Geometry & Topology

Volume 29 (2025)

The motivic lambda algebra and motivic Hopf invariant one problem

William Balderrama
Dominic Leon Culver
J D Quigley

msp

Geometry & Topology **29**:3 (2025) 1489–1570
DOI: 10.2140/gt.2025.29.1489
Published: 31 May 2025

The motivic lambda algebra and motivic Hopf invariant one problem

WILLIAM BALDERRAMA
DOMINIC LEON CULVER
J D QUIGLEY

We investigate forms of the Hopf invariant one problem in motivic homotopy theory over arbitrary base fields of characteristic not equal to 2. Maps of Hopf invariant one classically arise from unital products on spheres, and one consequence of our work is a classification of motivic spheres represented by smooth schemes admitting a unital product.

The classical Hopf invariant one problem was resolved by Adams, following his introduction of the Adams spectral sequence. We introduce the motivic lambda algebra as a tool to carry out systematic computations in the motivic Adams spectral sequence. Using this, we compute the E_2-page of the $\mathbb{R}$-motivic Adams spectral sequence in filtrations $f \leq 3$. This universal case gives information over arbitrary base fields.

We then study the 1-line of the motivic Adams spectral sequence. We produce differentials $d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2$ over arbitrary base fields, which are motivic analogues of Adams' classical differentials. Unlike the classical case, the story does not end here, as the motivic 1-line is significantly richer than the classical 1-line. We determine all permanent cycles on the $\mathbb{R}$-motivic 1-line, and explicitly compute differentials in the universal cases of the prime fields $\mathbb{F}_q$ and $\mathbb{Q}$, as well as $\mathbb{Q}_p$ and $\mathbb{R}$.

55T15; 14F42, 55Q25, 55Q45, 55S10

1 Introduction

Motivic homotopy theory is a homotopy theory for algebraic varieties, developed by Morel and Voevodsky [1999]. Since its conception and subsequent use by Voevodsky [2003; 2011] to resolve the Milnor and Bloch–Kato conjectures, an immense amount of work has gone into the theory, with applications to algebraic geometry, algebraic number theory, and algebraic topology.

Motivic stable homotopy theory is the home of $\mathbb{A}^1$-invariants on algebraic varieties, such as algebraic K-theory, motivic cohomology, and algebraic cobordism. The universal such invariants are motivic stable homotopy groups, and as such the internal structure of the motivic stable homotopy groups of spheres reflects the broad-scale structure of the motivic stable homotopy category. These motivic stable stems encode deep geometric and number-theoretic information; for example, Morel [2004] showed that the Milnor–Witt K-theory of a field appears in its stable stems, and Röndigs, Spitzweck and Østvær [Röndigs et al. 2019; 2021] have identified motivic stable stems in low Milnor–Witt stem in terms of variants of Milnor K-theory, Hermitian K-theory, and motivic cohomology.

Motivic homotopy theory was originally developed to apply ideas and tools from homotopy theory to problems in algebraic geometry and algebraic K-theory. Information now flows the other way as well. After p-completion, $\mathbb{C}$-motivic stable stems capture information about classical stable stems that is not seen using classical techniques. This has led to the highly successful program of Gheorghe, Isaksen, Wang and Xu [Isaksen 2019; Isaksen et al. 2023; Gheorghe et al. 2021], yielding groundbreaking advances in computations of classical stable homotopy groups of spheres. A similar program using $\mathbb{R}$-motivic stable stems to capture information about C_2-equivariant stable stems has also developed [Burklund et al. 2020; Belmont and Isaksen 2022; Dugger and Isaksen 2017a; 2017b; Guillou and Isaksen 2020; Belmont et al. 2021]. More recently, Bachmann, Kong, Wang and Xu [Bachmann et al. 2022] related F-motivic stable homotopy theory over a general field F to classical complex cobordism.

All of this has motivated a swath of explicit computations of motivic stable stems over particular base fields F. We refer the reader to [Isaksen and Østvær 2020] for a general survey, but mention the following 2-primary computations:

- $F=\mathbb{C}$ Dugger and Isaksen [2010] computed the $\mathbb{C}$-motivic stable stems through the 36 stem, and these computations were pushed out to the 90 stem in [Isaksen 2019; Isaksen et al. 2023].
- $F=\mathbb{R}$ Dugger and Isaksen [2017a] computed the first four Milnor–Witt stems over $\mathbb{R}$, and Belmont and Isaksen [2022] expanded on this to compute the first 11 Milnor–Witt stems over $\mathbb{R}$.
- $F=\mathbb{F}_q$ Wilson [2016] and Wilson and Østvær [2017] computed the motivic stable homotopy groups of finite fields in motivic weight zero through topological dimension 18.

There are still many mysteries contained in the motivic stable stems. All of the above computations were enabled by the *motivic Adams spectral sequence*, originally introduced by Morel [1999] and further developed by Dugger and Isaksen [2010]. This is a motivic analogue of the classical Adams spectral sequence, which was developed by Adams [1958; 1960] to resolve the *Hopf invariant one problem*. Adams used this spectral sequence to prove that the only elements of Hopf invariant one in the classical stable stems π_*^{cl} are the classical Hopf maps $\eta_{\mathrm{cl}} \in \pi_1^{\mathrm{cl}}$, $\nu_{\mathrm{cl}} \in \pi_3^{\mathrm{cl}}$, and $\sigma_{\mathrm{cl}} \in \pi_7^{\mathrm{cl}}$. This theorem has a number of implications, including classifications of which spheres can be made into H-spaces, which spheres are parallelizable, which 2-dimensional modules over the Steenrod algebra can be realized by cell complexes, which dimensions a finite-dimensional real division algebra can have, and more.

This paper is concerned with topics surrounding motivic analogues of the classical Hopf invariant one problem. There is an element η in the motivic stable stems, represented by the canonical map $\eta\colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$, which refines the classical complex Hopf map η_{cl}. Hopkins and Morel — see [Morel 2004] — showed that η is one of the generators of the Milnor–Witt K-theory of the base field. This motivic η behaves quite differently from the classical Hopf map; most famously, η is not nilpotent, and is generally not 2-torsion. Because η is not nilpotent, one may consider the η-inverted stable stems $\pi_{*,*}[\eta^{-1}]$. These are closely related to Witt K-theory [Bachmann 2022; Bachmann and Hopkins 2020], and have been the subject of thorough investigation [Andrews and Miller 2017; Guillou and Isaksen 2015; 2016; Ormsby and Röndigs 2020; Wilson 2018].

Using the theory of Cayley–Dickson algebras, Dugger and Isaksen [2013] have shown that the classical quaternionic and octonionic Hopf maps ν_{cl} and σ_{cl} also admit geometric refinements to motivic classes ν and σ. All of these motivic Hopf maps η, ν, and σ are maps of Hopf invariant one, but, unlike classically, they are not the only such maps. For example, the classical stable stems include into the weight 0 portion of the motivic stable stems, and η_{cl}, ν_{cl}, and σ_{cl} give rise to distinct examples of maps of Hopf invariant one in the motivic setting. If we reformulate the condition of a map α having nontrivial Hopf invariant as asking that the homology of the 2-cell complex with attaching map α not split as a module over the motivic Steenrod algebra, then the situation becomes even richer: for example, σ_{cl}^2 admits an $\mathbb{R}$-motivic refinement to a map of nontrivial Hopf invariant in this sense, closely related to the nonexistent Hopf map coming next in the sequence η, ν, σ.

All of this motivates the present work, the purpose of which is three-fold:

(1) to analyze the motivic Hopf invariant one problem and deduce geometric consequences;

(2) to advance our understanding of motivic stable stems over general base fields;

(3) to introduce the *motivic lambda algebra*, a new tool for motivic computations.

As mentioned above, Adams resolved the Hopf invariant one problem by introducing and studying the Adams spectral sequence. Morel [1999] and Dugger and Isaksen [2010] have already introduced the *F-motivic Adams spectral sequence*, which takes the form

$$E_2^{*,*,*} = \mathrm{Ext}_{\mathcal{A}^F}^{*,*,*}(\mathbb{M}^F, \mathbb{M}^F) \Rightarrow \pi_{*,*}^F.$$

Here $\mathcal{A}^F$ is the F-motivic Steenrod algebra [Voevodsky 2003; Hoyois et al. 2017], which acts on $\mathbb{M}^F$, the mod 2 motivic cohomology of $\mathrm{Spec}(F)$. This spectral sequence converges to $\pi^F_{*,*}$, the homotopy groups of the $(2,\eta)$-completed F-motivic sphere [Hu et al. 2011a; Kylling and Wilson 2019]. Implicit is the assumption that 2 is invertible in F.

In this paper, we bring the motivic Adams spectral sequence back to its classical roots, using it to study the motivic Hopf invariant one problem. We do not follow Adams' original approach. Instead, at least in broad outline, we follow J S P Wang's approach [1967], which proceeded by first gaining a good understanding of the E_2-page of the Adams spectral sequence. Importing this approach to motivic homotopy theory requires analyzing the E_2-page of the motivic Adams spectral sequence over general base fields in ranges beyond what is known by previous techniques.

To carry out this analysis, we bring another tool from classical stable homotopy theory into the motivic context: the *lambda algebra*. The classical lambda algebra Λ^{cl} is a certain differential graded algebra, originally constructed by Bousfield, Curtis, Kan, Quillen, Rector and Schlesinger [Bousfield et al. 1966], whose homology recovers the E_2-page of the Adams spectral sequence. The classical lambda algebra is now a standard member of the homotopy theorist's toolbox, and we cannot hope to list all of its applications, but the following are a selection:

(1) Wang's computation [1967] of the E_2-page of the Adams spectral sequence through the 3-line, and subsequent simplified resolution of the Hopf invariant one problem;

(2) some of the first automated computations of the E_2-page of the Adams spectral sequence, including products and Massey products [Tangora 1985; 1993; 1994; Curtis et al. 1987];

(3) the construction of Brown–Gitler spectra [1973], which played an important role in analyzing the *bo*-resolution [Mahowald 1981; Shimamoto 1984], the proof of the immersion conjecture [Cohen 1985], and more [Mahowald 1977; Goerss 1999; Hunter and Kuhn 1999];

(4) the algebraic Atiyah–Hirzebruch spectral sequence for $\mathbb{R}P^\infty$ [Wang and Xu 2016], used as input to their proof of the nonexistence of exotic smooth structures on the 61-sphere [Wang and Xu 2017];

(5) the only complete computations of the 4- and 5-lines of the Adams E_2-term [Chen 2011; Lin 2008].

We expect that the motivic lambda algebra will likewise become a useful member of the motivic homotopy theorist's toolbox. We focus in particular on developing the lambda algebra and applying this to the motivic Hopf invariant one problem. We consider both the unstable problem, with applications to H-space structures on motivic spheres, and the stable problem, which is concerned with the 1-line of the motivic Adams spectral sequence. The motivic situation is substantially richer than the classical situation, and requires us to develop a number of new techniques for motivic computations across general base fields.

Adams' resolution of the classical Hopf invariant one problem asserted the existence of differentials $d_2(h_{a+1}) = h_0 h_a^2$ in the Adams spectral sequence. There are classes h_a in the F-motivic Adams spectral sequence for any field F, corresponding to the motivic Hopf maps discussed above for $a \le 3$. Using

Betti realization, it is possible to lift Adams' differentials to the $\mathbb{C}$-motivic Adams spectral sequence. It follows that, if F admits a complex embedding, then h_{a+1} must support a nontrivial differential for $a \geq 3$. However, this is insufficient to determine the precise target of the differential, as well as to determine what happens over other base fields, particularly fields of positive characteristic. The techniques we develop are geared towards resolving this sort of issue. We use these to obtain a number of new results; let us give the following here, as it is the most pleasant to state.

Theorem A (Theorem 7.3.1) *For an arbitrary base field F of characteristic not equal to 2, there are differentials of the form*

$$d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2$$

in the F-motivic Adams spectral sequence, which are nonzero for $a \geq 3$.

It is worth making a couple remarks to distinguish this from the classical result.

Remark 1.0.1 Classically, there is at most one possible nontrivial target for a d_2-differential on h_{a+1}. As suggested by the target in Theorem A, the motivic situation is more complicated. For example, when $F = \mathbb{R}$, we show that, if $a \geq 4$, then the group of potential values of $d_2(h_{a+1})$ is given by $\mathbb{F}_2\{h_0 h_a^2, \rho h_1 h_a^2\}$. The general picture is similar, except there may be additional interference coming from the mod 2 Milnor K-theory of F. This computation requires new techniques for computing the cohomology of the motivic Steenrod algebra, which is much richer than the analogous classical computation. ◁

Remark 1.0.2 Even once we have carried out the algebraic work of identifying potential values of $d_2(h_{a+1})$, the classical proof does not directly generalize to yield Theorem A. In spirit, our proof follows Wang's classical inductive proof [1967]. The base case of Wang's induction is the differential $d_2(h_4) = h_0 h_3^2$, which follows easily from graded commutativity of stable stems. By contrast, our base case must include the differential $d_2(h_5) = (h_0 + \rho h_1)h_4^2$. Over $\mathbb{R}$, this differential may be deduced by combining complex and real Betti realization, but a completely different argument is required to obtain the differential for other fields. To obtain this differential over other base fields, we use a certain motivic Hasse principle to reduce to considering fields with simple mod 2 Milnor K-theory, then analyze how the classical Kervaire class θ_4 appears in the motivic stable stems. ◁

Remark 1.0.3 There is another elegant proof of the classical Adams differential $d_2(h_{a+1}) = h_0^2 h_a$, due to Bruner [1986b, Corollary 1.5], which makes use of power operations in the Adams spectral sequence. Tilson [2017] has explored analogues of Bruner's results in the $\mathbb{R}$-motivic setting, but so far these methods have only succeeded in determining the $\mathbb{R}$-motivic differential $d_2(h_{a+1})$ for $a \leq 3$. ◁

1.1 Brief overview

Now let us give a very brief overview of what we do in this paper, before giving a more thorough summary in Section 1.2. This paper has three main parts. These parts are not independent, but none rely on the hardest aspects of the others.

The first part is purely algebraic, and is the most computationally intensive. In Section 2, we introduce the F*-motivic lambda algebra* (Theorem B), and in Section 4 we use the $\mathbb{R}$-motivic lambda algebra to compute $\mathrm{Ext}_{\mathbb{R}}$ in filtrations $f \leq 3$ (Theorem C). The result is quite complicated, with eight infinite families of multiplicative generators and numerous relations between these. As we explain in Section 7.1, this gives information about Ext_F for any base field F once the mod 2 Milnor K-theory of F is known.

The second part is shorter, and does not rely on the above computation. In Section 6, after some preliminaries in Section 5, we consider the motivic analogue of the Hopf invariant one problem in its classical *unstable* formulation, concerning unstable 2-cell complexes with specified cup product, as well as concerning geometric applications such as to H-space structures on motivic spheres. Our analysis proceeds by a novel reduction to the classical case and other known results, by first formulating a certain motivic Lefschetz principle (Proposition 5.2.1), then using this to build unstable "Betti realization" functors over arbitrary algebraically closed fields (Proposition 5.3.2). One consequence of this analysis is a complete classification of motivic spheres which are represented by smooth schemes admitting a unital product (Theorem D).

The third part is our main homotopical contribution. In Section 7, we give a detailed study of the 1-line of the F-motivic Adams spectral sequence. This work has a direct geometric interpretation: permanent cycles on the 1-line of the motivic Adams spectral sequence classify how the motivic Steenrod algebra can act on the cohomology of a motivic 2-cell complex. This section does not rely on the full strength of our computation of $\mathrm{Ext}_{\mathbb{R}}$, and should be understandable by the reader familiar with prior work on the $\mathbb{R}$-motivic Adams spectral sequence. The main theorems in this section are Theorem A above, together with much more detailed information about the 1-line of the F-motivic Adams spectral sequence for the particular fields $F = \mathbb{R}$, $F = \mathbb{F}_q$ with q an odd prime power, $F = \mathbb{Q}_p$ with p any prime, and $F = \mathbb{Q}$ (Theorem E). As this includes all the prime fields, these computations give information that applies to an arbitrary base field. When $F = \mathbb{R}$, we completely determine all permanent cycles on the 1-line by comparison with a computation in Borel C_2-equivariant homotopy theory (Theorem F); both the equivariant computation and the method of comparison are of independent interest.

1.2 Summary of results

We now summarize our work in more detail. We begin with our introduction of the motivic lambda algebra. The nature of the classical lambda algebra Λ^{cl} [Bousfield et al. 1966] was greatly clarified by Priddy [1970], who introduced the notion of a *Koszul algebra* and showed that Λ^{cl} is the Koszul complex of the classical Steenrod algebra. We carry out the motivic analogue of this, producing the following.

Theorem B (Section 2.4) *There is a differential graded algebra* Λ^F, *the* F*-motivic lambda algebra*, *with the following properties*:

(1) Λ^F *may be described explicitly in terms of generators*, *relations*, *and monomial basis.*

(2) *There is a surjective and multiplicative quasiisomorphism* $C(\mathcal{A}^F) \to \Lambda^F$ *from the cobar complex of the* F*-motivic Steenrod algebra to* Λ^F*. In particular, there is an isomorphism*

$$H_*\Lambda^F \cong \mathrm{Ext}^*_F$$

compatible with all products and Massey products. Moreover, the squaring operation $\mathrm{Sq}^0 \colon \mathrm{Ext}^*_F \to \mathrm{Ext}^*_F$ *lifts to a map* $\theta \colon \Lambda^F \to \Lambda^F$ *of differential graded algebras.*

(3) Λ^F *generalizes the classical lambda algebra, in the sense that, if* F *is algebraically closed, then* $\Lambda^F[\tau^{-1}] = \Lambda^{\mathrm{cl}}[\tau^{\pm 1}]$*. In particular, it is considerably smaller than* $C(\mathcal{A}^F)$*.*

Here we have abbreviated $\mathrm{Ext}^{*,*,*}_{\mathcal{A}^F}(\mathbb{M}^F, \mathbb{M}^F)$ to Ext^*_F, where the single index refers to filtration, or homological degree, ie $\mathrm{Ext}^f_F = H^f(\mathcal{A}^F)$.

Remark 1.2.1 Several subtleties arise in the construction and identification of the motivic lambda algebra. We note two interesting points here:

(1) Priddy's notion [1970] of Koszul algebra is not general enough for our situation: $\mathcal{A}^F$ is generally not augmented as an $\mathbb{M}^F$-algebra, and $\mathbb{M}^F$ is generally not central in $\mathcal{A}^F$. This forces us to consider a more general notion of a Koszul algebra, as well as to find new arguments to prove that $\mathcal{A}^F$ is Koszul in this more general sense.

(2) As readers familiar with the motivic Adem relations might suspect, the elements τ and ρ of $\mathbb{M}^F$ appear in the relations defining the motivic lambda algebra, as well as in its differential and the endomorphism θ lifting Sq^0. Determining these formulas precisely is delicate and requires some careful arguments. ◁

Remark 1.2.2 As indicated above, we construct the F-motivic lambda algebra as a certain Koszul complex for the F-motivic Steenrod algebra. The Koszul story produces other complexes as well: for any $\mathcal{A}^F$-modules M and N with M projective over $\mathbb{M}^F$, there are complexes $\Lambda^F(M, N)$ serving as small models of the cobar complex computing $\mathrm{Ext}_{\mathcal{A}^F}(M, N)$. An amusing special case of this produces a lambda algebra Λ^{C_2} for the C_2-equivariant Steenrod algebra (Remark 2.3.5). ◁

We use the motivic lambda algebra to study Ext_F in low filtration. Before diving into our more extensive computations, we illustrate the structure of Λ^F with some simple examples in Section 3.1, showing how it may be used to give easy rederivations of some well-known low-dimensional relations in Ext_F. We then carry out our main algebraic computation in Section 4, where we prove the following. Note that $\mathrm{Ext}^0_{\mathbb{R}} = \mathbb{F}_2[\rho]$.

Theorem C *The structure of* $\mathrm{Ext}_{\mathbb{R}}$ *in filtrations* $f \leq 3$ *is as described in Section 4; in particular, the* $\mathbb{F}_2[\rho]$*-module structure is described in Theorem 4.2.12, including a description of multiplicative generators and the action of* Sq^0*, and the majority of the multiplicative structure is described in Theorem 4.3.7.*

Here we are justified in focusing on $\mathrm{Ext}_{\mathbb{R}}$ as it is, in a certain precise sense, the universal case (see Remark 2.2.8). We explain in Section 7.1 how to pass from information about $\mathrm{Ext}_{\mathbb{R}}$ to information about Ext_F for other base fields F.

Example 1.2.3 (Theorem 4.2.12(1)) The computation of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ is much more involved than the corresponding classical computation, and the result is much richer. We refer the reader to Section 4 for the full statements, but illustrate this here with the following sample. Classically, $\mathrm{Ext}_{\mathrm{cl}}^{\leq 3}$ is generated as an algebra by the classes h_a and c_a for $a \geq 0$. By contrast, a minimal multiplicative generating set of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ as an $\mathbb{F}_2[\rho]$-algebra is given by the classes in the following table:

multiplicative generator	ρ-torsion exponent
h_{a+1}	∞
c_{a+1}	∞
$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a$	2^a
$\tau^{2^a(8n+1)} h_{a+2}^2$	$2^{a+1}\cdot 3$
$\tau^{\lfloor 2^{a-1}(2(16n+1)+1)\rfloor} h_{a+3}^2 h_a$	$2^a\cdot 13$
$\tau^{2^a(4(4n+1)+1)} h_{a+3}^3$	$2^a\cdot 7$
$\tau^{\lfloor 2^{a-1}(16n+1)\rfloor} c_a$	$2^a\cdot 7$
$\tau^{2^{a+1}(8n+1)} c_{a+1}$	$2^{a+2}\cdot 3$
$\tau^{\lfloor 2^{a-1}(2(4n+1)+1)\rfloor} c_a$	$2^a\cdot 3$

Here $a, n \geq 0$, and the ρ-torsion exponent of a class α is the minimal r for which $\rho^r\alpha = 0$; the classes h_{a+1} and c_{a+1} are ρ-torsion-free. Note that all of the classes listed are named for their image in $\mathrm{Ext}_{\mathbb{C}}$, and are not themselves products. ◁

Example 1.2.4 Observe that the multiplicative generators h_a and c_a of $\mathrm{Ext}_{\mathrm{cl}}^{\leq 3}$ appear, with a shift, as ρ-torsion-free classes in $\mathrm{Ext}_{\mathbb{R}}$. This is a general phenomenon: Dugger and Isaksen [2017a, Theorem 4.1] produce an isomorphism $\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}] \simeq \mathrm{Ext}_{\mathrm{dcl}}[\rho^{\pm 1}]$; here $\mathrm{Ext}_{\mathrm{dcl}} = \mathrm{Ext}_{\mathrm{cl}}$ only given a motivic grading such that $\mathrm{Ext}_{\mathrm{cl}}^{s,f} = \mathrm{Ext}_{\mathrm{dcl}}^{2s+f,f,s+f}$. As we discuss in Section 3.2, this in fact refines to a splitting $\mathrm{Ext}_{\mathbb{R}} \cong \mathrm{Ext}_{\mathrm{dcl}}[\rho] \oplus \mathrm{Ext}_{\mathbb{R}}^{\rho\text{-tors}}$, where $\mathrm{Ext}_{\mathbb{R}}^{\rho\text{-tors}} \subset \mathrm{Ext}_{\mathbb{R}}$ is the subgroup of ρ-torsion; moreover, this splitting is modeled by a multiplicatively split inclusion $\widetilde{\theta}\colon \Lambda^{\mathrm{dcl}} \to \Lambda^{\mathbb{R}}$. The general shape of $\mathrm{Ext}_{\mathbb{R}}$ forced by this may be illustrated by the description of the 1-line

$$\mathrm{Ext}_{\mathbb{R}}^1 = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a\geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a})\{\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a : n \geq 0\}. \tag{1-1}$$

◁

As $\mathrm{Ext}_{\mathrm{cl}}^{\leq 3}$ is entirely understood by Wang's computation [1967], the hard work of Theorem C is in computing the ρ-torsion subgroup of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$. This is the most computationally intensive part of the paper, and proceeds by a direct case analysis of monomials in $\Lambda^{\mathbb{R}}$ in low filtration. In the end, we find that $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ carries the multiplicative generators listed in Example 1.2.3, and that there are many exotic relations between these generators. Our computation describes all of this.

With the algebraic computation of Theorem C in place, we turn to more homotopical topics, namely those surrounding the *Hopf invariant one problem*. There are (at least) *two* good motivic analogues of the Hopf invariant one problem: one which is unstable, concerning the construction of unstable 2-cell complexes with nontrivial cup product structure, and one which is stable, concerning the construction of stable 2-cell complexes with nontrivial $\mathcal{A}^F$-module structure. As we recall in Section 6.2, understanding the latter question is equivalent to understanding the 1-line of the F-motivic Adams spectral sequence; we get to this in Section 7, which we will discuss further below.

It is the former unstable formulation which has more direct geometric applications. For example, following [Dugger and Isaksen 2013] on the Hopf construction in motivic homotopy theory, it is directly tied up with the question of which unstable motivic spheres $S^{a,b}$ admit H-space structures (see Lemma 6.4.3). Here $S^{a,b}$ is the motivic sphere which is $\mathbb{A}^1$-homotopy equivalent to $\Sigma^{a-b}\mathbb{G}_m^{\wedge b}$. We discuss this unstable formulation in Section 6, which is independent of our other calculations. One pleasant consequence of this story is the following.

Theorem D (Theorem 6.4.5) *The only motivic spheres which are represented by smooth F-schemes admitting a unital product are $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$.*

The statement of Theorem D is directly analogous to the classical result that the only spheres admitting unital products are S^0, S^1, S^3, and S^7. Classically, the nonexistence of H-space structures on any other spheres may be reduced to the Hopf invariant one problem, which was then established by Adams. This reduction makes use of the instability condition that $\mathrm{Sq}^a(x) = x^2$ whenever $x \in H^a(X)$ for some space X. There is an analogous instability condition for the motivic cohomology of a motivic space, but it holds only in a smaller range than we would need; as a consequence, some additional input is needed to analyze the unstable motivic Hopf invariant one problem (see Remark 6.3.2).

This additional input is interesting in itself. It follows from the formulation of the unstable motivic Hopf invariant one problem that, at least for nonexistence, one may reduce to the case where F is algebraically closed. In Section 5.2, we explain how work of Wilson and Østvær [2017] implies a certain *Lefschetz principle* for suitable 2-primary categories of cellular motivic spectra. When combined with Mandell's p-adic homotopy theory [2001], this gives a 2-primary unstable "Betti realization" functor for any algebraically closed field F, which is well behaved with respect to the mod 2 cohomology of motivic cell complexes; see Section 5.3. This gives a direct relation between motivic and classical homotopy theory, and we are then able to analyze the unstable motivic Hopf invariant one problem using a combination of classical results, work of Dugger and Isaksen [2013] on the motivic Hopf construction, and work of Asok, Doran and Fasel [Asok et al. 2017] on smooth models of motivic spheres.

Finally, in Section 7, we turn to a study of the 1-line of the F-motivic Adams spectral sequence. After a few preliminaries, we begin by proving Theorem A, producing the differentials

$$d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2$$

valid for any F (Theorem 7.3.1). As we mentioned above, the main content of this theorem is not the fact that the classes h_{a+1} for $a \geq 3$ support nonzero differentials, but the exact value of the target of these differentials. We mention two interesting aspects of this computation here:

First, in order to get a more explicit handle on possible targets of $d_2(h_{a+1})$, we reduce to considering the case where F is a prime field, ie $F = \mathbb{F}_p$ with p odd or $F = \mathbb{Q}$. The latter case is then handled with the aid of a *Hasse principle*. We explain how work of Ormsby and Østvær [2013] on the structure of $\mathbb{M}^{\mathbb{Q}}$ may be used to give a concrete description of $\mathrm{Ext}_{\mathbb{Q}}$ and of the Hasse map

$$\mathrm{Ext}_{\mathbb{Q}} \to \mathrm{Ext}_{\mathbb{R}} \times \prod_{p \text{ prime}} \mathrm{Ext}_{\mathbb{Q}_p}, \tag{1-2}$$

in particular proving this map is injective (Proposition 7.1.3). In this way we reduce to computing the differentials $d_2(h_n)$ over the fields $\mathbb{F}_p$ with p odd, $\mathbb{Q}_p$ with p prime, and $\mathbb{R}$.

Second, the classical argument, using the fact that $2\sigma^2 = 0$, may be used to compute $d_2(h_4)$, but a new argument is required to produce the differential $d_2(h_5) = (h_0 + \rho h_1)h_4^2$ (Proposition 7.3.3). Once this differential is resolved, the rest follow by an inductive argument analogous to Wang's classical argument [1967]. After a further reduction when $F = \mathbb{R}$, the differential $d_2(h_5)$ may be resolved uniformly in the above choices of base field. In short, to resolve this differential, we lift the Hurewicz map $\pi_*^{\mathrm{cl}} \to \pi_{*,0}^F$ to a map $\mathrm{Ext}_{\mathrm{cl}}^{*,*} \to \mathrm{Ext}_F^{*,*,0}$ of spectral sequences (Proposition 5.1.1) and, by considering the effect of this on the Kervaire class θ_4, deduce that $(h_0 + \rho h_1)h_4^2$ must be hit by h_5.

The story does not stop with the differentials $d_2(h_{a+1})$, as Ext_F^1 contains many more classes than these; recall for instance $\mathrm{Ext}_{\mathbb{R}}^1$ from (1-1). Having resolved these differentials, we move on to giving an explicit analysis of the 1-line of the F-motivic Adams spectral sequence for a number of base fields F. Our main results may be summarized in the following.

Theorem E *The following are carried out in Section 7:*

1. *In Theorem 7.4.9, we compute all d_2-differentials out of $\mathrm{Ext}_{\mathbb{R}}^1$, as well as all permanent cycles in $\mathrm{Ext}_{\mathbb{R}}^1$.*
2. *In Theorem 7.5.3, for q a prime power satisfying $q \equiv 1 \pmod 4$, we compute all Adams differentials out of $\mathrm{Ext}_{\mathbb{F}_q}^1$, in particular giving all permanent cycles in $\mathrm{Ext}_{\mathbb{F}_q}^1$.*
3. *In Theorem 7.5.6, for q a prime power satisfying $q \equiv 3 \pmod 4$, we compute all d_2-differentials out of $\mathrm{Ext}_{\mathbb{F}_q}^1$, as well as all higher differentials in stems $s \leq 7$, in particular giving all permanent cycles in $\mathrm{Ext}_{\mathbb{F}_q}^1$ in stems $s \leq 7$.*
4. *In Theorem 7.6.2, for p an odd prime, we give as much information about $\mathrm{Ext}_{\mathbb{Q}_p}^1$ as was given for $\mathrm{Ext}_{\mathbb{F}_p}^1$.*
5. *In Theorem 7.6.6, we compute all d_2-differentials out of $\mathrm{Ext}_{\mathbb{Q}_2}^1$, as well as all higher differentials in stems $s \leq 7$, in particular giving all permanent cycles in $\mathrm{Ext}_{\mathbb{Q}_2}^1$ in stems $s \leq 7$.*
6. *In Theorem 7.7.1, we give the same information for $\mathrm{Ext}_{\mathbb{Q}}^1$ as was given for $\mathrm{Ext}_{\mathbb{Q}_2}^1$.*

Cases (2)–(6) of Theorem E proceed by a direct analysis, combining the Hopf differentials we produced in Theorem A with arithmetic differentials that may be obtained by comparison with the F-motivic Adams spectral sequence for integral motivic cohomology. The latter has been computed by Kylling [2015] for $F = \mathbb{F}_q$ with q an odd prime power, by Ormsby [2011] for $F = \mathbb{Q}_p$ with p an odd prime, and by Ormsby and Østvær [2013] for $F = \mathbb{Q}_2$ and $F = \mathbb{Q}$. Case (6), where $F = \mathbb{Q}$, may be read off the cases $F = \mathbb{R}$ and $F = \mathbb{Q}_p$, using our good understanding of the Hasse map (1-2). As with Ormsby and Østvær's computations over $\mathbb{Q}$, the final description of the set of d_2-cycles in $\mathrm{Ext}^1_{\mathbb{Q}}$ is quite intricate, but we feel that our techniques show that understanding the $\mathbb{Q}$-motivic Adams spectral sequence for $\pi^{\mathbb{Q}}_{*,*}$ is an accessible problem ripe for future investigation.

The $\mathbb{R}$-motivic computation requires more work. Recall the structure of $\mathrm{Ext}^1_{\mathbb{R}}$ from (1-1). Theorem A describes what happens on the ρ-torsion-free summand of this, but says nothing about the large quantity of ρ-torsion classes. It is possible to use similar methods to compute all d_2-differentials supported on this ρ-torsion summand, and we do so in Proposition 7.4.8. However, this is insufficient to determine which classes in $\mathrm{Ext}^1_{\mathbb{R}}$ are permanent cycles, as higher differentials may, and indeed must, occur.

We resolve this by comparison with *Borel C_2-equivariant homotopy theory*. Behrens and Shah [2020] formulate and prove an equivalence

$$(\mathcal{S}\mathrm{p}^{\mathrm{cell}}_{\mathbb{R}})^{\wedge}_{(2,\rho)}[\tau^{-1}] \simeq \mathrm{Fun}(BC_2, \mathcal{S}\mathrm{p}^{\wedge}_2)$$

between the τ-periodic $(2,\rho)$-complete cellular $\mathbb{R}$-motivic category and the 2-complete Borel C_2-equivariant category. Define

$$\mathrm{Ext}^{s,f,w}_{BC_2} = \mathrm{Ext}^{s-w,f}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, H^* P^{\infty}_w),$$

where P^{∞}_w is a stunted real projective space. These form the E_2-pages of the classical Adams spectral sequences for the stable cohomotopy groups of infinite stunted projective spaces. The equivalence of Behrens and Shah gives an effective method of computing these groups by "inverting τ" in $\mathrm{Ext}_{\mathbb{R}}$. The τ-periodic behavior of $\mathrm{Ext}_{\mathbb{R}}$ is plainly visible in our computation of $\mathrm{Ext}^{\leq 3}_{\mathbb{R}}$, allowing us to directly read off the structure of $\mathrm{Ext}^{\leq 3}_{BC_2}$ (Lemma 7.4.3). In particular,

$$\mathrm{Ext}^1_{BC_2} = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a})\{\tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a : n \in \mathbb{Z}\}$$

(compare (1-1)). We warn the reader that this naming of classes is incompatible with viewing Ext_{BC_2} as a collection of ordinary Adams spectral sequences; for example, h_0 does not detect 2, but instead the transfer $P^{\infty}_0 \to S^0$. We may use the relatively simple structure of these 1-lines to verify that $\mathrm{Ext}^1_{\mathbb{R}} \to \mathrm{Ext}^1_{BC_2}$ reflects permanent cycles (Lemma 7.4.4), and this reduces the identification of permanent cycles in $\mathrm{Ext}^1_{\mathbb{R}}$ to the identification of permanent cycles in $\mathrm{Ext}^1_{BC_2}$. The problem of ρ-torsion permanent cycles in $\mathrm{Ext}^1_{BC_2}$ turns out to be equivalent to the vector fields on spheres problem (Lemma 7.4.5), which was resolved by Adams [1962]. Together with known information regarding the ρ-torsion-free classes, this leads to the following classification of maps $\Sigma^c P^{\infty}_w \to S^0$ detected in Adams filtration 1.

Theorem F (Theorem 7.4.7) *For $a \geq 0$, write $a = c + 4d$ with $0 \leq c \leq 3$, and define $\psi(a) = 2^c + 8d$. Then the subgroup of permanent cycles in $\mathrm{Ext}^1_{BC_2}$ is given by*

$$\mathbb{F}_2[\rho]\{h_1, h_2, h_3, \rho h_4\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{\psi(a)})\{\rho^{2^a - \psi(a)} \tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a : n \in \mathbb{Z}\}.$$

Moreover, one may characterize maps $\Sigma^c P^\infty_w \to S^0$ detected by each of these classes.

1.3 Future directions

The classical lambda algebra has been applied broadly in stable homotopy theory. This suggests several natural directions for future work, and we list a few here.

1.3.1 Homological computations The homology of the classical lambda algebra can be computed algorithmically via a method known as the *Curtis algorithm*. This procedure was refined and implemented by Tangora [1985] to compute the cohomology of the Steenrod algebra through internal degree 56, as well as to compute products and Massey products [Tangora 1993; 1994]; further computations of Curtis, Goerss, Mahowald and Milgram [Curtis et al. 1987] pushed this out to describe the cohomology of the Steenrod algebra through stem 51. More recently, the Curtis algorithm was used by Wang and Xu [2016] to compute the algebraic Atiyah–Hirzebruch spectral sequence for $\mathbb{R}P^\infty$, providing the data necessary for their proof of the uniqueness of the smooth structure on the 61-sphere [Wang and Xu 2017].

Our method for computing $\mathrm{Ext}^{\leq 3}_{\mathbb{R}}$ is closely related to the homology algorithm of [Tangora 1985], only modified to take into account the $\mathbb{F}_2[\rho]$-module structure of $\Lambda^{\mathbb{R}}$, as well to incorporate some additional flexibility in choosing representatives for the sake of a more digestible manual computation. By ignoring this additional flexibility and incorporating the ideas of [loc. cit., Section 3.4], one obtains a Curtis algorithm for computing the homology of the $\mathbb{R}$-motivic lambda algebra, as well as of other motivic lambda complexes. The effectiveness of these procedures in higher dimensions remains to be seen.

In addition to its use in computer-assisted computations, the classical lambda algebra has also been used in [Lin 2008; Chen 2011] to completely compute the cohomology of the classical Steenrod algebra through filtration 5. In principle, there should be no obstruction to continuing our computation of $\mathrm{Ext}^{\leq 3}_{\mathbb{R}}$ to higher filtrations, other than the rather more involved calculations and bookkeeping that this would necessarily take.

1.3.2 Motivic Brown–Gitler spectra Brown–Gitler spectra [1973] have many applications in classical algebraic topology, including Mahowald's analysis [1981; Shimamoto 1984] of the *bo*-resolution, Cohen's solution [1985] of the immersion conjecture, and more [Mahowald 1977; Hunter and Kuhn 1999; Goerss 1999]. The classical lambda algebra was essential for constructing and analyzing Brown–Gitler spectra [1973; Shimamoto 1984] as above, as well as [Goerss et al. 1986]. Culver and Quigley [2021] introduced a motivic analogue of the *bo*-resolution, the kq-resolution, and analyzed it over algebraically closed fields of characteristic zero. The analysis of the kq-resolution over more general base fields would be greatly simplified by the existence of motivic Brown–Gitler spectra.

1.3.3 Unstable motivic Adams spectral sequences The classical lambda algebra Λ^{cl} has certain subcomplexes $\Lambda^{\mathrm{cl}}(n)$ which form the E_1-page of an unstable Adams spectral sequence:

$$E_1 \cong \Lambda^{\mathrm{cl}}(n) \Rightarrow \pi_* S^n.$$

Moreover, James's 2-local fiber sequence [1957]

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1},$$

which gives rise to the EHP sequence, is modeled by short exact sequences [Curtis 1971, Section 11]

$$0 \to \Lambda^{\mathrm{cl}}(n) \to \Lambda^{\mathrm{cl}}(n+1) \to \Sigma^n \Lambda^{\mathrm{cl}}(2n+1) \to 0,$$

which are useful for understanding both the unstable complexes $\Lambda^{\mathrm{cl}}(n)$ and the stable complex Λ^{cl}. It is natural to ask whether there are analogous subcomplexes of Λ^F related to a suitable motivic unstable Adams spectral sequence. The motivic situation seems to be much more delicate: it is not obvious how to define such subcomplexes of Λ^F, and the nature of the cohomology of motivic Eilenberg–Mac Lane spaces suggests that a motivic unstable Adams spectral sequence may not be as well behaved. A better understanding of these topics would shed light both on the nature of Λ^F and on unstable F-motivic homotopy theory.

1.4 Conventions

We maintain the following conventions throughout the paper:

(1) We work solely at the prime 2.

(2) We write F for a base field of characteristic not equal to 2.

(3) We write $\pi^F_{*,*}$ for the homotopy groups of the $(2,\eta)$-completed F-motivic sphere spectrum.

(4) Our homotopy and cohomology groups are bigraded by (s,w), where s is stem and w is weight.

(5) In particular, we write $S^{a,b}$ for the motivic sphere which is $\mathbb{A}^1$-homotopy equivalent to $\Sigma^{a-b}\mathbb{G}_m^{\wedge b}$.

(6) We write $H^{*,*}$ for reduced mod 2 F-motivic cohomology and H^* for reduced ordinary mod 2 cohomology.

(7) We write, for instance, $H^{*,*}(X_+)$ for the unreduced mod 2 motivic cohomology of X.

(8) We will use homological grading even for cohomology classes, in the sense that, if $x \in H^{a,b}(X)$, then we say $|x| = (-a,-b)$. This allows us to say, for instance, $|\tau| = (0,-1)$ and $|\rho| = (-1,-1)$, regardless or whether we are working with homology or cohomology.

(9) We write $\mathbb{M}^F = H^{*,*}(\mathrm{Spec}(F)_+)$ for the unreduced mod 2 motivic cohomology of a point.

(10) We write $\mathbb{M}^F_0$ for the portion of $\mathbb{M}^F$ concentrated on the line $s = w$, so that $\mathbb{M}^F = \mathbb{M}^F_0[\tau]$. (The ring $\mathbb{M}^F_0$ may be identified as the mod 2 Milnor K-theory of F, by work of Voevodsky; see [Isaksen and Østvær 2020, Section 2.1] for an overview of the structure of $\mathbb{M}^F$).

(11) We write Ext_F for the cohomology of the F-motivic Steenrod algebra, employing the grading conventions given in the following two points.

(12) We write Ext_F^f for the filtration f piece of Ext_F.

(13) We write $\mathrm{Ext}_F^{s,f,w} \subset \mathrm{Ext}_F^f$ for the subset of elements in filtration f with topological stem s and weight w.

(14) We use a subscript or superscript cl to denote classical objects; in particular, π_*^{cl} are the classical 2-completed stable stems, $\mathcal{A}^{\mathrm{cl}}$ is the classical mod 2 Steenrod algebra, and $\mathrm{Ext}_{\mathrm{cl}}$ is its cohomology.

(15) We take the binomial coefficient $\binom{a}{b}$ to be $\frac{a!}{b!(a-b)!}$ for $0 \le b \le a$, and to be zero otherwise.

Acknowledgements

The authors thank Uğur Yiğit for sharing a copy of his thesis, which proposes an alternative construction of a C_2-equivariant lambda algebra; Eva Belmont and Dan Isaksen for sharing data produced for their study of $\mathrm{Ext}_{\mathbb{R}}$; and the referees for helpful comments and suggestions. Balderrama thanks Nick Kuhn for some enlightening conversations about real projective spaces. Culver would like to thank the Max Planck Institute for providing a wonderful working environment and financial support while this paper was being written. Quigley thanks Eva Belmont, Dan Isaksen, and Inna Zakharevich for helpful discussions.

Part I The motivic lambda algebra

2 The motivic lambda algebra

In this section, we show that Priddy's construction [1970] of the lambda algebra as a certain Koszul complex can be extended to produce a motivic lambda algebra. As noted in Remark 1.2.1, a more refined notion of Koszulity is needed to handle the more exotic nature of the $\mathbb{M}^F$-algebra $\mathcal{A}^F$. The notion of a Koszul algebra has been generalized in various ways; see [Polishchuk and Positselski 2005] for an account of some developments in this area. We will use the formulation given in [Balderrama 2023, Section 3], as this gives a sufficiently general definition of Koszul algebra and explicit description of their associated Koszul complex. The reader familiar with Koszul algebras will find no surprises in this material.

In Section 2.2, we review the structure of the F-motivic Steenrod algebra $\mathcal{A}^F$. We show that $\mathcal{A}^F$ is in fact a Koszul algebra in Section 2.3, ultimately by reducing to Priddy's classical PBW criterion for Koszulity [Priddy 1970, Section 5]. The *F-motivic lambda algebra* Λ^F is then defined to be the Koszul complex of $\mathcal{A}^F$. We compute the structure of Λ^F explicitly, and introduce an endomorphism θ of Λ^F lifting the squaring operation Sq^0 on Ext_F. All of this structure is summarized in one place in Section 2.4.

2.1 Review of Koszul algebras

This section summarizes the definitions and facts from [Balderrama 2023, Section 3] regarding Koszul algebras which we will use to construct the motivic lambda algebra. We review this material in some detail, in order to specialize from the more abstract context considered there. Many of the results we need have appeared in varying levels of generality throughout the literature; in particular, the definition of Koszulity we use can be considered as a direct generalization of the homogeneous case considered by Rezk [2012, Section 4].

We fix throughout this subsection an associative algebra S to serve as our base ring, together with an associative algebra A which is an S-algebra in the sense of being equipped with an algebra map $S \to A$. Equivalently, A is a monoid in the category of S-bimodules. We abbreviate $\otimes = \otimes_S$.

We are most interested in the case where $S = \mathbb{M}^F$ and $A = \mathcal{A}^F$, and so, to avoid some subtle points regarding signs, we shall assume that S is of characteristic 2. In addition, we suppose throughout that A is projective as a left S-module.

Definition 2.1.1 Say that A is a *graded S-algebra* if we have chosen a decomposition $A = \bigoplus_{n\geq 0} A[n]$ of S-bimodules such that

(1) $S \cong A[0]$;

(2) the product on A restricts to $A[n] \otimes A[m] \to A[n+m]$.

Say that A is a *filtered S-algebra* if we have chosen a filtration $A \cong \operatorname{colim}_{n\to\infty} A_{\leq n}$ such that

(1) $S \cong A_{\leq 0}$;

(2) the product on A restricts to $A_{\leq n} \otimes A_{\leq m} \to A_{\leq n+m}$.

Finally, say that the filtration on a filtered S-algebra A is *projective* if (both A and) the associated graded algebra

$$\operatorname{gr} A := \bigoplus_{n\geq 0} A[n], \qquad A[n] := \operatorname{coker}(A_{\leq n-1} \to A_{\leq n})$$

are projective as left S-modules. ◁

Fix a left A-module M. Write $B^{\mathrm{un}}(A, A, M)$ and $B(A, A, M)$ for the unreduced and reduced bar resolutions of M relative to S; that is, for the unnormalized and normalized chain complexes associated to the standard monadic resolution of M with respect to the adjunction $\mathrm{L}\mathcal{M}\mathrm{od}_S \leftrightarrows \mathrm{L}\mathcal{M}\mathrm{od}_A$. These are projective left A-module resolutions provided that M is projective as a left S-module. If A is a filtered algebra, then $B^{\mathrm{un}}(A, A, M)$ is a filtered complex, with filtration defined by

$$B_n^{\mathrm{un}}(A, A, M)[\leq m] := \operatorname{Im}\Bigg(\bigoplus_{m_1+\cdots+m_n=m} A \otimes A_{\leq m_1} \otimes \cdots \otimes A_{\leq m_n} \otimes M \to B_n^{\mathrm{un}}(A, A, M)\Bigg), \tag{2-1}$$

and this descends to a filtration of $B(A, A, M)$; compare for instance [Priddy 1970, Section 10; Rezk 2012, Section 4; Balderrama 2023, Section 3.5]. If A is augmented, then this augmentation makes S into an A-bimodule, allowing us to form the bar complex $B(A) := S \otimes_A B(A, A, S)$ and consider the homology $H_*(A) := H_*(B(A))$, and the filtration of (2-1) descends to a filtration on $B(A)$. If A is graded, then A is naturally filtered by $A_{\le n} = \bigoplus_{i \le n} A[i]$; this filtration is split in the sense that $A \cong \operatorname{gr} A$, and likewise the filtration on $B(A)$ is split with $\operatorname{gr} B(A) = \bigoplus_{m \ge 0} B(A)[m]$. This then passes to a splitting $H_*(A) \cong \bigoplus_{m \ge 0} H_*(A)[m]$.

Definition 2.1.2 [Rezk 2012, Definition 4.4; Balderrama 2023, Definition 3.5.3] We say that A is a *homogeneous Koszul S-algebra* provided that

(1) A has been given the structure of a graded S-algebra;

(2) $H_n(A)[m] = 0$ for $n \ne m$.

We say that A is a *Koszul S-algebra* if

(1) A has been equipped with a projective filtration;

(2) $\operatorname{gr} A$ is a homogeneous Koszul S-algebra. ◁

Suppose now that A is projectively filtered, and fix a left A-module M which is flat as a left S-module. The filtration of (2-1) on $B(A, A, M)$ induced by that on A satisfies $\operatorname{gr} B(A, A, M) \cong A \otimes B(\operatorname{gr} A) \otimes M$, and so the convergent spectral sequence associated to this filtration is of signature

$$E^1_{p,q} = A \otimes H_q(\operatorname{gr} A)[p] \otimes M \Rightarrow H_q B(A, A, M), \quad d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1}. \tag{2-2}$$

Definition 2.1.3 Let M be an A-module which is flat as a left S-module. The *Koszul resolution* of M is the augmented chain complex

$$M \leftarrow K(A, A, M)$$

defined by

$$K_p(A, A, M) = E^1_{p,p} = A \otimes H_p(\operatorname{gr} A)[p] \otimes M,$$

with differential given by the d^1-differential of the spectral sequence (2-2). When M is projective as a left S-module, we define the *Koszul complex* $K_A(M, M')$ as the cochain complex

$$K_A(M, M') := \operatorname{Hom}_A(K(A, A, M), M') \cong \operatorname{Hom}_S(H_*(\operatorname{gr} A) \otimes M, M'),$$

with differential inherited from that on $K(A, A, M)$. ◁

Observe that, by construction, $K(A, A, M)$ is a subcomplex of $B(A, A, M)$, and dually $K_A(M, M')$ is a quotient complex of the cobar complex $C_A(M, M') := \operatorname{Hom}_A(B(A, A, M), M')$. When A is Koszul, the spectral sequence of (2-2) collapses into the Koszul complex $K(A, A, M)$, proving the following.

Theorem 2.1.4 (see [Priddy 1970, Theorem 3.8; Rezk 2012, Proposition 4.8; Balderrama 2023, Theorem 3.5.5]) *Suppose that A is a Koszul S-algebra, and fix left A-modules M and M'.*

1. *If M is flat over S, then there is an injective quasiisomorphism $K(A,A,M) \subset B(A,A,M)$.*
2. *If M is projective over S, then there is a surjective quasiisomorphism $C_A(M,M') \to K_A(M,M')$.*

In particular, if M is projective over S, the homology of $K_A(M,M')$ is isomorphic to $\mathrm{Ext}_A(M,M')$. □

This allows us to define Koszul complexes in the generality we need. We now recall some facts from [Balderrama 2023, Sections 3.6–3.7] describing the structure of Koszul complexes; these are direct analogues of [Priddy 1970, Theorem 4.6]. We begin by fixing some conventions.

Definition 2.1.5 Fix a left S-module M. Then the dual $M^\vee = \mathrm{L}\mathcal{M}\mathrm{od}_S(M,S)$ carries the structure of a *right* S-module by

$$(f \cdot s)(m) = f(m) \cdot s.$$

If M is in fact an S-bimodule, then $M^\vee$ also carries an S-bimodule structure, with left S-module structure

$$(s \cdot f)(m) = (f(m \cdot s)).$$

Now, if M is a left S-module and M' is an S-bimodule, then there is a comparison map

$$c\colon M^\vee \otimes M'^\vee \to (M' \otimes M)^\vee, \quad c(f \otimes f')(m' \otimes m) = f'(m' f(m)).$$

If M is finitely presented and projective as a left S-module, then this map is an isomorphism. In general, if M'' is another left S-module, then we write

$$M^\vee \mathbin{\hat{\otimes}} M'' := \mathrm{L}\mathcal{M}\mathrm{od}_S(M,M''),$$

so that, in particular,

$$M^\vee \mathbin{\hat{\otimes}} M'^\vee \cong (M' \otimes M)^\vee;$$

in good cases, this may be realized as a topological tensor product, as the notation suggests. ◁

The theory of Koszul algebras is closely related to the theory of quadratic algebras; let us fix some notation.

Definition 2.1.6 Fix an S-bimodule B and subbimodule $R \subset B \otimes B$. The *quadratic algebra* generated by the pair (B,R) is the algebra

$$T(B,R) := \bigoplus_{n \geq 0} T_n(B,R), \quad T_n(B,R) := \mathrm{coker}\Bigl(\sum_{i+j=n} B^{\otimes i-1} \otimes R \otimes B^{\otimes j-1} \to B^{\otimes n}\Bigr),$$

with multiplication inherited from the tensor algebra $T(B)$. Similarly, given a subbimodule $R' \subset B^\vee \mathbin{\hat{\otimes}} B^\vee$ dual to a quotient of $B \otimes B$, we define the completed quadratic algebra

$$\hat{T}(B^\vee,R') := \prod_{n \geq 0} \hat{T}_n(B^\vee,R'), \quad \hat{T}_n(B^\vee,R') := \mathrm{coker}\Bigl(\sum_{i+j=n} (B^\vee)^{\hat{\otimes} i-1} \mathbin{\hat{\otimes}} R' \mathbin{\hat{\otimes}} (B^\vee)^{\hat{\otimes} j-1} \to (B^\vee)^{\hat{\otimes} n}\Bigr).$$

Say that (B,R) is a *quadratic datum* if $T(B,R)$ is projective. In this case, the *dual quadratic datum* to (B,R) is the pair $(B^\vee, R^\perp)$, where $R^\perp = (T_2(B,R))^\vee$. ◁

The cohomology of a homogeneous Koszul algebra may be explicitly described as follows.

Theorem 2.1.7 (see [Priddy 1970, Theorem 2.5; Rezk 2012, Proposition 4.12; Balderrama 2023, Theorem 3.6.4]) (1) *Let (B,R) be a quadratic datum. Then $H^1(T(B,R))[1]\cong B^\vee$, and the inclusion $B^\vee\subset H^*(T(B,R))$ extends to an isomorphism $\widehat{T}(B^\vee,R^\perp)\cong\prod_{n\geq 0}H^n(T(B,R))[n]$.*

(2) *Let $A=\bigoplus_{n\geq 0}A[n]$ be a homogeneous Koszul algebra, and let $R=\ker(A[1]\otimes A[1]\to A[2])$. Then $A\cong T(A[1],R)$ is quadratic, and $H^*(A)\cong\widehat{T}(A[1]^\vee,R^\perp)$.* □

Now fix a quadratic algebra $A=T(A[1],R)$ and left A-modules M and M', supposing that M is projective as a left S-module. We may use Theorem 2.1.7 to describe the Koszul complex $K_A(M,M')$. Recall that

$$K_A^n(M,M')=\mathrm{L}\mathcal{M}\mathrm{od}_A(A\otimes H_n(A)[n]\otimes M,M')\cong\mathrm{L}\mathcal{M}\mathrm{od}_S(H_n(A)[n]\otimes M,M').$$

If we suppose that $H_*(A)$ is projective as a left S-module, as holds if A is Koszul, then there is an isomorphism $(H_n(A))^\vee\cong H^n(A)$ of S-bimodules. In this case, we have

$$K_A^n(M,M')\cong\mathrm{L}\mathcal{M}\mathrm{od}_S(M,H^n(\mathrm{gr}\,A)[n]\mathbin{\widehat{\otimes}}M')\cong\mathrm{L}\mathcal{M}\mathrm{od}_S\bigl(M,\widehat{T}_n(A[1]^\vee,R^\perp)\mathbin{\widehat{\otimes}}M'\bigr);$$

Thus $K_A^*(M,M')$ is completely described as a graded object by Theorem 2.1.7.

It remains to describe the differential on $K_A(M,M')$. Observe first that, if M'' is an additional A-module, then there are pairings

$$\wr\colon K_A^n(M,M')\otimes_{\mathbb{Z}}K_A^{n'}(M',M'')\to K_A^{n+n'}(M,M'').$$

This is a pairing of chain complexes compatible with analogous pairings on cobar complexes and, when A is Koszul, it is a chain-level lift of the standard composition product in Ext_A. In addition, it may be described in terms of the product structure on $\widehat{T}(A[1]^\vee,R^\perp)$ as follows (see [Balderrama 2023, Sections 3.2 and 3.7]). Write μ for the multiplication on $\widehat{T}(A[1]^\vee,R^\perp)$. Then, given $f\colon M\to\widehat{T}_n(A[1]^\vee,R^\perp)\mathbin{\widehat{\otimes}}M'$ and $g\colon M'\to\widehat{T}_{n'}(A[1]^\vee,R^\perp)\mathbin{\widehat{\otimes}}M''$, we have

$$f\wr g=(\mu\otimes 1)\circ(1\otimes g)\circ f.$$

In the special case where $M=M'$, these pairings give $K_A(M,M)$ the structure of a differential graded algebra, and give $K_A(M,M')$ the structure of a differential graded $K_A(M,M)$-$K_A(M',M')$-bimodule. Note that $K_A^1(M,M)=\mathrm{L}\mathcal{M}\mathrm{od}_S(A[1]\otimes M,M)$. The A-module structure on M restricts to an element $Q^M\in K_A^1(M,M)$, and we have the following.

Theorem 2.1.8 [Balderrama 2023, Theorem 3.7.1] *The differential on $K_A(M,M')$ is given by*

$$\delta\colon K_A^n(M,M')\to K_A^{n+1}(M,M'),\quad \delta(f)=Q^M\wr f-f\wr Q^{M'}.$$

In particular, if $M=M'$, then $\delta(f)$ is the commutator $[Q^M,f]$. □

This theorem describes Koszul complexes for a homogeneous Koszul algebra. Suppose now that A is an arbitrary Koszul S-algebra, and continue to fix left A-modules M and M' with M projective as a left S-module. The additive and multiplicative structure of the Koszul complexes $K_A(M,M')$ depend only on the algebra $\operatorname{gr} A$ and left S-modules M and M', and so are still described by Theorem 2.1.7. In practice, the differential on $K_A(M, M')$ may be identified using the following.

Let $qR = \ker(A_{\leq 1} \otimes A_{\leq 1} \to A_{\leq 2})$, and observe that $(A_{\leq 1}, qR)$ is a quadratic datum. Let $A^{\mathrm{big}} = \bigoplus_{n \geq 0} A_{\leq n}$. This is a graded algebra, and the inclusion $A_{\leq 1} \subset A^{\mathrm{big}}$ extends to a map $T(A_{\leq 1}, qR) \to A^{\mathrm{big}}$ of graded algebras.

Theorem 2.1.9 [Balderrama 2023, Theorem 3.7.3] (1) *$T(A_{\leq 1}, qR) \to A^{\mathrm{big}}$ is an isomorphism of graded algebras.*

(2) *A^{big} is a homogeneous Koszul algebra.*

(3) *The surjection $A^{\mathrm{big}} \to A$ gives rise to short exact sequences*

$$0 \to K_A^n(M, M') \to K_{A^{\mathrm{big}}}^n(M, M') \to K_A^{n-1}(M, M') \to 0,$$

which are split if A is augmented.

In particular, $K_A(M, M') \subset K_{A^{\mathrm{big}}}(M, M')$ is a subcomplex with differential on the target described by Theorem 2.1.8. □

2.2 The motivic Steenrod algebra

We will construct the motivic lambda algebra by applying the theory recalled in Section 2.1 to the mod 2 motivic Steenrod algebra, whose structure we now recall. The conventions of Section 1.4 are in force throughout this section.

We note in particular that, following these conventions, we take the somewhat unconventional approach of consistently using *homological grading*. Thus, for example, $\tau \in H^{0,1}(\mathrm{Spec}(F)_+)$, but we shall write $|\tau| = (0, -1)$, as this is how it will appear in the lambda algebra.

We begin by recalling the general structure of the base ring $\mathbb{M}^F = H^{*,*}(\mathrm{Spec}(F)_+)$.

Example 2.2.1 For any F, we have $\mathbb{M}^F = \mathbb{M}_0^F[\tau]$, where

$$|\tau| = (0, -1)$$

and $\mathbb{M}_0^F \subset \mathbb{M}^F$ is the subring concentrated on the line $s = w$, isomorphic to the Milnor K-theory of F taken mod 2. The following are some particular examples of the ring $\mathbb{M}_0^F$. We refer the reader to [Isaksen and Østvær 2020, Section 2.1] for further details.

- For $F = \overline{F}$ algebraically closed, such as $F = \mathbb{C}$, we have

$$\mathbb{M}_0^{\overline{F}} \cong \mathbb{F}_2$$

- For $F = \mathbb{R}$ the real numbers, we have

$$\mathbb{M}_0^{\mathbb{R}} \cong \mathbb{F}_2[\rho],$$

  where $|\rho| = (-1, -1)$.

- For $F = \mathbb{F}_q$ a finite field of odd prime-power order q, we have
$$\mathbb{M}_0^{\mathbb{F}_q} \cong \begin{cases} \mathbb{F}_q[u]/u^2 & \text{if } q \equiv 1 \pmod 4, \\ \mathbb{F}_q[\rho]/\rho^2 & \text{if } q \equiv 3 \pmod 4, \end{cases}$$
where $|\rho| = |u| = (-1,-1)$.
- For $F = \mathbb{Q}_p$ the p-adic rationals with p an arbitrary prime, we have
$$\mathbb{M}_0^{\mathbb{Q}_p} \cong \begin{cases} \mathbb{F}_2[\pi,u]/(\pi^2,u^2) & \text{if } q \equiv 1 \pmod 4, \\ \mathbb{F}_2[\pi,\rho]/(\rho^2,\rho\pi+\pi^2) & \text{if } q \equiv 3 \pmod 4, \\ \mathbb{F}_2[\pi,\rho,u]/(\rho^3,u^2,\pi^2,\rho u,\rho\pi,\rho^2+u\pi) & \text{if } q = 2, \end{cases}$$
where $|\rho| = |u| = |\pi| = (-1,-1)$.

See also Section 7.1 for a discussion of $\mathbb{M}^{\mathbb{Q}}$. ◁

Voevodsky [2003] (with minor corrections by Riou [2012]) and Hoyois, Kelly and Østvær [Hoyois et al. 2017] have constructed Steenrod squares
$$\mathrm{Sq}^a \colon H^{m,n}(X) \to H^{m+a,n+\lfloor a/2 \rfloor}(X)$$
for $a \geq 0$ and shown that they generate the algebra $\mathcal{A}^F$ of natural operations in mod 2 motivic cohomology. It is convenient to take the convention that $\mathrm{Sq}^a = 0$ for $a < 0$. The relations between these squares are generated by $\mathrm{Sq}^0 = 1$ together with the *Adem relations*:

Theorem 2.2.2 [Voevodsky 2003, Theorem 10.2; Riou 2012, théorème 4.5.1; Hoyois et al. 2017, Theorem 5.1] *Fix positive integers a and b with $a < 2b$.*

If a is even and b is odd, then
$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{0 \leq j \leq \lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{\substack{1 \leq j \leq \lfloor a/2 \rfloor \\ j \text{ odd}}} \binom{b-1-j}{a-2j} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j .$$

If a and b are odd, then
$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{\substack{1 \leq j \leq \lfloor a/2 \rfloor \\ j \text{ odd}}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j .$$

If a and b are even, then
$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{0 \leq j \leq \lfloor a/2 \rfloor} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j .$$

If a is odd and b is even, then
$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{\substack{0 \leq j \leq \lfloor a/2 \rfloor \\ j \text{ even}}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{\substack{1 \leq j \leq \lfloor a/2 \rfloor \\ j \text{ odd}}} \binom{b-1-j}{a-1-2j} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j .$$

In all cases, the bounds on summation are not necessary, but give regions where the given binomial coefficients may be nonzero. □

As with the classical Steenrod algebra, $\mathcal{A}^F$ admits an admissible basis.

Definition 2.2.3 Given a sequence $I = (r_1, \dots, r_k)$ with $r_i > 0$ for all $1 \le i \le k$, we abbreviate $\mathrm{Sq}^I = \mathrm{Sq}^{r_1} \dots \mathrm{Sq}^{r_k}$. Say that Sq^I is *admissible* if $r_i \ge 2r_{i+1}$ for all $1 \le i \le k-1$. ◁

Proposition 2.2.4 [Voevodsky 2003, Section 11] *$\mathcal{A}^F$ is freely generated as a left $\mathbb{M}^F$-module by the admissible squares Sq^I.* □

The mod 2 motivic cohomology $H^{*,*}(X_+)$ of any smooth scheme X carries the structure of a left $\mathcal{A}$-module. These actions satisfy the following *Cartan formulas*.

Proposition 2.2.5 [Voevodsky 2003, Proposition 9.6; Riou 2012, Proposition 4.4.2] *Let $a \ge 0$ and $x, y \in H^{*,*}(X_+)$. Then*

$$\mathrm{Sq}^{2a}(xy) = \sum_{r=0}^{a} \mathrm{Sq}^{2r}(x)\,\mathrm{Sq}^{2a-2r}(y) + \tau \sum_{s=0}^{a-1} \mathrm{Sq}^{2s+1}(x)\,\mathrm{Sq}^{2a-2s-1}(y),$$

$$\mathrm{Sq}^{2a+1}(xy) = \sum_{r=0}^{a} (\mathrm{Sq}^{2r+1}(x)\,\mathrm{Sq}^{2ai-2r}(y) + \mathrm{Sq}^{2r}(x)\,\mathrm{Sq}^{2a-2r+1}(y)) + \rho \sum_{s=0}^{a-1} \mathrm{Sq}^{2s+1}(x)\,\mathrm{Sq}^{2a-2s-1}(y).$$ □

The action of $\mathcal{A}^F$ on $\mathbb{M}^F$ is determined by these Cartan formulas and the following.

Proposition 2.2.6 [Voevodsky 2003; Röndigs and Østvær 2016, Appendix A] *The action of $\mathcal{A}^F$ on $\mathbb{M}^F$ satisfies*

$$\mathrm{Sq}^{\ge 1}(x) = 0 \quad \text{for } x \in \mathbb{M}^F_0, \qquad \mathrm{Sq}^1(\tau) = \rho, \qquad \mathrm{Sq}^{\ge 2}(\tau) = 0.$$ □

As in the classical case, the Cartan formulas of Proposition 2.2.5 may be encoded in a coproduct on the algebra $\mathcal{A}^F$. The resulting structure is not quite a Hopf algebra, but is dual to a Hopf algebroid structure on the dual Steenrod algebra $(\mathcal{A}^F)^\vee$. This complication arises in part due to the following. The Steenrod algebra $\mathcal{A}^F$ is an $\mathbb{M}^F$-algebra, by way of the homomorphism $\mathbb{M}^F \to \mathcal{A}^F$ sending an element $x \in \mathbb{M}^F$ to the stable operation given by left multiplication by x. However, $\mathbb{M}^F$ does not land in the center of $\mathcal{A}^F$; equivalently, $\mathcal{A}^F$ has nontrivial $\mathbb{M}^F$-bimodule structure. We may describe this structure explicitly as follows.

Proposition 2.2.7 *The $\mathbb{M}^F$-bimodule structure of $\mathcal{A}^F$ is determined by*

$$\mathrm{Sq}^n\, x = x\,\mathrm{Sq}^n \quad \text{for } x \in \mathbb{M}^F_0,$$
$$\mathrm{Sq}^{2n}\,\tau = \tau\,\mathrm{Sq}^{2n} + \rho\tau\,\mathrm{Sq}^{2n-1},$$
$$\mathrm{Sq}^{2n+1}\,\tau = \tau\,\mathrm{Sq}^{2n+1} + \rho\,\mathrm{Sq}^{2n} + \rho^2\,\mathrm{Sq}^{2n-1}.$$

Proof It suffices to show both sides of each equality coincide when evaluated on an arbitrary cohomology class. For example, for any X and $x \in H^{*,*}(X_+)$, we have

$$\begin{aligned}(\mathrm{Sq}^{2n}\,\tau)(x) = \mathrm{Sq}^{2n}(\tau x) &= \sum_{i+j=n} (\mathrm{Sq}^{2i}\,\tau)(\mathrm{Sq}^{2j}\,x) + \tau \sum_{i+j=n-1} (\mathrm{Sq}^{2i+1}\,\tau)(\mathrm{Sq}^{2j+1}\,x) \\ &= \tau\,\mathrm{Sq}^{2n}(x) + \rho\tau\,\mathrm{Sq}^{2n-1}(x)\end{aligned}$$

by Proposition 2.2.5. This proves the second equation, and the other cases are similar. □

Remark 2.2.8 Although we work in this section over an arbitrary base field F, there is a sense in which $F = \mathbb{R}$ represents the universal case: the class ρ may be defined over any field F, making $\mathbb{M}^F$ into an $\mathbb{M}^{\mathbb{R}}$-module, and in all cases we have

$$\mathcal{A}^F = \mathbb{M}^F \otimes_{\mathbb{M}^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}.$$

In fact, the formulas of Proposition 2.2.6 describe an action of $\mathcal{A}^{\mathbb{R}}$ on $\mathbb{M}^F$ for which

$$\mathrm{Ext}_F \cong \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}, \mathbb{M}^F),$$

and at least additively this depends only on the $\mathbb{F}_2[\rho]$-module structure of $\mathbb{M}^F_0$.

It is worth putting this observation in a slightly more general context. The Cartan formulas of Proposition 2.2.5 give the category of left $\mathcal{A}^{\mathbb{R}}$-modules a symmetric monoidal structure. If R is a monoid in this category, then the tensor product $R \otimes_{\mathbb{M}^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}$ may be equipped with a product with the property that

$$\mathrm{LMod}_{R\otimes_{\mathbb{M}^{\mathbb{R}}}\mathcal{A}^{\mathbb{R}}} \simeq \mathrm{LMod}_R(\mathrm{LMod}_{\mathcal{A}^{\mathbb{R}}});$$

this is the *semitensor product* of [Massey and Peterson 1965]. Moreover, we have

$$\mathrm{Ext}_{R\otimes_{\mathbb{M}^{\mathbb{R}}}\mathcal{A}^{\mathbb{R}}}(R, R) \cong \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}, R).$$

The algebras $\mathcal{A}^F$ are obtained in the case where $R = \mathbb{M}^F$. Another simple class of example is given by the algebras $\mathcal{A}^{\mathbb{R}}/(\rho^n, \tau^m)$, where n and m are such that τ^m is central in $\mathcal{A}^{\mathbb{R}}/(\rho^n)$. A more interesting example is the following: there is an isomorphism of algebras

$$\mathcal{A}^{C_2} \cong \mathbb{M}^{C_2} \otimes_{\mathbb{M}^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}},$$

where $\mathcal{A}^{C_2}$ is the C_2-equivariant Steenrod algebra, $\mathbb{M}^{C_2}$ is the C_2-equivariant cohomology of a point, and $\mathcal{A}^{\mathbb{R}}$ acts on $\mathbb{M}^{C_2}$ as described, for instance, in [Guillou et al. 2020, Section 2] (building on [Hu and Kriz 2001]). ◁

2.3 The motivic lambda algebra

We now produce the motivic lambda algebra. For simplicity of notation, we consider the base field F as fixed, and abbreviate

$$\mathcal{A} = \mathcal{A}^F, \quad \mathbb{M} = \mathbb{M}^F$$

throughout this subsection.

2.3.1 Koszulity of $\mathcal{A}$ We begin by showing that $\mathcal{A}$ is Koszul. The algebra $\mathcal{A}$ is a projectively filtered $\mathbb{M}$-algebra under the length filtration: $\mathcal{A}_{\leq n} \subset \mathcal{A}$ is the submodule generated by squares Sq^I where I is a sequence of length at most n. In particular,

$$\mathcal{A}_{\leq 1} = \mathbb{M}\{\mathrm{Sq}^a : a \geq 0\}$$

as a left $\mathbb{M}$-module, with the understanding that $\mathrm{Sq}^0 = 1$ in $\mathcal{A}$. By Definition 2.1.3, to show that $\mathcal{A}$ is Koszul we must show that $\operatorname{gr}\mathcal{A}$ is homogeneous Koszul. To show that the classical Steenrod algebra is Koszul, Priddy [1970, Theorem 5.3] developed a *PBW criterion* for Koszulity. We cannot apply this criterion directly, in part due to the nontrivial $\mathbb{M}$-bimodule structure of $\operatorname{gr}\mathcal{A}$. Our strategy is to filter this issue away, thereby reducing to Priddy's criterion.

Theorem 2.3.1 *$\mathcal{A}$ is a Koszul $\mathbb{M}$-algebra.*

Proof As $\mathcal{A}$ is a projectively filtered algebra, we need only show that $\operatorname{gr}\mathcal{A}$ is a homogeneous Koszul algebra, ie that $H_n(\operatorname{gr}\mathcal{A})[m] = 0$ for $n \neq m$. To that end, we define a new filtration $\overline{F}_\bullet \operatorname{gr}\mathcal{A}$ on $\operatorname{gr}\mathcal{A}$ by declaring $\overline{F}_{\leq m} \operatorname{gr}\mathcal{A} \subset \operatorname{gr}\mathcal{A}$ to be generated by elements of the form Sq^I, where $I = (r_1, \dots, r_k)$ is a sequence satisfying $r_1 + \cdots + r_k \leq m$. This filtration is multiplicative, and so we may form its associated graded algebra $\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A}$.

The same construction employed in Section 2.1 shows that the filtration $\overline{F}_\bullet \operatorname{gr}\mathcal{A}$ induces a filtration on the bar complex $B(\operatorname{gr}\mathcal{A})$ with associated graded $B(\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A})$. This filtration is compatible with the decomposition

$$B(\operatorname{gr}\mathcal{A}) \cong \bigoplus_{m \geq 0} B(\operatorname{gr}\mathcal{A})[m],$$

and so, for each m, there is a convergent spectral sequence

$$E_1^n = H_n B(\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A})[m] \Rightarrow H_n(\operatorname{gr}\mathcal{A})[m].$$

It is thus sufficient to verify that $\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A}$ is a homogeneous Koszul algebra with respect to the grading $\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A} = \bigoplus_{m \geq 0} \overline{\operatorname{gr}} \operatorname{gr}^m \mathcal{A}$. By passing from $\operatorname{gr}\mathcal{A}$ to $\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A}$, we have filtered away both the nontrivial $\mathbb{M}$-bimodule structure on $\operatorname{gr}\mathcal{A}$ described in Proposition 2.2.7 and the parts of the Adem relations involving ρ which appear in Theorem 2.2.2, and in the end we may identify

$$\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A} \cong \mathbb{M}^F \otimes_{\mathbb{F}_2[\tau]} \operatorname{gr}\mathcal{A}^{\mathbb{C}}.$$

From here, it is easily seen that the admissible basis of $\overline{\operatorname{gr}} \operatorname{gr}\mathcal{A}$ satisfies Priddy's PBW criterion [1970, Section 5.1]. It now follows from [loc. cit., Theorem 5.3] that $\overline{\operatorname{gr}} \operatorname{gr} A$ is Koszul; the assumption in [loc. cit.] that the base is a field is not needed so long as everything in sight is free over the base. □

Remark 2.3.2 When $F = \mathbb{R}$, the filtration $\overline{F}_\bullet \operatorname{gr} A$ coincides with the ρ-adic filtration of $\operatorname{gr} A$. The use of $\overline{F}$ allows us to apply our argument uniformly to arbitrary base fields, but we could have also proved Theorem 2.3.1 in the $\mathbb{R}$-motivic case, and deduced the general case from this. Indeed, everything in Section 2.1 is compatible with base change (see [Balderrama 2023, Lemma 3.5.7]), so Koszulity of $\mathcal{A}^{\mathbb{R}}$

implies that any algebra obtained from the construction of Remark 2.2.8 is Koszul. As an example not explicitly covered by the statement of Theorem 2.3.1, $\mathcal{A}^{C_2}$ is Koszul over $\mathbb{M}^{C_2}$. ◁

Definition 2.3.3 The *F-motivic lambda algebra* Λ^F is the Koszul complex $K_{\mathcal{A}^F}(\mathbb{M}^F, \mathbb{M}^F)$ associated to the Koszul $\mathbb{M}^F$-algebra $\mathcal{A}^F$, as defined in Definition 2.1.3, where $\mathcal{A}^F$ acts on $\mathbb{M}^F$ as described in Proposition 2.2.6. ◁

We shall abbreviate $\Lambda = \Lambda^F$ throughout the rest of this subsection. Theorem 2.1.4 now implies the following.

Theorem 2.3.4 *Let $C(\mathcal{A}) = C_{\mathcal{A}}(\mathbb{M}, \mathbb{M})$ denote the cobar complex of $\mathcal{A}$. Then there is a surjective multiplicative quasiisomorphism*

$$C(\mathcal{A}) \to \Lambda.$$

In particular,

$$H_*\Lambda \cong \mathrm{Ext}^*_{\mathcal{A}}(\mathbb{M}, \mathbb{M}),$$

and this isomorphism is compatible with all products and Massey products. □

Remark 2.3.5 More generally, the theory recalled in Section 2.1 produces and describes Koszul complexes $K_{\mathcal{A}}(M, M')$ modeling the cobar complex $C_{\mathcal{A}}(M, M')$ for any left $\mathcal{A}$-modules M and M' with M projective over $\mathbb{M}$. Classically, the case where $M = H^*(\mathbb{R}P^\infty)$ and $M' = \mathbb{F}_2$ is of particular importance. Another amusing example is given over $F = \mathbb{R}$ with the observation that $K_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}, \mathbb{M}^{C_2}) \cong K_{\mathcal{A}^{C_2}}(\mathbb{M}^{C_2}, \mathbb{M}^{C_2}) = \Lambda^{C_2}$ (see Remarks 2.2.8 and 2.3.2). ◁

2.3.2 The structure of the motivic lambda algebra We will now apply the theory recalled in Section 2.1 to describe Λ explicitly. First note that $\Lambda = \bigoplus_{m \geq 0} \Lambda[m]$ with $\Lambda[1] = (\mathcal{A}[1])^\vee$, where $\mathcal{A}[1] = \mathrm{coker}(\mathbb{M} \to \mathcal{A}_{\leq 1})$. As a left $\mathbb{M}$-module, we may identify

$$\mathcal{A}[1] = \mathbb{M}\{\mathrm{Sq}^r : r \geq 1\}.$$

Dualizing, we may identify

$$\Lambda[1] = \{\lambda_r : r \geq 0\}\mathbb{M}$$

as a right $\mathbb{M}$-module, where λ_r is dual to Sq^{r+1} in the given basis. Considering internal algebraic degrees yields $|\lambda_r| = \big(r+1, \lfloor \tfrac{1}{2}(r+1) \rfloor\big)$; following our conventions (Section 1.4), we subtract off the filtration from the algebraic stem to obtain the topological stem, and so instead write $|\lambda_r| = \big(r, \lceil \tfrac{1}{2} r \rceil\big)$.

We now begin by describing the multiplicative structure of Λ.

Proposition 2.3.6 *The left $\mathbb{M}$-module structure on $\Lambda[1]$ is determined by*

$$x\lambda_n = \lambda_n x \quad \text{for } x \in \mathbb{M}_0, \qquad \tau\lambda_{2n+1} = \lambda_{2n+1}\tau + \lambda_{2n+2}\rho, \qquad \tau\lambda_{2n} = \lambda_{2n}\tau + \lambda_{2n+1}\tau\rho + \lambda_{2n+2}\rho^2.$$

Proof This follows by dualizing Proposition 2.2.7. □

Proposition 2.3.7 *If a is odd or b is even, then*

$$\lambda_a\lambda_{2a+b+1} = \sum_{0\leq r<b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r},$$

and if a is even and b is odd, then

$$\lambda_a\lambda_{2a+b+1} = \sum_{0\leq r<b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r}\tau^{(r-1)\bmod 2} + \sum_{0\leq r\leq (b+1)/2} \lambda_{a+b+1-r}\lambda_{2a+1+r}\binom{\lfloor\frac{1}{2}b\rfloor-\lfloor\frac{1}{2}r\rfloor}{\lfloor\frac{1}{2}r\rfloor}\rho.$$

Proof By Theorem 2.1.7, the bimodule of relations defining Λ as a quadratic algebra with generating bimodule $\Lambda[1]$ may be identified as $\mathcal{A}[2]^\vee = \ker(\mathcal{A}[1]^\vee\otimes\mathcal{A}[1]^\vee\to R^\vee)$, where $R\subset\mathcal{A}[1]\otimes\mathcal{A}[1]$ is the projection of the subbimodule $qR\subset\mathcal{A}_{\leq 1}\otimes\mathcal{A}_{\leq 1}$ of Adem relations recalled in Theorem 2.2.2. It follows by direct computation that this kernel is generated by the indicated relations. □

Remark 2.3.8 Unless both a and b are even, the Adem relation expanding a product of the form $\lambda_a\lambda_b$ is exactly as in the classical lambda algebra. ◁

The additive structure of Λ may be understood just as in the classical case.

Definition 2.3.9 Given a sequence $I=(r_1,\dots,r_n)$, write $\lambda_I=\lambda_{r_1}\cdots\lambda_{r_n}$. Call the sequence I *coadmissible* if $2r_i\geq r_{i+1}$ for all $1\leq i\leq n-1$. ◁

Proposition 2.3.10 *Λ is freely generated as a right $\mathbb{M}$-module by classes of the form λ_I, where I is a coadmissible sequence.*

Proof The relations of Proposition 2.3.7 imply that the coadmissible classes λ_I generate Λ as a right $\mathbb{M}$-module, and we must only verify that they do so freely. Following Remarks 2.2.8 and 2.3.2, there is an isomorphism

$$\Lambda\cong\Lambda^{\mathbb{R}}\otimes_{\mathbb{M}^{\mathbb{R}}}\mathbb{M};$$

thus we may reduce to the case where $F=\mathbb{R}$. By construction, Λ is free as a right $\mathbb{M}$-module. Thus, to show that the coadmissible classes λ_I freely generate Λ over $\mathbb{M}$, it is sufficient to verify the same for $\Lambda/(\rho)[\tau^{-1}]$ over $\mathbb{M}/(\rho)[\tau^{-1}]$. There is an isomorphism $\Lambda/(\rho)[\tau^{-1}]\cong\Lambda^{\mathrm{cl}}\otimes_{\mathbb{F}_2}\mathbb{F}_2[\tau^{\pm1}]$, so this follows from the classical case. □

Finally, we describe the differential on Λ by applying Theorem 2.1.8.

Proposition 2.3.11 *The differential on Λ is determined by the Leibniz rule, together with*

$$\delta(x)=0\quad\text{for } x\in\mathbb{M}_0\qquad \delta(\tau)=\lambda_0\rho,\qquad \delta(\lambda_n)=\sum_{1\leq r\leq n/2}\lambda_{n-r}\lambda_{r-1}\binom{n-r}{r}.$$

Proof Recall the construction $\mathcal{A}^{\text{big}} = \bigoplus_{m\geq 0} \mathcal{A}_{\leq m}$ used in the statement of Theorem 2.1.9. By inspection, we find that $\mathcal{A}^{\text{big}}$ may be identified as the "big motivic Steenrod algebra", defined with generators and relations the same as $\mathcal{A}$ only without the stipulation that $\mathrm{Sq}^0 = 1$. Let $\Lambda^{\text{big}} = K_{\mathcal{A}^{\text{big}}}(\mathbb{M}, \mathbb{M})$, where $\mathcal{A}^{\text{big}}$ acts on $\mathbb{M}$ through the quotient $\mathcal{A}^{\text{big}} \to \mathcal{A}$, ie with Sq^0 acting by the identity.

Theorem 2.1.9 tells us that $\mathcal{A}^{\text{big}}$ is a homogeneous Koszul algebra, and that there is an inclusion $\Lambda \subset \Lambda^{\text{big}}$ of differential graded algebras. As $\mathcal{A}^{\text{big}}$ is homogeneous Koszul, Theorem 2.1.7 applies to show that Λ^{big} is generated by classes λ_r for $r \geq -1$, subject to relations of the same form as described for Λ in Propositions 2.3.6 and 2.3.7. The inclusion $\Lambda \subset \Lambda^{\text{big}}$ is the obvious one, identifying Λ as the subalgebra of Λ^{big} generated by the classes λ_r for $r \geq 0$.

Theorem 2.1.8 describes the differential on Λ^{big} as

$$\delta(f) = [Q, f] = Q \cdot f - f \cdot Q,$$

where $Q \in \Lambda^{\text{big}}[1] \cong (\mathcal{A}^{\text{big}}[1])^\vee$ is the map $\mathcal{A}^{\text{big}}[1] \cong \mathcal{A}_{\leq 1} \otimes \mathbb{M} \to \mathbb{M}$ induced by the action of $\mathcal{A}^{\text{big}}$ on $\mathbb{M}$. In the basis $\mathcal{A}^{\text{big}}[1] = \mathcal{A}_{\leq 1} = \mathbb{M}\{\mathrm{Sq}^r : r \geq 0\}$, this map is the projection onto Sq^0, which by definition is the class $\lambda_{-1} \in \Lambda^{\text{big}}$. So the differential on Λ^{big} is given by

$$\delta(f) = [\lambda_{-1}, f] = \lambda_{-1} f - f \lambda_{-1},$$

and $\Lambda \subset \Lambda^{\text{big}}$ is closed under this. The proposition follows upon expanding out this commutator using the relations defining the algebra Λ^{big}. □

Remark 2.3.12 The description of the differential on Λ as the commutator $\delta(f) = [\lambda_{-1}, f]$ has appeared classically as well; see [Bruner 1988, page 83]. ◁

2.3.3 A closed formula for $\delta(\tau^n)$ Proposition 2.3.11 gives a recursive process for computing $\delta(\tau^n)$. It is possible to solve this recursion, and we do so here. Recall that the pair $(\mathbb{M}, \mathcal{A}^\vee)$ carries the structure of a Hopf algebroid. In particular, $\mathcal{A}^\vee$ is a commutative ring, and $\mathcal{A}^\vee_{\leq 1}$ is a quotient of this ring. Now, the differential $\delta\colon \Lambda[0] \to \Lambda[1]$ may be described as the composite

$$\eta_R + \eta_L \colon \Lambda[0] = \mathbb{M} \to \mathcal{A}^\vee \to \mathcal{A}^\vee_{\leq 1} \to \operatorname{coker}(\mathbb{M} \to \mathcal{A}^\vee_{\leq 1}) = \Lambda[1],$$

where $\eta_L, \eta_R \colon \mathbb{M} \to \mathcal{A}^\vee$ are given by $\eta_R(m)(a) = \epsilon(ma)$ and $\eta_L(m)(a) = \epsilon(am)$, where $\epsilon\colon \mathcal{A} = \mathcal{A} \otimes_{\mathbb{M}} \mathbb{M} \to \mathbb{M}$ encodes the action of $\mathcal{A}$ on $\mathbb{M}$.

We may use this interpretation to compute $\delta(\tau^n)$. The full structure of the Hopf algebroid $(\mathbb{M}, \mathcal{A}^\vee)$ was determined by Voevodsky [2003]; however, we only need a small piece of this, which is easily computed by hand from the structure of $\mathcal{A}$ recalled in Section 2.2. We record this piece in the following.

Lemma 2.3.13 *There is an isomorphism of rings*

$$\mathcal{A}^\vee_{\leq 1} = \mathbb{M}[\tau_0, \xi_1]/(\tau_0^2 + \xi_1 \tau_0 \rho + \xi_1 \tau),$$

where the quotient map

$$\mathcal{A}^\vee_{\leq 1} \to \Lambda[1]$$

acts by

$$\tau_0^\epsilon \xi_1^n \mapsto \lambda_{2n-1+\epsilon}$$

for $\epsilon \in \{0,1\}$ and $n \geq 0$, with the interpretation that $\lambda_{-1} = 0$. Moreover, the maps $\eta_L, \eta_R \colon \mathbb{M} \to \mathcal{A}_{\leq 1}^\vee$ act by

$$\eta_R(x) = x \quad \text{for } x \in \mathbb{M}, \qquad \eta_L(x) = x \quad \text{for } x \in \mathbb{M}_0, \qquad \eta_L(\tau) = \tau + \tau_0\rho.$$

Proof The structure of the ring $\mathcal{A}_{\leq 1}^\vee$ may be read off the coproduct of $\mathcal{A}$, as given in Proposition 2.2.5, and its relation with our basis of $\Lambda[1]$ then follows by construction. The behavior of the left and right units may be read off the $\mathbb{M}$-bimodule structure of $\mathcal{A}_{\leq 1}$ as given in Proposition 2.3.11, together with knowledge of the counit map $\epsilon \colon \mathcal{A}_{\leq 1} \to \mathbb{M}$ given in Proposition 2.2.6. □

The main input to our computation of $\delta(\tau^n)$ is the following elementary computation.

Lemma 2.3.14 *In the ring $\mathcal{A}_{\leq 1}^\vee$, we have*

$$\tau_0^n = \sum_{\substack{\epsilon \in \{0,1\} \\ (n-\epsilon)/2 \leq i \leq n-1}} \tau_0^\epsilon \xi_1^i \binom{i+\epsilon-1}{n-i-1} \tau^{n-i-\epsilon} \rho^{2i-n+\epsilon}.$$

These bounds on i are not necessary, but give a region where the binomial coefficients may be nonzero.

Proof We first compute τ^n in the quotient ring

$$\mathbb{F}_2[\tau_0, \xi_1]/(\tau_0^2 + \xi_1\tau_0 + \xi_1)$$

of $\mathcal{A}_{\leq 1}^\vee$, in which both τ and ρ are set to 1. Clearly,

$$\tau_0^n = \sum_{0 \leq i \leq n} (\xi_1^i c_{n,i} + \tau_0 \xi_1^i d_{n,i})$$

for some $c_{n,i}, d_{n,i} \in \mathbb{F}_2$. The relation

$$\tau_0^n = \xi_1(\tau_0^{n-1} + \tau_0^{n-2})$$

gives rise to recurrence relations

$$c_{n,i} = c_{n-1,i-1} + c_{n-2,i-1}, \quad d_{n,i} = d_{n-1,i-1} + d_{n-2,i-1}.$$

Set $c'_{i,n} = c_{n+i,i}$ and $d'_{i,n} = d_{n+i,i}$. Then these relations become

$$c'_{i,n} = c'_{i-1,n-1} + c'_{i-1,n}, \quad d'_{i,n} = d'_{i-1,n-1} + d'_{i-1,n},$$

exactly as seen in Pascal's triangle. Paired with the initial conditions

$$c'_{i,0} = c'_{0,1} = d'_{1,0} = 0, \quad c'_{1,1} = 1 = d'_{0,1},$$

we find that

$$c'_{i,n} = \binom{i-1}{n-1}, \quad d'_{n,i} = \binom{i}{n-1},$$

and thus

$$c_{n,i}=\binom{i-1}{n-i-1},\quad d_{n,i}=\binom{i}{n-i-1}.$$

Plugging this back in, we find

$$\tau_0^n=\sum_{0\le i\le n}\left(\xi_1^i\binom{i-1}{n-i-1}+\tau_0\xi_1^i\binom{i}{n-i-1}\right)=\sum_{\substack{\epsilon\in\{0,1\}\\0\le i\le n}}\tau_0^\epsilon\xi_1^i\binom{i+\epsilon-1}{n-i-1}.$$

To compute τ_0^n in $\mathcal{A}^\vee_{\le 1}$ itself, recall that $|\tau|=(0,-1)$, $|\rho|=(-1,-1)$, $|\tau_0|=(1,0)$, $|\xi_1|=(2,1)$. Solving

$$|\tau_0^n|=|\tau_0^\epsilon\xi_1^i\tau^a\rho^b|$$

yields

$$a=n-i-\epsilon,\quad b=2i-n+\epsilon.$$

It follows that

$$\tau_0^n=\sum_{\substack{\epsilon\in\{0,1\}\\0\le i\le n}}\tau_0^\epsilon\xi_1^i\binom{i+\epsilon-1}{n-i-1}\tau^{n-i-\epsilon}\rho^{2i-n+\epsilon}$$

in $\mathcal{A}^\vee_{\le 1}$. For this binomial coefficient to be nonzero, we require

$$0\le i+\epsilon-1,\quad 0\le n-i-1,\quad n-i-1\le i+\epsilon-1,$$

giving the stated bounds on summation. □

Proposition 2.3.15 *The differential δ satisfies*

$$\delta(\tau^n)=\sum_{r\ge 0}\lambda_r\binom{n+\lfloor\frac{1}{2}r\rfloor}{r+1}\tau^{n-\lfloor r/2\rfloor-1}\rho^{r+1}.$$

Proof Following Lemma 2.3.13, to compute $\delta(\tau^n)$ one may compute

$$\tau^n+(\tau+\tau_0\rho)^n$$

in terms of the standard basis of $\mathcal{A}^\vee_{\le 1}=\mathbb{M}[\tau_0,\xi_1]/(\tau_0^2+\xi_1\tau_0\rho+\xi_1\tau)$. Moreover, it is sufficient to work in the quotient of $\mathcal{A}^\vee_{\le 1}$ wherein τ and ρ are set to 1, as the necessary quantity of τ's and ρ's may be recovered by comparing degrees, just as in the proof of Lemma 2.3.14. Using Lemma 2.3.14, we find

$$1+(1+\tau_0)^n=\sum_{1\le k\le n}\binom{n}{k}\tau_0^k=\sum_{1\le k\le n}\binom{n}{k}\sum_{\substack{\epsilon\in\{0,1\}\\i\ge 0}}\binom{i+\epsilon-1}{k-i-1}\tau_0^\epsilon\xi_1^i;$$

here we are free to omit the bounds of summation on i, as they merely recorded when certain binomial coefficients were zero. The coefficient of $\tau_0^\epsilon\xi_1^i$ in this sum is

$$\sum_{1\le k\le n}\binom{n}{k}\binom{i+\epsilon-1}{k-i-1}=\sum_{1\le k\le n}\binom{n}{k}\binom{i+\epsilon-1}{2i+\epsilon-k}=\binom{n+i+\epsilon-1}{2i+\epsilon};$$

here the first equality uses the standard identity $\binom{a}{b} = \binom{a}{a-b}$, and the second uses Vandermonde's identity. Adding in a sufficient number of ρ's and τ's, and converting to $\Lambda[1]$, we learn

$$\delta(\tau^n) = \sum_{\substack{\epsilon\in\{0,1\},\, i\geq 0\\ (i,\epsilon)\neq(0,0)}} \lambda_{2i+\epsilon-1}\binom{n+i+\epsilon-1}{2i+\epsilon}\tau^{n-i-\epsilon}\rho^{2i+\epsilon}.$$

Set $r = 2i+\epsilon-1$. Then $\lfloor \frac{1}{2}r \rfloor = i+\epsilon-1$, leading to the given description. □

2.3.4 Lift of Sq^0 The dual motivic Steenrod algebra $\mathcal{A}^\vee$ is a commutative Hopf algebroid, and thus its cohomology, which agrees by definition with $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M},\mathbb{M})$, is equipped with algebraic Steenrod operations [Bruner 1986a]. The purpose of this section is to lift the operation Sq^0 to an endomorphism of Λ. Our approach essentially follows the proof of [Palmieri 2007, Proposition 1.4].

Let $C(\mathcal{A}) = C_{\mathcal{A}}(\mathbb{M},\mathbb{M})$ denote the cobar complex of the algebra $\mathcal{A}$; this is by definition the same as the cobar complex of the Hopf algebroid $\mathcal{A}^\vee$. As $\mathcal{A}^\vee$ is a commutative ring, $C(\mathcal{A})$ is the Moore complex of a cosimplicial commutative ring, and the levelwise Frobenius on this cosimplicial commutative ring induces a map

$$\sigma\colon C(\mathcal{A}) \to C(\mathcal{A}).$$

This is a map of differential graded algebras, and Sq^0 is the map induced by σ in homology.

Theorem 2.3.16 *The map $\sigma\colon C(\mathcal{A}) \to C(\mathcal{A})$ induced by the levelwise Frobenius descends to a map*

$$\theta\colon \Lambda \to \Lambda$$

of differential graded algebras. This map is given on generators by

$$\theta(x) = x^2 \quad \text{for } x\in\mathbb{M}, \qquad \theta(\lambda_{2n-1}) = \lambda_{4n-1}, \qquad \theta(\lambda_{2n}) = \lambda_{4n+1}\tau + \lambda_{4n+2}\rho.$$

Proof Recall $\mathcal{A}^{\mathrm{big}}$ and Λ^{big} from the proof of Proposition 2.3.11. Let $C(\mathcal{A}^{\mathrm{big}})$ be the cobar complex for $\mathcal{A}^{\mathrm{big}}$ with respect to augmentation of $\mathcal{A}^{\mathrm{big}}$, so that $H_*C(\mathcal{A}^{\mathrm{big}}) = \Lambda^{\mathrm{big}}$ as algebras. The levelwise Frobenius gives a map

$$\sigma\colon C(\mathcal{A}^{\mathrm{big}}) \to C(\mathcal{A}^{\mathrm{big}})$$

of differential graded algebras and, by taking homology, this induces a map

$$\theta'\colon \Lambda^{\mathrm{big}} \to \Lambda^{\mathrm{big}}$$

of algebras. We claim that to produce θ it suffices to show that θ' restricts to an endomorphism of $\Lambda \subset \Lambda^{\mathrm{big}}$ satisfying the given formulas. Indeed, there is an inclusion $C(\mathcal{A}) \subset C(\mathcal{A}^{\mathrm{big}})$ of algebras, which does not respect differentials but does commute with the levelwise Frobenius σ. It would thus follow that the restriction θ of θ' to Λ is induced by the levelwise Frobenius on $C(\mathcal{A})$. In particular, this would show that $\sigma\colon C(\mathcal{A}) \to C(\mathcal{A})$ indeed descends to an algebra map $\theta\colon \Lambda\to\Lambda$. That θ moreover respects the differential is inherited from σ.

To understand θ', it suffices to understand its effect on the generators of Λ^{big}, ie to understand the map

$$\theta'\colon \Lambda^{\mathrm{big}}[1] \to \Lambda^{\mathrm{big}}[1].$$

Recall that $\Lambda^{\mathrm{big}}[1] = (\mathcal{A}^{\mathrm{big}}[1])^\vee = \mathcal{A}^\vee_{\leq 1}$. This ring was described in Lemma 2.3.13, and θ' acts on it by the Frobenius. We find that θ' satisfies the same formulas as described for θ, only with the addition that $\theta'(\lambda_{-1}) = \lambda_{-1}$. In particular, θ' does restrict to Λ, and this restriction satisfies the stated formulas. $\square$

2.4 Summary

For ease of reference, let us summarize what we have learned in one place. As always, F is a base field of characteristic not equal to 2.

2.4.1 Generators There is a differential graded algebra Λ^F, the *F-motivic lambda algebra*, together with a multiplicative quasiisomorphism $C(\mathcal{A}^F) \to \Lambda^F$, where $C(\mathcal{A}^F)$ is the reduced cobar complex of $\mathcal{A}^F$. We write $\Lambda^F = \bigoplus_{m\geq 0} \Lambda^F[m]$, where the differential on Λ^F is of the form $\delta\colon \Lambda^F[m] \to \Lambda^F[m+1]$.

The F-motivic lambda algebra Λ^F is an $\mathbb{M}^F$-bimodule algebra, generated by classes $\lambda_r \in \Lambda^F[1]$ for $r \geq 0$. In the trigrading (stem, filtration, weight), we have

$$|\tau| = (0,0,-1), \quad |\rho| = (-1,0,-1), \quad |\lambda_a| = \left(a, 1, \left\lceil \tfrac{1}{2}a \right\rceil\right).$$

A right $\mathbb{M}^F$-module basis of Λ^F is given by those $\lambda_{r_1}\cdots\lambda_{r_n}$ with $2r_i \geq r_{i+1}$ for $1 \leq i \leq n-1$.

2.4.2 Relations The F-motivic lambda algebra is a quadratic $\mathbb{M}^F$-bimodule algebra. Recall that $\mathbb{M}^F = \mathbb{M}^F_0[\tau]$. The $\mathbb{M}^F$-bimodule structure of Λ^F is determined by

$$x\lambda_n = \lambda_n x \quad \text{for } x \in \mathbb{M}_0^F, \qquad \tau\lambda_{2n+1} = \lambda_{2n+1}\tau + \lambda_{2n+2}\rho, \qquad \tau\lambda_{2n} = \lambda_{2n}\tau + \lambda_{2n+1}\tau\rho + \lambda_{2n+2}\rho^2,$$

and the quadratic relations are given as follows. Fix $a, b \geq 0$. If a is odd or b is even, then

$$\lambda_a\lambda_{2a+b+1} = \sum_{0\leq r<b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r};$$

and if a is even and b is odd, then

$$\lambda_a\lambda_{2a+b+1} = \sum_{0\leq r<b/2} \lambda_{a+b-r}\lambda_{2a+1+r}\binom{b-r-1}{r}\tau^{(r-1) \bmod 2} + \sum_{0\leq r\leq \lceil b/2\rceil} \lambda_{a+b+1-r}\lambda_{2a+1+r}\binom{\lfloor \frac{1}{2}b\rfloor - \lfloor \frac{1}{2}r\rfloor}{\lfloor \frac{1}{2}r\rfloor}\rho.$$

2.4.3 Differentials The differential on Λ is determined by the Leibniz rule, together with

$$\delta(x) = 0 \quad \text{for } x \in \mathbb{M}_0^F, \quad \delta(\tau) = \lambda_0\rho, \quad \delta(\lambda_n) = \sum_{1\leq r\leq n/2} \lambda_{n-r}\lambda_{r-1}\binom{n-r}{r}.$$

Moreover, we have

$$\delta(\tau^n) = \sum_{r\geq 0} \lambda_r \binom{n + \lfloor \frac{1}{2}r\rfloor}{r+1}\tau^{n-\lfloor r/2\rfloor -1}\rho^{r+1}.$$

2.4.4 The endomorphism θ The squaring operation $\mathrm{Sq}^0\colon \mathrm{Ext}_F^{s,f,w} \to \mathrm{Ext}_F^{2s+f,f,w+f}$ lifts to an endomorphism $\theta\colon \Lambda^F \to \Lambda^F$ of differential graded algebras, determined by

$$\theta(x) = x^2 \quad \text{for } x \in \mathbb{M}^F, \qquad \theta(\lambda_{2n-1}) = \lambda_{4n-1}, \qquad \theta(\lambda_{2n}) = \lambda_{4n+1}\tau + \lambda_{4n+2}\rho.$$

3 Some first examples, and the doubling map

3.1 First examples

Before continuing on, we give some basic examples illustrating the form of the motivic lambda algebra. In particular, we use Λ^F to define some classes in Ext_F, and reprove some well-known low-dimensional relations. This material is meant only to familiarize the reader with Λ^F; we give a more thorough and entirely self-contained investigation in Section 4.

Given a cycle $z \in \Lambda^F$, in this section we write $[z] \in \mathrm{Ext}_F$ for the corresponding cohomology class.

Lemma 3.1.1 *We have $\delta(\lambda_{2^a-1}) = 0$ for all $a \geq 0$.*

Proof The proof is identical to the proof of [Wang 1967, Proposition 2.2]. □

This allows us to define the following Hopf elements.

Definition 3.1.2 Let $h_a := [\lambda_{2^a-1}]$. ◁

Lemma 3.1.3 *If $\rho = 0$ in $\mathbb{M}^F$, such as when F is algebraically closed, then $\delta(\tau^n) = 0$ for all $n \geq 0$.*

Proof This is immediate from the differential $\delta(\tau) = \lambda_0\rho$. □

In general, if ρ is nilpotent in $\mathbb{M}^F$, then various powers of τ will be cycles in Λ^F. We shall write τ^n in place of $[\tau^n]$ in this case. We begin by considering some examples in the case where F is algebraically closed.

Proposition 3.1.4 *For F algebraically closed, there is a relation*

$$\tau \cdot h_1^3 = h_2 h_0^2.$$

Proof By definition, $\tau \cdot h_1^3 = [\lambda_1^3\tau]$ and $h_2h_0^2 = h_0^2h_2 = [\lambda_0^2\lambda_3]$. We have

$$\lambda_0^2\lambda_3 = \lambda_1^3\tau,$$

so these classes coincide in Ext_F. □

Proposition 3.1.5 *For F algebraically closed, there is a relation*

$$\tau \cdot h_1^4 = 0.$$

However, $h_1^n \neq 0$ for any n.

Proof Observe that $\lambda_0\lambda_1 = 0$, and thus $h_1 h_0 = 0$. Combined with Proposition 3.1.4, we find

$$\tau \cdot h_1^4 = \tau h_1^3 \cdot h_1 = h_2 h_0^2 \cdot h_1 = 0.$$

Alternatively, $\tau h_1^4 = [\lambda_1^4 \tau]$, and there is a differential

$$\delta(\lambda_2^2\lambda_1) = \lambda_1^4\tau.$$

On the other hand, for h_1^n to vanish, the class λ_1^n must be nullhomotopic, ie $\delta(x) = \lambda_1^n$ for some $x \in \Lambda$. The class x must live in stem $n+1$, weight n, and filtration $n-1$, and in this degree Λ is generated by the cycle $\lambda_3\lambda_1^{n-2}$. So no such x exists. □

Next we consider some examples relevant to base fields F over which ρ does not vanish. We begin by defining some classes. Note that the differential

$$\delta(\tau) = \lambda_0\rho$$

implies that $\delta(\tau^{2^n}) \equiv 0 \pmod{\rho^{2^n}}$. This allows for the following definition.

Definition 3.1.6 If $F = \mathbb{R}$, then
$$\tau^{2^{a-1}} h_a := \left[\frac{1}{\rho^{2^a}}\delta(\tau^{2^a})\right]$$
for $a \geq 1$. In general, $\tau^{2^{a-1}} h_a \in \mathrm{Ext}_F$ is defined by pushing these classes forward along the map $\Lambda^{\mathbb{R}} \to \Lambda^F$ induced by $\mathbb{M}^{\mathbb{R}} \to \mathbb{M}^F$ (see Remark 2.2.8). ◁

Remark 3.1.7 Following our convention that Λ^F is considered primarily as a right $\mathbb{M}^F$-module, it would be more natural to write $h_a\tau^{2^{a-1}}$ for the classes introduced above. We have chosen instead to work with naming conventions compatible with those in [Belmont and Isaksen 2022], as no confusion should arise. ◁

Remark 3.1.8 If $\tau^{2^{a-1}}$ is a cycle in Ext_F, then $\tau^{2^{a-1}} h_a = \tau^{2^{a-1}} \cdot h_a$. ◁

Example 3.1.9 We have

$$\tau h_1 = [\lambda_1\tau + \lambda_2\rho], \quad \tau^2 h_2 = [\lambda_3\tau^2 + \lambda_5\tau\rho^2 + \lambda_6\rho^3], \quad \tau^4 h_3 = [\lambda_7\tau^4 + \lambda_{11}\tau^2\rho^4 + \lambda_{13}\tau\rho^6 + \lambda_{14}\rho^7].$$

In fact, we may identify $\tau^{\lfloor 2^{a-1}\rfloor} h_a = [\tau^{2^a}\lambda_{2^a-1}]$ for all $a \geq 1$. ◁

The following relation was proved over $\mathbb{R}$ by Dugger and Isaksen [2017a, Proof of Lemma 6.2] using Massey products and May's convergence theorem. We may use the lambda algebra to provide an explicit direct proof.

Proposition 3.1.10 *There is a relation*

$$(h_0 + \rho h_1) \cdot \tau h_1 = 0.$$

Proof By definition,

$$h_0 \cdot \tau h_1 = [\lambda_0(\lambda_1\tau + \lambda_2\rho)], \quad \rho h_1 \cdot \tau h_1 = [\rho\lambda_1(\lambda_1\tau + \lambda_2\rho)].$$

Expanding, we have

$$\lambda_0(\lambda_1\tau+\lambda_2\rho)=\lambda_1^2\tau\rho+\lambda_1\lambda_2\rho^2+\lambda_2\lambda_1\rho^2, \quad \rho h_1(\lambda_1\tau+\lambda_2\rho)=\lambda_1^2\tau\rho+\lambda_1\lambda_2\rho^2.$$

But

$$\delta(\lambda_3\tau\rho+\lambda_4\rho^2)=\lambda_2\lambda_1\rho^2,$$

so $h_0\cdot\tau h_1=\rho h_1\cdot\tau h_1$. The proposition follows. □

The fact that $\delta(\tau^n)\equiv 0 \pmod{\rho}$ allows for the following definition.

Definition 3.1.11 If $F=\mathbb{R}$, then

$$\tau^{2n}h_0 := \Big[\frac{1}{\rho}\delta(\tau^{2n+1})\Big].$$

In general, $\tau^{2n}h_0\in \mathrm{Ext}_F$ is defined by pushing these classes forward along the map $\Lambda^{\mathbb{R}}\to\Lambda^F$ induced by $\mathbb{M}^{\mathbb{R}}\to\mathbb{M}^F$ (see Remark 2.2.8). ◁

Example 3.1.12 We have

$$\begin{aligned}
h_0&=[\lambda_0],\\
\tau^2h_0&=[\lambda_0\tau^2+\lambda_1\tau^2\rho+\lambda_3\tau\rho^3+\lambda_4\rho^4],\\
\tau^4h_0&=[\lambda_0\tau^4+\lambda_3\tau^3\rho^3+\lambda_4\tau^2\rho^4+\lambda_5\tau^2\rho^5+\lambda_7\tau\rho^7+\lambda_8\rho^8].
\end{aligned}$$ ◁

The following proposition was originally proved over $\mathbb{R}$ by Dugger and Isaksen [2017a, Proof of Lemma 6.2] using Massey products, May's convergence theorem, and analysis of the ρ-Bockstein spectral sequence. Using the lambda algebra, the proof amounts to checking that the products of cycle representatives are equal.

Proposition 3.1.13 *There is a relation*

$$\tau^2h_0\cdot h_1=\rho(\tau h_1)^2.$$

Proof We may directly compute

$$\begin{aligned}
\tau^2h_0\cdot h_1&=[(\lambda_0\tau^2+\lambda_1\tau^2\rho+\lambda_3\tau\rho^3+\lambda_4\rho^4)\lambda_1]\\
&=[\lambda_1^2\tau^2\rho+\lambda_2\lambda_1\tau\rho^2+\lambda_2^2\rho^3+\lambda_2\lambda_3\rho^4]\\
&=[\rho(\lambda_1\tau+\lambda_2\rho)^2]=\rho(\tau h_1)^2.
\end{aligned}$$ □

3.2 The doubling map

Dugger and Isaksen [2017a, Theorem 4.1] produce an isomorphism

$$\mathrm{Ext}_{\mathrm{cl}}[\rho^{\pm1}]\cong\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}],$$

which doubles internal degrees. We can lift this isomorphism to a quasiisomorphism of lambda algebras.

Proposition 3.2.1 *Let Λ^{dcl} denote the classic lambda algebra, only given a motivic grading where $|\lambda_n|$ has stem $2n+1$ and weight $n+1$. For any F, there is a retraction*

$$\Lambda^{\mathrm{dcl}} \xrightarrow{\widetilde{\theta}} \Lambda^F \to \Lambda^{\overline{F}} \xrightarrow{q} \Lambda^{\mathrm{dcl}}$$

with the following properties:

(1) *All maps shown are maps of differential graded algebras respecting θ.*

(2) *$\widetilde{\theta}$ is given on generators by $\widetilde{\theta}(\lambda_n) = \lambda_{2n+1}$.*

(3) *q is given on generators by $q(\tau) = 0$, $q(\lambda_{2n}) = 0$, and $q(\lambda_{2n+1}) = \lambda_n$.*

Now say $F = \mathbb{R}$, and write $\mathrm{Ext}_{\mathbb{R}}^{\rho\text{-tors}} \subset \mathrm{Ext}_{\mathbb{R}}$ for the ρ-torsion subgroup of $\mathrm{Ext}_{\mathbb{R}}$.

(4) *The map $\mathrm{Ext}_{\mathrm{dcl}}[\rho] \oplus \mathrm{Ext}_{\mathbb{R}}^{\rho\text{-tors}} \to \mathrm{Ext}_{\mathbb{R}}$ induced by $\widetilde{\theta}$ and the inclusion of ρ-torsion is an isomorphism.*

(5) *In particular, $\widetilde{\theta}$ extends to a quasiisomorphism $\Lambda^{\mathrm{dcl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\rho^{\pm 1}] \to \Lambda^{\mathbb{R}}[\rho^{-1}]$.*

Proof The assignments given in (2) and (3) are easily seen to extend to maps of differential graded algebras, proving (1), and that the resulting sequence is a retraction is clear. Evidently (4) implies (5), so we are left with proving (4).

It is equivalent to verify that the composite $\mathrm{Ext}_{\mathrm{dcl}}[\rho] \to \mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{R}} / \mathrm{Ext}_{\mathbb{R}}^{\rho\text{-tors}}$ is an isomorphism. This is a split inclusion of free $\mathbb{F}_2[\rho]$-modules, so for it to be an isomorphism it is sufficient to verify that it is an isomorphism after inverting ρ, and for this it is sufficient for the injection $\mathrm{Ext}_{\mathrm{dcl}}[\rho^{\pm 1}] \to \mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$ to be an isomorphism. By Dugger and Isaksen's isomorphism [2017a, Theorem 4.1] $\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}] \cong \mathrm{Ext}_{\mathrm{dcl}}[\rho^{\pm 1}]$, we find that our map $\mathrm{Ext}_{\mathrm{dcl}}[\rho^{\pm 1}] \to \mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$ is an injection between vector spaces of equal finite rank in each degree, and is thus an isomorphism. □

Remark 3.2.2 Proposition 3.2.1 has the following amusing corollary: there is a multiplicative injection

$$Q : \ker(\mathrm{Sq}^0 : \mathrm{Ext}_{\mathrm{cl}} \to \mathrm{Ext}_{\mathrm{cl}}) \to \mathrm{Ext}_{\mathbb{C}}^{\tau\text{-tors}},$$

acting in degrees as Sq^0 would. For example, as $\widetilde{\theta}\lambda_0^n = \lambda_1^n$, we find that $Q(h_0^n) = h_1^n$. This provides another explanation of the fact that h_1 is not nilpotent in $\mathrm{Ext}_{\mathbb{C}}$. It is natural to ask whether Q accounts for all indecomposable τ-torsion classes in $\mathrm{Ext}_{\mathbb{C}}$, but a counterexample is given by the class B_6 in stem 55 and filtration 7, as $\mathrm{Ext}_{\mathrm{cl}}^{24,7} = 0$. ◁

4 $\mathrm{Ext}_{\mathbb{R}}$ in filtrations $f \leq 3$

In this section, we use the $\mathbb{R}$-motivic lambda algebra to compute $\mathrm{Ext}_{\mathbb{R}}^f$ for $f \leq 3$. Throughout this section, we shall abbreviate

$$\Lambda = \Lambda^{\mathbb{R}}.$$

4.1 Preliminaries

We begin by describing our strategy for computing $\mathrm{Ext}_{\mathbb{R}}$. We rely on the following device, which uses ideas from Tangora's work [1985] on the classic lambda algebra to produce something like a chain-level lift of the ρ-Bockstein spectral sequence [Hill 2011]. While the algorithm is essentially standard, we include a detailed description since we were unable to find a reference with the algorithm in precisely the form we need in the sequel. We begin with some preliminary definitions.

Definition 4.1.1 Let $V = \mathbb{F}_2\{x_s : s \in S\}$ be a (locally) finite $\mathbb{F}_2$-vector space with ordered basis.

(1) The *leading term* of a class $x \in V$ is the largest term appearing when x is written as a sum of basis elements.

(2) We write $x < x'$ when the leading term of x is less than that of x'.

(3) Given another vector space $U = \mathbb{F}_2\{x_s : s \in T\}$ with ordered basis, map $\phi\colon V \to U$, and $s \in S$ and $t \in T$, we write
$$\phi(x_s + <) = y_t + <$$
for the *relation* that there exist some classes $u < x_s$ and $v < y_t$ for which $\phi(x_s + u) = y_t + v$. $\triangleleft$

The main technical lemma we need is the following. The reader is invited to skip this lemma on first reading; the details are not necessary to understand our computation, and we rephrase what we need in the context of Λ in Theorem 4.1.4.

Lemma 4.1.2 *Let (C, d) be a chain complex of locally finite and free $\mathbb{F}_2[\rho]$-modules, and suppose (for simplicity) that $H_*C[\rho^{-1}] = 0$. Choose an ordered basis $\mathbb{F}_2\{x_s : s \in T\}$ for $C/(\rho)$, and extend this to a basis $\mathbb{F}_2\{\rho^n x_s : (s,n) \in T \times \mathbb{N}\}$ for C, itself ordered by $\rho^n x_s < \rho^m x_t$ whenever $n > m$, or else $n = m$ and $s < t$. Let $\{\alpha_s : s \in B\}$ be a basis for $H_*(C/(\rho))$, indexed by a subset $B \subset T$ with the property that, for each α_s, there is some $z_s \in C$ with leading term x_s which projects to a cycle representative of α_s. Let $B_1 \subset B$ be the subset of those s for which x_s is the leading term of some cycle in C, and let $B_0 = B \setminus B_1$.*

There is then a unique injection $t\colon B_0 \to B$ such that
$$d(x_s + <) = \rho^{r(s)} x_{t(s)} + <$$
for all $s \in B_0$. Here $r(s) \geq 1$ is an integer uniquely determined by comparing the degrees of x_s and $x_{t(s)}$. Moreover, t restricts to a bijection $t\colon B_0 \cong B_1$, and there is an isomorphism
$$H_*C = \bigoplus_{s \in B_0} \mathbb{F}_2[\rho]/(\rho^{r(s)}),$$
where we may take the summand indexed by s to be generated by any class of the form $\rho^{-r(s)} \cdot d(x_s + <)$ with leading term $x_{t(s)}$.

Proof We begin by defining a function $t^{-1}\colon B_1 \to B$. Fix $b \in B_1$; we claim that there exists some $s \in B$ such that $d(x_s + <) = x_b + <$. The function t^{-1} will then be defined by declaring $t^{-1}(b)$ to be the minimal s for which $d(x_s + <) = x_b + <$.

Indeed, let z_b be a cycle with leading term x_b which projects to a cycle representative for α_b. As $H_*C[\rho^{-1}] = 0$, necessarily $\rho^r z_b$ is nullhomologous for some minimal $r \geq 1$. That is, there is some $y \in C$ not divisible by ρ such that $d(y) = \rho^r z_b$. If $y = x_s + <$ with $s \in B$, then we are done. Otherwise, as y is a cycle in $C/(\rho)$, necessarily y is homologous to some $x_s + u$ with $u < x_s$ and $s \in B$, in which case there exists some v with $d(v) = x_s + u + y$. We find that

$$d(x_s + <) = d(x_s + u) = d(x_s + u + d(v)) = d(y) = \rho^r z_b = \rho^r x_b + <,$$

as claimed. Thus we have produced the function t^{-1}.

Next we claim that t^{-1} restricts to a function $t^{-1}\colon B_1 \to B_0$. Indeed, suppose towards contradiction that there are some $b \in B_1$ such that $x_{t^{-1}(b)}$ is the leading term of some cycle. That is to say, suppose given $u, v < x_{t^{-1}(b)}$ such that

$$d(x_{t^{-1}(b)} + u) = x_b + <, \quad d(x_{t^{-1}(b)} + v) = 0.$$

Adding these together, we find

$$d(u + v) = x_b + <.$$

As $u + v < x_{t^{-1}(b)}$, this contradicts minimality of $t^{-1}(b)$. Thus we have a function $t^{-1}\colon B_1 \to B_0$.

Next we claim that t^{-1} is a bijection. It is a function between locally finite sets, and the assumption that $H_*C[\rho^{-1}] = 0$ implies that these sets have the same cardinality in each degree. So it is sufficient to verify that t^{-1} is an injection. Indeed, suppose towards contradiction that there were some $b < c$ in B_1 for which $t^{-1}(b) = s = t^{-1}(c)$. Thus there are $u, v < x_s$ such that

$$d(x_s + u) = x_b + <, \quad d(x_s + v) = x_c + <.$$

Adding these together, we find

$$d(u + v) = x_c + <.$$

As $u + v < x_s$, this contradicts minimality of $t^{-1}(c)$.

By taking the inverse of $t^{-1}\colon B_1 \to B_0$, we have thus proved the existence of a bijection $t\colon B_0 \to B_1$ with the property that $d(x_s + <) = x_{t(s)} + <$ for all $s \in B_0$. With this t, the given description of H_*C is clear; in effect, we have described how to choose a basis for C for which d is upper triangular, where, if a diagonal entry is divisible by ρ^r, so too are all entries above it. Compare the notion of a tag from [Tangora 1985].

It remains to verify uniqueness. Suppose towards contradiction that we have found some other injection $t'\colon B_0 \to B$ such that $d(x_s + <) = x_{t'(s)} + <$ for all $s \in B_0$. The condition that $t' \neq t$ means that there exists some $s \in B_0$ for which $d(x_s + <) = x_{t'(s)} + <$, but s is not minimal among possible $a \in B_0$ with $d(x_a + <) = x_{t'(s)} + <$. Choose such s with $t'(s)$ maximal, and let $a = t^{-1}(t'(s))$ be the minimal $a \in B_0$ with $d(x_a + <) = x_{t'(s)} + <$. So there are $u, v < x_a$ for which

$$d(x_a + u) = x_{t'(s)} + <, \quad d(x_a + v) = x_{t'(a)} + <.$$

Adding these together, we find that

$$d(u + v) = x_{t'(s)} + x_{t'(a)} + <,$$

where $u + v < x_a$. If $t'(a) < t'(s)$, then this reduces to

$$d(u + v) = x_{t'(s)} + <,$$

contradicting minimality of a. If $t'(s) < t'(a)$, then this reduces to

$$d(u + v) = x_{t'(a)} + <,$$

contradicting maximality of $t'(s)$. So there is no such t', proving that t is the unique injection satisfying the required property. □

We now specialize to the computation of $\mathrm{Ext}_{\mathbb{R}}$. Observe that by Proposition 3.2.1, we may reduce to considering only the ρ-torsion subgroup of $\mathrm{Ext}_{\mathbb{R}}$. In terms of Λ, this amounts to ignoring monomials of the form λ_I where I is a sequence of odd numbers. We will apply Lemma 4.1.2 to compute this ρ-torsion subgroup as follows.

We take as basis of $\Lambda/(\rho)$ the standard basis $\lambda_I \tau^n$ where I is coadmissible (Definition 2.3.9) and $n \geq 0$. We also need to order this basis. In the region where we will compute, our choice of order makes no difference, in the sense that all "error terms" appearing in "$+<$" will be divisible by ρ. But for concreteness let us say that $\lambda_I \tau^n < \lambda_J \tau^m$ if $n > m$, or else $n = m$ and $I < J$ lexicographically, ie if $I = (i_1, \dots, i_f)$ and $J = (j_1, \dots, j_f)$, then $i_1 < j_1$, or else $i_1 = j_1$ and $i_2 < j_2$, and so forth.

We must fix some further notation. Let $\{\alpha'_s : s \in S_0\}$ be a basis for $\mathrm{Ext}_{\mathrm{cl}}$, and write $\alpha_s \in \mathrm{Ext}_{\mathbb{C}}$ for the image of α'_s under the map induced by $\widetilde{\theta}\colon \Lambda^{\mathrm{dcl}} \to \Lambda^{\mathbb{C}}$ (see Proposition 3.2.1). Extend this to a minimal generating set $\{\alpha_s : s \in S\}$ for $\mathrm{Ext}_{\mathbb{C}}$ as an $\mathbb{F}_2[\tau]$-module. For $s \in S$, let n_s denote the τ-torsion exponent of α_s, so that $\{\alpha_s \tau^n : s \in S,\ n < n_s\}$ is an $\mathbb{F}_2$-basis for $\mathrm{Ext}_{\mathbb{C}}$. For each $s \in S$, choose a distinct coadmissible monomial $\lambda_{I(s)}$ which is the leading term of a cycle representative for α_s in $\Lambda_{\mathbb{C}}$, making this choice so that, if $s \in S_0$, then $\lambda_{I(s)}$ is in the image of $\widetilde{\theta}$. See the discussion following Proposition 4.2.1 for the particular choices we will take in our computation.

Let $B' = \{(s, n) : s \in S,\ n < n_s\}$. Given $b = (s, n) \in B'$, write $x_b = \lambda_{I(s)} \tau^n \in \Lambda^{\mathbb{R}}$. Let $B \subset B'$ be the subset of pairs not of the form $(s, 0)$ with $s \in S_0$. Let $B_1 \subset B$ be the subset of those b such that x_b is the leading term of some cycle, and let $B_0 = B \setminus B_1$. Let $B[f] \subset B$ be the subset of those b for which x_b is in filtration f, and extend this notation to all the indexing sets under consideration.

For our computation, we will produce, for every $b \in B_0[f]$ with $f \le 2$, some $t(b) \in B$ such that

$$\delta(x_b + <) = \rho^{r(b)} x_{t(b)} + <,$$

making this choice so that $t\colon B_0 \to B$ is injective. Here $r(b) \ge 1$ is some integer which may be determined by comparing the stems of x_b and $x_{t(b)}$.

Definition 4.1.3 In the above situation, we shall write $x_b \to x_{t(b)}\rho^{r(b)}$. ◁

Theorem 4.1.4 *Fix notation as above. Then*:

1. t *is uniquely determined* (*given our choice of ordered basis*).
2. t *restricts to bijections* $t\colon B_0[f] \cong B_1[f+1]$.
3. *The* ρ*-torsion subgroup of* $\mathrm{Ext}_{\mathbb{R}}^{f+1}$ *is isomorphic to*

$$\bigoplus_{b \in B_0[f]} \mathbb{F}_2[\rho]/(\rho^{r(b)}),$$

where the summand corresponding to $b \in B_0[f]$ *is generated by any class of the form*

$$\frac{\delta(x_b + <)}{\rho^{r(b)}}$$

with leading term $x_{t(b)}$.

Proof This follows by specializing Lemma 4.1.2 to the complementary summand of $\widetilde{\theta}\colon \Lambda^{\mathrm{cl}} \subset \Lambda$. □

Most notably, the ρ-torsion in $\mathrm{Ext}_{\mathbb{R}}^{f+1}$ is obtained by understanding differentials out of $\Lambda[f]$; this is significantly easier than finding cycles in $\Lambda[f+1]$ directly.

We end with two remarks, which could have been made in the more general context of Lemma 4.1.2.

Remark 4.1.5 More generally, $H^*(\mathcal{A}^{\mathbb{R}}/(\rho^m)) = H_*(\Lambda/(\rho^m))$ (denoted by $\mathrm{Ext}_{(m)}$ in Section 7) may be read off our computation as follows. For each $b \in B_0$, choose $u_b \in \Lambda$ such that $u_b < x_b$ and $\delta(x_b + u_b) = \rho^{r(b)} x_{t(b)} + <$, and let $z_b = \rho^{-r(b)} \cdot \delta(x_b + u_b)$. Then $H_*(\Lambda/(\rho^m))$ is given as follows:

1. For each $s \in S_0$, there is a summand of the form $\mathbb{F}_2[\rho]/(\rho^m)$, generated by the image of α_s.
2. For each $x_b \to \rho^{r(b)} x_{t(b)}$, there is a summand of the form $\mathbb{F}_2[\rho]/(\rho^{\min(m, r(b))})$, generated by the class with cycle representative z_s.
3. For each $x_b \to \rho^{r(b)} x_{t(b)}$, there is a summand of the form $\mathbb{F}_2[\rho]/(\rho^{m - \max(0, m - r(b))})$, generated by the class with cycle representative $\rho^{\max(0, m - r(b))}(x_b + u_b)$. ◁

Remark 4.1.6 Our approach to computing $\mathrm{Ext}_{\mathbb{R}}$ via Λ is closely related to the computation of $\mathrm{Ext}_{\mathbb{R}}$ via the ρ-Bockstein spectral sequence $\mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}$ [Hill 2011]. The precise relation is as follows. For $b = (s, n) \in B$, let $\alpha_b = \alpha_s \tau^n$, so that $\{\alpha_b : b \in B\}$ is a basis of $\mathrm{Ext}_{\mathbb{C}}$. Our ordering on Λ and choice of classes x_b gives B an order, thus making this into an ordered basis of $\mathrm{Ext}_{\mathbb{C}}$. Now, $x_b \to \rho^{r(b)} x_{t(b)}$ if and only if $d_{r(b)}(\alpha_b + <) = \rho^{r(b)} \alpha_{t(b)} + <$ in the ρ-Bockstein spectral sequence. ◁

The above discussion describes how we will compute $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ as an $\mathbb{F}_2[\rho]$-module. The computation gives more, as it produces explicit cocycle representatives for our generators of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$. We will use this in Section 4.3 to compute products in $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$.

4.2 $\mathrm{Ext}_{\mathbb{R}}^{f}$ for $f \leq 3$

We now proceed to the computation. We begin by understanding $\Lambda^{\mathbb{R}}/(\rho) \cong \Lambda^{\mathbb{C}}$.

Proposition 4.2.1 $\mathrm{Ext}_{\mathbb{C}}^{\leq 3}$ *is generated as a commutative* $\mathbb{F}_2[\tau]$*-algebra by classes* h_a *for* $a \geq 0$*, represented in* $\Lambda^{\mathbb{C}}$ *by* λ_{2^a-1}*, and* c_a *for* $a \geq 0$*, represented in* $\Lambda^{\mathbb{C}}$ *by* $\lambda_{2^a 3-1}\lambda_{2^{a+2}-1}^2$*. A full set of relations is given by*

$$h_{a+1}h_a = 0, \quad h_{a+2}^2 h_a = 0, \quad h_2 h_0^2 = \tau h_1^3, \quad h_{a+3}h_{a+1}^2 = h_{a+2}^3$$

for all $a \geq 0$*. This is free over* $\mathbb{F}_2[\tau]$*, with basis given by the classes in the following table*:

class	*constraints*
1	
h_a	$a \geq 0$
$h_a \cdot h_b$	$a \geq b \geq 0$ and $a \neq b+1$
$h_a \cdot h_b \cdot h_c$	$a \geq b \geq c \geq 0$ with $a \neq b+1$, $b \neq c+1$ and, if $b = c$ or $a = b$, then $a \neq c+2$
c_a	$a \geq 0$

The only such classes not in the image of $\widetilde{\theta}\colon \mathrm{Ext}_{\mathrm{dcl}} \to \mathrm{Ext}_{\mathbb{C}}$ *are those in which either* h_0 *or* c_0 *appears.*

Proof This is essentially well known, owing to work of Isaksen [2019] on the cohomology of the $\mathbb{C}$-motivic Steenrod algebra. Alternatively, one may compute $H_{\leq 3}(\Lambda^{\mathbb{C}}/(\tau))$ following Wang's approach [1967], and run the τ-Bockstein spectral sequence to recover $\mathrm{Ext}_{\mathbb{C}}^{\leq 3}$. One finds that $H_{\leq 3}(\Lambda^{\mathbb{C}}/(\tau))$ agrees with $\mathrm{Ext}_{\mathrm{cl}}^{\leq 3}$, with two exceptions:

(1) Instead of $h_0^2 \cdot h_2 = h_1^3$, one has $h_0^2 \cdot h_2 = 0$.

(2) There is a new cycle α represented by $\lambda_2^2\lambda_1$.

There is a τ-Bockstein differential $d_1(\alpha) = \tau h_1^4$, after which we recover the claimed $\mathbb{F}_2[\tau]$-module basis of $\mathrm{Ext}_{\mathbb{C}}^{\leq 3}$. The hidden extension $h_0^2 \cdot h_2 = \tau h_1^3$ was shown in Proposition 3.1.4; alternatively, it is the only relation compatible with $\mathrm{Sq}^0(h_0^2 \cdot h_2) = \tau^2 h_1^2 h_3 = \tau^2 h_2^3 = \mathrm{Sq}^0(\tau h_1^3)$. □

Proposition 4.2.1 describes a basis for $\mathrm{Ext}_{\mathbb{C}}^{\leq 3}$, thus giving our set $S[\leq 3]$. We must also choose lambda algebra representatives of these classes. We shall choose c_n to be represented by $\lambda_{2^n 3-1}\lambda_{2^{n+2}-1}^2$ and a product $h_{n_1} \cdots h_{n_k}$ with $n_1 \geq \cdots \geq n_k$ to be represented by $\lambda_{2^{n_1}-1} \cdots \lambda_{2^{n_k}-1}$. We warn that these representatives are not minimal; for example, we have chosen $\lambda_3\lambda_0$ as our representative for $h_2 h_0$, rather than the minimal representative $\lambda_2\lambda_1$. However, they are easily defined and convenient enough for our computation.

The following identity will be used frequently in consolidating various cases in our computation. It is an immediate consequence of the description of θ given in Theorem 2.3.16.

Lemma 4.2.2 *We have*

$$\theta^a(\lambda_0\tau^n) = \lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(2n+1)\rfloor} + O(\rho^{2^{a-1}})$$

for all $n \geq 0$, the error term being omitted when $a = 0$. □

Remark 4.2.3 Explicitly,

$$\lfloor 2^{a-1}(2n+1)\rfloor = \begin{cases} 2^{a-1}(2n+1) & \text{if } a \geq 1, \\ n & \text{if } a = 0. \end{cases}$$

This sort of pattern appears frequently throughout our computation, as a consequence of Lemma 4.2.2. ◁

We now produce the relation "→" described in Definition 4.1.3, proceeding filtration by filtration. To start, observe that $B_0[0] = \{\tau^n : n \geq 1\}$.

Proposition 4.2.4 *We have*

$$\delta(\tau^{2^a(2m+1)}) = \lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a} + O(\rho^{\lceil 2^a+2^{a-1}\rceil})$$

for all $a, m \geq 0$. In particular,

$$\tau^{2^a(2m+1)} \to \lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}.$$

Proof When $a = 0$, as τ^2 is a cycle mod ρ^2, we may compute

$$\delta(\tau^{2m+1}) = \delta(\tau)\tau^{2m} + O(\rho^2) = \lambda_0\tau^{2m}\rho + O(\rho^2),$$

as claimed. By Lemma 4.2.2, applying θ^a for $a \geq 1$ to this yields

$$\begin{aligned}\delta(\tau^{2^a(2m+1)}) &= (\lambda_{2^a-1}\tau^{2^{a-1}(4m+1)} + O(\rho^{2^{a-1}}))\rho^{2^a} + O(\rho^{2^{a+1}}) \\ &= \lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a} + O(\rho^{2^a+2^{a-1}}).\end{aligned}$$

Combining the cases $a = 0$ and $a \geq 1$ yields the proposition. □

Corollary 4.2.5 *The set $B_0[1]$ consists of those $\lambda_{2^a-1}\tau^n$ such that n is not of the form $2^{a-1}(4m+1)$ for any m.* □

We have located the following indecomposable classes.

Definition 4.2.6 For $a, n \geq 0$, we declare

$$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor}h_a$$

to be the class represented by

$$\rho^{-2^a}\cdot\delta(\tau^{2^a(2n+1)}).$$ ◁

We now compute out of $B_0[1]$.

Proposition 4.2.7 *For the combinations of a and b below, we have $\lambda_{2^b-1}\tau^{2^a(2m+1)} \to$ the following monomial*:

row	*case*	*target*
(1)	$a < b-1$ or $a = b$	$\lambda_{2^b-1}\lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(2)	$a > b+1$ and $b \neq 0$	$\lambda_{2^a-1}\lambda_{2^b-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(3)	$a = b-1$ and $m = 2n+1$	$\lambda^2_{2^b-1}\tau^{2^b(4n+1)}\rho^{2^b}$
(4)	$a = b+1$ and $b \neq 0$	$\lambda^2_{2^{b+1}-1}\tau^{\lfloor 2^{b-1}(8m+1)\rfloor}\rho^{2^b 3}$

Moreover, these cases are mutually exclusive and altogether exhaust $B_0[1]$.

Proof That these cases are mutually exclusive and altogether exhaust $B_0[1]$ is seen by direct inspection. As the monomials arising as targets are ρ-multiples of distinct elements of $B[2]$, it suffices to verify that for each claim of $x \to y$ we have $\delta(x + <) = y + <$.

(1) We have
$$\delta(\lambda_{2^b-1}\tau^{2^a(2m+1)}) = \lambda_{2^b-1}\lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a} + O(\rho^{2^a+2^{a-1}}).$$

(2) Note that
$$\tau^{2^a(2m+1)}\lambda_{2^b-1} = \lambda_{2^b-1}\tau^{2^a(2m+1)} + <,$$
as τ is central mod ρ. Now we have
$$\begin{aligned}\delta(\tau^{2^a(2m+1)}\lambda_{2^b-1}) &= (\lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a} + O(\rho^{2^a+2^{a-1}}))\lambda_{2^b-1}\\ &= \lambda_{2^a-1}\lambda_{2^b-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a} + O(\rho^{2^a+1}).\end{aligned}$$

(3) Note that
$$\theta^b(\lambda_0\tau^{2n+1}) = \lambda_{2^b-1}\tau^{\lfloor 2^{b-1}(4n+3)\rfloor} + O(\rho).$$
Now we have
$$\delta(\theta^b(\lambda_0\tau^{2n+1})) = \theta^b(\delta(\lambda_0\tau^{2n+1})) = \theta^b(\lambda_0^2\tau^{2n}\rho + O(\rho^2)) = \lambda^2_{2^b-1}\tau^{2^b(4n+1)}\rho^{2^b} + O(\rho^{2^b+1}).$$

(4) We have
$$\begin{aligned}\delta(\lambda_{2^b-1}\tau^{2^{b+1}(2m+1)}) &= \delta(\theta^{b-1}(\lambda_1\tau^{8m+4})) = \theta^{b-1}(\lambda_1\delta(\tau^4)\tau^{8m} + O(\rho^8))\\ &= \theta^{b-1}(\lambda_3^2\tau^{8m+1}\rho^6 + O(\rho^7)) = \lambda^2_{2^{b+1}-1}\tau^{2^{b-1}(8m+1)}\rho^{2^b 3} + O(\rho^{2^b 3+2^{b-1}}).\end{aligned}$$
Here the third equality uses the Adem relations $\lambda_1\lambda_3 = 0$ and $\lambda_1\lambda_5 = \lambda_3\lambda_3$ to determine the leading term of $\lambda_1\delta(\tau^4)$. □

Corollary 4.2.8 *The set $B_0[2]$ consists of those $\lambda_{2^b-1}\lambda_{2^c-1}\tau^n$ where $b = c$ or $b \geq c+2$, and where moreover*:

(1) $n \neq \lfloor 2^{b-1}(4m+1)\rfloor$ *and* $n \neq \lfloor 2^{c-1}(4m+1)\rfloor$ *for any* m.

(2) *If $b = c = 0$, then n is odd.*

(3) *If* $b=c\geq 1$, *then* $n\neq 2^b(4m+1)$ *for any* m.

(4) *If* $b=c\geq 2$, *then* $n\neq 2^{b-2}(8m+1)$ *for any* m. □

We have located the following indecomposable classes.

Definition 4.2.9 For $a,n\geq 0$, we declare

$$\tau^{2^a(8n+1)}h_{a+2}^2$$

to be the class represented by

$$\rho^{-2^{a+1}3}\cdot\delta(\lambda_{2^{a+1}-1}\tau^{2^{a+2}(2n+1)})$$ ◁

We now compute out of $B_0[2]$.

Proposition 4.2.10 *For* $b=c$ *or* $b\geq c+2$, *we have* $\lambda_{2^b-1}\lambda_{2^c-1}\tau^{2^a(2m+1)}\to$ *the following monomial*:

#	*case*	*target*
(1)	$b=c=0,\ a=-1,\ m=2n+1$	$\lambda_0^3\tau^{2n}\rho$
(2)	$b=c\geq 1,\ a=b-1,\ m=2n+1$	$\lambda_{2^b-1}^3\tau^{2^b(2n+1)}\rho^{2^b}$
(3)	$b=c\geq 0,\ a=c,\ m=2n+1$	$\lambda_{2^b-1}^3\tau^{\lfloor 2^{b-1}(4(2n+1)+1)\rfloor}$
(4)	$b=c\geq 1,\ a=b+1$	$\lambda_{2^{b-1}3-1}\lambda_{2^{b+1}-1}^2\tau^{\lfloor 2^{b-2}(16m+1)\rfloor}\rho^{2^{b-1}7}$
(5)	$b=c\geq 1,\ a=b+2$	$\lambda_{2^{b+2}-1}^2\lambda_{2^{b-1}-1}\tau^{\lfloor 2^{b-2}(2(16m+1)+1)\rfloor}\rho^{2^{b-1}13}$
(6)	$b=c\geq 1,\ a\geq b+3$	$\lambda_{2^a-1}\lambda_{2^b-1}^2\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(7)	$b=c\geq 2,\ a=b-2,\ m=4n+2$	$\lambda_{2^b-1}^3\tau^{\lfloor 2^{b-2}(2(4n+1)+1)\rfloor}\rho^{2^b}$
(8)	$b=c\geq 2,\ a=b-2,\ m=2n+1$	$\lambda_{2^{b-2}3-1}\lambda_{2^b-1}^2\tau^{\lfloor 2^{b-3}(2(4n+1)+1)\rfloor}\rho^{2^{b-2}3}$
(9)	$b=c\geq 3,\ a\leq b-3$	$\lambda_{2^b-1}^2\lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(10)	$b-2\geq c=0,\ a=0,\ m=2n+1$	$\lambda_{2^b-1}\lambda_0^2\tau^{2n}\rho$
(11)	$b-2=c\geq 1,\ a=b$	$\tau^{2^c(8n+1)}\lambda_{2^c3-1}\lambda_{2^{c+2}-1}^2\rho^{2^{c+1}3}$
(12)	$b-2=c\geq 1,\ a\geq b+2$	$\lambda_{2^a-1}\lambda_{2^b-1}\lambda_{2^c-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(13)	$b-3\geq c\geq 1,\ a\geq b,\ a\neq b+1$	$\lambda_{2^a-1}\lambda_{2^b-1}\lambda_{2^c-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(14)	$b-2\geq c\geq 1,\ c\leq a<b,\ a\notin\{c+1,b-1\}$	$\lambda_{2^b-1}\lambda_{2^a-1}\lambda_{2^c-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$
(15)	$b-2=c\geq 1,\ a=c-1,\ m=2n+1$	$\lambda_{2^{c+1}-1}^3\tau^{2^c(2n+1)}\rho^{2^c}$
(16)	$b-3\geq c\geq 1,\ a=c-1,\ m=2n+1$	$\lambda_{2^b-1}\lambda_{2^c-1}^2\tau^{2^c(2n+1)}\rho^{2^c}$
(17)	$b-2=c\geq 1,\ a=c+1,\ m=2n+1$	$\lambda_{2^b-1}^3\tau^{\lfloor 2^{b-3}(4(4n+1)+1)\rfloor}\rho^{2^{b-2}7}$
(18)	$b-3=c\geq 1,\ a=c+1$	$\lambda_{2^{c+2}-1}^3\tau^{\lfloor 2^{c-1}(8m+1)\rfloor}\rho^{2^c3}$
(19)	$b-4\geq c\geq 1,\ a=c+1$	$\lambda_{2^b-1}\lambda_{2^{c+1}-1}^2\tau^{\lfloor 2^{c-1}(8m+1)\rfloor}\rho^{2^c3}$
(20)	$b-3\geq c\geq 1,\ a=b-1,\ m=2n+1$	$\lambda_{2^b-1}^2\lambda_{2^c-1}\tau^{2^b(2n+1)}\rho^{2^b}$
(21)	$b-2\geq c\geq 1,\ a=b+1$	$\lambda_{2^{b+1}-1}^2\lambda_{2^c-1}\tau^{\lfloor 2^{b-1}(8m+1)\rfloor}\rho^{2^b3}$
(22)	$b-2\geq c\geq 2,\ a\leq c-2$	$\lambda_{2^b-1}\lambda_{2^c-1}\lambda_{2^a-1}\tau^{\lfloor 2^{a-1}(4m+1)\rfloor}\rho^{2^a}$

Moreover, these cases are mutually exclusive and altogether exhaust $B_0[2]$.

Proof That these cases are mutually exclusive and altogether exhaust $B_0[2]$ is seen by direct inspection. As the monomials arising as targets are ρ-multiples of distinct elements of $B[3]$, it suffices to verify that for each claim of $x \to y$ we have $\delta(x + <) = y + <$.

Each case represents a collection of families of monomials whose leading terms are connected by θ. Thus we may always reduce to the smallest possible c, except in cases (9) and (22), where doing so would place extra constraints on a. In addition, by working modulo the smallest power of ρ in which the proposed target does not vanish, we may always reduce to the smallest possible m.

We may further divide the list of cases provided into three types: those which require no calculations beyond those carried out in Proposition 4.2.7; cases (15) and (18); and the more interesting cases which do require additional calculation, producing new indecomposable classes in $\mathrm{Ext}^3_{\mathbb{R}}$. Here cases (15) and (18) are not really exceptional; they could be consolidated into cases (16) and (19), only this would require slightly modifying the setup of Section 4.1, and it is easier to just separate them out. The more interesting cases are (4), (5), (8), (11), and (17). The remaining less interesting cases may all be handled exactly the same way as the first two cases of Proposition 4.2.7 were handled. Thus we shall not handle them individually, and instead only illustrate this point with a verification of (21). With these reductions in place, the proposition is proved by the following calculations:

(4) Here we are claiming $\delta(\lambda_1^2\tau^4 + <) = \lambda_2\lambda_3^2\rho^7 + <$. In fact, $\delta(\lambda_1^2\tau^4) = \lambda_2\lambda_3^2\rho^7$ on the nose.

(5) Here we are claiming $\delta(\lambda_1^2\tau^8 + <) = \lambda_7^2\lambda_0\tau\rho^{13} + <$. Observe that $\delta(\lambda_1^2\tau^8) = \lambda_3^3\tau^4\rho^8 + O(\rho^{12})$, but $\lambda_3\tau^4\rho^4$ is already seen as a target in case (1). Thus some additional correction term must be added to $\lambda_1^2\tau^8$ to get down to $\lambda_7^2\lambda_0\tau\rho^{13}$. Such a correction term is given by

$$u = \lambda_3^2\tau^6\rho^4 + \lambda_3\lambda_5\tau^5\rho^6 + \lambda_3\lambda_6\tau^4\rho^7 + \lambda_5\lambda_7\tau^3\rho^{10} + (\lambda_5\lambda_8 + \lambda_6\lambda_7)\tau^2\rho^{11} + (\lambda_{11}\lambda_3 + \lambda_5\lambda_9)\tau^2\rho^{12} + (\lambda_8\lambda_7 + \lambda_7\lambda_8 + \lambda_6\lambda_9)\tau\rho^{13};$$

with this choice of u, we have $\delta(\lambda_1^2\tau^8 + u) = \lambda_7^2\lambda_0\tau\rho^{13} + O(\rho^{14})$.

(8) Here we are claiming $\delta(\lambda_3^2\tau^3 + <) = \lambda_2\lambda_3^2\tau\rho^3 + <$. Indeed, let

$$u = (\lambda_3\lambda_4 + \lambda_4\lambda_3)\tau^2\rho + \lambda_3\lambda_5\tau^2\rho^2 + \lambda_4\lambda_5\tau\rho^3;$$

then we have $\delta(\lambda_3^2\tau^3 + u) = \lambda_2\lambda_3^2\tau\rho^3 + O(\rho^4)$.

(11) Here we are claiming $\delta(\lambda_7\lambda_1\tau^8 + <) = \lambda_5\lambda_7^2\tau^2\rho^{12}$. Indeed, let

$$u = \lambda_9\lambda_7\tau^4\rho^8 + \lambda_9\lambda_{11}\tau^2\rho^{12};$$

then we have $\delta(\lambda_7\lambda_1\tau^8 + u) = \lambda_5\lambda_7^2\tau^2\rho^{12} + O(\rho^{14})$.

(15) Here we are claiming $\delta(\lambda_7\lambda_1\tau^3 + <) = \lambda_3^3\tau^2\rho^2 + <$. Indeed, let

$$u = \lambda_7\lambda_2\tau^2\rho + (\lambda_9\lambda_1 + \lambda_5\lambda_5)\tau^2\rho^2;$$

then we have $\delta(\lambda_7\lambda_1\tau^3 + <) = \lambda_3^3\tau^2\rho^2 + O(\rho^3)$.

(17) Here we are claiming $\delta(\lambda_7\lambda_1\tau^{12}+<)=\lambda_7^3\tau^5\rho^{14}+<$. Indeed, let

$$u=\lambda_{11}\lambda_3\tau^9\rho^6+(\lambda_{11}\lambda_4+\lambda_{12}\lambda_3+\lambda_7\lambda_8+\lambda_8\lambda_7)\tau^8\rho^7+\lambda_7^2\tau^9\rho^6+\lambda_9\lambda_7\tau^8\rho^8;$$

then we have $\delta(\lambda_7\lambda_1\tau^{12}+u)=\lambda_7^3\tau^5\rho^{14}+O(\rho^{15})$.

(18) Here we are claiming $\delta(\lambda_{15}\lambda_1\tau^4+<)=\lambda_7^3\tau\rho^6+<$. Indeed, let

$$u=(\lambda_{19}\lambda_3+\lambda_{11}\lambda_{11})\tau\rho^6;$$

then we have $\delta(\lambda_{15}\lambda_1\tau^4+u)=\lambda_7^3\tau\rho^6+O(\rho^7)$.

(21) Here we are claiming $\delta(\lambda_{2^b-1}\lambda_{2^c-1}\tau^{2^{b+1}(2m+1)}+<)=\lambda_{2^{b+1}-1}^2\lambda_{2^c-1}\tau^{2^{b-1}(8m+1)}\rho^{2^b3}+<$, at least provided $b-2\geq c\geq 1$. This case is intended to illustrate all the remaining cases, and is identical in form to case (2) of Proposition 4.2.7. Recall from Proposition 4.2.7 that

$$\delta(\lambda_{2^b-1}\tau^{2^{b+1}(2m+1)}+O(\rho))=\lambda_{2^{b+1}-1}^2\tau^{2^{b-1}(8m+1)}\rho^{2^b3}+O(\rho^{2^b3+1}).$$

As

$$\lambda_{2^b-1}\lambda_{2^c-1}\tau^{2^{b+1}(2m+1)}\equiv\lambda_{2^b-1}\tau^{2^{b+1}(2m+1)}\lambda_{2^c-1}\pmod{\rho},$$

it follows that

$$\begin{aligned}\delta(\lambda_{2^b-1}\lambda_{2^c-1}\tau^{2^{b+1}(2m+1)}+O(\rho))&=\delta(\lambda_{2^b-1}\tau^{2^{b+1}(2m+1)}\lambda_{2^c-1}+O(\rho))\\&=(\lambda_{2^{b+1}-1}^2\tau^{2^{b-1}(8m+1)}\rho^{2^b3}+O(\rho^{2^b3+1}))\lambda_{2^c-1}\\&=\lambda_{2^{b+1}-1}^2\lambda_{2^c-1}\tau^{2^{b-1}(8m+1)}\rho^{2^b3}+O(\rho^{2^b3+1}),\end{aligned}$$

which gives the desired relation. The remaining cases are either identical in form to this, or simpler in that they do not require one to first move τ around to reduce to a case already considered in Proposition 4.2.7. □

This produces the indecomposable classes

$$\tau^{2^{a-1}(2(16n+1)+1)}h_{a+3}^2h_a,\quad \tau^{2^a(4(4n+1)+1)}h_{a+3}^3,\quad \tau^{2^{a-1}(16n+1)}c_a,$$
$$\tau^{2^{a+1}(8n+1)}c_{a+1},\quad \tau^{2^{a-1}(2(4n+1)+1)}c_a$$

for $a,n\geq 0$, following the same recipe as employed in Definitions 4.2.6 and 4.2.9, only where one must employ θ-iterates of τ-multiples of the correction terms u given in Proposition 4.2.10.

Proposition 4.2.10 concludes the work necessary for our computation of the $\mathbb{F}_2[\rho]$-module structure of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$. Let us now summarize in one theorem what we have learned. We wish to give a minimal generating set of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ whose elements are products of the indecomposable classes we have found. Before doing so, let us treat the following subtlety.

By way of example, let $x=(1/\rho^{2^b})\delta(\lambda_{2^b-1}\tau^{2^{b-1}(4m+3)})$ with $b\geq 1$, and let $\alpha\in\mathrm{Ext}_{\mathbb{R}}$ be the class represented by x. Our computation in Proposition 4.2.7 combined with the recipe of Theorem 4.1.4 would yield α as an element of a minimal generating set for $\mathrm{Ext}_{\mathbb{R}}$. Observe that x has leading term

$\lambda_{2^b-1}^2\tau^{2^b(4m+1)}$. It follows quickly from this that x has the same leading term as the cocycle representative of $(\tau^{2^{b-1}(4m+1)}h_b)^2$ given by the product of those cocycle representatives for $\tau^{2^{b-1}(4m+1)}h_b$ given in Definition 4.2.6. However, this does not prove that $\alpha = (\tau^{2^{b-1}(4m+1)}h_b)^2$: we have not ruled out the possibility that $\alpha+\beta = (\tau^{2^{b-1}(4m+1)}h_b)^2$ for some nonzero β represented by a cycle $y < \lambda_{2^b-1}^2\tau^{2^b(4m+1)}$. This is still sufficient to deduce that we may, if necessary, replace α with $\alpha+\beta$ in our minimal generating set in order to obtain a minimal generating set built as products of indecomposables. It turns out that no such correction is necessary.

Lemma 4.2.11 *Write $\phi\colon \mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{C}}$ for the quotient. Fix classes α, β in $\mathrm{Ext}_{\mathbb{R}}^1$ or $\mathrm{Ext}_{\mathbb{R}}^2$, at least one of which is ρ-torsion and not both in $\mathrm{Ext}_{\mathbb{R}}^2$. Let r be minimal for which $\rho^r\alpha = 0$ or $\rho^r\beta = 0$. Fix $\gamma \in \mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ not divisible by ρ and such that $\rho^r\gamma = 0$, and suppose $\phi(\alpha)\cdot\phi(\beta) = \phi(\gamma)$. Then $\alpha\cdot\beta = \gamma$.*

Proof Under the given conditions, there is in fact a unique class in the degree of $\alpha\cdot\beta$ which is not divisible by ρ and is killed by ρ^r. This may be seen by direct inspection of the propositions preceding this. □

We may now state the main theorem of this section.

Theorem 4.2.12 (1) *A minimal multiplicative generating set for $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ as an $\mathbb{F}_2[\rho]$-algebra is given by the classes in the following table*:

multiplicative generator	*ρ-torsion exponent*
h_{a+1}	∞
c_{a+1}	∞
$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor}h_a$	2^a
$\tau^{2^a(8n+1)}h_{a+2}^2$	$2^{a+1}\cdot 3$
$\tau^{\lfloor 2^{a-1}(2(16n+1)+1)\rfloor}h_{a+3}^2h_a$	$2^a\cdot 13$
$\tau^{2^a(4(4n+1)+1)}h_{a+3}^3$	$2^a\cdot 7$
$\tau^{\lfloor 2^{a-1}(16n+1)\rfloor}c_a$	$2^a\cdot 7$
$\tau^{2^{a+1}(8n+1)}c_{a+1}$	$2^{a+2}\cdot 3$
$\tau^{\lfloor 2^{a-1}(2(4n+1)+1)\rfloor}c_a$	$2^a\cdot 3$

Here $a, n \geq 0$, and the ρ-torsion exponent of a class α is the minimal r for which $\rho^r\alpha = 0$; the classes h_{a+1} and c_{a+1} are ρ-torsion-free.

(2) *The operation Sq^0 acts on these classes by incrementing a in each row.*

(3) *The image of these classes under $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{C}}$ is as their name suggests.*

(4) *A minimal $\mathbb{F}_2[\rho]$-module generating set for $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ is given in the following table. In all cases, the ρ-torsion exponent of a given class is the minimal ρ-torsion exponent of the multiplicative generators it is written as a product of.*

$\mathbb{F}_2[\rho]$-*module generator*	*constraints*
1	
h_a	$a \geq 1$
$\tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a$	$a, n \geq 0$
$h_a \cdot h_b$	$a \geq b \geq 1$ and $a \neq b+1$
$h_a \cdot \tau^{\lfloor 2^{b-1}(4n+1) \rfloor} h_b$	$a \geq 1$ and $b, n \geq 0$, and $a \neq b \pm 1$
$\tau^{\lfloor 2^{a-1} \rfloor} h_a \cdot \tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a$	$a, n \geq 0$
$\tau^{2^a(8n+1)} h_{a+2}^2$	$a, n \geq 0$
$h_a \cdot h_b \cdot h_c$	$a \geq b \geq c \geq 1$ *with* $a \neq b+1$, $b \neq c+1$, *and if* $b = c$ *or* $a = b$ *then* $a \neq c+2$
$h_a \cdot h_b \cdot \tau^{\lfloor 2^{c-1}(4n+1) \rfloor} h_c$	$a \geq b \geq 1$ and $c, n \geq 0$ *with* $a \neq b+1$ *and* $c \notin \{a \pm 1, b \pm 1\}$, *and if* $a = b$ *then* $c \notin \{a-2, a, a+2\}$, *and if* $a = b+2$ *then* $c \neq a$
$h_a \cdot \tau^{\lfloor 2^{b-1} \rfloor} h_b \cdot \tau^{\lfloor 2^{b-1}(2n+1) \rfloor} h_b$	$a \geq 1$ *and* $b, n \geq 0$, *and* $a \notin \{b-2, b-1, b+1\}$
$h_0 \cdot h_0 \cdot \tau^{2n} h_0$	$n \geq 0$
$h_a \cdot \tau^{2^b(8n+1)} h_{b+2}^2$	$a \geq 1$ *and* $b, n \geq 0$, *and either* $a \leq b-1$ *or* $a \geq b+4$
$\tau^{\lfloor 2^{a-1}(2(16n+1)+1) \rfloor} h_{a+3}^2 h_a$	$a, n \geq 0$
$\tau^{2^a(4(4n+1)+1)} h_{a+3}^3$	$a, n \geq 0$
c_a	$a \geq 1$
$\tau^{\lfloor 2^{a-1}(16n+1) \rfloor} c_a$	$a, n \geq 0$
$\tau^{2^{a+1}(8n+1)} c_{a+1}$	$a, n \geq 0$
$\tau^{\lfloor 2^{a-1}(2(4n+1)+1) \rfloor} c_a$	$a, n \geq 0$

Proof All of this may be read off the preceding computations, using Lemma 4.2.11 with Proposition 4.2.1 if necessary to write a given class as a product of classes in the given generating set. □

We point out the following corollary.

Corollary 4.2.13 *The operation* $\rho \cdot \mathrm{Sq}^0$ *is injective on* $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$. □

Remark 4.2.14 As indicated in Remark 4.1.6, one may also read off our computation a description of all differentials in the ρ-Bockstein spectral sequence $\mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}$ emanating out of filtration at most 2. We leave this to the interested reader. ◁

4.3 Multiplicative structure

We now compute the multiplicative structure of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$. This material is mostly not needed for our study of the 1-line of the motivic Adams spectral sequence in Section 7; the exception is that we will use the relation Proposition 4.3.4(4) in the proof of Theorem 7.4.9.

Already Lemma 4.2.11 produces a large number of relations. For example, it implies that we may always shift powers of τ around in products that do not vanish in $\mathrm{Ext}_{\mathbb{C}}$, provided it makes sense to do so, yielding relations such as

$$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor}h_a \cdot \tau^{\lfloor 2^{b-1}(4m+1)\rfloor}h_b = h_a \cdot \tau^{\lfloor 2^{b-1}\left(4(m+2^{a-b-2}(4n+1))+1\right)\rfloor}h_b$$

for $a \geq b+2$. These were implicitly used in the proof of Theorem 4.2.12. The condition that the product does not vanish in $\mathrm{Ext}_{\mathbb{C}}$ is necessary; see Example 4.3.3 below.

We are left only with relations that would be realized as hidden extensions in the ρ-Bockstein spectral sequence. These arise from the possible failure of the relations $h_{a+1}h_a = 0$ and $h_{a+2}^2 h_a = 0$ to lift through $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{C}}$.

Remark 4.3.1 The following computations will involve some explicit calculations with cocycle representatives. For ease of reference, we collect some important cocycle representatives here:

class	cocycle representative
h_0	λ_0
h_1	λ_1
h_2	λ_3
h_3	λ_7
c_0	$\lambda_2\lambda_3^2$
c_1	$\lambda_5\lambda_7^2$
$\tau^{2^a}h_{a+1}$	$\rho^{-2^{a+1}} \cdot \delta(\tau^{2^{a+1}}) = \theta^{a+1}(\lambda_0) = \tau^{2^a}\lambda_{2^{a+1}-1} = \lambda_{2^{a+1}-1}\tau^{2^a} + O(\rho^{2^a})$
$\tau^2 h_0$	$\lambda_0\tau^2 + \lambda_1\tau^2\rho + \lambda_3\tau\rho^3 + \lambda_4\rho^4$
$\tau^4 h_0$	$\lambda_0\tau^4 + \lambda_3\tau^3\rho^3 + \lambda_5\tau^2\rho^5 + \lambda_7\tau\rho^7 + \lambda_8\rho^8$
τh_2^2	$\lambda_3^2\tau + (\lambda_3\lambda_4 + \lambda_4\lambda_3)\rho = \tau\lambda_3^2$
$\tau^9 h_2^2$	$\lambda_3^2\tau^9 + (\lambda_3\lambda_4 + \lambda_4\lambda_3)\tau^8\rho + \lambda_5\lambda_3\tau^8\rho^2 + O(\rho^{10})$

We will use these without further comment. ◁

We begin with some products in $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ which lift the relation $h_{a+1}h_a = 0$.

Proposition 4.3.2 (1) $h_{a+1} \cdot \tau^{\lfloor 2^{a-1}(4(2n+1)+1)\rfloor}h_a = \rho^{2^a} \cdot \tau^{2^a}h_{a+1} \cdot \tau^{2^a(4n+1)}h_{a+1}$.

(2) $h_{a+1} \cdot \tau^{\lfloor 2^{a-1}(8n+1)\rfloor}h_a = 0$.

(3) $\tau^{2^{a+1}(4n+1)}h_{a+2} \cdot h_{a+1} = \rho^{2^{a+1}} \cdot \tau^{2^a(8n+1)}h_{a+2}^2$.

(4) $\tau^{2^a(8n+1)}h_{a+2}^2 \cdot h_{a+1} = \rho^{2^a} \cdot \tau^{\lfloor 2^{a-1}(16n+1)\rfloor}c_a$.

(5) $\tau^{2^a(8n+1)}h_{a+2}^2 \cdot \tau^{2^a(4m+1)}h_{a+1} = \rho^{2^a} \cdot \tau^{\lfloor 2^{a-1}\left(2(4(m+2n))+1\right)\rfloor}c_a$.

(6) $h_{a+3} \cdot \tau^{2^a(16n+1)}h_{a+2}^2 = 0$.

(7) $h_{a+3} \cdot \tau^{2^a(8(2n+1)+1)}h_{a+2}^2 = \rho^{2^{a+3}} \cdot \tau^{2^a(4(4n+1)+1)}h_{a+3}^2$.

Proof In each of these, we may use Sq^0 to reduce to the case $a=0$. In all cases where the product does not vanish, the claimed value of the product is the unique nonzero class in its degree which is both ρ-torsion and divisible by ρ, so it suffices to verify the product working modulo the smallest power of ρ in which the claimed value does not vanish. In doing so, we may in each case reduce to $n=m=0$. With these reductions in place, the proposition is proved by the following computations:

(1) Here we are claiming $h_1\cdot\tau^2h_0=\rho\cdot\tau h_1\cdot\tau h_1$. Indeed, we may compute

$$(\lambda_0\tau^2+\lambda_1\tau^2\rho+\lambda_3\tau\rho^3+\lambda_4\rho^4)\cdot\lambda_1=\lambda_1^2\tau^2\rho+\lambda_2\lambda_1\tau\rho^2+\lambda_2\lambda_2\rho^3+\lambda_2\lambda_3\rho^4=\rho(\lambda_1\tau+\lambda_2\rho)^2=\rho\theta(\lambda_0)^2,$$

which represents $\rho\cdot\tau h_1\cdot\tau h_1$.

(2) There are no nonzero ρ-torsion classes in this degree, so the product must vanish.

(3) Here we are claiming $h_1\cdot\tau^2h_2=\rho^2\cdot\tau h_2^2$. Indeed, we may compute

$$\lambda_1\cdot\tau^2\lambda_3=\rho^2\cdot\tau\lambda_3^2$$

on the nose.

(4) Here we are claiming $h_1\cdot\tau h_2^2=\rho\cdot c_0$. Indeed, we may compute

$$\lambda_1\cdot\tau\lambda_3^2=\lambda_2\lambda_3^2\rho$$

on the nose.

(5) Here we are claiming $\tau h_1\cdot\tau h_2^2=\rho\cdot\tau c_0$. For this, it suffices to work mod ρ^2. Here we may compute

$$\tau\lambda_1\cdot\tau\lambda_2^2=\rho\cdot\lambda_2\lambda_3^2\tau+O(\rho^2),$$

and the claim follows.

(6) Here we have reduced to $a=0$ but not yet to $n=0$. The only nonzero ρ-torsion class in this degree is $\rho^6\tau^{16n+1}c_1$, so it suffices to work mod ρ^7. In doing so, we may now reduce to $n=0$. Indeed, we have

$$\tau\lambda_3^2\cdot\lambda_7=0,$$

and the claim follows.

(7) Here we are claiming $h_3\cdot\tau^9h_2^2=\rho^8\cdot\tau^5h_3^3$. For this, it suffices to work mod ρ^9. Here we may compute

$$(\lambda_3^2\tau^9+(\lambda_4\lambda_3+\lambda_3\lambda_4)\tau^8\rho+\lambda_5\lambda_3\tau^8\rho^2)\cdot\lambda_7=\lambda_7^3\tau^5\rho^8+O(\rho^9),$$

yielding the claim. □

Example 4.3.3 We have

$$\tau^2h_2\cdot h_1^2=\rho^3c_0,\quad h_2\cdot(\tau h_1)^2=0.$$

This serves as a warning that one cannot in general freely shift around powers of τ in products. ◁

We now give some products that lift the relation $h_{a+2}^2 h_a = 0$.

Proposition 4.3.4 (1) $h_{a+2}^2 \cdot \tau^{\lfloor 2^{a-1}(16n+1)\rfloor} h_a = 0$.

(2) $h_{a+2}^2 \cdot \tau^{\lfloor 2^{a-1}(4(2n+1)+1)\rfloor} h_a = \rho^{2^{a+1}} \cdot \tau^{\lfloor 2^{a-1}(2(4n+1)+1)\rfloor} c_a$.

(3) $h_{a+2}^2 \cdot \tau^{\lfloor 2^{a-1}(8(2n+1)+1)\rfloor} h_a = \rho^{2^a 3} \cdot \tau^{2^{a+1}} h_{a+2} \cdot \tau^{2^a(8n+1)} h_{a+2}^2$.

(4) $\tau^{2^{a+2}} h_{a+3} \cdot \tau^{2^{a+2}(4n+1)} h_{a+3} \cdot h_{a+1} = \rho^{2^{a+1}3} \cdot \tau^{2^a(4(4n+1)+1)} h_{a+3}^3$.

(5) $h_{a+1} \cdot h_{a+3} \cdot \tau^{2^{a+2}(4n+1)} h_{a+3} = \rho^{2^{a+2}} \cdot \tau^{2^{a+1}(8n+1)} c_{a+1}$.

(6) $\tau^{2^{a+1}(8n+1)} h_{a+3}^2 \cdot h_{a+1} = 0$.

Proof As in the proof of Proposition 4.3.2, we may use Sq^0 to reduce to the case $a = 0$, and in all cases where the product does not vanish may reduce to $n = 0$. With these reductions in place, the proposition is proved by the following computations:

(1) There are no nonzero ρ-torsion classes in this degree, so the product must vanish.

(2) Here we are claiming $h_2^2 \cdot \tau^2 h_0 = \rho^2 \cdot \tau c_0$. For this, it suffices to work mod ρ^3. Recall that $\tau^2 h_0$ is represented by $\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho + O(\rho^3)$. We may compute

$$(\lambda_0 \tau^2 + \lambda_1 \tau^2 \rho) \cdot \lambda_3^2 = \rho^2 \cdot \lambda_2 \lambda_3^2 \tau + O(\rho^3),$$

and the claim follows.

(3) Here we are claiming $h_2^2 \cdot \tau^4 h_0 = \rho^3 \cdot \tau^2 h_2 \cdot \tau h_2^2$. For this, it suffices to work mod ρ^4. Observe that

$$h_2 \cdot h_2 \cdot \tau^4 h_0 = h_2 \cdot \tau^2 h_2 \cdot \tau^2 h_0 = \tau^2 h_2 \cdot h_2 \cdot \tau^2 h_0 = \tau^2 h_2 \cdot \tau^2 h_2 \cdot h_0$$

by Lemma 4.2.11. We may now compute

$$\lambda_0 \cdot \tau^2 \lambda_3 \cdot \tau^2 \lambda_3 = \rho^3 \cdot \lambda_3^3 \tau^3 + O(\rho^4),$$

yielding the claim.

(4) Here we are claiming $\tau^4 h_3 \cdot \tau^4 h_3 \cdot h_1 = \rho^6 \cdot \tau^5 h_3^3$. For this, it suffices to work mod ρ^7. Here we may compute

$$\lambda_1 \cdot \tau^4 \lambda_7 \cdot \tau^4 \lambda_7 = \rho^6 \cdot \lambda_7^3 \tau^5 + O(\rho^7),$$

yielding the claim.

(5) Here we are claiming $h_1 \cdot h_3 \cdot \tau^4 h_3 = \rho^4 \cdot \tau^2 c_1$. For this, it suffices to work mod ρ^5. Here we may compute

$$\lambda_1 \cdot \tau^4 \lambda_7 \cdot \lambda_7 = \rho^4 \cdot \lambda_5 \lambda_7^2 \tau^2 + O(\rho^6),$$

yielding the claim.

(6) There are no nonzero ρ-torsion classes in this degree, so the product must vanish. □

The preceding propositions leave open three families of products. A complete resolution of these requires the following, which appeared as a conjecture in an earlier version of this work. We thank Dugger, Hill and Isaksen for supplying a proof.

Lemma 4.3.5 (Dugger, Hill and Isaksen) *There are relations*

(1) $\tau^{4m+1}h_1 \cdot \tau^{2l}h_0 = \tau h_1 \cdot \tau^{2(2m+l)}h_0$;

(2) $\tau^{4(4m+1)}h_3 \cdot \tau^{8l+1}h_2^2 = \tau^4 h_3 \cdot \tau^{8(2m+l)+1}h_2^2$;

(3) $\tau^{8m+1}h_2^2 \cdot \tau^{2l}h_0 = \tau h_2^2 \cdot \tau^{2(2m+l)}h_0$.

Proof These will be proved using Massey product-shuffling techniques. The Massey products we require are most easily computed using the ρ-Bockstein spectral sequence; see especially [Belmont and Isaksen 2022, Section 7.4] for a discussion of Massey products in $\mathrm{Ext}_{\mathbb{R}}$.

(1) By induction on m, it suffices to show

$$\tau^{2l}h_0 \cdot \tau^{4m+5}h_1 = \tau^{2l+4}h_0 \cdot \tau^{4m}h_1$$

for $m \geq 0$. Observe that

$$\tau^{4m+5}h_1 = \langle \rho^2, \rho^2\tau^2 h_2, \tau^{4m+1}h_1 \rangle, \quad \tau^{2l+4}h_0 = \langle \tau^{2l}, \rho^2, \rho^2\tau^2 h_1 \rangle$$

with no indeterminacy. We may therefore shuffle

$$\tau^{2l}h_0 \cdot \tau^{4m+5}h_1 = \tau^{2l}h_0 \langle \rho^2, \rho^2\tau^2 h_2, \tau^{4m+1}h_1 \rangle = \langle \tau^{2l}h_0, \rho^2, \rho^2\tau^2 h_2 \rangle \tau^{4m+1}h_1 = \tau^{2l+4}h_0 \cdot \tau^{4m+1}h_1.$$

(2) By induction on m, it suffices to show

$$\tau^{8l+1}h_2^2 \cdot \tau^{4(4m+1)+16}h_3 = \tau^{8l+17}h_2^2 \cdot \tau^{4(4m+1)}h_3$$

for $m \geq 0$. Observe that

$$\tau^{4(4m+1)+16}h_3 = \langle \rho^8, \rho^8\tau^8 h_4, \tau^{4(4m+1)}h_3 \rangle, \quad \tau^{8l+17}h_2^2 = \langle \tau^{8l+1}h_2^2, \rho^8, \rho^8\tau^8 h_4 \rangle$$

with no indeterminacy. We may therefore shuffle

$$\begin{aligned}\tau^{8l+1}h_2^2 \cdot \tau^{4(4m+1)+16}h_3 &= \tau^{8l+1}h_2^2 \langle \rho^8, \rho^8\tau^8 h_4, \tau^{4(4m+1)}h_3 \rangle \\ &= \langle \tau^{8l+1}h_2^2, \rho^8, \rho^8\tau^8 h_4 \rangle \tau^{4(4m+1)}h_3 = \tau^{8l+17}h_2^2 \cdot \tau^{4(4m+1)}h_3.\end{aligned}$$

(3) By induction on m, it suffices to show

$$\tau^{2l}h_0 \cdot \tau^{8m+9}h_2^2 = \tau^{2l+8}h_0 \cdot \tau^{8m+1}h_2^2$$

for $m \geq 0$. Observe that

$$\tau^{8m+9}h_2^2 = \langle \rho\tau^4 h_3, \rho^7, \tau^{8m+1}h_2^2 \rangle, \quad \tau^{2l+8}h_0 = \langle \tau^{2l}h_0, \rho\tau^4 h_0, \rho^7 \rangle$$

with no indeterminacy. We may therefore shuffle

$$\begin{aligned}\tau^{2l}h_0 \cdot \tau^{8m+9}h_2^2 &= \tau^{2l}h_0 \langle \rho\tau^4 h_3, \rho^7, \tau^{8m+1}h_2^2 \rangle \\ &= \langle \tau^{2l}h_0, \rho\tau^4 h_0, \rho^7 \rangle \tau^{8m+1}h_2^2 = \tau^{2l+8}h_0 \cdot \tau^{8m+1}h_2^2.\end{aligned}$$

□

From here, we have the following.

Proposition 4.3.6 *Write $2m+l+1 = 2^k(2n+1)$. Then the following hold*:

(1) $\tau^{2^a(4m+1)}h_{a+1}\cdot\tau^{\lfloor 2^{a-1}(4l+1)\rfloor}h_a = \rho^{2^a(2^{k+1}-1)}\cdot h_{a+1}\cdot\tau^{2^{a+k}(4n+1)}h_{a+k+1}$.

(2) $\tau^{2^{a+2}(4m+1)}h_{a+3}\cdot\tau^{2^a(8l+1)}h_{a+2}^2 = \rho^{2^{a+1}(2^{k+2}-3)}\cdot h_{a+1}\cdot h_{a+3}\cdot\tau^{2^{a+k+2}(4n+1)}h_{a+k+3}$.

(3) $\tau^{2^a(8m+1)}h_{a+2}^2\cdot\tau^{\lfloor 2^{a-1}(4l+1)\rfloor}h_a = \rho^{2^a(2^{k+1}-1)}\cdot h_{a+2}^2\cdot\tau^{2^{a+k}(4n+1)}h_{a+k+1}$.

Proof In each of these, we may use Sq^0 to reduce to the case $a=0$. By working modulo the smallest power of ρ in which the claimed product does not vanish, we may reduce to the case $n=0$. By Lemma 4.3.5, we may moreover reduce to the case $m=0$. The proposition is now proved by the following computations:

(1) Here we are claiming $\tau h_1\cdot\tau^{2(2^k-1)}h_0 = \rho^{2^{k+1}-1}\cdot h_1\cdot\tau^{2^k}h_{k+1}$. Recall that $\tau^{2(2^k-1)}h_0$ is represented by $\rho^{-1}\delta(\tau^{2(2^k-1)+1})$. Now, the Leibniz rule implies

$$\rho^{-1}\delta(\tau^{2(2^k-1)+1})\cdot\tau\lambda_1 = \rho^{-1}\delta(\tau^{2^{k+1}})\cdot\lambda_1 + \rho^{-1}\tau^{2(2^k-1)+1}\cdot\delta(\tau)\cdot\lambda_1.$$

The second summand vanishes, as $\delta(\tau)\cdot\lambda_1 = \rho\lambda_0\cdot\lambda_1 = 0$; the first represents $\rho^{2^{k+1}-1}\cdot\tau^{2^k}h_{k+1}\cdot h_1$, yielding the claimed relation.

(2) Here we are claiming $\tau^4 h_3\cdot\tau^{8(2^k-1)+1}h_2^2 = \rho^{2(2^{k+2}-3)}\cdot h_1\cdot h_3\cdot\tau^{2^{k+2}}h_{k+3}$. Recall that $\tau^{8(2^k-1)+1}h_2^2$ is represented by $\rho^{-6}\delta(\lambda_1\tau^{8(2^k-1)+4})$. Now, the Leibniz rule implies

$$\rho^{-6}\delta(\lambda_1\tau^{8(2^k-1)+4})\cdot\tau^4\lambda_7 = \rho^{-6}\lambda_1\cdot\delta(\tau^{2^{k+3}})\cdot\lambda_7 + \rho^{-6}\lambda_1\cdot\tau^{8(2^k-1)+4}\cdot\delta(\tau^4)\cdot\lambda_7.$$

The second term vanishes, as $\delta(\tau^4)\cdot\lambda_3 = \tau^2\lambda_3\cdot\lambda_7 = 0$; the first represents $\rho^{2(2^{k+2}-3)}\cdot h_1\cdot h_3\cdot\tau^{2^{k+2}}h_{k+3}$, yielding the claimed relation.

(3) Here we are claiming $\tau h_2^2\cdot\tau^{2(2^k-1)}h_0 = \rho^{2^{k+1}-1}\cdot h_2^2\cdot\tau^{2^k}h_{k+1}$. Recall that $\tau^{2(2^k-1)}h_0$ is represented by $\rho^{-1}\delta(\tau^{2(2^k-1)+1})$. Now, the Leibniz rule implies

$$\rho^{-1}\delta(\tau^{2(2^k-1)+1})\cdot\tau\lambda_3^2 = \rho^{-1}\delta(\tau^{2^{k+1}})\cdot\lambda_3^2 + \rho^{-1}\tau^{2(2^k-1)+1}\cdot\delta(\tau)\cdot\lambda_3^2.$$

The second term vanishes, as $\delta(\tau)\cdot\lambda_3^2 = \rho\lambda_0\cdot\lambda_3^2 = 0$. The first summand represents $\rho^{2^{k+1}-1}h_{k+1}\cdot h_2^2$, yielding the claimed relation. □

The relations above suffice to write any product in $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ in terms of the basis given in Theorem 4.2.12. Thus we have the following.

Theorem 4.3.7 *A full set of relations for $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ is given by those visible relations which may be deduced from Lemma 4.2.11 together with the products listed in Propositions 4.3.2, 4.3.4, and 4.3.6.* □

Part II The motivic Hopf invariant one problem

5 Some homotopical preliminaries

With the algebraic computation of Section 4 out of the way, we now proceed to more homotopical considerations. This brief section collects a couple of constructions that will be used in the following sections. Explicitly, Section 5.1 will be used in our computation of $d_2(h_5)$ in Section 7, and Section 5.3 will be used in our discussion of the unstable Hopf invariant one problem in Section 6.

5.1 The Hurewicz map

The constant functor $c\colon \mathcal{S}\mathrm{p}^{\mathrm{cl}} \to \mathcal{S}\mathrm{p}^F$ has a lax symmetric monoidal right adjoint c^*, described by

$$c^*(X) = \mathcal{S}\mathrm{p}^F(S^{0,0}, X).$$

In particular, the unit of $c^*(S^{0,0})$ gives a ring map

$$S^0 \to c^*(S^{0,0}),$$

and on homotopy groups this yields a Hurewicz map

$$c\colon \pi_*^{\mathrm{cl}} \to \pi_{*,0}^F.$$

Proposition 5.1.1 *For any F, there is map*

$$c\colon \mathrm{Ext}_{\mathrm{cl}}^{s,f} \to \mathrm{Ext}_F^{s,f,0}$$

of multiplicative spectral sequences, converging to the Hurewicz map

$$c\colon \pi_*^{\mathrm{cl}} \to \pi_{*,0}^F.$$

Moreover, c commutes with Sq^0 and satisfies $c(h_0) = h_0 + \rho h_1$.

Proof Write $H\mathbb{F}_2$ for the ordinary mod 2 Eilenberg–Mac Lane spectrum and $H\mathbb{F}_2^F$ for the motivic spectrum representing mod 2 motivic cohomology. Then $c^*(H\mathbb{F}_2^F) = H\mathbb{F}_2$, thereby giving maps

$$H\mathbb{F}_2^{\otimes n} \simeq c^*(H\mathbb{F}_2^F)^{\otimes n} \to c^*((H\mathbb{F}_2^F)^{\otimes n}).$$

Thus there is a map from the canonical Adams resolution of the sphere to the restriction along c^* of the canonical Adams resolution of the F-motivic sphere. On homotopy groups, this gives a map from the cobar complex of $\mathcal{A}^{\mathrm{cl}}$ to the weight 0 portion of the cobar complex of $\mathcal{A}^F$, and passing to homology we obtain a map

$$\mathrm{Ext}_{\mathrm{cl}}^{s,f} \to \mathrm{Ext}_F^{s,f,0}$$

which is multiplicative and commutes with Sq^0, and by construction this is a map of spectral sequences converging to the Hurewicz map. That $c(h_0) = h_0 + \rho h_1$ follows as these are the classes detecting 2 (see for instance [Isaksen and Østvær 2020, Remark 6.3]). □

5.2 The Lefschetz principle

The *Lefschetz principle* asserts, informally, that "everything" which is true over $\mathbb{C}$ is true over any algebraically closed field. In this subsection, we note how one may read off a certain motivic Lefschetz principle from [Wilson and Østvær 2017].

So far, we have primarily been concerned with F-motivic homotopy theory for F a field of characteristic not equal to 2. For this subsection, we extend our notation to apply also when F is some ring in which 2 is invertible. We shall write $S^{0,0}$ for the $H\mathbb{F}_2^F$-nilpotent completion of the F-motivic sphere spectrum. When F is a field, this is the $(2,\eta)$-completion of the F-motivic sphere spectrum, and, when F is an algebraically closed field, this reduces to a 2-completion [Hu et al. 2011a; Kylling and Wilson 2019]. Let $\mathcal{S}\mathrm{p}_2^F$ denote the category of modules over this completed F-motivic sphere spectrum. In addition, let $\mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}} \subset \mathcal{S}\mathrm{p}_2^F$ denote the cellular subcategory, ie the category generated by the spheres $S^{a,b}$ under colimits.

Proposition 5.2.1 *Let F be an algebraically closed field. Then there is an equivalence*

$$\mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}} \simeq \mathcal{S}\mathrm{p}_2^{\mathbb{C},\mathrm{cell}}.$$

Moreover, this is compatible on Adams spectral sequences with the isomorphism $\mathrm{Ext}_F \cong \mathrm{Ext}_{\mathbb{C}}$.

Proof First suppose that F is of odd characteristic p. We follow the methods of [Wilson and Østvær 2017, Section 6]. Let $W(F)$ be the ring of Witt vectors on F, and choose an algebraically closed field L of characteristic 0 together with embeddings

$$\mathbb{C} \to L \leftarrow W(F) \to F.$$

This gives rise to base change functors

$$\mathcal{S}\mathrm{p}^{\mathbb{C}} \to \mathcal{S}\mathrm{p}^{L} \leftarrow \mathcal{S}\mathrm{p}^{W(F)} \to \mathcal{S}\mathrm{p}^{F},$$

and, in particular, maps

$$\pi_{*,*}^{\mathbb{C}} \to \pi_{*,*}^{L} \leftarrow \pi_{*,*}^{W(F)} \to \pi_{*,*}^{F}. \tag{5-1}$$

Although $W(F)$ is not a field, Wilson and Østvær [2017] show that its Steenrod algebra and Adams spectral sequence are still well behaved, and [loc. cit., Corollary 6.3] that the above maps are modeled on motivic Adams spectral sequences by a zigzag of isomorphisms

$$\mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_{L} \leftarrow \mathrm{Ext}_{W(F)} \to \mathrm{Ext}_{F}.$$

It follows that (5-1) is a zigzag of isomorphisms. In particular, consider the zigzag

$$\mathcal{S}\mathrm{p}_2^{\mathbb{C},\mathrm{cell}} \to \mathcal{S}\mathrm{p}_2^{L,\mathrm{cell}} \leftarrow \mathcal{S}\mathrm{p}_2^{W(F),\mathrm{cell}} \to \mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}}.$$

This is a zigzag of colimit-preserving functors of compactly generated stable categories which are equivalences on subcategories of compact generators, and is thus a zigzag of equivalences. This yields the canonical equivalence $\mathcal{S}\mathrm{p}_2^{\mathbb{C},\mathrm{cell}} \simeq \mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}}$.

If F is of characteristic zero, then we may apply the same argument instead to a zigzag of the form

$$\mathbb{C} \to L \leftarrow F$$

with L algebraically closed. □

5.3 Betti realization

If X is a smooth scheme over $\mathbb{C}$, then the space of complex points of X is a complex manifold. This refines to give *Betti realization* functors [Morel and Voevodsky 1999] from $\mathbb{C}$-motivic spaces to ordinary spaces, and from $\mathbb{C}$-motivic spectra to ordinary spectra, with a number of nice properties. We may use the Lefschetz principle of Proposition 5.2.1 to obtain an analogue for an arbitrary algebraically closed field F.

Let S^0 denote the 2-completed sphere spectrum, and $\mathcal{S}\mathrm{p}_2^{\mathrm{cl}}$ the category of modules thereover.

Proposition 5.3.1 *Let F be an algebraically closed field. Then there is a symmetric monoidal "Betti realization" functor*

$$\mathrm{Be}\colon \mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}} \to \mathcal{S}\mathrm{p}_2^{\mathrm{cl}},$$

factoring through an equivalence from the category of modules over $S^{0,0}[\tau^{-1}]$ in $\mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}}$ to $\mathcal{S}\mathrm{p}_2^{\mathrm{cl}}$, with the following properties:

1. $\mathrm{Be}(\tau) = 1$. *In particular*, $\mathrm{Be}(S^{a,b}) = S^a$, *so that* Be *induces a map* $\pi^F_{s,w} \to \pi^{\mathrm{cl}}_s$, *and these patch together to an isomorphism* $\pi^F_{*,*}[\tau^{-1}] \cong \pi^{\mathrm{cl}}_*[\tau^{\pm 1}]$.
2. *The above isomorphism is modeled on Adams spectral sequences by the map*
$$\mathrm{Ext}_F \to \mathrm{Ext}_F[\tau^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}[\tau^{\pm 1}].$$
3. *The composite* $\mathrm{Be} \circ c\colon \mathcal{S}\mathrm{p}_2^{\mathrm{cl}} \to \mathcal{S}\mathrm{p}_2^{F,\mathrm{cell}} \to \mathcal{S}\mathrm{p}_2^{\mathrm{cl}}$ *is an equivalence. In particular, the map* $c\colon \mathrm{Ext}_{\mathrm{cl}} \to \mathrm{Ext}_F$ *of Proposition 5.1.1 extends to an equivalence* $\mathrm{Ext}_{\mathrm{cl}}[\tau^{\pm 1}] \to \mathrm{Ext}_F[\tau^{-1}]$.

Proof These facts are known of the Betti realization functor for $F = \mathbb{C}$ [Dugger and Isaksen 2010, Section 2], and the general case immediately follows from Proposition 5.2.1. □

Using Mandell's p-adic homotopy theory [2001], we may also produce an unstable analogue. Let F be an algebraically closed field. Note from [Hu et al. 2011b, Proposition 15] that the spectrum $H\mathbb{F}_2^F$ is cellular; moreover, $\mathrm{Be}(H\mathbb{F}_2^F) = H\mathbb{F}_2$, as can be seen by inspection of homotopy groups. Let $\mathcal{S}\mathrm{pc}(F)$ be the category of F-motivic spaces and $\mathcal{S}\mathrm{pc}_2$ be the category of 2-complete spaces.

Proposition 5.3.2 *Let F be an algebraically closed field, and define*

$$\mathrm{Be}\colon \mathcal{S}\mathrm{pc}(F) \to \mathcal{S}\mathrm{pc}_2, \quad \mathrm{Be}(X) = \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_2}\big(\mathrm{Be}((H\mathbb{F}_2^F)^{X_+}), \overline{\mathbb{F}}_2\big).$$

Then $\mathrm{Be}(S^{a,b}) = (S^a)^\wedge_2$, *and, at least when restricted to the full subcategory of* $\mathcal{S}\mathrm{pc}(F)$ *consisting of simply connected finite motivic cell complexes, the functor* Be *preserves finite colimits and satisfies*

$$H\mathbb{F}_2^{\mathrm{Be}(X)_+} \simeq \mathrm{Be}((H\mathbb{F}_2^F)^{X_+}).$$

Proof We begin by recalling two facts from Mandell's work [2001] on p-adic homotopy theory. Strictly speaking, Mandell states his main theorem at the level of homotopy categories; a reference explicitly treating the full homotopical version we use is [Lurie 2011, Section 3]. First, the functor

$$\mathcal{S}\mathrm{pc} \to \mathcal{C}\mathrm{Alg}_{H\overline{\mathbb{F}}_2}, \quad Y \mapsto H\overline{\mathbb{F}}_2^{Y_+},$$

is fully faithful when restricted to the full subcategory of connected 2-complete nilpotent spaces with locally finite mod 2 cohomology. In particular, if Y is a connected nilpotent space with locally finite mod 2 cohomology, then the unit map

$$Y \simeq \mathcal{S}\mathrm{pc}(*, Y) \to \mathcal{C}\mathrm{Alg}_{H\overline{\mathbb{F}}_2}(H\overline{\mathbb{F}}_2^{Y_+}, H\overline{\mathbb{F}}_2^{*_+}) \simeq \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_2}(H\mathbb{F}_2^{Y_+}, H\overline{\mathbb{F}}_2)$$

realizes the target as the 2-completion of Y. Second, the functor

$$\mathcal{C}\mathrm{Alg}_{H\mathbb{F}_2}^{\mathrm{op}} \to \mathcal{S}\mathrm{pc}, \quad R \mapsto \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_2}(R, H\overline{\mathbb{F}}_2),$$

lands in $\mathcal{S}\mathrm{pc}_2$ and preserves finite colimits when restricted to the full subcategory of $\mathbb{E}_\infty$-algebras R over $\mathbb{F}_2$ such that R_* is locally finite-dimensional, $R_0 = \mathbb{F}_2$, $R_1 = 0$, and the Dyer–Lashof operation Q^0 acts by the identity on R_*.

We now apply this to our situation. The stable Betti realization functor is symmetric monoidal, and thus $\mathrm{Be}((H\mathbb{F}_2^F)^{X_+})$ is indeed an $\mathbb{E}_\infty$-ring over $\mathbb{F}_2$. Moreover, as Sq^0 acts by the identity on $H^{*,*}(X)$, the Dyer–Lashof operation Q^0 acts by the identity on $\pi_* \mathrm{Be}((H\mathbb{F}_2^F)^{X_+})$. In particular, $\mathrm{Be}((H\mathbb{F}_2^F)^{S^{a,b}_+}) \simeq H\mathbb{F}_2^{S^a_+}$, and so the proposition follows by applying Mandell's theory. ◻

Remark 5.3.3 We have focused in this section on 2-primary motivic homotopy theory over a field F of characteristic not 2. However, our discussion applies in general to p-primary motivic homotopy theory over a field F of characteristic not p. ◁

6 The motivic Hopf invariant one problem

In this section, we formulate and discuss motivic analogues of the Hopf invariant one problem. The material in this section is not needed for Section 7.

6.1 The unstable Hopf invariant one problem

Classically, Adams' determination of the permanent cycles in $\mathrm{Ext}^1_{\mathrm{cl}}$ resolved the Hopf invariant one problem. The Hopf invariant one problem may be formulated motivically using the following.

Definition 6.1.1 Let $f: S^{2a-1,2b} \to S^{a,b}$ be an unstable map between motivic spheres; in particular, $a \geq b \geq 0$ and $a \geq 1$. Write $C(f)$ for the cofiber of f. The map f vanishes in mod 2 motivic cohomology for degree reasons, and thus there exists an isomorphism

$$H^{*,*}(C(f)_+) \cong \mathbb{M}^F\{1, x, y\}$$

of $\mathbb{M}^F$-modules, where $|x| = (-a, -b)$ and $|y| = (-2a, -2b)$. Say that f has *Hopf invariant one* if one may choose such generators x and y to satisfy

$$x^2 = y,$$

ie if $H^{*,*}(C(f)_+) \cong \mathbb{M}^F[x]/(x^3)$; otherwise $x^2 = 0$ and f has Hopf invariant zero. ◁

The unstable motivic Hopf invariant one problem is now the following question.

Question 6.1.2 For which (a, b) does there exist a map $f : S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one? ◁

This turns out to mostly reduce to the classical case, by way of the following.

Lemma 6.1.3 *Let $f : S^{2a-1,2b} \to S^{a,b}$ be an unstable F-motivic map. Then f has Hopf invariant one if and only if its base change to an algebraic closure of F is of Hopf invariant one.*

Proof This is immediate from the definitions. □

Proposition 6.1.4 *Fix an unstable F-motivic map $f : S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one. Then the Betti realization* (see Proposition 5.3.2) *of f is an odd multiple of 2, η, ν, or σ. In particular, $a \in \{1, 2, 4, 8\}$.*

Proof By Lemma 6.1.3, we may as well suppose that F is algebraically closed. Let $C(f)$ denote the cofiber of f and $C(\mathrm{Be}(f))$ the cofiber of $\mathrm{Be}(f)$. Then $\mathrm{Be}(C(f)) = C(\mathrm{Be}(f))$ by Proposition 5.3.2, and thus $H^*\big(C(\mathrm{Be}(f))_+\big) = H^*\big(\mathrm{Be}(C(f))_+\big) = \mathbb{F}_2[x]/(x^3)$ with $|x| = -a$. In other words, the map between 2-completed spheres $\mathrm{Be}(f) : S^{2a-1} \to S^a$ has Hopf invariant one. The proposition now follows from Adams' resolution [1960] of the Hopf invariant one problem. □

Proposition 6.1.4 is not a complete answer to Question 6.1.2, as we have not given any bounds on b, nor have we discussed the existence of maps of Hopf invariant one. Although we will not end up with a complete answer in general, there is more we can say. Before this, we recall what information is encoded in the 1-line of the F-motivic Adams spectral sequence.

6.2 The stable Hopf invariant one problem

Question 6.1.2 can be rephrased as asking when there exists an unstable 2-cell complex, with cells in dimension (a, b) and $(2a, 2b)$, such that in cohomology the bottom cell squares to the top cell. In the stable category, one no longer has cup squares; instead, one has Steenrod operations. Thus we may consider the stable motivic Hopf invariant one problem to be the following question.

Question 6.2.1 What $\mathcal{A}^F$-modules arise as the cohomology of 2-cell complexes? In particular, for which (a, b) does there exist a 2-cell complex, with cells in dimensions $(0, 0)$ and (a, b) and attaching map vanishing in mod 2 motivic cohomology, such that $H^{*,*}X = \mathbb{M}^F\{x, y\}$ is not split as an $\mathcal{A}^F$-module? ◁

This is a particular case of the *realization problem* for $\mathcal{A}^F$-modules, and is exactly what the 1-line of the F-motivic Adams spectral sequence encodes. The following is standard.

Proposition 6.2.2 *Fix a class $\epsilon \in \mathrm{Ext}_F^{a-1,1,b}$ classifying an extension $0 \to \mathbb{M}^F\{y\} \to E \to \mathbb{M}^F\{x\} \to 0$ of $\mathcal{A}^F$-modules with $|x| = (0,0)$ and $|y| = (-a,-b)$. Then the following are equivalent:*

1. *There is stable 2-cell complex C with cells in dimensions $(0,0)$ and (a,b) such that $H^{*,*}C \cong E$.*
2. *The class ϵ is a permanent cycle in the F-motivic Adams spectral sequence, and thus detects a stable class $\alpha \in \pi^F_{a-1,b}$.*

Explicitly, if $\epsilon \in \mathrm{Ext}_F^{a-1,1,b}$ detects $\alpha \in \pi^F_{a-1,b}$, then the cofiber $C(\alpha)$ satisfies $H^{*,*}C(\alpha) \cong E$; and, if C is a stable 2-cell complex with $H^{*,*}C = E$, then the fiber of the inclusion $S^{0,0} \to C$ is a map $\alpha\colon S^{a-1,b} \to S^{0,0}$ detected by $\epsilon \in \mathrm{Ext}_F^{a-1,1,b}$. □

As we will see in Section 7, the 1-line of the F-motivic Adams spectral sequence is already quite rich, and strongly depends on the base field F. Thus, in considering the stable Hopf invariant one problem, one may not reduce to the case where F is algebraically closed, unlike in the unstable case.

6.3 Relation between the unstable and stable motivic Hopf invariant one problems

We may now relate the unstable and stable questions, Questions 6.1.2 and 6.2.1.

Proposition 6.3.1 *Let $f\colon S^{2a-1,2b} \to S^{a,b}$ be a map of Hopf invariant one. Then the associated stable class $\alpha \in \pi^F_{a-1,b}$ is detected by a permanent cycle in $\mathrm{Ext}_F^{a-1,1,b}$ which, after base change to the algebraic closure of F, is one of*

$$h_0, \quad h_1, \quad \tau h_1, \quad h_2, \quad \tau h_2, \quad \tau^2 h_2, \quad h_3, \quad \tau h_3, \quad \tau^2 h_3, \quad \tau^3 h_3, \quad \tau^4 h_3.$$

In particular, if $\mathrm{Ext}_F^{a-1,1,b}$ does not contain any such permanent cycle, then there is no map $f\colon S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one.

Proof By Lemma 6.1.3, we may suppose that F itself is algebraically closed. By stabilizing Proposition 6.1.4, we find that $\mathrm{Be}(\alpha)$ is detected by h_1, h_2, or h_3 in $\mathrm{Ext}^1_{\mathrm{cl}}$. Recall from Proposition 5.3.1 that Betti realization is modeled on Adams spectral sequences by the map

$$\mathrm{Ext}_F \to \mathrm{Ext}_F[\tau^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}[\tau^{\pm 1}].$$

In particular, the structure of Ext_F (see Proposition 4.2.1) implies that α must be detected by a permanent cycle in Ext_F of the form $\tau^n h_0$, $\tau^n h_1$, $\tau^n h_2$, or $\tau^n h_3$ for some $n \geq 0$. As f is an unstable map, this class must have nonnegative weight, reducing to the listed classes. □

Remark 6.3.2 Our method of relating the unstable motivic Hopf invariant one problem to the stable motivic Hopf invariant one problem, going through the "Betti realization" functors of Section 5.3, may seem somewhat roundabout. This route was taken for the following reason: if $f\colon S^{2a-1} \to S^a$ is a map of Hopf invariant one, then the fact that $H^*(C(f))$ is nonsplit as an $\mathcal{A}^{\mathrm{cl}}$-module, and thus the associated stable class $\alpha \in \pi^{\mathrm{cl}}_{a-1}$ is detected in $\mathrm{Ext}^1_{\mathrm{cl}}$, follows from the instability condition $\mathrm{Sq}^a(x) = x^2$.

Motivically, the analogous instability condition asserts that, if X is a motivic space and $x \in H^{2a,a}(X_+)$, then $\mathrm{Sq}^{2a}(x) = x^2$ [Voevodsky 2003, Lemma 9.7]. Now suppose that $f\colon S^{2a-1,2b} \to S^{a,b}$ is an unstable map of Hopf invariant one, and write $H^{*,*}(C(f)_+) = \mathbb{M}^F[x]/(x^3)$ with $|x| = (-a,-b)$. If a is even and $b \le \frac{1}{2}a$, then one may set $c = \frac{1}{2}a - b$ and deduce $\mathrm{Sq}^a(\tau^c x) = \tau^{2c}x^2$, so that $H^{*,*}(C(f))$ is not split as an $\mathcal{A}^F$-module. If a is odd, then one may argue by appealing to an integral motivic Hopf invariant and graded commutativity, as in the classical case. Thus, it is to rule out the possibility of a map $f\colon S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one with $b > \frac{1}{2}a$ that we have taken our approach. ◁

Our computations in Section 7 show, for a variety of base fields F, when Ext^1_F contains a permanent cycle whose image over the algebraic closure is one of the classes listed in Proposition 6.3.1, yielding various nonexistence results. To obtain existence results, we must recall how maps of Hopf invariant one arise.

6.4 Geometric applications

Adams' resolution of the classical Hopf invariant one problem had geometric consequences; notably, it implied that the only spheres which admit H-space structures are S^0, S^1, S^3, and S^7. It makes sense to ask for the motivic analogue of this, ie to ask which spheres $S^{a,b}$ admit H-space structures.

This question is in some sense geometric, but we can also ask for something even more concrete. The spheres $S^{a,b}$ are certain sheaves on the Nisnevich site of smooth F-schemes, and so it is reasonable to ask when $S^{a,b}$ is in fact represented by a smooth F-scheme. This question was raised and studied by Asok et al. [2017]; in particular, they produce explicit smooth affine schemes representing $S^{a,\lceil a/2\rceil}$, as well as prove that $S^{a,b}$ is not represented by a smooth scheme for $a > 2b$. Motivated by this, we are led to ask the following question.

Question 6.4.1 For what pairs (a,b) is $S^{a,b}$ a motivic H-space? Of these, when is it represented by a smooth F-scheme which admits a unital product? ◁

Classically, the connection between the H-space structures and the Hopf invariant one problem is via the *Hopf construction*. This construction may also be carried out in the motivic category, and has been studied in this context in [Dugger and Isaksen 2013]. We recall the key points.

Definition 6.4.2 [Dugger and Isaksen 2013, Definition C.1] Let X, Y, and Z be pointed spaces, and let $h\colon X \times Y \to Z$ be a pointed map. The *Hopf construction of h* is the map $H(h)\colon X \star Y \to \Sigma Z$ obtained by taking homotopy colimits of the rows of the diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ * & \longleftarrow & Z & \longrightarrow & * \end{array}$$

◁

Here $\star$ is the join. Note that $S^{a,b} \star S^{c,d} \simeq S^{a+c+1,b+d}$; thus the Hopf construction may be used to construct maps between motivic spheres. Using the theory of Cayley–Dickson algebras, Dugger and Isaksen [2013, Section 4] used this to define *motivic Hopf maps* $\eta \in \pi^F_{1,1}$, $\nu \in \pi^F_{3,2}$, and $\sigma \in \pi^F_{7,4}$. As noted in [loc. cit., Remark 4.14], these motivic Hopf maps have Hopf invariant one. This is a general property of the Hopf construction, which we may summarize in the following.

Lemma 6.4.3 *If $\mu\colon S^{a-1,b} \times S^{a-1,b} \to S^{a-1,b}$ is an H-space product, then its Hopf construction $H(\mu)\colon S^{2a-1,2b} \to S^{a,b}$ has Hopf invariant one.*

Proof The proof of the analogous fact for topological spaces [Steenrod 1962, Section I.5] extends to motivic spaces. We summarize the key points.

Define the (mod 2) *degree* of a pointed map $S^{a,b} \to S^{a,b}$ of motivic spaces to be its induced map in reduced motivic cohomology. A pointed map $f\colon S^{a-1,b} \times S^{a-1,b} \to S^{a-1,b}$ of motivic spaces is said to have *degree* (α, β) if $f|_{S^{a-1,b}\times\{p_2\}}$ has degree α and $f|_{\{p_1\}\times S^{a-1,b}}$ has degree β. Since μ is an H-space product, its restrictions to $S^{a-1,b} \times \{p_2\}$ and $\{p_1\} \times S^{a-1,b}$ are homotopic to the identity, so μ has degree $(1,1)$. The lemma follows by showing that, more generally, the Hopf invariant, defined in the evident way, of the Hopf construction of a map of degree (α, β) is $\alpha \cdot \beta$.

Steenrod and Epstein's proof of [Steenrod 1962, Lemma 5.3] carries over to the motivic setting to complete the proof. The main point is that Steenrod and Epstein work with particular models of the cone, join, homotopy cofiber, and suspension in their proof, but any model would work, as all of their statements only depend on the homotopy types of the relevant spaces and homotopy classes of the relevant maps. More precisely, with notation as in their proof, one may replace E_1, E_2, E_+, and E_- by the cones on S_1, S_2, S, and S, respectively, to avoid any potential point-set issues. In particular, one regards E_1, E_2, E_+, and E_- as suspension data in the sense of [Dugger and Isaksen 2013, Remark 2.9] for the various suspensions appearing in the Hopf construction. In this language, the identifications of various pushouts in the proof of [Steenrod 1962, Lemma 5.3] are examples of induced orientations [Dugger and Isaksen 2013, Remark 2.10]. The proof carries through unchanged with these new choices of E_1, E_2, E_+, and E_-.

To be precise, their proof considers maps $S^{n-1} \times S^{n-1} \to S^{n-1}$ with $n > 1$ even and works integrally. Routine modifications extend this to arbitrary $n \geq 1$ provided one works mod 2 throughout. Classically, this is the adaption needed to incorporate the degree 2 map $S^1 \to S^1$, which is the Hopf construction of the standard product on $S^0 \cong C_2$. Motivically, this is the adaption needed for our lemma to hold for arbitrary unstable motivic spheres $S^{a-1,b}$, allowing especially for the uniform treatment of 2 and η. □

Remark 6.4.4 Under Definition 6.1.1, the map $h\colon S^{1,1} \to S^{1,1}$ represented by the squaring map on $\mathbb{G}_m$, sometimes called the "zeroth Hopf map" and stably detected by h_0, is *not* a map of Hopf invariant one. In the context of Lemma 6.4.3, this is justified by the fact that, for degree reasons, h is not the Hopf construction of an H-space structure on any motivic sphere. ◁

We can now summarize what is known in the following.

Theorem 6.4.5 *A motivic sphere is represented by a smooth F-scheme admitting a unital product if and only if it is one of*

$$S^{0,0}, \quad S^{1,1}, \quad S^{3,2}, \quad S^{7,4}.$$

In addition to the motivic spheres listed above, the following motivic spheres admit H-space structures:

$$S^{1,0}, \quad S^{3,0}, \quad S^{7,0}.$$

The only other motivic spheres that could possibly admit H-space structures are

$$S^{3,1}, \quad S^{7,3}, \quad S^{7,2}, \quad S^{7,1};$$

moreover, an H-space structure on such a sphere produces a permanent cycle in Ext_F whose image over the algebraic closure is τh_2, τh_3, $\tau^2 h_3$, or $\tau^3 h_3$, respectively.

Proof That the spheres $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$ are represented by smooth F-schemes admitting a unital product is given by [Dugger and Isaksen 2013]. The spheres $S^{1,0}$, $S^{3,0}$, and $S^{7,0}$ are the images of S^1, S^3, and S^7, respectively, under the unstable constant functor from spaces to motivic spaces, and so inherit H-space structures from their classical structures. That all the spheres listed are the only spheres which may admit H-space structures follows from Lemma 6.4.3 and Proposition 6.3.1, as does the final claim concerning the F-motivic Adams spectral sequence. Finally, Asok et al. [2017, Proposition 2.3.1] prove that, if $S^{a-1,b}$ is represented by a smooth F-scheme, then necessarily $2b \geq a-1$, and the only possible H-spaces satisfying this are $S^{0,0}$, $S^{1,1}$, $S^{3,2}$, and $S^{7,4}$, as listed. □

We note the following special case.

Corollary 6.4.6 *Suppose there is an $\mathbb{R}$-motivic map $f: S^{2a-1,2b} \to S^{a,b}$ of Hopf invariant one. Then (a,b) is one of*

$$(1,0), \quad (2,1), \quad (4,2), \quad (8,4), \quad (2,0), \quad (4,0), \quad (8,0).$$

Moreover, all of these are realized, and in fact

$$S^{0,0}, \quad S^{1,1}, \quad S^{3,2}, \quad S^{7,4}, \quad S^{1,0}, \quad S^{3,0}, \quad S^{7,0}$$

are all the $\mathbb{R}$-motivic spheres admitting H-space structures.

Proof This is immediate from Theorem 6.4.5, either appealing to the fact that $\mathrm{Ext}_{\mathbb{R}}$ vanishes in the degrees detecting the remaining possibilities, or else noting that the real points of $S^{a,b}$ are S^{a-b}, so that, if $S^{a,b}$ is an H-space, then $a-b \in \{0,1,3,7\}$. □

7 The 1-line of the motivic Adams spectral sequence

We now analyze the 1-line of the F-motivic Adams spectral sequence. We begin in Section 7.1 by explaining how to read off the structure of Ext_F for various fields F from our computation of $\mathrm{Ext}_{\mathbb{R}}$.

After some additional preliminaries in Section 7.2, we give a direct motivic analogue of the classical differentials in Section 7.3, proving $d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2$ for $a \geq 3$ over arbitrary base fields. We then proceed to give more detailed information about the 1-line for the particular fields F of the form $\mathbb{R}$, $\mathbb{F}_q$ with q an odd prime power, $\mathbb{Q}_p$ with p any prime, and $\mathbb{Q}$.

7.1 Computing Ext_F

As a general rule, Ext_F is largely understood once $\mathbb{M}^F$ and $\mathrm{Ext}_{\mathbb{R}}$ are both understood. Rather than formulate a precise statement, let us just describe Ext_F for the various particular fields F we shall encounter, namely those described in Example 2.2.1 as well as $F = \mathbb{Q}$.

Recall from Remark 2.3.2 that, for any field F, we may view $\mathbb{M}^F$ as an $\mathcal{A}^{\mathbb{R}}$-module, and there is an isomorphism

$$\mathrm{Ext}_F \cong \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}, \mathbb{M}^F).$$

Thus, the main point is to understand $\mathbb{M}^F$ as an $\mathcal{A}^{\mathbb{R}}$-module, and this is in fact determined by $\mathbb{M}_0^F$ as an $\mathbb{F}_2[\rho]$-module. For the examples of interest, we have the following. Abbreviate

$$\mathbb{M} = \mathbb{F}_2[\tau, \rho], \quad \mathbb{M}_{(r)} = \mathbb{M}/(\rho^r).$$

Lemma 7.1.1 *As $\mathcal{A}^{\mathbb{R}}$-modules, we have the following*:

(1) $\mathbb{M}^{\mathbb{R}} = \mathbb{M}$.

(2) *If $F = \bar{F}$ is algebraically closed, then* $\mathbb{M}^F = \mathbb{M}_{(1)}$.

(3) *If* $q \equiv 1 \pmod 4$, *then* $\mathbb{M}^{\mathbb{F}_q} = \mathbb{M}_{(1)}\{1, u\}$.

(4) *If* $q \equiv 3 \pmod 4$, *then* $\mathbb{M}^{\mathbb{F}_q} = \mathbb{M}_{(2)}$.

(5) *If* $p \equiv 1 \pmod 4$, *then* $\mathbb{M}^{\mathbb{Q}_p} = \mathbb{M}_{(1)}\{1, \pi, u, \pi u\}$.

(6) *If* $p \equiv 3 \pmod 4$, *then* $\mathbb{M}^{\mathbb{Q}_p} = \mathbb{M}_{(2)}\{1, \pi\}$.

(7) $\mathbb{M}^{\mathbb{Q}_2} = \mathbb{M}_{(3)}\{1\} \oplus \mathbb{M}_{(1)}\{u, \pi\}$.

(8) $\mathbb{M}^{\mathbb{Q}} = \mathbb{M}\{1\} \oplus \mathbb{M}_{(1)}\{[2]\} \oplus \mathbb{M}_{(1)}\{[p], a_p : p \equiv 1 \pmod 4\} \oplus \mathbb{M}_{(2)}\{u_p : p \equiv 3 \pmod 4\}$.

Proof All but the case $F = \mathbb{Q}$ may be read off the examples listed in Example 2.2.1. When $F = \mathbb{Q}$, the ring $\mathbb{M}^{\mathbb{Q}}$ is described in [Ormsby and Østvær 2013, Propositions 5.3 and 5.4], following [Milnor 1970]. Our description may be read off this upon setting $u_p = [p] + \rho$ for $p \equiv 3 \pmod 4$. □

For $r \geq 0$, define

$$\mathrm{Ext}_{(r)} = \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}, \mathbb{M}_{(r)}) = H_*(\Lambda^{\mathbb{R}}/(\rho^r)).$$

The $\mathbb{F}_2[\rho]$-module structure of $\mathrm{Ext}_{(r)}$ may be easily computed from $\mathrm{Ext}_{\mathbb{R}}$ via the long exact sequence associated to the cofiber of ρ^r. Even less work is necessary when $\mathrm{Ext}_{\mathbb{R}}$ has been computed by some method compatible with the ρ-Bockstein spectral sequence such as ours; see in particular Remark 4.1.5. Thus Theorem 4.2.12 allows us to read off $\mathrm{Ext}^f_{(r)}$ for $f \leq 2$, as well as the image of $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}^3_{(r)}$. This does not give the entirety of $\mathrm{Ext}^3_{(r)}$; however, we at least know that whatever remains is generated by classes which appear in the ρ-Bockstein spectral sequence as $\rho^k\alpha$ with $\alpha \in \mathrm{Ext}^3_{(1)}$ and $k < r$, and this is enough information for our purposes.

Lemma 7.1.1 describes for various F how Ext_F may be written as a direct sum of copies of various $\mathrm{Ext}_{(r)}$. For example, $\mathrm{Ext}_{\mathbb{Q}_2} = \mathrm{Ext}_{(3)}\{1\} \oplus \mathrm{Ext}_{(1)}\{u, \pi\}$. We may use this to prove a Hasse principle for $\mathrm{Ext}_{\mathbb{Q}}$.

Lemma 7.1.2 *The map*

$$\mathbb{M}^{\mathbb{Q}} \to \mathbb{M}^{\mathbb{Q}_p}$$

satisfies

$$[p] \mapsto \pi, \quad a_p \mapsto u\pi, \quad u_p \mapsto \pi + \rho.$$

Here the first is relevant for $p = 2$ *or* $p \equiv 1 \pmod 4$, *the second for* $p \equiv 1 \pmod 4$, *and the third for* $p \equiv 3 \pmod 4$.

Proof The behavior of these maps is described in [Ormsby and Østvær 2013, Proposition 5.3]. Our description follows immediately; note we have defined $u_p = [p] + \rho$ for $p \equiv 3 \pmod 4$. □

Proposition 7.1.3 *The Hasse map*

$$\mathrm{Ext}_{\mathbb{Q}} \to \mathrm{Ext}_{\mathbb{R}} \times \prod_p \mathrm{Ext}_{\mathbb{Q}_p}$$

is injective.

Proof By Lemma 7.1.1, we have

$$\mathrm{Ext}_{\mathbb{Q}} = \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{(1)}\{[2]\} \oplus \mathrm{Ext}_{(1)}\{[p], a_p : p \equiv 1 \pmod 4\} \oplus \mathrm{Ext}_{(2)}\{u_p : p \equiv 3 \pmod 4\}.$$

The summand $\mathrm{Ext}_{\mathbb{R}}$ maps isomorphically to $\mathrm{Ext}_{\mathbb{R}}$, and the maps $\mathrm{Ext}_{\mathbb{Q}} \to \mathrm{Ext}_{\mathbb{Q}_p}$ are determined by Lemma 7.1.2. In particular, it is easily seen that the maps

$$\mathrm{Ext}_{(1)}\{[2]\} \to \mathrm{Ext}_{\mathbb{Q}_2}, \quad \mathrm{Ext}_{(1)}\{[p], a_p\} \to \mathrm{Ext}_{\mathbb{Q}_p}, \quad \mathrm{Ext}_{(2)}\{u_p\} \to \mathrm{Ext}_{\mathbb{Q}_p}$$

are all split injections, and the proposition follows. □

The preceding discussion, together with our computation of $\mathrm{Ext}_{\mathbb{R}}$, describes what we will need of Ext_F in low filtrations and arbitrary stem. So that we may rule out various higher differentials in low stems for degree reasons, we record the following.

Lemma 7.1.4 $\mathrm{Ext}_{(1)}$ *is given in stems* $s \leq 6$ *by the module*

$$\mathbb{F}_2[\tau] \otimes \left(\mathbb{F}_2\{h_0^n : n \geq 0\} \oplus \mathbb{F}_2\{h_1, h_1^2, h_1^3, h_2, h_0h_1, h_2^2\}\right) \oplus \mathbb{F}_2[\tau]/(\tau)\{h_1^4, h_1^5, h_1^6\}.$$

Proof These groups have been computed in [Dugger and Isaksen 2010]. □

7.2 Existence of Hopf elements

Our computation of the F-motivic Adams differentials $d_2(h_{a+1})$ will follow a similar pattern to Wang's computation [1967] of the corresponding classical Adams differentials (differentials which were first computed in [Adams 1960]). This is an inductive argument, beginning with information about the Hopf elements which are known to exist. We record some of this information in this subsection.

Write $\epsilon \in \pi^F_{0,0}$ for the class represented by the twist map $S^{1,1} \otimes S^{1,1} \to S^{1,1} \otimes S^{1,1}$.

Lemma 7.2.1 *Fix $\alpha \in \pi^F_{a,b}$ and $\beta \in \pi^F_{c,d}$. Then there is an identity*

$$\alpha \cdot \beta = (-1)^{(a-b)(c-d)} \epsilon^{bd} \cdot \beta \cdot \alpha.$$

Moreover, $1-\epsilon$ is detected in Ext_F by h_0 and 2 by $h_0 + \rho h_1$.

Proof The claimed graded commutativity is given in [Morel 2004, Corollary 6.1.2]; see also [Isaksen and Østvær 2020, Section 6.1] for a discussion. That $1-\epsilon$ is detected by h_0 and 2 by $h_0 + \rho h_1$ is noted in [Isaksen and Østvær 2020, Remark 6.3]. □

Lemma 7.2.2 *For any field F, the class h_a is a permanent cycle for $a \in \{0, 1, 2, 3\}$.*

Proof The class h_0 is a permanent cycle by Lemma 7.2.1. Dugger and Isaksen [2013] construct the motivic Hopf elements η, ν, and σ, and indicate [loc. cit., Remark 4.14] that these are detected by h_1, h_2, and h_3, respectively; see also our discussion in Section 6.4. Thus these classes must be permanent cycles. □

7.3 Nonexistence of Hopf elements

The purpose of this subsection is to prove the following.

Theorem 7.3.1 *For an arbitrary base field F of characteristic not equal to 2, there are differentials of the form*

$$d_2(h_{a+1}) = (h_0 + \rho h_1) h_a^2$$

in the F-motivic Adams spectral sequence, which are nonzero for $a \geq 3$. ◁

By naturality, it suffices to produce these differentials in the case where F is a prime field, ie $F = \mathbb{F}_q$ or $F = \mathbb{Q}$, and when F is algebraically closed. Moreover, by the Hasse principal given in Proposition 7.1.3, the case $F = \mathbb{Q}$ may be deduced from the cases $F = \mathbb{Q}_p$ and $F = \mathbb{R}$ combined. All of these build on the case where F is algebraically closed, which may be treated as follows.

Proposition 7.3.2 *If $F = \overline{F}$ is algebraically closed, then*

$$d_2(h_{a+1}) = h_0 h_a^2.$$

This is nonzero for $a \geq 3$.

Proof The corresponding classical differentials are known due to [Adams 1960]. The proposition could be reduced to this by appealing to Proposition 5.3.1; however, we shall instead proceed as follows.

Wang [1967, Section 3] gives another proof of the classical differentials, combining only a minimal amount of homotopical input with a good understanding of $\mathrm{Ext}_{\mathrm{cl}}$. His argument may be applied essentially verbatim to produce the claimed $\overline{F}$-motivic differentials. It is this argument that may be adapted to work for other base fields, so to motivate our later computations let us recall this argument in full.

The proof proceeds by induction on a, where only the base case requires any homotopical input.

Consider the base case $a = 3$. The class h_3 is a permanent cycle, detecting the Hopf element σ; see Lemma 7.2.2. By Lemma 7.2.1, we find that $2\sigma^2 = 0$. As 2 is detected by h_0 over algebraically closed fields, it follows that $h_0h_3^2$ cannot survive the Adams spectral sequence. The structure of $\mathrm{Ext}_{\overline{F}}$ implies that $d_2(h_4) = h_0h_3^2$ is the only way for $h_0h_3^2$ to die.

Now suppose we have produced the differential $d_2(h_a) = h_0h_{a-1}^2$ for some $n \geq 4$. The relation $h_{a+1}h_a = 0$ together with the Leibniz rule implies

$$0 = d_2(h_{a+1}h_a) = d_2(h_{a+1}) \cdot h_a + h_{a+1} \cdot d_2(h_a).$$

Applying our inductive hypothesis and the relation $h_{a+1} \cdot h_{a-1}^2 = h_a^3$, this reduces to

$$(d_2(h_{a+1}) + h_0h_a^2) \cdot h_a = 0.$$

The algebraic structure of $\mathrm{Ext}_{\overline{F}}^3$ implies that $d_2(h_{a+1}) \in \mathbb{F}_2\{h_0h_a^2\}$, so it suffices to verify that $h_0h_a^3 \neq 0$ for $a \geq 4$. This follows from Wang's computation [1967, Proposition 3.4] by comparison along the map $\mathrm{Ext}_{\overline{F}} \to \mathrm{Ext}_{\overline{F}}[\tau^{-1}] \simeq \mathrm{Ext}_{\mathrm{cl}}[\tau^{\pm 1}]$. □

The base step for the inductive argument given in Proposition 7.3.2 works for arbitrary base fields, but the inductive step falls apart. This inductive step relies on the algebraic fact that, when working over an algebraically close field, multiplication by h_a is injective on the degree of $d_2(h_{a+1})$ for $a \geq 4$. Over other base fields, this fails for $a = 4$: this degree may contain elements of the form $\omega h_1 h_4^2$ where $\omega \in \mathrm{Ext}_F^{-1,0,-1}$ is a sum of elements such as ρ, π, and u, and

$$\omega h_1 h_4^2 \cdot h_4 = \omega h_1 \cdot h_4^3 = \omega h_1 \cdot h_3^2 \cdot h_5 = 0.$$

Luckily, the inductive step fails only for $a = 4$; once we have resolved $d_2(h_5)$, the remaining differentials will follow via the same argument. To resolve this differential, we proceed as follows.

Proposition 7.3.3 *Let F be a field of the form $\mathbb{F}_q$ for q odd, $\mathbb{Q}_p$ for any p, or $\mathbb{R}$. Then there is a differential*

$$d_2(h_5) = (h_0 + \rho h_1)h_4^2$$

in the F-motivic Adams spectral sequence.

Proof When $F = \mathbb{R}$, we first make the following reduction. Observe that $\mathrm{Ext}_{\mathbb{R}}$ in the degree of $d_2(h_5)$ is given by $\mathbb{F}_2\{h_0h_4^2, \rho h_1h_4^2\}$, and that neither of these classes are divisible by ρ^2. Thus it is sufficient to verify this differential in the Adams spectral sequence for the cofiber of ρ^2. By [Behrens and Shah 2020, Lemma 7.8], this cofiber is a ring spectrum, and so its Adams spectral sequence is multiplicative. Having made this reduction, the remainder of the argument is uniform in the given choices of F. For brevity of notation, in the following we shall write Ext_F for the object so named when $F = \mathbb{F}_q$ or $F = \mathbb{Q}_p$, and write the same for $\mathrm{Ext}_{(2)}$ when $F = \mathbb{R}$.

First observe that, as $\tau^4 \in \mathrm{Ext}_F^0$, the class τ^{16} is a square and thus a d_2-cycle. As τ^{16} acts injectively on Ext_F^f for $f \leq 3$, it suffices to show

$$d_2(\tau^{16}h_5) = (h_0 + \rho h_1)\tau^{16}h_4^2.$$

Consider the Hurewicz map $c\colon \pi_* \to \pi_{*,0}^F$. Let $\theta_4 \in \pi_{30}S^0$ be the Kervaire class, detected by h_4^2 and satisfying $2\theta_4 = 0$. By Proposition 5.1.1, we find that $c(\theta_4)$ is detected by $(\mathrm{Sq}^0)^4(h_0^2) = \tau^{16}h_4^2$. As $2 \cdot c(\theta_4) = 0$, the class $(h_0 + \rho h_1)\tau^{16}h_4^2$ cannot survive. The only possibility is that $d_2(\tau^{16}h_4) = (h_0 + \rho h_1)\tau^{16}h_4^2$, yielding the desired differential. □

Remark 7.3.4 When $F = \mathbb{R}$, the differential $d_2(h_5)$, and in fact all the differentials $d_2(h_{a+1})$, may also be produced as follows. By comparison with $\mathbb{C}$, one finds $d_2(h_5) \in h_0h_4^2 + \mathbb{F}_2\{\rho h_1h_4^2\}$. Thus it suffices to verify that $d_2(h_5)$ is not ρ-torsion. This is a consequence of the fact that the isomorphism $\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}] \simeq \mathrm{Ext}_{\mathrm{dcl}}[\rho^{\pm 1}]$ [Dugger and Isaksen 2017b, Theorem 4.1] commutes with Adams differentials. ◁

We need just one more algebraic fact for the proof of Theorem 7.3.1.

Lemma 7.3.5 *Let $\omega \in \mathrm{Ext}_F^0$ be nonzero. Then $\omega h_1h_a^3 \neq 0$ for all $a \geq 5$.*

Proof The class $h_0h_{a-1}^3$ is nonzero in $\mathrm{Ext}_{\mathrm{cl}}$ for $a \geq 5$ by [Wang 1967, Proposition 3.4]. Proposition 3.2.1 gives an injection $\mathrm{Ext}_{\mathrm{dcl}} \to \mathrm{Ext}_F$, and this extends by linearity to an injection $\mathrm{Ext}_F^0 \otimes_{\mathbb{F}_2} \mathrm{Ext}_{\mathrm{dcl}} \to \mathrm{Ext}_F$, as can be seen by using Lemma 7.1.1 to reduce to the injections $\mathrm{Ext}_{(r)}^0 \otimes_{\mathbb{F}_2} \mathrm{Ext}_{\mathrm{dcl}} \to \mathrm{Ext}_{(r)}$. The class $\omega h_1h_a^3$ is the image of $\omega \otimes h_0h_{a-1}^3$ under this map, yielding the claim. □

We may now give the following.

Proof of Theorem 7.3.1 As discussed, it suffices to consider only the cases where F is of the form $\mathbb{F}_q$ for some q odd, $\mathbb{Q}_q$ for some q, or $\mathbb{R}$. So let F be one of these. We now induct on a, with base cases $a = 3$ and $a = 4$.

First consider the case $a = 3$. By Lemma 7.2.2, the class h_3 is a permanent cycle detecting the class σ. By Lemma 7.2.1, $2\sigma^2 = 0$, and so $(h_0 + \rho h_1)h_3^2$ must be the target of a differential. The only possibility is that $d_2(h_4) = (h_0 + \rho h_1)h_3^2$.

The case $a = 4$ was handled in Proposition 7.3.3.

Now suppose inductively that we have produced the differential $d_2(h_a) = (h_0 + \rho h_1)h_{a-1}^2$ for some $a \geq 5$. Combining the Leibniz rule with the relation $h_{a+1}h_a = 0$, we find

$$0 = d_2(h_{a+1}h_a) = d_2(h_{a+1})h_a + h_{a+1}d_2(h_a).$$

Applying our inductive hypothesis and the relation $h_{a+1}h_{a-1}^2 = h_a^2$, we find

$$(d_2(h_{a+1}) + (h_0 + \rho h_1)h_a^2)h_a = 0.$$

It follows that $d_2(h_{a+1}) = (h_0 + \rho h_1)h_a^2 + x$ where x is some class killed by h_a. The only classes in this degree are $h_0 h_a^2$ and those of the form $\omega h_1 h_a^2$ where $\omega \in \mathrm{Ext}_F^0$. By comparison with $\overline{F}$, we find that x must be zero or a nonzero class of the form $\omega h_1 h_a^2$ with $\omega \in \mathrm{Ext}_F^{-1,0,-1}$. As $a \geq 5$, Lemma 7.3.5 implies that none of the latter are killed by h_a. Thus $x = 0$, yielding the desired differential. □

This concludes our uniform analysis of differentials out of Ext_F^1. The rest of this section is dedicated to studying the 1-line in more detail for particular fields F.

7.4 The real numbers

We now study the case $F = \mathbb{R}$ in more detail. Recall from Theorem 4.2.12 that

$$\mathrm{Ext}_{\mathbb{R}}^1 = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a \geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a})\{\tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a : n \geq 0\}.$$

Here recall that $2^{a-1}(4n+1) = 2n$ for $a = 0$. Theorem 7.3.1 allows one to understand the fate of the classes in the ρ-torsion-free summand, so we turn our attention to the ρ-torsion subgroup. We shall first pin down which of these ρ-torsion classes are permanent cycles, and then by separate methods compute all d_2-differentials on these ρ-torsion classes. A comparison reveals that there must be numerous higher differentials, but determining these is outside the scope of our computation. The first point of order is the following.

Definition 7.4.1 For $a \geq 0$, write $a = c + 4d$ with $0 \leq c \leq 3$, and define $\psi(a) = 2^c + 8d$ to be the a^{th} *Radon–Hurwitz number*. ◁

Proposition 7.4.2 *The class $\rho^r \tau^{2^{a-1}(4n+1)} h_a$ is a permanent cycle if and only if $r \geq 2^a - \psi(a)$.*

The proof of Proposition 7.4.2 requires some preliminaries. We proceed by comparison with Borel C_2-equivariant stable homotopy theory. Let Ext_{BC_2} denote the E_2-page of the Borel C_2-equivariant Adams spectral sequence [Greenlees 1988]. Explicitly,

$$\mathrm{Ext}_{BC_2}^{s,f,w} = \mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}^{s-w,f}(\mathbb{F}_2, H^* P_w^\infty);$$

this is just a combination of the ordinary Adams spectral sequences for the stable cohomotopy groups of infinite stunted projective space. By Lin's positive resolution [1980] of the Segal conjecture, this spectral sequence converges to $\pi_{*,*}^{C_2}$, the homotopy groups of the 2-completion of the C_2-equivariant sphere spectrum.

Betti realization followed by Borel completion yields a functor from the stable $\mathbb{R}$-motivic category to the Borel C_2-equivariant stable category $\mathrm{Fun}(BC_2, \mathcal{S}\mathrm{p})$, and Behrens and Shah [2020, Section 8] show that this may be understood as completing at ρ and inverting τ. Applying this to an Adams resolution, we find that

$$\mathrm{Ext}_{BC_2} = \lim_{n\to\infty} \mathrm{Ext}_{(2^n)}[\tau^{-2^n}].$$

The simple form of $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ allows us to immediately read off $\mathrm{Ext}_{BC_2}^{\leq 3}$.

Lemma 7.4.3 *$\mathrm{Ext}_{BC_2}^{\leq 3}$ is exactly as $\mathrm{Ext}_{\mathbb{R}}^{\leq 3}$ is described in Theorem 4.2.12, except n is allowed to be negative, and in place of the map $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{C}}$ is a map $\mathrm{Ext}_{BC_2} \to \mathrm{Ext}_{\mathbb{C}}[\tau^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}[\tau^{\pm 1}]$.* □

In particular,

$$\mathrm{Ext}_{BC_2}^1 = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a\geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a})\{\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a : n \in \mathbb{Z}\}.$$

We have introduced Ext_{BC_2} in order to make the following reduction.

Lemma 7.4.4 *Write $h\colon \mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{BC_2}$ for the canonical map of spectral sequences. Fix a ρ-torsion class $x \in \mathrm{Ext}_{\mathbb{R}}^1$. Then x is a permanent cycle if and only if $h(x)$ is a permanent cycle.*

Proof Clearly, if x is a permanent cycle, then the same must be true of $h(x)$. Conversely, suppose that $h(x)$ is a nontrivial permanent cycle; we claim that x is a permanent cycle.

Write Ext_{C_2} for the E_2-page of the C_2-equivariant Adams spectral sequence [Hu and Kriz 2001, Section 6], converging to the same target as Ext_{BC_2}. This splits additively as $\mathrm{Ext}_{C_2} = \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{\mathrm{NC}}$ for a certain summand $\mathrm{Ext}_{\mathrm{NC}}$ (see [Guillou et al. 2020, Section 2]), and h factors as $h = g \circ f : \mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{C_2} \to \mathrm{Ext}_{BC_2}$, the first map being the obvious inclusion and the second map killing the summand $\mathrm{Ext}_{\mathrm{NC}}$.

As $h(x)$ is a nontrivial permanent cycle, it detects a class α in Borel Adams filtration 1. The class α must then be detected in $\mathrm{Ext}_{C_2}^{\leq 1}$. By [Belmont et al. 2021], the map $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{C_2}$ is an isomorphism in the degrees under consideration, so the same must be true for $\mathrm{Ext}_{C_2} \to \mathrm{Ext}_{BC_2}$. As there is at most one nonzero ρ-torsion class in these degrees, the only possibility is that α is detected by $f(x)$ in $\mathrm{Ext}_{C_2}^1$, implying that $f(x)$ is a permanent cycle. As $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{C_2}$ is the inclusion of a summand, this implies that x is a permanent cycle, as claimed. □

Thus it suffices to understand permanent cycles in $\mathrm{Ext}_{BC_2}^1$. The main point is the following.

Lemma 7.4.5 *There exists a nonzero ρ-torsion class $\alpha \in \pi_{s,w}^{C_2}$ detected in Borel Adams filtration 1 if and only if the inclusion of the bottom cell of P_{w-s-1}^{w-1} is split, where P_k^n is the Thom spectrum of the k-fold Whitney sum of the tautological line bundle over the real projective space $\mathbb{R}P^n$.*

Proof First suppose given such a map α. The structure of $\mathrm{Ext}^1_{BC_2}$ implies that α must have ρ-torsion exponent $s+1$, and so there is a lift $\overline{\alpha}$ in the diagram

$$\begin{array}{ccc} \Sigma^{s-w+1} P^{w-1}_{w-s-1} & & \\ \uparrow{\scriptstyle \partial} & \overset{\overline{\alpha}}{\searrow} & \\ \Sigma^{s-w} P^{\infty}_{w} & \xrightarrow{\ \alpha\ } & S^0 \\ \uparrow & \nearrow{\scriptstyle \rho^{s+1}\alpha=0} & \\ \Sigma^{s-w} P^{\infty}_{w-s-1} & & \end{array}$$

As α and ∂ have Adams filtration 1, necessarily $\overline{\alpha}$ has Adams filtration 0. It follows that precomposing $\overline{\alpha}$ with the inclusion of the bottom cell $S^0 \to \Sigma^{s-w+1} P^{w-1}_{w-s-1}$ gives a map $S^0 \to S^0$ which is nonzero in mod 2 cohomology, and must therefore be an equivalence. In other words, $\overline{\alpha}$ splits off the bottom cell of P^{w-1}_{w-s-1}.

Conversely, if the inclusion of the bottom cell of P^{w-1}_{w-s-1} is split, then its splitting gives a nonzero map $\overline{\alpha}$ as above in Adams filtration 0. Let $\alpha = \overline{\alpha} \circ \partial$; we claim that α is a nonzero class detected in Adams filtration 1. Indeed, the cofibering $P^{w-1}_{w-s-1} \to P^{\infty}_{w-s-1} \to P^{\infty}_{w}$ gives an exact sequence

$$\mathrm{Ext}^0(\mathbb{F}_2, H^* P^{\infty}_{w}) \to \mathrm{Ext}^0(\mathbb{F}_2, H^* P^{\infty}_{w-s-1}) \to \mathrm{Ext}^0(\mathbb{F}_2, H^* P^{w-1}_{w-s-1}) \xrightarrow{\partial'} \mathrm{Ext}^1(\mathbb{F}_2, H^* P^{\infty}_{w}),$$

where ∂' models restriction along ∂ in the previous diagram. The first map is exactly

$$\rho^{s+1} \colon \mathrm{Ext}^{*,0,w}_{BC_2} \to \mathrm{Ext}^{*,0,w-s-1}_{BC_2}.$$

As $\mathrm{Ext}^0_{BC_2} = \mathbb{F}_2[\rho]$, we find that the kernel of ∂' consists of only that class represented by the inclusion $\mathbb{F}_2 \to H^0 P^{w-1}_{w-s-1}$. So ∂' is injective in the relevant degrees, implying that α is nonzero and of Adams filtration 1, as claimed. □

We may now give the following.

Proof of Proposition 7.4.2 By Lemma 7.4.4, it suffices to show that a class $\rho^r \tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a \in \mathrm{Ext}^1_{BC_2}$ is a permanent cycle if and only if $r \geq 2^a - \psi(a)$. By sparseness of $\mathrm{Ext}^1_{BC_2}$, the class $\rho^r \tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a$ is a permanent cycle if and only if there is some ρ-torsion class $\alpha \in \pi^{C_2}_{2^a - r - 1, -2^{a+1}n - r}$ detected in Borel Adams filtration 1. By Lemma 7.4.5, this holds if and only if inclusion of the bottom cell of $P^{-2^{a+1}n-r-1}_{-2^{a+1}n-2^a}$ is split. By James periodicity [1958; 1959], this holds if and only if the inclusion of the bottom cell of $P^{2^N - 2^{a+1}n - r - 1}_{2^N - 2^{a+1} - 2^a}$ is split for some sufficiently large $N \gg 0$; that is, we may assume ourselves to be working with suspension spectra of honest real projective spaces. When this happens was resolved by Adams' solution [1962, Theorem 1.2] of the vector fields on spheres problem, yielding the condition claimed. □

Corollary 7.4.6 *The classes $\tau^{\lfloor 2^{a-1}(4n+1) \rfloor} h_a$ are permanent cycles for $a \leq 3$.* □

Corollary 7.4.6 could also be proved more directly, applying the technique used in the proof of Theorem 7.3.1 or Proposition 7.4.8 below to reduce to the region considered by Belmont and Isaksen.

It is worth summarizing what we have learned from the proof of Proposition 7.4.2 about the stable cohomotopy groups of projective spaces.

Theorem 7.4.7 *The subgroup of permanent cycles in* $\mathrm{Ext}^1_{BC_2}$ *is given by*

$$\mathbb{F}_2[\rho]\{h_1, h_2, h_3, \rho h_4\} \oplus \bigoplus_{a\geq 0} \mathbb{F}_2[\rho]/(\rho^{\psi(a)})\{\rho^{2^a-\psi(a)}\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a : n \in \mathbb{Z}\}.$$

A choice of maps $\Sigma^c P_w^\infty \to S^0$ *detected by these permanent cycles is given by the following*:

(1) *For all* $r \geq 0$, *there are maps*

$$P^\infty_{1-r} \to P^\infty_1 \xrightarrow{\eta} S^0, \quad \Sigma P^\infty_{2-r} \to \Sigma P^\infty_2 \xrightarrow{\nu} S^0, \quad \Sigma^3 P^\infty_{4-r} \to \Sigma^3 P^\infty_3 \xrightarrow{\sigma} S^0.$$

Here η, ν, *and* σ *are equivariant refinements of the Hopf maps with the same names. These composites are detected by* $\rho^r h_1$, $\rho^r h_2$, *and* $\rho^r h_3$, *respectively.*

(2) *For all* $r \geq 0$, *there is a map*

$$\Sigma^7 P^\infty_{7-r} \to \Sigma^7 P^\infty_7 \xrightarrow{\mathrm{Sq}(\sigma)} S^0,$$

where $\mathrm{Sq}(\sigma)$ *is the symmetric square of* $\sigma\colon S^7 \to S^0$. *This composite is detected by* $\rho^{1+r} h_4$.

(3) *For all* $a \geq 0$, $n \in \mathbb{Z}$, *and* $1 \leq r \leq \psi(a)$, *there is a map*

$$\Sigma^{2^a(2n+1)-1} P^\infty_{-2^a(2n+1)+r} \xrightarrow{\partial} \Sigma^{2^a(2n+1)} P^{-2^a(2n+1)+r-1}_{-2^a(2n+1)} \xrightarrow{s} S^0.$$

Here ∂ *is the cofiber of the map* $\Sigma^{2^a(2n+1)-1} P^\infty_{-2^a(2n+1)} \to \Sigma^{2^a(2n+1)-1} P^\infty_{-2^a(2n+1)+r}$, *and* s *is any map that splits off the bottom cell of* $P^{-2^a(2n+1)+r-1}_{-2^a(2n+1)}$. *This composite is detected by* $\rho^{2^a-r}\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a$.

Proof Recall that

$$\mathrm{Ext}^1_{BC_2} = \mathbb{F}_2[\rho]\{h_a : a \geq 1\} \oplus \bigoplus_{a\geq 0} \mathbb{F}_2[\rho]/(\rho^{2^a})\{\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a : n \in \mathbb{Z}\}.$$

We have just analyzed which classes in the ρ-torsion summand are permanent cycles, leading to exactly the claimed ρ-torsion permanent cycles with representatives as described in (3). Lemma 7.2.2 implies that h_1, h_2, and h_3 are permanent cycles, and these detect the maps described in (1). Theorem 7.3.1 shows that $\rho^n h_a$ supports a d_2-differential for $a \geq 5$ and $n \geq 0$, and that h_4 supports a d_2-differential but ρh_4 does not. We are left with verifying that ρh_4 is a permanent cycle detecting the map $\mathrm{Sq}(\sigma)$. Indeed, taking geometric fixed points yields an isomorphism $\pi^{C_2}_{*,*}[\rho^{-1}] \cong \pi^{\mathrm{cl}}_*[\rho^{\pm 1}]$ which sends $\mathrm{Sq}(\alpha)$ to α for any $\alpha \in \pi^{\mathrm{cl}}_*$. This isomorphism is modeled on Adams spectral sequences by $\mathrm{Ext}_{C_2}[\rho^{-1}] \cong \mathrm{Ext}_{\mathbb{R}}[\rho^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}[\rho^{\pm 1}]$. As ρh_4 is the only class in its degree lifting $h_3 \in \mathrm{Ext}^1_{\mathrm{cl}}$, it must be that ρh_4 detects $\mathrm{Sq}(\sigma)$. □

Proposition 7.4.2 implies that the classes $\tau^{2^{a-1}(4n+1)} h_a$ must support Adams differentials for $a \geq 4$. Although we do not compute all these differentials, we do give the following.

Proposition 7.4.8 *For all* $n \geq 0$ *and* $a \geq 3$, *there is a differential*

$$d_2(\tau^{2^a(4n+1)} h_{a+1}) = (h_0 + \rho h_1)(\tau^{2^{a-1}(4n+1)} h_a)^2.$$

Proof We give separate arguments for the case $a=3$ and $a>3$. First consider the case $a=3$. The class $\tau^{4(4n+1)}h_3$ is a permanent cycle by Corollary 7.4.6, detecting a class which we might call $\tau^{4(4n+1)}\sigma$. By Lemma 7.2.1, $2\cdot(\tau^{4(4n+1)}\sigma)^2=0$, and so $(h_0+\rho h_1)\cdot(\tau^{4(4n+1)}h_3)^2$ must die. This class is not divisible by ρ, and the only non-ρ-divisible classes that may hit it are $\tau^8 h_4$ and $\tau^8 h_4+\rho^{16}h_5$. By Theorem 7.3.1, if $d_2(\tau^8 h_4+\rho^{16}h_5)=(h_0+\rho h_1)\cdot(\tau^{4(4n+1)}h_3)^2$, then $d_2(\tau^8 h_4)=(h_0+\rho h_1)\cdot(\tau^{4(4n+1)}h_3+h_4)^2$. This is not possible as $\tau^8 h_4$ is ρ-torsion and this target is not. Thus, in fact, $d_2(\tau^8 h_4)=(h_0+\rho h_1)\cdot(\tau^{4(4n+1)}h_3)^2$, as claimed.

Next consider the case $a>3$. The ρ-torsion subgroup of $\mathrm{Ext}_{\mathbb{R}}$ in the degree of $d_2(\tau^{2^a(4n+1)}h_{a+1})$ is given by $\mathbb{F}_2\{h_0,\rho h_1\}\otimes\mathbb{F}_2\{(\tau^{2^{a-1}(4n+1)}h_a)^2\}$. These classes are not divisible by ρ^2, and so it suffices to verify the differential in the Adams spectral sequence for the cofiber of ρ^2. By [Behrens and Shah 2020, Lemma 7.8], this cofiber is a ring spectrum, so its Adams spectral sequence is multiplicative. As τ^2 is a cycle, τ^4 is a d_2-cycle, so we reduce to showing $d_2(h_{a+1})=(h_0+\rho h_1)h_a^2$. This was shown in Theorem 7.3.1. □

We may summarize what we have learned as follows.

Theorem 7.4.9 *The nontrivial d_2-differentials out of the 1-line of the $\mathbb{R}$-motivic Adams spectral sequence are exactly those given in the following table*:

source	*target*	*constraints*
h_4	$h_0h_3^2$	
$\rho^r h_a$	$\rho^r(h_0+\rho h_1)h_{a-1}^2$	$a\geq 5, r\geq 0$
$\rho^r\tau^{2^{a-1}(4n+1)}h_a$	$\rho^r(h_0+\rho h_1)(\tau^{2^{a-2}(4n+1)}h_{a-1})^2$	$n\geq 0, a\geq 4, 0\leq r\leq 2^{a-1}-1$

The 1-line of the E_3-page of the $\mathbb{R}$-motivic Adams spectral sequence has a basis given by the elements in the following table:

$\mathbb{F}_2[\rho]$-module generator	*constraints*	*ρ-torsion exponent*
h_a	$a\in\{1,2,3\}$	∞
ρh_4		∞
$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor}h_a$	$n\geq 0$ *and* $a\in\{0,1,2,3\}$	2^a
$\rho^{2^{a-1}-1}\tau^{2^{a-1}(4n+1)}h_a$	$n\geq 0$ *and* $a\geq 4$	$2^{a-1}+1$

Those classes in $\mathrm{Ext}^1_{\mathbb{R}}$ which are permanent cycles are given in the following table:

$\mathbb{F}_2[\rho]$-module generator	*constraints*	*ρ-torsion exponent*	*stem*
h_a	$a\in\{1,2,3\}$	∞	2^a-1
ρh_4		∞	14
$\tau^{\lfloor 2^{a-1}(4n+1)\rfloor}h_a$	$n\geq 0$ *and* $a\in\{0,1,2,3\}$	2^a	2^a-1
$\rho^{2^a-\psi(a)}\tau^{2^{a-1}(4n+1)}h_a$	$n\geq 0, a\geq 4$	$\psi(a)$	$\psi(a)-1$

Proof All of this is immediate from Theorem 7.3.1, Propositions 7.4.2 and 7.4.8, Theorem 7.4.7, and the ρ-torsion exponents of the generators of $\mathrm{Ext}_{\mathbb{R}}^3$ given in Theorem 4.2.12, with the following exception: Proposition 7.4.8 produces differentials $d_2(\tau^{8(4n+1)}h_4) = (h_0+\rho h_1)(\tau^{4(4n+1)}h_3)^2$, and one must use Proposition 4.3.4(4) to check that this target has ρ-torsion exponent 7. □

7.5 Finite fields

We now study the case where F is a finite field. For the most part, this case follows by combining Theorem 7.3.1 with differentials out of Ext_F^0 that may be deduced from [Kylling 2015]. By naturality, our discussion in this subsection gives information for F an arbitrary field of odd characteristic.

We will need the following definition.

Definition 7.5.1 For an integer q, let $\nu_2(q)$ denote the 2-adic valuation of q, ie

$$q = 2^{\nu_2(q)}(2n+1)$$

for some integer n, and let

$$\varepsilon(q) = \nu_2(q-1), \quad \lambda(q) = \nu_2(q^2-1). \qquad \triangleleft$$

We now split into cases based on congruence of the order of the field mod 4.

7.5.1 $q \equiv 1 \pmod 4$

Fix a prime power q such that $q \equiv 1 \pmod 4$. We work over $F = \mathbb{F}_q$. Recall that $\mathrm{Ext}_{\mathbb{F}_q} = \mathrm{Ext}_{(1)}\{1, u\}$. In particular,

$$\mathrm{Ext}_{\mathbb{F}_q}^1 = \mathbb{F}_2[\tau]\{1,u\} \otimes \mathbb{F}_2\{h_a : a \geq 0\}.$$

The class u is a permanent cycle for degree reasons, and we have already computed the differential on all the classes h_a. However the story does not stop there; instead, we have the following.

Lemma 7.5.2 *There are differentials*

$$d_{\varepsilon(q)+s}(\tau^{2^s}) = u\tau^{2^s-1}h_0^{\varepsilon(q)+s}$$

for all $s \geq 0$.

Proof Kylling [2015, Lemma 4.2.1] produces identical differentials in the $\mathbb{F}_q$-motivic Adams spectral sequence for $H\mathbb{Z}$. The claimed differentials follow by naturality. □

This may be combined with Theorem 7.3.1 to easily compute all differentials out of the 1-line.

Theorem 7.5.3 *For* $q \equiv 1 \pmod 4$, *the* 1*-line of the* $\mathbb{F}_q$*-motivic Adams spectral sequence supports only the nontrivial differentials given in the following table*:

source	d_r	target	constraints
$\tau^n h_0$	$d_{\varepsilon(q)+\nu_2(n)}$	$\tau^{n-1} h_0^{\varepsilon(q)+\nu_2(n)+1}$	$n \geq 1$
$\tau^{2n+1} h_2$	d_2	$u\tau^{2n} h_2 h_0^2$	$n \geq 0$, $\varepsilon(q) = 2$
$\tau^{2n+1} h_3$	d_2	$u\tau^{2n} h_3 h_0^2$	$n \geq 0$, $\varepsilon(q) = 2$
$\tau^{2n+1} h_3$	d_3	$u\tau^{4n+1} h_3 h_0^3$	$n \geq 0$, $\varepsilon(q) = 3$
$\tau^{4n+2} h_3$	d_3	$u\tau^{4n+1} h_3 h_0^3$	$n \geq 0$, $\varepsilon(q) = 2$
$\tau^n h_b$	d_2	$\tau^n h_0 h_{b-1}^2 + d_2(\tau^n) h_b$	$n \geq 0$, $b \geq 4$
$u\tau^n h_b$	d_2	$u\tau^n h_0 h_{b-1}^2$	$n \geq 0$, $b \geq 4$

After these have been run, the 1*-line of the* E_∞*-page of the* $\mathbb{F}_q$*-motivic Adams spectral sequence has a basis given by the elements in the following table*:

class	constraints
h_0	
$\tau^n h_1$	$n \geq 0$
$\tau^n h_2$	$n \geq 0$, *where if* $\varepsilon(q) = 2$ *then* $n \equiv 0 \pmod 2$
$\tau^n h_3$	$n \geq 0$, *where if* $\varepsilon(q) = 2$ *then* $n \equiv 0 \pmod 4$, *and if* $\varepsilon(q) = 3$ *then* $n \equiv 0 \pmod 2$
$u\tau^n h_b$	$n \geq 0$, $b \in \{0, 1, 2, 3\}$

Proof The first four families of differentials follow immediately from Lemmas 7.5.2 and 7.2.2, and the remaining two by combining Lemma 7.5.2 with Theorem 7.3.1. Note in particular that $d_2(\tau^n) \equiv 0 \pmod u$, and thus $d_2(\tau^n h_b) \neq 0$ for $b \geq 4$. The second table may be easily read off the first, provided we verify that we have not missed any differentials, ie that the classes listed in the second table are indeed permanent cycles. For degree reasons, the only possible nontrivial differentials on the classes $\tau^n h_b$ with $b \in \{1, 2, 3\}$ would be of the form

(1) $d_r(\tau^n h_1) \stackrel{?}{=} \tau^{n-1} h_0^{r-1}$,

(2) $d_2(\tau^n h_2) \stackrel{?}{=} u\tau^{n-1} h_0^2 h_2$,

(3) $d_2(\tau^n h_3) \stackrel{?}{=} u\tau^{n-1} h_0^2 h_3$,

(4) $d_3(\tau^n h_3) \stackrel{?}{=} u\tau^{n-1} h_0^3 h_3$

with $n \geq 1$. The first is impossible for $n = 1$ as h_0 detects 2 and thus no power of h_0 may be killed, and is impossible for $n \geq 2$ as the class $\tau^{n-1} h_0^{r-1}$ must support the differential given the first row of the first table. The remaining three differentials may occur, and when they occur is accounted for in the given tables. □

7.5.2 $\boldsymbol{q \equiv 3 \pmod 4}$ Now fix a prime power q such that $q \equiv 3 \pmod 4$. We work over $F = \mathbb{F}_q$. Recall that $\mathrm{Ext}_{\mathbb{F}_q} = \mathrm{Ext}_{(2)}$.

Lemma 7.5.4 *We may identify*

$$\mathrm{Ext}^0_{\mathbb{F}_q} = \mathbb{F}_2[\tau^2, \rho, \tau\rho]/(\rho^2 = \rho \cdot (\tau\rho) = (\tau\rho)^2 = 0),$$

and $\mathrm{Ext}^1_{\mathbb{F}_q}$ *is the tensor product of* $\mathbb{F}_2[\tau^2]$ *with*

$$\mathbb{F}_2\{h_0, \rho\tau \cdot h_0\} \oplus \mathbb{F}_2\{h_1, \rho \cdot h_1, \rho\tau \cdot h_1, \tau h_1\} \oplus \mathbb{F}_2\{h_b, \rho \cdot h_b, \rho\tau \cdot h_b : b \geq 2\}.$$

Proof This follows quickly from our computation of $\mathrm{Ext}_{\mathbb{R}}$, following the recipe of Remark 4.1.5. Alternatively, one may compute the ρ-Bockstein spectral sequence

$$\mathrm{Ext}_{(1)}[\rho]/(\rho^2) \Rightarrow \mathrm{Ext}_{(2)}$$

directly (see [Wilson and Østvær 2017]); the only relevant differential is $d_1(\tau) = \rho h_0$. □

As in the previous case, powers of τ support arbitrarily long differentials.

Lemma 7.5.5 *There are differentials*

$$d_{\lambda(q)+s}(\tau^{2^{s+1}}) = \rho\tau^{2^{s+1}-1} h_0^{\lambda(q)+s}$$

for all $s \geq 0$. *On the other hand*, $\rho\tau$ *is a permanent cycle.*

Proof The class $\rho\tau$ is a permanent cycle for degree reasons. Kylling [2015, Lemma 4.2.2] produces identical differentials in the $\mathbb{F}_q$-motivic Adams spectral sequence for $\mathbb{F}_q$-motivic $H\mathbb{Z}$. The claimed differentials follow by naturality. □

Theorem 7.5.6 *For* $q \equiv 3 \pmod 4$, *the* 1*-line of the* $\mathbb{F}_q$*-motivic Adams spectral sequence supports the differentials given in the following table*:

source	d_r	target	constraints
$\tau^{2n}h_0$	$d_{\lambda(q)+\nu_2(n)}$	$\rho\tau^{2n-1}h_0^{\lambda(q)+\nu_2(n)+1}$	$n \geq 1$
$\tau^{4n+2}h_3$	d_3	$\rho\tau^{4n+1}h_0^3 h_3$	$n \geq 0$, $\lambda(q) = 3$
$\tau^{2n}h_b$	d_2	$\tau^{2n}(h_0 + \rho h_1)h_{b-1}^2$	$n \geq 1$, $b \geq 4$
$\rho\tau^{2n+1}h_b$	d_2	$\rho\tau^{2n+1}h_0 h_{b-1}^2$	$n \geq 0$, $b \geq 4$

After the d_2*-differentials have been run, the* 1*-line of the* E_3*-page of the* $\mathbb{F}_q$*-motivic Adams spectral sequence has a basis given by the classes in the following table*:

class	constraints
h_0	
$\rho^{\epsilon} \cdot \tau^{2n}h_b$	$n \geq 0$, $\epsilon \in \{0,1\}$, $b \in \{1,2,3\}$
$\rho\tau^{2n+1}h_b$	$n \geq 0$, $b \in \{0,1,2,3\}$
$\rho^{\epsilon}\tau^{4n+1}h_1$	$n \geq 0$, $\epsilon \in \{0,1\}$
$\tau^{2n}h_0$	$n \geq 1$
$\rho\tau^{2n}h_b$	$n \geq 0$, $b \geq 4$

Of these, all the classes in the first region are permanent cycles, with the exception that $\tau^{4n+2}h_3$ supports a d_3-differential if $\lambda(q) = 3$. The classes $\tau^{2n}h_0$ for $n \geq 1$ are not permanent cycles, and we leave open the fate of the classes $\rho\tau^{2n}h_b$ for $n \geq 1$ and $b \geq 4$.

Proof The given differentials follow quickly by combining Theorem 7.3.1 with Lemma 7.5.5, and this accounts for all d_2-differentials. Note in particular that τ^2 is a d_2-cycle as $\lambda(q) \geq 3$ whenever $q \equiv 3 \pmod 4$. Thus the given E_3-page may be produced by linearly propagating the differentials of Theorem 7.3.1. Note also that $d_2(\rho\tau^{2n}h_b) = \rho\tau^{2n}(h_0 + \rho h_1)h_{b-1}^2 = 0$ for all $n \geq 0$ and $b \geq 4$, yielding the classes in the final row of the second table.

It remains only to verify that the permanent cycles provided are indeed permanent cycles. As ρ and $\rho\tau$ are permanent cycles for degree reasons, we may reduce to considering only the classes $\tau^{2n}h_b$, $\rho\tau^{2n+1}h_0$, and $\tau^{4n+1}h_1$ for $b \in \{1,2,3\}$ and $n \geq 0$. For degree reasons, the only possible nontrivial differentials supported by these classes would be of the form

(1) $d_2(\tau^{2n}h_b) \overset{?}{=} \rho\tau^{2n-1}h_0^2h_b$ for $b \in \{2,3\}$,

(2) $d_3(\tau^{2n}h_3) \overset{?}{=} \rho\tau^{2n-1}h_0^3h_3$

with $n \geq 1$. The first does not hold, as τ^2 and h_b are d_2-cycles. The second holds only when $\lambda(q) = 3$, and this is accounted for in the theorem statement. □

7.6 The p-adic rationals

We now work over $F = \mathbb{Q}_p$, the p-adic rationals. This is very similar to the case where $F = \mathbb{F}_q$, only where the additional input necessary to understand differentials out of $\mathrm{Ext}^0_{\mathbb{Q}_p}$ comes from work of Ormsby [2011] for p odd and Ormsby and Østvær [2013] for $p = 2$. The case where p is odd turns out to entirely reduce to what we have already done.

Lemma 7.6.1 *There are the following differentials in the $\mathbb{Q}_p$-motivic Adams spectral sequence*:

(1) *If $p \equiv 1 \pmod 4$, then $d_{a(q)+s}(\tau^{2^s}) = u\tau^{2^s-1}h_0^{a(q)+s}$;*

(2) *If $p \equiv 3 \pmod 4$, then $d_{\lambda(q)+s}(\tau^{2^{s+1}}) = \rho\tau^{2^{s+1}-1}h_0^{\lambda(q)+s}$.*

Proof Ormsby [2011, Theorem 5.2] produces identical differentials in the $\mathbb{Q}_p$-motivic Adams spectral sequence for the Brown–Peterson spectrum $\mathrm{BP}\langle 0\rangle$. The claimed differentials follow by naturality. □

We may summarize the situation as follows.

Theorem 7.6.2 *Fix an odd prime p, and consider the facts outlined about the $\mathbb{F}_p$-motivic Adams spectral sequence in Theorems 7.5.3 and 7.5.6. The same facts hold for the $\mathbb{Q}_p$-motivic Adams spectral sequence upon tensoring with $\mathbb{F}_p\{1,\pi\}$.*

Proof The class π is a permanent cycle for degree reasons, and the differentials given in Lemma 7.6.1 agree with those given in Lemmas 7.5.2 and 7.5.5. All of the work carried out over $\mathbb{F}_p$ then goes through verbatim, only where everything in sight has a twin copy indexed by π. □

Remark 7.6.3 The somewhat awkward phrasing of Theorem 7.6.2 is necessary as we did not wish to repeat two verbatim copies of both Theorems 7.5.3 and 7.5.6, but we have not shown that the 1-line of the $\mathbb{Q}_p$-motivic Adams spectral sequence is a direct sum of two copies of the 1-line of the $\mathbb{F}_p$-motivic Adams spectral sequence. The possible failure of this arises from the fact that when $p \equiv 3 \pmod 4$, the classes $\rho\tau^{2n}h_b$ for $b \geq 4$ could support different higher differentials over $\mathbb{F}_p$ and $\mathbb{Q}_p$. ◁

The case where $p = 2$ requires a separate analysis. Recall that

$$\mathrm{Ext}_{\mathbb{Q}_2} = \mathrm{Ext}_{(3)}\{1\} \oplus \mathrm{Ext}_{(1)}\{u, \pi\}.$$

Lemma 7.6.4 *We may identify*

$$\mathrm{Ext}^0_{(3)} = \mathbb{F}_2(\tau^4, \rho\tau^2, \rho^2\tau, \rho^2\tau^3, \rho) \subset \mathbb{F}_2[\tau, \rho]/(\rho^3),$$

and $\mathrm{Ext}^1_{(3)}$ *is the tensor product of* $\mathbb{F}_2[\tau^4]$ *with the direct sum of the modules*

$$\mathbb{F}_2\{h_0, \tau^2h_0, \rho^2\tau h_0, \rho^2\tau^3h_0\},$$
$$\mathbb{F}_2\{1, \rho\} \otimes \mathbb{F}_2\{\tau h_1\} \oplus \mathbb{F}_2\{\rho\tau^3h_1\} \oplus \mathbb{F}_2\{1, \rho, \rho^2, \rho\tau^2, \rho^2\tau^2, \rho^2\tau^3\} \otimes \mathbb{F}_2\{h_1\},$$
$$\mathbb{F}_2\{1, \rho, \rho^2, \rho^2\tau, \rho^2\tau^3, \rho\tau^2, \rho^2\tau^2\} \otimes \mathbb{F}_2\{h_b : b \geq 2\}.$$

Proof As with Lemma 7.5.4, this follows from our computation of $\mathrm{Ext}_{\mathbb{R}}$ via the recipe in Remark 4.1.5, or via the ρ-Bockstein spectral sequence; here the relevant ρ-Bockstein differentials are $d_1(\tau) = \rho h_0$ and $d_2(\tau^2) = \rho^2\tau h_1$. □

Lemma 7.6.5 *The classes*

$$\tau^{4n+1}\rho^2, \quad \tau^{2n}\rho, \quad \tau^{4n+3}\rho^2, \quad \pi\tau^n, \quad u, \quad u\tau^{2n+1}$$

are permanent cycles. There are differentials

$$d_{4+r}(\tau^{2^{r+2}}) = \pi\tau^{2^{r+2}-1}h_0^{4+r}, \quad d_{3+r}(u\tau^{2^{r+1}}) = \rho^2\tau^{2^{r+1}-1}h_0^{3+r}, \quad d_{3+r}(\tau^{2^{r+1}}h_0) = \pi\tau^{2^{r+1}-1}h_0^{4+r}$$

for all $r \geq 0$.

Proof Ormsby and Østvær [2013, Lemma 5.7] compute differentials in the $\mathbb{Q}_2$-motivic Adams spectral sequence for $\mathrm{BP}\langle 0\rangle$. The claimed facts follow by comparison. □

Theorem 7.6.6 *The* 1*-line of the* $\mathbb{Q}_2$*-motivic Adams spectral sequence supports the following nontrivial differentials*:

source	d_r	target	constraints
$\tau^{2n}h_0$	$d_{3+\nu_2(n)}$	$\pi\tau^{2n-1}h_0^{4+\nu_2(n)}$	$n \geq 1$
$\tau^{4n}h_b$	d_2	$\tau^{4n}(h_0 + \rho h_1)h_{b-1}^2$	$n \geq 0, b \geq 4$
$\rho\tau^{2n}h_b$	d_2	$\rho^2\tau^{2n}h_1h_{b-1}^2$	$n \geq 0, b \geq 5$
$u\tau^n h_b$	d_2	$u\tau^n h_0h_{b-1}^2$	$n \geq 0, b \geq 4$
$\pi\tau^n h_b$	d_2	$\pi\tau^n h_0h_{b-1}^2$	$n \geq 0, b \geq 4$
$u\tau^{4n+2}h_3$	d_3	$\rho^2\tau^{4n+1}h_0^3h_3$	$n \geq 0$

After all the d_2-differentials have been run, the 1-line of the E_3-page of the $\mathbb{Q}_2$-motivic Adams spectral sequence has a basis given by the classes in the following table:

class	constraints
h_0	
$\rho^\delta \tau^{4n} h_b$	$n \geq 0, \delta \in \{0,1,2\}, b \in \{1,2,3\}$
$\rho^2 \tau^{2n+1} h_0$	$n \geq 0$
$\rho^\epsilon \tau^{4n+1} h_1$	$n \geq 0, \epsilon \in \{0,1\}$
$\rho^{1+\epsilon} \tau^{4n+3} h_1$	$n \geq 0, \epsilon \in \{0,1\}$
$\rho^{1+\epsilon} \tau^{4n+2} h_1$	$n \geq 0, \epsilon \in \{0,1\}$
$u h_0$	
$u \tau^{2n+1} h_0$	$n \geq 0$
$u \tau^n h_b$	$n \geq 0, b \in \{1,2\}$
$u \tau^{2n+1} h_3$	$n \geq 0$
$u \tau^{4n} h_3$	$n \geq 0$
$\pi \tau^n h_b$	$n \geq 0, b \in \{0,1,2,3\}$
$u^\epsilon \tau^{2n} h_0$	$n \geq 1, \epsilon \in \{0,1\}$
$u \tau^{4n+2} h_3$	$n \geq 0$
$\rho^{1+\epsilon} \tau^{4n} h_4$	$n \geq 0, \epsilon \in \{0,1\}$
$\rho^2 \tau^{4n} h_b$	$n \geq 0, b \geq 5$

Of these, the classes in the first region are permanent cycles, the classes $u^\epsilon \tau^{2n} h_0$ with $n \geq 1$ and $\epsilon \in \{0,1\}$, as well as $u\tau^{4n+2} h_3$ with $n \geq 0$, support higher differentials, and we leave open the fate of the classes $\rho^{1+\epsilon}\tau^{4n} h_4$ and $\rho^2\tau^{4n} h_b$ for $n \geq 0$, $\epsilon \in \{0,1\}$, and $b \geq 5$.

Proof The given differentials follow by combining Theorem 7.3.1 with Lemma 7.6.5. For example,

$$d_2(\rho\tau^{2n} h_b) = \rho\tau^{2n} \cdot d_2(h_b) = \rho\tau^{2n} \cdot (h_0 + \rho h_1) h_{b-1}^2 = \rho^2 \tau^{2n} h_1 h_{b-1}^2$$

for $b \geq 4$, which is nonzero precisely when $b \geq 5$; as another example,

$$d_3(u\tau^{4n+2} h_3) = d_3(u\tau^2) \cdot \tau^{4n} h_3 = \rho^2 \tau \cdot \tau^{4n} h_3 = \rho^2 \tau^{4n+1} h_3.$$

We must verify that all d_2-differentials are accounted for in this table; the claimed description of the E_3-page follows quickly. We must also verify that the classes we give as permanent cycles are indeed permanent cycles. It suffices to verify the latter.

We may cut down the number of classes to consider by taking into account the classes which are products of the permanent cycles given in Lemma 7.6.5 with some other class. After this reduction, degree considerations rule out all differentials except for possibly

(1) $d_r(\tau^{4n+1} h_1) \stackrel{?}{=} \tau^{4n} h_0^{r+1}$,

(2) $d_r(\rho\tau^{4n+3} h_1) \in \mathbb{F}_2\{u, \pi\} \otimes \mathbb{F}_2\{\tau^{4n+2} h_0^{r+1}\}$,

(3) $d_r(\rho\tau^{4n+2} h_1) \stackrel{?}{=} \rho^2 \tau^{4n+1} h_0^{r+1}$,

(4) $d_r(u\tau^{2n}h_1) \overset{?}{=} \rho^2\tau^{2n-1}h_0^{r+1}$

with $n \geq 0$, and in the fourth case $n \geq 1$. In all cases, the possible nonzero targets are present and not boundaries in Ormsby and Østvær's computation [2013] of the Adams spectral sequence for the $\mathbb{Q}_2$-motivic $\mathrm{BP}\langle 0\rangle$, so by naturality they cannot be boundaries in the Adams spectral sequence for the sphere. Thus these possible nonzero differentials are in fact not possible, yielding the theorem. □

7.7 The rational numbers

We end by considering the case $F = \mathbb{Q}$. By naturality, this gives information over arbitrary fields of characteristic zero. Recall the functions ε and λ defined in Definition 7.5.1.

Theorem 7.7.1 *The* 1*-line of the* E_3*-page of the* $\mathbb{Q}$*-motivic Adams spectral sequence is given by a direct sum of that for the* $\mathbb{R}$*-motivic Adams spectral sequence with the classes in the following table*, *where* p *ranges through all primes*:

class	constraints
$\tau^n h_b[2]$	$n \geq 0, b \in \{0,1,2,3\}$
$h_0[p]$	$p \equiv 1 \pmod 4$
$\tau^n h_1[p]$	$p \equiv 1 \pmod 4, n \geq 0$
$\tau^{2n} h_2[p]$	$p \equiv 1 \pmod 4, n \geq 0$
$\tau^{2n+1} h_2[p]$	$p \equiv 1 \pmod 4, n \geq 0, \varepsilon(p) \geq 3$
$\tau^{4n} h_3[p]$	$p \equiv 1 \pmod 4, n \geq 0$
$\tau^{4n+2} h_3[p]$	$p \equiv 1 \pmod 4, n \geq 0, \varepsilon(p) \geq 3$
$\tau^{2n+1} h_3[p]$	$p \equiv 1 \pmod 4, n \geq 0, \varepsilon(p) \geq 4$
$\tau^n h_b a_p$	$p \equiv 1 \pmod 4, n \geq 0, b \in \{0,1,2,3\}$
$h_0 u_p$	$p \equiv 3 \pmod 4$
$\tau^{2n} h_b u_p$	$p \equiv 3 \pmod 4, n \geq 0, b \in \{1,2\}$
$\tau^{4n} h_3 u_p$	$p \equiv 3 \pmod 4, n \geq 0$
$\tau^{4n+2} h_3 u_p$	$p \equiv 3 \pmod 4, n \geq 0, \lambda(p) \geq 4$
$\rho\tau^{2n} h_b u_p$	$p \equiv 3 \pmod 4, n \geq 0, b \in \{1,2,3\}$
$\rho\tau^{2n+1} h_b u_p$	$p \equiv 3 \pmod 4, n \geq 0, b \in \{1,2,3,4\}$
$\rho^\epsilon\tau^{4n+1} h_1 u_p$	$p \equiv 3 \pmod 4, n \geq 0, \epsilon \in \{0,1\}$
$\tau^{2n} h_0[p]$	$p \equiv 1 \pmod 4, n \geq 1$
$\tau^{2n+1} h_0[p]$	$p \equiv 1 \pmod 4, n \geq 1, \varepsilon(p) \geq 3$
$\tau^{4n+2} h_3[p]$	$p \equiv 1 \pmod 4, n \geq 0, \varepsilon(p) = 2$
$\tau^{2n+1} h_3[p]$	$p \equiv 1 \pmod 4, n \geq 0, \varepsilon(p) = 3$
$\tau^{4n+2} h_3 u_p$	$p \equiv 3 \pmod 4, n \geq 0, \lambda(p) = 3$
$\tau^{2n} h_0 u_p$	$p \equiv 3 \pmod 4, n \geq 1$
$\rho\tau^{2n} h_b u_p$	$p \equiv 3 \pmod 4, n \geq 0, b \geq 4$

Moreover, we have the following information about higher differentials. The classes in the first region of this table are permanent cycles, as are the classes h_a and $\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a$ for $a \leq 3$. The classes in the second region of this table support higher differentials, as do the classes in $\mathrm{Ext}^1_{\mathbb{R}}$, which must support higher differentials by Theorem 7.4.9. We leave open the fate of the classes in the third region of this table, as well as the possibility of exotic higher differentials on the classes ρh_4 and $\rho^{2^a-\psi(a)}\tau^{2^{a-1}(4n+1)} h_a$ for $a \geq 4$.

Proof Recall the splitting

$$\mathrm{Ext}_{\mathbb{Q}} = \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{(1)}\{[2]\} \oplus \mathrm{Ext}_{(1)}\{[p], a_p : p \equiv 1 \pmod 4\} \oplus \mathrm{Ext}_{(2)}\{u_p : p \equiv 3 \pmod 4\}$$

implied by Lemma 7.1.1. As in the proof of Proposition 7.1.3, each of these summands is itself either $\mathrm{Ext}_{\mathbb{R}}$ or an identifiable summand of some corresponding $\mathrm{Ext}_{\mathbb{Q}_p}$; for p odd, this summand looks like $\mathrm{Ext}_{\mathbb{F}_p}$. We may thus read the given table off the information given in Theorems 7.4.9, 7.6.2 (with Theorems 7.5.3 and 7.5.6), and 7.6.6, provided we verify the following claim: if $\alpha[p] \in \mathrm{Ext}^1_{\mathbb{Q}}$ is a class in stem $s \leq 6$, then $\alpha[p]$ or αu_p is a d_r-cycle if and only if it projects to a d_r-cycle in the $\mathbb{Q}_p$-motivic Adams spectral sequence; and, likewise, if $\alpha \in \mathrm{Ext}^1_{\mathbb{R}}$ is a class in stem $s \leq 7$, then α is a d_r-cycle in the $\mathbb{Q}$-motivic Adams spectral sequence if and only if it projects to a d_r-cycle in the $\mathbb{R}$-motivic Adams spectral sequence.

As in the proofs of Theorems 7.5.3, 7.5.6, and 7.6.6, differentials on the classes $\alpha[p]$ and αu_p in stems $s \leq 6$ are completely determined by the structure of differentials on the classes $[p]\tau^{2^i}$ and $u_p\tau^{2^i}$ in the $\mathbb{Q}$-motivic Adams spectral sequence for $\mathrm{BP}\langle 0\rangle$, together with the fact that h_0, h_1, h_2, and h_3 are permanent cycles. The $\mathbb{Q}$-motivic Adams spectral sequence for $\mathrm{BP}\langle 0\rangle$ was computed in [Ormsby and Østvær 2013, Theorem 5.8]. We find that differentials on the classes $[p]\tau^{2^i}$ and $u_p\tau^{2^i}$ in the $\mathbb{Q}$-motivic Adams spectral sequence for $\mathrm{BP}\langle 0\rangle$ are entirely detected over $\mathbb{Q}_p$, and our first claim follows. That the classes $h_a \in \mathrm{Ext}^1_{\mathbb{R}}$ for $a \leq 3$ are permanent cycles was seen in Lemma 7.2.2, and the classes $\tau^{\lfloor 2^{a-1}(4n+1)\rfloor} h_a \in \mathrm{Ext}^1_{\mathbb{R}}$ must be permanent cycles for $a \leq 3$ as there is no room for exotic higher differentials. □

References

[Adams 1958] **J F Adams**, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. 32 (1958) 180–214 MR Zbl

[Adams 1960] **J F Adams**, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. 72 (1960) 20–104 MR Zbl

[Adams 1962] **J F Adams**, *Vector fields on spheres*, Ann. of Math. 75 (1962) 603–632 MR Zbl

[Andrews and Miller 2017] **M Andrews**, **H Miller**, *Inverting the Hopf map*, J. Topol. 10 (2017) 1145–1168 MR Zbl

[Asok et al. 2017] **A Asok**, **B Doran**, **J Fasel**, *Smooth models of motivic spheres and the clutching construction*, Int. Math. Res. Not. 2017 (2017) 1890–1925 MR Zbl

[Bachmann 2022] **T Bachmann**, *η-periodic motivic stable homotopy theory over Dedekind domains*, J. Topol. 15 (2022) 950–971 MR Zbl

[Bachmann and Hopkins 2020] **T Bachmann**, **M J Hopkins**, *η-periodic motivic stable homotopy theory over fields*, preprint (2020) arXiv 2005.06778

[Bachmann et al. 2022] **T Bachmann**, **H J Kong**, **G Wang**, **Z Xu**, *The Chow t-structure on the ∞-category of motivic spectra*, Ann. of Math. 195 (2022) 707–773 MR Zbl

[Balderrama 2023] **W Balderrama**, *Algebraic theories of power operations*, J. Topol. 16 (2023) 1543–1640 Zbl

[Behrens and Shah 2020] **M Behrens**, **J Shah**, *C_2-equivariant stable homotopy from real motivic stable homotopy*, Ann. K-Theory 5 (2020) 411–464 MR Zbl

[Belmont and Isaksen 2022] **E Belmont**, **D C Isaksen**, *$\mathbb{R}$-motivic stable stems*, J. Topol. 15 (2022) 1755–1793 MR Zbl

[Belmont et al. 2021] **E Belmont**, **B J Guillou**, **D C Isaksen**, *C_2-equivariant and $\mathbb{R}$-motivic stable stems, II*, Proc. Amer. Math. Soc. 149 (2021) 53–61 MR Zbl

[Bousfield et al. 1966] **A K Bousfield**, **E B Curtis**, **D M Kan**, **D G Quillen**, **D L Rector**, **J W Schlesinger**, *The mod $-$ p lower central series and the Adams spectral sequence*, Topology 5 (1966) 331–342 MR Zbl

[Brown and Gitler 1973] **E H Brown, Jr**, **S Gitler**, *A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra*, Topology 12 (1973) 283–295 MR Zbl

[Bruner 1986a] **R R Bruner**, *The homotopy theory of H_∞ ring spectra*, from "H_∞ ring spectra and their applications" (R R Bruner, J P May, J E McClure, M Steinberger, editors), Lecture Notes in Math. 1176, Springer (1986) Chapter IV, 88–128 MR Zbl

[Bruner 1986b] **R R Bruner**, *The Adams spectral sequence of H_∞ ring spectra*, from "H_∞ ring spectra and their applications" (R R Bruner, J P May, J E McClure, M Steinberger, editors), Lecture Notes in Math. 1176, Springer (1986) Chapter VI, 169–214 MR Zbl

[Bruner 1988] **R Bruner**, *An example in the cohomology of augmented algebras*, J. Pure Appl. Algebra 55 (1988) 81–84 MR Zbl

[Burklund et al. 2020] **R Burklund**, **J Hahn**, **A Senger**, *Galois reconstruction of Artin–Tate $\mathbb{R}$-motivic spectra*, preprint (2020) arXiv 2010.10325

[Chen 2011] **T-W Chen**, *Determination of $\mathrm{Ext}^{5,*}_{\mathscr{A}}(\mathbb{Z}/2,\mathbb{Z}/2)$*, Topology Appl. 158 (2011) 660–689 MR Zbl

[Cohen 1985] **R L Cohen**, *The immersion conjecture for differentiable manifolds*, Ann. of Math. 122 (1985) 237–328 MR Zbl

[Culver and Quigley 2021] **D L Culver**, **J D Quigley**, *kq-resolutions, I*, Trans. Amer. Math. Soc. 374 (2021) 4655–4710 MR Zbl

[Curtis 1971] **E B Curtis**, *Simplicial homotopy theory*, Adv. Math. 6 (1971) 107–209 MR Zbl

[Curtis et al. 1987] **E B Curtis**, **P Goerss**, **M Mahowald**, **R J Milgram**, *Calculations of unstable Adams E_2 terms for spheres*, from "Algebraic topology" (H R Miller, D C Ravenel, editors), Lecture Notes in Math. 1286, Springer (1987) 208–266 MR Zbl

[Dugger and Isaksen 2010] **D Dugger**, **D C Isaksen**, *The motivic Adams spectral sequence*, Geom. Topol. 14 (2010) 967–1014 MR Zbl

[Dugger and Isaksen 2013] **D Dugger**, **D C Isaksen**, *Motivic Hopf elements and relations*, New York J. Math. 19 (2013) 823–871 MR Zbl

[Dugger and Isaksen 2017a] **D Dugger**, **D C Isaksen**, *Low-dimensional Milnor–Witt stems over $\mathbb{R}$*, Ann. K-Theory 2 (2017) 175–210 MR Zbl

[Dugger and Isaksen 2017b] **D Dugger**, **D C Isaksen**, *$\mathbb{Z}/2$-equivariant and $\mathbb{R}$-motivic stable stems*, Proc. Amer. Math. Soc. 145 (2017) 3617–3627 MR Zbl

[Gheorghe et al. 2021] **B Gheorghe**, **G Wang**, **Z Xu**, *The special fiber of the motivic deformation of the stable homotopy category is algebraic*, Acta Math. 226 (2021) 319–407 MR Zbl

[Goerss 1999] **P G Goerss**, *Hopf rings, Dieudonné modules, and $E_*\Omega^2 S^3$*, from "Homotopy invariant algebraic structures" (J-P Meyer, J Morava, W S Wilson, editors), Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 115–174 MR Zbl

[Goerss et al. 1986] **P G Goerss**, **J D S Jones**, **M E Mahowald**, *Some generalized Brown–Gitler spectra*, Trans. Amer. Math. Soc. 294 (1986) 113–132 MR Zbl

[Greenlees 1988] **J P C Greenlees**, *Stable maps into free G-spaces*, Trans. Amer. Math. Soc. 310 (1988) 199–215 MR Zbl

[Guillou and Isaksen 2015] **B J Guillou**, **D C Isaksen**, *The η-local motivic sphere*, J. Pure Appl. Algebra 219 (2015) 4728–4756 MR Zbl

[Guillou and Isaksen 2016] **B J Guillou**, **D C Isaksen**, *The η-inverted $\mathbb{R}$-motivic sphere*, Algebr. Geom. Topol. 16 (2016) 3005–3027 MR Zbl

[Guillou and Isaksen 2020] **B J Guillou**, **D C Isaksen**, *The Bredon–Landweber region in C_2-equivariant stable homotopy groups*, Doc. Math. 25 (2020) 1865–1880 MR Zbl

[Guillou et al. 2020] **B J Guillou**, **M A Hill**, **D C Isaksen**, **D C Ravenel**, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}*, Tunis. J. Math. 2 (2020) 567–632 MR Zbl

[Hill 2011] **M A Hill**, *Ext and the motivic Steenrod algebra over $\mathbb{R}$*, J. Pure Appl. Algebra 215 (2011) 715–727 MR Zbl

[Hoyois et al. 2017] **M Hoyois**, **S Kelly**, **P A Østvær**, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. 19 (2017) 3813–3849 MR Zbl

[Hu and Kriz 2001] **P Hu**, **I Kriz**, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence*, Topology 40 (2001) 317–399 MR Zbl

[Hu et al. 2011a] **P Hu**, **I Kriz**, **K Ormsby**, *Convergence of the motivic Adams spectral sequence*, J. K-Theory 7 (2011) 573–596 MR Zbl

[Hu et al. 2011b] **P Hu**, **I Kriz**, **K Ormsby**, *Remarks on motivic homotopy theory over algebraically closed fields*, J. K-Theory 7 (2011) 55–89 MR Zbl

[Hunter and Kuhn 1999] **D J Hunter**, **N J Kuhn**, *Mahowaldean families of elements in stable homotopy groups revisited*, Math. Proc. Cambridge Philos. Soc. 127 (1999) 237–251 MR Zbl

[Isaksen 2019] **D C Isaksen**, *Stable stems*, Mem. Amer. Math. Soc. 1269, Amer. Math. Soc., Providence, RI (2019) MR Zbl

[Isaksen and Østvær 2020] **D C Isaksen**, **P A Østvær**, *Motivic stable homotopy groups*, from "Handbook of homotopy theory" (H Miller, editor), CRC Press, Boca Raton, FL (2020) 757–791 MR Zbl

[Isaksen et al. 2023] **D C Isaksen**, **G Wang**, **Z Xu**, *Stable homotopy groups of spheres: from dimension 0 to 90*, Publ. Math. Inst. Hautes Études Sci. 137 (2023) 107–243 MR Zbl

[James 1957] **I M James**, *On the suspension sequence*, Ann. of Math. 65 (1957) 74–107 MR Zbl

[James 1958] **I M James**, *Cross-sections of Stiefel manifolds*, Proc. Lond. Math. Soc. 8 (1958) 536–547 MR Zbl

[James 1959] **I M James**, *Spaces associated with Stiefel manifolds*, Proc. Lond. Math. Soc. 9 (1959) 115–140 MR Zbl

[Kylling 2015] **J I Kylling**, *Hermitian K-theory of finite fields via the motivic Adams spectral sequence*, master's thesis, University of Oslo (2015) Available at https://www.duo.uio.no/bitstream/handle/10852/45345/JonasIrgensKylling.pdf?sequence=11&isAllowed=y

[Kylling and Wilson 2019] **J I Kylling**, **G M Wilson**, *Strong convergence in the motivic Adams spectral sequence*, preprint (2019) arXiv 1901.03399

[Lin 1980] **W H Lin**, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc. 87 (1980) 449–458 MR Zbl

[Lin 2008] **W-H Lin**, $\mathrm{Ext}_A^{4,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ *and* $\mathrm{Ext}_A^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topology Appl. 155 (2008) 459–496 MR Zbl

[Lurie 2011] **J Lurie**, *Derived algebraic geometry*, *XIII*: *Rational and p-adic homotopy theory*, preprint (2011) Available at https://www.math.ias.edu/~lurie/papers/DAG-XIII.pdf

[Mahowald 1977] **M Mahowald**, *A new infinite family in ${}_2\pi_*{}^s$*, Topology 16 (1977) 249–256 MR Zbl

[Mahowald 1981] **M Mahowald**, *bo-resolutions*, Pacific J. Math. 92 (1981) 365–383 MR Zbl

[Mandell 2001] **M A Mandell**, *E_∞ algebras and p-adic homotopy theory*, Topology 40 (2001) 43–94 MR Zbl

[Massey and Peterson 1965] **W S Massey**, **F P Peterson**, *The cohomology structure of certain fibre spaces, I*, Topology 4 (1965) 47–65 MR Zbl

[Milnor 1970] **J Milnor**, *Algebraic K-theory and quadratic forms*, Invent. Math. 9 (1970) 318–344 MR Zbl

[Morel 1999] **F Morel**, *Suite spectrale d'Adams et invariants cohomologiques des formes quadratiques*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999) 963–968 MR Zbl

[Morel 2004] **F Morel**, *An introduction to $\mathbb{A}^1$-homotopy theory*, from "Contemporary developments in algebraic K-theory" (M Karoubi, A O Kuku, C Pedrini, editors), ICTP Lect. Notes 15, Abdus Salam Int. Cent. Theoret. Phys., Trieste, Italy (2004) 357–441 MR Zbl

[Morel and Voevodsky 1999] **F Morel**, **V Voevodsky**, *$\mathbb{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR Zbl

[Ormsby 2011] **K M Ormsby**, *Motivic invariants of p-adic fields*, J. K-Theory 7 (2011) 597–618 MR Zbl

[Ormsby and Østvær 2013] **K M Ormsby**, **P A Østvær**, *Motivic Brown–Peterson invariants of the rationals*, Geom. Topol. 17 (2013) 1671–1706 MR Zbl

[Ormsby and Röndigs 2020] **K Ormsby**, **O Röndigs**, *The homotopy groups of the η-periodic motivic sphere spectrum*, Pacific J. Math. 306 (2020) 679–697 MR Zbl

[Palmieri 2007] **J H Palmieri**, *The lambda algebra and* Sq^0, from "Proceedings of the School and Conference in Algebraic Topology" (J Hubbuck, N H V Hung, L Schwartz, editors), Geom. Topol. Monogr. 11, Geom. Topol. Publ., Coventry (2007) 201–216 MR Zbl

[Polishchuk and Positselski 2005] **A Polishchuk**, **L Positselski**, *Quadratic algebras*, Univ. Lect. Ser. 37, Amer. Math. Soc., Providence, RI (2005) MR Zbl

[Priddy 1970] **S B Priddy**, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970) 39–60 MR Zbl

[Rezk 2012] **C Rezk**, *Rings of power operations for Morava E-theories are Koszul*, preprint (2012) arXiv 1204.4831

[Riou 2012] **J Riou**, *Opérations de Steenrod motiviques*, preprint (2012) arXiv 1207.3121

[Röndigs and Østvær 2016] **O Röndigs**, **P A Østvær**, *Slices of hermitian K-theory and Milnor's conjecture on quadratic forms*, Geom. Topol. 20 (2016) 1157–1212 MR Zbl

[Röndigs et al. 2019] **O Röndigs**, **M Spitzweck**, **P A Østvær**, *The first stable homotopy groups of motivic spheres*, Ann. of Math. 189 (2019) 1–74 MR Zbl

[Röndigs et al. 2021] **O Röndigs**, **M Spitzweck**, **P A Østvær**, *The second stable homotopy groups of motivic spheres* (2021) arXiv 2103.17116 To appear in Duke Math. J.

[Shimamoto 1984] **D H Shimamoto**, *An integral version of the Brown–Gitler spectrum*, Trans. Amer. Math. Soc. 283 (1984) 383–421 MR Zbl

[Steenrod 1962] **N E Steenrod**, *Cohomology operations*, Ann. of Math. Stud. 50, Princeton Univ. Press (1962) MR Zbl

[Tangora 1985] **M C Tangora**, *Computing the homology of the lambda algebra*, Mem. Amer. Math. Soc. 337, Amer. Math. Soc., Providence, RI (1985) MR Zbl

[Tangora 1993] **M C Tangora**, *Some remarks on products in* Ext, from "Algebraic topology" (M C Tangora, editor), Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 419–429 MR Zbl

[Tangora 1994] **M C Tangora**, *Some Massey products in* Ext, from "Topology and representation theory" (E M Friedlander, M E Mahowald, editors), Contemp. Math. 158, Amer. Math. Soc., Providence, RI (1994) 269–280 MR Zbl

[Tilson 2017] **S Tilson**, *Squaring operations in the $RO(C_2)$-graded and real motivic Adams spectral sequences*, preprint (2017) arXiv 1702.04632

[Voevodsky 2003] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. 98 (2003) 1–57 MR Zbl

[Voevodsky 2011] **V Voevodsky**, *On motivic cohomology with $\mathbb{Z}/l$-coefficients*, Ann. of Math. 174 (2011) 401–438 MR Zbl

[Wang 1967] **J S P Wang**, *On the cohomology of the* mod − 2 *Steenrod algebra and the non-existence of elements of Hopf invariant one*, Illinois J. Math. 11 (1967) 480–490 MR Zbl

[Wang and Xu 2016] **G Wang**, **Z Xu**, *The algebraic Atiyah–Hirzebruch spectral sequence of real projective spectra*, preprint (2016) arXiv 1601.02185

[Wang and Xu 2017] **G Wang**, **Z Xu**, *The triviality of the 61-stem in the stable homotopy groups of spheres*, Ann. of Math. 186 (2017) 501–580 MR Zbl

[Wilson 2016] **G M Wilson**, *Motivic stable stems over finite fields*, PhD thesis, Rutgers University (2016) Available at `https://www.proquest.com/docview/1844381634`

[Wilson 2018] **G M Wilson**, *The eta-inverted sphere over the rationals*, Algebr. Geom. Topol. 18 (2018) 1857–1881 MR Zbl

[Wilson and Østvær 2017] **G M Wilson**, **P A Østvær**, *Two-complete stable motivic stems over finite fields*, Algebr. Geom. Topol. 17 (2017) 1059–1104 MR Zbl

WB: *Department of Mathematics, University of Virginia*
Charlottesville, VA, United States
DLC: *Max-Planck-Institut für Mathematik*
Bonn, Germany
JDQ: *Department of Mathematics, Cornell University*
Ithaca, NY, United States
Current address: *Department of Mathematics, University of Virginia*
Charlottesville, VA, United States

williamb@math.uni-bonn.de, dominic.culver@gmail.com, mbp6pj@virginia.edu

Proposed: Marc Levine Received: 19 July 2022
Seconded: Stefan Schwede, Haynes R Miller Revised: 21 August 2023

Geometry & Topology

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2025 is US $865/year for the electronic version, and $1210/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Geometry & Topology

Volume 29 Issue 3 (pages 1115–1691) 2025