)\cdot D(u_1-u_2)\,dx \\ &\geq \frac{\min\{1, p-1\}}{2^{p-2}} \int_{\widetilde{\Omega}^0} (|Du_1| + |Du_2|)^{p-2}|Du_1 - Du_2|^2\,dx. \end{aligned}$$

This implies $u_1 \equiv u_2$. It is straightforward to see that the minimizer of (1.3) with $\varepsilon = 0$ is unique, due to the convexity of I_p and $\mathscr{A}^0$. It suffices to show that the minimizer u_0 satisfies (1.4) when $p \geq (n+1)/2$ and satisfies (1.5) when $p < (n+1)/2$. We show this by taking different test function $v \in \mathscr{A}^0$ in the equation

$$0 = \frac{d}{dt} I_p[u_0 + tv]\Big|_{t=0}. \tag{2.8}$$

First we take $v \in C_c^\infty(\widetilde{\Omega}^0)$. Then (2.8) reads as

$$0 = \int_{\widetilde{\Omega}^0} |Du_0|^{p-2} Du_0 \cdot Dv \, dx.$$

This implies

$$-\operatorname{div}(|Du_0|^{p-2} Du_0) = 0 \quad \text{in } \widetilde{\Omega}^0.$$

Next, we take $v \in C_c^\infty(\Omega)$ such that $v = 1$ in $\overline{\mathcal{D}_1^0 \cup \mathcal{D}_2^0}$. From (2.8) and integration by parts, we have

$$\begin{aligned} 0 &= \int_{\widetilde{\Omega}^0} |Du_0|^{p-2} Du_0 \cdot Dv \, dx \\ &= -\int_{\widetilde{\Omega}^0} \operatorname{div}(|Du_0|^{p-2} Du_0) v \, dx + \sum_{i=1}^2 \int_{\partial \mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu \, dS \\ &= \sum_{i=1}^2 \int_{\partial \mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu \, dS. \end{aligned}$$

For the case when $p \geq (n+1)/2$, it remains to show that u_0 equals to the same constant on $\overline{\mathcal{D}_1^0}$ and $\overline{\mathcal{D}_2^0}$. Assume that $u_0 = U_1$ in $\overline{\mathcal{D}_1^0}$ and $u_0 = U_2$ in $\overline{\mathcal{D}_2^0}$ with $U_1 \neq U_2$. Then by the fundamental theorem of calculus,

$$U_1 - U_2 = \int_{h_2(x')}^{h_1(x')} D_n u_0(x) \, dx_n.$$

Taking the absolute value and raising to the power of p on the both sides, by Hölder's inequality, (1.7), and (1.9), we have

$$|U_1 - U_2|^p \leq C|x'|^{2(p-1)} \int_{h_2(x')}^{h_1(x')} |D_n u_0(x)|^p \, dx_n.$$

This implies

$$\int_{|x'|<1/2} \frac{|U_1 - U_2|^p}{|x'|^{2(p-1)}} \, dx' \leq C \int_{\Omega_{1/2}^0} |Du_0|^p \, dx.$$

The left-hand side diverges since $p \geq (n+1)/2$ and $U_1 \neq U_2$, which leads to a contradiction. Therefore, $U_1 = U_2$.

For the case when $p < (n+1)/2$, we need to show the flux on each of $\partial\mathcal{D}_i^0$ vanishes. We will only show the flux vanishes on $\partial\mathcal{D}_1^0$, as a similar argument applies to $\partial\mathcal{D}_2^0$. Let v be a function compactly supported in Ω such that $v = 1$ in $\overline{\mathcal{D}_1^0}$, $v = 0$ in $\overline{\mathcal{D}_2^0}$,

$$v(x', x_n) = \left(\frac{2x_n - (h_1(x') + h_2(x'))}{h_1(x') - h_2(x')}\right)_+, \quad x \in \Omega_{1/2}^0,$$

and v is smooth in $\widetilde{\Omega}^0 \setminus \Omega_{1/2}^0$. Then $v \in \mathscr{A}^0$. Indeed, we only need to verify

$$\int_{\Omega_{1/2}^0} |Dv|^p \, dx \leq C \int_{|x'|<1/2} \int_{\frac{h_1(x')+h_2(x')}{2}}^{h_1(x')} \frac{1}{|x'|^{2p}} \, dx_n dx' \leq C \int_0^{1/2} r^{n-2p} \leq C$$

since $n - 2p > -1$. Taking this v in (2.8) and integrating by parts, we have

$$\int_{\partial\mathcal{D}_1^0} |Du_0|^{p-2} Du_0 \cdot \nu \, dS = 0.$$

The theorem is proved. □

Next, we show that u_ε converges to u_0 in the following sense.

Theorem 2.5 *Let $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (1.2), and $u_0 \in W^{1,p}(\Omega)$ be the minimizer of (1.3) with $\varepsilon = 0$. Then as $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$, and $u_\varepsilon \to u_0$ strongly in $C^{1,\beta}(K)$ for some $\beta > 0$ and any*

$$K \subset\subset \Omega \setminus \Big(\bigcup_{0<\varepsilon\leq\varepsilon_0} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup \{0\} \Big) \quad \textit{with } \varepsilon_0 > 0.$$

As a consequence, as $\varepsilon \to 0$, $U_1^\varepsilon \to U_1$ and $U_2^\varepsilon \to U_2$ when $p < (n+1)/2$, and $U_1^\varepsilon, U_2^\varepsilon \to U_0$ when $p \geq (n+1)/2$, where U_0, U_1, U_2 are the constants in (1.4) and (1.5).

Proof First we take an arbitrary function $w \in W^{1,p}(\Omega)$ such that $w = \varphi$ on $\partial\Omega$ and $Dw = 0$ in $\mathcal{B}$, where $\mathcal{B} \subset \Omega$ is an open set containing $\cup_{0\leq\varepsilon\leq c}\overline{\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon}$, where $0 < c < \operatorname{dist}(\mathcal{D}_1^0 \cup \mathcal{D}_2^0, \partial\Omega)$. Therefore, $w \in \mathscr{A}^\varepsilon$ for all $\varepsilon \in [0, c]$, where $\mathscr{A}^\varepsilon$ is the admissible set defined in (1.3). By Theorem 2.4,

$$\|Du_\varepsilon\|_{L^p(\Omega)} \leq \|Dw\|_{L^p(\Omega)}.$$

This together with the Poincaré inequality implies that $\|u_\varepsilon\|_{W^{1,p}(\Omega)}$ is bounded uniformly in ε. Then there exists a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ and a function $u_* \in W^{1,p}(\Omega)$, such that $u_{\varepsilon_j} \rightharpoonup u_*$ weakly in $W^{1,p}(\Omega)$, and $u_{\varepsilon_j} \to u_*$ strongly in $L^p(\Omega)$, as $j \to \infty$. From (2.6), we know that $U_1^{\varepsilon_j}$ and $U_2^{\varepsilon_j}$, the values of u_{ε_j} in $\mathcal{D}_1^{\varepsilon_j}$ and $\mathcal{D}_2^{\varepsilon_j}$, are uniformly bounded. Then there exists a subsequence of $\{\varepsilon_j\}_{j\in\mathbb{N}}$, still denoted by $\{\varepsilon_j\}_{j\in\mathbb{N}}$, so that $U_i^{\varepsilon_j} \to U_i$ for some constants U_i, $i = 1, 2$. Therefore, for any $x \in \mathcal{D}_i^0$, we have

$$u_*(x) = \lim_{j\to\infty} U_i^{\varepsilon_j} = U_i, \quad i = 1, 2.$$

On the other hand, for any $k, l \in \mathbb{Z}_+$, we denote

$$\mathcal{K}_{k,l} := \Omega\backslash\Big(\bigcup_{0<\varepsilon\le 1/k} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup B_{1/l}(0)\Big).$$

By the classical $C^{1,\alpha}$ estimate, when $\varepsilon_j \le 1/k$, we have

$$\|u_{\varepsilon_j}\|_{C^{1,\alpha}(\overline{\mathcal{K}}_{k,l})} \le C(k, l).$$

This implies that there is a subsequence that converges in $C^{1,\beta}(\overline{\mathcal{K}}_{k,l})$ for any $\beta < \alpha$. We can apply the Cantor diagonal argument to select a subsequence, still denoted by $\{\varepsilon_j\}_{j\in\mathbb{N}}$, such that

$$u_{\varepsilon_j} \to u_{**} \ \text{ in } \ C^{1,\beta}(K) \ \text{ as } \ j \to \infty \tag{2.9}$$

for any $K \subset\subset \Omega\backslash\Big(\bigcup_{0<\varepsilon\le\varepsilon_0} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup \{0\}\Big)$ with $\varepsilon_0 > 0$ and some $C^{1,\beta}$ function u_{**}. Therefore, u_{**} is a weak solution to the p-Laplace equation in $\widetilde{\Omega}^0$. Since $u_\varepsilon \to u_*$ strongly in $L^p(\Omega)$, we have $u_* \equiv u_{**}$. It remains to show that $u_* \equiv u_0$, which implies the convergence of $\{u_\varepsilon\}$.

When $p \ge (n + 1)/2$, by the same argument as in the proof of Theorem 2.4, we know that $U_1 = U_2$. It remains to prove that

$$\int_{\partial\mathcal{D}_1^0\cup\partial\mathcal{D}_2^0} |Du_*|^{p-2} Du_* \cdot \nu = 0. \tag{2.10}$$

Let Ω' be an open set such that $\mathcal{D}_1^0 \cup \mathcal{D}_2^0 \subset\subset \Omega' \subset\subset \Omega$. Since u_{ε_j} is the solution to (1.2), when j is sufficiently large, by integration by parts in $\Omega'\backslash\overline{\mathcal{D}_1^{\varepsilon_j} \cup \mathcal{D}_2^{\varepsilon_j}}$, we have

$$\int_{\partial\Omega'} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0.$$

Therefore, by (2.9), we also have

$$\int_{\partial\Omega'} |Du_*|^{p-2} Du_* \cdot \nu = 0. \tag{2.11}$$

Since u_* is a weak solution to the p-Laplace equation in $\widetilde{\Omega}^0$, by integration by parts again in $\Omega'\backslash\overline{\mathcal{D}_1^0 \cup \mathcal{D}_2^0}$, (2.11) directly implies (2.10).

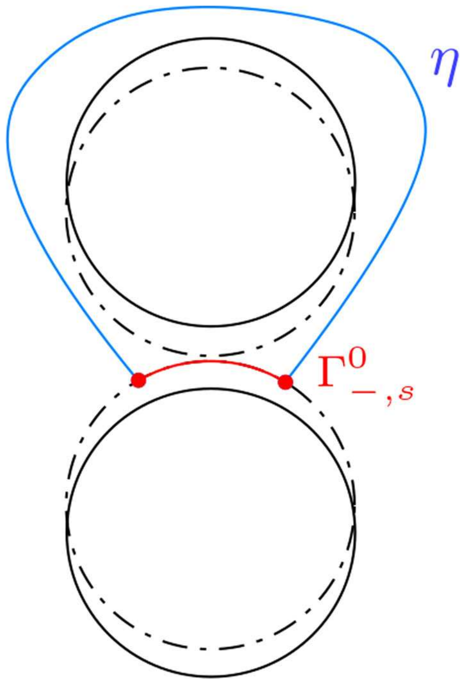

Fig. 1 An illustration of $\Gamma^0_{-,s} \cup \eta$

When $p < (n+1)/2$, it remains to prove that

$$\int_{\partial \mathcal{D}_i^0} |Du_*|^{p-2} Du_* \cdot \nu = 0, \quad i = 1, 2.$$

We will prove it only for $i = 1$. Fix a small $s \in (0, 1/2)$, we take a smooth surface η so that $\mathcal{D}_1^\varepsilon$ is surrounded by $\Gamma^0_{-,s} \cup \eta$. See Fig. 1.

Since $\int_{\partial \mathcal{D}_1^{\varepsilon_j}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0$, by integration by parts, we have

$$-\int_{\Gamma^0_{-,s}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu + \int_\eta |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0.$$

Note that the minus sign appears because ν on $\Gamma^0_{-,s}$ is pointing upwards, while ν on η is pointing away from $\mathcal{D}_1^\varepsilon$. By (2.7), we have $|Du_{\varepsilon_j}(x)| \leq C(\varepsilon_j + |x'|^2)^{-1}$. Therefore,

$$\left| \int_{\Gamma^0_{-,s}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu \right| \leq C \int_{|x'|<s} \frac{1}{|x'|^{2p-2}}\, dx' \leq C s^{n-2p+1},$$

where we used $2p - 2 < n - 1$. By (2.9), we know that

$$\int_\eta |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu \to \int_\eta |Du_*|^{p-2} Du_* \cdot \nu \quad \text{as} \quad j \to \infty.$$

Therefore,

$$\left|\int_{\eta} |Du_*|^{p-2} Du_* \cdot \nu\right| \leq C s^{n-2p+1}.$$

Similarly by (2.5), we have $|Du_*(x)| \leq C|x'|^{-2}$ and

$$\left|\int_{\Gamma^0_{-,s}} |Du_*|^{p-2} Du_* \cdot \nu\right| \leq C s^{n-2p+1}.$$

By integration by parts, we have

$$\begin{aligned}\left|\int_{\mathcal{D}_1^0} |Du_*|^{p-2} Du_* \cdot \nu\right| &= \left|\int_{\eta} |Du_*|^{p-2} Du_* \cdot \nu - \int_{\Gamma^0_{-,s}} |Du_*|^{p-2} Du_* \cdot \nu\right| \\ &\leq C s^{n-2p+1}.\end{aligned}$$

Sending $s \to 0$ and using $p < (n+1)/2$, we have

$$\int_{\mathcal{D}_1^0} |Du_*|^{p-2} Du_* \cdot \nu = 0.$$

Finally, by the uniqueness of solution to (1.4) and (1.5), we can conclude that $u_* \equiv u_0$, and the full sequence u_ε converges to u_0 in the corresponding topology. □

3 Mean oscillation estimates

In this section, we give the proof of Proposition 1.5 using mean oscillation estimates. Throughout this section, unless otherwise specified, we use C to denote positive constants depending only on n, p, c_1, and c_2, which could differ from line to line. Here c_1 and c_2 are the same constants in (1.8) and (1.9), respectively. For simplicity, we denote $u := u_\varepsilon$ and we omit the superscript ε throughout this section when there is no confusion.

First, we fix a point $\bar{x} \in \Omega_{1/2}$ and derive some mean oscillation estimates of Du on a ball intersecting Ω_1, namely $\Omega_r(\bar{x})$, for different radii r.

3.1 Mean oscillation estimates for small *r*

We recall a classical interior mean oscillation estimate when $B_r(\bar{x}) \subset \Omega_1$. Estimates of this type, with different exponents involved, were developed in [16, 22, 38].

Lemma 3.1 *Let $u \in W^{1,p}(\Omega_1)$ be a solution to* (1.2). *There exist constants $C > 1$ and $\alpha \in (0,1)$ depending only on n and p, such that $u \in C^{1,\alpha}(\Omega_1)$ and for every*

$B_r(\bar{x}) \subset \Omega_1$ and $\rho \in (0, r]$, we have

$$\phi(\bar{x}, \rho) \leq C \left(\frac{\rho}{r}\right)^{\alpha} \phi(\bar{x}, r),$$

where we denote

$$\phi(\bar{x}, r) = \left(\fint_{\Omega_r(\bar{x})} |Du - (Du)_{\Omega_r(\bar{x})}|^p \right)^{\frac{1}{p}}. \tag{3.1}$$

3.2 Mean oscillation estimates for intermediate *r*

Next, we consider the case when $B_r(\bar{x})$ intersects with only one of Γ_+ and Γ_-. In this case, we derive mean oscillation estimates around any $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$.

Without loss of generality, let $\hat{x} \in \Gamma_- \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$. Then by (1.8) and (1.9), there exists a constant $c = c(n, c_1, c_2) \in (0, \min\{c_1/4, 1/4\})$, such that $B(\hat{x}, r) \cap \Gamma_+ = \emptyset$ for any $r \in (0, c\underline{\delta}(\hat{x}))$. Here we recall (1.10). We first choose a coordinate system $y = (y', y_n)$ such that $y(\hat{x}) = 0$, the direction of axis y_n is the normal vector at $\hat{x} \in \Gamma_-$ pointing upwards. Note that the coordinate y is a rotation (plus a transition) of the coordinate x, namely $y = T(x - \hat{x})$ for some rotation matrix $T \in \mathbb{R}^{n\times n}$, which maps the normal vector of Γ_- at $\hat{x}$ pointing upward, namely $\nu = (-Dh_2(\hat{x}'), 1)/\sqrt{1 + |Dh_2(\hat{x}')|^2}$ in the x-coordinate, to the unit vector $\mathbf{e}_{y_n} = (0, \ldots, 0, 1)$ in y-coordinate. Therefore, by (1.7) and (1.9), there exists a constant $C = C(n, c_2) > 0$, such that

$$|T - I_n| \leq C|\hat{x}'| \quad \text{and} \quad |T^{-1} - I_n| \leq C|\hat{x}'|. \tag{3.2}$$

Thus there exists a constant $c_3 = c_3(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\})$ such that $\Omega_{R_0}(\hat{x}) = \{\hat{x} + T^{-1}y : y \in B_{R_0} : y_n > \chi(y')\}$, where $R_0 = c_3\underline{\delta}(\hat{x}) \in (0, 1/4)$ and $\chi : \{y' \in \mathbb{R}^{n-1} : |y'| < R_0\} \to \mathbb{R}$ is a C^2 function in the y-coordinate system such that

$$\chi(0') = 0, \quad D_{y'}\chi(0') = 0, \quad \|\chi\|_{C^2} \leq C\|h_2\|_{C^2}, \tag{3.3}$$

for some constant $C = C(n) > 0$. Then we let

$$z = \Lambda(y) = (y', \ y_n - \chi(y')).$$

Since Γ_- is C^2, by (3.3) there exist constants

$$c_4 = c_4(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\}), \tag{3.4}$$

$C = C(n, c_1, c_2) > 0$, and $R_1 = c_4\underline{\delta}(\hat{x})$ such that

$$|D_{y'}\chi(y')| \leq C\,|y'| \leq 1/2 \quad \text{if} \quad |y'| \leq 2R_1,$$
$$\Omega_{r/2}(\hat{x}) \subset \Lambda^{-1}(B_r^+) \subset \Omega_{2r}(\hat{x}) \quad \forall\, r \in (0, 2R_1],$$

and thus

$$|D\Lambda(y) - I_n| \leq C\,|y'| \leq 1/2 \quad \text{if} \quad |y'| \leq 2R_1. \tag{3.5}$$

Therefore, there exist positive constants $c(n)$ and $c'(n)$ depending only on n, such that for any $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$ and $0 < r \leq c_4\underline{\delta}(\hat{x})$,

$$c(n)r^n \leq |\Omega_r(\hat{x})| \leq c'(n)r^n. \tag{3.6}$$

Note that

$$\det(D\Lambda) \equiv 1.$$

Then $u_1(z) := u(\Lambda^{-1}(z))$ satisfies the following equation with constant Dirichlet boundary condition

$$\begin{cases} -\operatorname{div}_z\left(|A^T D_z u_1|^{p-2} A A^T D_z u_1\right) = 0 & \text{in } B_{R_1}^+, \\ u_1 = U_2^{\varepsilon} & \text{on } B_{R_1} \cap \partial\mathbb{R}_+^n, \end{cases} \tag{3.7}$$

where we denote

$$A := A(z) := (a_{ij}(z)) := D\Lambda(\Lambda^{-1}(z)).$$

Next we extend the equation to the whole ball B_{R_1}. We take the even extension of a_{nn} and a_{ij}, $i, j = 1, 2, \ldots, n-1$, with respect to $z_n = 0$, and take the odd extension of a_{in} and a_{ni}, $i = 1, 2, \ldots, n-1$, with respect to $z_n = 0$. Then we reflect u_1 with respect to $z_n = 0$. Namely, we define $u_1(z) = 2\,U_2^{\varepsilon} - u_1(z', -z_n)$ for $z \in B_{R_1}^-$. We still denote these functions by u_1 and A after the extension. Because of the Dirichlet boundary condition, it is easily seen that u_1 satisfies

$$-\operatorname{div}_z\left(\mathbf{A}(z, D_z u_1)\right) = 0 \quad \text{in } B_{R_1}, \tag{3.8}$$

where the nonlinear operator $\mathbf{A}$ is defined as

$$\mathbf{A}(z,\xi) = |A^T\xi|^{p-2}AA^T\xi \qquad \text{for } z \in B_{R_1},\ \xi \in \mathbb{R}^n.$$

By (3.5), similar to [20, Lemma 2.3], there exists a constant $C = C(n, p, c_1, c_2) > 0$, such that for any $z \in B_{R_1}$ and $\xi \in \mathbb{R}^n$,

$$|\mathbf{A}(z,\xi) - |\xi|^{p-2}\xi| \le C\,|z'|\,|\xi|^{p-1}. \tag{3.9}$$

Assume that $r \in (0, R_1]$. We let $v_1 \in u_1 + W_0^{1,p}(B_r)$ be the unique solution to

$$\begin{cases} -\operatorname{div}_z(|D_z v_1|^{p-2}D_z v_1) = 0 & \text{in } B_r, \\ v_1 = u_1 & \text{on } \partial B_r. \end{cases} \tag{3.10}$$

By testing (3.10) and (3.8) with $v_1 - u_1$ and using (3.9), we have the comparison estimate

$$\fint_{B_r} |D_z u_1 - D_z v_1|^p \le C r^{\min\{2,p\}} \fint_{B_r} |D_z u_1|^p, \tag{3.11}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. For detailed proof of (3.11), see [22, Eq. (4.35)] when $p \in (1, 2)$ and [23, Lemma 3.4] when $p \ge 2$.

Applying Lemma 3.1 and the comparison estimate (3.11), we have

Lemma 3.2 *Suppose that $u_1 \in W^{1,p}(B^+_{R_1})$ is a solution to (3.7). Then for any $\mu \in (0, 1)$ and $r \in (0, R_1]$, we have*

$$\begin{aligned}&\left(\fint_{B^+_{\mu r}} |D_{z'}u_1|^p + |D_{z_n}u_1 - (D_{z_n}u_1)_{B^+_{\mu r}}|^p\right)^{1/p} \\ &\le C\mu^\alpha \left(\fint_{B^+_r} |D_{z'}u_1|^p + |D_{z_n}u_1 - (D_{z_n}u_1)_{B^+_r}|^p\right)^{1/p} \\ &\quad + C_\mu r^{\theta_p}\left(\fint_{B^+_r} |D_z u_1|^p\right)^{1/p},\end{aligned} \tag{3.12}$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 3.1, C_μ is a constant depending on μ, n, p, c_1, and c_2, and C is a constant depending on n, p, c_1, and c_2.

Proof By Lemma 3.1, (3.11), and the triangle inequality, we have

$$\begin{aligned}
&\left(\fint_{B_{\mu r}} |D_z u_1 - (D_z u_1)_{B_{\mu r}}|^p\right)^{1/p}\\
&\leq C\left(\fint_{B_{\mu r}} |D_z v_1 - (D_z v_1)_{B_{\mu r}}|^p\right)^{1/p} + C\left(\fint_{B_{\mu r}} |D_z u_1 - D_z v_1|^p\right)^{1/p}\\
&\leq C\mu^\alpha\left(\fint_{B_r} |D_z v_1 - (D_z v_1)_{B_r}|^p\right)^{1/p} + C\mu^{-\frac{n}{p}}\left(\fint_{B_r} |D_z u_1 - D_z v_1|^p\right)^{1/p}\\
&\leq C\mu^\alpha\left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p\right)^{1/p} + C\mu^{-\frac{n}{p}}\left(\fint_{B_r} |D_z u_1 - D_z v_1|^p\right)^{1/p}\\
&\leq C\mu^\alpha\left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p\right)^{1/p} + C_\mu r^{\theta_p}\left(\fint_{B_r} |D_z u_1|^p\right)^{1/p}. \qquad (3.13)
\end{aligned}$$

Since u_1 is even in z_n, (3.13) directly implies (3.12). The proof is completed. □

We now define

$$\psi(\hat{x}, r) = \left(\fint_{\Omega_r(\hat{x})} |D_{y'} u|^p + |D_{y_n} u - (D_{y_n} u)_{\Omega_r(\hat{x})}|^p\right)^{1/p}. \qquad (3.14)$$

Following a similar argument as in the proof of [20, Lemma 2.5], we have

Lemma 3.3 *Suppose that u is a solution to (1.2) and $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$. Then there exist constants $C > 0$ depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0, 1/4)$ and $r \in (0,\ c_4\underline{\delta}(\hat{x})]$, it holds that*

$$\psi(\hat{x}, \mu r) \leq C\mu^\alpha \psi(\hat{x}, r) + C_\mu r^{\theta_p}\left(\fint_{\Omega_r(\hat{x})} |Du|^p\right)^{1/p}, \qquad (3.15)$$

where $\theta_p = \min\{1, 2/p\}$, $\alpha \in (0, 1)$ is the same constant as in Lemma 3.1, $c_4 = c_4(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\})$ is the same constant as in (3.4) and ψ is defined in (3.14).

By iteration, Lemma 3.3 also implies

Corollary 3.4 *Let u, $\hat{x}$, c_4, α, and θ_p be as in Lemma 3.3 and $\alpha_1 \in (0, \min\{\alpha, \theta_p\})$. Then there exists a constant $C > 0$ depending only on n, p, c_1, c_2, and α_1, such that for any $0 < \rho \leq r \leq c_4\underline{\delta}(\hat{x})$, it holds that*

$$\psi(\hat{x}, \rho) \leq C\left(\frac{\rho}{r}\right)^{\alpha_1} \psi(\hat{x}, r) + C\,\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_r(\hat{x}))}.$$

Proof We choose $\mu = \mu(n, p, c_1, c_2, \alpha_1) \in (0, 1/6)$ sufficiently small such that $C\mu^{\alpha-\alpha_1} < 1$, where C is the same constant in (3.15). Then Lemma 3.3 implies that for any $r \in (0,\ c_4\underline{\delta}(\hat{x})]$, we have

$$\psi(\hat{x}, \mu r) \leq \mu^{\alpha_1}\psi(\hat{x}, r) + C_{\alpha_1} r^{\theta_p}\|Du\|_{L^\infty(\Omega_r(\hat{x}))}, \tag{3.16}$$

where $C_{\alpha_1} > 0$ is a constant depending only on n, p, c_1, c_2, and α_1. By iteration, from (3.16) we get

$$\begin{aligned}\psi(\hat{x}, \mu^j r) &\leq \mu^{\alpha_1 j}\psi(\hat{x}, r) + C_{\alpha_1}\sum_{i=1}^{j}\mu^{\alpha_1(i-1)}(\mu^{j-i}r)^{\theta_p}\|Du\|_{L^\infty(\Omega_r(\hat{x}))}\\ &= \mu^{\alpha_1 j}\psi(\hat{x}, r) + C_{\alpha_1}\sum_{i=1}^{j}\mu^{\alpha_1(j-1)}\mu^{(j-i)(\theta_p-\alpha_1)}r^{\theta_p}\|Du\|_{L^\infty(\Omega_r(\hat{x}))}\\ &\leq \mu^{\alpha_1 j}\psi(\hat{x}, r) + C_{\alpha_1}(\mu^j r)^{\alpha_1}\|Du\|_{L^\infty(\Omega_r(\hat{x}))}.\end{aligned} \tag{3.17}$$

Here in the last inequality we used the facts that $\alpha_1 < \theta_p$ and $r \in (0, 1)$.

Now for any $0 < \rho \leq r \leq c_4\underline{\delta}(\hat{x})$, let j be the integer such that $\mu^{j+1} < \rho/r \leq \mu^j$. Then by (3.17) with $\mu^{-j}\rho$ in place of r, we get

$$\begin{aligned}\psi(\hat{x}, \rho) &\leq \mu^{\alpha_1 j}\psi(\hat{x}, \mu^{-j}\rho) + C_{\alpha_1}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\mu^{-j}\rho}(\hat{x}))}\\ &\leq C_{\alpha_1}\left(\frac{\rho}{r}\right)^{\alpha_1}\psi(\hat{x}, r) + C_{\alpha_1}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_r(\hat{x}))},\end{aligned}$$

where $C_{\alpha_1} > 0$ is a constant depending only on n, p, c_1, c_2, and α_1. The proof is completed. □

3.3 Mean oscillation estimates for large *r*

Finally, we consider the case when $B_r(\bar{x})$ could potentially intersects with both Γ_+ and Γ_-. In this case, we fix $\bar{x} \in \Omega_{1/2}$ and assume $\frac{c_4}{216}\underline{\delta}(\bar{x}) \leq r \leq c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$, where $\underline{\delta}(\bar{x})$ is defined in (1.10), c_4 is the same constant as in (3.4), and c_5 is a constant which will be determined later. We define the map $\mathcal{Z} = \tilde{\Lambda}_{\bar{x}}(x)$ by

$$\begin{cases}\mathcal{Z}' = x',\\ \mathcal{Z}_n = (h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big(\dfrac{x_n - h_2(x') + \varepsilon/2}{h_1(x') - h_2(x') + \varepsilon} - \dfrac{1}{2}\Big).\end{cases} \tag{3.18}$$

Thus $\tilde{\Lambda}_{\bar{x}}$ is invertible in Ω_1,

$$Q_1 := \tilde{\Lambda}_{\bar{x}}(\Omega_1) = \Big\{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1,\ |\mathcal{Z}_n| < \frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big\},$$

and

$$\tilde{\Gamma}_{\pm} := \tilde{\Lambda}_{\bar{x}}(\Gamma_{\pm}) = \Big\{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1,\ \mathcal{Z}_n = \pm\frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big\}.$$

Then $u_2(\mathcal{Z}) := u(\tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z}))$ satisfies the following equation with constant Dirichlet boundary conditions

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}}\Big(|B^T D_{\mathcal{Z}} u_2|^{p-2}(\det(B))^{-1} B B^T D_{\mathcal{Z}} u_2\Big) = 0 & \text{in } Q_1, \\ u_2 = U_1^{\varepsilon} \quad \text{on } \tilde{\Gamma}_+, \quad u_2 = U_2^{\varepsilon} & \text{on } \tilde{\Gamma}_-, \end{cases}$$

where we denote

$$B := B_{\bar{x}} := B(\mathcal{Z}) := (b_{ij}(\mathcal{Z})) := D\tilde{\Lambda}_{\bar{x}}(\tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})).$$

For $\mathcal{Z} \in Q_1$, let $x = \tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})$. Then

$$\begin{aligned} b_{ii}(\mathcal{Z}) &= 1 \quad \text{for } i \in \{1, 2, \ldots, n-1\}, \\ b_{ij}(\mathcal{Z}) &= 0 \quad \text{for } i \in \{1, 2, \ldots, n-1\},\ j \in \{1, 2, \ldots, n\},\ i \neq j, \\ b_{nj}(\mathcal{Z}) &= \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{(h_1(x') - h_2(x') + \varepsilon)^2} \\ &\quad \cdot\Big[D_{x_j} h_2(x')\big(x_n - h_1(x') - \frac{\varepsilon}{2}\big) - D_{x_j} h_1(x')\big(x_n - h_2(x') + \frac{\varepsilon}{2}\big)\Big] \end{aligned}$$

for $j \in \{1, 2, \ldots, n-1\}$, and

$$b_{nn}(\mathcal{Z}) = \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{h_1(x') - h_2(x') + \varepsilon}.$$

Therefore,

$$\det(B(\mathcal{Z})) = b_{nn}(\mathcal{Z}) = \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{h_1(\mathcal{Z}') - h_2(\mathcal{Z}') + \varepsilon}$$

is a function independent of $\mathcal{Z}_n$. Assume

$$\frac{c_4}{216}\delta(\bar{x}) \leq r \leq \frac{1}{8}\delta(\bar{x})^{\frac{1}{2}} \tag{3.19}$$

and let $\bar{\mathcal{Z}} = \tilde{\Lambda}_{\bar{x}}(\bar{x})$. When $\mathcal{Z} \in \mathbb{R}^n$ satisfies $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \leq r$, by the triangle inequality and (3.19), we always have

$$|Z'| \leq |\bar{\mathcal{Z}}'| + r \leq |\bar{x}'| + r < 1. \tag{3.20}$$

Then for any $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le r$ and $x = \tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})$, by the triangle inequality and (3.19), we have

$$|x'| \le r + |\bar{x}'| \le \big(1 + \sqrt{216/c_4}\big) r^{\frac{1}{2}} \quad \text{and} \quad |x'|^2 \ge \frac{1}{2}|\bar{x}'|^2 - r^2 \ge \frac{1}{4}(|\bar{x}'|^2 - \varepsilon).$$

Thus, using (1.7), (1.8), and (1.9), we infer that for $j = 1, 2, \ldots, n-1$ and some constants $C > 0$ depending only on n, p, c_1, and c_2,

$$\begin{aligned}
|b_{nj}(\mathcal{Z})| &\le 2c_2 \frac{|x'|(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)}{h_1(x') - h_2(x') + \varepsilon} \le 2c_2 \frac{|x'|(2c_2|\bar{x}'|^2 + \varepsilon)}{c_1|x'|^2 + \varepsilon} \\
&\le C|x'| \le Cr^{\frac{1}{2}} \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}, \\
|b_{nn}(\mathcal{Z}) - 1| &= \left| \frac{\int_0^1 \frac{d}{dt}\big(h_1(tx' + (1-t)\bar{x}') - h_2(tx' + (1-t)\bar{x}')\big)dt}{h_1(x') - h_2(x') + \varepsilon} \right| \\
&\le 2c_2 \frac{(|x'| + |\bar{x}'|)|x' - \bar{x}'|}{c_1|x'|^2 + \varepsilon} \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}},
\end{aligned} \tag{3.21}$$

and similarly,

$$\left|\big(\det(B(\mathcal{Z}))\big)^{-1} - 1\right| = \left|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\right| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}. \tag{3.22}$$

Therefore, when (3.19) holds and $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le r$, we have for some constant $C = C(n, p, c_1, c_2) > 0$,

$$|B(\mathcal{Z}) - I_n| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}. \tag{3.23}$$

In particular, there exists a constant

$$c_5 = c_5(n, p, c_1, c_2) \in (0, 1/8), \tag{3.24}$$

such that if $\frac{c_4}{216}\underline{\delta}(\bar{x}) \le c_5 \underline{\delta}(\bar{x})^{\frac{1}{2}}$ and $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le c_5 \underline{\delta}(\bar{x})^{\frac{1}{2}}$, it also holds that

$$|B(\mathcal{Z}) - I_n| \le 1/2 \quad \text{and} \quad \left|(\det(B(\mathcal{Z})))^{-1} - 1\right| \le 1/2. \tag{3.25}$$

Next we extend u_2 and B to the whole cylinder $\mathcal{C}_1 := \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1\}$. We denote $H := h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon$. We take the even extension of b_{nn}, and $b_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = H/2$, and take the odd extension of b_{in} and b_{ni}, $i = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = H/2$. Then we take the periodic extension of B in the $\mathcal{Z}_n$ axis, so that the period is equal to $2H$. Then we inductively reflect u_2 with respect to $\pm(k+\frac{1}{2})H$ for $k \in \mathbb{N}$. Namely, for $\mathcal{Z} \in \mathcal{C}_1$ and $k \in \mathbb{Z}$, we define

$$u_2(\mathcal{Z}) := \begin{cases} 2kU_1^\varepsilon - 2kU_2^\varepsilon + u_2(\mathcal{Z}_n - 2kH), & \text{if } |\mathcal{Z}_n - 2kH| \le \dfrac{H}{2}, \\ (2k+2)U_1^\varepsilon - 2kU_2^\varepsilon - u_2((2k+1)H - \mathcal{Z}_n), & \text{if } |\mathcal{Z}_n - (2k+1)H| \le \dfrac{H}{2}. \end{cases}$$

Then because of the Dirichlet boundary conditions, it is easily seen that u_2 satisfies

$$-\operatorname{div}_{\mathcal{Z}}\left(\mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2)\right) = 0 \quad \text{in } \mathcal{C}_1, \tag{3.26}$$

where the nonlinear operator $\mathbf{B}$ is defined as

$$\mathbf{B}(\mathcal{Z},\xi) = (\det(B(\mathcal{Z})))^{-1}|B^T\xi|^{p-2}BB^T\xi \qquad \text{for } \mathcal{Z} \in \mathcal{C}_1,\ \xi \in \mathbb{R}^n,$$

and

$$(\det(B(\mathcal{Z})))^{-1} = \left(b_{nn}(\mathcal{Z})\right)^{-1} = \frac{h_1(\mathcal{Z}') - h_2(\mathcal{Z}') + \varepsilon}{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}.$$

Similar to (3.9), using (3.20), (3.22), (3.23), (3.25), and (3.20), we obtain that for any $r \in \left[\frac{c_4}{216}\underline{\delta}(\bar{x}), c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}\right]$, $\mathcal{Z} \in B_r(\bar{\mathcal{Z}})$, and $\xi \in \mathbb{R}^n$,

$$|\mathbf{B}(\mathcal{Z},\xi) - |\xi|^{p-2}\xi| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}|\xi|^{p-1}, \tag{3.27}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. Now we let $v_2 \in u_2 + W_0^{1,p}(B_r(\bar{\mathcal{Z}}))$ be the unique solution to

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}}\left(|D_{\mathcal{Z}}v_2|^{p-2}D_{\mathcal{Z}}v_2\right) = 0 & \text{in } B_r(\bar{\mathcal{Z}}), \\ v_2 = u_2 & \text{on } \partial B_r(\bar{\mathcal{Z}}). \end{cases}$$

Using (3.27), similar to (3.11), we have the following comparison estimate

$$\fint_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}}u_2 - D_{\mathcal{Z}}v_2|^p \le C\left(\frac{r}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\right)^{\min\{2,p\}} \fint_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}}u_2|^p, \tag{3.28}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2.

For $\bar{x} \in \Omega_{1/2}$ and $r \in (0, \frac{1}{8}\underline{\delta}(\bar{x})^{\frac{1}{2}})$, we define

$$\tilde{\phi}(\bar{x}, r) = \left(\fint_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}}u_2 - (D_{\mathcal{Z}}u_2)_{B_r(\bar{\mathcal{Z}})}|^p\right)^{1/p}. \tag{3.29}$$

Then following the same proofs as those of Lemma 3.3 and Corollary 3.4 with (3.28) in place of (3.11), we have

Lemma 3.5 *Suppose that* $\bar{x} \in \Omega_{1/2}$ *and* u_2 *is a solution to* (3.26). *Then there exist constants* $C > 0$ *depending only on* n, p, c_1, *and* c_2, *and* $C_\mu > 0$ *depending on* n, p, c_1, c_2, *and* μ, *such that for any* $\mu \in (0, 1)$ *and* $r \in [\frac{c_4}{216}\underline{\delta}(\bar{x}), c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}]$, *it holds that*

$$\tilde{\phi}(\bar{x}, \mu r) \le C\mu^\alpha \tilde{\phi}(\bar{x}, r) + C_\mu \Big(\frac{r}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\Big)^{\theta_p} \Big(\fint_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}} u_2|^p\Big)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α *is the same constant as in Lemma* 3.1, c_4, c_5 *are the same constants as in* (3.4) *and* (3.24), *and* $\tilde{\phi}$ *is defined in* (3.29). *Moreover, for any* $\alpha_1 \in (0, \min\{\alpha, \theta_p\})$, *there exists a constant* $C > 0$ *depending only on* n, p, c_1, c_2, *and* α_1, *such that for any* $\frac{c_4}{216}\underline{\delta}(\bar{x}) \le \rho \le r \le c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$, *it holds that*

$$\tilde{\phi}(\bar{x}, \rho) \le C\left(\tfrac{\rho}{r}\right)^{\alpha_1} \tilde{\phi}(\bar{x}, r) + C\left(\frac{\rho}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\right)^{\alpha_1} \|D_{\mathcal{Z}} u_2\|_{L^\infty(B_r(\bar{\mathcal{Z}}))}. \tag{3.30}$$

3.4 Mean oscillation decay estimates

Now we deduce mean oscillation decay estimates by connecting the three different cases of radii r, when $\underline{\delta}(\bar{x})$ is sufficiently small. We let

$$\alpha_1 = \frac{1}{2}\min\left\{\alpha, \frac{2}{p}\right\} \tag{3.31}$$

and assume

$$\underline{\delta}(\bar{x}) \le \min\left\{\left(\frac{c_5}{10c_4}\right)^2, \frac{1}{4 + 4c_2}\right\}, \tag{3.32}$$

where α, c_4, and c_5 are the same constants as in Lemmas 3.1, 3.3, and 3.5, respectively.

Let $\bar{x} \in \Omega_{1/2}$. By (1.7), (1.9) and (3.32), we have

$$\operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \le \frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon) \le c_2|\bar{x}'|^2 + \varepsilon \le 1/4, \tag{3.33}$$

and thus

$$\operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) = \operatorname{dist}(\bar{x}, \partial\Omega_1). \tag{3.34}$$

Lemma 3.6 *Let $\bar{x} \in \Omega_{1/2}$, u be a solution to (1.2), and $r = \frac{1}{8}\underline{\delta}(\bar{x})^{1/2}$. Let ϕ and $\tilde{\phi}$ be defined as in (3.1) and (3.29) and let α_1 and c_4 be the same constants as in (3.31) and (3.4). Assume that (3.32) holds. Then there exists a constant $C > 0$ depending only on n, p, c_1, and c_2, such that the following holds:*

(i) *For any $\rho \in (0, \frac{c_4}{6}\underline{\delta}(\bar{x})]$,*

$$\phi(\bar{x}, \rho) \leq C \left(\frac{\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{3.35}$$

(ii) *For any $\rho \in [\frac{c_4}{216}\underline{\delta}(\bar{x}), r]$,*

$$\tilde{\phi}(\bar{x}, \rho) \leq C \left(\frac{\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{3.36}$$

Proof First, we prove assertion (ii). Note that $B_r(\bar{\mathcal{Z}}) \cap Q_1 \subset \tilde{\Lambda}_{\bar{x}}(\Omega_{\bar{x},r})$. By the definition of the extended solution u_2, (3.36) clearly holds for $\rho \in [c_5\underline{\delta}(\bar{x})^{1/2}, r]$, where $c_5 \in (0, 1/8)$ is the same constant as in (3.24). On the other hand, (3.30) directly implies (3.36) for $\rho \in [\frac{c_4}{216}\underline{\delta}(\bar{x}), c_5\underline{\delta}(\bar{x})^{1/2})$.

Next, we give the proof of assertion (i). We consider the following three cases:

$$\begin{aligned}
&\mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x}),\\
&\rho < \mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \frac{c_4}{6}\underline{\delta}(\bar{x}),\\
&\rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x}) < \mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-).
\end{aligned}$$

Case 1: $\mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$. Since $\bar{x} \in \Omega_{1/2}$, by (3.33) and the triangle inequality, we can choose $\hat{x} \in \Gamma_+ \cup \Gamma_-$ with $|\hat{x}| \leq 3/4$, such that $\mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) = |\hat{x} - \bar{x}|$, and thus $\Omega_\rho(\bar{x}) \subset \Omega_{2\rho}(\hat{x}) \subset \Omega_{3\rho}(\bar{x})$.

Since $|\hat{x} - \bar{x}| \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$, by the triangle inequality, we have

$$|\hat{x}'|^2 \leq 2|\bar{x}'|^2 + 2\Big(\frac{c_4}{6}\Big)^2\underline{\delta}(\bar{x})^2, \quad |\bar{x}'|^2 \leq 2|\hat{x}'|^2 + 2\Big(\frac{c_4}{6}\Big)^2\underline{\delta}(\bar{x})^2,$$

which also implies

$$\frac{1}{3}\underline{\delta}(\bar{x}) \leq \underline{\delta}(\hat{x}) \leq 3\underline{\delta}(\bar{x}) \tag{3.37}$$

since $c_4 \in (0, 1)$. By (3.37) and the fact that $|\hat{x} - \bar{x}| \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$, we also have

$$2|\hat{x} - \bar{x}| \leq 2\rho \leq c_4\underline{\delta}(\hat{x}). \tag{3.38}$$

Let $R_1 = c_4\underline{\delta}(\hat{x})$. Then $\Omega_{R_1}(\hat{x}) \subset \Omega_{\frac{3}{2}R_1}(\bar{x})$. Thus we can apply Corollary 3.4 at $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$ and use (3.6) to obtain

$$\begin{aligned}\phi(\bar{x}, \rho) &\leq C\psi(\hat{x}, 2\rho) \leq C\left(\frac{\rho}{R_1}\right)^{\alpha_1} \psi(\hat{x}, R_1) + C\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{R_1}(\hat{x}))} \\ &\leq C\left(\frac{\rho}{R_1}\right)^{\alpha_1} \psi(\hat{x}, R_1) + C\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}. \end{aligned} \tag{3.39}$$

By using (3.2) and the change of variables $x \to y$, we have

$$\begin{aligned}\psi(\hat{x}, R_1) &\leq \left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p} \\ &\quad + C|\hat{x}'|\,\|Du\|_{L^\infty(\Omega_{R_1}(\hat{x}))} \\ &\leq \left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p} \\ &\quad + CR_1^{1/2}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}. \end{aligned} \tag{3.40}$$

By (3.37), (3.32), and the fact that $c_5 \in (0, 1/8)$, it holds that

$$\frac{c_4}{2}\underline{\delta}(\bar{x}) \leq \frac{3}{2}R_1 \leq \frac{9c_4}{2}\underline{\delta}(\bar{x}) \leq \frac{c_5}{2}\underline{\delta}(\bar{x})^{1/2} \leq \frac{1}{2}r. \tag{3.41}$$

Since $\Omega_{R_1}(\hat{x}) \subset \Omega_{\frac{3}{2}R_1}(\bar{x})$, by using (3.6), (3.22)–(3.25), (3.41), the change of variables $x \to \mathcal{Z}$, and the triangle inequality, we also have

$$\begin{aligned}&\left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p} \\ &\leq C\frac{|\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))|^{1/p}}{|\Omega_{R_1}(\hat{x})|^{1/p}}\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p} \\ &\quad + C\frac{R_1}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))} \\ &\leq C\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p} \\ &\quad + CR_1^{1/2}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}. \end{aligned} \tag{3.42}$$

Without loss of generality, we assume $\hat{x} \in \Gamma_-$ and thus $\hat{\mathcal{Z}} := \tilde{\Lambda}_{\bar{x}}(\hat{x}) \in \tilde{\Gamma}_-$. We denote $B_R^+(\hat{\mathcal{Z}}) := B_R(\hat{\mathcal{Z}}) \cap \{\mathcal{Z} \in \mathbb{R}^n : \mathcal{Z}_n > \hat{\mathcal{Z}}_n\}$ for any $R > 0$. By (3.25) and (3.38), we have $B_{2R_1}(\hat{\mathcal{Z}}) \subset B_{3R_1}(\bar{\mathcal{Z}}) \subset B_r(\bar{\mathcal{Z}}) \subset \mathcal{C}_1 = \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1\}$. Since $c_4 \leq \min\{c_1/8, 1/8\}$, by (1.8) and (3.37), we know that $B_{2R_1}(\hat{\mathcal{Z}}) \cap Q_1 \equiv B_{2R_1}^+(\hat{\mathcal{Z}})$.

Again by (3.25), we also have $B^+_{R_1/2}(\hat{\mathcal{Z}}) \subset \tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x})) \subset B^+_{2R_1}(\hat{\mathcal{Z}})$. Therefore, by the triangle inequality and the definition of u_2,

$$\begin{aligned}&\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p}\\ &\le C\left(\fint_{B^+_{2R_1}(\hat{\mathcal{Z}})} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{B^+_{2R_1}(\hat{\mathcal{Z}})}|^p\right)^{1/p}\\ &= C\left(\fint_{B_{2R_1}(\hat{\mathcal{Z}})} |D_{\mathcal{Z}}u_2 - (D_{\mathcal{Z}}u_2)_{B_{2R_1}(\hat{\mathcal{Z}}_0)}|^p\right)^{1/p} \le C\tilde{\phi}(\bar{x}, 3R_1). \quad (3.43)\end{aligned}$$

Combining (3.39), (3.40), (3.42), and (3.43), we have

$$\begin{aligned}&\phi(\bar{x}, \rho)\\ &\le C\left(\frac{\rho}{R_1}\right)^{\alpha_1}\left(\tilde{\phi}(\bar{x}, 3R_1) + R_1^{1/2}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}\right) + C\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}.\end{aligned} \quad (3.44)$$

Note that by (3.37), $R_1^{-\alpha_1}R_1^{1/2} \le R_1^{-\alpha_1/2} \le Cr^{-\alpha_1}$. Thus (3.36) with $3R_1$ in place of ρ and (3.44) directly imply (3.35).

Case 2: $\rho < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \le \frac{c_4}{6}\underline{\delta}(\bar{x})$. Let $R_2 = \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-)$. Then we can apply the estimate (3.35) in Case 1 with R_2 in place of ρ to obtain

$$\phi(\bar{x}, R_2) \le C\left(\frac{R_2}{r}\right)^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \quad (3.45)$$

By (3.34), $B_{R_2}(\bar{x}) \subset \Omega_1$, and we can apply Lemma 3.1 to get

$$\phi(\bar{x}, \rho) \le C\left(\frac{\rho}{R_2}\right)^{\alpha_1}\phi(\bar{x}, R_2). \quad (3.46)$$

Combining (3.45) and (3.46) yields (3.35).

Case 3: $\rho \le \frac{c_4}{6}\underline{\delta}(\bar{x}) < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-)$. Let $R_3 = \frac{c_4}{6}\underline{\delta}(\bar{x})$. Then by (3.34), $B_{R_3}(\bar{x}) \subset \Omega_1$. By Lemma 3.1, we have

$$\phi(\bar{x}, \rho) \le C\left(\frac{\rho}{R_3}\right)^{\alpha_1}\phi(\bar{x}, R_3). \quad (3.47)$$

Similar to Case 1, by using (3.22)–(3.25), change of variables, and the triangle inequality, we obtain

$$\begin{aligned}\phi(\bar{x}, R_3) &\le C\frac{|\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))|^{1/p}}{|B_{R_3}(\bar{x})|^{1/p}}\left(\fint_{\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))}|D_{\mathcal{Z}}u_2-(D_{\mathcal{Z}}u_2)_{\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))}|^p\right)^{1/p}\\ &\quad +C\,\frac{R_3}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\|Du\|_{L^\infty(B_{R_3}(\bar{x}))}\\ &\le C\tilde{\phi}(\bar{x}, 2R_3)+\frac{CR_3}{r}\|Du\|_{L^\infty(B_{R_3}(\bar{x}))}. \end{aligned} \tag{3.48}$$

By (3.36) with $2R_3$ in place of ρ, we also have

$$\tilde{\phi}(\bar{x}, 2R_3) \le C\left(\frac{R_3}{r}\right)^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{3.49}$$

Combining (3.47), (3.48), and (3.49) yields (3.35). The proof is completed. □

It is straightforward to see the following lower bound of $|\Omega_\rho(\bar{x})|$ from the proof of Lemma 3.6 (assertion (i), case 1) and (3.6), which would be useful in the proof of Proposition 1.5.

Lemma 3.7 *Let $\bar{x} \in \Omega_{1/2}$ and $c_4 = c_4(n, c_1, c_2) \in (0, 1)$ be the same constant as in (3.4). Assume that (3.32) holds. Then there exists a constant $c > 0$ depending only on n, such that for any $\rho \in (0, \frac{c_4}{6}\underline{\delta}(\bar{x}))$, it holds that*

$$|\Omega_\rho(\bar{x})| \ge c\rho^n.$$

3.5 Proof of Proposition 1.5

Now we are ready to prove Proposition 1.5.

Proof of Proposition 1.5 Let $x_0 \in \Omega_{1/4}$ and we prove the proposition around $x = x_0$. We recall $\underline{\delta}(x_0) = \varepsilon + |x_0'|^2$. Let $x \in \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4}$. By the triangle inequality,

$$|x'|^2 \le 2|x_0'|^2 + \underline{\delta}(x_0)/8, \quad |x_0'|^2 \le 2|x'|^2 + \underline{\delta}(x_0)/8.$$

Therefore, for any $x \in \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4}$, it holds that

$$\frac{\underline{\delta}(x_0)}{3} \le \underline{\delta}(x) \le 3\underline{\delta}(x_0). \tag{3.50}$$

We denote

$$c_0 := \frac{1}{3}\min\left\{\left(\frac{c_5}{10c_4}\right)^2, \frac{1}{4+4c_2}\right\},$$

where c_4 and c_5 are the same constants as in (3.4) and (3.24). Thus $c_0 > 0$ depends only on n, p, c_1, and c_2. When $\underline{\delta}(x_0) > c_0$, by (3.50), we can apply classical results of Hölder regularity of the gradient for the p-Laplace equation to get (1.21) with $x = x_0$, and some $\beta = \beta(n, p) \in (0, 1)$ and $C = C(n, p, c_1, c_2) > 0$. Let $x_1, x_2 \in \Omega_{x_0, \sqrt{\underline{\delta}(x_0)}/4}$ and we denote

$$\rho := |x_1 - x_2|.$$

It suffices to show that when $\underline{\delta}(x_0) \le c_0$,

$$|Du(x_1) - Du(x_2)| \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})} \tag{3.51}$$

holds for some constant $C > 0$ depending only on n, p, c_1, and c_2, where $\alpha_1 \in (0, 1)$ is the same constant as in (3.31). From now on, we assume

$$\underline{\delta}(x_0) \le c_0 = \frac{1}{3}\min\big\{(\frac{c_5}{10c_4})^2, \frac{1}{4+4c_2}\big\}. \tag{3.52}$$

By (3.50), we have

$$\frac{\underline{\delta}(x_0)}{3} \le \underline{\delta}(x_1) \le 3\underline{\delta}(x_0) \quad \text{and} \quad \frac{\underline{\delta}(x_0)}{3} \le \underline{\delta}(x_2) \le 3\underline{\delta}(x_0) \tag{3.53}$$

and therefore (3.52) also implies that (3.32) holds for $\bar{x} = x_1, x_2 \in \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4} \subset \Omega_{1/2}$. Thus, we can apply both Lemmas 3.6 and 3.7 with $\bar{x} = x_1$ and with $\bar{x} = x_2$.

We consider two different cases: $\rho \le \frac{c_4}{36}\underline{\delta}(x_0)$ and $\rho > \frac{c_4}{36}\underline{\delta}(x_0)$.

Case 1: $\rho \le \frac{c_4}{36}\underline{\delta}(x_0)$. By (3.53), we also have

$$\rho \le \frac{c_4}{12}\underline{\delta}(x_1) \quad \text{and} \quad \rho \le \frac{c_4}{12}\underline{\delta}(x_2). \tag{3.54}$$

For any $x \in \Omega_\rho(x_2)$, by the triangle inequality

$$\begin{aligned}|Du(x_1) - Du(x_2)| \le\; &|Du(x_1) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x_2) - (Du)_{\Omega_\rho(x_2)}|\\ &+ |Du(x) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x) - (Du)_{\Omega_\rho(x_2)}|.\end{aligned}$$

We then take the L^p average over $x \in \Omega_\rho(x_2) \subset \Omega_{2\rho}(x_1)$ and use Lemma 3.7, the Lebesgue differentiation theorem, and the triangle inequality to get

$$\begin{aligned}&|Du(x_1) - Du(x_2)|\\ &\quad\le |Du(x_1) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x_2) - (Du)_{\Omega_\rho(x_2)}| + C\phi(x_1, 2\rho) + C\phi(x_2, \rho)\\ &\quad\le C\sum_{j=0}^{\infty}\phi(x_1, 2^{1-j}\rho) + C\sum_{j=0}^{\infty}\phi(x_2, 2^{-j}\rho),\end{aligned} \tag{3.55}$$

where ϕ is the mean oscillation of Du defined in (3.1).

By (3.53) and the triangle inequality, we know that $\Omega_{x_1,r} \subset \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2} \subset \Omega_1$, where $r = \frac{1}{8}\underline{\delta}(x_1)^{1/2}$. Since (3.54) holds, we can apply (3.35) with x_1 in place of $\bar{x}$ and $2^{1-j}\rho$ in place of ρ to obtain

$$\begin{aligned}\sum_{j=0}^{\infty}\phi(x_1, 2^{1-j}\rho) &\le C\sum_{j=0}^{\infty}\left(\frac{2^{1-j}\rho}{r}\right)^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_1,r})}\\ &\le C\left(\frac{\rho}{r}\right)^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}\\ &\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.\end{aligned} \tag{3.56}$$

Similarly, we also have

$$\sum_{j=0}^{\infty}\phi(x_2, 2^{-j}\rho) \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{3.57}$$

Combining (3.55), (3.56), and (3.57) yields (3.51).

Case 2: $\rho > \frac{c_4}{36}\underline{\delta}(x_0)$. By (3.53), we also have

$$\rho > \frac{c_4}{108}\underline{\delta}(x_1) \quad \text{and} \quad \rho > \frac{c_4}{108}\underline{\delta}(x_2). \tag{3.58}$$

With x_1 in place of $\bar{x}$ in Sect. 3.3, we denote the new coordinate by $\xi = \tilde{\Lambda}_{x_1}(x)$ and set $\xi_1 := (\xi_1', \xi_1^n) := \tilde{\Lambda}_{x_1}(x_1)$, where $\tilde{\Lambda}_{x_1}$ is defined as in (3.18). Similarly, with x_2 in place of $\bar{x}$, we denote another coordinate by $\eta := (\eta_2', \eta_2^n) := \tilde{\Lambda}_{x_2}(x)$ and set $\eta_2 = \tilde{\Lambda}_{x_2}(x_2)$. Let $u_2^{(1)}$ and $u_2^{(2)}$ be the extended solutions in the coordinates ξ and η defined as in Sect. 3.3, respectively. As in Sect. 3.3, we also define the mean oscillation of extended solutions in the two coordinates ξ and η by

$$\tilde{\phi}(x_1, r) = \left(\fint_{B_r(\xi_1)}|D_\xi u_2^{(1)} - (D_\xi u_2^{(1)})_{B_r(\xi_1)}|^p\right)^{1/p}$$

and

$$\tilde{\phi}(x_2, r) = \left(\fint_{B_r(\eta_2)}|D_\eta u_2^{(2)} - (D_\eta u_2^{(2)})_{B_r(\eta_2)}|^p\right)^{1/p}.$$

Let us first briefly describe our ideas to prove (3.51) in this case. By the triangle inequality,

$$\begin{aligned}|Du(x_1) - Du(x_2)| \le{}& |Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| + |Du(x_2) - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|\\ &+|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|,\end{aligned} \tag{3.59}$$

where $c_6 > 0$ is a constant which will be specified later. We will estimate the third term on the right hand side of (3.59) using careful change of variables $\xi \to \eta$ and estimate the first two terms using iteration of Lemma 3.6. A delicate transition from the original coordinates x to the new coordinates ξ (or η) is also needed when the radius (inside the iteration procedure) is at the scale of $\underline{\delta}(x_0)$.

Note that by (3.18), $\Phi := \tilde{\Lambda}_{x_2}\tilde{\Lambda}_{x_1}^{-1}$ is indeed a dilation of the coordinate ξ in the ξ^n direction, namely,

$$\eta = (\eta', \eta^n) = \Phi(\xi) = \Big(\xi', \ \frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon}\,\xi^n\Big).$$

By (1.7), (1.8), and (3.53), we have

$$\frac{c_1}{9c_2} \le \frac{c_1|x_2'|^2 + \varepsilon}{c_2|x_1'|^2 + \varepsilon} \le \frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} \le \frac{c_2|x_2'|^2 + \varepsilon}{c_1|x_1'|^2 + \varepsilon} \le \frac{9c_2}{c_1}. \tag{3.60}$$

This implies that Φ and Φ^{-1} are bounded independent of ε. Moreover, similar to (3.21), using (3.53) one can also show that

$$\begin{aligned} \Big|\frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} - 1\Big| &\le 2c_2\frac{(|x_1'| + |x_2'|)|x_1' - x_2'|}{c_1|x_1'|^2 + \varepsilon} \le C\underline{\delta}(x_0)^{-1/2}\rho,\\ \Big|\frac{h_1(x_1') - h_2(x_1') + \varepsilon}{h_1(x_2') - h_2(x_2') + \varepsilon} - 1\Big| &\le 2c_2\frac{(|x_1'| + |x_2'|)|x_1' - x_2'|}{c_1|x_2'|^2 + \varepsilon} \le C\underline{\delta}(x_0)^{-1/2}\rho, \end{aligned} \tag{3.61}$$

where $C > 0$ is a constant depending only on c_1 and c_2. Note that (3.61) directly implies

$$|D\Phi - I_n| \le C\underline{\delta}(x_0)^{-1/2}\rho \quad \text{and} \quad |D\Phi^{-1} - I_n| \le C\underline{\delta}(x_0)^{-1/2}\rho. \tag{3.62}$$

By the definitions of the extended solutions $u_2^{(1)}$ and $u_2^{(2)}$, we also know that for any $\xi, \eta \in \mathbb{R}^n$ with $|\xi'| < 1$ and $|\eta'| < 1$, we have

$$u_2^{(1)}(\xi) = u_2^{(2)}(\Phi(\xi)), \quad u_2^{(2)}(\eta) = u_2^{(1)}(\Phi^{-1}(\eta)).$$

Without loss of generality, we assume

$$\frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} \ge 1,$$

and thus we have $\Phi^{-1}(B_\rho(\eta_2)) \subset B_\rho(\xi_2)$, where we denote $\xi_2 := (\xi_2', \xi_2^n) := \Phi^{-1}(\eta_2)$. Clearly $\xi_1' = x_1'$ and $\xi_2' = x_2'$. Therefore, by using (1.7), (1.9), (3.58), and the triangle inequality, for any $\xi \in \Phi^{-1}(B_\rho(\eta_2))$, it holds that

$$\begin{aligned}|\xi-\xi_1| &\le |\xi-\xi_2|+|\xi_2-\xi_1| \le \rho+|\xi_2'-\xi_1'|+|\xi_2^n-\xi_1^n|\\ &\le 2\rho+h_1(x_1')-h_2(x_1')+\varepsilon\\ &\le 2\rho+c_2|x_1'|^2+\varepsilon\le c_6\rho,\end{aligned}$$

where $c_6>2$ is a constant depending only on n, c_1, and c_2. Thus $\Phi^{-1}(B_\rho(\eta_2))\subset B_\rho(\xi_2)\subset B_{c_6\rho}(\xi_1)$.

Let $c_5\in(0,1/8)$ be the same constant as in Lemma 3.5. From now on, we assume

$$\rho\le\min\{c_5/4,\,1/(64c_6)\}\underline{\delta}(x_0)^{1/2}, \tag{3.63}$$

since otherwise (3.51) clearly holds. Note that (3.58), (3.63), and (3.53) directly imply

$$\frac{c_4}{108}\underline{\delta}(x_2)<\rho\le c_5\underline{\delta}(x_2)^{1/2}<\frac{1}{8}\underline{\delta}(x_2)^{1/2} \tag{3.64}$$

and

$$\frac{c_4}{108}\underline{\delta}(x_1)<c_6\rho\le\frac{1}{16}\underline{\delta}(x_1)^{1/2}. \tag{3.65}$$

Now we estimate the three terms on the right-hand side of (3.59) separately.

We first estimate the term $|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|$ in (3.59). For any $\xi\in\Phi^{-1}(B_\rho(\eta_2))$, by the triangle inequality,

$$\begin{aligned}&|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|\\ &\quad\le|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}-D_\xi u_2^{(1)}(\xi)|+|D_\xi u_2^{(1)}(\xi)-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|.\end{aligned}$$

We then take the L^p average over $\xi\in\Phi^{-1}(B_\rho(\eta_2))\subset B_{c_6\rho}(\xi_1)$ and use (3.60), (3.62), and the triangle inequality to get

$$\begin{aligned}&|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|\\ &\le C\tilde\phi(x_1,c_6\rho)+\left(\fint_{\Phi^{-1}(B_\rho(\eta_2))}|D_\xi u_2^{(1)}(\xi)-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|^p\,d\xi\right)^{1/p}\\ &\le C\tilde\phi(x_1,c_6\rho)+\left(\fint_{B_\rho(\eta_2)}|D_\eta u_2^{(2)}(\eta)\,D\Phi-(D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|^p\,d\eta\right)^{1/p}\\ &\le C\tilde\phi(x_1,c_6\rho)+C\tilde\phi(x_2,\rho)+C\underline{\delta}(x_0)^{-1/2}\rho\,\|D_\eta u_2^{(2)}\|_{L^\infty(B_\rho(\eta_2))}.\end{aligned} \tag{3.66}$$

By (3.64) and (3.25) we know that $|D\tilde\Lambda_{x_2}|\le C$ in $\Omega_{x_2,\rho}$ and thus

$$\|D_\eta u_2^{(2)}\|_{L^\infty(B_\rho(\eta_2))}\le C\|Du\|_{L^\infty(\Omega_{x_2,\rho})}\le C\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{3.67}$$

By (3.65), we can apply (3.36) with x_1 in place of $\bar{x}$ and $c_6\rho$ in place of ρ, and use (3.53) to get

$$\begin{aligned}\tilde{\phi}(x_1, c_6\rho) &\leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4})}\\ &\leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.\end{aligned} \tag{3.68}$$

Similarly, we have

$$\begin{aligned}\tilde{\phi}(x_2, \rho) &\leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_2,\sqrt{\underline{\delta}(x_0)}/4})}\\ &\leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.\end{aligned} \tag{3.69}$$

Combining (3.66), (3.67), (3.68) and (3.69), we obtain

$$|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}| \leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{3.70}$$

Next, we estimate the term $|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}|$ in (3.59). We define $\rho_j := c_6 2^{-j}\rho$ for $j \in \mathbb{N}$. Because of (3.58), we let $j_1 \geq 1$ be the integer such that

$$\rho_{j_1} \geq \frac{c_4}{216}\underline{\delta}(x_1), \quad \rho_{j_1+1} < \frac{c_4}{216}\underline{\delta}(x_1). \tag{3.71}$$

Then by using the triangle inequality and Lemma 3.7, we have

$$\begin{aligned}&|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}|\\ &\leq |Du(x_1) - (Du)_{\Omega_{\rho_{j_1+1}}(x_1)}| + |(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|\\ &\quad + |(D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)} - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}|\\ &\leq C\sum_{j=j_1+1}^{\infty}\phi(x_1,\rho_j) + C\sum_{j=0}^{j_1}\tilde{\phi}(x_1,\rho_j) + |(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|.\end{aligned} \tag{3.72}$$

Using Lemma 3.7 and Hölder's inequality, we obtain

$$\begin{aligned}&|(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|\\ &\leq C\Big(\fint_{\Omega_{\rho_{j_1}}(x_1)} |Du(x) - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|^p\,dx\Big)^{1/p}\\ &\leq C\Big(\fint_{\tilde{\Lambda}_{x_1}(\Omega_{\rho_{j_1}}(x_1))} |D_\xi u_2^{(1)}(\xi)\,B(\xi) - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|^p\,d\xi\Big)^{1/p},\end{aligned} \tag{3.73}$$

where as in Sect. 3.3, we denote

$$B(\xi) = D\tilde{\Lambda}_{x_1}(\tilde{\Lambda}_{x_1}^{-1}(\xi)),$$

and use (3.25) in the last line. By (3.23) with x_1 in place of $\bar{x}$, from (3.73) we deduce

$$\begin{aligned} |(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}| &\le C\tilde{\phi}(x_1, 2\rho_{j_1}) + \frac{C\rho_{j_1}}{\underline{\delta}(x_1)^{1/2}}\|Du\|_{L^\infty(\Omega_{x_1,\rho_{j_1}})} \\ &\le C\tilde{\phi}(x_1, \rho_{j_1-1}) + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,c_6\rho})}. \end{aligned} \tag{3.74}$$

Here in the last inequality we also used (3.71) and (3.58). Combining (3.72), (3.74) and using (3.63), we get

$$\begin{aligned} &|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| \\ &\le C\sum_{j=j_1+1}^{\infty} \phi(x_1, \rho_j) + C\sum_{j=0}^{j_1} \tilde{\phi}(x_1, \rho_j) + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})}. \end{aligned} \tag{3.75}$$

We recall (3.58) and (3.65). Therefore, by applying Lemma 3.6 with x_1 in place of $\bar{x}$ and ρ_j in place of ρ, and using (3.75) and (3.53), we obtain

$$\begin{aligned} &|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_2)}| \\ &\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})} + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})} \\ &\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \end{aligned} \tag{3.76}$$

Similarly, it also holds that

$$|Du(x_2) - (D_\xi u_2^{(2)})_{B_\rho(\xi_2)}| \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{3.77}$$

Combining (3.59), (3.70), (3.76), and (3.77) yields (3.51). The proof is completed. □

4 Estimates of $U_1^\varepsilon - U_2^\varepsilon$

In the following, we use the $C^{1,\beta}$ estimate we derive in Proposition 1.5 to obtain an asymptotic expansion of Du_ε in terms of $U_1^\varepsilon - U_2^\varepsilon$, for arbitrary constants U_1^ε and U_2^ε (Proposition 4.1). When $p \ge (n+1)/2$, for the specific U_1^ε and U_2^ε in (1.2), $U_1^\varepsilon - U_2^\varepsilon$ will converges to 0 as $\varepsilon \to 0$, and we compute in Theorem 4.4 the rate of convergence using information of the flux $\mathcal{F}$ defined in (1.6). When $p < (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converges to $U_1 - U_2$ as shown in Theorem 2.5.

Proposition 4.1 *Let h_1, h_2 be C^2 functions satisfying* (1.7)–(1.9)*, $p > 1$, $n \geq 2$, $\varepsilon \in [0, 1/4)$, U_1^ε, U_2^ε be arbitrary constants with $|U_i^\varepsilon| \leq \|\varphi\|_{L^\infty(\partial\Omega)}$, and $u_\varepsilon \in W^{1,p}(\widetilde{\Omega}^\varepsilon)$ be a solution of*

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ u_\varepsilon = U_i^\varepsilon & \text{on } \partial\mathcal{D}_i^\varepsilon,\ i = 1, 2, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega. \end{cases}$$

There exist positive constants $\beta \in (0, 1)$, C_1 and C_2 depending only on n, p, c_1, and c_2, such that

$$Du_\varepsilon(x) = \left(0', \frac{U_1^\varepsilon - U_2^\varepsilon}{\delta(x)}\right) + \mathbf{f}_1(x, \varepsilon) \quad \text{for } x \in \overline{\Omega}_{1/4}^\varepsilon, \tag{4.1}$$

where δ is defined in (1.11)*,*

$$|\mathbf{f}_1(x, \varepsilon)| \leq C_1 \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}} \right). \tag{4.2}$$

Proof By mean value theorem, we know that for any $x = (x', x_n) \in \overline{\Omega}_{1/4}^\varepsilon$, there exist $\zeta(x') \in (-\frac{\varepsilon}{2} + h_2(x'), \frac{\varepsilon}{2} + h_1(x'))$ and $y(x) = (x', \zeta(x')) \in \Omega_{1/4}^\varepsilon$, such that

$$D_n u_\varepsilon(y(x)) = (U_1^\varepsilon - U_2^\varepsilon)/\delta(x).$$

Let

$$\mathbf{f}_1(x, \varepsilon) := (D_{x'} u_\varepsilon(x), D_n u_\varepsilon(x) - D_n u_\varepsilon(y(x))). \tag{4.3}$$

By Propositions 1.5 and 2.2, we have

$$\begin{aligned} |D_n u_\varepsilon(x) - D_n u_\varepsilon(y(x))| &\leq C|x_n - \zeta(x')|^\beta \delta(x)^{-\beta/2} \|Du_\varepsilon\|_{L^\infty(\Omega_{x,\sqrt{\delta(x)}/2})} \\ &\leq C_1 \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}} \right). \end{aligned} \tag{4.4}$$

Let $z(x) = (x', \varepsilon/2 + h_1(x'))$. Since $u_\varepsilon \equiv U_1^\varepsilon$ on $\overline{\mathcal{D}}_1^\varepsilon$, by Proposition 2.2, we have

$$\begin{aligned} |D_{x'} u(z(x))| &\leq C|x'||Du(z(x))| \\ &\leq C_1|x'| \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}} \right). \end{aligned} \tag{4.5}$$

Similar to (4.4), we also have

$$|D_{x'} u(x) - D_{x'} u(z(x))| \leq C_1 \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon + |x'|}}} \right). \tag{4.6}$$

Therefore, by (4.5), (4.6), and the triangle inequality, we have

$$|D_{x'}u(x)| \leq C_1\left(\frac{|U_1^{\varepsilon}-U_2^{\varepsilon}|}{\delta^{1-\beta/2}(x)}+\|\varphi\|_{L^{\infty}(\partial\Omega)}e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}}\right).$$

This completes the proof of Proposition 4.1. □

Following the proof of [26, Proposition 2.1] with slight modification, we have the following proposition, which will be proved in the Appendix to make this article self-contained.

Proposition 4.2 *Let $n \geq 2$, $p \geq (n+1)/2$, h_1, h_2 be C^2 functions satisfying (1.7)–(1.9), $\varepsilon \in (0,1)$, and $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (1.2). Then there exist positive constants C_1, C_2 depending only on n, p, c_1, c_2, and $\|\varphi\|_{L^\infty}$, such that for any $r \in (0,1)$,*

$$\left|\mathcal{F}-\lim_{\varepsilon\to 0_+}\int_{\Gamma^{\varepsilon}_{-,r}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu\right| \leq C_1 e^{-\frac{C_2}{r}}, \tag{4.7}$$

where $\mathcal{F}$ is given in (1.6).

Remark 4.3 The proof of Proposition 4.2 relies on the facts that $|Du_0| \leq C_1e^{-\frac{C_2}{r}}$ and $\|u_\varepsilon - u_0\|_{C^{1,\alpha}(K)} \to 0$ as $\varepsilon \to 0$ for any $K \subset\subset \Omega\setminus\Big(\bigcup_{0<\varepsilon\leq\varepsilon_0}(\mathcal{D}_1^\varepsilon\cup\mathcal{D}_2^\varepsilon)\cup\{0\}\Big)$ with $\varepsilon_0 > 0$, where u_0 is the minimizer of (1.3) with $\varepsilon = 0$. See Proposition 2.2 and Theorem 2.5. However, Du_0 may be unbounded when $p < (n+1)/2$. This fact was overlooked in [14, 15].

With the help of Propositions 2.2 and 4.2, we are able to derive the rate of convergence for $U_1^\varepsilon - U_2^\varepsilon$ when $\varepsilon \to 0$.

Theorem 4.4 *Let $p \geq (n+1)/2$, U_1^ε and U_2^ε be the constants in (1.2). Then it holds that*

$$\lim_{\varepsilon\to 0_+}(U_1^\varepsilon - U_2^\varepsilon)\Theta(\varepsilon)^{-1} = \operatorname{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)}, \tag{4.8}$$

where $\Theta(\varepsilon)$ is given in (1.12), K is given in (1.13), and $\mathcal{F}$ is given in (1.6).

Proof By Proposition 4.1, we have

$$\left|Du_\varepsilon\cdot\nu(y)-\frac{U_1^\varepsilon-U_2^\varepsilon}{\delta(y)}\right| \leq C_1\Big(\frac{|U_1^\varepsilon-U_2^\varepsilon|}{\delta^{1-\beta/2}(y)}+e^{-\frac{C_2}{\sqrt{\varepsilon+|y'|}}}\Big) \quad \text{for} \ \ y \in \Gamma^{\varepsilon}_{-,1/4}, \tag{4.9}$$

where δ is defined in (1.11).

We will show that every sequence converges to the same limit.

First, Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a decreasing sequence such that $\varepsilon_j \to 0$ as $j \to \infty$ and $U_1^{\varepsilon_j} \geq U_2^{\varepsilon_j}$ for every $j \in \mathbb{N}$. By (4.9), for any $\tau \in (0, 1/2)$ and $r \in (0, 1/4)$, we have

$$(1-\tau)\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1-C\delta_j^{\beta/2}(y)) - C(\tau)e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}} \leq |Du_{\varepsilon_j}|^{p-2}Du_{\varepsilon_j}\cdot\nu(y)$$
$$\leq (1+\tau)\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1+C\delta_j^{\beta/2}(y)) + C(\tau)e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}} \quad \text{for } y \in \Gamma^{\varepsilon_j}_{-,1/4},$$

where $\delta_j(y) := \varepsilon_j + h_1(y') - h_2(y')$, $C, C_2 > 0$ depends only on n, p, c_1, c_2, β, $\|\varphi\|_{L^\infty}$, and $\mathrm{dist}(\mathcal{D}_1^{\varepsilon_j} \cup \mathcal{D}_2^{\varepsilon_j}, \partial\Omega)$, and $C(\tau)$ additionally depends on τ. By the change of variables $dS = \sqrt{1+|D_{x'}h_2(y')|^2}dy'$, (1.7), and (1.9), we have for any $r < 1/4$,

$$\begin{aligned}
&(1-\tau)\int_{|y'|<r}\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1-C\delta_j^{\beta/2}(y))\sqrt{1+|D_{x'}h_2(y')|^2}\,dy'\\
&\quad - C(\tau)\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy'\\
&\leq \int_{\Gamma^{\varepsilon_j}_{-,r}} |Du_{\varepsilon_j}|^{p-2}Du_{\varepsilon_j}\cdot\nu\, dS\\
&\leq (1+\tau)\int_{|y'|<r}\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1+C\delta_j^{\beta/2}(y))\sqrt{1+|D_{x'}h_2(y')|^2}\,dy'\\
&\quad + C(\tau)\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy'.
\end{aligned} \tag{4.10}$$

Note that

$$\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy' \leq Cr^{(n-1)}. \tag{4.11}$$

Moreover, for $p \geq (n+1)/2$,

$$\lim_{r\to 0_+}\lim_{\varepsilon\to 0_+}\int_{|y'|<r}\Big(\frac{\Theta(\varepsilon)}{\delta(y)}\Big)^{p-1}dy' = \frac{1}{K}. \tag{4.12}$$

The verification of (4.12) follows from direct computations, which will be given Lemma A.1. By taking the limit as $j \to \infty$ in (4.10) first, then taking $r \to 0$ and $\tau \to 0$, from (4.7), (4.10), (4.11), and (4.12) we get

$$\lim_{j\to\infty}\big((U_1^{\varepsilon_j}-U_2^{\varepsilon_j})\Theta(\varepsilon_j)^{-1}\big)^{p-1} = K\mathcal{F}. \tag{4.13}$$

Similarly, if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a decreasing sequence such that $\varepsilon_j \to 0$ as $j \to \infty$ and $U_1^{\varepsilon_j} \le U_2^{\varepsilon_j}$ for every $j \in \mathbb{N}$, then we have

$$\lim_{j\to\infty} \big((U_2^{\varepsilon_j} - U_1^{\varepsilon_j})\Theta(\varepsilon_j)^{-1}\big)^{p-1} = -K\mathcal{F}. \tag{4.14}$$

Therefore, we conclude that if $\mathcal{F} > 0$, then for any decreasing sequence $\varepsilon_j \to 0_+$, there exists $j_0 \in \mathbb{N}$ such that $U_1^{\varepsilon_j} \ge U_2^{\varepsilon_j}$ for every $j \ge j_0$, since otherwise there exists a decreasing subsequence $\{\varepsilon_{j_k}\}$ such that $U_1^{\varepsilon_{j_k}} < U_2^{\varepsilon_{j_k}}$, then from (4.14) we should have $\mathcal{F} \le 0$. Thus if $\mathcal{F} > 0$, (4.13) implies (4.8). Similarly, (4.8) also holds when $\mathcal{F} < 0$. For the remaining case when $\mathcal{F} = 0$, let $\{\varepsilon_j\}$ be any decreasing sequence such that $\varepsilon_j \to 0$ as $j \to \infty$. Then there exists a decreasing subsequence $\{\varepsilon_{j_k}\}$ such that either $U_1^{\varepsilon_{j_k}} \le U_2^{\varepsilon_{j_k}}$ holds for every $k \in \mathbb{N}$ or $U_1^{\varepsilon_{j_k}} \ge U_2^{\varepsilon_{j_k}}$ holds for every $k \in \mathbb{N}$. Thus (4.13) (or (4.14)) implies that

$$\lim_{k\to\infty} (U_1^{\varepsilon_{j_k}} - U_2^{\varepsilon_{j_k}})\Theta(\varepsilon_{j_k})^{-1} = 0.$$

Therefore, (4.8) is also true when $\mathcal{F} = 0$. The proof of the theorem is completed. □

5 Proofs of Theorem 1.1 and Proposition 1.4

In this section, we give the proofs of Theorem 1.1 and Proposition 1.4.

Proof of Theorem 1.1 First, we consider the case when $p \ge (n+1)/2$. Let

$$f_0(\varepsilon) := (U_1^\varepsilon - U_2^\varepsilon)/\Theta(\varepsilon) - \operatorname{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)}. \tag{5.1}$$

By Theorem 4.4 and (4.2), we know that

$$\lim_{\varepsilon\to 0} f_0(\varepsilon) = 0,$$

and

$$|\mathbf{f}_1(x,\varepsilon)| \le C_1\Big(\delta(x)^{\beta/2-1}\Theta(\varepsilon)(|K\mathcal{F}|^{1/(p-1)} + |f_0(\varepsilon)|) + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}}\Big),$$

where $\mathbf{f}_1$ is defined as in (4.3). Then (1.15) follows from (4.1), (5.1), and the above.

Next, we consider the case when $1 < p < (n+1)/2$. We define

$$g_0(\varepsilon) := U_1^\varepsilon - U_2^\varepsilon - (U_1 - U_2)$$

and $\mathbf{g}_1(x,\varepsilon) = \mathbf{f}_1(x,\varepsilon)$ as in (4.3). By Theorem 2.5,

$$\lim_{\varepsilon\to 0} g_0(\varepsilon) = 0.$$

Similar as above, (1.16) follows. □

Proof of Proposition 1.4 For the case when $p \geq (n+1)/2$, by the symmetry of the domain and uniqueness of the solution, we know that $u_0(x', x_n) = -u_0(x', -x_n)$. Therefore, $U_0 = u(x', 0) = 0$. By the strong maximum principle, $u_0 > 0$ in $\widetilde{\Omega}^0 \cap \{x_n > 0\}$. Then by the Hopf lemma, $Du_0 \cdot \nu > 0$ on $\partial \mathcal{D}_1^0$, and hence $\mathcal{F} > 0$.

For the case when $1 < p < (n+1)/2$, we prove by contradiction. Assume $U_1 \leq U_2$. Since $u_0(x', x_n) = -u_0(x', -x_n)$, we know that $u_0(x', 0) = 0$ and $U_1 = -U_2$. Therefore, $U_1 \leq 0$ and u_0 achieves minimum in B_5^+ on $\partial \mathcal{D}_1^0$. Hence $Du_0 \cdot \nu > 0$ on $\partial \mathcal{D}_1^0$ by the Hopf lemma. This implies

$$\int_{\partial \mathcal{D}_1^0} |Du_0|^{p-2} Du_0 \cdot \nu > 0,$$

which contradicts to $(1.5)_3$. □

Appendix A

In the first part of the appendix, we prove Proposition 4.2. The proof essentially follows those of [26, Proposition 2.1] and [15, Lemma 5.1]. Our estimate is sharper due to a better estimate on $|Du_0|$ (Proposition 2.2).

Proof of Proposition 4.2 Similar to the proof of Theorem 2.5, for small $r \in (0, 1/2)$, we take a smooth surface η so that $\mathcal{D}_1^\varepsilon$ is surrounded by $\Gamma_{-,s}^0 \cup \eta$. See Fig. 1. We denote the surface

$$\Sigma_r^\varepsilon := \left\{ x \in \mathbb{R}^n : |x'| = r, -\frac{\varepsilon}{2} + h_2(x') < x_n < h_2(x') \right\}.$$

Since $\int_{\partial \mathcal{D}_1^0} |Du_0|^{p-2} Du_0 \cdot \nu = \mathcal{F}$ and $\int_{\partial \mathcal{D}_1^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu = 0$, by integration by parts, we have

$$-\int_{\Gamma_{-,r}^0} |Du_0|^{p-2} Du_0 \cdot \nu + \int_\eta |Du_0|^{p-2} Du_0 \cdot \nu = \mathcal{F}, \tag{A.1}$$

and

$$-\int_{\Gamma_{-,r}^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu + \int_{\Sigma_r^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu + \int_\eta |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu = 0. \tag{A.2}$$

Note that the minus signs appear because ν on $\Gamma_{-,r}^0$ and $\Gamma_{-,r}^\varepsilon$ are defined to be pointing upwards, while ν on η and Σ_r^ε are pointing away from $\mathcal{D}_1^\varepsilon$. By (2.5), we have

$|Du_0(x)| \le C_1 e^{-\frac{C_2}{r}}$ in $\overline{\Omega_r^0}$, and hence

$$\left|\int_{\Gamma^0_{-,r}} |Du_0|^{p-2} Du_0 \cdot \nu\right| \le C_1 e^{-\frac{C_2}{r}} \tag{A.3}$$

for some positive ε-independent constants C_1 and C_2. By Theorem 2.5, we have

$$\lim_{\varepsilon\to 0_+} \int_\eta |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu = \int_\eta |Du_0|^{p-2} Du_0 \cdot \nu. \tag{A.4}$$

By (2.7), we have $|Du_\varepsilon(x)| \le C(\varepsilon + |x'|^2)^{-1}$ in $\Omega^0_{1/2}$. Therefore,

$$\begin{aligned}\left|\int_{\Sigma_r^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu\right| &\le C \int_{|x'|=r} \int_{-\varepsilon/2+h_2(x')}^{h_2(x')} \Big(\frac{1}{\varepsilon + |x'|^2}\Big)^{p-1} dx_n dS\\ &\le C \frac{\varepsilon r^{n-2}}{(\varepsilon + r^2)^{p-1}}.\end{aligned} \tag{A.5}$$

Finally, (4.7) follows directly from (A.1)–(A.5). Proposition 4.2 is proved. □

In the following, we verify (4.12).

Lemma A.1 (4.12) *holds when* $p \ge (n+1)/2$.

Proof We only give the proof for the case when $n \ge 3$. The case $n = 2$ follows similarly and is simpler. After a rotation of coordinates if necessary, we may assume that

$$D^2_{x'}(h_1 - h_2)(0') = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{n-1}).$$

First, we replace $\delta(y)$ in the denominator with the quadratic polynomial $\varepsilon + \sum_{i=1}^{n-1} \lambda_i y_i^2/2$. By (1.8), (1.9), and the fact that h_1, h_2 are C^2, we estimate

$$\begin{aligned}&\left|\int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\delta(y)}\Big)^{p-1} - \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy'\right|\\ &= \Theta(\varepsilon)^{p-1} \left|\int_{|y'|<r} \frac{\Big(\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2\Big)^{p-1} - \delta(y)^{p-1}}{\Big[\delta(y)\Big(\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2\Big)\Big]^{p-1}} dy'\right|\\ &\le C\Theta(\varepsilon)^{p-1} \int_{|y'|<r} \frac{h(r)|y'|^2(\varepsilon + |y'|^2)^{p-2}}{(\varepsilon + |y'|)^{2p-2}} dy'\\ &\le Ch(r) \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy',\end{aligned}$$

where $h(r)$ is the modulus of continuity of $D^2_{x'}(h_1 - h_2)$, and hence $h(r) \to 0$ as $r \to 0$, and C is some positive constant independent of ε and r. Therefore, it suffices to show that for any $r > 0$,

$$\lim_{\varepsilon \to 0_+} \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2} \Big)^{p-1} dy' = \frac{1}{K}. \tag{A.6}$$

In the spherical coordinates, for $y' \in \mathbb{R}^{n-1}$, we write

$$y_1 = \sqrt{\frac{2}{\lambda_1}} s \cos\theta_1, \quad y_2 = \sqrt{\frac{2}{\lambda_2}} s \sin\theta_1 \cos\theta_2, \quad y_3 = \sqrt{\frac{2}{\lambda_3}} s \sin\theta_1 \sin\theta_2 \cos\theta_3, \ldots,$$
$$y_{n-2} = \sqrt{\frac{2}{\lambda_{n-2}}} s \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2},$$
$$y_{n-1} = \sqrt{\frac{2}{\lambda_{n-1}}} s \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \sin\theta_{n-2},$$

where $s \in [0, \infty)$, $\theta_1, \theta_2, \ldots, \theta_{n-3} \in [0, \pi]$ and $\theta_{n-2} \in [0, 2\pi)$. For convenience of notation, we denote $\Sigma := [0, \pi]^{n-3} \times [0, 2\pi)$. By this change of variables,

$$\int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2} \Big)^{p-1} dy' = \frac{2^{\frac{n-1}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)}} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + s^2} \Big)^{p-1} s^{n-2} J(\theta)\, ds d\theta, \tag{A.7}$$

where

$$\varphi(\theta) = \Big(\frac{2}{\lambda_1} \cos^2\theta_1 + \frac{2}{\lambda_2} \sin^2\theta_1 \cos^2\theta_2 + \cdots + \frac{2}{\lambda_{n-1}} \sin^2\theta_1 \cdots \sin^2\theta_{n-2} \Big)^{\frac{1}{2}},$$

and

$$J(\theta) = \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \cdots \sin\theta_{n-3}.$$

Note that

$$\sqrt{\frac{2}{\max \lambda_i}} \le \varphi(\theta) \le \sqrt{\frac{2}{\min \lambda_i}}.$$

When $p > (n+1)/2$, $\Theta(\varepsilon) = \varepsilon^{\frac{2p-n-1}{2(p-1)}}$. By the change of variables $t = \varepsilon^{-1}s^2$, the right-hand side of (A.7) becomes

$$\frac{2^{\frac{n-3}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r^2}{\varphi^2(\theta)\varepsilon}} \frac{t^{\frac{n-3}{2}}}{(1+t)^{p-1}}\, dt\, J(\theta) d\theta.$$

Since $(n-3)/2 - (p-1) = (n-2p-1)/2 < -1$, the integral converges as $\varepsilon \to 0$. Therefore,

$$\begin{aligned}
&\lim_{\varepsilon \to 0_+} \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy' \\
&= \frac{2^{\frac{n-3}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^\infty \frac{t^{\frac{n-3}{2}}}{(1+t)^{p-1}}\, dt\, J(\theta) d\theta \\
&= \frac{2^{\frac{n-3}{2}} |\mathbb{S}^{n-2}|}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} B\Big(\frac{n-1}{2}, \frac{2p-n-1}{2}\Big),
\end{aligned} \tag{A.8}$$

where B is the beta function. Recalling the identities

$$|\mathbb{S}^{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\big(\frac{n-1}{2}\big)}, \quad B\Big(\frac{n-1}{2}, \frac{2p-n-1}{2}\Big) = \frac{\Gamma\big(\frac{n-1}{2}\big)\Gamma\big(p - \frac{n-1}{2}\big)}{\Gamma(p-1)}, \tag{A.9}$$

and plugging them into (A.8), we have proved (A.6) for the case when $p > (n+1)/2$.

When $p = (n+1)/2$, $\Theta(\varepsilon) = |\ln \varepsilon|^{-\frac{1}{p-1}}$. By the change of variables $w = \varepsilon^{-1/2}s$, the right-hand side of (A.7) becomes

$$\begin{aligned}
&\frac{2^{\frac{n-1}{2}} |\ln \varepsilon|^{-1}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} \frac{w^{n-2}}{(1+w^2)^{\frac{n-1}{2}}}\, dw\, J(\theta)\, d\theta \\
&= \frac{2^{\frac{n-1}{2}} |\ln \varepsilon|^{-1}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} \frac{w}{1+w^2}\, dw\, J(\theta)\, d\theta \\
&\quad + \frac{2^{\frac{n-1}{2}} |\ln \varepsilon|^{-1}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} \left(\frac{w^{n-2}}{(1+w^2)^{\frac{n-1}{2}}} - \frac{w}{1+w^2}\right) dw\, J(\theta)\, d\theta \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By direct computations,

$$\begin{aligned}
\mathrm{I} &= \frac{2^{\frac{n-3}{2}} |\ln \varepsilon|^{-1}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} d \ln(1+w^2) J(\theta)\, d\theta \\
&= \frac{2^{\frac{n-3}{2}} |\ln \varepsilon|^{-1}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \Big[\ln\big(\varepsilon + \frac{r^2}{\varphi^2(\theta)}\big) - \ln \varepsilon\Big] J(\theta)\, d\theta.
\end{aligned}$$

Therefore, by (A.9),

$$\lim_{\varepsilon\to 0_+} \mathrm{I} = \frac{2^{\frac{n-3}{2}}|\mathbb{S}^{n-2}|}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}} = \frac{(2\pi)^{\frac{n-1}{2}}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}. \tag{A.10}$$

To estimate II, we split the integral over $(0, \frac{r}{\varphi(\theta)\sqrt{\varepsilon}})$ into $(0, 1)$ and $(1, \frac{r}{\varphi(\theta)\sqrt{\varepsilon}})$, and denote them by II_1 and II_2, respectively. It is easily seen that $|\mathrm{II}_1| \le C|\ln\varepsilon|^{-1}$. To estimate II_2, we have

$$|\mathrm{II}_2| \le C|\ln\varepsilon|^{-1}\int_\Sigma\int_1^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} \frac{w\left[w^{n-3}-(1+w^2)^{\frac{n-3}{2}}\right]}{(1+w^2)^{\frac{n-1}{2}}}\,dw\,J(\theta)\,d\theta.$$

By the mean value theorem, there exists a $\xi\in(w^2, 1+w^2)$, such that

$$w^{n-3}-(1+w^2)^{\frac{n-3}{2}} = -\frac{n-3}{2}\xi^{\frac{n-5}{2}}.$$

Note that $(w^2, 1+w^2)\subset(w^2, 2w^2)$ when $w\ge 1$. Therefore,

$$\left|\int_1^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}} \frac{w\left[w^{n-3}-(1+w^2)^{\frac{n-3}{2}}\right]}{(1+w^2)^{\frac{n-1}{2}}}\,dw\right| \le C\int_1^\infty \frac{w^{n-4}}{(1+w^2)^{\frac{n-1}{2}}} \le C,$$

which implies $|\mathrm{II}_2|\le C|\ln\varepsilon|^{-1}$. Hence, by (A.10) and the estimate $|\mathrm{II}_1|+|\mathrm{II}_2|\le C|\ln\varepsilon|^{-1}$, we have proved (A.6) for the case when $p=(n+1)/2$. □

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Ammari, H., Ciraolo, G., Kang, H., Lee, H., Yun, K.: Spectral analysis of the Neumann–Poincaré operator and characterization of the stress concentration in anti-plane elasticity. Arch. Ration. Mech. Anal. **208**(1), 275–304 (2013)
2. Ammari, H., Davies, B., Yu, S.: Close-to-touching acoustic subwavelength resonators: eigenfrequency separation and gradient blow-up. Multiscale Model. Simul. **18**(3), 1299–1317 (2020)
3. Ammari, H., Kang, H., Lee, H., Lee, J., Lim, M.: Optimal estimates for the electric field in two dimensions. J. Math. Pures Appl. (9) **88**(4), 307–324 (2007)
4. Ammari, H., Kang, H., Lim, M.: Gradient estimates for solutions to the conductivity problem. Math. Ann. **332**(2), 277–286 (2005)

5. Antontsev, S.N., Rodrigues, J.F.: On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara Sez. VII Sci. Mat. **52**(1), 19–36 (2006)
6. Babuška, I., Andersson, B., Smith, P.J., Levin, K.: Damage analysis of fiber composites. I. Statistical analysis on fiber scale. Comput. Methods Appl. Mech. Eng. **172**(1–4), 27–77 (1999)
7. Bao, E., Li, H.G., Li, Y.Y., Yin, B.: Derivative estimates of solutions of elliptic systems in narrow regions. Quart. Appl. Math. **72**(3), 589–596 (2014)
8. Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect conductivity problem. Arch. Ration. Mech. Anal. **193**(1), 195–226 (2009)
9. Bao, E., Li, Y.Y., Yin, B.: Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. Commun. Partial Differ. Equ. **35**(11), 1982–2006 (2010)
10. Bonnetier, E., Triki, F.: Pointwise bounds on the gradient and the spectrum of the Neumann–Poincaré operator: the case of 2 discs, Multi-scale and high-contrast PDE: from modelling, to mathematical analysis, to inversion, pp. 81–91 (2012)
11. Bonnetier, E., Triki, F.: On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D. Arch. Ration. Mech. Anal. **209**(2), 541–567 (2013)
12. Brander, T., Ilmavirta, J., Kar, M.: Superconductive and insulating inclusions for linear and non-linear conductivity equations. Inverse Probl. Imaging **12**(1), 91–123 (2018)
13. Capdeboscq, Y., Yang Ong, S.C.: Quantitative Jacobian determinant bounds for the conductivity equation in high contrast composite media. Discrete Contin. Dyn. Syst. Ser. B **25**(10), 3857–3887 (2020)
14. Ciraolo, G., Sciammetta, A.: Gradient estimates for the perfect conductivity problem in anisotropic media. J. Math. Pures Appl. (9) **127**, 268–298 (2019)
15. Ciraolo, G., Sciammetta, A.: Stress concentration for closely located inclusions in nonlinear perfect conductivity problems. J. Differ. Equ. **266**(9), 6149–6178 (2019)
16. DiBenedetto, E., Manfredi, J.: On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. Am. J. Math. **115**(5), 1107–1134 (1993)
17. Dong, H., Li, H.G.: Optimal estimates for the conductivity problem by Green's function method. Arch. Ration. Mech. Anal. **231**(3), 1427–1453 (2019)
18. Dong, H., Li, Y.Y., Yang, Z.: Optimal gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. J. Eur. Math. Soc. (to appear). (2021). https://doi.org/10.4171/JEMS/1432
19. Dong, H., Li, Y.Y., Yang, Z.: Gradient estimates for the insulated conductivity problem: the non-umbilical case (2022). arXiv:2203.10081
20. Dong, H., Yang, Z., Zhu, H.: The insulated conductivity problem with p-Laplacian. Arch. Ration. Mech. Anal. **247**(5), 95 (2023)
21. Dong, H., Zhang, H.: On an elliptic equation arising from composite materials. Arch. Ration. Mech. Anal. **222**(1), 47–89 (2016)
22. Duzaar, F., Mingione, G.: Gradient estimates via linear and nonlinear potentials. J. Funct. Anal. **259**(11), 2961–2998 (2010)
23. Duzaar, F., Mingione, G.: Gradient estimates via non-linear potentials. Am. J. Math. **133**(4), 1093–1149 (2011)
24. Garroni, A., Kohn, R.V.: Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459**(2038), 2613–2625 (2003)
25. Gorb, Y.: Singular behavior of electric field of high-contrast concentrated composites. Multiscale Model. Simul. **13**(4), 1312–1326 (2015)
26. Gorb, Y., Novikov, A.: Blow-up of solutions to a p-Laplace equation. Multiscale Model. Simul. **10**(3), 727–743 (2012)
27. Idiart, M.I.: The macroscopic behavior of power-law and ideally plastic materials with elliptical distribution of porosity. Mech. Res. Commun. **35**(8), 583–588 (2008)
28. Kang, H., Lim, M., Yun, K.: Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities. J. Math. Pures Appl. (9) **99**(2), 234–249 (2013)
29. Kang, H., Lim, M., Yun, K.: Characterization of the electric field concentration between two adjacent spherical perfect conductors. SIAM J. Appl. Math. **74**(1), 125–146 (2014)
30. Kim, J., Lim, M.: Electric field concentration in the presence of an inclusion with eccentric core-shell geometry. Math. Ann. **373**(1–2), 517–551 (2019)
31. Levy, O., Kohn, R.V.: Duality relations for non-Ohmic composites, with applications to behavior near percolation. J. Statist. Phys. **90**(1–2), 159–189 (1998)

32. Li, H.G.: Asymptotics for the electric field concentration in the perfect conductivity problem. SIAM J. Math. Anal. **52**(4), 3350–3375 (2020)
33. Li, H.G., Li, Y.Y., Yang, Z.: Asymptotics of the gradient of solutions to the perfect conductivity problem. Multiscale Model. Simul. **17**(3), 899–925 (2019)
34. Li, H.G., Wang, F., Xu, L.: Characterization of electric fields between two spherical perfect conductors with general radii in 3D. J. Differ. Equ. **267**(11), 6644–6690 (2019)
35. Li, Y.Y., Yang, Z.: Gradient estimates of solutions to the conductivity problem with flatter insulators. Anal. Theory Appl. **37**(1), 114–128 (2021)
36. Li, Y.Y., Yang, Z.: Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. Math. Ann. **385**(3–4), 1775–1796 (2023)
37. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**(11), 1203–1219 (1988)
38. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. Commun. Partial Differ. Equ. **16**(2–3), 311–361 (1991)
39. Lim, M., Yun, K.: Blow-up of electric fields between closely spaced spherical perfect conductors. Commun. Partial Differ. Equ. **34**(10–12), 1287–1315 (2009)
40. Ruzicka, M.: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
41. Suquet, P.-M.: Overall potentials and extremal surfaces of power law or ideally plastic composites. J. Mech. Phys. Solids **41**(6), 981–1002 (1993)
42. Vázquez, J.L.: A strong maximum principle for some quasilinear elliptic equations. Appl. Math. Optim. **12**(3), 191–202 (1984)
43. Weinkove, B.: The insulated conductivity problem, effective gradient estimates and the maximum principle. Math. Ann. **385**(1–2), 1–16 (2023)
44. Yun, K.: Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. SIAM J. Appl. Math. **67**(3), 714–730 (2007)
45. Yun, K.: Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections. J. Math. Anal. Appl. **350**(1), 306–312 (2009)
46. Yun, K.: An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions. J. Differ. Equ. **261**(1), 148–188 (2016)