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TIGHT CONTACT STRUCTURES ON SOME FAMILIES OF SMALL SEIFERT FIBER SPACES

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Abstract. Suppose K is a knot in a 3-manifold Y, and that Y admits a pair of distinct contact structures. Assume that K has Legendrian representatives in each of these contact structures, such that the corresponding Thurston-Bennequin framings are equivalent. This paper provides a method to prove that the contact structures resulting from Legendrian surgery along these two representatives remain distinct. Applying this method to the situation where the starting manifold is $-\Sigma(2,3,6m+1)$ and the knot is a singular fiber, together with convex surface theory we can classify the tight contact structures on certain families of Seifert fiber spaces.

1. Introduction

Classifying (non-isotopic positive) tight contact structures on 3-manifolds (especially Seifert fiber spaces) has been studied for a while. Usually one can get an upper bound on the number of such structures from convex surface theory but it can be difficult to actually construct different ones. We first observe that if we start with two different contact structures on some manifold Y whose Ozsváth–Szabó contact invariants [5] [6] are different then we can produce non-isotopic contact structures on a new 3-manifold obtained by surgery on a knot in Y. Note that this is not a new theorem, as it follows easily from the literature for example the work by Ghiggini [1], but we have not seen it explicitly stated.

Theorem 1.1. *Let ξ_1 and ξ_2 be two contact structures on a 3-manifold Y. Take any smooth knot K in Y. Let K_i, $i = 1, 2$ be a Legendrian representative of K in ξ_i. We denote (Y_i^-, ξ_i^-) to be the contact 3-manifold obtained by taking the -1 contact surgery (Legendrian surgery) on (Y, ξ_i)*

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respectively. Moreover, we write $c(\xi_i)$, $c(\xi_i^-)$ to be the contact invariant of the correspond contact manifold. Then

$$c(\xi_1) \neq c(\xi_2) \quad \textit{implies} \quad c(\xi_1^-) \neq c(\xi_2^-) \tag{1.1}$$

A version of the above Theorem is used in the beautiful paper by Ghiggini and Van Horn-Morris [3], where they obtained all the contact structure on the "wrongly"-oriented Brieskorn homology sphere $-\Sigma(2,3,6m-1)$ by doing Legendrian surgery, starting from different contact structures on the 3-manifold Y given by 0-surgery on the right handed trefoil. The way they distinguish the contact structures is by using the contact invariant, however those different structures on Y all have vanishing contact invariant. Thus they need to use twisted coefficients, and by explicitly analysing the contact structures and open book decomposition, they show those vanishing ordinary contact invariants are all non-zero and distinct with twisted coefficients—and after doing Legendrian surgery on those different contact structures on Y, the resulting ones on $-\Sigma(2,3,6m-1)$ all have distinct untwisted contact invariant.

Theorem 1.1 can be thought of as showing that "the property of having distinct contact invariants is preserved under Legendrian surgery."

Using the main Theorem together with some 3-manifolds whose tight contact structures are classified, along with some convex surface theory, we are able to further classify tight contact structures on other 3-manifolds. For example, as starting manifolds we can consider the family of Brieskorn spheres $-\Sigma(2,3,6m+1)$, whose tight contact structures have been classified by Tosun.

THEOREM 1.2 (Tosun [8]). *For $m \geq 1$, Brieskorn sphere $-\Sigma(2,3,6m+1)$ has exactly $\frac{m(m+1)}{2}$ tight contact structures which are all strongly fillable and with pairwise distinct non-zero contact invariant.*

By applying Theorem 1.1 to the case where $Y = -\Sigma(2,3,6m+1)$, and K is a singular fiber, we are able to conclude the following classification theorem.

THEOREM 1.3. *For $1 \leq m$ and $1 \leq n < 18m+4$ the Seifert fibered 3-manifolds $M(-2; \frac{3m+n-2}{6m+2n-5}, \frac{2}{3}, \frac{5m+1}{6m+1})$ have exactly*

$$\sum_{a=1}^{m} (n+3(a-1))a = \frac{(2m+n-2)(m+1)m}{2}$$

different tight contact structures which are all strongly fillable.

Moreover, similar to Theorem 1.3 by using another singular fiber we can get another family of Seifert fiber space whose tight contact structures we can classify.

THEOREM 1.4. *For* $1 \le m$ *and* $1 \le n < 12m+3$ *the Seifert fibered 3-manifolds* $M\big(-2; \frac{1}{2}, \frac{4(m-1)+2n+1}{6(m-1)+3n+1}, \frac{5m+1}{6m+1}\big)$ *have exactly*

$$\sum_{a=1}^{m}(n+2(a-1))a = \frac{(4m-4+3n)(m+1)m}{6}$$

different tight contact structures which are all strongly fillable.

In these theorems, the notation $M(e_0; r_1, r_2, r_3)$ indicates a Seifert fibered 3-manifold with three exceptional fibers, and having the indicated Seifert invariants. Note that Tosun has classified tight contact structures on some other small Seifert fiber spaces with $e_0 = -2$ ([8, Theorem 1.1]), but the two families mentioned in above theorem are not included in those results.

We will first see the proof of Theorem 1.1 in the next section. In Section 3 we will see how to get those Seifert fiber space mentioned in Theorem 1.3 from $-\Sigma(2,3,6m+1)$, and how to use Theorem 1.1 together with another lemma due to Jonathan Simone [7] to obtain a lower bound on the number of different tight contact structures. Last in Section 4 we show that number is equal to an upper bound coming from convex surface theory.

2. Proof of Theorem 1.1

The proof is actually quite simple just by manipulating with the naturality and vanishing result of contact invariant under the Legendrian surgery.

PROOF OF THEOREM 1.1. The Theorem is clearly true if Y_1^- is not smoothly diffeomorphic to Y_2^-, so we suppose $Y_1^- = Y_2^- = Y^-$ in other words we suppose the smooth surgeries are the same. Denote $c(\xi_i)$ and $c(\xi_i^-)$ to be the contact invariant in $\widehat{HF}(-Y)$ and $\widehat{HF}(-Y^-)$ correspond to ξ_i and ξ_i^-, then by the hypothesis $c(\xi_1) \neq c(\xi_2)$ we may assume $c(\xi_1) \neq 0$. Let W be the cobordism from Y to Y^- induced by the smooth surgery on the knot, and let $\overline{W}$ be the opposite cobordism from $-Y^-$ to $-Y$. Legendrian surgery induces a map [6] $F_{\overline{W}} : \widehat{HF}(-Y^-) \to \widehat{HF}(-Y)$. Denote by $\mathfrak{t}_i$ the canonical *Spin*^{c} structure on $\overline{W}$ induced by the Weinstein structure coming from thinking of W as a Weinstein handle attached to K_i in ξ_i, respectively. By [1, Lemma 2.11], for any *Spin*^{c} structure $\mathfrak{s}$ on $\overline{W}$ we have

$$F_{\overline{W},s}(c(\xi_i^-)) = \begin{cases} c(\xi) & \text{if } s = t_i \\ 0 & \text{if } s \neq t_i. \end{cases} \tag{2.1}$$

We will argue by contradiction that $c(\xi_1^-) \neq c(\xi_2^-)$. Suppose $c(\xi_1^-) = c(\xi_2^-)$. Since smoothly the two Legendrian surgeries are the same surgery on the

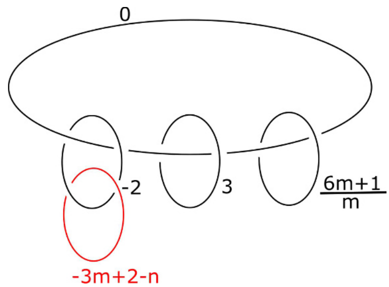

Fig. 1: The surgery diagram without red circle and red framing describe $M_m = -\Sigma(2,3,6m+1)$, and describe $M_m^n = M(0; \frac{3m+n-2}{6m+2n-5}, -\frac{1}{3}, -\frac{m}{6m+1})$ when we consider the red circle and red framing.

same knot, they induce the same map on Heegaard Floer homology. However there are two possibilities of the relation between their corresponding $Spin^c$ structures, either $\mathfrak{t}_1 = \mathfrak{t}_2$ or not.

If $\mathfrak{t}_1 = \mathfrak{t}_2$ then by equation (2.1) we have

$$c(\xi_1) = F_{\overline{W},\mathfrak{t}_1}(c(\xi_1^-)) = F_{\overline{W},\mathfrak{t}_2}(c(\xi_2^-)) = c(\xi_2)$$

which is a contradiction. On the other hand if $\mathfrak{t}_1 \neq \mathfrak{t}_2$ then again by (2.1) we have $c(\xi_1) = F_{\overline{W},\mathfrak{t}_1}(c(\xi_1^-)) = F_{\overline{W},\mathfrak{t}_1}(c(\xi_2^-)) = 0$ which is a contradiction again. Therefore $c(\xi_1^-) \neq c(\xi_2^-)$. $\square$

3. Construction of Seifert fiber spaces and contact structures

Now we want to use the Theorem 1.1 in a specific example, but before that we need to see how tight contact structures on those Seifert spaces mentioned in Theorem 1.3 come from the tight contact structures on $-\Sigma(2,3,6n+1)$ in more detail.

Consider the diagram of Figure 1 (without the red part) for $M_m = -\Sigma(2,3,6m+1)$. We may also describe M_m as the Seifert manifold $M\left(0; \frac{1}{2}, -\frac{1}{3}, -\frac{m}{6m+1}\right)$; in particular we can decompose it as

$$(\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (V_1 \cup V_2 \cup V_3)$$

where Σ is a pair of pants (i.e. 3 punctured sphere) and V_i's are solid torus neighborhood of the singular fibers F_i. The A_i are homeomorphisms from ∂V_i to $-\partial(\Sigma \times S^1)$. We also choose identifications on $V_i \cong D^2 \times S^1$, $\partial V_i \cong \mathbb{R}^2/\mathbb{Z}^2$ such that $(1,0)^T$ corresponds to meridian, and $-\partial(M_m \backslash V_i) \cong \mathbb{R}^2/\mathbb{Z}^2$ such that $(0,1)^T$ corresponds to S^1 fiber and $(1,0)^T$ corresponds to $-\partial(pt \times \Sigma)$. After we have determined the identifications we can represent the maps A_i using matrices as follows

$$A_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 6m+1 & 6 \\ m & 1 \end{pmatrix}.$$

We observe that the singular fiber F_1 is represented by the red circle in Figure 1, which is the meridian of the -2 framed circle in the figure. The Seifert space $M_m^n = M\big(0; \frac{3m+n-2}{6m+2n-5}, -\frac{1}{3}, -\frac{m}{6m+1}\big)$ is obtained by taking the surgery along the red circle with the diagram framing $-3m+2-n$. It's not hard to see that M_m^n is equivalent to the Seifert fibered 3-manifold $M\big(-2; \frac{3m+n-2}{6m+2n-5}, \frac{2}{3}, \frac{5m+1}{6m+1}\big)$ in Theorem 1.3 by manipulating the Seifert invariants. Next let's see how the framing on the red circle relates to the different contact structures on M_m.

One way to think about those different contact structures of M_m is by putting them into a triangle with m rows, where there are a-th many vertices at the a-th row (from top to bottom), and each vertex represents a contact structure on M_m. If a vertex is in the a-th row then the corresponding contact structure has the maximal twisting number of regular fiber equal to $-6(m-a)-1$ ([8, Claim 4.2]). Given those different maximal twisting numbers of regular fiber together with the attaching map we are able to determine the maximal twisting number of singular fiber corresponding to each row.

In particular if we put the V_1 into a standard contact neighborhood with boundary slope $\frac{1}{n_1}$, then the boundary slope on $-\partial(M_m \backslash V_1)$ is $s_1 = -\frac{n_1}{2n_1+1}$. If the maximal twisting number of regular fiber is $-6(m-a)-1$ we have $2n_1+1 = -6(m-a)-1$, so $n_1 = -3(m-a)-1$

To put this another way, we denote twisting number of F_1 to be $\mathrm{tw}(F_1) =$ contact framing – torus framing where the torus framing is determined by the longitude $(0,1)^T$ in V_1. Then on the a-th row of the triangle we have that the maximal twisting number of regular fiber is $-6(m-a)-1$ and the maximal twisting number of F_1 is $\mathrm{tw}(F_1) = -3(m-a)-1 = -3m+3a-1$.

Moreover, since $A_1(0,1)^T = (1,0)^T$ we can see that the framing of F_1 corresponding to the choice of V_1 exactly corresponds to the 0 framing of the meridian of -2 framed circle in the diagram. Hence if we start with a contact structure on M_m lying in the a-th row of the triangle, the $(-3m+2-n)$ framing on the red circle in the diagram corresponds to $(-3m+2-n)$ torus framed surgery, which by the twisting number calculation is the smooth coefficient corresponding to $(-3a+3-n)$ contact surgery (for example, starting with the contact structure the top row, we are considering contact $-n$ surgery). Observe that performing contact $(-3a+3-n)$ surgery involves $(3a-3+n)$ different choices of stabilization to become a Legendrian surgery. Thus each tight contact structure having twisting number $-6(m-a)-1$

gives rise to $(3a-3+n)$ (possibly) different Legendrian surgeries. There are a-th many different tight structures for each $1 \leq a \leq m$ so there are potentially a total of $\sum_{a=1}^{m}(n+3(a-1))a$ different tight contact structures on M_m^n given by Legendrian surgery on the singular fiber F_1. In summary, we have the following table.

tight contact structure on M_m	maximal twisting number of regular fiber	maximal twisting number of F_1	contact surgery coefficient	different ways of stabilizing contact surgery ξ
ξ_1^1	$-6(m-1)-1$	$-3m+3-1$	$-n$	n
$\xi_2^1 \quad \xi_2^2$	$-6(m-2)-1$	$-3m+6-1$	$-n-3$	$n+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\xi_m^1 \quad \xi_m^2 \ldots \xi_m^m$	-1	-1	$-n-3(m-1)$	$n+3(m-1)$

To show all of those resulting contact structure on M_m^n are all different we need Theorem 1.1 and the following Lemma.

LEMMA 3.1 (J. Simone [7]). *If (Y,ξ) is weakly symplectically fillable and $(W.J_i)$ is a Stein cobordism from (Y,ξ) to (Y',ξ_i) for $i=1,2$ such that the $Spin^c$ structures induced by J_1 and J_2 are not isomorphic, then ξ_1 and ξ_2 are non-isotopic tight contact structures.*

Now we are able to show the following proposition.

PROPOSITION 3.2. *All tight contact structures on M_m^n mentioned above are non-isotopic and strongly fillable.*

PROOF. First by Theorem 1.2 all tight contact structures on M_m are strongly fillable. Since different stabilization have different rotation number which implies the $Spin^c$ structures induced by Legendrian surgeries are different, it follows from Lemma 3.1 that for any contact structure ξ_i^j if we choose different stabilization, Legendrian surgery will result in different contact structures. Moreover, since Legendrian surgery preserves strong fillability, the contact structures we obtained are all strongly fillable.

Furthermore, for two different contact structures ξ_i^j and ξ_p^q their contact invariants are different and non-zero by Theorem 1.2, and by construction we are considering the same smooth surgery on F_1. By Theorem 1.1 any pair of Legendrian surgeries on smoothly isotopic knots in ξ_i^j and ξ_p^q will give different contact structure on M_m^n. Therefore none of the tight contact structures on M_m^n mentioned above are isotopic. $\square$

4. Upper bound

Our way to determine an upper bound on the number of distinct tight contact structures is a typical application of convex surface theory.

PROPOSITION 4.1. *M_m^n has at least $\sum_{a=1}^{m}(3a-3+n)a$ different tight contact structure when $1 \le m$ and $0 \le n < 18m+3$*

PROOF. The proof is presented assuming the reader is familiar with convex surface theory [4]. The argument is parallel to the ones in [2],[3] and [8].

We first rewrite the Seifert notation for M_m^n as $M\big(0; -\frac{1}{3}, -\frac{m}{6m+1}, \frac{3m+n-2}{6m+2n-5}\big)$ (we exchange the fiber order here). As before we can describe M_m^n as $M_m^n = (\Sigma \times S^1) \cup_{A_1 \cup A_2 \cup A_3} (V_1 \cup V_2 \cup V_3)$ where Σ is a pair of pants (i.e. 3 punctured sphere) and V_i's are solid torus neighborhood of the singular fibers F_i. Again the A_i are maps from ∂V_i to $-\partial(\Sigma \times S^1)$. We again choose identifications on $V_i \cong D^2 \times S^1$, $\partial V_i \cong \mathbb{R}^2/\mathbb{Z}^2$ such that $(1,0)^T$ correspond to the meridian, and $-\partial M_m^n \backslash V_i \cong \mathbb{R}^2/\mathbb{Z}^2$ such that $(0,1)^T$ corresponds to the S^1 fiber and $(1,0)^T$ corresponds to the $-\partial(pt \times \Sigma)$. Then we can choose

$$A_1 = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6m+1 & 6 \\ m & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 6m+2n-5 & 2 \\ -3m-n+2 & -1 \end{pmatrix},$$

we also have the inverse of the A_i

$$A_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}, \; A_2^{-1} = \begin{pmatrix} 1 & -6 \\ -m & 6m+1 \end{pmatrix}, \; A_3^{-1} = \begin{pmatrix} -1 & -2 \\ 3m+n-2 & 6m+2n-5 \end{pmatrix}.$$

We would like to first determine the maximal twisting number of tight contact structure on M_m^n.

CLAIM 4.2. *If ξ is a tight contact structure on M_m^n then* $\mathrm{tw}(\xi) \ge -6m+5$.

PROOF. As a standard starting point of arguments like this, we begin with very negative twisting number n_i on V_i so the boundary slope on V_i are $\frac{1}{n_i}$ and when measured on $-\partial(M_m^n \backslash V_i)$ it becomes

$$s_1 = \frac{n_1}{3n_1-1}, \quad s_2 = \frac{mn_2+1}{(6m+1)n_2+6}, \quad s_3 = \frac{(-3m-n+2)n_3-1}{(6m+2n-5)n_3+2}$$

Note that when $3n_1 - 1 \ne (6m+1)n_2 + 6$, and taking the Legendrian ruling slope on each $\partial(M_m^n \backslash V_i)$ to be infinite using Giroux flexibility, if we consider the vertical annulus $\mathcal{A}$ whose Legendrian boundaries are Legendrian rulings on $\partial(M_m^n \backslash V_1)$ and $\partial(M_m^n \backslash V_2)$, the Imbalance Principle [4] implies the existence of a bypass on one side or the other of $\mathcal{A}$—and then the Twist Number Lemma [4] allows us to increase the the corresponding twisting number by one. Iterating the procedure, so long as the assumption $3n_1 - 1 \ne (6m+1)n_2 + 6$ continues to hold then the Twist Number Lemma will apply to allow us to increase n_1 up to $-2m+2$ and n_2 to -1. If this happens, then at this point $3n_1 - 1 = (6m+1)n_2 + 6 = -6m+5$. Hence if we let $\mathrm{tw}(\xi) = t$ we need to show that if we increase n_1 and n_2 to a point at which $3n_1 - 1 = (6m + 1)n_2 + 6 = t$, then $t \geq -6m + 5$.

Assume $3n_1 - 1 = (6m + 1)n_2 + 6$ and write this common value as t. Observe that this implies that there is some integer k with $n_1 = -(6m+1)k - (2m-2)$, and $n_2 = -3k - 1$, and $t = -3(6m + 1)k - (6m - 5)$. Since $n_1, n_2 < 0$, we have $k \geq 0$. If $k = 0$ then we are done, so we suppose $k \geq 1$. We consider the annulus $\mathcal{A}$ again: since we assume that we have achieved the twisting number, the dividing set $\mathcal{A}$ consists of parallel arcs connecting the boundary components of $\mathcal{A}$. If we cut along $\mathcal{A}$ and round the edge we get a smooth manifold $M^n_m \backslash (V_1 \cup V_2 \cup \mathcal{A})$ such that the boundary is smoothly isotopic to $\partial(M^n_m \backslash V_3)$. Moreover, by the Edge Rounding lemma [4] the slope is

$$s_{\partial(M^n_m \backslash (V_1 \cup V_2 \cup \mathcal{A}))} = \frac{n_1}{3n_1 - 1} + \frac{mn_2 + 1}{(6m + 1)n_2 + 6} - \frac{1}{3n_1 - 1}$$
$$= \frac{(9m + 1)k + (3m - 2)}{3(6m + 1)k + (6m - 5)}.$$

By applying A_3^{-1} to the corresponding vector, we find that when measured in ∂V_3 the slope is

$$s_{\partial V_3} = -\frac{(12m + n - 1)k - n}{k - 1} = -(12m + n - 1) - \frac{12m - 1}{k - 1}.$$

Now if $k = 1$, we have $t = -3(6m + 1) - (6m - 5) = -24m + 2$. Since $1 \leq m$, we see $t \leq -22$. On the other hand by the above calculation, when $k = 1$ the slope on ∂V_3 is infinite and is $-\frac{1}{2}$ on $-\partial(M^n_m \backslash V_3)$ which implies the maximal twisting number is at least -2. This contradicts the maximality of t.

If $k \geq 2$, we have $t \leq -6(6m + 1) - (6m - 5) = -42m - 1$. Using the above slope again and since $1 \leq m$ and $1 \leq n$, we have $s_{\partial V_3} < -12$. Moreover, since we start with boundary slope $\frac{1}{n_3}$ with n_3 very negative, by using [4, Corollary 4.8] we can find a torus (between a small standard torus around F_3 and the torus obtained above by cutting and rounding) with slope -1. This is $\frac{3m+n-3}{-6m-2n+7}$ when measured on $-\partial(M^n_m \backslash V_3)$. However if $n < 18m + 4$ we have $t \leq -42m - 1 < -6m - 2n + 7$ contradicting the maximality of t. □

It was proved by Wu [9] for Seifert space with $e_0 = -2$ the maximal twisting number is less than 0. So we left to analyze the situation when $-6m + 5 \leq \mathrm{tw}(\xi) \leq -1$

We have already arranged that $n_2 = -1$ and $n_1 = -2m + 2$. If there is no bypass on either V_1 or V_2 we are in the above situation where $k = 0$, i.e. $\mathrm{tw}(\xi) = -6m + 5$. Since V_1, V_2 are standard neighborhoods there is a unique tight contact structure on V_1 and V_2. Moreover, the tight contact structure on neighborhood of $\mathcal{A}$ is uniquely determined by the dividing set which are just horizontal arcs. Finally, the slope on ∂V_3 is $-n$ (strictly, this is the slope on the boundary of $M_m^n \backslash (V_1 \cup V_2 \cup \mathcal{A})$, viewed from V_3). By the classification of tight contact structures on solid torus (Theorem 2.3 [4]) there are exactly n different tight contact structure on V_3. Hence together there are at most n different tight contact structure on M_m^n when $\mathrm{tw}(\xi) = -6m + 5$.

If there exists a bypass on V_1 by the Twist Number Lemma we can increase n_1 to $-2m + 3$, and then consider the dividing set on $\mathcal{A}$. Again by the Imbalance Principle there exists a bypass on $-\partial(M_m^n \backslash V_2)$ with ruling slope ∞. Since $s_2 = \frac{-m+1}{-6m-5} = \frac{m-1}{6m-5}$, after attaching bypass and applying [4, Lemma 3.15] the new boundary slope become $s_2 = \frac{(m-1)-1}{6(m-1)-5} = \frac{m-2}{6m-11}$. Using the Imbalance Principle and the Twist Number Lemma again we increase n_1 to $-2m + 4$. (Same situation happens if there exists a bypass on V_2)

If there is no further bypass on V_1 or V_2 we are in the situation where $\mathrm{tw}(\xi) = -6(m - 1) + 5$. We cut and round edges to get

$$s_{\partial(M_m^n \backslash (V_1 \cup V_2 \cup \mathcal{A}))} = \frac{n_1}{3n_1 - 1} + \frac{(m-1)-1}{-(6m-1)+5} - \frac{1}{3n_1 - 1} = \frac{3m-5}{6m-11}$$

and again by applying A_3^{-1} we get the slope $-3 - n$ on ∂V_3. Moreover, since $s_2 = \frac{m-2}{6m-11}$ by applying A_2^{-1} we got slope -2 on ∂V_2. Therefore since there is 1 tight contact structure on V_1 and $\mathcal{A}$, 2 tight contact structure on V_2 and $3 + n$ on V_3 there are at most $2 \times (3 + n)$ tight contact structure on M_m^n with $\mathrm{tw}(\xi) = -6m + 11$.

Inductively we can do this until $n_1 = 0$. i.e. we have $(m - 1)$ times of choosing the existence of a bypass on V_1 (or V_2). Suppose we can do it till the l-th time, which means that at the l-th step we increase n_1 from $-2m + 2l$ to $-2m + 2l + 2$, the slope on s_2 changes from $\frac{(m-(l-1))-1}{6(m-(l-1))-5}$ to $\frac{(m-l)-1}{6(m-l)-5}$ and the maximal twisting number is $\mathrm{tw}(\xi) = -6(m - l) + 5$. Cut, round, and apply A_3^{-1} again: we find slope $-3l - 1 - n$ on ∂V_3. Applying A_2^{-1} gives slope $-l - 1$ on ∂V_2. Hence there are at most $(3l + 1 + n)(l + 1)$ tight contact structures on M_m^n with $\mathrm{tw}(\xi) = -6(m - l) + 5$.

In summary we have the following table:

$\mathrm{tw}(\xi)$	n_1	slope on ∂V_3	Slope on ∂V_2
$-6m+5$	$-2m+2$	$-n$	-1
$-6(m-1)+5$	$-2m+4$	$-n-3$	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
-1	0	$-n-3(m-1)$	$-m$

Therefore in total we have at most $\sum_{a=1}^{m}(3a-3+n)a = \frac{(2m+n-2)(m+1)m}{2}$ for $1 \leq n < 18m+4$. $\square$

An exactly parallel argument works for Theorem 1.4; we just do the surgery along the singular fiber F_2 in $-\Sigma(2,3,6m+1)$ instead of F_1. We obtain another family of Seifert fiber spaces as stated in the Theorem, and analogous proofs follow for determining the contact structures.

Remark 4.3. The above types of argument could potentially also be used for Legendrian surgery on the third singular fiber, and other 3-manifolds with contact structures having different contact invariants (For example $-\Sigma(2,3,6n-1)$.) However the framing calculation and the convex surface argument is more subtle, so the author chooses to not present them here.

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